# ATIYAH-BOTT LOCALIZATION IN EQUIVARIANT WITT COHOMOLOGY

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ABSTRACT. Let N be a normalizer of the diagonal torus  $T_1 \cong \mathbb{G}_m$  in  $\mathrm{SL}_2$ . We prove localization theorems for  $\mathrm{SL}_2^n$  and  $N^n$  for equivariant cohomology with coefficients in the (twisted) Witt sheaf, along the lines of the classical localization theorems for equivariant cohomology for a torus action. We also have an analog of the Bott residue formula for  $\mathrm{SL}_2^n$  and N. In the case of an  $\mathrm{SL}_2^n$ -action, there is a rather serious restriction on the orbit type. For an N-action, there is no restriction for the localization result, but for the Bott residue theorem, one requires a certain type of decomposition of the fixed points for the  $T_1$ -action.

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# Introduction

Localization to the fixed point set in equivariant cohomology and the Bott residue theorem give very effective tools for computing cohomology and characteristic classes for manifolds with an action of a finite group or a torus. See for

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example [2], [3], [4] [5], [23]. In the algebraic setting, this yields similar computations for Chow rings and  $CH^*$ -valued Chern classes; this has been developed by Edidin-Graham [9] and Thomason [26] (for algebraic K-theory). As part of a larger program for refining such invariants to motivic cohomology theories on schemes over more general bases, yielding for example invariants in the Grothendieck-Witt ring of the base scheme, we would like to add such localization methods to the toolbox.

The simplest theory that yields such quadratic information is based on the sheaf of Witt rings. When working over a field k, this will yield invariants in the Witt ring W(k); combined with the classical numerical invariants in  $\mathbb{Z} = \mathrm{CH}^0(k)$ , this lifts canonically to invariants in the Grothendieck-Witt ring  $\mathrm{GW}(k)$ .

However, one cannot use a torus action to achieve a Witt ring version of the classical localization theorems. The reason for this is very simple: localization for the action of a torus  $T \cong \mathbb{G}_m^n$  is based on the isomorphism

$$H_{BT}^*(pt,\mathbb{Z}) \cong \mathbb{Z}[t_1,\ldots,t_n]$$

where  $t_i = c_1(L_i)$  and  $L_i$  is the line bundle on BT corresponding to the character  $\chi_i: T \to \mathbb{G}_m$  given by the *i*th projection. In the algebraic setting, one uses instead the T-equivariant Chow groups of Totaro [28] and Edidin-Graham [8], with similar result

$$CH^*(BT) = \mathbb{Z}[t_1, \dots, t_n].$$

Localization to the fixed point subscheme of a T-action on a k-scheme X requires inverting a non-zero element  $P_X \in \mathrm{CH}^d(BT)$  for some  $d \geq 1$ . The Witt-sheaf analog would require inverting some  $P_X \in H^d(BT, \mathcal{W})$  for some  $d \geq 1$ , but the Witt-sheaf cohomology of BT is given by

$$H^d(BT, \mathcal{W}) = \begin{cases} W(k) & \text{for } d = 0\\ 0 & \text{else,} \end{cases}$$

and thus the localization method fails.

One can use instead the group  $SL_2^n$ . Ananyevskiy [1, Introduction] has computed  $H^*(BSL_2^n, \mathcal{W})$  as

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) = W(k)[e_1, \dots, e_n]$$

where  $e_i \in H^2(\mathrm{BSL}_2^n, \mathcal{W})$  is the Euler class of the rank 2 bundle corresponding to the *i*th projection  $\mathrm{SL}_2^n \to \mathrm{SL}_2$  followed by the usual inclusion  $\mathrm{SL}_2 \subset \mathrm{GL}_2$ . This is thus perfectly parallel to the Chow ring of  $B\mathbb{G}_m^n$  and one can hope for a corresponding localization theorem. Unfortunately, the situation is not so simple and one finds some rather strong restrictions for an  $\mathrm{SL}_2^n$ -action on a k-scheme K that are necessary to have an analog of the classical localization results.

For a group-scheme G over k, let  $\mathbf{Sch}^G/k$  denote the category of quasi-projective k-schemes with a G-action. For  $X \in \mathbf{Sch}^G/k$  with a G-linearized invertible sheaf  $\mathcal{L}$ , there are well-defined equivariant cohomology and Borel-Moore homology for the  $\mathcal{L}$ -twisted sheaf of Witt groups,

$$H_G^*(X, \mathcal{W}(\mathcal{L}), H_{G,*}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L}));$$

See Definition 4.3 and Definition 4.8.

**Definition 1.** For X in  $\mathbf{Sch}^{\mathrm{SL}_{2}^{n}}/k$ , we call the  $\mathrm{SL}_{2}^{n}$ -action *localizing* if the following conditions hold:

Let  $O \subset X$  be a  $\operatorname{SL}_2^n$ -orbit, with quotient  $\operatorname{SL}_2^n \setminus O = \operatorname{Spec} k_O$ ; we consider O as an object in  $\operatorname{\mathbf{Sch}}^{\operatorname{SL}_2^n}/k_O$ . Then there is a morphism  $O \to (\operatorname{SL}_2^n/H) \times_k k_O$  in

 $\mathbf{Sch}^{\mathrm{SL}_{2}^{n}}/k_{O}$ , where  $H \subset \mathrm{SL}_{2}^{n}$  is a closed subgroup of one of the two following types:

- i) H is a maximal parabolic subgroup
- ii) H is a "diagonal" subgroup  $G_{ij} \subset SL_2^n$ ,

$$G_{ij} = \{(g_1, \dots, g_n) \mid g_i = g_i\}$$

for some  $i \neq j$ .

**Theorem 2** (Localization for  $SL_2^n$ : Theorem 6.7). Let k be a field of characteristic  $\neq 2$  and let X be in  $\mathbf{Sch}^{SL_2^n}/k$ . Let  $\mathcal L$  be a  $SL_2^n$ -linearized invertible sheaf on X and let  $i: X^{SL_2^n} \to X$  be the inclusion of the  $SL_2^n$ -fixed points. Suppose that the  $SL_2^n$ -action is localizing. Let

$$e_* = \prod_{i=1}^n e_i \cdot \prod_{1 \le i \le j \le n} e_i - e_j \in H^{* \ge 2}(\mathrm{BSL}_2^n, \mathcal{W})$$

Then the push-forward map

$$i_*: H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2,*}(X^{\operatorname{SL}^n_2}, \mathcal{W}(i^*\mathcal{L}))[1/e_*] \to H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2,*}(X, \mathcal{W}(\mathcal{L}))[1/e_*]$$

is an isomorphism. Moreover

$$H^{\mathrm{B.M.}}_{\mathrm{SL}^n_0,*}(X^{\mathrm{SL}^n_2},\mathcal{W}(i^*\mathcal{L})) \cong H^{\mathrm{B.M.}}_*(X^{\mathrm{SL}^n_2},\mathcal{W}(i^*\mathcal{L})) \otimes_{W(k)} W(k)[e_1,\ldots,e_n]$$

The situation is somewhat nicer for an action by  $N^n$ , where  $N \subset \operatorname{SL}_2$  is the normalizer of the diagonal torus  $T_1 \subset \operatorname{SL}_2$ . N is generated by the diagonal torus  $T_1 \subset \operatorname{SL}_2$  and an addition element  $\sigma$ 

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have computed  $H^*(BN, W)$  in [18]. This is slightly larger than  $H^*(BSL_2, W) = W(k)[e]$ , but agrees with  $H^*(BSL_2, W)$  in positive degrees and the two are isomorphic after inverting e; similarly,

$$H^*(BN^n, \mathcal{W})[1/\prod_{i=1}^n e_i] = H^*(\mathrm{BSL}_2^n, \mathcal{W})[1/\prod_{i=1}^n e_i] = W(k)[e_1^{\pm 1}, \dots, e_n^{\pm 1}]$$

so  $N^n$  is another suitable candidate for a localization theorem. Here we have a close analogy with the classical case.

There is a fly in the ointment, arising from the fact that  $N^n$  is not connected. Our first localization theorem localizes the Borel-Moore homology of a scheme X with  $N^n$  action to the  $T_1^n$ -fixed points.

**Theorem 3** (Localization for  $N^n$ : Theorem 7.10). Let k be a field of characteristic  $\neq 2$  and let X be in  $\mathbf{Sch}^{N^n}/k$ . Let  $\mathcal{L}$  be a G-linearized invertible sheaf on X and let  $i: X^{T_1^n} \to X$  be the inclusion of the  $T_1^n$ -fixed points. Then there is a nonzero homogeneous element  $P_X \in \mathbb{Z}[e_1, \ldots, e_n]$  such that, after inverting  $P_X$ , the push-forward map

$$i_*: H_{G,*}^{\operatorname{B.M.}}(X^{T_1^n}, \mathcal{W}(i^*\mathcal{L})) \to H_{G,*}^{\operatorname{B.M.}}(X, \mathcal{W}(\mathcal{L}))$$

is an isomorphism.

For the case n = 1, we have a finer result, under an additional assumption.

**Definition 4.** 1. Let X be a quasi-projective k-scheme with N-action. Let  $\bar{\sigma}$  be the image of  $\sigma$  in  $N/T_1$ . We let  $|X|^N$  denote the union of the integral components  $Z \subset X^{T_1}$  such that the generic point  $z \in Z$  is fixed by  $\bar{\sigma}$ , and let  $X_{\mathrm{ind}}^{T_1}$  be the union of the integral components  $Z \subset X^{T_1}$  such that  $\bar{\sigma} \cdot Z \cap Z = \emptyset$ .

- 2. We call the N-action semi-strict if  $X_{\text{red}}^{T_1} = |X|^N \cup X_{\text{ind}}^{T_1}$
- 3. If the N-action on X is semi-strict, we say the N-action on X is strict if  $|X|^N \cap X_{\text{ind}}^{T_1} = \emptyset$  and we can decompose  $|X|^N$  as a disjoint union of two N-stable closed subschemes

$$|X|^N = X^N \coprod X_{\mathrm{fr}}^{T_1}$$

where the  $N/T_1$ -action on  $X_{\rm fr}^{T_1}$  is free.

In the case of a semi-strict N-action, we have

**Theorem 5** (Localization for an N-action: Theorem 8.6). Let k be a field of characteristic  $\neq 2$  and let X be in  $\mathbf{Sch}^N/k$ . Suppose the N-action is semi-strict. Let  $\mathcal{L}$  be an N-linearized invertible sheaf on X and let  $i:|X|^N \to X$  be the inclusion. Then there is an integer M>0 such that the push-forward map

$$i_*: H_{N,*}^{\mathrm{B.M.}}(|X|^N, \mathcal{W}(i^*\mathcal{L}))[1/Me] \to H_{G,*}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L}))[1/Me]$$

is an isomorphism.

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Remark. The reader will have noticed that the integer M in this last result might be even, in which case M might be a nilpotent in W(k) and the theorem tells us nothing. However, if the base-field k admits a real embedding, this will not be the case, even if one would lose 2-primary information by inverting M. Similar remarks hold for the polynomial  $P_X$  in Theorem 3.

We have an analog of the Bott residue formula in the case of a localizing  $SL_2^n$ -action. In case k has characteristic zero, let Y be a connected k-scheme with trivial  $SL_2^n$ -action and let V be a  $SL_2^n$ -linearized vector bundle on Y. Then there is a "generic representation type"  $[V^{gen}]$ , defined as an isomorphism class of a  $SL_2^n$ -representation over k, and an associated generic Euler class  $e_{SL_2^n}^{gen} \in H^*(BSL_2^n, \mathcal{W})$ , well-defined up to multiplication by a unit in W(k).

**Theorem 6** (Bott Residue Theorem for  $\operatorname{SL}_2^n$ : Theorem 9.5). Let  $G = \operatorname{SL}_2^n$  and suppose k has characteristic zero. Take  $X \in \operatorname{\mathbf{Sch}}^{\operatorname{SL}_2^n}/k$  and let  $\mathcal L$  be a  $\operatorname{SL}_2^n$ -linearized invertible sheaf on X. We suppose that the action is localizing, and that the inclusion  $i:X^{\operatorname{SL}_2^n}\to X$  is a regular embedding. Let  $i_j:X_j^{\operatorname{SL}_2^n}\to X$ ,  $j=1,\ldots,s$ , be the connected components of  $X^{\operatorname{SL}_2^n}$ , with normal bundle  $N_{i_j}$ . Suppose in addition that  $e_{\operatorname{SL}_2^n}^{\operatorname{gen}}(N_{i_j})\neq 0$  for each j. Let P be as in Theorem 2. Then

$$e_{\mathrm{SL}_2^n}(N_{i_j}) \in H^*_{\mathrm{SL}_2^n}(X_j^{\mathrm{SL}_2^n}, \mathcal{W}(\det^{-1}N_{i_j}))[(Pe_{\mathrm{SL}_2^n}^{gen}(N_{i_j}))^{-1}]$$

is invertible. Letting  $e^{gen}_{\mathrm{SL}^n_2}(N_i) = \prod_{j=1}^s e^{gen}_{\mathrm{SL}^n_2}(N_{i_j})$ , the inverse of the isomorphism

$$i_*: H^{\operatorname{B.M.}}_{G*}(X^{\operatorname{SL}^n_2}, \mathcal{W}(i^*\mathcal{L}))[(Pe^{gen}_{\operatorname{SL}^n_2}(N_i))^{-1}] \xrightarrow{\sim} H^{\operatorname{B.M.}}_{G*}(X, \mathcal{W}(\mathcal{L})[Pe^{gen}_{\operatorname{SL}^n_2}(N_i))^{-1}]$$

is the map

$$x \mapsto \prod_{j=1}^{s} i_{j}^{!}(x) \cap e_{\mathrm{SL}_{2}^{n}}(N_{i_{j}})^{-1},$$

where we use the evident isomorphism

$$H_{G*}^{\mathrm{B.M.}}(X^{\mathrm{SL}_2^n}, \mathcal{W}(i^*\mathcal{L})) = \prod_{j=1}^s H_{G*}^{\mathrm{B.M.}}(X_j^{\mathrm{SL}_2^n}, \mathcal{W}(i_j^*\mathcal{L})).$$

In the case of an  $X \in \mathbf{Sch}^N/k$ , we need to assume that the action is semi-strict, and that  $i: |X|^N \to X$  is a regular embedding.

**Theorem 7** (Bott Residue Theorem for N: Theorem 9.5). Let k be a field of characteristic  $\neq 2$ . Take  $X \in \mathbf{Sch}^N/k$  and let  $\mathcal{L}$  be an N-linearized invertible sheaf on X. We suppose the N-action is semi-strict, and the inclusion  $i:|X|^N \to X$  is a regular embedding. For each connected component  $i_j:|X|_j^N \to X$ , let  $N_{i_j}$  be the normal bundle of  $i_j$  Then there is an M>0 such that

$$e_N(N_{i_j}) \in H_N^*(|X|_j^N, W)[1/Me]$$

is invertible, and  $i_*$  defines an isomorphism

$$i_*: H_{N,*}^{\mathrm{B.M.}}(|X|^N, \mathcal{W}(i^*\mathcal{L}))[1/Me)] \xrightarrow{\sim} H_{N,*}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L})[1/Me]$$

with inverse the map

$$x \mapsto \prod_{j} i_{j}^{!}(x)/e_{N}(N_{i_{j}})$$

Just as in the classical case, for  $G = \mathrm{SL}_2^n, N^n$ , we have the usual description of the G-equivariant Witt Borel-Moore homology (or cohomology for X smooth) for a trivial G-action,

$$H_{G,*}^{\operatorname{B.M.}}(X,\mathcal{W}) \cong H_*^{\operatorname{B.M.}}(X,\mathcal{W}) \otimes_{W(k)} H^{-*}(BG,\mathcal{W})$$

See Corollary 5.3.

If  $|X|^N \neq X^N$ , one would also want to compute  $H_{N,*}^{\mathrm{B.M.}}(|X|^N, \mathcal{W}(i^*\mathcal{L}))$ . By using localization for Borel-Moore homology, one can reduce to the case of a strict N-action, with  $|X|^N = X^N \coprod X_{\mathrm{fr}}^{T_1}$ , and in this case, one would need to compute  $H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{fr}}^{T_1}, \mathcal{W}(\mathcal{L}))$ . In general, I found no nice answer for this, but using localization again, one should be able to reduce to the case of dimension zero. This is just the case of Spec  $F(\sqrt{a})$ , F a field, with  $\bar{\sigma}$  acting by the usual conjugation (and  $T_1$  acting trivially). This case is discussed in §10. The answer (Corollary 10.12) is a bit complicated but there is an explicit computation (Corollary 10.14) of the push-forward

$$H_N^*(\operatorname{Spec} F(\sqrt{a}), \mathcal{W}) \to H_N^*(\operatorname{Spec} F, \mathcal{W})$$

which suffices for computing the push-forward

$$H_{N,0}^{\mathrm{B.M.}}(X,\mathcal{W}) \to W(k)$$

for X proper over k and F a finite extension of k. Possibly this situation does not occur much in practice, but the reader may find the computation amusing.

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#### 1. Quotients and fixed points

We work over a noetherian separated base-scheme B of finite Krull dimension and let G be a linear algebraic group scheme over B, flat over B, with a fixed embedding  $i: G \to \operatorname{GL}_N/B$  realizing G as a closed subgroup scheme of some  $\operatorname{GL}_N/B$ .

We let  $\operatorname{\mathbf{Sch}}^G/B$  denote the category of G-quasi-projective B-schemes, that is, an object is a quasi-projective B-scheme X with a G-action  $G \times X \to X$  that extends to a G-action  $G \times \mathbb{P}(V)_B \to \mathbb{P}(V)_B$  with respect to some vector bundle V and some locally closed immersion  $i: X \to \mathbb{P}(V)_B$ ; in other words, the G-action on X is G-linearized. Morphisms are G-equivariant morphisms of G-schemes (not necessarily respecting the G-linearization). We write  $\operatorname{\mathbf{Sm}}^G/B$  for the full subcategory of  $\operatorname{\mathbf{Sch}}^G/B$  with objects those X that are smooth over G.

Given a vector bundle V or coherent sheaf  $\mathcal{F}$  on some  $X \in \mathbf{Sch}^G/B$ , we refer to a G-action on V or  $\mathcal{F}$  over the given G-action on X as a G-linearization of V or  $\mathcal{F}$ , and refer to V or  $\mathcal{F}$  with such an action as G-linearized.

We recall some definitions from [14]. For  $X \in \operatorname{\mathbf{Sch}}^G/B$ , we say that G acts trivially on X if the action morphism  $G \times_B X \to X$  is the projection to X. For general  $X \in \operatorname{\mathbf{Sch}}^G/B$ , we have the G-fixed subscheme  $X^G \hookrightarrow X$ , this being the maximal G-stable closed subscheme Y of X on which G acts trivially.

We recall some results due to Thomason [25].

# **Proposition 1.1.** Let $X \in \mathbf{Sch}^G/B$ be reduced.

- 1. There is a stratification of X into locally closed reduced G-stable subschemes,  $X = \coprod_{i=1}^r X_i$ , such that for each i the quotient  $q_i : X_i \to G \backslash X_i$  (as an fppf sheaf) exists in  $\mathbf{Sch}/B$ . Moreover  $X_i \to G \backslash X_i$  is a geometric quotient as well. If G is smooth over B, then we may take the stratification as above with each  $q_i$  smooth.
- 2. Suppose that G is a split torus over B,  $G \cong \mathbb{G}_m^n/B$ . Then there is a stratification of X as in (1) such that for each i,  $X_i$  is affine, and there is a diagonalizable subgroup scheme  $T_i'$  of G, acting trivially on  $X_i$ , such that the quotient torus  $T_i'' := G/T_i' \cong \mathbb{G}_m^{n_i}/B$  acts freely on  $X_i$ , making  $X_i$  into a trivial  $T_i''$ -torsor over  $G \setminus X_i = T_i'' \setminus X_i$ . If B is excellent, we may take the stratification so that in addition,  $X_i$  is regular.

