1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the* constant presheaf *associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let $\mathcal{A}^{pre}$ be the constant presheaf. There is an obvious morphism of sheaves $\mathcal{A}^{pre} \to \mathcal{A}$ which sends an element $a \in \mathcal{A}^{pre}(U) = A$ to the constant map $U \to A$. This induces a morphism from the sheafification of $\mathcal{A}^{pre}$ to $\mathcal{A}$ which we claim is an isomorphism. To see that it is an isomorphism we need only check the stalks, and since stalks are preserved under sheafification, we need only check that $\mathcal{A}^{pre} \to \mathcal{A}$ induces an isomorphism on the stalks. Clearly, the stalks of $\mathcal{A}^{pre}$ are A. Now consider a representative of the stalk of $\mathcal{A}$ at P. That is, an open set $U \ni P$ and section $s : U \to A$. The preimage $s^{-1}(s(P))$ of the value of s at P is an open subset of U on which the restriction of s is constant. Hence, every element of the stalk can be represented using a constant section and therefore $\mathcal{A}_P = A$.

Exercise 1.2.

a *For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ show that for each point P, $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.*

b *Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P.*

c *Show that a sequence $\cdots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.*

Solution.

a Recall that filitered colimits commute with finite limits in the category of sets. Now kernel is a finite limit and stalk is a filtered colimit. Image is the kernel of $\mathscr{G} \to \operatorname{coker} \phi$.

b ϕ is injective if and only if its kernel is zero, and *phi* is surjective if and only if its cokernel is zero if and only if its image is $\mathscr{G}$. So these follow from part (a).

c Exactness can be stated as $\operatorname{im} \phi^i = \ker \phi^{i+1}$ and so it follows from part (a).

Exercise 1.3.

a *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that ϕ s surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\phi(t_i) = s|_{U_i}$ for all i.*

b *Give an example of a surjective morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.*

Solution.

a This is equivalent to saying that ϕ is surjective on each stalk.

b Consider the sheaf of holomorphic functions on $\mathbb{C} - \{0\}$ and the map $f \mapsto exp(f)$. For every holomorphic function defined on some open set of $\mathbb{C} - \{0\}$ we can write it locally as $f = \log g$ for some f so this morphism is surjective on stalks. Globally, we cannot.

Exercise 1.4. *a Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.*

b Use part (a) to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{im}\phi$ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Solution. a Sheafification preserves stalks, now use Exercise 1.2(a).

b The image is defined as the sheafification of the "presheaf image" which is certainly a subpresheaf of $\mathscr{G}$. Sheafification preserves injective morphisms and sheaves and so the image is a subsheaf of $\mathscr{G}$.

Exercise 1.5. *Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.*

Solution. Exercise 1.2(b) and Proposition 1.1.

Exercise 1.6. *a Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

b Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Solution. a Sheafification is a left adjoint and therefor preserves colimits (i.e. preserves surjections).

b The forgetful functor is the right adjoint to sheafification. Since it is a right adjoint it preserves kernels and so $0 \to \mathscr{F}' \to \mathscr{F}$ is exact as a sequence of presheaves. That is, $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$. Then use part (a).

Exercise 1.7. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

a Show that $\operatorname{im}\phi \cong \mathscr{F}/\ker\phi$.

b Show that $\operatorname{coker}\phi \cong \mathscr{G}/\operatorname{im}\phi$.

Solution. Follows from Exercises 1.6 and 1.4(b).

Exercise 1.8. *For any open subset $U \subseteq X$, show that the functor $\Gamma(U, -)$ from sheaves on X to abelian groups is a left exact functor.*

Solution. Since U is an open subset, there is a morphism of sites $i : X_{open} \to U_{open}$ with underlying functor the inclusion $U_{open} \to X_{open}$. We also have a morphism of sites defined by the continuous morphism $p : U \to \bullet$ of U to a point. Global sections of a sheaf $\mathscr{F}$ on U is then $p_* i_* \mathscr{F}$. Since both p_* and i_* are right adjoints, they are left exact.

Exercise 1.9. Direct sum. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf.*

Solution. A consequence of the forgetful functor preserving limits.

Exercise 1.10. Direct Limit. *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limite to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a diret limit in the category of sheaves on X.*

Solution. Sheafification is a left adjoint and so preserves colimits.

