Chapter 14

The Grothendieck Riemann–Roch theorem

The goal of Riemann–Roch theorems is to relate the dimension of the space of solutions of an analytic or algebraic problem — typically realized as the space $H^0(\mathcal{F})$ of global sections of a coherent sheaf $\mathcal{F}$ on a compact analytic or projective algebraic variety X — to topological invariants, expressed in terms of polynomials in the Chern classes of the sheaf and of the tangent bundle of X. In practice, the formulas deal not with $h^0(\mathcal{F})$ but with the Euler characteristic $\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F})$ of $\mathcal{F}$, so the strength and importance of Riemann–Roch theorems, which are very great in the case of curves and surfaces, decline as the dimension of X, and with it the number of potentially nonzero cohomology groups, grows. Nevertheless, Riemann–Roch theorems have played an important role in the history of algebraic geometry.

Our goal in this chapter is to state, explain and apply a version of the Riemann–Roch theorem proved by Grothendieck that deals not just with a sheaf on a variety X but with families of such sheaves. To clarify its context, we start this chapter with older versions of the theorem. Although some of these were first proven in an analytic context, we will stick with the category of projective algebraic varieties. Good references for the simplest forms of these theorems are Sections IV.1 and V.1 of Hartshorne [1977].

Convention: To simplify notation in this chapter, we sometimes identify a class in $A_0(X)$ with its degree when X is a projective algebraic variety.

14.1 The Riemann–Roch formula for curves and surfaces

14.1.1 Curves

The original Riemann–Roch formula deals with a smooth projective curve C over $\mathbb{C}$. It says in particular that the dimension $h^0(K_C)$ of the space of regular 1-forms on C, an

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algebraic/analytic invariant, is equal to the topological genus $g(C) = 1 - \chi_{\text{top}}(C)/2$. To express this in modern language and suggest the generalizations to come, we invoke Serre duality, which says that

$$h^0(K_C) = h^1(\mathcal{O}_C),$$

and the Hopf index theorem for the topological Euler characteristic, which says that $\chi_{\text{top}}(C) = c_1(\mathcal{T}_C)$. In these terms, we can state the Riemann–Roch theorem as the formula

$$\chi(\mathcal{O}_C) := h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = \frac{c_1(\mathcal{T}_C)}{2}.$$

From this formula, and the additivity of the Euler characteristic, it is easy to prove the Riemann–Roch formula for any line bundle on C, as we will see now.

To start, recall that for a coherent sheaf $\mathcal{F}$ on an arbitrary projective variety the Euler characteristic is defined to be

$$\chi(\mathcal{F}) = \sum (-1)^i h^i(\mathcal{F}).$$

This formula is *additive on exact sequences*: if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of coherent sheaves on a projective variety, then the resulting long exact sequence in cohomology

$$\cdots \longrightarrow H^{i+1}(\mathcal{F}'') \longrightarrow H^i(\mathcal{F}') \longrightarrow H^i(\mathcal{F}) \longrightarrow H^i(\mathcal{F}'') \longrightarrow H^{i-1}(\mathcal{F}') \longrightarrow \cdots$$

yields $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Returning to the case of a smooth curve, suppose that $\mathcal{L} = \mathcal{O}_C(D)$ for an effective divisor D of degree $c_1(\mathcal{L}) = d$. From the sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_D \longrightarrow 0,$$

it follows that

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_C) + \chi(\mathcal{L}|_D) = c_1(\mathcal{L}) + \frac{c_1(\mathcal{T}_C)}{2}F = d + 1 - g,$$

and a similar sequence extends this formula to line bundles of the form $\mathcal{O}_C(D - E)$, that is, arbitrary line bundles. (See Appendix D for a fuller discussion of this theorem and its consequences.)

It is not hard to go from this to the version for an arbitrary coherent sheaf $\mathcal{F}$ on C, valid for a smooth curve over any field (see Section 14.2.1 for the definition of the Chern classes of coherent sheaves):

Theorem 14.1 (Riemann–Roch for curves). *If $\mathcal{F}$ is a coherent sheaf on a smooth curve C, then*

$$\chi(\mathcal{F}) = c_1(\mathcal{F}) + \operatorname{rank}(\mathcal{F})\frac{c_1(\mathcal{T}_C)}{2}.$$

14.1.2 Surfaces

To state a Riemann–Roch theorem for a smooth projective surface S, we start again from a special case,

$$\chi(\mathcal{O}_S) = \frac{c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)}{12},$$

usually referred to as *Noether's formula* (see Bădescu [2001, Chapter 5] for references). From this, the prior Riemann–Roch for curves, and sequences of the form

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{L}(D)|_D \longrightarrow 0$$

for smooth effective divisors $D \subset S$, we can deduce the version for line bundles:

$$\chi(\mathcal{L}) = \frac{c_1(\mathcal{L})^2 + c_1(\mathcal{L})c_1(\mathcal{T}_C)}{2} + \frac{c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)}{12}.$$

For example, to prove the formula for $\mathcal{L} = \mathcal{O}_X(D)$ when D is a smooth curve on S, we use the sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)|_D \longrightarrow 0.$$

From the additivity of the Euler characteristic, we get

$$\begin{aligned}\chi(\mathcal{O}_S(D)) &= \chi(\mathcal{O}_S) + \chi(\mathcal{O}_S(D)|_D)\\ &= \frac{c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)}{12} + \chi(\mathcal{O}_S(D)|_D).\end{aligned}$$

To evaluate the last term, observe that $\mathcal{O}_S(D)|_D$ is a line bundle of degree $D \cdot D$ on the curve D, which by the adjunction formula has genus

$$g(D) = \frac{D \cdot D + D \cdot K_S}{2} + 1.$$

Using Riemann–Roch for curves we obtain

$$\begin{aligned}\chi(\mathcal{O}_S(D)|_D) &= D \cdot D - \frac{D \cdot D + D \cdot K_S}{2}\\ &= \frac{D \cdot D + D \cdot c_1(\mathcal{T}_S)}{2},\end{aligned}$$

and the Riemann–Roch formula above follows for $\mathcal{L} = \mathcal{O}_S(D)$.

As in the previous case of curves, this can be extended to apply to arbitrary coherent sheaves on S:

Theorem 14.2. *If $\mathcal{F}$ is a coherent sheaf on a smooth projective surface S, then*

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F}) + c_1(\mathcal{F})c_1(\mathcal{T}_C)}{2} + \operatorname{rank}(\mathcal{F})\frac{c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)}{12}.$$

14.2 Arbitrary dimension

Much of the content of the formulas in Section 14.1 above was known to 19th century algebraic geometers, although the formulas were expressed without cohomology, and only for line bundles (represented by divisors). In the 20th century these formulas were extended to sheaves on varieties of arbitrary dimension by Hirzebruch. One key to this extension was the introduction of cohomology groups in general, and the recognition that the left-hand side of all the classical formulations of Riemann–Roch represented Euler characteristics of sheaves.

Equally important was understanding how to express the polynomials in the Chern classes that appear in the right-hand side of these formulas in a way that generalized to arbitrary dimensions. We digress to introduce the two useful power series in the Chern classes that are needed, the *Chern character* and the *Todd class*.

14.2.1 The Chern character

We have seen many uses of the Whitney formula (Theorem 5.3); we have also used special cases of the formula for the Chern classes of the tensor product of vector bundles, which we deemed too complicated to write down in closed form in general. But there is a better way to make sense of these two formulas, discovered by Hirzebruch [1966]; together, they say that a certain power series in the Chern classes, the *Chern character*, defines a ring homomorphism. To explain this useful fact, we first recall the definition of the Grothendieck ring of vector bundles:

The set of isomorphism classes $[\mathcal{A}]$ of vector bundles $\mathcal{A}$ of finite rank on a variety X forms a semigroup under direct sum that we will call $\mathrm{Bun}(X)$. The Euler characteristic χ defines a homomorphism of semigroups $\mathrm{Bun}(X) \to \mathbb{Z}$ and, as we have already remarked, this map is also additive on exact sequences of bundles $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. It is interesting to ask what other such maps there may be, and a natural step in investigating this is to form the *Grothendieck group* $K(X)$ of vector bundles on a variety X. This is defined as the free abelian group on the set of isomorphism classes $[\mathcal{A}]$ of vector bundles $\mathcal{A}$ on X, modulo relations $[\mathcal{A}] + [\mathcal{C}] = [\mathcal{B}]$ for every short exact sequence of vector bundles $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. The natural map of semigroups $\mathrm{Bun}(X) \to K(X)$ is *universal*, in the sense that any map from $\mathrm{Bun}(X)$ to a group that is additive on short exact sequences factors uniquely through the map to $K(X)$. A bonus of this construction is that, since tensoring with a vector bundle preserves exact sequences, $K(X)$ has a natural ring structure, where the product is given by tensor product.

The *Chern character* is a way of combining the Chern classes to produce a ring homomorphism

$$\mathrm{Ch} : K(X) \to A(X) \otimes \mathbb{Q}.$$

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To see how such a combination could be defined (and how the rational coefficients arise), consider first the case of line bundles. If $\mathcal{L}$ and $\mathcal{M}$ are two line bundles, then the first Chern class of the tensor product $c(\mathcal{L} \otimes \mathcal{M})$ is $c_1(\mathcal{L}) + c_1(\mathcal{M})$, so if we set

$$\mathrm{Ch}([\mathcal{L}]) = e^{c_1(\mathcal{L})} = 1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2} + \frac{c_1(\mathcal{L})^3}{6} + \cdots,$$

then

$$\begin{aligned} \mathrm{Ch}(\mathcal{L} \otimes \mathcal{M}) &= e^{c_1(\mathcal{L})+c_1(\mathcal{M})} \\ &= e^{c_1(\mathcal{L})} e^{c_1(\mathcal{M})} \\ &= \mathrm{Ch}([\mathcal{L}])\,\mathrm{Ch}([\mathcal{M}]). \end{aligned}$$

Note that the apparently infinite sums are actually finite, since $A^i(X)$ vanishes for $i > \dim X$.

If now $\mathcal{E} = \bigoplus \mathcal{L}_i$ is a direct sum of line bundles, then for Ch to preserve sums we must extend the definition above by setting

$$\mathrm{Ch}(\mathcal{E}) = \sum e^{c_1(\mathcal{L}_i)}.$$

The coefficients of this power series are symmetric in the "Chern roots" $c_1(\mathcal{L}_i)$, and thus can be expressed in terms of the elementary symmetric functions of these quantities — that is, in terms of the Chern classes of $\mathcal{E}$.