*Proof.* For (1), it follows from [25, Proposition 4.7] that there is a non-empty open G-stable subscheme U of X such that the fppf quotient sheaf  $G \setminus U$  is represented by a separated scheme of finite type over B, with quotient map  $q: U \to G \setminus U$  smooth if G is smooth over B. By [24, Proposition 1(2), Théorème 3(1)], this implies that  $q: U \to G \setminus U$  is a geometric quotient. Replacing  $G \setminus U$  with a non-empty affine open subscheme V and replacing U with  $q^{-1}(V)$ , we may assume that  $G \setminus U$  is affine, hence in  $\mathbf{Sch}/B$ . Replacing X with the complement  $X \setminus U$  with its induced G-action, the result follows by noetherian induction.

Part (2) follows from Thomason's generic slice theorem for a torus action, [25, Proposition 4.10 and Remark 4.11].  $\Box$ 

Remark 1.2. Suppose  $B = \operatorname{Spec} k$ , and that G is linearly reductive over k. Take  $X \in \mathbf{Sm}^G/k$ . By [14, Theorem 5.4],  $X^G$  is smooth over k. Examples include

 $G = \operatorname{SL}_2^n$  and k of characteristic zero, or  $G = \mathbb{G}_m^n$ , and k arbitrary. Letting  $N \subset \operatorname{SL}_2$  be the normalizer of the diagonal torus,  $N^n$  is linearly reductive over k if  $\operatorname{char} k \neq 2$ .

**Definition 1.3.** Let G be a smooth, finite type group-scheme over B, and let  $X \in \mathbf{Sm}^G/B$  be a smooth B-scheme with G-action  $\rho: G \times_B X \to X$ . Consider  $G \times_B X$  as a group scheme over X via the projection and let  $G_X \subset G \times_B X$  be the isotropy group of the diagonal section  $\Delta_X \subset X \times_B X$ . We have the morphism

$$\rho_{\delta}: G \times_B X \to X \times_B X$$

defined via the translation action of  $G \times_B X$  on  $\Delta_X$ . We call X a homogeneous space for G if the induced morphism

$$\bar{\rho}_{\delta}: G \times_B X/G_X \to X \times_B X$$

is an isomorphism.

Remark 1.4. Let F be a field. If  $X \in \mathbf{Sm}^G/F$  is a G-homogeneous space, let  $F_X \subset \Gamma(X, \mathcal{O}_X)$  be the subring of G-invariant functions,

$$F_X := \Gamma(X, \mathcal{O}_X)^G$$

Then  $G \setminus X = \operatorname{Spec} F_X$ .

Since X has a F'-point x over a finite separable extension of F, we see that  $F_X$  is a subring of  $\Gamma(G_{F'}, \mathcal{O}_{G_{F'}})^{G_{F'}}$ . Since G is smooth over F, we may assume that  $G_{F'}$  is an extension

$$1 \to G_{F'}^0 \to G_{F'} \to G_{F'}/G_{F'}^0 \to 1$$

with  $G_{F'}^0$  geometrically connected, and  $G_{F'}/G_{F'}^0$  isomorphic to a finite union of copies of Spec F', as F'-scheme. Thus  $\Gamma(G_{F'}, \mathcal{O}_{G_{F'}})^{G_{F'}}$  is a finite separable extension of F, and hence  $F_X$  is a finite separable extension of F, and X is a smooth  $F_X$ -scheme.

**Definition 1.5.** Let G be a smooth, finite type group-scheme over B and take  $X \in \mathbf{Sch}^G/B$  with structure morphism  $\pi: X \to B$ . Form a stratification  $X = \coprod_{i=1}^r X_i$  as in Proposition 1.1 and let  $q_i: X_i \to G \setminus X_i$  be the corresponding quotient map. For each  $y \in G \setminus X_i$ , we have the subscheme  $q_i^{-1}(y)$  of X, with its induced  $G_{k(y)}$ -action, defining an object  $q_i^{-1}(y) \in \mathbf{Sm}^{G_{k(y)}}/k(y)$ , and refer to  $q_i^{-1}(y) \in \mathbf{Sch}^{G_{k(y)}}/k(y)$  as an *orbit* of G in X. For  $O := q_i^{-1}(y)$ , we denote the subfield  $q_i^*k(y) \subset \Gamma(O, \mathcal{O}_O)$  by  $k_O$ , and refer to the orbit  $O \in \mathbf{Sm}^{G_{k_O}}/k_O$ .

We note that the notion of orbit O and corresponding field  $k_O$  are independent of the choice of stratification  $\coprod_{i=1}^r X_i$  of X. Also, an orbit  $O \in \mathbf{Sm}^{G_{k_O}}/k_O$  is a homogeneous space for  $G_{k_O}$ .

# 2. Some representation theory

**Lemma 2.1.** Let G be an affine linearly reductive algebraic group scheme over a field k. We suppose that G is either reductive and k-split, or, for L an extension field of k with separable closure  $L^{sep}$ , that each  $L^{sep}$ -representation of G is  $L^{sep}$ -conjugate to a k-representation of G. Let  $\mathcal{O}$  be a local k-algebra, essentially of finite type over k and let  $f: G \to \operatorname{GL}_n$  be an  $\mathcal{O}$ -representation. Then there is a k-representation  $f_0: G \to \operatorname{GL}_n$  such that f is  $\mathcal{O}$ -conjugate to the base-extension of  $f_0$ . Moreover,  $f_0$  is unique up to k-conjugation.

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I am indebted to Philippe Gille for the argument.

*Proof.* Let L denote the residue field of  $\mathcal{O}$ , with separable closure  $L^{sep}$ , and let  $\bar{f}: G \to \operatorname{GL}_n$  be the L-representation given by the reduction of f. In the case of a k-split, reductive group scheme, the theory of Tits algebras [27] shows that  $\bar{f}$  is  $L^{sep}$ -conjugate to a k-representation  $f_0: G \to \operatorname{GL}_n$ ; if G is not of this type, then this follows from our assumption on G.

Let  $C_{f,f_0}^{\mathcal{O}}$  be the transporter scheme over  $\mathcal{O}$ ,

$$C_{f,f_0}^{\mathcal{O}}(R) = \{ g \in \mathrm{GL}_n(R) \mid gfg^{-1} = f_0 \}$$

and define the group-scheme  $C_{f_0,f_0}^{\mathcal{O}}$  over  $\mathcal{O}$  similarly. Let  $\mathcal{O}^{sh}$  be the strict henselization of  $\mathcal{O}$ . Margaux's theorem [21, Corollary 4.9] implies that f is  $\mathcal{O}^{sh}$ -conjugate to  $f_0$ . This shows that  $C_{f,f_0}^{\mathcal{O}}$  is a  $C_{f_0,f_0}^{\mathcal{O}}$ -torsor for the étale topology, so is classified by an element of  $H_{\text{\'et}}^1(\mathcal{O}, C_{f_0,f_0}^{\mathcal{O}})$ . By Schur's lemma,  $C_{f_0,f_0}^{\mathcal{O}}$  is a product of Weil restrictions of group-schemes  $GL_{n_i}$ , so  $H_{\text{\'et}}^1(\mathcal{O}, C_{f_0,f_0}^{\mathcal{O}}) = \{*\}$ . Thus  $C_{f,f_0}^{\mathcal{O}}$  admits a section over  $\mathcal{O}$ .

Applying the argument to the k-representation  $f_0$  gives the uniqueness: if  $\bar{f}$  is  $L^{sep}$ -conjugate to a second k-representation  $f'_0$ , then  $C^k_{f'_0,f_0}$  is a trivial  $C^k_{f_0,f_0}$ -torsor, so  $f_0$  and  $f'_0$  are k-conjugate.

**Definition 2.2.** Take  $Y \in \mathbf{Sch}^G/k$  with trivial G-action, and let  $\mathcal{V}$  be a G-linearized locally free coherent sheaf on Y. We say that  $\mathcal{V}$  is G-trivialized on some open subscheme  $j:U\to Y$  if there is a k-representation  $V(f_0),\ f_0:G\to \mathrm{GL}_n$ , and an isomorphism  $\phi:j^*\mathcal{V}\xrightarrow{\sim} \mathcal{O}_U\otimes_k V(f_0)$ . We call U a G-trivializing open subscheme and  $\phi$  a G-trivialization.

Remark 2.3. It follows from Lemma 2.1 that if G satisfies the hypotheses of that Lemma, then each point  $y \in Y$  admits a G-trivializing open neighborhood  $y \in U$ .

Let  $N \subset SL_2$  be the normalization of the diagonal torus  $T_1 \subset SL_2$ 

$$T_1(R) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in R^{\times} \right\}$$

The group scheme  $N \subset \mathrm{SL}_2$  is an extension of  $T_1 \cong \mathbb{G}_m$  by  $\mathbb{Z}/2$ , and the  $1 \in \mathbb{Z}/2$  lifts to the order four element

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We let  $\bar{\sigma}$  denote the image of  $\sigma$  in  $N/T_1$ .

Suppose that  $\operatorname{char} k \neq 2$ . Then N is linearly reductive and has the family of irreducible representations  $\rho_m: N \to \operatorname{GL}_2, \ m=1,2,\ldots$ , defined by

$$\rho_m\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}$$
$$\rho_m(\sigma) = \begin{pmatrix} 0 & 1 \\ (-1)^m & 0 \end{pmatrix}$$

We let  $\rho_0: N \to \mathbb{G}_m$  denote the trivial character and  $\rho_0^-: N \to \mathbb{G}_m$  the sign character,  $N \to N/T_1 \cong \{\pm 1\}$ .

The set of representations  $\{\rho_m \mid m \geq 0\} \cup \{\rho_0^-\}$  gives a complete set of representatives for all irreducible k-representations of N. Each  $\rho_m$  gives the bundle

 $\tilde{\mathcal{O}}(m) := V(\rho_m) \times^N EN$  on BN, where  $V(\rho_m)$  is the vector space of the representation  $\rho_m$ . We define  $\tilde{\mathcal{O}}(0)^- := V(\rho_0^-) \times^N EN$ .

For m > 0, the basis  $e_1 \wedge e_2$  of  $V(\rho_m)$  defines an isomorphisms  $\det \rho_m \cong \rho_0$ ,  $\det \tilde{\mathcal{O}}(m) \cong \tilde{\mathcal{O}}(0)$  for m odd and  $\det \rho_m \cong \rho_0^-$ ,  $\det \tilde{\mathcal{O}}(m) \cong \tilde{\mathcal{O}}(0)^-$  for m even.

Remark 2.4. For k a field of characteristic  $\neq 2$ , the group  $N^n$  is linearly reductive and the classification of the irreducible representations of N described above shows that the hypotheses of Lemma 2.1 are satisfied.

We extend the notion of an N-trivialization to certain non-trivial N-actions.

**Definition 2.5.** Take  $Y \in \mathbf{Sch}^N/k$  and let  $\mathcal{V}$  be an N-linearized locally free coherent sheaf on Y. Suppose that  $T_1 \subset N$  acts trivially on Y, giving the action of  $N/T_1 \cong \mathbb{Z}/2$  on Y, with  $\sigma \in N$  acting by an involution  $\tau : Y \to Y$ .

1. We have the decomposition of  $\mathcal{V}$  into weight subspaces for the  $T_1$ -action

$$\mathcal{V} = \bigoplus_m \mathcal{V}_m$$

where  $T_1$  acts on  $\mathcal{V}_m$  via the character  $\chi_r(t) = t^m$ . Define the moving part of  $\mathcal{V}$ ,  $\mathcal{V}^m$ , by

$$\mathcal{V}^{\mathfrak{m}} := \bigoplus_{m \neq 0} \mathcal{V}_m$$

2. Given a k-representation  $f: N \to \operatorname{GL}_n$  and  $U \subset Y$  an N-stable open subscheme, the  $\mathcal{O}_U$ -semi-linear extension of f is the sheaf  $\mathcal{O}_U \otimes_k V(f)$  on U, where  $T_1$  acts  $\mathcal{O}_U$ -linearly via f

$$t \cdot (x \otimes v) := x \otimes f(t)(v)$$

and  $\sigma$  acts as

$$\sigma \cdot (x \otimes v) = \tau^*(x) \otimes f(\sigma)(v).$$

We write this N-linearized sheaf as  $\mathcal{O}_U \otimes_k^{\tau,\sigma} V(f)$ , and call it the  $\mathcal{O}_U$  semi-linear extension of f.

3. We say that V is N-trivialized on an N-stable open subscheme  $j: U \to Y$  if there is a k-representation  $f: N \to \mathrm{GL}_n$  such that  $j^*V$  is isomorphic to the  $\mathcal{O}_U$ -semi-linear extension of f,

$$j^*\mathcal{V} \cong \mathcal{O}_U \otimes_k^{\tau,\sigma} V(f).$$

**Lemma 2.6.** Suppose k has characteristic  $\neq 2$ , let Y, V, and  $\tau : Y \to Y$  be as in Definition 2.5. Take  $y \in Y$ .

- 1. There exists an N-stable open neighborhood  $j:U\to Y$  of y and an N-trivialization of  $j^*\mathcal{V}^{\mathfrak{m}}$ ,  $j^*\mathcal{V}^{\mathfrak{m}}\cong\mathcal{O}_U\otimes_k^{\tau,\sigma}V(f^{\mathfrak{m}})$ . Moreover, we may take  $f^{\mathfrak{m}}$  to be a sum of copies of the representations  $\rho_m$ , m>0.
- 2. Suppose that  $N/T_1$  acts freely on Y. Then there exists an N-stable open neighborhood  $j: U \to Y$  of y and an N-trivialization of  $j^*V$ ,  $j^*V \cong \mathcal{O}_U \otimes_k^{\tau,\sigma} V(f)$ . Moreover, we may take f to be a sum of copies of the representations  $\rho_m$ ,  $r \geq 0$ .

*Proof.* Decompose  $\mathcal{V}$  into a finite sum of weight spaces for the  $T_1$ ,

$$\mathcal{V} = \bigoplus_m \mathcal{V}_m$$

where t acts on  $\mathcal{V}_m$  by the character  $\chi_m(t) = t^m$ . We can find an N-stable open neighborhood U of y with each  $\mathcal{V}_m \cong \mathcal{O}_U^{m_r}$  as  $\mathcal{O}_U$ -module; since  $\sigma \cdot t = t^{-1}\sigma$ , the action by  $\sigma$  restricts to  $\mathcal{O}_Y$ -linear isomorphisms

$$\phi_{m,\sigma}: \tau^* \mathcal{V}_m \xrightarrow{\sim} \mathcal{V}_{-m}$$

Thus we have the decomposition of  $\mathcal{V}$  into N-stable subsheaves

$$\mathcal{V} = \mathcal{V}_0 \oplus \oplus_{m>0} (\mathcal{V}_m \oplus \mathcal{V}_{-m})$$

Suppose m > 0. Choosing a frame  $e_1, \ldots, e_{r_m}$  for  $j^*\mathcal{V}_m$  and let  $e'_j := \phi_{m,\sigma}(\tau^*e_j)$ . Then  $e'_1, \ldots, e'_{r_m}$  gives a framing for  $\mathcal{V}_{-m}$ , and  $\phi_{-m,\sigma}(\tau^*e'_j) = (-1)^m e_j$ . This gives the isomorphism of  $j^*(\mathcal{V}_m \oplus \mathcal{V}_{-m})$  with the  $\mathcal{O}_U$ -semi-linear extension of  $\rho_m^{n_m}$ , with  $T_1$  acting  $\mathcal{O}_U$ -linearly. This gives the isomorphism of N-linearized sheaves on U

$$j^*\mathcal{V}^{\mathfrak{m}} \cong \mathcal{O}_U \otimes (\bigoplus_{m>0} \rho_m^{r_m})$$

proving (1)

We now take m=0; we may assume that  $U=\operatorname{Spec} R$  is affine and that the  $N/T_1$ -action on U is free. Let  $q:U\to \bar U:=N\backslash U$  be the quotient map, which is an étale morphism of degree two, with  $\bar U=\operatorname{Spec} \bar R$ , where  $\bar R\subset R$  is the ring of  $\sigma$ -invariants in R.

Choose a framing  $e_1, \ldots, e_{r_0}$  for  $j^*\mathcal{V}_0$ , and write

$$\phi_{0,\sigma}(\tau^*e_j) = \sum_i a_{ij}e_i$$

This gives us the matrix  $A = (a_{ij})$  with  $A \cdot A^{\tau}$  the  $r_0 \times r_0$  identity matrix. By Hilbert's Theorem 90 for  $GL_n$  and the étale extension  $\bar{R} \hookrightarrow R$ , after shrinking U if necessary, there is a  $B \in GL_n(R)$  with  $A = B \cdot (B^{\tau})^{-1}$ . Changing our framing by  $B = (b_{ij})$ , by taking  $f_j = \sum_i b_{ij} e_i$ , we have

$$\phi_{0,\sigma}(\tau^* f_j) = f_j$$

giving the isomorphism  $j^*\mathcal{V}_0 \cong \mathcal{O}_U \otimes^{\tau,\sigma} V(\rho_0^{r_0})$ . We take  $f := \bigoplus_{m \geq 0} \rho_m^{m_r}$ , proving (2).

Construction 2.7 (Generic representation class). Let G be group scheme over k, satisfying the hypotheses of Lemma 2.1. Let  $\mathcal{V}$  be a locally free coherent sheaf with G-linearization on some connected  $Y \in \mathbf{Sch}^G/k$ .

Case 1: G acts trivially on Y. Let y be a point of Y. By Remark 2.3, there is a G-stable open neighborhood  $j_{U_y}:U_y\hookrightarrow Y$  of  $y\in Y$  and a G-trivialization  $\psi_y:j_{U_y}^*\mathcal{V}\stackrel{\sim}{\to} \mathcal{O}_U\otimes_k V(f)$  for some k-representation  $f_0$  of G. Taking the canonical decomposition of V(f) into isotypical components gives a corresponding decomposition of  $j_{U_y}^*\mathcal{V}$ . By the uniqueness part of Lemma 2.1, the fact that Y is connected implies we have a global decomposition of  $\mathcal{V}$  into isotypical components, indexed by the irreducible k-representations of G,

$$\mathcal{V} = \bigoplus_{\phi} \mathcal{V}_{\phi}$$

such that on each G-trivializing open subscheme  $j_U: U \to Y$ , we have

$$j_U^* \mathcal{V}_\phi \cong \mathcal{O}_U \otimes_k V(\phi)^{n_\phi}$$

where  $V(\phi)$  is the irreducible k-representation indexed by  $\phi$ .

Again using the connectivity of Y, we see that the k-representation  $\bigoplus_{\phi} V(\phi)^{n_{\phi}}$  is uniquely determined by  $\mathcal{V}$ , up to isomorphism of k-representations of G. We denote the isomorphism class of the k-representation  $\bigoplus_{\phi} V(\phi)^{n_{\phi}}$  of G by  $[\mathcal{V}^{gen}]$ .

Case 2.  $G = N, T_1$  acts trivially on Y and  $\mathcal{V} = \mathcal{V}^{\mathfrak{m}}$ . In this case we first decompose

 $\mathcal{V}$  into  $T_1$  isotypical components  $\mathcal{V} = \bigoplus_{m \neq 0} \mathcal{V}_m$  as in the proof of Lemma 2.6(1). For each N-trivializing open subscheme  $j_U : U \to Y$ , we have the isomorphism

$$j_U^* \mathcal{V} \cong \mathcal{O}_U \otimes_k^{\tau,\sigma} \oplus_{m>0} \mathcal{O}_U \otimes_k^{\tau,\sigma} V(\rho_m)^{n_m}.$$

Since this local description follows uniquely from the decomposition into  $T_1$ -isotypical components, the isomorphism class of the k-representation  $\bigoplus_{m>0} V(\rho_m)^{n_m}$  is uniquely determined by  $\mathcal{V}$ . Again, we denote the isomorphism class  $\bigoplus_{m>0} V(\rho_m)^{n_m}$  by  $[\mathcal{V}^{gen}]$ . Case 3.  $G=N, T_1$  acts trivially on Y and the map  $q:Y\to N\backslash Y=\langle \bar{\sigma}\rangle\backslash Y$  is an étale degree two cover. In this case we first decompose  $\mathcal{V}$  into  $T_1$  isotypical components  $\mathcal{V}=\bigoplus_m \mathcal{V}_m$  as in the proof of Lemma 2.6(2), and proceed as in Case 2. For each N-trivializing open subscheme  $j_U:U\to Y$ , we have the isomorphism

$$j_U^* \mathcal{V} \cong \mathcal{O}_U \otimes_{\iota}^{\tau,\sigma} V(\rho_0)^{n_0} \oplus \oplus_{m>0} \mathcal{O}_U \otimes_{\iota}^{\tau,\sigma} V(\rho_m)^{n_m}.$$

Since this local description follows uniquely from the decomposition into  $T_1$ -isotypical components, the isomorphism class of the k-representation  $V(\rho_0)^{n_0} \oplus \oplus_{m>0} V(\rho_m)^{n_m}$  is uniquely determined by  $\mathcal{V}$ . Again, we denote the isomorphism class  $V(\rho_0)^{n_0} \oplus \oplus_{m>0} V(\rho_m)^{n_m}$  by  $[\mathcal{V}^{gen}]$ .