Exercise 1.11. *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.*

Solution. This is again a consequence of finite limits commuting with filtered colimits. Since each $\mathscr{F}_n$ is a sheaf, given an open set U and a cover $\{U_i \to U\}$ we can write $\mathscr{F}_n(U)$ as the limit $\varprojlim \mathscr{F}_n(U_{ij})$ where the limit is indexed by double intersections with inclusions as morphisms.

If the space is Noetherian, then we can choose the cover to be finite, hence, the limit is finite. So now for any open cover $\{U_i \to U\}$ (which we can choose to be finite)

$$\begin{aligned}\varprojlim_{ij}(\varinjlim_n \mathscr{F}_n)(U_{ij}) &= \varprojlim_{ij}(\varinjlim_n \mathscr{F}_n(U_{ij})) = \varinjlim_n(\varprojlim_{ij} \mathscr{F}_n(U_{ij})) \\ &= \varinjlim_n \mathscr{F}_n(U) = (\varinjlim_n \mathscr{F}_n)(U)\end{aligned}$$

Exercise 1.12. Inverse limit. *Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the section wise inverse limit is a sheaf.*

Solution. Same as the previous solution but since arbitrary limits commute, we don't need to assume the cover to be finite.

Exercise 1.13. Espace Étalé of a Presheaf. *Show that the sheaf $\mathscr{F}^+$ associated to a presheaf $\mathscr{F}$ can be described as follows: for any open sets $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of continuous sections of Spé$(\mathscr{F})$ over U.*

Solution. Let U be an open subset of X, and consider $s \in \mathscr{F}^+(U)$. We must show that $s : U \to$Spé$(\mathscr{F})$ is continuous. Let $V \subseteq$Spé$(\mathscr{F})$ be an open subset and consider the preimage $s^{-1}V$. Suppose $P \in X$ is in the preimage of V. Since $s(Q) \in \mathscr{F}_Q$ for each point $Q \in X$, we see that $P \in U$. This means that there is an open neighbourhood U' of P contained in P and a section $t \in \mathscr{F}(U')$ such that for all $Q \in U'$, the germ $t_{U'}$ of t at U' is equal to $s(U')$. That is, $s|_{U'} = t$.

So we have $s|_{U'}^{-1}(V) = t^{-1}(V)$, which is open since by definition of the topology on Spé($\mathscr{F}$), t is continuous. So there is an open neighbourhood $t^{-1}(V)$ of P that is contained in the preimage. The point P was arbitrary, and so we have shown that every point in the preimage $s^{-1}V$ has an open neighbourhood contained in the preimage $s^{-1}V$. Hence, it is the union of these open neighbourhoods, and therefore open itself. So s is continuous.

Now suppose that $s : U \to$Spé($\mathscr{F}$) is a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. First we show that for any open V and any $t \in \mathscr{F}(V)$, the set $t(V) \subset$Spé($\mathscr{F}$) is open. To see this, recall that the topology on Spé($\mathscr{F}$) is defined as the strongest such that every morphism of this kind is continuous. If we have a topology $\mathscr{U}$ (where $\mathscr{U}$ is the collection of open sets) on Spé($\mathscr{F}$) such that each $t \in \mathscr{F}(U)$ is continuous, and $W \in$Spé($\mathscr{F}$) has the property that $t^{-1}W$ is open in X for any $t \in \mathscr{F}(V)$ and any open V, then the topology generated by $\mathscr{U} \cup \{W\}$ also has the property that each $t \in \mathscr{F}(U)$ is continuous. So since we are taking the strongest topology such that each $t \in \mathscr{F}(U)$ is continuous, if a subset $W \subset$Spé($\mathscr{F}$) has the property that $t^{-1}W$ is open in U for each $t \in \mathscr{F}(U)$, then W is open in Spé($\mathscr{F}$). Now fix one $s \in \mathscr{F}(U)$ and consider $t \in F(V)$. For a point $x \in t^{-1}s(U)$, it holds that $s(x) = t(x)$. That is, the germs of t and s are the same at x. This means that there is some open neighbourhood W of x contained in both U and V such that $s|_W = t|_W$, and hence $s = t$ for every $y \in W$ so $W \subset t^{-1}s(U)$. Since every point in $t^{-1}s(U)$ has an open neighbourhood in $t^{-1}s(U)$, we see that $t^{-1}s(U)$ is open and therefore by the reasoning just discussed we see that $s(U)$ is open in Spé($\mathscr{F}$).

