D-Modules Draft 1

Definitions, Theorems, Remarks, and Notable Examples

Isaac Martin

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Contents

1 D-modules over Affine n-space

Here we cover the basic theory of modules over the Weyl algebra, or in other words, the theory of D-modules in the case where $X = \mathbb{A}^n_K$.

1.1 Weyl Algebra

Let K be a field of characteristic 0. We construct the Weyl algebra in two ways and prove that these constructions produce isomorphic rings.

Definition 1.1. Let K be a field of characteristic 0 and let $K[X] = K[x_1,...,x_n] = \Gamma(X, \mathcal{O}_X)$ be the polynomial ring over K in n variables, and let $X = \mathbb{A}^n_K = \mathbb{A}^n$. Consider the algebra of K-linear operators $\mathrm{End}_K(K[X])$ and more specifically the operators $\hat{x}_i, \partial_j \in \mathrm{End}_K(K[X])$ for $1 \leq, i, j \leq n$. These are defined

$$\hat{x}_i : K[X] \longrightarrow K[X],\ f \mapsto x_i \cdot f$$

and

$$\partial_j : K[X] \longrightarrow K[X],\ f \mapsto \frac{\partial f}{\partial x_j}.$$

These are both linear operators, and they satisfy the relation

$$[\partial_j, \hat{x}_i] = \partial_j \hat{x}_i - \hat{x}_i \partial_j = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$ and is otherwise 0.

Since $K[\hat{x}] \cong K[x]$ as rings, we typically drop the hat notation and simply write x_i for $\hat{x}_i$. For any two operators $A, B \in \mathrm{End}(R)$ we write $[A,B] = AB - BA$. The commutator is a K-bilinear map on $\mathrm{End}(R)$.

We can also write the Weyl algebra down as a quotient of a free algebra in $2n$ generators over K.

Definition 1.2. The free algebra $K\{x_1,...,x_{2n}\}$ in $2n$ generators is the set of K-linear combinations of words in $x_1,...,x_{2n}$. Multiplication is given by concatenation on monomials and then extended to arbitrary elements by the distributive property. We have a homomorphism

$$\phi : K\{x_1,...,x_{2n}\} \longrightarrow A_n$$

given by $x_i \mapsto x_i$ and $x_{i+n} \mapsto \partial_i$ for $1 \leq i \leq n$. Let J be the two-sided ideal of $K\{x_1,...,x_{2n}\}$ generated by $[x_{i+n}, x_i] - 1$ for $1 \leq i \leq n$. Each of these generators is mapped to zero in A_n by the relations in Definition (1.1), so $J \subseteq \ker\phi$. We therefore obtain a map $\hat{\phi} : Kx_1,...,x_{2n}/J \longrightarrow A_n$ induced by ϕ.

Theorem 1.3. *The map $\hat{\phi}$ is an isormophism.*

To summarize, in A_n,

- x_i and x_j commute
- ∂_i and ∂_j commute
- $[\partial_i, x_j] = \delta_{ij}$, that is, ∂_i and x_j commute unless $i = j$.

Example 1.4. Given a polynomial $f \in K[x]$, we can think of f as an operator in $\mathrm{End}_K(K[x])$ by the map $x \mapsto \hat{x}$, and the operator f is simply given by multiplication by f. I claim that the commutator of f with ∂ satisfies the following relation: $[\partial, f] = f'$ where f' is the derivative of f. To see this, it suffices to show that $[\partial, x^n] = nx^{n-1}$ for $n \in \mathbb{Z}_{\geq 0}$, since $[-,-]$ is K-bilinear.

We show this by induction. The commutator relation $[\partial, x] = 1$ serves as the base case, so suppose $[\partial, x^k] = kx^{k-1}$ for $1 \leq k < n$. Then

$$\begin{aligned}\partial x^n &= (\partial x)x^{n-1}\\ &= (1 - x\partial)x^{n-1}\\ &= x^{n-1} - x\partial x^{n-1}\\ &= x^{n-1} - x \cdot (n-1)x^{n-2} = n \cdot x^{n-1},\end{aligned}$$

giving us the result.