Remark 2.8. Let k a field of characteristic zero. Then  $G = \operatorname{SL}_2^n$  is linearly reductive over k, so we may apply the above construction. In this case, we may use the set  $\{\operatorname{Sym}^{m_1}(F_1) \otimes \ldots \otimes \operatorname{Sym}^{m_n}(F_n), 0 \leq m_i\}$  as a set of representatives for the irreducible representations of G, where  $F_i$  is the G-representation given by the ith projection  $\operatorname{SL}_2^n \to \operatorname{SL}_2$  followed by the standard inclusion  $\operatorname{SL}_2 \hookrightarrow \operatorname{GL}_2$ . Each such representation comes with a canonical trivialization of  $\operatorname{det} \phi$ .

For k a field of characteristic  $\neq 2$ , and  $G = N^n$ , the tensor products of the representations  $\rho_m$ ,  $\rho_0^-$  of N similarly give a complete set of representative for the irreducible representations of  $N^n$ , so for each such representation  $\phi$  of  $N^n$ , we have a unique multi-index  $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$  and a canonical isomorphism  $\det \phi \cong (\rho_0^-)^{\otimes \epsilon_1} \circ p_1 \otimes \ldots \otimes (\rho_0^-)^{\otimes \epsilon_n} \circ p_n$ , where  $p_i : N^n \to N$  is the *i*th projection.

# 3. Cohomology and Borel-Moore homology

We recall from [6] the general set-up for twisted cohomology and twisted Borel-Moore homology and then specialize to the case of the Witt sheaf. We refer the reader to [6] for details and proofs of the material in this section.

We fix a noetherian separated base-scheme B of finite Krull dimension.

Let  $\mathbf{Sch}/B$  denote the category of quasi-projective B-schemes and  $\mathbf{Sm}/B$  the full subcategory of smooth B-schemes; for  $Z \in \mathbf{Sch}/B$  we let  $\pi_Z : Z \to B$  denote the structure morphism. We have the motivic stable homotopy category  $\mathrm{SH}(-)$  with its six functor formalism on  $\mathbf{Sch}/B$ .

Let  $\mathcal{K}(Z)$  denote the K-theory space of perfect complexes on Z. For  $p:V\to Z$  a vector bundle with zero-section  $s:Z\to V$ , we have the auto-equivalence of  $\mathrm{SH}(Z)$   $\Sigma^V:=p_\#s_*$ , with inverse  $\Sigma^{-V}:=s^!p^*$ . The association  $V\mapsto \Sigma^V$  extends to a functor

$$\Sigma^-: \mathcal{K}(Z) \to \operatorname{Aut}(\operatorname{SH}(Z)), \ v \mapsto \Sigma^v$$

sending a locally free sheaf  $\mathcal{V}$  to the operator  $\Sigma^{\mathbb{V}(\mathcal{V})}$ , where  $\mathbb{V}(\mathcal{V}) \to Z$  is the vector bundle  $\operatorname{Spec}_{\mathcal{O}_Z}\operatorname{Sym}^*\mathcal{V}$ .

In addition, for each distinguished triangle in  $D^{perf}(Z)$ 

$$\mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to \mathcal{V}'[1]$$

we have the canonical isomorphisms

$$\Sigma^{\mathcal{V}} \cong \Sigma^{\mathcal{V}' \oplus \mathcal{V}'')} \cong \Sigma^{\mathcal{V}'} \circ \Sigma^{\mathcal{V}''} \cong \Sigma^{\mathcal{V}''} \circ \Sigma^{\mathcal{V}'}$$

We write  $\Sigma^{a,b}$  for  $\Sigma^{a-b}_{S^1}\Sigma^b_{\mathbb{G}_m}$  and we have for  $r\in\mathbb{N}$  the canonical isomorphisms

$$\Sigma^{\mathcal{O}_Z^r} \cong \Sigma^r_{\mathbb{P}^1} \cong \Sigma^{2r,r}$$

on SH(Z).

For  $f: Z \to W$  a smooth morphism in  $\operatorname{\mathbf{Sch}}/B$ , we have functor  $f_\#: \mathcal{H}(Z) \to \mathcal{H}(W)$  with right adjoint  $f^*: \mathcal{H}(W) \to \mathcal{H}(Z)$ .  $f_\#$  is the functor on presheaves induced by the pull-back functor  $f^{-1}: \operatorname{\mathbf{Sm}}/W \to \operatorname{\mathbf{Sm}}/Z$ ,  $f^{-1}(Y \to W) := Y \times_W Z \xrightarrow{p_2} Z$ , and the functors  $f_\#$ ,  $f^*$  on  $\operatorname{SH}(-)$  are the respective T-stabilizations of the unstable  $f_\#$ ,  $f^*$ . Thus, for  $q: X \to Y$  a morphism in  $\operatorname{\mathbf{Sm}}/B$ , we have the induced morphism  $q: \pi_{X\#}(1_X) \to \pi_{Y\#}(1_Y)$ , giving us the functor  $h_B: \operatorname{\mathbf{Sm}}/B \to \operatorname{SH}(B)$ . Alternatively, the morphism  $q: \pi_{X\#}(1_X) \to \pi_{Y\#}(1_X) \to \pi_{Y\#}(1_Y)$  arises from the isomorphisms  $\pi_{X!}\pi_X^! \cong \pi_{X\#}\pi_X^*$ ,  $\pi_{Y!}\pi_Y^! \cong \pi_{Y\#}\pi_Y^*$ ,  $1_X \cong \pi_X^*1_B$ ,  $1_Y \cong \pi_Y^*1_B$ , and the natural transformation

$$\pi_{X!}\pi_X^! \cong \pi_{Y!}f_!f^!\pi_Y^! \xrightarrow{\eta_!^!} \pi_{Y!}\pi_Y^!$$

applied to  $1_B$ ; here  $\eta_1^!$  is the counit of the adjunction  $f_! \dashv f^!$ .

3.1.  $\mathcal{E}$ -cohomology, Borel-Moore homology, products. For  $Z \in \mathbf{Sch}/B$  and  $v \in \mathcal{K}(Z)$ , we have the twisted Borel-Moore motive

$$Z/B(v)_{\mathrm{B.M.}} := \pi_{Z!}(\Sigma^v 1_Z) \in \mathrm{SH}(B)$$

and for  $\mathcal{E} \in SH(B)$  the twisted  $\mathcal{E}$ -Borel-Moore homology

$$\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^{a,b}Z/B(v)_{\mathrm{B.M.}},\mathcal{E}).$$

We write  $Z/B_{\text{B.M.}}$  for  $Z/B(0)_{\text{B.M.}}$ ,  $\mathcal{E}_{a,b}^{\text{B.M.}}(Z/B)$  for  $\mathcal{E}_{a,b}^{\text{B.M.}}(Z/B,0)$  and  $\mathcal{E}^{\text{B.M.}}(Z/B,v)$  for  $\mathcal{E}_{0,0}^{\text{B.M.}}(Z/B,v)$ . We often drop the -/B from the notation if the base-scheme is understood.

Define the twisted  $\mathcal{E}$ -cohomology by

$$\mathcal{E}^{a,b}(Z,v) := \operatorname{Hom}_{\operatorname{SH}(Z)}(1_Z, \Sigma^{a,b} \Sigma^v \pi_Z^* \mathcal{E}) = \operatorname{Hom}_{\operatorname{SH}(B)}(1_B, \Sigma^{a,b} \pi_{Z*}(\Sigma^v \pi_Z^* \mathcal{E})).$$

If  $p:W\to Z$  is a morphism in  $\operatorname{\mathbf{Sch}}/B$ , we will often write  $W/B(v)_{\mathrm{B.M.}}$  for  $W/B(p^*v)_{\mathrm{B.M.}}$ , and similarly use the notation  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(W/B,v)$ ,  $\mathcal{E}^{a,b}(W,v)$ , etc. For  $\mathcal{E}=1_B$ , we write  $H_{a,b}^{\mathrm{B.M.}}$  and  $H^{a,b}$  for  $(1_B)_{a,b}^{\mathrm{B.M.}}$  and  $(1_B)^{a,b}$ .

Sending  $Z \in \mathbf{Sch}/B$  to  $\mathcal{E}^{a,b}(Z,v)$  defines a functor

$$\mathcal{E}^{a,b}(-,v): \mathbf{Sch}/B^{\mathrm{op}} \to \mathbf{Ab}$$

with pullback map  $f^*: \mathcal{E}^{a,b}(W,v) \to \mathcal{E}^{a,b}(W',v)$  for a Z-morphism  $f: W' \to W$  induced by the functor  $f^*: \mathrm{SH}(W) \to \mathrm{SH}(W')$ .

If  $\mathcal{E}$  is a commutative ring spectrum in SH(B) (i.e. and commutative monoid object), the multiplication in  $\mathcal{E}$  induces associative external products

$$\mathcal{E}^{a,b}(Z,v)\times\mathcal{E}^{a',b'}(Z',v')\to\mathcal{E}^{a+a',b+b'}(Z\times_BZ',v+v')$$

pulling back by the diagonal gives  $\mathcal{E}^{*,*}(Z)$  the structure of a bi-graded associative ring with unit and makes  $\mathcal{E}^{*,*}(Z,v)$  a bigraded  $\mathcal{E}^{*,*}(Z)$ -module. The multiplication also induces the map (see [6, 2.1.10])

$$\pi_Z^! \mathcal{E} \wedge \pi_Z^* \mathcal{E} \to \pi_Z^! \mathcal{E}.$$

Via the adjunction isomorphism

$$\mathcal{E}_{a.b}^{\mathrm{B.M.}}(Z/B,v) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^{a,b}Z/B(v)_{\mathrm{B.M.}},\mathcal{E}) = \mathrm{Hom}_{\mathrm{SH}(Z)}(\Sigma^{a,b}\Sigma^{v}1_{Z},\pi_{Z}^{!}\mathcal{E})$$

this multiplication map gives us the cap product pairing

$$\cap: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v) \times \mathcal{E}^{c,d}(Z,w) \to \mathcal{E}_{a-c,b-d}^{\mathrm{B.M.}}(Z/B,v-w)$$

defined by

$$\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v) \times \mathcal{E}^{c,d}(Z,w) =$$

$$\mathrm{Hom}_{\mathrm{SH}(Z)}(\Sigma^{a,b}\Sigma^{v}1_{Z},\pi_{Z}^{!}\mathcal{E}) \times \mathrm{Hom}_{\mathrm{SH}(Z)}(\Sigma^{-w}1_{Z},\Sigma^{c,d}\pi_{Z}^{*}\mathcal{E})$$

$$\to \mathrm{Hom}_{\mathrm{SH}(Z)}(\Sigma^{a-c,b-d}\Sigma^{v-w}1_{Z},\pi_{Z}^{!}\mathcal{E})$$

$$= \mathcal{E}_{a,c,b-d}^{\mathrm{B.M.}}(Z/B,v-w)$$

3.2. **Purity isomorphisms.** For  $X \in \mathbf{Sm}/B$ ,  $v \in \mathcal{K}(X)$ , the adjunction  $\pi_{\#} \dashv \pi_{X}^{*}$  gives the isomorphism

$$\mathcal{E}^{a,b}(X,v) := \operatorname{Hom}_{\operatorname{SH}(B)}(\pi_{X\#}(\Sigma^{-v}1_X), \Sigma^{a,b}\mathcal{E}).$$

More generally, for  $\mathcal{F} \in SH(X)$ , we set  $\mathcal{E}^{a,b}(\mathcal{F},v) := Hom_{SH(B)}(\pi_{X\#}(\Sigma^{-v}\mathcal{F}), \Sigma^{a,b}\mathcal{E})$ . For example, if  $i: Z \to X$  is a closed immersion in  $\mathbf{Sch}/B$ , we have the cohomology with support

$$\mathcal{E}_Z^{a,b}(X,v) := \operatorname{Hom}_{\operatorname{SH}(B)}(\pi_{X\#}(\Sigma^{-v}i_*1_Z), \Sigma^{a,b}\mathcal{E}).$$

The purity isomorphism  $\pi_{X\#} \cong \pi_{X!} \circ \Sigma^{\Omega_{X/B}}$  gives the Pioncaré duality isomorphism of cohomology with Borel-Moore homology

$$(3.1) \ \mathcal{E}^{a,b}(X,v) \cong \operatorname{Hom}_{\operatorname{SH}(B)}(\pi_{X!}(\Sigma^{\Omega_{X/B}-v}1_X),\Sigma^{a,b}\mathcal{E}) = \mathcal{E}^{\operatorname{B.M.}}_{-a,-b}(X/B,\Omega_{X/B}-v)$$

Similarly if  $i: Z \to X$  is a closed immersion in  $\mathbf{Sm}/B$ , we have  $i_* = i_!$ , giving the isomorphism

$$(3.2) \quad \mathcal{E}_{Z}^{a,b}(X,v) \cong \operatorname{Hom}_{\operatorname{SH}(B)}(\pi_{X!}(\Sigma^{\Omega_{X/B}-v}i_{!}1_{Z}),\Sigma^{a,b}\mathcal{E})$$

$$\cong \operatorname{Hom}_{\operatorname{SH}(B)}(\pi_{Z!}(\Sigma^{i^{*}\Omega_{X/B}-i^{*}v}1_{Z}),\Sigma^{a,b}\mathcal{E})$$

$$= \mathcal{E}_{-a-b}^{\operatorname{B.M.}}(Z/B,i^{*}\Omega_{X/B}-i^{*}v).$$

Using the exact sequence

$$0 \to \mathcal{N}_i \to i^* \Omega_{X/B} \to \Omega_{Z/B} \to 0$$

with  $\mathcal{N}_i$  the conormal sheaf, this yields the purity isomorphism

(3.3) 
$$\mathcal{E}_Z^{a,b}(X,v) \cong \mathcal{E}^{a,b}(Z,v-\mathcal{N}_i).$$

3.3. Thom isomorphisms, oriented spectra. If  $\mathcal{E}$  is an oriented ring spectrum in SH(B), and for  $v \in D^{perf}(X)$  of virtual rank r, we have the Thom isomorphism (depending on the choice of orientation)  $\Sigma^v \pi_X^! \mathcal{E} \cong \Sigma^{-2r, -r} \pi^! \mathcal{E}$ , giving the isomorphism

$$\mathcal{E}_{a-2r,b-r}^{\mathrm{B.M.}}(Z/B,v) \cong \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B).$$

If  $\mathcal{E}$  is an SL-oriented ring spectrum, we have the Thom isomorphism (also depending on the choice of orientation)  $\Sigma^{-2,-1}\Sigma^{\det v}\pi_X^!\mathcal{E}\cong\Sigma^{-2r,-r}\Sigma^v\pi^!\mathcal{E}$ , where  $\det v$  the virtual determinant of v, giving the isomorphism

$$\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B, v - \mathcal{O}_Z^r) \cong \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B, \det v - \mathcal{O}_Z)$$

3.4. **Euler class.** For  $p: V \to Z$  a vector bundle on some  $Z \in \mathbf{Sch}/B$ ,  $V = \mathbb{V}(\mathcal{V})$ , we have the Euler class  $e(V) \in H^{0,0}(Z,\mathcal{V})$  defined as follows (see [6, §3.1] for details): Evaluating the natural transformation  $\mathrm{Id}_{\mathrm{SH}(V)} \to s_* s^*$  at  $1_V$  gives the map  $\alpha_s: 1_V \to s_* 1_Z$ . The section  $s: Z \to V$  to p gives the map  $h(s): 1_Z \to p_\#(1_V)$ ; composed with  $p_\#(\alpha_s)$  gives the map  $p_\#(\alpha_s) \circ h(s): 1_Z \to p_\# s_* 1_Z = \Sigma^{\mathcal{V}}(1_Z)$ , equivalently,

$$p_{\#}(\alpha_s) \circ h(s) : 1_Z \to \Sigma^{\mathcal{V}} \pi_Z^*(1_B).$$

Define  $e(V) := [p_{\#}(\alpha_s) \circ h(s)] \in H(Z, \mathcal{V})$ . For  $\mathcal{E} \in SH(B)$  a commutative ring spectrum, applying the unit map  $\epsilon_{\mathcal{E}} : 1_B \to \mathcal{E}$  gives the Euler class  $e(V) \in \mathcal{E}(Z, \mathcal{V})$ . If  $\mathcal{E}$  is oriented and V has rank r, this is  $e(V) \in \mathcal{E}^{2r,r}(Z)$  and if  $\mathcal{E}$  is SL-oriented, this is  $e(V) \in \mathcal{E}^{2r,r}(Z, \det \mathcal{V} - \mathcal{O}_Z)$ .

3.5. Functorialities. For  $p:Z\to W$  a proper morphism in  $\mathbf{Sch}/B$ , and  $v\in\mathcal{K}(W)$ , we have the *proper pull-back* 

$$p^*: W/B(v)_{\mathrm{B.M.}} \to Z/B(v)_{\mathrm{B.M.}}$$

inducing the proper push-forward on Borel-Moore homology

$$p_* := (p^*)^* : \mathcal{E}_{a.b}^{\mathrm{B.M.}}(Z/B, v) \to \mathcal{E}_{a.b}^{\mathrm{B.M.}}(W/B, v)$$

For  $q: P \to W$  a smooth morphism in  $\mathbf{Sch}/B$ , we have the *smooth push-forward* 

$$q_!: P/B(v+\Omega_q) \to W/B(v)$$

inducing the *smooth pullback* on Borel-Moore homology

$$q! := (q_!)^* : \mathcal{E}_{a,b}^{\mathrm{B.M.}}(W/B, v) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(P/B, v + \Omega_q).$$

If  $q: P \to W$  is a vector bundle on W (or even a torsor for a vector bundle on W), then q! is an isomorphism.

If  $i: Z \to W$  is a regular immersion in  $\mathbf{Sch}/B$ , we have the Gysin pullback map

$$i^!:\mathcal{E}^{\mathrm{B.M.}}_{a,b}(W/B,v)\to\mathcal{E}^{\mathrm{B.M.}}_{a,b}(Z/B,v-\mathcal{N}_i)$$

where  $\mathcal{N}_i$  is the conormal sheaf of i. If  $q: P \to W$  is a vector bundle and  $0_P: W \to P$  is the zero-section, then q! and  $0_P!$  are inverse isomorphisms.

We call a morphism  $f: Z \to W$  an lci morphism if f admits a factorization as  $f = q \circ i$ , with  $i: Z \to P$  a regular immersion and  $q: P \to W$  a smooth morphism (both in  $\mathbf{Sch}/B$ ). Since  $Z \to B$  in  $\mathbf{Sch}/B$  is quasi-projective over B, an lci morphism in  $\mathbf{Sch}/B$  is a smoothable lci morphism in the sense of [6]. Let  $L_f$  be the relative cotangent complex of f. The lci pullback

$$f^!: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(W/B,v) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_f)$$

is defined as  $f^! := i^! \circ q^!$ , using the distinguished triangle

$$i^*\Omega_q \to L_f \to \mathcal{N}_i[1] \to i^*\Omega_p[1]$$

to define the isomorphism  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_f)\cong\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+\Omega_p-\mathcal{N}_i)$ , and is independent of the choice of factorization. Moreover, for composable lci morphisms  $g:W\to U,\,f:Z\to W$ , after using the distinguished triangle

$$f^*L_q \to L_{qf} \to L_f \to f^*L_q[1]$$

to identify  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_f+L_g)$  with  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_{gf})$ , we have  $(gf)^!=f^!g^!$ . See [6, Theorem 4.2.1] for these facts. We will usually unify the notation, writing  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_q)$  for  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+\Omega_q)$  in the case of a smooth morphism q and  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v+L_i)$  for  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z/B,v-\mathcal{N}_i)$  in the case of a regular embedding i.