Now let $s : U \to$Spé($\mathscr{F}$) be a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. For every point $x \in U$, the image of x under s is some germ (t, W) in the stalk of $\mathscr{F}$ at x. That is, an open neighbouhood W of x (which we can assume is contained in U) and $t \in \mathscr{F}(W)$. Since s is continuous, and we have seen that $t(W)$ is open, it follows that $s^{-1}(t(W))$ is open in X. This means there is an open neighbourhood W' of x on which $t|_{W'} = s|_{W'}$. Since s is locally representable by sections of $\mathscr{F}$, it is a well defined section of $\mathscr{F}^+$.

Exercise 1.14. Support. *Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The* support *of s, denoted* $\operatorname{Supp} s$*, is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that* $\operatorname{Supp} s$ *is a closed subset of U. We define the* support *of $\mathscr{F}$,* $\operatorname{Supp} \mathscr{F}$*, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.*

Solution. We show that the complement of the support V is open. For every point $P \in V$, since P is not in the support the germ of the section s is zero. This means there is a neighourhood V_P of P on which s vanishes. Note that $V_P \cap \operatorname{Supp} s = \varnothing$ since an intersection would imply $s_Q = 0$ for all Q in the intersection. Now $V = \cup V_P$, a union of opens, therefore V is open.

An example of a sheaf whose support is not closed is $j_!\mathscr{F}$ from Exercise 1.19(b).

Exercise 1.15. Sheaf $\mathscr{H}om$. *Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set* $\hom(\mathscr{F}|_U, \mathscr{G}|_U)$ *of morphisms of* *the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \hom(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf.*

Solution. Suppose we have an open set U, a cover $\{U_i \to U\}$, and a set of natural transformations $\phi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$, that agree on restrictions to the U_{ij}. We define a natural transformation $\mathscr{F}|_U \to \mathscr{G}|_U$. Given an open subset $V \subset U$ we want a morphism $\mathscr{F}(V) \to \mathscr{G}(V)$. Since $\{V \cap U_i \to V\}$ is a cover of V, we can write $\mathscr{F}(V)$ and $\mathscr{F}(G)$ as a limit over $\{V \cap U_{ij}\}$ and we already have morphisms $\mathscr{F}(V \cap U_{ij}) \to \mathscr{G}(V \cap U_{ij})$ from our initial data. It doesn't matter which morphism we choose since the requirement that the ϕ_i agree on restrictions means they will be the same. So now we have a morphism $\mathscr{F}(V) \to \mathscr{G}(V)$ between the limits, it remains to show that these actually form a natural transformation, but this can be seen by drawing the appropriate diagram

$$\begin{tikzcd}
F(V) \arrow[rr] \arrow[dr] \arrow[dd] & & \prod F(V_i) \arrow[rr] \arrow[dr] \arrow[dd] & & \prod F(V_{ij}) \arrow[dr] \arrow[dd] & \\
& F(W) \arrow[rr] \arrow[dd] & & \prod F(W_i) \arrow[rr] \arrow[dd] & & \prod F(W_{ij}) \arrow[dd] \\
G(V) \arrow[rr] \arrow[dr] & & \prod G(V_i) \arrow[rr] \arrow[dr] & & \prod G(V_{ij}) \arrow[dr] & \\
& G(W) \arrow[rr] & & \prod G(W_i) \arrow[rr] & & \prod G(W_{ij})
\end{tikzcd}$$

Exercise 1.16. Flasque Sheaves. *A sheaf $\mathscr{F}$ on a topological space X is* flasque *if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.*

a *Show that a constant sheaf on an irreducible topological space is flasque.*

b *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.*

c *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.*

d *If $f : X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.*

e *Let $\mathscr{F}$ be any sheaf on X. Show that the sheaf of discontinuous sections dis $\mathscr{F}$ is flasque and that there is a natural injective morphism of $\mathscr{F}$ into des $\mathscr{F}$.*

Solution. a If X is irreducible then every open set is connected and we have already seen that a constant sheaf takes every connected open subset to the same set/group. So all the restrictions are identity morphisms and hence, it is flasque.

b The only thing to prove is that $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Let $f \in \mathscr{F}''(U)$. Since the morphism $\mathscr{F} \to \mathscr{F}''$ is a surjective morphism of sheaves there is a cover $\{U_i\}$ of U on which the restriction of f lifts to an element $\{f_i\}$ of $\mathscr{F}(U_i)$.