We define $\mathrm{Ch}(\mathcal{E})$ in general by using these expressions: If $\mathcal{E}$ is any vector bundle, we write $c(\mathcal{E}) = \prod(1 + \alpha_i)$, and then define

$$\mathrm{Ch}(\mathcal{E}) = \sum e^{\alpha_i};$$

in other words, the k-th graded piece $\mathrm{Ch}_k(\mathcal{E})$ of the Chern character is

$$\mathrm{Ch}_k(\mathcal{E}) = \sum \frac{\alpha_i^k}{k!},$$

expressed as a polynomial in the elementary symmetric functions of the α_i and applied to the Chern classes $c_i(\mathcal{E})$. The first few cases are

$$\begin{aligned} \mathrm{Ch}_0(\mathcal{E}) &= \mathrm{rank}(\mathcal{E}), \\ \mathrm{Ch}_1(\mathcal{E}) &= c_1(\mathcal{E}), \\ \mathrm{Ch}_2(\mathcal{E}) &= \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2}. \end{aligned}$$

The splitting principle implies that this formula does indeed give a ring homomorphism: First, Whitney's formula shows that if

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of vector bundles, then the Chern roots of $\mathcal{E}$ are the Chern roots of $\mathcal{E}'$ together with the Chern roots of $\mathcal{E}''$, so the definition above yields $\mathrm{Ch}(\mathcal{E}) = \mathrm{Ch}(\mathcal{E}') + \mathrm{Ch}(\mathcal{E}'')$. Further, the Chern roots of $\mathcal{E}' \otimes \mathcal{E}''$ are the pairwise sums of the Chern

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roots of $\mathcal{E}'$ and those of $\mathcal{E}''$, so the definition in terms of Chern roots again immediately yields the product formula $\mathrm{Ch}(\mathcal{E}' \otimes \mathcal{E}'') = \mathrm{Ch}(\mathcal{E}')\,\mathrm{Ch}(\mathcal{E}'')$, as required.

Since the Chern character is equivalent data to the rational Chern class, this yields a formula for the rational Chern class of a tensor product. The result is quite convenient for machine computation, but the conversion of polynomials in the power sums to polynomials in the elementary symmetric polynomials is complicated enough that it is not so useful for computation by hand; see Exercise 14.11 for an example.

Coherent sheaves

Let X be a smooth projective variety and $\mathcal{F}$ a coherent sheaf of X. By the Hilbert syzygy theorem, we can resolve $\mathcal{F}$ by locally free sheaves; that is, we can find an exact sequence

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

in which all the sheaves $\mathcal{E}_i$ are locally free. We can use this to extend the definitions of Chern classes and Chern characters to all coherent sheaves, in the only way possible that makes the Whitney formula and the product formula hold in general: We define the Chern polynomial and Chern character by

$$c(\mathcal{F}) = \prod_{i=0}^{n} c(\mathcal{E}_i)^{(-1)^i} \quad \text{and} \quad \mathrm{Ch}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Ch}(\mathcal{E}_i).$$

Of course for these definitions to make sense we need to know that they are independent of the resolution chosen; the verification is left as Exercise 14.12.

Caution: If $Y \subset X$ is a Cartier divisor, then from the sequence

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we see the Chern class of the sheaf $\mathcal{O}_Y$, viewed as a coherent sheaf on X, is simply

$$c(\mathcal{O}_Y) = \frac{c(\mathcal{O}_X)}{c(\mathcal{O}_X(-Y))} = \frac{1}{1-[Y]} = 1 + [Y].$$

The equality $c(\mathcal{O}_Y) = 1 + [Y]$ is emphatically *not* true for subvarieties $Y \subset X$ of codimension $c > 1$: In general, even when X and Y are smooth the Chern class of $\mathcal{O}_Y$ may have components of codimensions greater than c, and even the component in $A^c(X)$ differs from $[Y]$ by a factor of $(-1)^{c-1}(c-1)!$. This is a consequence of the Grothendieck Riemann–Roch theorem below; for examples, see Exercises 14.13–14.14.

The information in the Chern classes

Up to torsion, giving the Chern classes of a bundle $\mathcal{E}$ is equivalent to giving the class of $\mathcal{E}$ in $K(X)$:

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Theorem 14.3 (Grothendieck). *If X is a smooth projective variety, then the map*

$$\mathrm{Ch} : K(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

is an isomorphism of rings.

For more information, see Fulton [1984, Example 15.2.16b].

Strikingly, there is an analogous statement in the category of differentiable manifolds: If we define the topological K-group of a manifold M to be the group of formal linear combinations of $\mathcal{C}^\infty$ vector bundles modulo relations coming from exact sequences, with ring structure given as above by tensor products, then for a suitable filtration of the K-group the Chern character gives an isomorphism

$$\mathrm{gr}\, K(M) \otimes \mathbb{Q} \cong H^{2*}(X, \mathbb{Q}),$$

where the term on the right is the ring of even-degree rational cohomology classes (see Griffiths and Adams [1974]).

14.2.2 The Todd class

In the case of curves and surfaces we derived the Riemann–Roch formula by starting with an expression for the Euler characteristic $\chi(\mathcal{O}_X)$ of our variety X in terms of the Chern classes of the tangent bundle $\mathcal{T}_X$ of X. The *Todd class* of a vector bundle gives us a way of doing this in general: It is a polynomial in the Chern classes, with rational coefficients, such that for any smooth variety X we have

$$\chi(\mathcal{O}_X) = \{\mathrm{Td}(\mathcal{T}_X)\}_0;$$

that is, if X is n-dimensional, $\chi(\mathcal{O}_X) = \mathrm{Td}_n(\mathcal{T}_X)$, the degree of the n-th graded piece of the Todd class.

This is the property of these classes that led Todd to their definition (see Exercise 14.15). In fact the Todd class may be characterized as the only formula in the Chern classes that, when evaluated on the Chern classes of the tangent bundle $\mathcal{T}_{\mathbb{P}}$ of any product $\mathbb{P}$ of projective spaces, has zero-dimensional component $\{\mathrm{Td}(\mathcal{T}_{\mathbb{P}})\}_0 = 1$ (Hirzebruch [1966, p. 3]).

The naturality of this characterization does not imply that the formula will look simple! Suppose $\mathcal{E}$ is a vector bundle/locally free sheaf of rank n on a smooth variety X, and formally factor its Chern class:

$$c(\mathcal{E}) = \prod_{i=1}^{n}(1 + \alpha_i).$$

We define the *Todd class* of $\mathcal{E}$ to be

$$\mathrm{Td}(\mathcal{E}) = \prod_{i=1}^{n} \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

written as a power series in the elementary symmetric polynomials $c_i(\mathcal{E})$ of the α_i.

This definition, together with Whitney's formula and the splitting principle, immediately implies a multiplicative property: If

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of vector bundles, then, as before, the Chern roots of $\mathcal{E}$ are the Chern roots of $\mathcal{E}'$ together with the Chern roots of $\mathcal{E}''$, so the definition above at once yields $\mathrm{Td}(\mathcal{E}) = \mathrm{Td}(\mathcal{E}')\,\mathrm{Td}(\mathcal{E}'')$.

To calculate the first few terms of the Todd class, write

$$1 - e^{-\alpha} = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6} - \frac{\alpha^4}{24} + \cdots,$$

so

$$\frac{1 - e^{-\alpha}}{\alpha} = 1 - \frac{\alpha}{2} + \frac{\alpha^2}{6} - \frac{\alpha^3}{24} + \frac{\alpha^4}{120} - \cdots;$$

inverting this, we get

$$\frac{\alpha}{1 - e^{-\alpha}} = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{12} - \frac{\alpha^4}{720} + \cdots,$$

so

$$\mathrm{Td}(\mathcal{E}) = \prod_{i=1}^{n}\left(1 + \frac{\alpha_i}{2} + \frac{\alpha_i^2}{12} - \frac{\alpha_i^4}{720} + \cdots\right).$$

Rewriting the first few of these in terms of the symmetric polynomials of the α_i — that is, the Chern classes of $\mathcal{E}$ — we get formulas for the first few terms of the Todd class:

$$\begin{aligned}
\mathrm{Td}_0(\mathcal{E}) &= 1,\\
\mathrm{Td}_1(\mathcal{E}) &= \sum \frac{\alpha_i}{2} = \frac{c_1(\mathcal{E})}{2},\\
\mathrm{Td}_2(\mathcal{E}) &= \frac{1}{12}\sum \alpha_i^2 + \frac{1}{4}\sum_{i<j} \alpha_i\alpha_j = \frac{c_1^2(\mathcal{E}) + c_2(\mathcal{E})}{12},\\
\mathrm{Td}_3(\mathcal{E}) &= \frac{1}{24}\sum_{i \neq j} \alpha_i\alpha_j^2 = \frac{c_1(\mathcal{E})c_2(\mathcal{E})}{24}.
\end{aligned}$$

14.2.3 Hirzebruch Riemann–Roch

We now have the language necessary to express the Hirzebruch Riemann–Roch theorem, one formula that specializes to all the Riemann–Roch theorems stated above and their higher-dimensional analogs. In fact, given the definitions above, it is remarkably simple:

Theorem 14.4 (Hirzebruch's Riemann–Roch formula). *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then*

$$\chi(\mathcal{F}) = \{\mathrm{Ch}(\mathcal{F})\,\mathrm{Td}(\mathcal{T}_X)\}_n.$$

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This formula was first stated and proved in the setting of algebraic varieties over $\mathbb{C}$ by Hirzebruch [1966]; it was generalized to differentiable manifolds and elliptic differential operators by Atiyah and Singer; see Palais [1965]. The generalization to varieties over arbitrary fields is a special case of the work of Grothendieck, which we will describe next.

14.3 Families of bundles

Grothendieck's version of the Riemann–Roch theorem introduces a fundamental new idea into the mix.

Briefly, suppose we have a family $\{X_b\}_{b\in B}$ of schemes, and a family of sheaves $\mathcal{F}_b$ on X_b — in other words, a morphism $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X. As we see in Appendix B, the vector spaces $H^0(\mathcal{F}_b)$ form, at least for b in an open subset $U \subset B$, the fibers of a sheaf on B; this is the direct image $\pi_*\mathcal{F}$. We can think of the "classical" Hirzebruch Riemann–Roch theorem, applied to the sheaf $\mathcal{F}_b$ on the general fiber X_b, as a formula for the dimension $h^0(\mathcal{F}_b)$ (that is, the rank of the sheaf $\pi_*\mathcal{F}$), with "error terms" coming from the dimensions $h^i(\mathcal{F}_b)$ of the higher cohomology groups (i.e., the ranks of the higher direct images $R^i\pi_*\mathcal{F}$). The Grothendieck version of the Riemann–Roch theorem describes the way in which the spaces $H^0(\mathcal{F}_b)$ fit together as b varies as measured by the Chern classes of $\pi_*\mathcal{F}$.