Example 3.1. For  $B = \operatorname{Spec} k$ , the cohomology of the sheaf of Witt groups  $\mathcal{W}$  is represented by the Eilenberg-MacLane spectrum  $\operatorname{EM}(\mathcal{W}) \in \operatorname{SH}(k)$  via canonical isomorphisms for  $X \in \operatorname{\mathbf{Sm}}/k$  and  $r \in \mathbb{Z}$ 

$$H^a_{\mathrm{Nis}}(X, \mathcal{W}) \cong \mathrm{EM}^{a,0}(\mathcal{W})(X) \cong \mathrm{EM}^{a+r,r}(\mathcal{W})(X)$$

Moreover, for  $\mathcal{L}$  an invertible sheaf on X, there is a twisted version of the Witt sheaf,  $\mathcal{W}(\mathcal{L})$ , defined as the sheaf of  $\mathcal{L}$ -valued non-degenerate bilinear forms, modulo  $\mathcal{L}$ -valued hyperbolic forms, and there are canonical isomorphisms

$$H_{\mathrm{Nis}}^{a}(X, \mathcal{W}(\mathcal{L})) \cong \mathrm{EM}^{a,0}(\mathcal{W})(X, \mathcal{L} - \mathcal{O}_X)$$

and

$$H_{\mathrm{Nis}}^{a}(X, \mathcal{W}(\det v)) \cong \mathrm{EM}^{a,0}(\mathcal{W})(X, v - \mathcal{O}_{X}^{r}).$$

for  $v \in \mathcal{K}(X)$  of virtual rank r.

We define the  $\mathcal{L}$ -twisted  $\mathcal{W}$ -Borel-Moore homology of  $Z \in \mathbf{Sch}/k$  as

$$H_a^{\mathrm{B.M.}}(Z, \mathcal{W}(\mathcal{L})) := \mathrm{EM}(\mathcal{W})_{a,0}^{\mathrm{B.M.}}(Z/B, \mathcal{L} - \mathcal{O}_Z)$$

For  $i: Z \to X$  a closed immersion in X smooth of dimension  $d_X$  over k, the isomorphism (3.2) thus yields the isomorphism

$$H_a^{\mathrm{B.M.}}(Z/B, \mathcal{W}(\mathcal{L})) \cong H_Z^{d_X - a}(X, \mathcal{W}(\det^{-1} \mathcal{N}_i \otimes \mathcal{L}))$$

with  $H_Z^*(X, -)$  the usual cohomology with support and  $\mathcal{N}_i$  the conormal sheaf.

By an abuse of notation, we denote the twisted EM(W)-cohomology of some  $Z \in \mathbf{Sch}/k$  and invertible sheaf  $\mathcal{L}$  on Z as  $H^a_{Nis}(Z, \mathcal{W}(\mathcal{L}))$ :

$$H^a_{\mathrm{Nis}}(Z, \mathcal{W}(\mathcal{L})) := \mathrm{EM}(\mathcal{W})^{a,0}(Z, \mathcal{L} - \mathcal{O}_Z) \cong \mathrm{EM}(\mathcal{W})^{a+b,b}(Z, \mathcal{L} - \mathcal{O}_Z).$$

In case Z is smooth over k, this agrees with the twisted Witt-sheaf cohomology. For  $V \to Z$  a rank r-vector bundle, this gives us the Euler class

$$e(V) \in \mathrm{EM}(\mathcal{W})^{2r,r}(Z,\det^{-1}V - \mathcal{O}_Z) = H^r(Z,\mathcal{W}(\det^{-1}V)).$$

where det V is the invertible sheaf of sections of the line bundle  $\Lambda^r V$ , and det<sup>-1</sup>  $V := (\det V)^{-1}$ .

3.6.  $\mathcal{E}$ -cohomology with supports, products and cap products. Let  $j: U \to Z$  be an open immersion with closed complement  $i: T \to Z$ , let  $f: Z' \to Z$  be a morphism, giving the cartesian diagram

$$U' \xrightarrow{j'} Z'$$

$$\downarrow \qquad \qquad \downarrow_f$$

$$U \xrightarrow{j} Z,$$

and let  $i': T' \to Z'$  be the closed complement of U'. We have the cohomology with support

$$\mathcal{E}_T^{c,d}(Z,w) := \operatorname{Hom}_{\operatorname{SH}(Z)}(\Sigma^v i_{T!} 1_T, \Sigma^{c,d} \pi_Z^* \mathcal{E})$$

The Borel-Moore homology with support

$$\mathcal{E}_{a.b}^{\mathrm{B.M.}\ T'}(Z'/B,v) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^{a,b}\pi_{W!}\Sigma^v i_{T'!}1_{T'},\mathcal{E})$$

is isomorphic to  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(T'/B,v)$  via the isomorphisms

$$\operatorname{Hom}_{\operatorname{SH}(B)}(\Sigma^{a,b}\pi_{W!}\Sigma^{v}1_{T'},\mathcal{E}) \cong \operatorname{Hom}_{\operatorname{SH}(T')}(\Sigma^{a,b}\Sigma^{v}1_{T'},i_{T'}^{!}\pi_{W}^{!}\mathcal{E})$$

$$\cong \operatorname{Hom}_{\operatorname{SH}(T')}(\Sigma^{a,b}\Sigma^{v}1_{T'},\pi_{T'}^{!}\mathcal{E}) = \mathcal{E}_{a,b}^{\operatorname{B.M.}}(T'/B,v).$$

The cap product extends to a cap product

$$\mathcal{E}^{\mathrm{B.M.}}_{a,b}(Z'/B,v)\times\mathcal{E}^{c,d}_T(Z,w)\to\mathcal{E}^{\mathrm{B.M.}}_{a-c,b-d}(T'/B,v-w)$$

by first applying  $f^*: \mathcal{E}^{c,d}_T(Z,w) \to \mathcal{E}^{c,d}_{T'}(Z',w)$ . Noting that  $i_{T'!}1_{T'} \wedge_{T'}1_{T'} = i_{T'!}1_{T'}$ , we have the pairing

$$\begin{split} \operatorname{Hom}_{\operatorname{SH}(Z')}(\Sigma^{a,b}\Sigma^{v}1_{Z'},\pi_{Z'}^{!}\mathcal{E}) \times \operatorname{Hom}_{\operatorname{SH}(Z')}(\Sigma^{-w}i_{T'!}1_{T'},\Sigma^{c,d}\pi_{Z'}^{*}\mathcal{E}) \\ \to \operatorname{Hom}_{\operatorname{SH}(Z')}(\Sigma^{a-c,b-d}\Sigma^{v-w}i_{T'!}1_{T'},\pi_{Z'}^{!}\mathcal{E}) &\cong \mathcal{E}_{a-c,b-d}^{\operatorname{B.M.}}(T'/B,v-w). \end{split}$$

Let  $i_{T*}: \mathcal{E}^{c,d}_T(Z,w) \to \mathcal{E}^{c,d}(Z,w)$  be the map induced by the counit  $1_Z \to i_{T*}i_T^*1_W = i_{T!}1_T$  (i.e., "forget supports"). We also have the proper push-forward map  $i_{T'*}$  on Borel-Moore homology. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z'/B,v) \times \mathcal{E}_{T}^{c,d}(Z,w) & \stackrel{\cap}{\longrightarrow} \mathcal{E}_{a-c,b-d}^{\mathrm{B.M.}}(T'/B,v-w) \ . \\ & & \downarrow^{i_{T'*}} \\ \\ \mathcal{E}_{a,b}^{\mathrm{B.M.}}(Z'/B,v) \times \mathcal{E}^{c,d}(Z,w) & \stackrel{\cap}{\longrightarrow} \mathcal{E}_{a-c,b-d}^{\mathrm{B.M.}}(Z'/B,v-w) \end{array}$$

For  $T'' \subset Z'$  a second closed subset, the products in  $\mathcal{E}$ -cohomology similarly define products in cohomology with support

$$\mathcal{E}_{T''}^{a,b}(Z',v) \times \mathcal{E}_{T}^{c,d}(Z,w) \to \mathcal{E}_{T'\cap T''}^{a+c,b+d}(Z',v+w)$$

and if Z' is smooth, these products are compatible with the cap product

$$\mathcal{E}^{\mathrm{B.M.}}_{a.b}(T'',v) \times \mathcal{E}^{c,d}_T(Z,w) \to \mathcal{E}^{\mathrm{B.M.}}_{a-c.b-d}(T' \cap T'',v-w)$$

via the suitable purity isomorphisms (3.2).

# 3.7. Refined Gysin morphism. Let

$$X_0 \xrightarrow{\iota'} X$$

$$\downarrow p$$

$$\downarrow p$$

$$\downarrow p$$

$$\downarrow p$$

$$\downarrow p$$

be a cartesian diagram in  $\mathbf{Sch}/B$ , with  $\iota$  a regular immersion. We recall from [6, Definition 4.2.5] the refined Gysin morphism

$$\iota_p^!: \mathbb{H}^{\mathrm{B.M.}}(X, v) \to \mathbb{H}^{\mathrm{B.M.}}(X_0, v + L_\iota)$$

This is actually defined for  $\iota$  a (smoothable) lci morphism, but we will not need this.

3.8. Compatibilities. We list the various compatibilities of the operations of lci pullback, refined Gysin morphism and proper push-forward in the following proposition.

**Proposition 3.2.** Take  $Y \in \mathbf{Sch}/B$  with  $v \in \mathcal{K}(Y)$ .

1 (compatibility with proper push-forward and smooth pullback). Let

$$W_0 \xrightarrow{\iota''} W$$

$$\downarrow^{q_2} \qquad \downarrow^{p_2} X$$

$$\downarrow^{q_1} \qquad \downarrow^{p_1} Y$$

$$\downarrow^{q_1} \qquad \downarrow^{p_1} Y$$

be a commutative diagram in  $\mathbf{Sch}/B$ , with both squares cartesian, and with  $\iota$  a regular immersion.

1a. Suppose  $p_2$  is proper. Then the diagram

$$H^{\mathrm{B.M.}}(W,v) \xrightarrow{\iota_{p}^{!}} H^{\mathrm{B.M.}}(W_{0},v+L_{\iota})$$

$$\downarrow^{p_{2*}} \qquad \qquad \downarrow^{q_{2*}}$$

$$H^{\mathrm{B.M.}}(X,v) \xrightarrow{\iota_{p_{1}}^{!}} H^{\mathrm{B.M.}}(X_{0},v+L_{\iota})$$

commutes.

1b. Suppose  $p_2$  is smooth. Then the diagram

$$H^{\mathrm{B.M.}}(W, v + L_{p_2}) \xrightarrow{\iota_p^!} H^{\mathrm{B.M.}}(W_0, v + L_{\iota} + L_{q_2})$$

$$\downarrow_{p_2^!} \qquad \qquad \downarrow_{p_1^!} \qquad \qquad \downarrow_{p_1^!} \qquad \qquad H^{\mathrm{B.M.}}(X, v) \xrightarrow{\iota_{p_1}^!} H^{\mathrm{B.M.}}(X_0, v + L_{\iota})$$

commutes.

2 (functoriality). Let

$$X_{1} \xrightarrow{\iota'_{1}} X_{0} \xrightarrow{\iota'_{0}} X$$

$$\downarrow^{p_{1}} \qquad \downarrow^{p_{0}} \qquad \downarrow^{p}$$

$$Y_{1} \xrightarrow{\iota_{1}} Y_{0} \xrightarrow{\iota_{0}} Y$$

be a commutative diagram in  $\mathbf{Sch}/B$ , with both squares cartesian, and with  $\iota_0$  and  $\iota_1$  regular immersions. Then

$$\iota_1^! \circ \iota_0^! = (\iota_0 \circ \iota_1)^! : H^{\operatorname{B.M.}}(X, v) \to H^{\operatorname{B.M.}}(X_1, v + L_{\iota_0 \circ \iota_1}).$$

Here we use the distinguished triangle

$$\iota_1^*L_{\iota_0} \to L_{\iota_0 \circ \iota_1} \to L_{\iota_1} \to \iota_1^*L_{\iota_0}[1]$$

to identify  $H^{\text{B.M.}}(X_1, v + L_{t_1} + L_{t_0})$  with  $H^{\text{B.M.}}(X_1, v + L_{t_0 \circ t_1})$ .

3 (excess intersection formula). Let

$$W_0 \xrightarrow{\iota''} W$$

$$\downarrow^{q_2} \qquad \downarrow^{p_2} X_0 \xrightarrow{\iota'} X$$

$$\downarrow^{q_1} \qquad \downarrow^{p_1} Y_0 \xrightarrow{\iota} Y$$

be a commutative diagram in  $\operatorname{\mathbf{Sch}}/B$ , with both squares cartesian, and with  $\iota$  and  $\iota'$  both regular immersions. We have a natural surjection of locally free sheaves  $q_1^*\mathcal{N}_{\iota} \to \mathcal{N}_{\iota'}$  with locally free kernel  $\mathcal{E}$ , giving the Euler class  $e(q_2^*E) \in H^{\operatorname{B.M.}}(W_0, \mathcal{E})$  with  $E = \mathbb{V}(\mathcal{E})$ . Then for  $\alpha \in H^{\operatorname{B.M.}}(W, v)$ , we have

$$\iota^{!}(\alpha) = \iota'^{!}(\alpha) \cdot e(q_{2}^{*}E) \in H^{\text{B.M.}}(W_{0}, v + L_{\iota}) = H^{\text{B.M.}}(W_{0}, v + L_{\iota'} - \mathcal{E}),$$

where we use the exact sequence

$$0 \to \mathcal{E} \to q_1^* \mathcal{N}_{\iota} \to \mathcal{N}_{\iota'} \to 0$$

to give the canonical isomorphism

$$H^{\mathrm{B.M.}}(W_0, v + L_\iota) \cong H^{\mathrm{B.M.}}(W_0, v + L_{\iota'} - \mathcal{E}))$$

4. (Fundamental classes, Poincaré duality) Let  $[B] \in H^{0,0}(B) = H^{\mathrm{B.M.}}(B/B)$  be the class of  $\mathrm{Id}_{1_B}$ . Let  $p_X: X \to B$  be an lci scheme over B, in  $\mathbf{Sch}/B$  (an lci scheme for short). Define the fundamental class  $[X] \in H^{\mathrm{B.M.}}(X/B, L_{X/B})$  by

$$[X] := p_X^!([B]).$$

i. For  $f: Y \to X$  an lci morphism of lci-schemes, we have  $f^!([X]) = [Y]$ . ii. Let  $\mathcal{E} \in SH(B)$  be a commutative ring spectrum and take  $v \in \mathcal{K}(X)$ . Then the cap product with [X]

$$[X] \cap -: \mathcal{E}^{a,b}(X,v) \to \mathcal{E}^{\mathrm{B.M.}}_{-a,-b}(X,L_{X/B}-v)$$

is an isomorphism for  $X \in \mathbf{Sm}/B$ , and is equal to the purity isomorphism.

Remark 3.3 (Relative pull-back). 1. Let

$$X_0 \xrightarrow{q} X_1$$

$$\downarrow^{f_0} \qquad \downarrow^{f_1}$$

$$B_0 \xrightarrow{p} B_1$$

be a Tor-independent cartesian square in  $\operatorname{\mathbf{Sch}}/B$ , with  $B_0$  and  $B_1$  smooth over B. Take  $v \in \mathcal{K}(X)$ . We define the relative pull-back

$$p!: \mathcal{E}^{\mathrm{B.M.}}(X_1/B_1, v) \to \mathcal{E}^{\mathrm{B.M.}}(X_0/B_0, v)$$

as follows. Since  $B_0$  and  $B_1$  are smooth over B, the map p is lci. Since the square is Tor-independent and cartesian, q is an lci morphism and  $L_q = f^*L_p$ , giving the lci pull-back  $q!: \mathcal{E}^{\mathrm{B.M.}}(X_1/B_1, v) \to \mathcal{E}^{\mathrm{B.M.}}(X_0/B_1, v + L_p)$ .

The purity isomorphisms

$$p_{X_0!} = p_{B_1!} \circ (pf_0)_! \cong p_{B_1\#} \circ \Sigma^{-L_{B_1/B}} \circ (pf_0)_!$$

and

$$p_{X_0!} = p_{B_0!} \circ f_{0!} \cong p_{B_0\#} \circ \Sigma^{-L_{B_0/B}} \circ f_{0!}$$

gives us the isomorphisms

$$\mathcal{E}^{\text{B.M.}}(X_0/B_1, v + L_p) \cong \mathcal{E}^{\text{B.M.}}(X_0/B, v + L_p + L_{B_1/B})$$

and

$$\mathcal{E}^{\text{B.M.}}(X_0/B_0, v) \cong \mathcal{E}^{\text{B.M.}}(X_0/B, v + L_{B_0/B})$$

The distinguished triangle

$$p^*L_{B_1/B} \to L_{B_0/B} \to L_p \to p^*L_{B_1/B}$$

gives the isomorphism

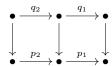
$$\mathcal{E}^{\text{B.M.}}(X_0/B, v + L_p + L_{B_1/B}) \cong \mathcal{E}^{\text{B.M.}}(X_0/B, v + L_{B_0/B}).$$

Putting these together gives the isomorphism

$$\vartheta_p: \mathcal{E}^{\mathrm{B.M.}}(X_0/B_1, v + L_p) \xrightarrow{\sim} \mathcal{E}^{\mathrm{B.M.}}(X_0/B_0, v)$$

and we define  $p! := \vartheta_p \circ q!$ .

2. The relative lci pull-back maps are functorial with respect to adjacent cartesian squares



when defined. This follows from the functoriality of pull-back for lci morphisms and the identities

$$q_2^!\circ\vartheta_{p_1}=\vartheta_{p_1}q_2^!$$

and

$$\vartheta_{p_1p_2}=\vartheta_{p_2}\circ\vartheta_{p_1},$$

this latter after using the canonical isomorphism  $\mathcal{E}^{\text{B.M.}}(-/B, v + L_{p_1} + L_{p_2}) \cong \mathcal{E}^{\text{B.M.}}(-/B, v + L_{p_1p_2})$ . The easy proofs of these identities are left to the reader.

# 4. EQUIVARIANT BOREL-MOORE WITT HOMOLOGY

In this section we recall the construction of the G-equivariant  $\mathcal{E}$ -cohomology and G-equivariant  $\mathcal{E}$ -Borel-Moore homology functors for  $\mathcal{E} \in \mathrm{SH}(B)$  by Di Lorenzo and Mantovani [7, §1.2]. As described in loc. cit. the theory is only for  $\mathcal{E}$  being the Eilenberg-MacLane spectrum of a homotopy module, and the theory we present here is in essence no more general than this, as  $\mathcal{E}$  needs a certain "boundedness" property for the resulting theory to be reasonable. The general theory is currently being constructed by Mantovani [20]. The theory we give here is a twist of that of Di Lorenzo and Mantovani; this modification is made so that the theory will compatible with the operation of forgetting the G-action on the Borel-Moore homology.

The construction of equivariant cohomology and Borel-Moore homology in the algebraic setting follows the method of Totaro [28], Edidin-Graham [8], and Morel-Voevodsky [22, §4]. Let  $F \cong \mathbb{A}^n_B$  be the fundamental representation of  $\operatorname{GL}_n := \operatorname{GL}_n/B$ . For  $j \geq 0$ , let  $V_j = F^{n+j}$ ; we identify  $V_j$  with the scheme of  $n \times n + j$ 

matrices with the usual left  $GL_n$ -action. Let  $W_j \subset V_j$  be the closed subscheme of matrices of rank < n and let  $E_j GL_n := V_j \setminus W_j$ . We have  $\operatorname{codim}_{V_j} W_j = j + 1$  and  $GL_n$  acts freely on  $E_j GL_n$  with quotient the Grassmannian  $\operatorname{Gr}_B(n, n+j)$ .

