I Basic Theory

1 Lecture 1

Eg. $f(x_1,\dots,x_r)\in\mathbb{Z}[x_1,\dots,x_r]$, $f(x_1,\dots,x_r)=0$?

$$\begin{aligned} f(x_1,\dots,x_r) &\equiv 0 \mod p \\ \text{"} \quad & \text{"} \mod p^2 \\ \vdots \quad & \quad \vdots \\ \text{"} \quad & \text{"} \mod p^n \end{aligned}$$

Local fields packages all this information together.

§1 Absolute values

Definition 1.1: Let K be a field. An absolute value on K is a function $|\ |: K\to\mathbb{R}_{\geq 0}$ such that:

(i) $|x|=0$ iff $x=0$

(ii) $|xy|=|x||y|$ $\forall x,y\in K$

(iii) $|x+y|\leq|x|+|y|$ $\forall x,y\in K$ (triangle inequality)

We say $(K,|\cdot|)$ is a valued field.

Eg. • $K=\mathbb{Q},\mathbb{R},\mathbb{C}$ with usual abs. val. $|a+ib|$

2 $=\sqrt{a^2+b^2}$. Write $|\ |_\infty$ for this abs. val.

• K any field. Trivial absolute value

$$|x|=\begin{cases} 0 & x=0 \\ 1 & x\neq 0\end{cases}$$

- $K=\mathbb{Q}$, p prime

For $0\neq x\in\mathbb{Q}$, write $x=p^n\frac{a}{b}$, where $(a,p)=1$, $(b,p)=1$

The p-adic absolute value is defined to be

$$|x|_p=\begin{cases}0 & x=0\\ p^{-n} & x=p^n\frac{a}{b}\end{cases}$$

Axioms: (i) clear

Write $y=p^m\frac{c}{d}$

(ii) $|xy|_p=\left|p^{m+n}\dfrac{ac}{bd}\right|_p=p^{-m-n}$
$=|x|_p|y|_p$

(iii) wlog. $m\geqslant n$

$$|x+y|_p=\left|p^n\left(\frac{ad+p^{m-n}bc}{cd}\right)\right|_p$$
$$\leqslant p^{-n}=\max(|x|_p,|y|_p)$$

3

An abs. val $|\cdot|$ on K induces a metric $d(x,y)=|x-y|$ on K, hence induces a topology on K.

Defn 1.2: Let $|\cdot|$, $|\cdot|'$ be absolute values on a field K. We say $|\cdot|$, $|\cdot|'$ are equivalent if they induce the same topology.

An equiv. class of abs. values is called a place.

Proposition 1.3: Let $|\cdot|$, $|\cdot|'$ be (non-trivial) abs. values on K. The following are equivalent (TFAE)

(i) $|\cdot|$ and $|\cdot|'$ are equivalent.

(ii) $|x| < 1 \iff |x|' < 1 \quad \forall x \in K$.

(iii) $\exists c \in \mathbb{R}_{>0}$ s.t. $|x|^c = |x|' \quad \forall x \in K$.

Proof: (i) $\Rightarrow$ (ii)

$$|x| < 1 \iff x^n \to 0 \text{ w.r.t. } |\cdot|$$
$$\overset{(i)}{\iff} x^n \to 0 \text{ w.r.t. } |\cdot|'$$
$$\iff |x|' < 1$$

(ii) $\Rightarrow$ (iii)

$$|x|^c = |x|'$$
$$\iff c \log|x| = \log|x|'$$
$$\iff c = \frac{\log|x|'}{\log|x|} \text{ if } |x| \neq 1$$

Let $a \in K^\times$ s.t. $|a| < 1$ (exists since $|\,|$ non-trivial)

We need to show that

$$\forall x \in K^\times \quad \frac{\log|x|}{\log|a|} = \frac{\log|x|'}{\log|a|'}$$

Assume $\dfrac{\log|x|}{\log|a|} < \dfrac{\log|x|'}{\log|a|'}$

Choose $m, n \in \mathbb{Z}$ s.t. $\dfrac{\log|x|}{\log|a|} < \dfrac{m}{n} < \dfrac{\log|x|'}{\log|a|'}$

Then we have $n \log|x| < m \log|a|$

$$n \log|x|' > m \log|a|'$$

$\Rightarrow \left|\dfrac{x^n}{a^m}\right| < 1$ and $\left|\dfrac{x^n}{a^m}\right|' > 1$ ⨳

Similar to $\log|x| > \log|x|'$

Similarly for $\frac{\log|b|}{\log|a|} = \frac{\log|b|'}{\log|a|'}$ $\square$

(iii) $\Rightarrow$ (i)

Remark: $|\cdot|_\infty^2$ on $\mathbb{C}$ is not an abs. val. by our defn. Some authors replace Δ ineq. with

$$|x+y|^\beta \le |x|^\beta + |y|^\beta \text{ for some fixed } \beta \in \mathbb{R}_{>0}$$

5 This course: Mainly interested in.

Defn 1.4: An absolute value $|\cdot|$ on K is said to be **non-archimedean** if it satisfies the **ultrametric inequality**:

$$(*) \quad |x+y| \le \max(|x|, |y|).$$

If $|\cdot|$ is not non-archimedean, then it is **archimedean**.