14.3.1 Grothendieck Riemann–Roch

Theorem 14.5 (Grothendieck's Riemann–Roch formula)**.** *If $\pi : X \to B$ is a projective morphism with X and B smooth, and $\mathcal{F}$ is a coherent sheaf on X, then*

$$\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F}) = \pi_*\left[\frac{\operatorname{Ch}(\mathcal{F})\cdot\operatorname{Td}(\mathcal{T}_X)}{\pi^*\operatorname{Td}(\mathcal{T}_B)}\right] \in A(B)\otimes\mathbb{Q}.$$

This was first stated and proved in Borel and Serre [1958]; a shorter and more natural argument may be found in Fulton [1984, Section 15.2]

Hirzebruch's Riemann–Roch is Grothendieck's Riemann–Roch in the special case when B is a single point.

There are equivalent formulations of the Grothendieck Riemann–Roch theorem. For example, using the push-pull formula we can rewrite it in the form

$$\left(\sum_{i=0}^{n}(-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F})\right)\operatorname{Td}(\mathcal{T}_B) = \pi_*[\operatorname{Ch}(\mathcal{F})\cdot\operatorname{Td}(\mathcal{T}_X)].$$

Also, in case the map π is a submersion — that is, the differential $d\pi$ is surjective everywhere — then from the short exact sequence

$$0 \longrightarrow \mathcal{T}^v_{X/B} \longrightarrow \mathcal{T}_X \longrightarrow \pi^*\mathcal{T}_B \longrightarrow 0$$

for the relative tangent bundle of π and the multiplicativity of the Todd class we have:

Corollary 14.6. *If $\pi : X \to B$ is a projective morphism and a submersion, and $\mathcal{F}$ is a coherent sheaf on X, then*

$$\sum_{i=0}^{n} (-1)^i \operatorname{Ch}(R^i\pi_*\mathcal{F}) = \pi_*[\operatorname{Ch}(\mathcal{F}) \cdot \operatorname{Td}(\mathcal{T}^v_{X/B})].$$

When applying these formulas it is crucial to know when the sheaves $R^i\pi_*\mathcal{F}$ have fibers at an arbitrary point b equal to $H^i(\mathcal{F}|_{\pi^{-1}(b)})$. The theorem on cohomology and base change (Theorem B.5 in Appendix B) gives conditions under which this happens. Given this, it is possible to use the Grothendieck Riemann–Roch formula to calculate the Chern classes of many of the bundles whose Chern classes we calculated by ad hoc methods, e.g., filtrations and the splitting principle, earlier in this book. Indeed, virtually all of the bundles we have introduced and analyzed in the preceding chapters can be defined as direct images, and the Grothendieck Riemann–Roch formula gives an expression of their Chern classes. In the next section we will work this out in a simple case. Other examples are suggested in Exercises 14.17–14.19.

Following this example, the remainder of this chapter will be concerned with two applications of the Grothendieck Riemann–Roch formula in situations where we do not have alternative ways of calculating the Chern classes of the bundles in question. In Section 14.4, we will describe an application of the formula to the geometry of vector bundles on projective space, and in Section 14.5 an application to the geometry of families of curves.

14.3.2 Example: Chern classes of $\operatorname{Sym}^3 \mathcal{S}^*$ on $\mathbb{G}(1,3)$

Recall that to compute the number of lines on a smooth cubic surface $S \subset \mathbb{P}^3$ in Chapter 6 we introduced a vector bundle $\mathcal{E}$ on the Grassmannian $G = \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. Informally, we described $\mathcal{E}$ by saying that, for each line $L \subset \mathbb{P}^3$, the fiber of $\mathcal{E}$ at the point $[L] \in G$ was the vector space of homogeneous cubic polynomials on L; that is,

$$\mathcal{E}_{[L]} = H^0(\mathcal{O}_L(3)).$$

We explained that the number of lines, computed with multiplicity, was equal to $c_4(\mathcal{E})$. (The result that every smooth cubic surface actually has 27 distinct lines required further argument.)

Back then, we showed how $\mathcal{E}$ may be realized as a direct image, but did not use that construction to calculate its Chern class. (Rather, we calculated the Chern classes of $\mathcal{E}$ by realizing that $\mathcal{E} = \mathrm{Sym}^3 \mathcal{S}^*$, the third symmetric power of the dual of the universal subbundle on G, and using the splitting principle.) We now have a tool that will allow us to calculate the Chern classes of $\mathcal{E}$ directly from its construction; as an illustration of the general technique, we will carry this out explicitly here.

To set this up, recall that the universal family of lines over G is the incidence correspondence

$$\Phi = \{(L, p) \in G \times \mathbb{P}^3 \mid p \in L\};$$

we will let $\alpha : \Phi \to G$ and $\beta : \Phi \to \mathbb{P}^3$ be the projection maps. The bundle $\mathcal{E}$ is the direct image of $\mathcal{L} = \beta^* \mathcal{O}_{\mathbb{P}^3}(3)$.

To compute the Chern classes of $\mathcal{E}$ we first observe that the restriction of $\mathcal{O}_L(3)$ of $\mathcal{L}$ to each fiber $\Phi_{[L]} = \alpha^{-1}([L]) = L \cong \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(3)$, which has no higher cohomology. From the theorem on cohomology and base change (Theorem B.5), it follows that the direct image

$$\mathcal{E} = \alpha_* \mathcal{L} = \alpha_*(\beta^* \mathcal{O}_{\mathbb{P}^3}(3))$$

is locally free, with fiber $H^0(\mathcal{O}_L(3))$ at $[L]$.

Because of the vanishing of the higher cohomology of $\mathcal{L}$ on the fiber of α, the higher direct images $R^i \alpha_*(\mathcal{L})$ are 0 for $i > 0$, so the Grothendieck Riemann–Roch theorem becomes a formula for the Chern character of $\mathcal{E}$:

$$\mathrm{Ch}(\mathcal{E}) = \alpha_*(\mathrm{Ch}(\mathcal{L}) \cdot \mathrm{Td}(\mathcal{T}^v_{\Phi/G})).$$

To evaluate this explicitly requires the following steps:

(a) Describe the Chow ring $A(\Phi)$.
(b) Describe the direct image map $\alpha_* : A(\Phi) \to A(G)$.
(c) Calculate the Chern character of $\mathcal{L}$ and the Todd class of the relative tangent bundle $\mathcal{T}^v_{\Phi/G}$.
(d) Take the direct image of their product, to arrive at $\mathrm{Ch}(\mathcal{E})$.
(e) Finally, convert this back into the Chern classes of $\mathcal{E}$.

The necessary result for step (a) is Proposition 9.10: $\Phi = \mathbb{P}\mathcal{S}$ is the projectivization of the universal subbundle on G, so that we have

$$A(\Phi) = A(G)[\zeta]/(\zeta^2 - \sigma_1 \zeta + \sigma_{1,1}),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1))$.

As for step (b), we have

$$\alpha_*(1 + \zeta + \zeta^2 + \cdots) = s(\mathcal{S}) = \frac{1}{c(\mathcal{S})} = \frac{1}{1 - \sigma_1 + \sigma_{1,1}} = 1 + \sigma_1 + \sigma_2;$$

in other words,

$$\alpha_*\zeta = 1, \quad \alpha_*(\zeta^2) = \sigma_1, \quad \alpha_*(\zeta^3) = \sigma_2 \quad \text{and} \quad \alpha_*(\zeta^4) = 0.$$

(The last equation can also be seen directly: $\zeta^4 = 0$ in $A(\Phi)$, since ζ is the pullback of the hyperplane class on $\mathbb{P}^3$.) By the push-pull formula, this allows us to evaluate the pushforward of any product of a power of ζ with the pullback of a class from G.

To compute the Chern character of $\mathcal{L}$ for step (c), we first observe that, since the fiber of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ at a point $(L, p) \in \Phi$ is the dual of the one-dimensional vector subspace of $\mathbb{C}^4$ corresponding to p, we have

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)) = \beta^* c_1(\mathcal{O}_{\mathbb{P}^3}(1)).$$

In particular, it follows that

$$c_1(\mathcal{L}) = 3\zeta,$$

and so

$$\mathrm{Ch}(\mathcal{L}) = 1 + 3\zeta + \tfrac{9}{2}\zeta^2 + \tfrac{27}{6}\zeta^3,$$

since higher powers of ζ vanish.

In Section 11.1.2 we saw how to find the Chern classes of the relative tangent bundle of Φ over G: If we denote by $\mathcal{U}$ the tautological subbundle on $\Phi = \mathbb{P}\mathcal{S}$, and by $\mathcal{Q}$ the tautological quotient bundle, we have

$$\mathcal{T}^v_{\Phi/G} = \mathcal{U}^* \otimes \mathcal{Q}.$$

From the exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \alpha^*\mathcal{S} \longrightarrow \mathcal{Q} \longrightarrow 0$$

we see that

$$c_1(\mathcal{Q}) = c_1(\alpha^*\mathcal{S}) - c_1(\mathcal{U}) = -\sigma_1 + \zeta,$$

and hence

$$c_1(\mathcal{T}^v_{\Phi/G}) = c_1(\mathcal{U}^* \otimes \mathcal{Q}) = \zeta + c_1(\mathcal{Q}) = -\sigma_1 + 2\zeta.$$

Plugging this into the formula for the Todd class, we have

$$\mathrm{Td}(\mathcal{T}^v_{\Phi/G}) = 1 + \frac{2\zeta - \sigma_1}{2} + \frac{(2\zeta - \sigma_1)^2}{12} - \frac{(2\zeta - \sigma_1)^4}{720}.$$

For step (d) we take the product

$$\begin{aligned}
\mathrm{Ch}(\mathcal{L})\,\mathrm{Td}(\mathcal{T}^v_{\Phi/G}) = 1 &+ \tfrac{1}{2}(8\zeta - \sigma_1) + \tfrac{1}{12}(94\zeta^2 - 22\sigma_1\zeta + \sigma_1^2) \\
&+ \tfrac{1}{12}(120\zeta^3 - 39\sigma_1\zeta^2 + 3\sigma_1^2\zeta) \\
&+ \tfrac{1}{720}(-2668\sigma_1\zeta^3 + 246\sigma_1^2\zeta^2 + 8\sigma_1^3\zeta - \sigma_1^4) \\
&+ \tfrac{1}{720}(198\sigma_1^2\zeta^3 + 24\sigma_1^3\zeta^2 - 3\sigma_1^4\zeta).
\end{aligned}$$

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Applying the direct image map, we find that by Grothendieck Riemann–Roch

$$\mathrm{Ch}(\mathcal{E}) = 4 + 6\sigma_1 + (7\sigma_2 - 3\sigma_{1,1}) - 3\sigma_{2,1} + \tfrac{1}{3}\sigma_{2,2}.$$