Let  $p_j: V_{j+1} = V_j \times F \to V_j$  be the projection on the first j factors, and let  $i_j: V_j \to V_{j+1} = V_j \times F$  be the 0-section. We have  $E_j \operatorname{GL}_n \times F \subset E_{j+1} \operatorname{GL}_n$ . Let  $\eta_j: E_j \operatorname{GL}_n \times F \to E_{j+1} \operatorname{GL}_n$  be the corresponding open immersion, and  $p_j: E_j \operatorname{GL}_n \times F \to E_j \operatorname{GL}_n$ ,  $i_j^0: E_j \operatorname{GL}_n \to E_j \operatorname{GL}_n \times F$  the projection and 0-section. Let  $i_j: E_j \operatorname{GL}_n \to E_{j+1} \operatorname{GL}_n$  be the composition  $\eta_j \circ i_j^0$ .

We consider  $E_j$   $GL_n$  as an object of  $\mathbf{Sm}^{GL_n}/B$ . Let G be a closed subgroup scheme of  $GL_n$ , flat over B. We let  $E_jG$  be the object of  $\mathbf{Sm}^G/B$  formed by restricting the  $GL_n$ -action on  $E_j$   $GL_n$  to G. The quotients  $B_jG := G\backslash E_jG$  exist as quasi-projective B-schemes and the quotient map  $\pi_{G,j}: E_jG \to B_jG$  is an étale G-torsor. Let  $N_{G,j}:=G\backslash E_jG\times F$ . We have the corresponding regular embeddings  $i_{G,j}: B_jG \to B_{j+1}G$ ,  $i_{G,j}^G: B_jG \to N_{G,j}$ , open immersion  $\eta_{G,j}: N_{G,j} \to B_{j+1}G$ , and smooth morphism  $p_{G,j}: B_{j+1}G \to B_jG$ , making  $B_{j+1}^0G$  the vector bundle over  $B_jG$  corresponding via descent to the representation  $G \hookrightarrow GL_n$ , and with zero-section  $i_{G,j}^0$ . We note that the vector bundle  $N_{G,j}$  is canonically isomorphic to the normal bundle of  $i_{G,j}$ .

We let  $\mathbf{Sch}_q^G/B$  be the full subcategory of  $\mathbf{Sch}^G/B$  consisting of those X for which all the fppf quotients  $G\backslash X\times_B E_jG$  are represented in  $\mathbf{Sch}/B$ ; here G acts diagonally on  $X\times_B E_jG$ . In case  $B=\mathrm{Spec}\,k$ , k a field, we have  $\mathbf{Sch}_q^G/B=\mathbf{Sch}^G/B$ , by [8, Lemma 9, Proposition 23].

Take  $X \in \mathbf{Sch}_q^G/B$ . We denote the quotient  $G \setminus X \times_B E_j G$  by  $X \times^G E_j G$ . Let  $N_{X,G,j} \subset X \times_B^G E_{j+1} G$  be the open subscheme  $G \setminus X \times_B E_j G \times F$ . The maps  $i_j$ ,  $p_j$  induce maps

$$i_{X,G,j}: X \times^G E_j G \to X \times^G E_{j+1} G, \ p_{X,G,j}: N_{X,G,j} \to X \times^G E_j G,$$

and we have the open immersion  $\eta_{X,G,j}: N_{X,G,j} \to X \times_B^G E_{j+1}G$ . This gives  $p_{X,G,j}: N_{X,G,j} \to X \times_B^G E_jG$  the structure of a vector bundle with 0-section  $i_{X,G,j}^0: X \times_B^G E_jG \to N_{X,G,j}$  and we have  $i_{X,G,j} = \eta_{X,G,j} \circ i_{X,G,j}^0$ .

 $i_{X,G,j}^0: X \times_B^G E_j G \to N_{X,G,j}$  and we have  $i_{X,G,j} = \eta_{X,G,j} \circ i_{X,G,j}^0$ . The structure map  $\pi_X: X \to B$  induces the maps  $\pi_{X,G,j}: X \times_G^G E_j G \to B_j G$  and  $\pi_{X,G,j}^0: N_{X,G,j} \to N_{G,j}$ . We write  $EG, BG, X \times_G^G EG$ , etc., for the corresponding Ind-objects.

# **Lemma 4.1.** Suppose G is smooth over B. Then $B_jG$ is smooth over B.

*Proof.* For  $G = \operatorname{GL}_n/B$ ,  $B_jG$  is the Grassmannian  $\operatorname{Gr}_B(n,n+j)$ , which is smooth over B. For a general  $G \subset \operatorname{GL}_n/B$ , smooth over B, we have the étale locally trivial fiber bundle  $B_jG \to \operatorname{Gr}_B(n,n+j)$  with fiber  $\operatorname{GL}_n/G$ , and étale locally trivial fiber bundle  $\operatorname{GL}_n \to \operatorname{GL}_n/G$  with fiber G. Since G is smooth over B, so is  $\operatorname{GL}_n/G$  and thus so is  $B_jG$ .

Remark 4.2. Morel-Voevodsky [22, §4.3] give a general description of the construction outlined above. This is not a uniquely defined construction, but it is shown in loc. cit. that the resulting ind-object in the unstable motivic homotopy category  $\mathcal{H}(B)$  is independent of choices.

We suppose for the remainder of this section that G is a smooth group-scheme over B.

We let  $K^G(X)$  denote the K-theory space of perfect, G-linearized complexes on X and take  $v \in \mathcal{K}^G(X)$ . This gives us for each j the G-linearized perfect complex  $p_1^*v$  on  $X \times E_iG$ , and by descent the perfect complex  $v_i$  on  $X \times^G E_iG$ . Sending vto  $v_j$  defines the continuous map, natural in X,

$$(-)_i: \mathcal{K}^G(X) \to \mathcal{K}(X \times^G E_j G).$$

Similarly, for  $V \to X$  a G-linearized vector bundle, we have the vector bundle  $V_i \to X \times^G E_i G$  induced from the G-linearized vector bundle  $p_1^* V \to X \times_B E_i G$  by descent. If  $V = \mathbb{V}(\mathcal{V})$  for a G-linearized locally free sheaf  $\mathcal{V}$  on X, then  $V_j = \mathbb{V}(\mathcal{V}_j)$ .

As  $\mathcal{E}$ -cohomology  $\mathcal{E}^{a,b}(Y,v)$  is contravariantly functorial in (Y,v), the ind-system  $X \times^G EG := \{X \times^G E_j G\}_j$  gives the pro-system  $\{\mathcal{E}^{a,b}(X \times^G E_j G, v_j), i_{X,G,j}^*\}_j$  for  $v \in \mathcal{K}^G(X)$ .

**Definition 4.3.** We define

$$\mathcal{E}_{G}^{a,b}(X,v) = \lim_{\stackrel{\leftarrow}{j}} \mathcal{E}^{a,b}(X \times^{G} E_{j}G, v_{j}).$$

We write  $\mathcal{E}_G(X, v)$  for  $\mathcal{E}_G^{0,0}(X, v)$  and  $\mathcal{E}^{**}(BG, v)$  for  $\mathcal{E}_G^{**}(B, v)$ . For  $B = \operatorname{Spec} k$ ,  $\mathcal{E} = \operatorname{EM}(\mathcal{W})$ , and  $\mathcal{L}$  a G-linearized invertible sheaf on X, we set

$$H_G^a(X, \mathcal{W}(\mathcal{L})) := \mathrm{EM}(\mathcal{W})_G^{a,0}(X, \mathcal{L} - \mathcal{O}_X),$$

with  $\mathcal{O}_X$  having the canonical G-linearization. We write  $H^a(BG, \mathcal{W}(\mathcal{L}))$  for  $H^a_G(\operatorname{Spec} k, \mathcal{W}(\mathcal{L}))$ 

Remark 4.4. 1. The contravariant functoriality of  $\mathcal{E}$ -cohomology induces a corresponding functoriality for  $\mathcal{E}_G^{a,b}(X,v)$  and hence for  $H_G^*(X,\mathcal{W}(\mathcal{L}))$ .

2. The definition we give here is not the correct one for general  $\mathcal{E}$ , due at the very least to the usual lack of exactness of the inverse limit. However, as we shall see below (Remark 4.20), for X smooth, the inverse system used to define  $H_G^a(X, \mathcal{W}(\mathcal{L}))$ is eventually constant, so this problem does not arise. In addition, the exactness shows up only in the localization sequence for a decomposition of a scheme into a closed subscheme and open complement, and this takes place in the setting of Borel-Moore homology. In this case, the fact that the corresponding inverse system is eventually constant is given by Lemma 4.17 and Lemma 4.18 below; see also [7, §1.2] for the case of the Eilenberg-MacLane spectrum of a homotopy module.

**Proposition 4.5.** Let  $V \to X$  be a G-linearized vector bundle on X, and let V be the locally free sheaf of sections of  $V^{\vee}$ , with its canonical G-linearization. Then the collection of Euler classes  $\{e(V_i) \in \mathcal{E}(X \times^G E_i G, V_i)\}_i$  yields a well-defined element  $e_G(V) \in \mathcal{E}_G(X, \mathcal{V})$ . Moreover, the assignment  $(V \to X) \mapsto e_G(V) \in \mathcal{E}_G(X, \mathcal{V})$  is natural in the vector bundle  $V \to X$ : given a morphism  $f: Y \to X$ , a vector bundle  $W \to Y$  and an isomorphism of vector bundles  $\tilde{f}: W \to f^*V$ , the induced map  $(f, \tilde{f})^* : \mathcal{E}_G(X, \mathcal{V}) \to \mathcal{E}_G(Y, \mathcal{W}) \text{ sends } e_G(V) \text{ to } e_G(W).$ 

*Proof.* For each j we have the canonical isomorphisms  $i_{X,G,j}^*V_{j+1} \cong V_j$ ,  $i_{X,G,j}^*V_{j+1} \cong V_j$  $\mathcal{V}_j$ , giving the identity

$$i_{X,G,j}^*(e(V_{j+1})) = e(V_j) \in \mathcal{E}(X \times^G E_j G, \mathcal{V}_j),$$

which proves the first assertion. For the second, the data  $(f, \tilde{f})$  gives for each j the identity

$$(f \times^G \operatorname{Id}, \tilde{f} \times^G \operatorname{Id})^*(e(V_j)) = e(W_j)$$

whence follows  $(f, \tilde{f})^*(e_G(V)) = e_G(W)$ .

Example 4.6. Let  $\rho: G \to \operatorname{GL}_N$  be a representation of G over B, with total space  $V(\rho)$ . We consider  $V(\rho)$  as a G-linearized vector bundle on B, giving the Euler class  $e_G(\rho): e_G(V(\rho)) \in \mathcal{E}_G(B, \mathcal{V}(\rho)) = \mathcal{E}(BG, \mathcal{V}(\rho))$ . Here  $\mathcal{V}(\rho)$  is the G-linearized sheaf of sections of  $V(\rho)^{\vee}$ .

**Definition 4.7** (Generic Euler class). Let G be group scheme over k, satisfying the hypotheses of Lemma 2.1. Let V be a G-linearized vector bundle on some connected  $Y \in \mathbf{Sch}^G/k$ , with G-linearized locally free sheaf of section V. We suppose we are in Case 1, Case 2 or Case 3 of Construction 2.7 for V, so the generic representation class  $[V^{gen}]$  is defined.

Suppose  $\mathcal{V}^{gen}$  has even rank 2r (this is always the case in Case 2). Choose a representative representation  $V^{gen}$  for  $[\mathcal{V}^{gen}]$ , giving us the Euler class  $e_G(V^{gen}) \in H^{2r}(BG,\mathcal{W}(\mathcal{L}))$ , where  $\mathcal{L} = \det^{-1}\mathcal{V}^{gen}$ . Using the canonical isomorphism  $\mathcal{W}(\mathcal{L}^{\otimes 2}) \cong \mathcal{W}$ , we have  $e_G(V^{gen})^2 \in H^{4r}(BG,\mathcal{W})$  and we define  $[e_G^{gen}(V)] \subset H^{4r}(BG,\mathcal{W})$  to be the subset

$$[e_G^{gen}(V)] = \{u \cdot e_G(V^{gen})^2 \mid u \in W(k)^{\times}\} \subset H^{4r}(BG, \mathcal{W}).$$

Note that  $[e_G^{gen}(V)]$  is independent of the choice of representative  $V^{gen}$  for  $[\mathcal{V}^{gen}]$  and depends only on the isomorphism class of V as a G-linearized vector bundle on Y. Similarly, the localization

$$H^*(BG, \mathcal{W})[[e_G^{gen}(V)]^{-1}]$$

is equal to  $H^*(BG,\mathcal{W})[1/e^{gen_G}(V)]$  for any  $e^{gen}_G(V) \in [e^{gen}_G(V)]$ , and we will often use  $H^*(BG,\mathcal{W})[1/e^{gen}_G(V)]$  to denote  $H^*(BG,\mathcal{W})[[e^{gen}_G(V)]^{-1}]$ . Finally, if we have a trivialization of  $\det V^{gen}$ , then choosing one gives us  $e_G(V^{gen}) \in H^{2r}(BG,\mathcal{W})$  and we have

$$H^*(BG, \mathcal{W})[[e_G^{gen}(V)]^{-1}] = H^*(BG, \mathcal{W})[e_G(V^{gen})^{-1}].$$

Having defined G-equivariant twisted cohomology, we now proceed to define the G-equivariant twisted Borel-Moore homology.

Fix  $X \in \mathbf{Sch}^G/B$ , and  $v \in \mathcal{K}^G(X)$ . For each j, we have the Tor-independent cartesian diagram

$$X \times^{G} E_{j}G \xrightarrow{i_{X,G,j}} X \times^{G} E_{j+1}G$$

$$\downarrow \qquad \qquad \downarrow$$

$$B_{j}G \xrightarrow{i_{G,j}} B_{j+1}G$$

with  $i_{G,j}$  a regular embedding. Moreover,  $i_{X,G,j}^*v_{j+1} \cong v_j$ . This gives us the relative pull-back map

$$i_{G,j}^!: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_{j+1}G/B_{j+1}G, v_{j+1}) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_{j}G/B_{j}G, v_{j})$$

and thus gives us the pro-system  $\{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X\times^G E_jG/B_jG,v_j),i_{G,j}^!\}_j$ .

Similarly, we have the Tor-independent cartesian square

$$X \times E_{j}G \xrightarrow{i_{X,j}} X \times E_{j+1}G$$

$$\downarrow \qquad \qquad \downarrow$$

$$E_{j}G \xrightarrow{i_{j}} E_{j+1}G$$

giving the pro-system  $\{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G, v), i_j^!\}_j$ .

**Definition 4.8.** For  $X \in \mathbf{Sch}^G/B$  and  $v \in \mathcal{K}^G(X)$ , define

$$\mathcal{E}^{\mathrm{B.M.}}_{G,a,b}(X/BG,v) := \lim_{\stackrel{\longleftarrow}{i}} \mathcal{E}^{\mathrm{B.M.}}_{a,b}(X \times^G E_jG/B_jG,v_j).$$

We write  $\mathcal{E}_{G}^{\mathrm{B.M.}}(X/BG,v)$  for  $\mathcal{E}_{G,0,0}^{\mathrm{B.M.}}(X/BG,v)$ ,  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(BG,v)$  for  $\mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(B/BG,v)$ , and  $H_{G,a,b}^{\mathrm{B.M.}}(X/BG,v)$  for  $(1_B)_{G,a,b}^{\mathrm{B.M.}}(X/BG,v)$ , etc. For  $B = \operatorname{Spec} k$ , and  $\mathcal{L}$  a G-linearized invertible sheaf on X, define

$$H_{G,a}^{\mathrm{B.M.}}(X,\mathcal{W}(\mathcal{L})) := \mathrm{EM}(\mathcal{W})_{a,0}^{\mathrm{B.M.}}(X/BG,\mathcal{L}-\mathcal{O}_X)$$

and write  $H_a^{\mathrm{B.M.}}(BG,\mathcal{W}(\mathcal{L}))$  for  $H_{G,a}^{\mathrm{B.M.}}(\mathrm{Spec}\,k,\mathcal{W}(\mathcal{L}))$ . If we need to explicitly indicate the base-field k, we write  $H_{G,a}^{\mathrm{B.M.}}(X/k,\mathcal{W}(\mathcal{L}))$  for  $H_{G,a}^{\mathrm{B.M.}}(X,\mathcal{W}(\mathcal{L}))$ .

Lemma 4.9. 1. The transverse cartesian square

$$X \times E_{j}G \xrightarrow{\pi_{X,G,j}} X \times^{G} E_{j}G$$

$$\downarrow \qquad \qquad \downarrow$$

$$E_{j}G \xrightarrow{\pi_{G,j}} B_{j}G$$

induces the relative smooth pullback map

$$\pi_{G,j}^!: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, v_j) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G, v)$$

which gives rise to a map of pro-systems

$$\pi_{G,*}^!: \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, v_j), i_{G,j}^!\}_j \to \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G, v), i_j^!\}_j$$

2. The transverse cartesian square

$$\begin{array}{ccc} X \times E_j G \xrightarrow{p_1} X \\ \downarrow & \downarrow \\ E_j G \xrightarrow{p_{E_j G}} B \end{array}$$

induces the relative smooth pullback map

$$p_{E_jG}^!:\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v)\to\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X\times E_jG/E_jG,v).$$

giving a map of pro-systems

$$p_{E_*G}^!: \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v)\}_j \rightarrow \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_jG/E_jG,v), i_j^!\}_j$$

*Proof.* The maps (1) and (2) are special cases of relative smooth pull-back (see Remark 3.3). The fact that the maps in (1) and (2) induce maps of pro-systems follows from the functoriality of relative lci pull-back, as noted in Remark 3.3.

**Proposition 4.10.** Let  $f: X \to Y$  be morphism in  $\mathbf{Sch}^G/B$  and take  $v \in \mathcal{K}^G(Y)$ . Consider the the sequence of induced morphisms

$$f_j := f \times^G \mathrm{Id} : Y \times^G E_j G \to X \times^G E_j G$$

1. Suppose f is an lci morphism. Then the  $f_i$  are all lci morphisms and induce a map of pro-systems

$$\{f_i^!\}_j: \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(Y \times^G E_j G/B_j G, v_j)\}_j \to \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, v_j + L_{f,j})\}_j$$

giving rise to a well-defined map

$$f^!: \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(Y/BG,v) \to \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X/BG,v+L_f)$$

2. Suppose f is a proper morphism. Then the  $f_j$  are all proper morphisms and induce a map of pro-systems

$$\{f_{j*}\}_j: \{\mathcal{E}_{a,b}^{\operatorname{B.M.}}(X\times^G E_jG/B_jG,v_j)\}_j \to \{\mathcal{E}_{a,b}^{\operatorname{B.M.}}(Y\times^G E_jG/B_jG,v_j)\}_j,$$

giving rise to a well-defined map

$$f_*: \mathcal{E}^{\mathrm{B.M.}}_{G,a,b}(X/BG,v) \to \mathcal{E}^{\mathrm{B.M.}}_{G,a,b}(Y/BG,v)$$

3. For general f, the maps  $f_j$  induce a map of pro-systems

$$f_j^*: \{\mathcal{E}^{a,b}(Y \times^G E_j G, v_j)\}_j \to \{\mathcal{E}^{a,b}(X \times^G E_j G, v_j)\}_j$$

 $giving\ rise\ to\ a\ well-defined\ map$ 

$$f^*: \mathcal{E}_G^{a,b}(Y,v) \to \mathcal{E}_G^{a,b}(X,v).$$

4. Given  $v, w \in \mathcal{K}^G(Y)$ , the products

$$\mathcal{E}^{a,b}(Y \times^G E_j G, v_j) \times \mathcal{E}^{c,d}(Y \times^G E_j G, w_j) \xrightarrow{\cup} \mathcal{E}^{a+c,b+d}(Y \times^G E_j G, v_j + w_j)$$

and cap products

$$\begin{split} \mathcal{E}^{\mathrm{B.M.}}_{a,b}(Y \times^G E_j G/B_j G, v_j) \times \mathcal{E}^{c,d}(Y \times^G E_j G, w_j) \\ \xrightarrow{\cap} \mathcal{E}^{\mathrm{B.M.}}_{a-c,b-d}(Y \times^G E_j G/B_j G, v_j - w_j) \end{split}$$

define maps of pro-systems, inducing a product

$$\mathcal{E}_{G}^{a,b}(Y,v) \times \mathcal{E}_{G}^{c,d}(Y,w) \xrightarrow{\cup} \mathcal{E}_{G}^{a+c,b+d}(Y,v+w)$$

and cap product

$$\mathcal{E}^{\mathrm{B.M.}}_{G,a,b}(Y/BG,v)\times\mathcal{E}^{c,d}_G(Y,w)\xrightarrow{\cap}\mathcal{E}^{\mathrm{B.M.}}_{G,a-c,b-d}(Y/BG,v-w).$$

5. (Fundamental classes, Poincaré duality) Let  $[B]_G \in H_G^{0,0}(B) = H_G^{\mathrm{B.M.}}(B/BG)$  be the family of classes of  $\mathrm{Id}_{1_{B_jG}}$ . Let  $p_X: X \to B$  be an lci scheme over B, in  $\mathbf{Sch}^G/B$ . Define the fundamental class  $[X]_G \in H_G^{\mathrm{B.M.}}(X/BG, L_{X/B})$  by

$$[X]_G := p_X^!([B]_G).$$

i. For  $f: Y \to X$  an lci morphism of lci-schemes in  $\mathbf{Sch}^G/B$ , we have  $f^!([X]_G) = [Y]_G$ .

ii. Let  $\mathcal{E} \in SH(B)$  be a commutative ring spectrum and take  $v \in \mathcal{K}^G(X)$ . The cap product with  $[X]_G$ 

$$[X]_G \cap -: \mathcal{E}_G^{a,b}(X,v) \to \mathcal{E}_{G,-a,-b}^{\mathrm{B.M.}}(X,L_{X/B}-v)$$

is an isomorphism for  $X \in \mathbf{Sm}^G/B$ .

