Lecture 23

K non-arch. local. π unif. $|k| = q$.

Definition 20.2: $f(X)$ Lubin-Tate series for π.

F_f Lubin-Tate formal gp.

The π^n-torsion group is

$$\mu_{f,n} = \{ x \in \bar{m} \mid \pi^n \cdot_{F_f} x = 0 \}$$
$$= \{ x \in \bar{m} \mid f_n(x) = \underbrace{f \circ \dots \circ f}_{n \text{ times}}(x) = 0 \}$$

Facts: • $\mu_{f,n}$ is an $\mathcal{O}_K$-module.

• $\mu_{f,n} \subseteq \mu_{f,n+1}$ $\forall n$.

Eg. $K = \mathbb{Q}_p$, $f(X) = (X+1)^p - 1$

$$[p^n]_{F_f}(X) = \underbrace{f \circ \dots \circ f}_{n \text{ times}} = (X+1)^{p^n} - 1 \quad (\text{Eg. induct on } n)$$

Thus $\mu_{f,n} = \{ \zeta_{p^n}^i - 1 \mid i = 0, 1, \dots, p^n - 1 \}$.

Now let $f(X) = \pi X + X^q$ Lubin-Tate series

Then $f_n(X) = f \circ f_{n-1}(X)$

$$= f_{n-1}(X)(\pi + f_{n-1}(X)^{q-1})$$

Set $h_n(X) := \dfrac{f_n(X)}{f_{n-1}(X)} = (\pi + f_{n-1}(X)^{q-1})$. $(f_0(X$

Proposition 20.3:

(i) $h_n(X)$ is a separable Eisenstein polynomial of degree $q^{n-1}(q-1)$.

(ii) $\mu_{f,n}$ is a free $\mathcal{O}_K/\pi^n\mathcal{O}_K$ - module of rank 1.

Proof: (i) $h_1(X) = \pi + X^{q-1}$

Clear that $h_n(X)$ monic of degree $q^{n-1}(q-1)$

$f(X) \equiv X^q \mod \pi \Rightarrow f_{n-1}(X)^{q-1} \equiv X^{q^{n-1}(q-1)} \mod \pi$.

Since $f_{n-1}(X)$ has 0 constant term

$h_n(X) = \pi + f_{n-1}(X)^{q-1}$ has constant term π.

Thus $h_n(X)$ is Eisenstein.

Since $h_n(X)$ irreducible, $h_n(X)$ is separable if

char $K = 0$ or if char $K = p$ and $h_n'(X) \neq 0$.

Assume char $K = p$, induct on n.

$h_1(X) = \pi + X^{q-1}$ is separable.

Suppose $h_{n-1}(X), \ldots, h_1(X)$ are separable.

Then $f_{n-1}(X) = h_{n-1}(X) \cdots h_1(X) X$ is separable.

(product of irred. polynomials of different degrees)

$h_n(X) = (\pi + f_{n-1}(X)^{q-1})$

$$\Rightarrow h_n'(X) = \underbrace{(q-1)}_{\neq 0} \underbrace{f_{n-1}'(X)}_{\neq 0} \underbrace{f_{n-1}(X)^{q-2}}_{\neq 0}$$

$\Rightarrow h_n(X)$ is separable.

ii) α a root of $h_n(X)$. Since $h_n(X)$, $f_{n-1}(X)$

are coprime, $\alpha \in \mu_{f,n} \setminus \mu_{f,n-1}$. Then the map

$$\tilde{\varphi} : \mathcal{O}_K \to \mu_{f,n}$$
$$a \mapsto a \cdot_{F_f} \alpha.$$

is an $\mathcal{O}_K$-module homomorphism with $\pi^n \mathcal{O}_K \subseteq \ker \tilde{\varphi}$

As $\alpha \in \mu_{f,n} \setminus \mu_{f,n-1}$, $\pi^{n-1} \cdot_{F_f} \alpha \neq 0$

thus $\pi^n \mathcal{O}_K = \ker \tilde{\varphi}$.

Thus $\tilde{\varphi}$ induces an injection

$$\varphi : \mathcal{O}_K / \pi^n \mathcal{O}_K \to \mu_{f,n}.$$

Since $f_n(X)$ is separable.

$$|\mu_{f,n}| = \deg f_n(X) = q^n = |\mathcal{O}_K / \pi^n \mathcal{O}_K|$$

Thus φ iso. by counting. $\square$

Proposition 20.4: g another Lubin-Tate series for

(i) $\mu_{f,n} \cong \mu_{g,n}$ as $\mathcal{O}_K$-modules.

(ii) $K(\mu_{f,n}) = K(\mu_{g,n})$.

Proof: Let $\theta \in \mathrm{Hom}_{\mathcal{O}_K}(F_f, F_g)$ iso. of formal $\mathcal{O}_K$-modules. Then θ induces iso. $\theta : (\bar{m}, +_{F_f}) \xrightarrow{\sim} (\bar{m}, +_{F_g})$ of $\mathcal{O}_K$-modules, and hence

$$\mu_{f,n} \cong \mu_{g,n}$$

Since $\mu_{f,n}$ algebraic, $K(\mu_{f,n})/K$ finite, hence complete. $\theta(X) \in \mathcal{O}_K[\![X]\!]$

$\Rightarrow$ For $x \in \mu_{f,n}$, $\theta(x) \in K(\mu_{f,n})$

$\Rightarrow K(\mu_{g,n}) \subseteq K(\mu_{f,n})$

Same argument for θ^{-1} gives $K(\mu_{f,n}) \subseteq K(\mu_{g,n})$

$$\Rightarrow K(\mu_{f,n}) = K(\mu_{g,n}) \quad . \square$$

Definition 20.5: $K_{\pi,n} := K(\mu_{f,n})$

Lubin-Tate extension of degree n associated to π.

