1 Lecture 3

Defn 2·8: A ring R is called a discrete valuation ring (DVR) if it is a PID with exactly one non-zero prime ideal (necessarily maximal).

Lemma 2·9: (i) Let v be a discrete valuation on K. Then $\mathcal{O}_K$ is a DVR.

(ii) Let R be a DVR. Then there exists a valuation v on $K := \mathrm{Frac}(R)$ s.t. $R = \mathcal{O}_K$.

Proof: (i) $\mathcal{O}_K$ is a PID by Lemma 2·6.

Let $0 \neq I \subseteq \mathcal{O}_K$ an ideal, then $I = (x)$.

If $x = \pi^n u$ for π a uniformizer. Then (x) is prime iff $n = 1$ and $I = (\pi) = \mathfrak{m}$.

(ii) Let R be a DVR with maximal ideal $\mathfrak{m}$.

Then $\mathfrak{m} = (\pi)$ some $\pi \in R$. By unique factorization of PID's, we may write any $x \in R \setminus \{0\}$ uniquely as $\pi^n u$, $n \geq 0$, $u \in R^\times$.

2 Then any $y \in K \setminus \{0\}$ can be written uniquely as $\pi^m u$, $u \in R^\times$, $m \in \mathbb{Z}$.

Define $v(\pi^m u) = m$: easy to check v

is a valuation and $\mathcal{O}_K = R$.

Examples: $\cdot\ \mathbb{Z}_{(p)}$, $k[[t]]$ are DVR's (k field).

§3 The p-adic numbers (p prime)

Recall: $\mathbb{Q}_p$ completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$

Ex sheet 1: $\mathbb{Q}_p$ is a field.

$|\cdot|_p$ extends to $\mathbb{Q}_p$ and the associated valuation is discrete. $\rightsquigarrow (\mathbb{Q}_p, |\cdot|_p)$ is a complete discretely valued field

Definition 3.1: The ring of p-adic integers $\mathbb{Z}_p$ is the valuation ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$

Facts: $\cdot\ \mathbb{Z}_p$ is a DVR, max. ideal $p\mathbb{Z}_p$, non-zero ideals are given by $p^n\mathbb{Z}_p$.

Proposition 3.2: $\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ inside $\mathbb{Q}_p$.

In particular $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ w.r.t. $|\cdot|_p$

Proof: Need to show $\mathbb{Z}$ dense in $\mathbb{Z}_p$.

$\mathbb{Q}$ is dense in $\mathbb{Q}_p$. Since $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ is open (closed balls are open), $\mathbb{Z}_p \cap \mathbb{Q}$ is dense in $\mathbb{Z}_p$.

$$\mathbb{Z}_p \cap \mathbb{Q} = \{x \in \mathbb{Q} \mid |x|_p \leq 1\} = \{\tfrac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$$

$$= \mathbb{Z}_{(p)}$$

localization at (p).

Thus suffices to show $\mathbb{Z}$ dense in $\mathbb{Z}_{(p)}$.

Let $\frac{a}{b} \in \mathbb{Z}_{(p)}$, $a, b \in \mathbb{Z}$, $p \nmid b$.

For $n \in \mathbb{N}$, choose $y_n \in \mathbb{Z}$ s.t. $b y_n \equiv a \mod p^n$.

Then $y_n \to \frac{a}{b}$ as $n \to \infty$

In particular part: $\mathbb{Z}_p$ complete and $\mathbb{Z} \subseteq \mathbb{Z}_p$ dense.

§ Inverse limits

Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets / groups / rings together with homomorphisms $\varphi_n : A_{n+1} \to A_n$. (transition maps.

The **inverse limit** of $(A_n)_{n=1}^{\infty}$ is the set / group / ring

$$\begin{array}{ccccc} A_{n+1} & \to & A_n & \to & A_{n-1} \\ a_{n+1} & \mapsto & a_n & \mapsto & a_{n-1} \end{array}$$

$$\varprojlim_n A_n = \{ (a_n) \in A_n \mid \varphi_n(a_{n+1}) = a_n \} \subseteq \prod_{n=1}^{\infty} A_n$$

4 Fact: If A_n is a group / ring $\forall n$, then $\varprojlim_n A_n$ is a group / ring. Define group / ring operations componentwise

Let $\theta_m : \varprojlim_n A_n \to A_m$ denote the natural projection.

The inverse limit satisfies the following **universal property.**

Proposition 3.3: For any set / group / ring B together with homomorphisms $\psi_n : B \to A_n$ such that

$$\begin{tikzcd} B \arrow[r, "\psi_{n+1}"] \arrow[dr, "\psi_n"'] & A_{n+1} \arrow[d, "\varphi_n"] \\ & A_n \end{tikzcd}$$

commutes $\forall n$.

Then there exists a unique homomorphism $\psi : B \to \varprojlim_n A_n$ s.t. $\theta_n \circ \psi = \psi_n$.

Proof: Define $\Psi: B \to \prod_{n=1}^{\infty} A_n$ by

$$b \mapsto \prod_{n=1}^{\infty} \psi_n(b).$$

Then $\psi_n = \varphi_n \circ \psi_{n+1} \Rightarrow \Psi(b) \in \varprojlim_n A_n$.