$$\begin{tikzcd}
\prod_{ij} \mathscr{F}'(U_{ij}) \arrow[r] & \prod_{ij} \mathscr{F}(U_{ij}) \arrow[r] & \prod_{ij} \mathscr{F}''(U_{ij}) \\
\prod_{i} \mathscr{F}'(U_{i}) \arrow[r] \arrow[u] & \prod_{i} \mathscr{F}(U_{i}) \arrow[r] \arrow[u] & \prod_{i} \mathscr{F}''(U_{i}) \arrow[u] \\
\mathscr{F}'(U) \arrow[r] \arrow[u] & \mathscr{F}(U) \arrow[r] \arrow[u] & \mathscr{F}''(U) \arrow[u]
\end{tikzcd}$$

Since f is a global section on U, the restriction to $\prod \mathscr{F}''(U_{ij})$ is zero and so the element $\{f_i|_{U_{ij}} - f_j|_{U_{ij}}\}$ gets sent to zero horizontally. Since sectionwise exactness in the middle is given automatically, this element pulls back horizontally to some $\{g_{ij}\}$. We have assumed $\mathscr{F}'$ to be flasque and so there is a $\{g_i\}$ in the preimage of $\{g_{ij}\}$.

$$\begin{tikzcd}
\{g_{ij}\} \arrow[r, dotted] & \{f_i|_{U_{ij}} - f_j|_{U_{ij}}\} \arrow[r, dotted] & 0 & \cdot & 0 & \cdot \\
\{g_i\} \arrow[u, dotted] & \{f_i\} \arrow[r, dotted] \arrow[u, dotted] & \{f|_{U_i}\} \arrow[u, dotted] & \cdot & \{f_i - g_i\} \arrow[r, dotted] \arrow[u, dotted] & \{f|_{U_i}\} \\
\cdot & \cdot & f \arrow[u, dotted] & \cdot & h \arrow[u, dotted] & \cdot
\end{tikzcd}$$

Now by commutivity of the diagram, $\{g_i - f_i\} \in \prod_i \mathscr{F}(U_i)$ is in the vertical kernel, and therefore lifts to some global section $h \in \mathscr{F}(U)$. Now if we push h up and to the right we get the restriction of f. So the image of h in $\mathscr{F}''(U)$ hasthe same restriction as f. Since $\mathscr{F}''$ is a sheaf this means that $h = f$ in $\mathscr{F}''(U)$. So we have found an element in the preimage of f.

c Let $V \subseteq U$ be open sets in the topological space X and consider the following diagram.

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] \arrow[u] & \mathscr{F}(V) \arrow[r] \arrow[u] & \mathscr{F}''(V) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[u] & \mathscr{F}(U) \arrow[r] \arrow[u] & \mathscr{F}''(U) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We know that the rows are exact by the previous part, and the columns are exact by the assumption that $\mathscr{F}'$ and $\mathscr{F}$ are flasque. It is now a straightforward diagram chase to find a preimage in $\mathscr{F}''(U)$ to anything in $\mathscr{F}''(V)$.

d For opens $V \subseteq U$ of Y we have

$$\big(f_*\mathscr{F}(V) \to f_*\mathscr{F}(U)\big) = \big(\mathscr{F}(f^{-1}V) \to \mathscr{F}(g^{-1}U)\big)$$

which is surjective.

e In general, for a subset I of set J indexing objects in some category C, the morphism $\prod_J X_j \to \prod_I X_i$ induced by the inclusion $I \subset J$ is surjective. The restriction morphisms are special cases of this where the indexing sets are points in the opens. The natural injective morphism $\mathscr{F} \to dis\ \mathscr{F}$ is clear. It is injective since two sections of a sheaf are the same if and only if they agree on stalks. If $\mathscr{F}$ were to be a nonseparated presheaf the morphism wouldn't be injective.

Exercise 1.17. Skyscraper Sheaves. *Let $i_P(A)$ be the skyscraper sheaf of a space X at a point P for an abelian group A. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$ and 0 elsewhere. Show that this sheaf can be described as $i_*(A)$, the pushforward of the constant sheaf A on $\overline{\{P\}}$.*

Solution. Let Q be a point. If $Q \notin \overline{\{P\}}$ then there is a neighbourhood $U \ni Q$ which doesn't contain P. Every element in the stalk can be represented by (s, W) where $W \subseteq U$ and there for $s = 0$. So the stalk is zero. Conversely, if $Q \in \overline{P}$, then every open neighbourhood of Q contains P, and so every group in the limit defining the stalk is A, with transition morphisms identities. Therefore the stalk is A.