It remains to convert this to the Chern class of $\mathcal{E}$ (step (e)). We have

$$c_1(\mathcal{E}) = \mathrm{Ch}_1(\mathcal{E}) = 6\sigma_1$$

and

$$\begin{aligned} c_2(\mathcal{E}) &= \tfrac{1}{2}\mathrm{Ch}_1(\mathcal{E})^2 - \mathrm{Ch}_2(\mathcal{E}) \\ &= 18\sigma_1^2 - (7\sigma_2 - 3\sigma_{1,1}) \\ &= 11\sigma_2 + 21\sigma_{1,1}. \end{aligned}$$

Similarly,

$$\begin{aligned} c_3(\mathcal{E}) &= \tfrac{1}{6}\mathrm{Ch}_1(\mathcal{E})^3 - \mathrm{Ch}_1(\mathcal{E})\,\mathrm{Ch}_2(\mathcal{E}) + 2\,\mathrm{Ch}_3(\mathcal{E}) \\ &= 36\sigma_1^3 - 6\sigma_1(7\sigma_2 - 3\sigma_{1,1}) - 6\sigma_{2,1} \\ &= 72\sigma_{2,1} - 24\sigma_{2,1} - 6\sigma_{2,1} \\ &= 42\sigma_{2,1}, \end{aligned}$$

and, finally, the payoff!

$$\begin{aligned} c_4(\mathcal{E}) &= \tfrac{1}{24}\mathrm{Ch}_1(\mathcal{E})^4 - \tfrac{1}{2}\mathrm{Ch}_1(\mathcal{E})^2\,\mathrm{Ch}_2(\mathcal{E}) + \tfrac{1}{2}\mathrm{Ch}_2(\mathcal{E})^2 + 2\,\mathrm{Ch}_1(\mathcal{E})\,\mathrm{Ch}_3(\mathcal{E}) - 6\,\mathrm{Ch}_4(\mathcal{E}) \\ &= 54\sigma_1^4 - 18\sigma_1^2(7\sigma_2 - 3\sigma_{1,1}) + \tfrac{1}{2}(7\sigma_2 - 3\sigma_{1,1})^2 - 36\sigma_1\sigma_{2,1} - 2\sigma_{2,2} \\ &= (108 - 72 + 29 - 36 - 2)\sigma_{2,2} \\ &= 27\sigma_{2,2}; \end{aligned}$$

we have calculated again the number of lines on a cubic surface, counted with multiplicities, under the assumption that the number is finite.

We have also illustrated a fact well-known to practicing algebraic geometers: One should almost never use Grothendieck Riemann–Roch to calculate the Chern classes of a bundle if there is an alternative!

14.4 Application: jumping lines

In this section, we will describe the notion of *jumping lines*, an invariant used to study the geometry of vector bundles on projective space. To keep the notation simple, we will deal just with the case of vector bundles of rank 2 on $\mathbb{P}^2$. The generalization to rank-2 bundles on $\mathbb{P}^n$ is indicated in Exercises 14.20–14.22.

We start by recalling that, though a vector bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^2$ may be indecomposable (see for example Exercise 5.41), its restriction to any line $L \cong \mathbb{P}^1 \subset \mathbb{P}^2$ can be expressed as a direct sum

$$\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$$

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of line bundles, and of course

$$c_1(\mathcal{E}|_L) = a_1 + a_2.$$

This *splitting type* (a_1, a_2) of $\mathcal{E}$ on L provides a useful way to analyze vector bundles: For every pair $a = (a_1, a_2)$ with $a_1 < a_2$ and $a_1 + a_2 = c_1(\mathcal{E}|_L)$, we define a subset $\Gamma_a \subset \mathbb{G}(1,2) = \mathbb{P}^{2*}$ by

$$\Gamma_a = \{L \in \mathbb{P}^{2*} \mid \mathcal{E}|_L \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)\}.$$

As we will see in a moment, these will be locally closed subsets of $\mathbb{P}^{2*}$. In particular, the decomposition of $\mathcal{E}|_L$ will be constant for L in an open dense subset of $\mathbb{P}^{2*}$; the lines outside this open set are called *jumping lines*. Together the loci Γ_a give a stratification of $\mathbb{P}^{2*}$ whose geometry is an important invariant of $\mathcal{E}$; the closures of the strata Γ_a are called *loci of jumping lines*. It follows at once that the loci of jumping lines will not change (except for indexing) if we replace $\mathcal{E}$ with a bundle of the form $\mathcal{E}(n)$.

14.4.1 Loci of bundles on $\mathbb{P}^1$ with given splitting type

To continue the analysis we need some basic facts about how vector bundles on $\mathbb{P}^1$ behave in families. Let B be a connected scheme. By a *family of vector bundles on $\mathbb{P}^1$ with base B* we mean a vector bundle $\mathcal{E}$ on the product $B \times \mathbb{P}^1$. The rank of the bundle $\mathcal{E}_t$ on $\mathbb{P}^1$ that is the restriction of $\mathcal{E}$ to the fiber $\{t\} \times \mathbb{P}^1$ is locally constant in t since $\mathcal{E}$ is locally trivial, and by the theorem on cohomology and base change (Theorem B.5) the Euler characteristic, and thus the degree, is too. Since we have assumed that B is connected, both these are constant. Denote the common rank by r and the common degree by d.

The actual decomposition $\mathcal{E}_t \cong \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i(t))$, however, may vary with t; we want to describe the possible variation. After rearranging the summands, the decomposition of $\mathcal{E}_t$ is given by an increasing sequence of integers with sum d:

$$a = (a_1, \dots, a_r) \quad \text{with} \quad a_1 \le a_2 \le \cdots \le a_r \in \mathbb{Z} \quad \text{and} \quad \sum a_i = d.$$

We partially order such sequences by initial subsums: we say

$$a \le b \quad \text{if} \quad \sum_{i=1}^{j} a_i \le \sum_{i=1}^{j} b_i \quad \text{for all} \quad j = 1, \dots, r;$$

and we say that $a < b$ if $a \le b$ and $a \ne b$. For example, if $a_{i+1} > a_i + 1$ for any $i < r$, then we can form a larger sequence by the replacements

$$a_i \leftarrow a_i + 1, \qquad a_{i+1} \leftarrow a_{i+1} - 1.$$

Thus there is a unique maximal sequence for a given r and d, determined by the condition $|a_i - a_j| \le 1$ for all i, j. Equivalently, this largest sequence is the "most balanced:" it is the unique sequence of the form $(a, \dots, a, a+1, \dots, a+1)$. On the other hand, there is no minimal sequence; for example, with $r = 2, d = 0$ we have $(0,0) > (-1,1) > (-2,2) > \cdots$. But if we impose an upper bound on a_r, say $a_r \le e$, then the partially ordered set is finite. If $e \gg d$ the set has unique minimal element $(-(r-1)e + d, e, \dots, e)$.

As a measure of the deviation of a given sequence a from the most balanced, we will set

$$u(a) = \sum_{i<j} \max\{a_j - a_i - 1, 0\}.$$

The quantity $u(a)$ should be thought of as the "expected codimension of the locus of bundles of splitting type a," as explained by the following result:

Theorem 14.7. *Let $\mathcal{E}$ be as above a vector bundle on $B \times \mathbb{P}^1$. If for each sequence a we set*

$$\Gamma_a = \{t \in B \mid \mathcal{E}_t \cong \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)\},$$

then:

(a) For each a, the locus

$$\Psi_a = \bigcup_{a' \le a} \Gamma_{a'}$$

is closed in B (so that in particular Γ_a is a locally closed subset of B).

(b) The codimension of Γ_a in B is at most $u(a)$.

We will prove (a) completely, but we prove (b) only for $r = 2$, the case we will use.

Proof: (a) Consider the function on B

$$\mu(t) = \max\{m \mid h^0(\mathcal{E}_t(-m)) > 0\}.$$

Since $h^0(\mathcal{E}_t(-m))$ is upper-semicontinuous in t, the function μ is as well; this shows that the degree $a_r(t)$ of the largest summand of $\mathcal{E}_t$ is upper-semicontinuous. Applying this to the bundle $\bigwedge^k(\mathcal{E}_t)$, we see that the function

$$\mu_k(t) = \max\{m \mid h^0((\textstyle\bigwedge^k \mathcal{E}_t)(-m)) > 0\} = a_r(t) + \cdots + a_{r-k+1}(t)$$

is likewise upper-semicontinuous, and correspondingly

$$d - \mu_{r-k}(t) = a_1(t) + \cdots + a_k(t)$$

is lower-semicontinuous, establishing part (a).

(b) ($r = 2$) Suppose that $\mathcal{E}$ is a bundle of rank 2 on $B \times \mathbb{P}^1$ and let $b \in B$ be a point. The function $u(a)$ is invariant under the addition of a fixed quantity to all the a_i, so we can twist the bundle $\mathcal{E}$ by the pullback of a line bundle on $\mathbb{P}^1$ without affecting the truth of the statement; after such a twist we can assume that the fiber at b has the form $\mathcal{E}_b \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ for some $n \geq 0$. If $n = 0$ there is nothing to prove, so we may assume $n \geq 1$, and we must show that the locus $\Gamma_{(0,n)}$ has codimension at most $u(0,n) = n - 1$ near b.

Since $h^1(\mathcal{E}_b) = 0$, the section $(1, 0)$ of $\mathcal{E}_b$ extends to a nowhere-zero section of $\mathcal{E}$ in a neighborhood of the fiber $\{b\} \times \mathbb{P}^1$. Replacing B by a suitably small open neighborhood of $b \in B$, then, we have a sequence

$$0 \longrightarrow \mathcal{O}_{B\times\mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{B\times\mathbb{P}^1}(n) \longrightarrow 0,$$

where we write $\mathcal{O}_{B\times\mathbb{P}^1}(n)$ for the pullback of $\mathcal{O}_{\mathbb{P}^1}(n)$ to $B \times \mathbb{P}^1$.