Remark: (i) $K_{\pi,n}$ does not depend on f, by Proposition 20.4.

(ii) $K_{\pi,n} \subseteq K_{\pi,n+1}$. $\forall n$

Theorem 20.6: $K_{\pi,n}/K$ totally ramified Galois of degree $q^{n-1}(q-1)$

Proof: (i) By Proposition 20.4, may choose $f(X) = \pi X + X^q$.

$K_{\pi,n}/K$ Galois since $K_{\pi,n}$ splitting field of $f_n(X)$.

Let α a root of $h_n(X) := \frac{f_n(X)}{f_{n-1}(X)}$.

STS. $K(\alpha) = K(\mu_{f,n}) = K_{\pi,n}$, since α root of Eisenstein polynomial of deg. $q^{n-1}(q-1)$.

"$\subseteq$" clear.

"$\supseteq$" By Prop 20.4, every element x of $\mu_{f,n}$ is of the form $a \cdot_{F_f} \alpha$ for some $a \in \mathcal{O}_K$.

$(\alpha \in \mu_{f,n} \setminus \mu_{f,n-1})$.

$K(\alpha)$ complete and $[a]_{F_f}(X) \in \mathcal{O}_K[\![X]\!]$

$\Rightarrow x = [a]_{F_f}(\alpha) \in K(\alpha)$.

$\Rightarrow K(\alpha) \supseteq K(\mu_{f,n})$. $\square$

$f(x) = \pi X + X^q$

- Theorem 20.7: There are isomorphisms

$$\psi_n : \mathrm{Gal}(K_{\pi,n}/K) \xrightarrow{\simeq} \left(\mathcal{O}_K/\pi^n\right)^\times \cong \mathcal{O}_K^\times/U_K^{(n)}.$$

characterized by

$$(*) \quad \psi_n(\sigma) \cdot_{F_f} x = \sigma(x), \quad \forall x \in \mu_{f,n}, \sigma \in \mathrm{Gal}(K_{\pi,n}$$

Proof: Let $\sigma \in \mathrm{Gal}(K_{\pi,n}/K)$. We show that

$\sigma \in \mathrm{Aut}_{\mathcal{O}_K}(\mu_{f,n})$.

Note: σ preserves $\mu_{f,n}$, and σ acts continuously on $K(\mu_{f,n})$.

Since $F_f(X,Y) \in \mathcal{O}_K[\![X]\!]$ and $[a]_{F_f} \in \mathcal{O}_K[\![X]\!]$ for all $a \in \mathcal{O}_K$, we have

continuity of σ
$$\begin{cases} \sigma(x +_{F_f} y) = \sigma(x) +_{F_f} \sigma(x) & \forall x, y \in \mu_{f,n} \\ \sigma(a \cdot_{F_f} x) = a \cdot_{F_f} \sigma(x) & \forall x \in \mu_{f,n}, a \in \mathcal{O}_K \end{cases}$$

Thus $\sigma \in \mathrm{Aut}_{\mathcal{O}_K}(\mu_{f,n})$

This induces a group hom.

$$\mathrm{Gal}(K_{\pi,n}/K) \hookrightarrow \mathrm{Aut}_{\mathcal{O}_K}(\mu_{f,n}),$$

injective since $K_{\pi,n} = K(\mu_{f,n})$

non-canonical

since $\mu_{f,n} \cong \mathcal{O}_K/\pi^n$

$$\mathrm{Aut}_{\mathcal{O}_K}(\mu_{f,n}) \underset{\text{canonical}}{\cong} \mathrm{Aut}_{\mathcal{O}_K}(\mathcal{O}_K/\pi^n) \cong (\mathcal{O}_K/\pi^n)^\times$$

Obtain: $\Psi_n : \mathrm{Gal}(K_{\pi,n}) \hookrightarrow (\mathcal{O}_K/\pi^n)^\times$ defined by

$\Psi_n(\sigma) \in (\mathcal{O}_K/\pi^n)^\times$ unique element s.t.

$$\Psi_n(\sigma) \cdot_{F_f} x = \sigma(x) \quad \forall x \in \mu_{f,n}.$$

$$[K_{\pi,n} : K] = q^{n-1}(q-1) = |(\mathcal{O}_K/\pi^n)^\times|$$

$\Rightarrow \Psi_n$ surj. by counting.

Let g be another Lubin-Tate series.

$$\Psi_n' : \mathrm{Gal}(K_{\pi,n}) \xrightarrow{\sim} (\mathcal{O}_K/\pi^n)^\times$$

Theorem 20.6 $\Rightarrow \exists\, \theta : F_f \to F_g$ iso of formal $\mathcal{O}_K$-modules.

Induces iso. $\theta : \mu_{f,n} \xrightarrow{\sim} \mu_{g,n}$ of $\mathcal{O}_K$-modules

$\theta \in \mathcal{O}_K[[X]] \Rightarrow$

$$\theta(\sigma(x)) = \sigma(\theta(x)) \quad \forall x \in \mu_{f,n},\ \sigma \in \mathrm{Gal}(K_{\pi,n}/K)$$

$\Rightarrow \theta(\Psi_n(\sigma) \cdot_{F_f} (x)) = \Psi_n'(\sigma) \cdot_{F_g} \theta(x)$

$\Rightarrow \Psi_n(\sigma) \cdot_{F_g} \theta(x) = \Psi_n'(\sigma) \cdot_{F_g} \theta(x).$

$\Rightarrow \Psi_n(\sigma) = \Psi_n'(\sigma).$ $\square$

Lecture 24