Let $\mathcal{A}$ be the constant sheaf of $\overline{\{P\}}$. If $U \subset X$ doesn't contain P then $i^{-1}U = \varnothing$ and so $i_*(A)(U) = \mathcal{A}(i^{-1}U) = \mathcal{A}(\varnothing) = 0$. If U does contain P, then $i^{-1}U = \overline{\{P\}}$ and so $i_*(A)(U) = \mathcal{A}(i^{-1}U) = \mathcal{A}(\overline{\{P\}}) = \Gamma(\mathcal{A}, \overline{\{P\}})$. It remains only to show that $\overline{\{P\}}$ is connected so that $\Gamma(\mathcal{A}, \overline{\{P\}}) = A$ but this follows from it having a unique generic point.

Exercise 1.18. Adjoint Property of f^{-1}. *Show that f^{-1} is the left adjoint to f_*.*

Solution. If we denote f^{-1}_{pre} the functor that sends a sheaf $\mathscr{F}$ to the PREsheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ then we have

$$\hom_{Sh(Y)}(af^{-1}_{pre}\mathscr{F}, \mathscr{G}) \cong \hom_{PreSh(Y)}(f^{-1}_{pre}\mathscr{F}, \mathscr{G}) \cong \hom_{Sh(X)}(\mathscr{F}, f_*\mathscr{G})$$

The theory of Kan extensions shows that f^{-1}_{pre} is the left adjoint to f_* and we already know that sheafification a is the left adjoint to the forgetful functor. So the composition $f^{-1} = af^{-1}_{pre}$ is left adjoint to the "composition" f_*.

Exercise 1.19. Extending a Sheaf by Zero. *Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.*

a *Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$.*

b *Let $\mathscr{F}$ be a sheaf on U. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$.*

c *For $\mathscr{F} \in Sh(X)$ show that there is an exact sequence of sheaves on X*

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

Solution. a If $P \notin Z$ and $(U, s) \in (i_*\mathscr{F})_P$ then there is an open subset V of U containing P but not intersecting Z. Since V doesn't intersect Z, $(i_*\mathscr{F})(V) = 0$ and so $s|_V = 0$, hence $(U, s) = 0$. Now suppose that $P \in Z$ via $k : P \to Z$. The stalk $(i_*\mathscr{F})_P$ is the group of global sections of $(ik)^*(i_*\mathscr{F}) = k^*i^*i_*\mathscr{F} = k^*\mathscr{F}$ which is the stalk $\mathscr{F}_P$.

b If $P \notin U$ then every open set containing P is not contained in U and so every group in the diagram that defines the limit $(j_!(\mathscr{F}))_P$ is zero. Hence $(j_!(\mathscr{F}))_P$ is zero. Alternatively, if $P \in U$ then for every open set $P \in V$ there is an open set $P \in V' \subseteq U$ and so every element (V, s) of the stalk is equivalent to an element $(V', s|_{V'})$ of the stalk $\mathscr{F}_P$.

c We just need to show that the sequence is exact on each of the stalks. From the previous two parts of the exercise however, depending on whether P is in U or Z we either get an isomorphism followed by a zero object, or the zero object followed by an isomorphism. So the sequence is exact on the stalks.

Exercise 1.20. Subsheaf with Supports. *Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support is contained in Z.*

a *Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is denoted $\mathscr{H}^0_Z(\mathscr{F})$.*

b *Let $U = X - Z$, and let $j : U \to X$ be the inclusion. Show that there is an exact sequence of sheaves on X*

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Solution. a Since it is a subpresheaf of a sheaf we know that it is separated. Let U be an open subset of X and $\{U_i\}$ a cover of U. Suppose that s_i is a section of $\Gamma_{Z\cap U_i}(U_i, \mathscr{F}|_{U_i})$ for each i and that the restrictions of s_i and s_j to U_{ij} agree. Since all this takes place in a subpresheaf of a sheaf there is some $s \in \mathscr{F}(U)$ whose restriction to the U_i are s_i. The only thing to check is that s has support inside Z. Suppose that $P \in U\backslash Z$. Since the

U_i cover U, the point P is in one of the U_i. Since s_i is the restriction of s to U_i, the germ s_P agrees with $(s_i)_P$ which is zero. Hence, $s_P = 0$ for all $P \in U \backslash Z$. So $s \in \Gamma_{Z \cap U}(U, \mathscr{F}|_U)$ and hence, $\mathscr{H}^0_Z(\mathscr{F})$ is a sheaf.

b First note that if $U \cap Z = \varnothing$ then $\mathscr{H}^0_Z(\mathscr{F})(U) = 0$ since $\mathscr{H}^0_Z(\mathscr{F})(U)$ is the group of sections whose support is contained in Z, but $Z \cap U = \varnothing$ and so $\mathscr{H}^0_Z(\mathscr{F})(U)$ is the group of sections with empty support. Since $\mathscr{F}$ is a sheaf, any section whose germ is zero at every point is trivial itself, and so $\mathscr{H}^0_Z(\mathscr{F})(U) = 0$.