Now, a sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0$$

splits if and only if there exists a bundle map $\varphi : \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{E}$ such that $\alpha \circ \varphi$ is the identity on $\mathcal{O}_{\mathbb{P}^1}(n)$. Accordingly, consider the exact sequence of bundles on $B \times \mathbb{P}^1$

$$\begin{aligned} 0 \to \mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}) \to \mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{E}) \\ \to \mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n)) \to 0 \end{aligned}$$

and the coboundary map

$$\pi_*\big(\mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n))\big) \xrightarrow{\delta} R^1\pi_*(\mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}))$$

appearing in the associated long exact sequence of sheaves on B. If we let $\sigma = \delta(\mathrm{id})$ be the image in

$$R^1\pi_*(\mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}))$$

of the identity section of $\pi_*\big(\mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}(n))\big)$, then the zero locus $(\sigma) \subset B$ of σ will be contained in the stratum $\Gamma_{(0,n)}$; since

$$R^1\pi_*(\mathcal{H}om(\mathcal{O}_{B\times\mathbb{P}^1}(n), \mathcal{O}_{B\times\mathbb{P}^1}))$$

is locally free of rank $n - 1$, it follows that the codimension of $\Gamma_{(0,n)}$ in B is at most $n - 1$, as required. □

One can also realize the Γ_a as pullbacks of strata in a family of vector bundles directly: After suitable twisting, every member $\mathcal{E}_b$ of the family will be an extension of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r-1} \longrightarrow \mathcal{E}_b \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0,$$

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and the family over B will locally be a pullback of a family defined over

$$B' := \mathrm{Ext}^1_{\mathbb{P}_1}(\mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}^{r-1}_{\mathbb{P}^1}) \cong \mathbb{A}^{(r-1)d}.$$

The codimension of the locus Γ_a in B' is exactly $u(a)$. For a general study, including a conjecture on the equations of the strata Γ_a in B', see Eisenbud and Schreyer [2008, Example 5.2].

14.4.2 Jumping lines of bundles of rank 2 on $\mathbb{P}^2$

Let $\mathcal{E}$ be a vector bundle of rank 2 on $\mathbb{P}^2$. The nature of the locus of jumping lines depends on whether $c_1(\mathcal{E}) \in A^1(\mathbb{P}^2) \cong \mathbb{Z}$ is even or odd. In the even case the expected dimension of the locus in 1, and we will compute the degree of the curve of jumping lines; for a complete treatment see Barth [1977]. In the odd case the expected dimension is 0, and we will compute the degree of this finite scheme; for a complete treatment see Hulek [1979].

Vector bundles with even first Chern class

Let $c_1(\mathcal{E}) = 2k\zeta$, where $\zeta \in A^1(\mathbb{P}^2)$ is the class of a line. We want to think of the restrictions of $\mathcal{E}$ to each line in $\mathbb{P}^2$ in turn as a family of vector bundles on $\mathbb{P}^1$, parametrized by the Grassmannian $\mathbb{G}(1,2) \cong \mathbb{P}^{2*}$. Let

$$\Phi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\}$$

be the universal line, and let $\pi_1 : \Phi \to \mathbb{P}^{2*}$ and $\pi_2 : \Phi \to \mathbb{P}^2$ be the projections. We can view $\pi_1 : \Phi \to \mathbb{P}^{2*}$ as the projectivization of the universal rank-2 subbundle $\mathcal{S}$ on $\mathbb{P}^{2*} = \mathbb{G}(1,2)$. We let

$$\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)) \in A^1(\Phi)$$

be the tautological class; note that this is also the pullback to Φ of the hyperplane class $\zeta \in A^1(\mathbb{P}^2)$ on $\mathbb{P}^2$. Similarly, we denote by α both the hyperplane class on $\mathbb{P}^{2*}$ and its pullback to Φ. The Chow ring of Φ is given by

$$A(\Phi) = \mathbb{Z}[\alpha, \zeta]/(\alpha^3, \zeta^2 - \alpha\zeta + \alpha^2).$$

Note that $\zeta^3 = 0$; this follows either from the relations above or from the fact that ζ is the pullback of a class from $A^1(\mathbb{P}^{2*})$. We see also that the pushforward map $\pi_{1*} : A(\Phi) \to A(\mathbb{P}^{2*})$ is given by

$$\alpha^2 \mapsto 0, \quad \alpha\zeta \mapsto \alpha, \quad \zeta^2 \mapsto \alpha.$$

To realize the restrictions of $\mathcal{E}$ to the lines in $\mathbb{P}^2$ as a family, we consider the pullback bundle

$$\mathcal{F} = \pi_2^*(\mathcal{E})$$

on Φ.

We now assume that the "expected" codimensions of the loci Γ_a of Theorem 14.7 are actually attained; that is, that for an open dense subset U of $L \in \mathbb{P}^{2*}$, the restriction of $\mathcal{F}$ to the fiber over L splits as

$$\mathcal{E}|_L = \mathcal{F}|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k),$$

that there is a codimension-1 locus — a curve — $C \subset \mathbb{P}^{2*}$ of lines L such that

$$\mathcal{E}|_L = \mathcal{F}|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(k-1),$$

and that no other splitting types occur.

To get some information about C we replace $\mathcal{E}$ by $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$ and consider $\mathcal{F}' = \pi_2^* \mathcal{E}'$. The restriction of $\mathcal{E}'$ to L will split as

$$\mathcal{E}'|_L \cong \begin{cases} \mathcal{O}_L(-1)^{\oplus 2} & \text{if } L \in U, \\ \mathcal{O}_L \oplus \mathcal{O}_L(-2) & \text{if } L \in C, \end{cases}$$

and thus

$$h^0(\mathcal{E}'|_L) = h^1(\mathcal{E}'|_L) = \begin{cases} 0 & \text{if } L \in U, \\ 1 & \text{if } L \in C. \end{cases}$$

By the theorem on cohomology and base change (Theorem B.5), the direct image $\pi_{1*}(\mathcal{F}')$ is 0: over the open subset $U \subset \mathbb{P}^{2*}$ there are no sections, and, since $\pi_{1*}(\mathcal{F}')$ is torsion-free, it follows that $\pi_{1*}(\mathcal{F}') = 0$. However, the jump in the cohomology groups $H^i(\mathcal{E}'|_L)$ is reflected in the higher direct image $R^1\pi_{1*}(\mathcal{F}')$; this will be a sheaf supported on the curve C. It follows that

$$c_1(R^1\pi_{1*}(\mathcal{F}'))$$

will give us the degree of this curve (counting each component with some positive multiplicity). The class $c_1(R^1\pi_{1*}(\mathcal{F}'))$ is something we can calculate from Grothendieck Riemann–Roch.

To make the computation, we first need the Todd class of the relative tangent bundle of $\pi_1 : \Phi \to \mathbb{P}^{2*}$. From Theorem 11.4,

$$c_1(\mathcal{T}^v_{\Phi/\mathbb{P}^{2*}}) = -\alpha + 2\zeta,$$

and correspondingly

$$\mathrm{Td}(\mathcal{T}^v_{\Phi/\mathbb{P}^{2*}}) = 1 + \tfrac{1}{2}(-\alpha + 2\zeta) + \tfrac{1}{12}(-\alpha + 2\zeta)^2$$

(note that since $T^v_{\Phi/\mathbb{P}^{2*}}$ is a line bundle, there is no cubic term). If we write the Chern class of the bundle $\mathcal{F}'$ as

$$c(\mathcal{F}') = -2\zeta + e\zeta^2,$$

then we have

$$\begin{aligned} \mathrm{Ch}(\mathcal{F}') &= \mathrm{rank}(\mathcal{F}') + c_1(\mathcal{F}') + \tfrac{1}{2}(c_1^2(\mathcal{F}') - 2c_2(\mathcal{F}')) \\ &= 2 - 2\zeta + (2 - e)\zeta^2. \end{aligned}$$

Now the Grothendieck Riemann–Roch formula tells us that

$$\begin{aligned}[C] &= \mathrm{Ch}_1(R^1\pi_{1*}(\mathcal{F}')) \\ &= -\pi_{1*}\{(2-2\zeta+(2-e)\zeta^2)\big(1+\tfrac{1}{2}(-\alpha+2\zeta)+\tfrac{1}{12}(-\alpha+2\zeta)^2\big)\}_2 \\ &= -\pi_{1*}\big(\tfrac{1}{6}(-\alpha+2\zeta)^2-\zeta(-\alpha+2\zeta)+(2-e)\zeta^2\big) \\ &= (e-1)\zeta;\end{aligned}$$

or, in other words, the degree of the curve of jumping lines, counted with multiplicities, is $e-1$. Given that $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$, we have

$$e = c_2(\mathcal{E}') = c_2(\mathcal{E}) - (k^2-1)$$

and so for our original bundle $\mathcal{E}$ we have:

Proposition 14.8. *If $\mathcal{E}$ is a vector bundle of rank 2 on $\mathbb{P}^2$ with even first Chern class $2k\zeta$, and the restriction of $\mathcal{E}$ to a general line $L \subset \mathbb{P}^2$ is balanced (that is, $\mathcal{E}|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k)$), then the degree of the curve of jumping lines, counted with multiplicities as above, is $c_2(\mathcal{E}) - k^2 = c_2(\mathcal{E}) - c_1^2(\mathcal{E})/4$.*

In fact there is a natural scheme structure on the curve C, defined by a Fitting invariant of $R^1\pi_{1*}(\mathcal{F}')$, that determines the multiplicities. See Maruyama [1983] for a treatment and further references. Although we have not described the multiplicities or given conditions under which they are equal to 1, Proposition 14.8 has nontrivial content: For example, we may deduce that the degree of C is at most $c_2(\mathcal{E}) - c_1^2(\mathcal{E})/4$, and if $c_2(\mathcal{E}) - c_1^2(\mathcal{E})/4 \neq 0$ we may conclude that the curve of jumping lines is nonempty. We give an explicit example below.

Vector bundles with odd first Chern class

We now assume that $c_1(\mathcal{E}) = (2k+1)\zeta$ and, as before, that the "expected" codimensions of the loci Γ_a of Theorem 14.7 are attained. Thus, over an open dense subset U of $L \in \mathbb{P}^{2*}$, the restriction of $\mathcal{F}$ to the fiber over L splits as

$$\mathcal{E}|_L = \mathcal{F}|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(k+1).$$

But now our assumption means that the locus of lines L such that

$$\mathcal{E}|_L = \mathcal{F}|_{\pi_2^{-1}(L)} \cong \mathcal{O}_L(k-1) \oplus \mathcal{O}_L(k+2)$$

is a finite set Γ. Again, no other splitting types occur.

To get some information about Γ, we again twist the vector bundle $\mathcal{E}$ so that the jump in splitting type is reflected in the ranks of the cohomology groups of its restriction to lines. Specifically, we replace $\mathcal{E}$ by $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-k-1)$, so that the restriction of the bundle $\mathcal{E}'$ to L will split as

$$\mathcal{E}'|_L \cong \begin{cases} \mathcal{O}_L \oplus \mathcal{O}_L(-1) & \text{if } L \in U, \\ \mathcal{O}_L(1) \oplus \mathcal{O}_L(-2) & \text{if } L \in \Gamma. \end{cases}$$

We now have

$$h^0(\mathcal{E}'|_L) = \begin{cases} 1 & \text{if } L \in U, \\ 2 & \text{if } L \in \Gamma, \end{cases}$$

and correspondingly

$$h^1(\mathcal{E}'|_L) = \begin{cases} 0 & \text{if } L \in U, \\ 1 & \text{if } L \in \Gamma. \end{cases}$$

As before, this means that the sheaf $R^1\pi_{1*}(\mathcal{F}')$ is supported on the exceptional locus, in this case Γ, and thus the length of this sheaf counts the number of points in Γ, with some nonzero multiplicities.