*Proof.* (1) Note that each map  $f \times \text{Id} : Y \times_B E_j G \to X \times_B E_j G$  is an lci morphism, and we have the cartesian diagram

$$Y \times_B E_j G \xrightarrow{f \times \mathrm{Id}} X \times_B E_j G$$

$$\downarrow^{q_j^X} \qquad \qquad \downarrow^{q_j^X}$$

$$Y \times^G E_j G \xrightarrow{f_j} X \times^G E_j G$$

Since the map  $q_j^X$  is faithfully flat, this implies that  $f_j$  is an lci morphism, so  $f_j^!$  is defined.

By the functoriality of (-)!, we have

$$i_{G,j}^! \circ f_{j+1}^! = f_j^! \circ i_{G,j}^!$$

as maps  $\mathcal{E}_{**}^{\mathrm{B.M.}}(X \times^G E_{j+1}G/B_{j+1}G, v_{j+1}) \to \mathcal{E}_{**}^{\mathrm{B.M.}}(Y \times^G E_jG/B_jG, v_j)$ , which proves (1).

For (2), we note that the diagram

$$Y \times_B E_j G \xrightarrow{\operatorname{Id} \times i_{G,j}} Y \times_B E_{j+1} G$$

$$\downarrow^{f \times \operatorname{Id}} \qquad \qquad \downarrow^{f \times \operatorname{Id}}$$

$$X \times_B E_j G \xrightarrow{f \times \operatorname{Id}} X \times_B E_{j+1} G$$

is cartesian and Tor-independent. Using faithful flatness again, this implies that the diagram

$$Y \times^{G} E_{j}G \xrightarrow{i_{Y,G,j}} Y \times^{G} E_{j+1}G$$

$$\downarrow^{f_{j}} \qquad \downarrow^{f_{j+1}}$$

$$X \times^{G} E_{j}G \xrightarrow{i_{X,G,j}} X \times^{G} E_{j+1}G$$

is also cartesian and Tor-independent. By the excess intersection formula [6, Proposition 4.2.2], this says that we can identify the Gysin pullbacks  $i_{X,G,j}^!$  and  $i_{X,G,j}^!$  with the refined Gysin pullback  $i_{X,G,j}^!$ , on  $\mathcal{E}_{**}^{\mathrm{B.M.}}(-/B,-)$ . By Proposition 3.2, this gives the identity

$$i_{Y,G,j}^! \circ f_{j+1*} = f_{j*} \circ i_{X,G,j}^!$$

(3) follows immediately from the functoriality of -\*.

For (4), for  $i: Z \to W$  a regular immersion in  $\mathbf{Sch}/B$ , and  $\alpha, \beta$  in  $\mathcal{E}$ -cohomology and  $\gamma$  in  $\mathcal{E}$ -Borel-Moore homology, all on W, and we have the identity in  $\mathcal{E}$ -cohomology

$$i^*(\alpha) \cdot i^*(\beta) = i^*(\alpha \cdot \beta)$$

and the identity in  $\mathcal{E}$ -Borel-Moore homology

$$i^!(\gamma) \cap i^*(\beta) = i^!(\gamma \cap \beta).$$

This proves (4).

(5) follows by applying Proposition 3.2(4) to each of the  $B_jG$ -schemes  $X \times^G E_jG$ .

Remark 4.11. Having the operations of  $f^!$  for lci f and  $f_*$  for proper f on the equivariant Borel-Moore homology  $\mathcal{E}_{G,**}^{\mathrm{B.M.}}(-/-,-)$ , and products in cohomology and cap products in Borel-Moore homology all the relations discussed in §3.8 in the non-equivariant case pass to the equivariant case. Indeed, we have seen that the various properties of a morphism f, such as being proper or lci, pass to the induced morphism on each each term in the pro-system defining the equivariant theory. The same holds for properties such as "f is smooth", "f is étale", etc., and also for a square of morphisms to commute, to be cartesian or to be cartesian and Tor-independent. Thus, each relation discussed in §3.8 in the non-equivariant case can be checked on the individual terms in the relevant pro-system, which then becomes a question in the the non-equivariant case.

For example, the products in equivariant cohomology make  $\mathcal{E}_{G}^{**}(X,*)$  a tri-graded ring (the third grading being the twist), and make  $\mathcal{E}_{G,**}^{\text{B.M.}}(X/BG,*)$  a tri-graded  $\mathcal{E}_{G}^{-*,-*}(X,-*)$ -module, we have a projection formula

$$f_*(\alpha \cap f^*\beta) = f_*(\alpha) \cap \beta$$

for f a proper map,  $\alpha$  in Borel-Moore homology and  $\beta$  in cohomology, we have the commutativity  $f^!g_*=g'_*f'^!$  in a Tor-independent cartesian square

$$Z \xrightarrow{g'} W$$

$$f' \downarrow \qquad \qquad \downarrow f$$

$$X \xrightarrow{g} Y$$

Remark 4.12. Suppose the G-action on X is trivial. Then we have the canonical isomorphism  $X \times^G E_j G \cong X \times_B B_j G$ , giving us the cartesian square

$$X \times^{G} E_{j}G \longrightarrow X$$

$$\downarrow^{\pi_{X,G,j}} \qquad \downarrow^{p_{X}}$$

$$B_{j}G \xrightarrow{p_{B_{j}G}} B$$

with  $p_{B_jG}$  smooth. The relative pullback maps  $p_{B_jG}^!$  thus gives the map of prosystems

$$p_{B_*G}^!: \{\mathcal{E}_{a,b}^{\text{B.M.}}(X/B, v)\}_j \to \{\mathcal{E}_{a,b}^{\text{B.M.}}(X \times^G E_j G/B_j G, v_j)\}_j$$

giving the map

$$p_{BG,X}^!: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v) \to \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X/BG,v).$$

**Definition 4.13.** Take  $\mathcal{E} \in SH(B)$ .

- 1. We call  $\mathcal{E}$  bounded if for each fixed  $X \in \mathbf{Sch}^G/B$ , integers (a,b), and  $v \in \mathcal{K}^G(X)$ , the pro-systems  $\{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, v_j), i_{G,j}^!\}_j$  and  $\{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G, v), i_j^!\}_j$  are eventually constant.
- 2. We say that  $\mathcal{E}$  is  $strongly\ bounded$  if, given integers a,b,d, there is a constant C such that, given  $Y \to B$  in  $\mathbf{Sm}/B$ , and a dense open immersion  $j: U \to V$  in  $\mathbf{Sch}/Y$  with closed complement W, such that W has codimension  $\geq c$  in V and V has relative dimension  $\leq d$  over Y, the restriction map

$$j^*: \mathcal{E}_{a,b}^{\operatorname{B.M.}}(V/Y,v) \to \mathcal{E}_{a,b}^{\operatorname{B.M.}}(U/Y,v)$$

is an isomorphism for all  $v \in \mathcal{K}(V)$  of virtual rank 0.

3. We say that  $\mathcal{E}$  is uniformly strongly bounded if  $\mathcal{E}$  is strongly bounded, the constant C depends only on d and a-b, C=C(d,a-b), and  $C(d,a-b)\geq C(d,a-b+1)$ 

Remark 4.14. We have already remarked that Definition 4.3) for equivariant cohomology should be considered as provisional. Definition 4.8 is also not in general the correct definition for similar reasons. A better definition would be to consider the pro-object of (derived) mapping spaces in the infinity categories  $\mathbf{SH}(B_iG)$ ,

$$\{\operatorname{Maps}_{\mathbf{SH}(B_jG)}(\pi_{X,G,j!}\Sigma^{a,b}\Sigma^v 1_{X\times^G E_jG},\mathcal{E})\}_j,$$

define

$$\operatorname{Maps}_{a,b}^G(X/BG,v,\mathcal{E}) := \lim_j \operatorname{Maps}_{\mathbf{SH}(B_jG)}(\pi_{X,G,j!}\Sigma^{a,b}\Sigma^v 1_{X\times^G E_jG},\mathcal{E})$$

(in the infinity category of spectra) and then set

$$\mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X/BG,v) := \pi_0 \mathrm{Maps}_{a,b}^G(X/BG,v,\mathcal{E}).$$

Equivariant  $\mathcal{E}$ -cohomology would be similarly defined via the pro-system

$$\{\operatorname{Maps}_{\mathbf{SH}(X\times^G E_iG)}(1_{X\times^G E_iG}, \Sigma^{a,b}\Sigma^v p_{X\times^G E_iG}^*\mathcal{E})\}_j,$$

giving the mapping spectrum

$$\operatorname{Maps}_{G}^{a,b}(X, v, \mathcal{E}) := \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X \times^{G} E_{j} G)} (1_{X \times^{G} E_{j} G}, \Sigma^{a,b} \Sigma^{v} p_{X \times^{G} E_{j} G}^{*} \mathcal{E}),$$

and one would set  $\mathcal{E}_G^{a,b}(X,v) := \pi_0 \operatorname{Maps}_G^{a,b}(X,v,\mathcal{E}).$ 

However, for  $\mathcal{E}$  bounded, the refined definition of equivariant Borel-Moore homology agrees with the naive one (see Lemme 4.15 below). In particular, this is the case for  $B = \operatorname{Spec} k$  and  $\mathcal{E} = \operatorname{EM}(\mathcal{M}_*)$ ,  $\mathcal{M}_*$  a homotopy module.

Using the Poincaré duality isomorphism of Proposition 4.10, we see that the refined definition of  $\mathcal{E}$ -cohomology agrees with the naive one for  $\mathcal{E}$  bounded and X smooth over B.

# **Lemma 4.15.** Suppose $\mathcal{E}$ is bounded.

1. The map of pro-systems

$$p_{E_*G}^!: \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v)\}_j \rightarrow \{\mathcal{E}_{a,b}^{\mathrm{B.M.}}(X\times E_jG/E_jG,v), i_j^!\}_j$$

of Lemma 4.9 induces an isomorphism

$$p_{E_*G}^*: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v) \to \lim_{j} \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G,v)$$

2. The natural map

$$\pi_0 \operatorname{Maps}_{a,b}^G(X/BG, v, \mathcal{E}) \to \mathcal{E}_{G,a,b}^{\operatorname{B.M.}}(X/BG, v)$$

is an isomorphism.

*Proof.* We may assume that v has virtual rank zero; replacing  $\mathcal{E}$  with  $\Sigma^{-a,-b}\mathcal{E}$ , we may assume a=b=0.

For (1), by [22, §4, Proposition 2.3], the projection  $p_1: X \times EG \to X$  becomes an isomorphism in  $\mathcal{H}(X)$ , and thus

$$p_1^*: \mathcal{E}^{\mathrm{B.M.}}(X,v) = \mathrm{Hom}_{\mathrm{SH}(X)}(1_X, \Sigma^{-v} p_X^! \mathcal{E}) \to \mathrm{Hom}_{\mathrm{SH}(X)}(\Sigma_{\mathbb{P}^1}^{\infty} X \times EG_+, \Sigma^{-v} p_X^! \mathcal{E})$$
 is an isomorphism.

We have

$$\operatorname{Hom}_{\operatorname{SH}(X)}(\Sigma^{\infty}_{\mathbb{P}^{1}}X \times EG_{+}, \Sigma^{-v}p_{X}^{!}\mathcal{E}) \cong \pi_{0} \operatorname{holim}_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(\Sigma^{\infty}_{\mathbb{P}^{1}}X \times E_{j}G_{+}, \Sigma^{-v}p_{X}^{!}\mathcal{E})$$

$$\cong \pi_{0} \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(p_{1\#}(1_{X \times E_{j}G}), \Sigma^{-v}p_{X}^{!}\mathcal{E})$$

$$\cong \pi_{0} \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(1_{X \times E_{j}G}, \Sigma^{-v}p_{1}^{*}p_{X}^{!}\mathcal{E})$$

$$\cong \pi_{0} \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(\Sigma^{v}1_{X \times E_{j}G}, p_{2}^{!}p_{E_{j}G}^{*}\mathcal{E})$$

$$\cong \pi_{0} \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(\Sigma^{v}1_{X \times E_{j}G}, p_{2}^{!}p_{E_{j}G}^{*}\mathcal{E})$$

$$\cong \pi_{0} \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(p_{2!}(\Sigma^{v}1_{X \times E_{j}G}), p_{E_{j}G}^{*}\mathcal{E})$$

But  $\pi_a \operatorname{Maps}_{\mathbf{SH}(X)}(p_{2!}(\Sigma^v 1_{X \times E_j G}), p_{E_j G}^* \mathcal{E}) = \mathcal{E}_{a,0}^{\operatorname{B.M.}}(X \times E_j G/E_j G, v)$ , so the assumption that  $\mathcal{E}$  is bounded implies that the inverse system

$$\{\pi_1 \operatorname{Maps}_{\mathbf{SH}(X)}(p_{2!}(\Sigma^v 1_{X \times E_i G}), p_{E_i G}^* \mathcal{E})\}$$

is eventually constant, so the natural map

$$\pi_0 \lim_{j} \operatorname{Maps}_{\mathbf{SH}(X)}(p_{2!}(\Sigma^{v} 1_{X \times E_{j}G}), p_{E_{j}G}^* \mathcal{E})$$

$$\to \lim_{j} \pi_0 \operatorname{Maps}_{\mathbf{SH}(X)}(p_{2!}(\Sigma^{v} 1_{X \times E_{j}G}), p_{E_{j}G}^* \mathcal{E})$$

$$= \lim_{j} \mathcal{E}^{B.M.}(X \times E_{j}G/E_{j}G, v)$$

is an isomorphism. Thus

$$\mathcal{E}^{\mathrm{B.M.}}(X,\mathcal{V}) \to \lim_{i} \mathcal{E}^{\mathrm{B.M.}}(X \times E_{j}G/E_{j}G,v)$$

is an isomorphism.

The proof of (2) is the same: the assumption that  $\mathcal{E}$  is bounded implies that the natural map

$$\pi_0 \operatorname{Maps}_{a,b}^G(X/BG, v, \mathcal{E}) \to \lim_j \pi_0 \operatorname{Maps}_{\mathbf{SH}(B_jG)}(\pi_{X,G,j!}(\Sigma^v 1_{X \times^G E_jG}), p_{B_jG}^* \mathcal{E})$$

$$= \lim_j \mathcal{E}^{\operatorname{B.M.}}(X \times^G E_jG/B_jG, v) = \mathcal{E}_G^{\operatorname{B.M.}}(X/BG, v)$$

is an isomorphism.

**Definition 4.16.** Suppose  $\mathcal{E}$  is bounded.

Define the "forget G" map for  $X \in \mathbf{Sch}^G/B$ ,  $v \in \mathcal{K}^G(X)$ ,

$$\rho_{X,G}: \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X/BG,v) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v),$$

by the zig-zag diagram

$$\mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X/BG,v) \xrightarrow{\lim \pi_{G,*}^!} \lim_{j} \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times E_j G/E_j G,v) \xleftarrow{\sim} \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X/B,v).$$

**Lemma 4.17.** If  $\mathcal{E}$  is strongly bounded, then  $\mathcal{E}$  is bounded.

Proof. We check the boundedness condition on a fixed  $X \in \mathbf{Sch}^G/B$  and for  $\mathcal{E}_{a,b}^{\mathrm{B.M.}}(-,v)$  for given a,b and  $v \in \mathcal{K}^G(X)$ . We may assume that X is connected, so v has a constant virtual rank r on X; replacing v with  $v + \mathcal{O}_X[1]^r$  and changing (a,b) to (a+2r,b+r), we may assume from the start that v has virtual rank zero. Take an integer d with  $\dim_B X \leq d$ .

Recall that we have represented G as a closed subgroup scheme of  $\operatorname{GL}_n/B$ , and that  $E_jG$  is an open subscheme of  $\mathbb{A}_B^{n(n+j)}$  with closed complement  $W_j$  of codimension j+1. Moreover, letting  $F=\mathbb{A}^n$  with the standard action of  $\operatorname{GL}_n$ , we have the open immersion  $\eta_j: E_jG \times F \to E_{j+1}G$ , with closed complement  $W_j \times F \setminus W_{j+1}$  of codimension j+1.

G acts freely on  $E_jG$  and on  $E_{j+1}G$ , so taking the quotients, we have induced open immersion  $\eta_{X,G,j}: X\times^G(E_jG\times F)\to X\times^GE_{j+1}G$  with closed complement  $G\setminus (W_j\times F\setminus W_{j+1})$  also of codimension j+1. On the other hand,  $p_{X,G,j}: X\times^G(E_jG\times F)\to X\times^GE_jG$  is the vector bundle  $N_{X,G,j}\to X\times^GE_jG$ , so the pullback map

$$p_{X,G,j}^!: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G (E_jG \times F)/B_{j+1}G, v) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_jG/B_jG, v)$$

is an isomorphism.

Suppose  $\mathcal{E}$  is strongly bounded. Note that  $X \times^G E_{j+1}G$  has dimension  $\leq d$  over  $B_{j+1}G$ , independent of j. Take the integer c as in Definition 4.13 for a, b, d. Then for  $j+1 \geq c$ , the restriction map

$$\eta_{X,G,j}^*: \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G E_{j+1}G/B_{j+1}G, v) \to \mathcal{E}_{a,b}^{\mathrm{B.M.}}(X \times^G (E_jG \times F)/B_{j+1}G, v)$$
 is an isomorphism, hence  $\mathcal{E}$  is bounded.

**Lemma 4.18.** Take  $B = \operatorname{Spec} k$ , k a perfect field and let  $\mathcal{M}_*$  be a homotopy module. Then  $\mathcal{E} := \operatorname{EM}(\mathcal{M}_*)$  is uniformly strongly bounded.

*Proof.* Take  $Y \in \mathbf{Sm}/k$ ,  $j: U \to V$  a dense open immersion in  $\mathbf{Sch}/Y$  with closed complement W, and with  $\dim_Y V \leq d$ . Take  $v \in \mathcal{K}(V)$  of virtual rank zero. We may assume that Y is integral, let  $d_Y = \dim_k Y$ . Choose a closed immersion  $V \subset X$  with  $X \in \mathbf{Sm}/k$  of dimension  $d_X$  over k. We have

$$\begin{split} \mathcal{E}_{a,b}^{\text{B.M.}}(V/Y,v) &= \mathcal{E}_{a,b}^{\text{B.M.}}(V/k,v + \Omega_{Y/k}) = \mathcal{E}_{a+2d_Y,b+d_Y}^{\text{B.M.}}(V/k,v + \Omega_{Y/k} - \mathcal{O}_V^{d_Y}) \\ &= \mathcal{E}_V^{2d_X - a - 2d_Y,d_X - b - d_Y}(X,v + \Omega_{Y/k} - \mathcal{O}_V^{d_Y}) \\ &= H_V^{d_X - d_Y + b - a}(X,\mathcal{M}_{d_X - b - d_Y}(\mathcal{L})) \end{split}$$

where  $\mathcal{L} = \det(v + \Omega_{Y/k})$ . Similarly, U is a closed subscheme of  $X \setminus W$ ,

$$\mathcal{E}^{\mathrm{B.M.}}_{a,b}(U/Y,\mathcal{V}) = H^{d_X-d_Y+b-a}_U(X \setminus W, \mathcal{M}_{d_X-b-d_Y}(\mathcal{L}))$$

and the restriction from V to U is induced by the restriction from X to  $X \setminus W$ .