Now for a point $P \notin Z$, any section of the stalk (V, s) can be represented by $(V', s|_{V'})$ with $V' \cap Z = \varnothing$ (take $V' = V \cap Z^c$). But this means that $s|_{V'} = 0$ and so the stalk of $\mathscr{H}^0_Z(\mathscr{F})$ ar $P \notin Z$ is zero. As has been noted in the previous exercise, for $P \in U$ the stalks $\mathscr{F}_P$ and $(j_*(\mathscr{F}|_U))_P$ are the same, and so the sequence is exact on stalks at these points.

Now consider a point $P \in Z$ and an element $(V, s) \in \mathscr{F}_P$. If this element gets sent to zero it means that there is some open subset $V' \subseteq V$ such that $s|_{V'}$ is zero in $j_*(\mathscr{F}|_U)(V') = \mathscr{F}(U \cap V')$. If $U \cap V' \neq \varnothing$ we have $(V, s) \sim (U \cap V', s|_{U \cap V'})$ and so our original germ was trivial. If $U \cap V' = \varnothing$ then $V' \subseteq Z$ and so $(V', s|_{V'})$ is an element of $(\mathscr{H}^0_Z(\mathscr{F}))_P$.

Now consider an element (V, s) of $(\mathscr{H}^0_Z(\mathscr{F}))_P$, with $P \in Z$ still, and its image in $(j_*(\mathscr{F}|_U))_P$. This is (V, t) where t is the image of s under the map $\mathscr{H}^0_Z(\mathscr{F})(V) \to j_*(\mathscr{F}|_U)(V) = \mathscr{F}(U \cap V)$ which is essentially restriction of s to $U \cap V$. Since the support of S is contained in Z, restricting it to something contained in $U = Z^c$ will give zero. Hence, $(V, t) = 0$. So the sequence is exact in the middle term.

As has just been noted, for any open set V the morphism $\mathscr{F}(V) \to j_*(\mathscr{F}|_U) = F(U \cap V)$ is restriction, and so if $\mathscr{F}$ is flasque, the rightmost arrow is surjective as a morphism of presheaves. This means that it is also surjective as a morphism of sheaves.

Exercise 1.21. Some Examples of Sheaves on Varieties. *Let X be a variety over an algebraically closed field k. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.*

a *Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf.*

b *If Y is a subvariety, then the quotient sheaf $\mathcal{O}_Y/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$ where $i : Y \to X$ is the inclusion and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.*

c *Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$*

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F})$ *is not surjective.*

d *Again let* $X = \mathbb{P}^1$*, and let* $\mathcal{O}$ *be the sheaf of regular functions. Let* $\mathscr{K}$ *be the constant sheaf on* X *associated to the function field* K *of* X*. Show that there is a natural injection* $\mathcal{O} \to \mathscr{K}$*. Show that the quotient sheaf* $\mathscr{K}/\mathcal{O}$ *is isomorphic to the direct sum of sheaves* $\sum_{P \in X} i_P(I_P)$ *where* I_p *is the group* $K/\mathcal{O}_P$*, and* $i_P(I_P)$ *denotes the skyscraper sheaf given by* I_P *at the point* P*.*

e *Finally show that in the case of (d) the sequence*

$$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}_X) \to 0$$

is exact.

Solution. a Let $\{U_i\}$ be an open cover of an open subset U, and suppose we are given sections $f_i \in \mathscr{I}_Y(U_i)$ that agree on their restrictions to the intersections $U_i \cap U_j$. Since $\mathscr{I}_Y$ is a subpresheaf of a sheaf, we know that we can find a section $f \in \mathcal{O}_X(U)$ whose restrictions to U_i are the f_i, we just need to check that it is indeed in $\mathscr{I}_Y(U)$. That is, that the function $f : U \to k$ vanishes at all points of $Y \cap U$. If P is a point of $Y \cap U$ then since $\{U_i\}$ is a cover, P is contained in some U_i. The restriction of f to U_i is f_i, so $f(P) = f_i(P)$ which is zero since $f_i \in \mathscr{I}_Y(U_i)$. Hence, f vanishes at all points of $U \cap Y$ and is therefore in $\mathscr{I}_Y(U)$.