An important difference between this case and that of even first Chern class is that here the direct image $\pi_{1*}(\mathcal{F}')$ is nonzero. By the theorem on cohomology and base change it is locally free of rank 1 away from Γ; in fact, it is locally free everywhere, as we will now show.

Let $L \in U$ be a point in an open subset of $\mathbb{P}^{2*}$ such that the splitting of $\mathcal{F}'|_L$ is $\mathcal{F}'|_L = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and the splitting of $\mathcal{F}'|_{L'}$ for $L' \in U$ other than L is $\mathcal{F}'|_{L'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\Phi_U}(-2)^2 \longrightarrow \mathcal{O}_{\Phi_U}(-2) \oplus \mathcal{O}_{\Phi_U}(-1)^3 \longrightarrow \mathcal{F}' \longrightarrow 0$$

(see Exercise 14.26). Applying $R\pi_{1*}$, we get an exact sequence of sheaves on U of the form

$$0 \longrightarrow \pi_{1*}\mathcal{F}' \longrightarrow R^1\pi_{1*}\mathcal{O}_{\Phi_U}(-2)^2 \longrightarrow R^1\pi_{1*}\mathcal{O}_{\Phi_U}(-2) \longrightarrow R^1\pi_{1*}\mathcal{F}' \longrightarrow 0.$$

This can be written as

$$0 \longrightarrow \pi_{1*}\mathcal{F}' \longrightarrow \mathcal{O}_U^2 \xrightarrow{(f,g)} \mathcal{O}_U \longrightarrow R^1\pi_{1*}\mathcal{F}' \longrightarrow 0,$$

where the common zero locus of f and g is the locus of jumping lines. Given that the locus of jumping lines has codimension 2, it follows that $\pi_{1*}\mathcal{F}'$ is locally free of rank 1, and thus

$$\pi_{1*}(\mathcal{F}') \cong \mathcal{O}_{\mathbb{P}^{2*}}(m)$$

for some m; the value of m will also emerge in the calculation.

Note that in a neighborhood of a point $p \in \Gamma \subset \mathbb{P}^{2*}$ corresponding to a line $L \subset \mathbb{P}^2$ the comparison map

$$\pi_{1*}(\mathcal{F}')|_p \to H_0(\mathcal{F}'|_L)$$

will be zero: none of the sections of $\mathcal{F}'|_L$ extend to a neighborhood of L.

We now apply Corollary 14.6. If we write

$$c(\mathcal{E}') = 1 - \zeta + e\zeta^2,$$

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then the Chern character of $\mathcal{E}'$, and of its pullback $\mathcal{F}'$ to Φ, is

$$\mathrm{Ch}(\mathcal{F}') = 2 - \zeta + \left(\tfrac{1}{2} - e\right)\zeta^2.$$

By Corollary 14.6,

$$\begin{aligned}\mathrm{Ch}(\pi_{1*}\mathcal{F}') - \mathrm{Ch}(R^1\pi_{1*}\mathcal{F}')& \\ &= \pi_{1*}\left[\left(2 - \zeta + \left(\tfrac{1}{2} - e\right)\zeta^2\right)\left(1 + \tfrac{1}{2}(-\alpha + 2\zeta) + \tfrac{1}{12}(-\alpha + 2\zeta)^2\right)\right] \\ &= 1 - e\alpha + \frac{e}{2}\alpha^2.\end{aligned}$$

From the isomorphism $\pi_{1*}(\mathcal{F}') \cong \mathcal{I}_\Gamma(m)$, we get

$$\mathrm{Ch}(\pi_{1*}\mathcal{F}') = 1 + m\alpha + \frac{m^2}{2}\alpha^2.$$

Since

$$\mathrm{Ch}(R^1\pi_{1*}\mathcal{F}') = \mathrm{Ch}(\mathcal{O}_\Gamma) = [\Gamma] = \gamma\alpha^2,$$

we have

$$1 - e\alpha + \frac{e}{2}\alpha^2 = 1 + m\alpha + \left(\frac{m^2}{2} - 2\gamma\right)\alpha^2.$$

From the degree-1 terms, we see that $m = -e$, and then equating the degree-2 terms we find that the degree of Γ is

$$\gamma = \frac{e^2 - e}{2}.$$

To express this in terms of the Chern classes of our original bundle $\mathcal{E}$, we observe that, by the splitting principle, for an arbitrary bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^2$ with first Chern class $c_1(\mathcal{E}) = (2k+1)\zeta$, the degree of the second Chern class of the twist $\mathcal{E}' = \mathcal{E}(-k-1)$ is

$$\begin{aligned}e &= c_2(\mathcal{E}) - (k+1)c_1(\mathcal{E})\zeta + (k+1)^2\zeta^2 \\ &= c_2(\mathcal{E}) - k(k+1).\end{aligned}$$

Proposition 14.9. *Let $\mathcal{E}$ be a vector bundle of rank 2 on $\mathbb{P}^2$ with odd first Chern class $(2k+1)\zeta$. Under the hypotheses introduced above, the locus $\Gamma \subset \mathbb{P}^{2*}$ of jumping lines is a set of $(e^2 - e)/2$ points counted with multiplicities, where $e = c_2(\mathcal{E}) - k(k+1)$.*

As in the case of Proposition 14.8, this statement implicitly invokes a scheme structure on Γ that we have not described; again, however, even in the absence of this we can deduce that the number of jumping lines is at most $(e^2 - e)/2$, and if $e^2 - e \neq 0$ we may conclude that the locus of jumping lines is nonempty.

14.4.3 Examples

Suppose that F_0, F_1 and F_2 are general homogeneous polynomials of degree d on $\mathbb{P}^2$, and let $\mathcal{E}$ be the kernel of the bundle map

$$\mathcal{O}_{\mathbb{P}^2}(-d)^{\oplus 3} \xrightarrow{(F_0, F_1, F_2)} \mathcal{O}_{\mathbb{P}^2}.$$

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Note that since F_0, F_1 and F_2 have no common zeros this is a surjection, so that $\mathcal{E}$ is locally free of rank 2.

By Whitney's formula we have

$$c(\mathcal{E}) = \frac{1}{(1 - d\zeta)^3} = 1 + 3d\zeta + 6d^2\zeta^2 \in A(\mathbb{P}^2).$$

For odd degree $d = 2k + 1$, Proposition 14.9 suggests that the number of jumping lines should be the binomial coefficient $\binom{3k^2+3k+1}{2}$.

For example, when $d = 1$ (hence $k = 0$) this number is 0. It is easy to see that this is correct: If $L = \mathbb{P}^1$ is a line then (after applying a suitable element of GL_3 to $\mathcal{O}_{\mathbb{P}^2}(-d)^{\oplus 3}$, that is, replacing F_0, F_1 and F_2 with independent linear combinations) we may assume that L is given by the equation $F_2 = 0$; then F_0, F_1 necessarily restrict to independent linear forms $\overline{F}_1$, $\overline{F}_2$ on L. Thus

$$\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \xrightarrow{(\overline{F}_0, \overline{F}_1, 0)} \mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Put differently, the bundle $\mathcal{E}$ is invariant under the transitive group PGL(3) (it is the cotangent bundle of $\mathbb{P}^2$), and this group must preserve the locus of jumping lines.

When $d = 3$, Proposition 14.9 suggests that the number of jumping lines should be 21; we will see how to verify this count in Exercise 14.24.

For even degree $d = 2k$, we would expect from Proposition 14.8 that the curve of jumping lines, with multiplicity, will have degree $3k^2 = \frac{3}{4}d^2$. In case $d = 2$ this number is 3; in Exercise 14.23 we explain how to verify that it is actually a nonsingular cubic.

14.5 Application: invariants of families of curves

We close this chapter by describing one of the most important applications of the Grothendieck Riemann–Roch formula, a result that is central to the study of the moduli space of curves.

Consider one-parameter families of curves of genus g; that is, flat morphisms $\alpha : X \to B$ from a surface X to a curve B, with fibers curves of genus g. How can we measure how much the isomorphism class of X_b is varying with $b \in B$?

In order to address this meaningfully, we have to make some restrictions on the type of curves that appear as fibers of α (as the footnote below will illustrate). For simplicity, in the present discussion we will assume that X and B are smooth, and that the fibers X_b of α are irreducible with at worst nodes as singularities. These conditions could be relaxed: we could drop the requirement that X and B be smooth, and we could weaken the hypothesis that X_b be irreducible to require only that X_b be *stable* (see for example Harris and Morrison [1998]) — but our goal here is just to acquaint the reader with the basic ideas.

A parenthetical note: This discussion touches on a topic that, had we the energy and expertise and you the willingness to countenance an 800-page book, we would take up: intersection theory on *algebraic stacks*. Briefly, the category of stacks is an enlargement of the category of schemes — schemes form a full subcategory of stacks — in which we can find parameter spaces that may not exist in the category of schemes. In the present setting, the correct framework for the calculations we are about to undertake is to view δ, λ and κ as classes in $A^1(\overline{\mathcal{M}}_g)$, where $\overline{\mathcal{M}}_g$ is the *moduli stack of stable curves*. A discussion of these ideas can be found in Harris and Morrison [1998, Chapters 2 and 3].

We consider three natural ways of quantifying the variation in the family:

(a) *Number of nodes*: If the family α were trivial, that is, if α were the projection of a product $C \times B$ to B, there would be no singular fibers. Thus the number of singular fibers is a measure of nontriviality.[1] It turns out (given that the total space X is smooth) to be natural to count a singular fiber with multiplicity equal to the number of nodes that appear in it. Thus we let $\delta(\alpha)$ be the total number of nodes appearing in the fibers of α. (Again, we are being unnecessarily restrictive here; it is possible to assign multiplicities without the assumption that X is smooth, but it requires additional complication.)

(b) *Degree of the Hodge bundle*: Recall that any projective curve C has a *dualizing sheaf* ω_C, which is the cotangent bundle if the curve is smooth and is defined in general to be $\omega_C := \operatorname{Ext}^{n-1}_{\mathbb{P}^n}(\mathcal{O}_C, \omega_{\mathbb{P}^n})$ for any embedding $C \subset \mathbb{P}^n$. Moreover, the dimension of $H^0(\omega_C)$ is equal to the arithmetic genus $p_a(C)$ of C.

