

D-Modules, Unit F -Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

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1 Questions

Question 1.1. What is the point of D-modules? Why do people care about them? What sort of questions do they answer? What insights do they provide?

2 Some Non-Commutative Algebra

\mathcal{D} -modules requires non-commutative algebra. Necessary facts are found here.

2.1 Filtered rings and modules

This subsection follows Ginzburg's notes quite closely, see [BIBTEX SETUP, GINZBURG D-MODULES Page 3].

Definition 2.1 (*Filtered Ring*). Let A be an associative ring with unit. We call A a *filtered ring* if an increasing filtration $\dots \subset A_i \subset A_{i+1} \subset \dots$ by additive subgroups is given such that

$$(i) \ A_i A_j \subset A_{i+j}$$

$$(ii) \ 1 \in A_0,$$

$$(iii) \ \bigcup A_i = A, \text{ i.e. the filtration is } \textit{exhausting}.$$

Typically, either (a) \mathbb{N} or (b) \mathbb{Z} is chosen for the index set. In the former case A is said to be *positively filtered*. Note that (a) can be viewed as a special case of (b) by setting $A_{-1} = 0$. In the latter case we will consider the topology induced by the filtration by taking $\{A_i\}_{i \in \mathbb{Z}}$ to be the base of open sets, and we then impose two additional conditions:

$$(iv) \ \bigcap A_i = \{0\}, \text{ i.e. the topology defined by } \{A_i\} \text{ is } \textit{separating}$$

1. A is complete with respect to this topology.

Finally, we denote by $\text{gr}A$ the associated graded ring $\bigoplus A_i/A_{i-1}$.

3 Differential Operators and D-Modules

Definition 3.1 (Quasi-coherent #1). Fix X a scheme over k , \mathcal{O}_X the structure sheaf, \mathcal{F} a sheaf of \mathcal{O}_X -modules. We call \mathcal{F} a *quasi-coherent* sheaf of \mathcal{O}_X -modules (or simply an \mathcal{O}_X -modules) if it satisfies the condition

$$\text{If } U \subseteq X \text{ an open affine, } f \in \mathcal{O}_X(U), \text{ and } U_f = \{u \in U \mid f(u) \neq 0\},$$

then $\mathcal{F}(U_f) = \mathcal{F}(U)_f = \mathcal{O}_X(U_f) \otimes_{\mathcal{O}_X(U)} \mathcal{F}$.

Definition 3.2 (Quasi-coherent #2). Let (X, \mathcal{O}_X) be a scheme. A sheaf of \mathcal{O}_X -modules \mathcal{F} is quasi-coherent if X can be covered by affine opens $U_i = \text{Spec } A_i$ such that for each i there exists an A_i module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. We say \mathcal{F}_i is coherent if each M_i can be taken to be finitely generated.

Remark 3.3. If A is a ring and M an A -module, the sheaf associated to M is denoted by \tilde{M} and is formed as follows. For each $\mathfrak{p} \in \text{Spec } A$, $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$ is the localization with respect to \mathfrak{p} . Given an open set $U \subseteq \text{Spec } A$, define

$$\tilde{M}(U) = \left\{ s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \mid s(\mathfrak{p}) \in M_{\mathfrak{p}}, \text{ and locally } s = \frac{m}{f}, m \in M, f \in A \right\}.$$

More verbosely, this last condition means that for each $\mathfrak{p} \in U$ there is a neighborhood $V \subseteq U$ of \mathfrak{p} such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_{\mathfrak{q}}$.

Alternatively, one may define

$$\tilde{M}(U_f) = M_f,$$

and then

$$\tilde{M}(U) = \varinjlim_{U_f \subseteq U} \tilde{M}(U_f).$$

Note that U_f is implied to be a distinguished open in one of the U_i , so really we need to take the limit above over all U_f in all U_i which intersect U nontrivially. This is a non-issue if U is affine.

Lemma 3.4. The following are equivalent conditions for \mathcal{F} a sheaf of \mathcal{O}_X modules:

- (a) \mathcal{F} is the direct limit of its coherent subschemes
- (b) For any Zariski open affine subset $U \subseteq X$ and any $f \in \mathcal{O}(U)$ one has $\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$.

A *quasi-coherent* sheaf is then one which satisfies these conditions.

Lemma 3.5 (Noether Normalization Lemma). Let k be a field, A a finitely generated k -algebra. Then there exists algebraically independent elements y_1, \dots, y_d in A for some positive d such that A is finitely generated as a module over $k[y_1, \dots, y_n]$.

Remark 3.6. The Noether normalization lemma provides a way to define differential operators using a manifold-esque coordinate approach. I prefer the following coordinate-free approach provided by Gröthendieck, however.

Definition 3.7 (Differential Operators). Let A be a commutative ring. For any pair of A -modules M, N we define the module $\mathcal{D}iff_A^k(M, N)$ inductively as follows:

$$(i) \mathcal{D}iff_A^0(M, N) = \text{Hom}_A(M, N)$$

$$(ii) \mathcal{D}iff_A^{k+1}(M, N) = \left\{ \text{additive maps } u : M \rightarrow N \mid \forall a \in A, (au - ua) \in \mathcal{D}iff_A^k(M, N) \right\}$$

It follows from the definition that $\mathcal{D}iff_A^k(M, N) \subset \mathcal{D}iff_A^{k+1}(M, N)$. We define

$$\mathcal{D}iff_A(M, N) := \bigcup_k \mathcal{D}iff_A^k(M, N).$$

In the case that $M = N$, we write $\mathcal{D}iff_A(M)$ and note that it is a filtered almost commutative ring.

4 D-Modules

Definition 4.1. A \mathcal{D} -module is a sheaf over the sheaf \mathcal{D}_X of regular differential operators over a variety (scheme, manifold, analytic complex manifold) which is quasi-coherent as an \mathcal{O}_X -module.