The fact that $\mathscr{I}_Y$ is a separated presheaf comes from the fact that every presheaf of a separated presheaf is separated, and every sheaf is separated.

b Let U be an open subset of X. If $f \in \mathcal{O}_X(U)$ is a regular function on U, then it is a function $U \to k$ that is locally representable as a quotient of polynomials. Restricting to $Y \cap U$ gives a section of $\mathcal{O}_Y(U \cap Y) = i_*(U \cap Y)$ and so we obtain a morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$. We want to see that the sequence

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

is exact. It follows from the definition of $\mathscr{I}_Y$ that the sequence is exact at $\mathscr{I}_Y$ and at $\mathcal{O}_X$. To see exactness at $i_*\mathcal{O}_Y$, consider a point $P \in X$ and the morphism of stalks $(\mathcal{O}_X)_P \to (i_*\mathcal{O}_Y)_P$. If $P \notin Y$ then $(i_*\mathcal{O}_Y)_P$ is zero, as there is an open neighbourhood of P not intersecting Y, so the morphism of stalks is surjective. Suppose $P \in Y$. An element of $(i_*\mathcal{O}_Y)_P$ is represented by a rational function on the ambient affine or projective space, which doesn't have a pole at P. This ambient space also includes X, and so this rational function also represents an element of $(\mathcal{O}_X)_P$. Hence, $(\mathcal{O}_X)_P \to (i_*\mathcal{O}_Y)_P$ is surjective, and so the sequence of sheaves is exact.

c Recall that $\mathbb{P}^1$ is the set of linear subspaces of k^2. Since the projective general linear group is transitive on pairs of distinct points, we can assume that $P = (0,1)$, and $Q = (1,1)$ and therefore, the sequence will be exact if and only if it is exact on its restriction to $\mathbb{A}^1 = \{(0,a) | a \in k\}$ where $P = 0$ and $Q = 1$.

Now the sequence on the stalk at a point R falls into three cases: either $R = P$, $R = Q$ or $R \neq P, Q$. In case $R \neq P, Q$, there is an open set U containing R which does not contain P and Q: the complement of the closed subset defined by $x(x-1) \in k[x]$. On this open set we have $\mathscr{F}(U) = 0$, by definition the skyscraper sheaves, and $\mathscr{I}_Y(U) = \mathcal{O}_X(U)$, by definition of $\mathscr{I}_Y$. Hence, the sequence is exact for any point in U.

If $R = P, Q$ then the sequence is the same so suppose that $R = Q$. The stalk of $\mathcal{O}_X$ at Q is the ring of rational functions whose denominator doesn't vanish at Q. That is, $(\mathcal{O}_X)_Q = \{\frac{f}{g} | g(1) \neq 0\}$. The ideal $(\mathscr{I}_Y)_Q$ is the subset of functions whose numerator does vanish at Q, that is $(\mathscr{I}_Y)_Q = \{\frac{f}{g} | g(1) \neq 0, f(1) = 0\}$. The quotient is isomorphic to k via evaluation at 1, which is the stalk of $\mathscr{F}$ at Q. So the sequence is exact at Q, and by symmetry, also at P.

On global sections however, the sequence is

$$0 \to 0 \to k \to k \oplus k \to 0$$

which cannot be exact.

d By definition, a regular function on U is a function that is represented locally by a rational function, that is, a section of $\mathscr{K}(U)$. More explicitely, a regular function on U is a function $f : U \to k$, such that there is an open cover $\{U_i\}$ of U on which $f|_{U_i}$ is a rational function with no poles in U_i. Since the f_i are restrictions of f as functions, they agree as functions on the intersections U_{ij}, and therefore define a section of $\mathscr{K}(U)$, the sheafification of $U \mapsto K$.