Eg. • $|\cdot|_\infty$ on $\mathbb{R}$ is archimedean.

• $|\cdot|_p$ is a non-archimedean abs. val. on $\mathbb{Q}$.

Lemma 1.5: Let $(K, |\cdot|)$ non-arch. $x, y \in K$. If $|x| < |y|$, then $|x-y| = y$.

Proof: $|x-y| \le \max(|x|, |y|) = |y|$.

$|y| \le \max(|x|, |x-y|)$

$\Rightarrow |y| \le |x-y|$. $\square$

Convergence is easier for non-arch. $|\cdot|$.

Proposition 1.6: Let $(K, |\cdot|)$ non-arch. and $(x_n)_{n=1}^{\infty}$ a sequence in K. If $|x_n - x_{n+1}| \to 0$, then $(x_n)_{n=1}^{\infty}$ is Cauchy.

In particular, if K is in addition complete, then $(x_n)_{n=1}^{\infty}$ converges.

Proof: For $\varepsilon > 0$, choose N s.t. $|x_n - x_{n+1}| < \varepsilon$ for $n > N$.

Then for $N < n < m$, $|x_n - x_m| = |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + \dots + (x_{m-1} - x_m)|$

$$< \varepsilon$$

$\Rightarrow (x_n)_{n=1}^{\infty}$ Cauchy.

In particular part: clear. $\square$

Eg. $p = 5$. Construct sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{Q}$ s.t.

(i) $x_n^2 + 1 \equiv 0 \bmod 5^n$

(ii) $x_n \equiv x_{n+1} \bmod 5^n$

as follows.

Take $x_1 = 2$. Suppose have constructed x_n.

Let $x_n^2 + 1 = a5^n$ and set $x_{n+1} = x_n + b5^n$

Then $x_{n+1}^2 + 1 = x_n^2 + 2b5^n + b^2 5^{2n} + 1$

$$= a5^n + 2b5^n + \underline{b^2 5^{2n}} \equiv 0 \bmod 5^n$$

We choose b s.t. $a + 2b \equiv 0 \bmod 5$;

then we have $x_{n+1}+1 \equiv 0 \mod 5^{n+1}$ as desired.

Now (ii) $\Rightarrow (x_n)_{n=1}^{\infty}$ is Cauchy.

Suppose $x_n \to l \in \mathbb{Q}$

Then $x_n^2 \to l^2$

But i) $\Rightarrow x_n^2 \to -1 \Rightarrow l^2 = -1$ #.

Thus $(\mathbb{Q}, |\cdot|_5)$ is not complete.

Defn 1.7: The p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t $|\ |_p$.

Analogy with $\mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{Q} \arrow[ur, "|\cdot|_\infty"] \arrow[dr, "|\cdot|_p"'] & \\ & \mathbb{Q}_p \end{tikzcd}$$

For $x \in K$ and $r \in \mathbb{R}_{>0}$, define

$$B(x,r) = \{ y \in K \mid |x-y| < r \}$$

$$\bar{B}(x,r) = \{ y \in K \mid |x-y| \leq r \}$$

Lemma 1.5: Let $(K, |\cdot|)$ be non-archimedean.

(i) If $z \in B(x,r)$, then $B(z,r) = B(x,r)$

(ii) If $z \in \bar{B}(x,r)$, then $\bar{B}(z,r) = \bar{B}(x,r)$

(iii) $B(x,r)$ is closed

(iv) $\bar{B}(x,r)$ is open

Proof: i) $|x-y| \leq r \Rightarrow |z-y| = |(z-x)+(x-y)|$

$$\leq \max(|z-x|, |x-y|)$$

$$= r$$

Thus $B(x,r) \subseteq B(z,r)$

Reverse inclusion follows by symmetry.

(ii) same as i)

(iii) If $u \notin B(x,r)$, then $B(x,r) \cap B(u,r) = \emptyset$

[illegible]

(iv) If $z \in \overline{B}(x,r)$, then $B(z,r) \subseteq B(z,r) = B(x,r)$.

$\square$

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