It is useful to fix a basis for the Weyl algebra, but for arbitrary n, the notation becomes cumbersome. To remedy this, we use multi-indices. For $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, we denote by $x^\alpha \partial^\beta$ the element $x_1^{\alpha_1} ... x_n^{\alpha_n} \partial_1^{\beta_1} ... \partial_n^{\beta_n} \in A_n$. As it turns out, the set of all elements of this form is a K-basis for A_n

Proposition 1.5. The set $\mathbf{B} = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ is a basis of A_n as a vector space over K. This is called the *canonical basis* of A_n and an operator $D \in A_n$ written as a linear combination of elements in $\mathbf{B}$ is said to be in *canonical form*.

1.2 Basic Properties of the Weyl Algebra

Despite the noncommutative of A_n, one might be tempted to draw comparisons between the Weyl algebra and a ring of polynomials, especially given that A_n admits such a nice basis. In particular, one might wonder if A_n admits any meaningful graded structure. The answer turns out to be "sort of". The goal of this section is primarily to define and examine this approximation of a graded structure. To do this, we first define the *degree* of an element in A_n. We then notice that this fails to define a $K[X]$-grading for A_n and provide a workaround.

We also say some words about the ideal structure of A_n and the case in which $\mathrm{char}\, K = p > 0$.

Definition 1.6. The *length* of a multindex $\alpha \in \mathbb{N}^n$ is denoted $|\alpha|$ and is defined

$$|\alpha| = |\alpha_1| + ... + |\alpha_n|.$$

Let D be an operator A_n. The *degree* of D, $\deg(D)$, is the largest length of the multi-indices $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ appears with non-zero coefficient in the canonical form of D. We define $\deg(0) = -\infty$.

It is important to remember that the definition of degree depends on the canonical basis for A_n. The degree of $x\partial$ can simply be read off as 2, but ∂x must first be written $x\partial + 1$ in order to see that $\deg(\partial x) = 2$.

The function $\deg : A_n \longrightarrow \mathbb{N}$ reproduces the multiplicative and additive structure of $\deg : K[X] \longrightarrow \mathbb{N}$:

Theorem 1.7. *Let $D, D' \in A_n$.*

(1) $\deg(DD') = \deg(D) + \deg(D')$

(2) $\deg(D + D') \leq \max\{\deg(D), \deg(D')\}$

(3) $\deg[D,D'] \leq \deg(D) + \deg(D') - 2$.

Note that equality holds in (2) when $\deg(D) \neq \deg(D')$ but that there is risk of cancellation otherwise, as is the case with rings of polynomials.

Proof: We refer to [**Gieseker75**] for the proof of (1) and (3) COMPLETE PROOF OF (3) MANUALLY. □

One might expect a graded structure to naturally fall from this definition of degree. The issue, as always, is one of noncommutativity. The element $x_1\partial_1$ ought to homogeneous, but it is the difference $\partial x - 1$ of two elements with non-equal degree. There is no way to define a collection of pairwise disjoint $K[X]$-submodules of A_n whose direct sum recovers A_n. Nonetheless, we can still find a collection of A_n submodules which resemble a grading on A_n. We will call this a *filtration* of A_n, and it turns out that this filtration will come with a natural associated graded $K[X]$-module whose properties will yield new information about A_n.

We first define arbitrary filtered rings and modules.

Definition 1.8. Let R be a K-algebra and M an R-module. A collection $\mathcal{F} = \{F_i\}_{i\geq 0}$ of K-vector spaces is said to be a *filtration* of R if

(i) $F_0 \subset F_1 \subset F_2 \subset \ldots \subset R$

(ii) $R = \bigcup_{i\geq 0} F_i$

(iii) $F_i \cdot F_j \subseteq F_{i+j}$.

If R has a filtration it is called a *filtered algebra*. Similarly, if R is a filtered algebra, then a *filtration of M* is a family $\Gamma = \{\Gamma_0\}_{i\geq 0}$ of K-vector spaces satisfying

(i) $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq M$,

(ii) $M = \bigcup_{i\geq 0} \Gamma_i$

(iii) $B_i\Gamma_j \subseteq \Gamma_{i+j}$.

Such a module is said to be *filtered*. In this section, we additionally adopt the convention that

(4) Γ_i is a finite-dimensional K-algebra for each $i \geq 0$,

which will become important in our discussion of dimension.