We compute  $H_V^{d_X-d_Y+b-a}(X, \mathcal{M}_{d_X-b-d_Y}(\mathcal{L}))$  by using the Rost-Schmid complex on X with supports in V, and similarly for U.

The Rost-Schmid complex for  $\mathcal{M}_*(\mathcal{L})$  on X with supports in V computes the cohomology  $H_V^p(X,\mathcal{M}_q(\mathcal{L}))$  (see [12, 13]; we use the results of [12] to view the homotopy module  $\mathcal{M}_*$  as a Milnor-Witt cycle module, and then the Rost-Schmid complex is Feld's Milnor-Witt cycle complex [13, §5.2]). Let  $X^{(q)} \cap V$  denote the set of codimension q points x of X with  $x \in V$ . The corresponding Rost-Schmid complex is

$$RS_{V}(X, \mathcal{M}_{q}(\mathcal{L}))^{*} := \bigoplus_{x \in X^{(0)} \cap V} \mathcal{M}_{q}(k(x); \mathcal{L} \otimes k(x)) \to \dots$$
$$\to \bigoplus_{x \in X^{(j)} \cap V} \mathcal{M}_{q-j}(k(x); \mathcal{L} \otimes \det^{-1} m_{x}/m_{x}^{2}) \to \dots$$

with the "general" term written here in degree j. One has the isomorphism

$$H_V^p(X, \mathcal{M}_q(\mathcal{L})) \cong H^p(RS_V(X, \mathcal{M}_q(\mathcal{L}))^*).$$

The complex  $RS_U(X \setminus W, \mathcal{M}_q(\mathcal{L}))^*$  is defined similarly.

Thus  $H_V^{d_X-d_Y+b-a}(X,\mathcal{M}_{d_X-b-d_Y}(\mathcal{L}))$  only involves the points of codimension  $d_X-d_Y+b-a,d_X-d_Y+b-a+1$  and  $d_X-d_Y+b-a-1$  on X. If W has codimension  $\geq c$  on V, then since  $\dim_k V=\dim_Y V+d_Y\leq d+d_Y$ , then W has dimension  $\leq d+d_Y-c$  over k, hence has codimension  $\geq c+d_X-d_Y-d$  on X. Thus if

$$c \ge b - a + d + 2$$

removing W from X will induce an isomorphism

$$H_V^{d_X-d_Y+b-a}(X,\mathcal{M}_{d_X-b-d_Y}(\mathcal{L})) \to H_U^{d_X-d_Y+b-a}(X\setminus W,\mathcal{M}_{d_X-b-d_Y}(\mathcal{L}))$$

Taking C(d, a-b) := d - (a-b) + 2, we see that  $\mathcal{E}$  is uniformly strongly bounded.  $\square$ 

Remark 4.19. These last two results, Lemma 4.17 and Lemma 4.18, in somewhat different form, are due to Di Lorenzo and Mantovani [7, Lemma 1.2.8, 1.2.9]. The definition of equivariant Borel-Moore homology via the limit mapping spectrum is from [20], where the deeper properties of this theory are discussed. We will only be looking at the case  $\mathcal{E} = \mathrm{EM}(\mathcal{W})$ , which recovers Witt-sheaf cohomology, so the naive version defined as the limit will suffice for our purposes.

 $Remark\ 4.20.\ 1.$  By Lemma 4.17 and Lemma 4.18, we have the canonical isomorphism

$$H_{G,a}^{\operatorname{B.M.}}(X,\mathcal{W}(\mathcal{L})) \cong H_a^{\operatorname{B.M.}}(X \times^G E_j G/B_j G,\mathcal{W}(\mathcal{L}_j))$$

for all j >> 0.

2. For  $X \in \mathbf{Sch}^G/B$ , let  $\mathcal{L}$  and  $\mathcal{M}$  be G-linearized invertible sheaves on X. From (1), the cap product maps

$$H_a^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, \mathcal{W}(\mathcal{M}_j) \times H^n(X \times^G E_j G, \mathcal{W}(\mathcal{L}_j))$$

$$\xrightarrow{\cap} H_{a-n}^{\mathrm{B.M.}}(X \times^G E_j G/B_j G, \mathcal{W}(\mathcal{L}_j \otimes \mathcal{M}_j))$$

give rise a well-defined cap products

$$\cap: H^{\mathrm{B.M.}}_{G,a}(X,\mathcal{W}(\mathcal{M})) \times H^n_G(X,\mathcal{W}(\mathcal{L})) \to H^{\mathrm{B.M.}}_{G,a-n}(X,\mathcal{W}(\mathcal{L} \otimes \mathcal{M})).$$

Using the pullback  $H^{-*}(BG, \mathcal{W}) \to H_G^{-*}(X, \mathcal{W})$ , this makes  $H_{G,*}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{M}))$  into a graded  $H^{-*}(BG, \mathcal{W})$ -module.

**Proposition 4.21.** Let  $i: Z \to X$  be a closed G-stable subscheme with complement  $j: V \to X$ . Suppose  $\mathcal{E}$  is bounded. Then for  $v \in \mathcal{K}^G(X)$ , the maps  $i_*$ ,  $j^*$  induce a long exact sequence of  $\mathcal{E}^{**}(BG)$ -modules

$$\dots \to \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(Z,v) \xrightarrow{i_*} \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(X,v)$$

$$\xrightarrow{j^*} \mathcal{E}_{G,a,b}^{\mathrm{B.M.}}(V,v) \xrightarrow{\delta} \mathcal{E}_{G,a-1,b}^{\mathrm{B.M.}}(Z,v) \to \dots$$

If we take  $\mathcal{E} = EM(\mathcal{W})$ , then for a G-linearized invertible sheaf  $\mathcal{L}$  on X, we have the long exact sequence of  $H^*(BG, \mathcal{W})$ -modules

$$\dots \to H_{G,a}^{\mathrm{B.M.}}(Z, \mathcal{W}(\mathcal{L}(i^*\mathcal{L}))) \xrightarrow{i_*} H_{G,a}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L}))$$

$$\xrightarrow{j^*} H_{G,a}^{\mathrm{B.M.}}(V, \mathcal{W}(j^*\mathcal{L})) \xrightarrow{\delta} H_{G,a-1}^{\mathrm{B.M.}}(Z, \mathcal{W}(i^*\mathcal{L})) \to \dots$$

*Proof.* The assertion for  $\mathcal{E} = \mathrm{EM}(\mathcal{W})$  follows from the general case of bounded  $\mathcal{E}$  by Lemma 4.17 and Lemma 4.18. To prove the general case, we use the long exact sequence in  $\mathcal{E}_a^{\mathrm{B.M.}}(-/B_jG,v_j)$  associated to the closed immersion  $Z\times^G E_jG\to X\times^G E_jG$  with open complement  $V\times^G E_jG\to X\times^G E_jG$  for j>>0 to give the desired long exact sequence in any given finite range of values of a (for fixed b).  $\square$ 

#### 5. Witt cohomology of BN and $BSL_2$

In this section we work over a fixed base field k of characteristic  $\neq 2$ . We let  $F \cong \mathbb{A}^2_k$  be the fundamental 2-dimensional representation of  $\mathrm{SL}_2$ , defined via left matrix multiplication on column vectors.

Recall from [18, Proposition 5.5] the computation of the Witt cohomology of  $BSL_2$  and BN.

**Theorem 5.1.** Let  $V \to \mathrm{BSL}_2$  be the rank 2 bundle associated to F, and let  $e \in H^2(\mathrm{BSL}_2, \mathcal{W})$  be the corresponding Euler class e(V).

1.  $H^*(BSL_2, W)$  is the polynomial algebra

$$H^*(BSL_2, \mathcal{W}) = W(k)[e].$$

2. The map  $p^*: H^*(BSL_2, \mathcal{W}) \to H^*(BN, \mathcal{W})$  induced by the inclusion  $N \hookrightarrow SL_2$  is injective; we write  $e \in H^*(BN, \mathcal{W})$  for  $p^*e$ . There is a canonical element  $x \in H^0(BN, \mathcal{W})$  with

$$H^*(BN, W) = W(k)[x, e]/((1+x) \cdot e, x^2 - 1)$$

3. PicBN  $\cong \mathbb{Z}/2$ . If  $\gamma \in \text{PicBN}$  is the non-trivial element, there is a rank two bundle  $\mathcal{T}$  on BN with determinant  $\gamma$ , giving the Euler class  $e(\mathcal{T}) \in H^2(BN, \mathcal{W}(\gamma))$ . Moreover,  $H^*(BN, \mathcal{W}(\gamma))$  is a free W(k)[e]-module with generator  $e(\mathcal{T})$ , and  $x \cdot e(\mathcal{T}) = -e(\mathcal{T})$ .

#### Corollary 5.2.

$$H^*(\mathrm{BSL}_2, \mathcal{W})[e^{-1}] \to H^*(BN, \mathcal{W})[e^{-1}]$$

is an isomorphism and  $H^*(BN, \mathcal{W}(\gamma))[e^{-1}]$  is a free  $H^*(BSL_2, \mathcal{W})[e^{-1}]$ -module with generator  $e(\mathcal{T})$ .

*Proof.* Since 
$$x \cdot e = -e$$
, this says that  $x$  maps to  $-1$  in  $H^*(BN, \mathcal{W})[e^{-1}]$ .

Since EM(W) is bounded, we have the "forget G" map

Suppose  $X \in \mathbf{Sch}^G/B$  has the trivial G-action, giving us the pull-back map of Remark 4.12,

$$p_{BG,X}^!: H_a^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L})) \to H_{G,a}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L})).$$

**Corollary 5.3.** We take  $G = \operatorname{SL}_2$  or G = N. Suppose  $X \in \operatorname{\mathbf{Sch}}^G/k$  has the trivial G-action and let  $\mathcal{L}$  be an invertible sheaf on X, also with trivial G-action. Let  $\chi: G \to \mathbb{G}_m$  be a character, and let  $\mathcal{L}(\chi)$  denote the corresponding G-linearized invertible sheaf on X. Then  $\pi^!_{BG/X}$  and the cap product define an isomorphism

$$\cap \circ \pi^{!}_{BG,X}: H^{-*}(BG,\mathcal{W}(\chi)) \otimes_{W(k)} H^{\mathrm{B.M.}}_{*}(X,\mathcal{W}(\mathcal{L})) \to H^{\mathrm{B.M.}}_{G,*}(X,\mathcal{W}(\mathcal{L}(\chi))).$$

Moreover, the products in  $H^*$  define isomorphisms

$$H^*(BG^n, \mathcal{W}) \cong H^*(BG, \mathcal{W})^{\otimes_{W(k)} n}$$

for 
$$G = SL_2, N$$
.

*Proof.* The second assertion follows from the first using induction, Lemma 4.17 and Lemma 4.18, and the purity isomorphism.

For the first assertion, we note that  $H^n(BG, \mathcal{W}(\chi))$  is a free finite rank W(k)module for each n; this follows from Theorem 5.1. We have the Quillen-type spectral
sequence

$$E_{p,q}^{1} = \bigoplus_{x \in X_{(p)}} H_{p+q}^{\mathrm{B.M.}}(k(x)/k, \mathcal{W}(\mathcal{L} \otimes k(x))) \Rightarrow H_{p+q}^{\mathrm{B.M.}}(X/B, \mathcal{W}(\mathcal{L}))$$

and a corresponding sequence for  $H_G^{\mathrm{B.M.}}(-,\mathcal{L}(\chi))$ 

$$E^{1}_{G,p,q} = \bigoplus_{x \in X_{(p)}} H^{\mathrm{B.M.}}_{G,p+q}(k(x)/k, \mathcal{W}(\mathcal{L}(\chi) \otimes k(x))) \Rightarrow H^{\mathrm{B.M.}}_{G,p+q}(X/B, \mathcal{W}(\mathcal{L}(\chi)).$$

Since  $H^*(BG, \mathcal{W}(\chi))$  is a flat W(k)-module, the first of these gives rise to a spectral sequence

$$\mathbb{E}^{1}_{p,q} = \bigoplus_{n} H^{n}(BG, \mathcal{W}(\chi)) \otimes_{W(k)} \bigoplus_{x \in X_{(p)}} H^{\mathrm{B.M.}}_{p+q+n}(k(x)/k, \mathcal{W}(\mathcal{L} \otimes k(x)))$$
$$\Rightarrow \bigoplus_{n} H^{n}(BG, \mathcal{W}(\chi)) \otimes_{W(k)} H^{\mathrm{B.M.}}_{p+q}(X/B, \mathcal{W}(\mathcal{L})),$$

with differentials of the form  $\mathrm{Id}\otimes d_r^{p,q+n}$ . The maps  $\cap\circ\pi_{BG,Z}^!$  for Z a subscheme of X induce a map of spectral sequences

$$\cap \circ \pi^!_{BG,-} : \mathbb{E}^*_{*,*} \to E^*_{G,*,*}$$

For  $x \in X_{(p)}$ , we have

$$H_{p+q}^{\mathrm{B.M.}}(k(x)/k, \mathcal{W}(\mathcal{L} \otimes k(x))) \cong \begin{cases} 0 & \text{for } q \neq 0 \\ W(k(x)) & \text{for } q = 0 \end{cases}$$

and it follows from Proposition 5.1 and the purity isomorphism that cap product induces isomorphisms

$$H^{n}(BG, \mathcal{W}(\chi)) \otimes_{W(k)} H_{p}^{\mathrm{B.M.}}(k(x)/k, \mathcal{W}(\mathcal{L} \otimes k(x)))$$

$$\stackrel{\sim}{\longrightarrow} H^{n}(BG_{k(x)}, \mathcal{W}(\chi)) \cong H_{G, n-n}^{\mathrm{B.M.}}(k(x)/k, \mathcal{W}(\chi)).$$

Thus  $\cap \circ \pi_(BG, -)^! : \mathbb{E}^*_{*,*} \to E^*_{G,*,*}$  is an isomorphism on the  $E^1$ -terms and since the spectral sequences are strongly convergent, this proves the first assertion.  $\square$ 

We have the group scheme  $\operatorname{SL}_2^n$  over k, with ith factor  $\operatorname{SL}_2^{(i)}$ . We let  $F_i$  denote the representation of  $\operatorname{SL}_2^n$  induced by F via the ith projection. For  $m_1, \ldots, m_n$  nonnegative integers, let  $\operatorname{Sym}^{m_1, \ldots, m_n}$  denote the  $\operatorname{SL}_2^n$ -representation  $\operatorname{Sym}^{m_1}(F_1) \otimes \ldots \otimes \operatorname{Sym}^{m_n}(F_n)$ ; in characteristic zero, these are exactly the irreducible representations of  $\operatorname{SL}_2^n$ .

For a rank r k-representation  $\phi$  of a linear algebraic group-scheme G, we have the corresponding G-linearized vector bundle  $V(\phi)$  on Spec k, and the associated Euler class  $e_G(V(\phi)) \in H^r(BG, \mathcal{W}(\det^{-1}\phi))$ ; we often write  $e_G(\phi)$  for  $e_G(V(\phi))$ .

**Proposition 5.4.** 1. The Euler class  $e_{SL_2^n}(Sym^{m_1,...,m_n})$  is zero if and only if all the  $m_i$  are even.

2. If at least one  $m_i$  is odd, then the localization map

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) \to H^*(\mathrm{BSL}_2^n, \mathcal{W})[e_{\mathrm{SL}_2^n}(\mathrm{Sym}^{m_1, \dots, m_n})^{-1}]$$

is injective.

*Proof.* Recall that  $\operatorname{Sym}^m(F)$  has rank m+1, so  $\operatorname{Sym}^{m_1,\dots,m_n}$  has odd rank if and only if all the  $m_i$  are even. For a vector bundle  $V \to Y$  over some smooth k-scheme Y, e(V) = 0 in  $H^*(Y, \mathcal{W}(\det^{-1}(V)))$  if V has odd rank ([17, Proposition 3.4]), so  $e_{\operatorname{SL}_n^n}(\operatorname{Sym}^{m_1,\dots,m_n}) = 0$  if all the  $m_i$  are even.

For the remaining assertions, we may assume that  $m_1$  is odd. We have the map  $Bi: \mathrm{BSL}_2 \to \mathrm{BSL}_2^n$  induced by the inclusion  $i: \mathrm{SL}_2 \to G$  as the first factor. Since  $i^*\mathrm{Sym}^{m_1,\dots,m_n} = \mathrm{Sym}^{m_1}(F)^r$  with  $r = \prod_{j=2}^n (m_j + 1)$ , we have  $Bi^*e_{\mathrm{SL}_2^n}(\mathrm{Sym}^{m_1,\dots,m_n}) = e_{\mathrm{SL}_2}(\mathrm{Sym}^{m_1})^r$ . By [18, Theorem 8.1],  $e_{\mathrm{SL}_2}(\mathrm{Sym}^{m_1}) = m_1!! \cdot e^{m_1+1} \in H^*(\mathrm{BSL}_2, \mathcal{W}) = W(k)[e]$ , where  $m_1!! := m_1(m_1 - 2) \cdots 3 \cdot 1$ .

Since  $H^*(BSL_2^n, \mathcal{W}) = W(k)[e_1, \dots, e_n]$ , the localization map

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) \to H^*(\mathrm{BSL}_2^n, \mathcal{W})[e(\mathrm{Sym}^{m_1, \dots, m_n})^{-1}]$$

is injective if and only if  $e_{\operatorname{SL}_2^n}(\operatorname{Sym}^{m_1,\dots,m_n})$  acts as a non-zero divisor on W(k). We can detect this after passing to  $W(k)[e_1,\dots,e_n]/(e_2,\dots,e_n)=W(k)[e]=H^*(\operatorname{BSL}_2,\mathcal{W})$ , which reduces us to the case n=1. Since  $e_{\operatorname{SL}_2}(\operatorname{Sym}^{m_1})=m_1!!\cdot e^{m_1+1}$ , we need to show that  $m_1!!$  is a non-zero divisor on W(k). This follows from the fact that  $m_1!!$  is an odd integer and all the  $\mathbb{Z}$ -torsion in W(k) is 2-primary. In particular,  $e_{\operatorname{SL}_2^n}(\operatorname{Sym}^{m_1,\dots,m_n})\neq 0$ , finishing the proof of (1).

Corollary 5.5. Suppose k has characteristic zero. Let  $\phi$  be a representation of  $\operatorname{SL}_2^n$  such that  $e_{\operatorname{SL}_2^n}(\phi) \in H^*(\operatorname{BSL}_2^n, \mathcal{W})$  is non-zero. Then the localization map

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) \to H^*(\mathrm{BSL}_2^n, \mathcal{W})[e_{\mathrm{SL}_2^n}(\phi)^{-1}]$$

is injective.

*Proof.* Since the characteristic is zero,  $BSL_2^n$  is linearly reductive and thus  $\phi$  is a direct sum of the irreducible representations  $Sym^{m_1,...,m_n}$ , so

$$V(\phi) \cong \bigoplus_i V(\operatorname{Sym}^{m_1^{(i)}, \dots, m_n^{(i)}})$$

Thus  $e_{\mathrm{SL}_2^n}(\phi) = \prod_i e(\mathrm{Sym}^{m_1^{(i)},\dots,m_n^{(i)}})$ , so

$$e_{\mathrm{SL}_{2}^{n}}(\phi) \neq 0 \Rightarrow e_{\mathrm{SL}_{2}^{n}}(\mathrm{Sym}^{m_{1}^{(i)},...,m_{n}^{(i)}}) \neq 0 \ \forall i.$$

The result then follows from Proposition 5.4.

