D-Modules, Unit F-Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

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Contents

1 Some Non-Commutative Algebra

$\mathcal{D}$-modules requires non-commutative algebra. Necessary facts are found here.

1.1 Filtered rings and modules

This subsection follows Ginzburg's notes quite closely, see [BIBTEX SETUP, GINZBURG D-MODULES Page 3].

Definition 1.1 (*Filtered Ring*)**.** Let A be an associative ring with unit. We call A a *filtered ring* if an increasing filtration $... \subset A_i \subset A_{i+1} \subset ...$ by additive subgroups is given such that

(i) $A_iA_j \subset A_{ij}$

(ii) $1 \in A_0$,

(iii) $\bigcup A_i = A$, i.e. the filtration is *exhausting*.

Typically, either (a) $\mathbb{N}$ or (b) $\mathbb{Z}$ is chosen for the index set. In the former case A is said to be *positively filtered*. Note that (a) can be viewed as a special case of (b) by setting $A_{-1} = 0$. In the latter case we will consider the topology induced by the filtration by taking $\{A_i\}_{i\in\mathbb{Z}}$ to be the base of open sets, and we then impose two additional conditions:

(iv) $\bigcap A_i = \{0\}$, i.e. the topology defined by $\{A_i\}$ is *separating*

1. A is complete with respect to this topology.

Finally, we denote by $\mathrm{gr} A$ the associated graded ring $\bigoplus A_i/A_{i-1}$.

2 Differential Operators and D-Modules

Definition 2.1 (Quasi-coherent #1)**.** Fix X a scheme over k, $\mathcal{O}_X$ the structure sheaf, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We call $\mathcal{F}$ a *quasi-coherent* sheaf of $\mathcal{O}_X$-modules (or simply an $\mathcal{O}_X$-modules) if it satisfies the condition

$$\text{If } U \subseteq X \text{ an open affine, } f \in \mathcal{O}_X(U), \text{ and } U_f = \{u \in U \mid f(u) \neq 0\},$$

then $\mathcal{F}(U_f) = \mathcal{F}(U)_f = \mathcal{O}_X(U_f) \otimes_{\mathcal{O}_X(U)} \mathcal{F}$.

Definition 2.2 (Quasi-coherent #2)**.** Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if X can be covered by affine opens $U_i = \operatorname{Spec} A_i$ such that for each i there exists an A_i module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. We say $\mathcal{F}_i$ is coherent if each M_i can be taken to be finitely generated.

Remark 2.3. If A is a ring and M an A-module, the sheaf associated to M is denoted by $\tilde{M}$ and is formed as follows. For each $\mathfrak{p} \in \operatorname{Spec} A$, $M_\mathfrak{p} = A_\mathfrak{p} \otimes_A M$ is the localization with respect to $\mathfrak{p}$. Given an open set $U \subseteq \operatorname{Spec} A$, define

$$\tilde{M}(U) = \left\{ s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} M_\mathfrak{p} \;\middle|\; s(\mathfrak{p}) \in M_\mathfrak{p}, \text{ and locally } s = \frac{m}{f}, m \in M, f \in A \right\}.$$

More verbosely, this last condition means that for each $\mathfrak{p} \in U$ there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_\mathfrak{q}$.

Alternatively, one may define

$$\tilde{M}(U_f) = M_f,$$

and then

$$\tilde{M}(U) = \varinjlim_{U_f \subseteq U} \tilde{M}(U_f).$$

Note that U_f is implied to be a distinguished open in one of the U_i, so really we need to take the limit above over all U_f in all U_i which intersect U nontrivially. This is a non-issue if U is affine.

Lemma 2.4. The following are equivalent conditions for $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules:

(a) $\mathcal{F}$ is the direct limit of its coherent subschemes

(b) For any Zariski open affine subset $U \subseteq X$ and any $f \in \mathcal{O}(U)$ one has $\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$.

A *quasi-coherent* sheaf is then one which satisfies these conditions.

Lemma 2.5 (Noether Normalization Lemma)**.** Let k be a field, A a finitely generated k-algebra. Then there exists algebraically independent elements $y_1, \ldots, y_d$ in A for some positive d such that A is finitely generated as a module over $k[y_1, \ldots, y_n]$.

Remark 2.6. The Noether normalization lemma provides a way to define differential operators using a manifold-esque coordinate approach. I prefer the following coordinate-free approach provided by Gröthendieck, however.

Definition 2.7 (Differential Operators)**.** Let A be a commutative ring. For any pair of A-modules M, N we define the module $\operatorname{Diff}_A^k(M, N)$ inductively as follows:

(i) $\mathrm{Diff}_A^0(M,N) = \mathrm{Hom}_A(M,N)$

(ii) $\mathrm{Diff}_A^{k+1}(M,N) = \left\{ \text{ additive maps } u : M \longrightarrow N \;\middle|\; \forall a \in A, (au - ua) \in \mathrm{Diff}_A^k(M,N) \right\}$

It follows from the definition that $\mathrm{Diff}_A^k(M,N) \subset \mathrm{Diff}_A^{k+1}(M,N)$. We define

$$\mathrm{Diff}_A(M,N) := \bigcup_k \mathrm{Diff}_A^k(M,N),$$

and it turns out that it is a filtered almost commutative ring.