The map is clearly unique (determined by $\psi_n = \varphi_n \circ \psi_{n+1}$) and is a homomorphism (of sets/groups/rings). $\square$

Defn 3.4: Let $I \subseteq R$ be an ideal. The I-adic completion of R is the ring

$$\hat{R} := \varprojlim_n R/I^n$$

where $R/I^{n+1} \to R/I^n$ is the natural projection.

Note there is a natural map $i: R \to \hat{R}$ by universal property ($\exists$ maps $R \to R/I^n$). We say R is I-adically complete if i is an iso.

Fact: $\ker(i: R \to \hat{R}) = \bigcap_{n=1}^{\infty} I^n$.

Let $(K, |\cdot|)$ be a non-archimedean valued field and $\pi \in \mathcal{O}_K$ s.t. $|\pi| < 1$.

Proposition 3.5: Assume K is complete w.r.t. $|\cdot|$.

(i) Then $\mathcal{O}_K \cong \varprojlim_n \mathcal{O}_K/\pi^n \mathcal{O}_K$. ($\mathcal{O}_K$ π-adically complete)

(ii) Every $x \in \mathcal{O}_K$ can be written uniquely as $x = \sum_{i=0}^{\infty} a_i \pi^i$, $a_i \in A$, where $A \subseteq \mathcal{O}_K$ is a set of coset representatives for $\mathcal{O}_K/\pi\mathcal{O}_K$. Moreover any power series $\sum_{i=0}^{\infty} a_i \pi^i$, $a_i \in A$ converges.

Proof: (i) K complete $+ \mathcal{O}_K$ closed $\Rightarrow \mathcal{O}_K$ complete.

$x \in \bigcap_{n=1}^{\infty} \pi^n \mathcal{O}_K \Rightarrow v(x) \geq n v(\pi) \ \forall n \Rightarrow x = 0$

hence $\mathcal{O}_K \to \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$ is injective.

Let $(x_n)_{n=1}^{\infty} \in \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$ and for each n, let $y_n \in \mathcal{O}_K$ be a lifting of $x_n \in \mathcal{O}_K/\pi^n\mathcal{O}_K$. Let v be the valuation on K normalized so that $v(\pi)=1$. Then $y_n - y_{n+1} \in \pi^n \mathcal{O}_K$ so that $v(y_n - y_{n+1}) \geq n$. Thus $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{O}_K$; Let $y_n \to y \in \mathcal{O}_K$. Then y maps to $(x_n)_{n=1}^{\infty}$ in $\varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$. Thus $\mathcal{O}_K \to \varprojlim_n \mathcal{O}_K/\pi^n\mathcal{O}_K$ is surjective.

(ii) Ex sheet □

Warning: If $(K, |\cdot|)$ not discretely valued, $\mathcal{O}_K$ not nec. m-adically complete.

Corollary 3.6: K is as in part (ii) of Prop 3.5, then every $x \in K$ can be written uniquely as $\sum_{i=n}^{\infty} a_i \pi^i$, $a_i \in A$. Conversely, any such expression $\sum_{i=n}^{\infty} a_i \pi^i$ defines an element of K.

Proof: Apply 3.5(ii) to $\pi^{-n}x$, where $n \in \mathbb{Z}$ s.t. $\pi^{-n}x \in \mathcal{O}_K$

Corollary 3.7: (i) $\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

(ii) Every element $x \in \mathbb{Q}_p$ can be written uniquely as $\sum_{i=n}^{\infty} a_i p^i$, $a_i \in \{0, 1, \dots, p-1\}$.

Proof: (i) It suffices by Prop. 3·5 to show that $\mathbb{Z}_p / p^n \mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$.

Let $f_n : \mathbb{Z} \to \mathbb{Z}_p / p^n \mathbb{Z}_p$ be the natural map.

We have
$$\ker(f_n) = \{x \in \mathbb{Z} \mid |x|_p \leq p^{-n}\}$$
$$= p^n \mathbb{Z}.$$

Thus $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p / p^n \mathbb{Z}_p$ is injective.

Let $\bar{c} \in \mathbb{Z}_p / p^n \mathbb{Z}_p$ and $c \in \mathbb{Z}_p$ a lift.

Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, $\exists x \in \mathbb{Z}$ s.t.

$x \in c + p^n \mathbb{Z}_p \leftarrow$ open in $\mathbb{Z}_p$.

Then $f_n(x) = \bar{c}$

$\Rightarrow \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p / p^n \mathbb{Z}_p$ is surjective.

(ii) Follows directly from Prop. 3·5(ii) using $\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{F}_p$. $\square$

Eg. $\frac{1}{1-p} = 1 + p + p^2 + p^3 + \dots$ in $\mathbb{Q}_p$.

Remark: Prop 3·5 implies $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$ both in bijection with
$$\{(a_i)_{i=-\infty}^{\infty} \mid a_i \in \{0, \dots, p-1\},\ a_i = 0 \text{ for } i \ll -\infty\}$$
- ring structures very different.