By the theorem on cohomology and base change (Theorem B.5), the quantity

$$\chi(\mathcal{O}_{X_b}) = h^0(\mathcal{O}_{X_b}) - h^1(\mathcal{O}_{X_b}) = 1 - p_a(X_b)$$

is independent of the point $b \in B$, so $h^0(\omega_{X_b})$ is independent of b. The *relative dualizing sheaf*

$$\omega_{X/B} = \omega_X \otimes \alpha^* \omega_B^{-1}$$

of the family restricts on each fiber X_b to ω_{X_b}. Thus

$$\mathcal{E} := \alpha_*(\omega_{X/B})$$

is a vector bundle on B of rank g. It is called the *Hodge bundle* of the family.

Set

$$\lambda(\alpha) = c_1(\mathcal{E}),$$

the *degree of the Hodge bundle*. If the family were trivial then $\mathcal{E}$ would be the trivial bundle, so $\lambda(\alpha)$ would be 0.

[1] This would not be true if we did not assume that the fibers were irreducible: You could take a trivial family $B \times C \to B$ and blow-up a point in $B \times C$ to arrive at a family of curves of constant modulus with one singular fiber. The logic would be valid if we assumed all fibers to be *stable curves*; in the present context we avoid getting into a discussion of this notion by assuming all fibers to be irreducible.

(c) *Self-intersection of the relative canonical divisor*: Set

$$\kappa(\alpha) = c_1(\omega_{X/B})^2.$$

If the family were trivial then the relative cotangent bundle $\omega_{X/B}$ would be the pullback of ω_C via projection on the second factor, so its self-intersection number $\kappa(\alpha)$ would be 0.

The *Mumford relation* is a linear relation among these three numerical invariants:

Theorem 14.10. *If $\alpha : X \to B$ is a morphism from a smooth projective surface X to a smooth projective curve B whose fibers are irreducible curves g having at most nodes as singularities, then*

$$\lambda(\alpha) = \frac{\kappa(\alpha) + \delta(\alpha)}{12}.$$

We will give the proof in Section 14.5.2; see Harris and Morrison [1998, Section 3E] for a proof in greater generality and Mumford [1983] for many extensions. First, however, we show how to compute an example; further examples are in the exercises, and together these show that the Mumford relation is the only linear relation satisfied by δ, κ and λ.

Though there are no other linear relations, these invariants satisfy some subtle inequalities. For example, if $X \to B$ is a one-parameter family of irreducible curves of genus g having at most nodes as singularities and not all singular, then

$$\delta \leq \left(8 + \frac{4}{g}\right)\lambda;$$

this is sharp, as shown by the example of Exercise 14.31. For a discussion of these questions, with references to the literature, see Harris and Morrison [1998].

14.5.1 Example: pencils of quartics in the plane

We will compute the invariants δ, κ, λ for a general pencil

$$\{C_t = V(t_0 F + t_1 G) \subset \mathbb{P}^2\}$$

of plane quartic curves. Set

$$X = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid p \in C_t\}.$$

We observe that X is smooth — the projection $\beta : X \to \mathbb{P}^2$ on the second factor expresses X as the blow-up of the plane at the 16 base points $F = G = 0$ of the pencil. We showed in Section 7.1 that every fiber C_t of $\alpha : X \to \mathbb{P}^1$ is either smooth or is irreducible with a single node; thus the family $\alpha : X \to \mathbb{P}^1$ satisfies the conditions of our discussion.

In Chapter 7 we computed that the number of nodes in the fibers of α is

$$\delta(\alpha) = 27.$$

To compute $\kappa(\alpha)$, let L be the preimage under β of a line in $\mathbb{P}^2$ and let E be the sum of the 16 exceptional divisors of β, the preimages of the base points in $\mathbb{P}^2$ of the family α. Let $l, e \in A^1(X)$ denote their classes. In these terms, the class of the fibers C_t of the projection α — that is, the pullback under α of the class η of a point in $\mathbb{P}^1$ — is given by

$$[C_t] = \alpha^*(\eta) = 4l - e.$$

In particular, this implies that

$$\alpha^* K_{\mathbb{P}^1} = -2\alpha^*(\eta) = -8l + 2e.$$

On the other hand, from the blow-up map $\beta : X \to \mathbb{P}^2$ we see that

$$K_X = \beta^*(K_{\mathbb{P}^2}) + E = -3L + E.$$

Thus, the first Chern class of the relative dualizing sheaf $\omega_{X/\mathbb{P}^1}$ is

$$c_1(\omega_{X/\mathbb{P}^1}) = c_1(K_X) - \alpha^* c_1(K_{\mathbb{P}^1}) = 5l - e$$

and the invariant $\kappa(\alpha)$ is given by

$$\kappa(\alpha) = c_1(\omega_{X/\mathbb{P}^1})^2 = 25 - 16 = 9.$$

To compute λ we describe the Hodge bundle: Since $5L - E = (4L - E) + L$, we can write the dualizing sheaf as

$$\omega_{X/\mathbb{P}^1} = \mathcal{O}_X(5L - E) = \alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1).$$

Now, the pushforward $\alpha_*(\beta^*\mathcal{O}_{\mathbb{P}^2}(1))$ is simply the trivial bundle on $\mathbb{P}^1$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$; by Proposition B.7 we have then

$$\begin{aligned}\alpha_*(\omega_{X/\mathbb{P}^1}) &= \alpha_*(\alpha^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathcal{O}_{\mathbb{P}^2}(1))\\ &= \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1))\\ &\cong \mathcal{O}_{\mathbb{P}^1}(1)^3,\end{aligned}$$

and correspondingly

$$\lambda(\alpha) = c_1(\alpha_*(\omega_{X/\mathbb{P}^1})) = 3.$$

Another, more concrete way to arrive at this is suggested in Exercise 14.27.

To summarize, we have

$$\delta(\alpha) = 27, \quad \kappa(\alpha) = 9 \quad \text{and} \quad \lambda(\alpha) = 3,$$

verifying Mumford's relation in this case.

Further examples of one-parameter families of curves for which we can verify the Mumford relation are given in Exercises 14.28–14.30.

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14.5.2 Proof of the Mumford relation

Proof of Theorem 14.10: The Grothendieck Riemann–Roch theorem tells us that

$$\sum_i (-1)^i \operatorname{Ch}(R^i \alpha_* \omega_{X/B}) = \alpha_*[\operatorname{Ch}(\omega_{X/B}) \operatorname{Td}(\mathcal{T}^v_{X/B})],$$

and we recall that $\operatorname{Td}(\mathcal{T}^v_{X/B}) = \operatorname{Td}(\mathcal{T}_X)/\alpha^*(\operatorname{Td}(\mathcal{T}_B))$. Thus

$$\begin{aligned} c_1(\alpha_*\omega_{X/B}) &= \operatorname{Ch}_1(R^0\alpha_*\omega_{X/B}) \\ &= \operatorname{Ch}_1(R^1\alpha_*\omega_{X/B}) + \left\{\alpha_*(\operatorname{Ch}(\omega_{X/B}) \operatorname{Td}(\mathcal{T}^v_{X/B}))\right\}_0 \\ &= c_1(R^1\alpha_*\omega_{X/B}) + \alpha_*\{\operatorname{Ch}(\omega_{X/B}) \operatorname{Td}(\mathcal{T}^v_{X/B})\}_0. \end{aligned}$$

We will compute the terms on the right.

We first claim that $R^1\alpha_*(\omega_{X/B}) = \mathcal{O}_B$, and thus $c_1(R^1\alpha_*(\omega_{X/B})) = 0$. This follows from a generalization of Serre duality due to Grothendieck, which we now describe:

Recall that Serre duality says that for any invertible sheaf $\mathcal{F}$ on a smooth curve C, we have a natural identification

$$H^1(\mathcal{F}) = \operatorname{Hom}(\mathcal{F}, \omega_C)^* = H^0(\mathcal{H}om(\mathcal{F}, \omega_C))^*.$$

Although this is sometimes stated just for nonsingular curves, it holds for any purely one-dimensional scheme C. Moreover, the identification is so natural that, properly formulated, it applies to families of such schemes: If $\alpha : X \to B$ is a family of projective curves of genus g and $\mathcal{F}$ is an invertible sheaf on X whose cohomology groups have constant rank on the fibers of α, then there is a natural isomorphism

$$R^1\alpha_*(\mathcal{F}) \cong \alpha_*(\mathcal{H}om(\mathcal{F}, \omega_{X/B}))^*;$$

see Barth et al. [2004, Section III.12]. If we apply this to the case of $\mathcal{F} = \omega_{X/B}$, we get

$$R^1\alpha_*(\omega_{X/B}) = \alpha_*(\mathcal{O}_X) = \mathcal{O}_B.$$

We next compute the necessary Todd classes. Recall the *Hopf index theorem* (Theorem 5.20): The degree of the top Chern class $c_2(\mathcal{T}_X)$ of the tangent bundle of the smooth projective surface X is the topological Euler characteristic of X. Also, the topological Hurwitz formula (Section 7.7) says that the topological Euler characteristic of X is the product of the Euler characteristics $2-2g$ of the general fiber of α and the Euler characteristic of the base curve B, plus the total number δ of nodes in the fibers of α. Combining these, we see that

$$\begin{aligned} c_2(\mathcal{T}_X) &= (2-2g)c_1(\mathcal{T}_B) + \delta \\ &= -c_1(\omega_{X/B}) \cdot \alpha^* c_1(\mathcal{T}_B) + \delta \\ &= (c_1(\mathcal{T}_X) - c_1\alpha^*\mathcal{T}_B) \cdot \alpha^* c_1(\mathcal{T}_B) + \delta. \end{aligned}$$

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Abbreviating the expression $\omega_{X/B}$ for the relative dualizing sheaf to simply ω, we can thus express the ratio $c(\mathcal{T}_X)/\alpha^* c(\mathcal{T}_B)$ as

$$\begin{aligned}\frac{c(\mathcal{T}_X)}{\alpha^* c(\mathcal{T}_B)} &= (1 + c_1(\mathcal{T}_X) + c_2(\mathcal{T}_X))(1 - \alpha^* c_1(\mathcal{T}_B)) \\ &= 1 - c_1(\omega) + \delta,\end{aligned}$$

where we have used the equality $c_1(\alpha^*(\mathcal{T}_B))^2 = \alpha^*(c_1(\mathcal{T}_B)^2) = 0$. Substituting these classes into the formulas for the Todd classes of degrees 0, 1, and 2, we get

$$\frac{\operatorname{Td}(\mathcal{T}_X)}{\alpha^* \operatorname{Td}(\mathcal{T}_B)} = 1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12}.$$

We now have everything we need to apply Corollary 14.6, and we conclude that

$$\begin{aligned}\lambda(\alpha) &= c_1(\alpha_* \omega) \\ &= \left\{\alpha_*\left(\left(1 + c_1(\omega) + \frac{c_1(\omega)^2}{2}\right)\left(1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12}\right)\right)\right\}_0 \\ &= \alpha_*\left(\frac{c_1(\omega)^2}{2} - \frac{c_1(\omega)^2}{2} + \frac{c_1(\omega)^2 + \delta}{12}\right) \\ &= \frac{\kappa(\alpha) + \delta(\alpha)}{12}.\end{aligned}$$

□

14.6 Exercises

Exercise 14.11. (a) Find the Chern characters of the universal bundles $\mathcal{S}$ and $\mathcal{Q}$ on $\mathbb{G} = \mathbb{G}(1,3)$.