The morphism $\mathscr{K} \to \sum_{P \in X} i_P(I_P)$ should be clear. To show exactness it is enough to show exactness on the stalks. The sequence on a stalk takes the form

$$0 \to \mathcal{O}_P \to \mathscr{K}_P \to (\sum_{Q \in X} i_Q(I_Q))_P \to 0$$

Since $\mathscr{K}$ is a constant sheaf, it takes the value K at every stalk. On the right, we have a sum of stalks of skyscraper sheaves, all of which vanish except $Q = P$ which by definition is $K/\mathcal{O}_P$. Hence, the sequence is

$$0 \to \mathcal{O}_P \to K \to K/\mathcal{O}_P \to 0$$

which is exact.

e The global sections functor is left exact so we only need to show that $\Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}_X)$ is surjective. Using the description of $\mathscr{K}/\mathcal{O}$ from the previous part as $\sum i_P(I_P)$, our task is the following: given a rational function $f \in K$ and a point P, find another rational function $f' \in K$ such that $f' \in \mathcal{O}_Q$ for every $Q \neq P$ and $f' - f \in \mathcal{O}_P$.

Using the isomorphism $K \cong k(x)$, we can write $f = \frac{\alpha(x)}{\beta(x)} = \frac{\prod_{i=1}^{n}(x-a_i)}{\prod_{i=1}^{m}(x-b_i)}$ and then the points in $\mathbb{A}^1 \subset \mathbb{P}^1$ for which $f \notin \mathcal{O}_Q$ are those corresponding to b_i, and $f \notin \mathcal{O}_\infty$ if $m < n$. Infact, write f as $f = x^{-\nu}\frac{\alpha}{\beta'}$ with $x \not| \alpha, \beta$. Since $PGL(1)$ is transitive on points, without loss of generality we can assume that our point P is $0 \in \mathbb{A}^1$. If $\nu \leq 0$ then choosing $f' = 1$ satisfies the required conditions. If $\nu > 0$, then choose $f' = \frac{\sum_{i=0}^{\nu} c_i}{x^\nu}$ with c_i defined iteratively via $c_0 = \frac{\alpha_0}{\beta_0}$ and $c_i = \beta_0^{-1}(a_i - \sum_{j=0}^{i-1} c_j\beta_{i-j})$ where α_i, β_i are the coefficients for $\alpha = \sum \alpha_i x^i$ and $\beta' = \sum \beta_i x^i$ respectively. Our thus chosen f' satisfies the requirement that $f' \in \mathcal{O}_Q$ for all $Q \neq P$ and so consider $f - f'$. We have $f - f' = \frac{\alpha}{x^\nu \beta'} - \frac{\sum_{i=0}^{\nu} c_i}{x^\nu} = \frac{\alpha - \beta' \sum_{i=0}^{\nu} c_i}{x^\nu \beta'}$. The ith coefficient of the numerator for $i \leq \nu$ is $\alpha_i - \sum_{j=0}^{i} c_j \beta_{i-j}$ which is zero due to our careful choice of the c_i. So the x^ν in the denominator vanishes and we see that $f - f' \in \mathcal{O}_P$ since $x \not| \beta'$.

Exercise 1.22. Glueing Sheaves. *Let X be a topological space, let $\{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \cong \mathscr{F}_j|_{U_i \cap U_j}$ such that (1) for each i we have $\phi_{ii} = id$, and (2) for each i, j, k we have $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \cong \mathscr{F}_i$ such that for each i, j we ahve $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$.*

Solution. Let $\iota_i : U_i \to X$ and $\iota_{ij} : U_i \cap U_j \to X$ be the inclusions of the open sets and define $\mathscr{G}_i = \iota_{i*}\mathscr{F}_i$ and $\mathscr{G}_{ij} = \iota_{ij*}(\mathscr{F}_i|_{U_{ij}})$. Restriction induces morphisms $\mathscr{G}_i \to \mathscr{G}_{ij}$ and restriction composed with ϕ_{ji} gives morphisms $\mathscr{G}_j \to \mathscr{G}_{ij}$. Define $\mathscr{F} = \varprojlim \mathscr{G}$ to be the inverse limit of the system of $\mathscr{G}_i$'s and $\mathscr{G}_{ij}$'s. This comes naturally with morphisms $\mathscr{F} \to \mathscr{G}_i$. By considering stalks we see that the morphisms $\mathscr{F}|_{U_i} \to \mathscr{G}_i|_{U_i} = \mathscr{F}_i$ are isomorphisms, since on stalks, every morphism of the system that we took the limit over becomes either zero or an isomorphism, and the isomorphisms are compatible due to the cocycle condition. If there were to be another sheaf $\mathscr{F}'$ with isomorphisms as stated in the question, this would define a cone of the system. So there would be a morphism $\mathscr{F}' \to \mathscr{F}$ and considering stalks shows that this would be an isomorphism.