D-Modules, Unit F-Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

Isaac Martin

Last compiled February 28, 2022

Contents

1 D-module

Here we cover basic definitions and theorems in the theory of D-modules with a heavy emphasis on examples.

1.1 Weyl Algebra

Let K be a field of characteristic 0. We construct the Weyl algebra in two ways and prove that these constructions produce isomorphic rings.

Definition 1.1. Let K be a field of characteristic 0 and let $K[X] = K[x_1,...,x_n] = \Gamma(X, \mathcal{O}_X)$ be the polynomial ring over K in n variables, and let $X = \mathbb{A}^n_K = \mathbb{A}^n$. Consider the algebra of K-linear operators $\mathrm{End}_K(K[X])$ and more specifically the operators $\hat{x_i}, \partial_j \in \mathrm{End}_K(K[X])$ for $1 \leq, i, j \leq n$. These are defined

$$\hat{x_i} : K[X] \longrightarrow K[X],\ f \mapsto x_i \cdot f$$

and

$$\partial_j : K[X] \longrightarrow K[X],\ f \mapsto \frac{\partial f}{\partial x_j}.$$

These are both linear operators, and they satisfy the relation

$$[\partial_j, \hat{x_i}] = \partial_j \hat{x_i} - \hat{x_i} \partial_j = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$ and is otherwise 0.

Since $K[\hat{x}] \cong K[x]$ as rings, we typically drop the hat notation and simply write x_i for $\hat{x_i}$. For any two operators $A, B \in \mathrm{End}(R)$ we write $[A,B] = AB - BA$. The commutator is a K-bilinear map on $\mathrm{End}(R)$.

We can also write the Weyl algebra down as a quotient of a free algebra in $2n$ generators over K.

Definition 1.2. The free algebra $K\{x_1,...,x_{2n}\}$ in $2n$ generators is the set of K-linear combinations of words in $x_1,...,x_{2n}$. Multiplication is given by concatenation on monomials and then extended to arbitrary elements by the distributive property. We have a homomorphism

$$\phi : K\{x_1,...,x_{2n}\} \longrightarrow A_n$$

given by $x_i \mapsto x_i$ and $x_{i+n} \mapsto \partial_i$ for $1 \leq i \leq n$. Let J be the two-sided ideal of $K\{x_1,...,x_{2n}\}$ generated by $[x_{i+n}, x_i] - 1$ for $1 \leq i \leq n$. Each of these generators is mapped to zero in A_n by the relations in Definition (1.1), so $J \subseteq \ker \phi$. We therefore obtain a map $\hat{\phi} : Kx_1,...,x_{2n}/J \longrightarrow A_n$ induced by ϕ.

Theorem 1.3. *The map $\hat{\phi}$ is an isormophism.*

To summarize, in A_n,

- x_i and x_j commute
- ∂_i and ∂_j commute
- $[\partial_i, x_j] = \delta_{ij}$, that is, ∂_i and x_j commute unless $i = j$.

Example 1.4. Given a polynomial $f \in K[x]$, we can think of f as an operator in $\mathrm{End}_K(K[x])$ by the map $x \mapsto \hat{x}$, and the operator f is simply given by multiplication by f. I claim that the commutator of f with ∂ satisfies the following relation: $[\partial, f] = f'$ where f' is the derivative of f. To see this, it suffices to show that $[\partial, x^n] = nx^{n-1}$ for $n \in \mathbb{Z}_{\geq 0}$, since $[-,-]$ is K-bilinear.