(b) Use this to find the Chern character of the tangent bundle $\mathcal{T}_{\mathbb{G}} = \mathcal{S}^* \otimes \mathcal{Q}$.

(c) Use this in turn to find the Chern class of $\mathcal{T}_{\mathbb{G}}$.

Exercise 14.12. Verify that the definition of the Chern class of a coherent sheaf given in Section 14.2.1 is well-defined; that is, $c(\mathcal{F})$ does not depend on the choice of resolution.

Exercise 14.13. Let $p \in \mathbb{P}^n$ be a point. Using the Koszul complex, show that the Chern class of the structure sheaf $\mathcal{O}_p$, viewed as a coherent sheaf on $\mathbb{P}^n$, is

$$c(\mathcal{O}_p) = 1 + (-1)^{n-1}(n-1)!\,[p].$$

Exercise 14.14. Let $C \subset \mathbb{P}^3$ be a smooth curve. Find the Chern class of the structure sheaf $\mathcal{O}_C$, viewed as a coherent sheaf on $\mathbb{P}^3$, when:

(a) C is a twisted cubic.

(b) C is an elliptic quartic curve.

(c) C is a rational quartic curve.

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Exercise 14.15. Consider the three varieties $X_1 = \mathbb{P}^3$, $X_2 = \mathbb{P}^1 \times \mathbb{P}^2$ and $X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(a) In each case, calculate the degrees of the classes $c_3(\mathcal{T}_{X_i})$, $c_1(\mathcal{T}_{X_i})c_2(\mathcal{T}_{X_i})$ and $c_1(\mathcal{T}_{X_i})^3$.
(b) Show that the resulting 3×3 matrix is nonsingular.
(c) Show that the Euler characteristic $\chi(\mathcal{O}_{X_i})$ is 1 for each i.
(d) Given that the Euler characteristic of the structure sheaf of a smooth projective threefold X is expressible as a polynomial of degree 3 in the Chern classes of its tangent bundle, show from the above examples that the polynomial must be

$$\mathrm{Td}_3(c_1, c_2, c_3) = \frac{c_1 c_2}{24}.$$

Exercise 14.16. Verify that the formula of Theorem 14.4 gives the classical Riemann–Roch formula for $n = 1$ or 2, and write down the analogous formula for $n = 3$.

Exercise 14.17. In Section 7.2 we introduced a vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ whose fiber at a point $p \in \mathbb{P}^2$ is the space $H^0(\mathcal{O}_{\mathbb{P}^2}(d)/\mathcal{I}_p^2(d))$ of homogeneous polynomials of degree d, modulo those vanishing to order 2 or more at p; we also described $\mathcal{E}$ as a direct image. Use this and the Grothendieck Riemann–Roch formula to calculate the Chern classes of $\mathcal{E}$.

Exercise 14.18. In Chapter 11 we introduced a vector bundle $\mathcal{E}$ on the universal line

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\}$$

whose fiber at a point $(L, p) \in \Phi$ is the space $H^0(\mathcal{O}_L(d)/\mathcal{I}_p^5(d))$ of homogeneous polynomials of degree d on L, modulo those vanishing to order 5 or more at p; we also described $\mathcal{E}$ as a direct image. Use this and the Grothendieck Riemann–Roch formula to calculate the Chern classes of $\mathcal{E}$.

Exercise 14.19. Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree d. Construct a bundle $\mathcal{E}$ on the dual plane $\mathbb{P}^{2*}$ whose fiber at a point L is the space of sections of the structure sheaf $\mathcal{O}_\Gamma$ of the intersection $\Gamma = C \cap L$, and use the Grothendieck Riemann–Roch formula to calculate the Chern classes of $\mathcal{E}$.

Exercise 14.20. Let $\mathcal{E}$ be an indecomposable vector bundle of rank 2 on $\mathbb{P}^3$ with first Chern class 0, and let

$$\Gamma_1 = \{L \in \mathbb{G}(1,3) \mid \mathcal{E}|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k) \text{ with } k \geq 1\}.$$

Find the class of the divisor $\Gamma_1 \subset \mathbb{G}(1,3)$.

Exercise 14.21. With $\mathcal{E}$ as in the preceding problem, let

$$\Gamma_2 = \{L \in \mathbb{G}(1,3) \mid \mathcal{E}|_L \cong \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k) \text{ with } k \geq 2\}.$$

Find the class of the locus $\Gamma_2 \subset \mathbb{G}(1,3)$, assuming it has the expected dimension 1.

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Exercise 14.22. Now let $\mathcal{E}$ be a vector bundle of rank 2 on $\mathbb{P}^3$ with first Chern class $c_1(\mathcal{E}) = \zeta$ (the hyperplane class), and let

$$\Phi_i = \{L \in \mathbb{G}(1,3) \mid \mathcal{E}|_L \cong \mathcal{O}_L(k+1) \oplus \mathcal{O}_L(-k) \text{ with } k \geq i\}$$

for $i = 1, 2$. Find the classes of the loci $\Phi_i \subset \mathbb{G}(1,3)$, assuming they have the expected codimension $2i$.

Exercise 14.23. Let F_0, F_1, F_2 be three general quadratic forms in three variables, and define a bundle $\mathcal{E}$ on $\mathbb{P}^2$ as the kernel of the surjection

$$\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \xrightarrow{(F_0, F_1, F_2)} \mathcal{O}_{\mathbb{P}^2}.$$

Prove the locus C of jumping lines of $\mathcal{E}$ is a nonsingular cubic curve in $\mathbb{P}^2$ as follows:

(a) Show that C is also the locus of jumping lines of

$$\mathcal{E}^* = \operatorname{coker}\Big(\mathcal{O}_{\mathbb{P}^2} \xrightarrow{(F_0, F_1, F_2)} \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 3}\Big).$$

(b) Show that $\mathcal{E}^*|_L$ contains a copy of $\mathcal{O}_L(2)$ as a summand if and only if some linear combination of F_0, F_1, F_2 vanishes identically on L.

(c) We may represent each F_i as a general symmetric 3×3 matrix of scalars Q_i. Introducing new variables z_0, z_1, z_2, we see from the previous item that the curve of jumping lines is a double cover of the smooth cubic curve defined in coordinates z_0, z_1, z_2 by the equation $\det\big(\sum_i z_i Q_i\big) = 0$.

Exercise 14.24. Let $\mathcal{E}$ be defined as in Exercise 14.23, but now take the F_i to be general cubics (and replace the occurrences of $\mathcal{O}_{\mathbb{P}^2}(-2)$ with $\mathcal{O}_{\mathbb{P}^2}(-3)$). Show that the jumping lines of 0 are exactly the lines in $\mathbb{P}^2$ contained in some curve of the form $\sum a_i F_i = 0$. Show that there are exactly 21 of these by observing that this is the degree of the seven-dimensional variety of reducible plane cubics in the $\mathbb{P}^9$ of all plane cubics.

Exercise 14.25. As an example, let $q : \Bbbk^4 \times \Bbbk^4 \to \Bbbk$ be a nondegenerate skew-symmetric bilinear form. We can define a bundle of rank 2 on $\mathbb{P}^3$ by setting

$$0_p = \langle p\rangle^{\perp}/\langle p\rangle.$$

Describe the locus of jumping lines for such a bundle.

Exercise 14.26. As in Section 14.4.2, let $L \in U$ be a point in an open subset of $\mathbb{P}^{2*}$ such that the splitting of $\mathcal{F}'|_L$ is $\mathcal{F}'|_L = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and the splitting of $\mathcal{F}'|_{L'}$ for $L' \in U$ other than L is $\mathcal{F}'|_{L'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Show that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\Phi_U}(-2)^2 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{O}_{\Phi_U}(1) \longrightarrow 0$$

and a resolution

$$0 \longrightarrow \mathcal{O}_{\Phi_U}(-2)^2 \longrightarrow \mathcal{O}_{\Phi_U}(-1)^3 \longrightarrow \mathcal{O}_{\Phi_U}(1) \longrightarrow 0,$$

and use these to deduce the presentation

$$0 \longrightarrow \mathcal{O}_{\Phi_U}(-2)^2 \longrightarrow \mathcal{O}_{\Phi_U}(-2) \oplus \mathcal{O}_{\Phi_U}(-1)^3 \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Exercise 14.27. We will show how to arrive at the result $\lambda(\alpha) = 3$ of Section 14.5.1 more concretely. Choose affine coordinates t on $\mathbb{A}^1 \subset \mathbb{P}^1$ and (x, y) on $\mathbb{A}^2 \subset \mathbb{P}^2$, and write the equation of C_t as $f_t(x, y)$, where f_t is a quartic polynomial in x and y whose coefficients are linear in t.

(a) Show that the differential

$$\varphi_t = \frac{dx}{(\partial/\partial y) f_t(x, y)}$$

is regular on C_t, as are $x\varphi_t$ and $y\varphi_t$.

(b) Show that these differentials give rise to sections φ, $x\varphi$ and $y\varphi$ of the Hodge bundle $\mathcal{E} = \alpha_*(\omega_{X/B})$ that are everywhere linearly independent for $t \neq \infty$, and that in a neighborhood of $t = \infty$ the sections $t\varphi$, $tx\varphi$ and $ty\varphi$ are everywhere regular and linearly independent.

(c) Deduce that the Hodge bundle $\mathcal{E} = \alpha_*(\omega_{X/B})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$, and in particular that $\lambda(\alpha) = 3$.

Exercise 14.28. Let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves of degree d. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 14.29. Let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of plane sections of a smooth surface $S \subset \mathbb{P}^3$ of degree d. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 14.30. Let $\{C_t \subset \mathbb{P}^1 \times \mathbb{P}^1\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of bidegree (a, b) in $\mathbb{P}^1 \times \mathbb{P}^1$. Calculate the numerical invariants δ, κ and λ for the family, and verify the Mumford relation.

Exercise 14.31. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a general curve of bidegree $(2, 2g + 2)$, and let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over C. Viewing $X \to \mathbb{P}^1$ as a family of hyperelliptic curves of genus g via projection on the first factor, calculate the invariants δ, κ and λ for the family; verify the Mumford relation, and also show that the inequality

$$\delta \leq \left(8 + \frac{4}{g}\right)\lambda$$

stated in Section 14.5 is sharp.