# 6. Localization for $\mathrm{SL}_2^n$ -actions

In this section we prove our localization theorem for  $SL_2^n$ . We assume our base-field k has characteristic  $\neq 2$ .

**Lemma 6.1.** Let  $SL_2$  act on  $\mathbb{P}^1$  by the standard action, giving the N-action on  $\mathbb{P}^1$  by restriction. Then

$$H_{\mathrm{SL}_2}^*(\mathbb{P}^1, \mathcal{W})[e^{-1}] = 0 = H_N^*(\mathbb{P}^1, \mathcal{W})[e^{-1}].$$

Proof. Let  $G = \operatorname{SL}_2$  and let  $\mathcal{G} \subset G \times \mathbb{P}^1$  be the isotropy subgroup of the diagonal section in  $\mathbb{P}^1 \times \mathbb{P}^1$ ; the fiber over  $p \in \mathbb{P}^1$  is thus the isotropy subgroup  $G_p$ . Let  $L \subset F \times \mathbb{P}^1$  be the rank one sub-bundle with fiber over  $p \in \mathbb{P}^1$  the 1-dimension subspace of  $F_{k(p)}$  which is  $\mathcal{G}_p$ -invariant. This gives us the rank 1 G-sub-bundle  $\mathcal{L} \subset EG \times \mathbb{P}^1 \times F$ , giving us the exact sequence of bundles on  $G \setminus EG \times \mathbb{P}^1$ 

$$0 \to G \backslash \mathcal{L} \to G \backslash \mathcal{F} \to G \backslash (\mathcal{F}/\mathcal{L}) \to 0$$

where  $\mathcal{F} = EG \times \mathbb{P}^1 \times F$ . Since  $G \setminus \mathcal{L}$  has rank one, we have  $e(G \setminus \mathcal{L}) = 0$  in  $H^2(G \setminus EG \times \mathbb{P}^1, \mathcal{W}(\det^{-1}(G \setminus \mathcal{L})))$ , and thus

$$e(G \setminus \mathcal{F}) = e(G \setminus \mathcal{L}) \cdot e(G \setminus (\mathcal{F}/\mathcal{L})) = 0.$$

Since  $e(G \setminus \mathcal{F}) = \pi^* e$ , where  $\pi : G \setminus EG \times \mathbb{P}^1 \to BG$  is the canonical projection, we have  $H^*_{\operatorname{SL}_2}(\mathbb{P}^1, \mathcal{W})[e^{-1}] = 0$ .

The same proof shows that 
$$H_N^*(\mathbb{P}^1, \mathcal{W})[e^{-1}] = 0.$$

Let  $p_i: \mathrm{SL}_2^n \to \mathrm{SL}_2$  be the projection on the *i*th factor.

**Lemma 6.2.** Take  $G = \operatorname{SL}_2^n$ , let  $L \supset k$  be a field extension and let  $P \subset G \times_k L$  be a maximal parabolic subgroup. Then there is a split Borel subgroup  $B \subset \operatorname{SL}_2/L$  such that  $P = \pi_i^{-1}(B)$  for some i.

Proof. Replacing k with L, we may assume L=k. Let  $\operatorname{SL}_2^{(i)}\subset\operatorname{SL}_2^n$  be the ith factor in  $\operatorname{SL}_2^n$ . Let  $\pi:\operatorname{SL}_2^n\to\operatorname{SL}_2^n/P$  be the quotient map. We claim that  $\pi(\operatorname{SL}_2^{(i)})=\operatorname{SL}_2^{(i)}/(P\cap\operatorname{SL}_2^{(i)})$  is closed in  $\operatorname{SL}_2^n/P$ . If not, take a closed point x in the closure but not in  $\pi(\operatorname{SL}_2^{(i)})$ . We pass to k(x), and changing notation, may assume the k(x)=k. The isotropy group of x in  $\operatorname{SL}_2^n$  is conjugate to P but since  $\operatorname{SL}_2^{(i)}$  is normal, the isotropy group of x in  $\operatorname{SL}_2^{(i)}$  is conjugate in  $\operatorname{SL}_2^{(i)}$  to  $\operatorname{SL}_2^{(i)}\cap P$ , and thus the orbit  $\operatorname{SL}_2^{(i)}\cdot x$  has the same dimension as  $\pi(\operatorname{SL}_2^{(i)})$ . This is impossible, since each component of the  $\operatorname{SL}_2^{(i)}$ -stable closed subscheme  $\pi(\operatorname{SL}_2^{(i)}) \setminus \pi(\operatorname{SL}_2^{(i)})$  has strictly smaller dimension than that of  $\pi(\operatorname{SL}_2^{(i)})$ . This implies that  $\pi(\operatorname{SL}_2^{(i)})$  is proper over k.

Thus if  $P \cap \operatorname{SL}_2^{(i)}$  is a proper subgroup of  $\operatorname{SL}_2^{(i)}$ , then  $P \cap \operatorname{SL}_2^{(i)}$  is a Borel subgroup B of  $\operatorname{SL}_2^{(i)}$ . Over some finite extension K of k, the torus T in B splits, so we have the eigenspace decomposition  $K^2 = Kv \oplus Kv'$  with respect to  $T_K$ . Letting  $U \subset B$  be the unipotent radical, one of v, v', say v is an eigenvector for  $U_K$ , but since U already has a unique (up to scalar) eigenvector  $w \in k^2$ , we have  $v = \lambda \cdot w$  for some  $\lambda \in K$ , and thus w is an eigenvector for T. Letting  $B' \subset \operatorname{SL}_2$  be the isotropy subgroup of  $[w] \in \mathbb{P}^1(k)$ , we have  $T \subset B'$  and  $U \subset B'$  and since B is generated by T and U, we have  $B \subset B'$  so B is the split Borel subgroup B'.

Since P is a maximal parabolic subgroup, there is exactly one index i for which  $P \cap \operatorname{SL}_2^{(i)}$  is not the whole group  $\operatorname{SL}_2^{(i)}$ , which proves the lemma.

**Definition 6.3.** Take indices  $i \neq j$ ,  $1 \leq i, j \leq n$ , and let  $G_{ij} \subset SL_2^n$  be the subgroup scheme with points  $(g_1, \ldots, g_n)$ ,  $g_i = g_j$ .

**Proposition 6.4.** For  $G = SL_2^n$ ,  $e_i - e_j \in H^2(BG, W)$  maps to zero in  $H^2_G(G/G_{i,j}, W)$ .

*Proof.* For notational simplicity, we take i=1, j=2. Sending  $(h_1, \ldots, h_n) \in G$  to  $h_2h_1^{-1} \in \operatorname{SL}_2$  gives an isomorphism of the homogeneous space  $G/G_{1,2}$  with  $\operatorname{SL}_2$ , with G acting on  $\operatorname{SL}_2$  by

$$(g_1,\ldots,g_n)\cdot g=g_2gg_1^{-1}$$

This shows that  $G/G_{1,2} \times^G EG \cong \mathrm{BSL}_2^{n-1}$ , with the canonical map  $G/G_{1,2} \times^G EG \xrightarrow{\pi} BG = \mathrm{BSL}_2^n$  being the 1,2-diagonal map  $\pi = (p_1, p_1) \times p_2 \times \ldots \times p_{n-1}$ . Clearly  $\pi^*(e_1) = \pi^*(e_2)$ , whence the result.

Let  $K \supset k$  be a field extension. A closed subgroup scheme  $G \subset \operatorname{SL}_2^n/K$  is a diagonal subgroup scheme if there are indices  $i \neq j$  such that G is K-conjugate to  $G_{ij} \times_k K$ .

We recall the notion of an orbit for a G-action on a scheme X, Definition 1.5.

**Definition 6.5.** Take  $X \in \mathbf{Sch}^{\mathrm{SL}_2^n}/k$  We call the  $\mathrm{SL}_2^n$ -action on X localizing if for each orbit  $O \subset X$ ,  $\mathrm{SL}_2^n \setminus O = \mathrm{Spec}\,k_O$  of positive dimension, there is a closed subgroup scheme  $P_O \subset \mathrm{SL}_2^n/k_O$ , with  $P_O$  either a maximal parabolic or a diagonal subgroup scheme, and a morphism  $O \to \mathrm{SL}_2^n/P_O$  in  $\mathbf{Sch}^{\mathrm{SL}_2^n/k_O}/k_O$ .

Let  $p_i : \mathrm{BSL}_2^n \to \mathrm{BSL}_2$  be the projection on the *i*-factor and let  $e_i = p_i^*(e) \in H^2(\mathrm{BSL}_2^n, \mathcal{W})$ . We have the Künneth formula (Corollary 5.3)

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) \cong H^*(\mathrm{BSL}_2, \mathcal{W})^{\otimes_{W(k)} n} \cong W(k)[e_1, \dots, e_n],$$

and the isomorphism of modules over  $H^*(\mathrm{BSL}_2^n, \mathcal{W}) \cong W(k)[e_1, \dots, e_n],$ 

$$H^*_{\mathrm{BSL}_2^n}((\mathbb{P}^1)^n,\mathcal{W}) \cong H^*_{\mathrm{SL}_2}(\mathbb{P}^1,\mathcal{W})^{\otimes_{W(k)}n}.$$

**Proposition 6.6.** Take  $X \in \mathbf{Sch}^{\mathrm{SL}_2^n}/k$ . Let  $e_* := \prod_{i=1} e_i \cdot \prod_{1 \leq i < j \leq n} e_i - e_j \in H^*(BG, \mathcal{W})$  and suppose that

i. the  $\operatorname{SL}_2^n$ -action on X is localizing ii.  $X^{\operatorname{SL}_2^n} = \emptyset$ .

Let  $\mathcal{L}$  be a  $\mathrm{SL}_2^n$ -linearized invertible sheaf on X. Then  $H^{\mathrm{B.M.}}_{\mathrm{SL}_2^n}(X,\mathcal{W}(\mathcal{L}))[1/e_*]=0$ .

*Proof.* If k is not perfect, of characteristic p>2, taking a base-change to the perfect closure  $k^{perf}$  of k induces an isomorphism  $H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,*}(-/k,\mathcal{W}(-))\to H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,*}(-/k^{perf},\mathcal{W}(-))$ , so if the result holds over  $k^{perf}$ , then the result holds for k. Thus we may assume that k is perfect.

We first show that for each orbit  $O \subset X$ , we have  $H^{\text{B.M.}}_{\mathrm{SL}^n_2/k_O}(O/k_O, \mathcal{W})[1/e_*] = 0$ .

By assumption, there is a morphism  $f: O \to (\operatorname{SL}_2^n/k_O)/P$  in  $\operatorname{\mathbf{Sch}}^{\operatorname{SL}_2^n/k_O}/k_O$  for  $P \subset \operatorname{SL}_2^n/k_O$  a closed subgroup scheme that is either a maximal parabolic or a diagonal subgroup scheme. Suppose that P is a maximal parabolic. Then  $(\operatorname{SL}_2^n/k_O)/P$  as  $k_O$  scheme with  $\operatorname{SL}_2^n/k_O$ -action is isomorphic  $\mathbb{P}^1_{k_O}$ , with the action given by the projection on the ith factor followed by the standard action of  $\operatorname{SL}_2/k_O$  on  $\mathbb{P}^1_{k_O}$ .

It follows from Lemma 6.1 that, under the pullback map for  $O \to \operatorname{Spec} k$ ,  $e_i \in H^2(\operatorname{BSL}_2^n, \mathcal{W})$  is sent to zero in  $H^2_{\operatorname{SL}_2^n/k_O}((\operatorname{SL}_2^n/k_O)/P, \mathcal{W})$ . Thus  $e_*$  maps to zero in  $H^*_{\operatorname{SL}_2^n/k_O}(O, \mathcal{W})$  and hence  $H^*_{\operatorname{SL}_2^n/k_O}(O, \mathcal{W})[1/e_*] = 0$ .

Similarly, since  $e_i - e_j$  maps to zero in  $H^2_{\mathrm{SL}_2^n}(\mathrm{SL}_2^n/G_{ij},\mathcal{W})$  by Proposition 6.4, we see that  $H^*_{\mathrm{SL}_2^n/k_O}(O,\mathcal{W})[1/e_*] = 0$  if there is an equivariant morphism  $O \to (\mathrm{SL}_2^n/G_{ij})_{k_O}$ .

 $(\operatorname{SL}_2^n/G_{ij})_{k_O}.$  As  $H^{\operatorname{B.M.}}_{\operatorname{SL}_2^n/k_O,*}(O/k_O,\mathcal{W})[1/e_*]$  is a module over  $H^*_{\operatorname{SL}_2^n/k_O}(O,\mathcal{W})[1/e_*]$ , this shows that  $H^{\operatorname{B.M.}}_{\operatorname{SL}_2^n/k_O,*}(O/k_O,\mathcal{W})[1/e_*] = 0$  for each orbit  $O \subset X$ .

Following Proposition 1.1, we may assume that X has a stratification  $X = \coprod_{i=1}^r X_i$  such that the quotients  $\operatorname{SL}_2^n \backslash X_i$  all exist. Using the localization sequence for  $H^{\operatorname{B.M.}}_{\operatorname{SL}_2^n,*}(-,\mathcal{W}(\mathcal{L}))$  (Proposition 4.21), we reduce to the case in which the quotient  $\pi: X \to Y := \operatorname{SL}_2^n \backslash X$  exists.

Using localization for  $H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,*}(-,\mathcal{W}(\mathcal{L}))$  again, and passing to the colimit in the usual way, we form the strongly convergent Quillen-type spectral sequence

$$E^1_{p,q} = \oplus_{y \in Y_{(p)}} H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2, p+q}(\pi^{-1}(y)/k, \mathcal{W}(\mathcal{L})) \Rightarrow H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2, p+q}(X, \mathcal{W}(\mathcal{L})).$$

Here we define  $H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,p+q}(\pi^{-1}(y)/k,\mathcal{W}(\mathcal{L}))$  as the colimit of  $H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,p+q}(\pi^{-1}(U)/k,\mathcal{W}(\mathcal{L}))$  as U runs over neighborhoods of y in Y. Since k is perfect, we have

$$H^{\mathrm{B.M.}}_{\mathrm{SL}^n,p+q}(\pi^{-1}(y)/k,\mathcal{W}(\mathcal{L})) \cong H^{\mathrm{B.M.}}_{\mathrm{SL}^n,/k(y),q}(\pi^{-1}(y)/k(y),\mathcal{W}(\mathcal{L}))$$

rewriting our spectral sequence as

$$E_{p,q}^1 = \bigoplus_{y \in Y_{(p)}} H_{\operatorname{SL}_2^n/k(y),q}^{\operatorname{B.M.}}(\pi^{-1}(y)/k(y), \mathcal{W}(\mathcal{L})) \Rightarrow H_{\operatorname{SL}_2^n,p+q}^{\operatorname{B.M.}}(X,\mathcal{W}(\mathcal{L})).$$

As inverting  $e_*$  is exact, this gives the spectral sequence

$$E_{p,q}^{1} = \bigoplus_{y \in Y_{(p)}} H_{\mathrm{SL}_{n}^{n}/k(y),q}^{\mathrm{B.M.}}(\pi^{-1}(y)/k(y), \mathcal{W}(\mathcal{L}))[1/e_{*}] \Rightarrow H_{\mathrm{SL}_{n}^{n},p+q}^{\mathrm{B.M.}}(X,\mathcal{W}(\mathcal{L}))[1/e_{*}].$$

Since each  $\pi^{-1}(y)$  is an  $\mathrm{SL}_2^n$ -orbit O with quotient  $\mathrm{Spec}\,k_O = \mathrm{Spec}\,k(y)$ , this shows that  $H^{\mathrm{B.M.}}_{\mathrm{SL}_2^n,*}(X,\mathcal{W}(\mathcal{L}))[1/e_*] = 0$ , as claimed.

**Theorem 6.7** (Atiyah-Bott localization for  $SL_2^n$ ). Let k be a field of characteristic  $\neq 2$ . Take  $X \in \mathbf{Sch}^{SL_2^n}/k$ , let  $\mathcal{L}$  be a  $SL_2^n$ -linearized invertible sheaf on X, and suppose that the  $SL_2^n$ -action on X is localizing. Let  $e_* \in W(k)[e_1, \ldots, e_n]$  be the polynomial

$$e_* = \prod_{i=1}^n e_i \cdot \prod_{1 \le j < i \le n} e_i - e_j$$

Then the inclusion  $i: X^{\operatorname{SL}_2^n} \to X$  induces an isomorphism

$$i_*: H^{\operatorname{B.M.}}_*(X^{\operatorname{SL}_2^n}, \mathcal{W}(i^*\mathcal{L})) \otimes_{W(k)} W(k)[e, 1/e_*] \to H^{\operatorname{B.M.}}_{\operatorname{SL}_2^n, *}(X, \mathcal{W}(\mathcal{L}))[1/e_*]$$

*Proof.* Let  $U = X \setminus X^{\mathrm{SL}_2^n}$ . By Proposition 6.6, we have  $H^{\mathrm{B.M.}}_{\mathrm{SL}_2^n,*}(X,\mathcal{W}(\mathcal{L}))[1/e_*] = 0$ . Using the localization sequence (Proposition 4.21) for  $H^{\mathrm{B.M.}}_{\mathrm{SL}_2^n,*}$  with respect to the closed immersion  $i: X^{\mathrm{SL}_2^n} \to X$ , and inverting  $e_*$ , it follows that

$$i_*: H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2,*}(X^{\operatorname{SL}^n_2}, \mathcal{W}(i^*\mathcal{L}))[1/e_*] \to H^{\operatorname{B.M.}}_{\operatorname{SL}^n_2,*}(X, \mathcal{W}(\mathcal{L}))[1/e_*]$$

is an isomorphism.

Since the  $SL_2^n$ -action on  $X^{SL_2^n}$  is trivial, we have the Künneth isomorphism (Corollary 5.3, note that  $SL_2^n$  has only the trivial character)

$$H^{\mathrm{B.M.}}_{\mathrm{SL}^n_2,*}(X^{\mathrm{SL}^n_2},\mathcal{W}(i^*\mathcal{L})) \cong H^{\mathrm{B.M.}}_*(X^{\mathrm{SL}^n_2},\mathcal{W}(i^*\mathcal{L})) \otimes_{W(k)} H^{-*}(\mathrm{BSL}^n_2,\mathcal{W}).$$

Also,

$$H^*(\mathrm{BSL}_2^n, \mathcal{W}) = W(k)[e_1, \dots, e_n]$$

again by Theorem 5.1 and Corollary 5.3. This completes the proof.

# 7. Homogeneous spaces and localization for $N^n$

We continue to assume that our base-field k has characteristic  $\neq 2$ .

We consider the subgroup  $N^n \subset \operatorname{SL}_2^n$ , the normalizer of the diagonal torus  $T_n = T_1^n \subset \operatorname{SL}_2^n$ . We write  $\iota(t) \in N$  for the matrix  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ ; for  $\lambda \in k^{\times}$ , we write  $\lambda \cdot \sigma$  for  $\iota(\lambda) \cdot \sigma$ . We let  $\iota_j : N \to N^n$  be the inclusion as the jth factor; we write  $\iota_j(t)$  for  $\iota_j(\iota(t))$ . We set  $\sigma_j := \iota_j(\sigma)$ . We use the standard coordinates  $t_1, \ldots, t_n$  on  $T_n$ , that is

$$t_i(\iota_j(t)) = \begin{cases} t & \text{for } i = j\\ 1 & \text{otherwise.} \end{cases}$$

For  $I \subset \{1, \ldots, n\}$ , let  $\sigma_I = \prod_{i \in I} \sigma_i \in N^n$  and for  $\lambda_* = (\lambda_1, \ldots, \lambda_n) \in T_n$ , let  $\lambda_I = \prod_{i \in I} \iota_i(\lambda_i)$ . For  $\lambda \in k^{\times}$ , we let  $\lambda_I = \prod_{i \in I} \iota_i(\lambda)$ . We write  $T_I$  for the subtorus  $\prod_{i \in I} \iota_i(T_1)$  of  $T_n$ .

For  $m_* := (m_1, \ldots, m_n)$  an n-tuple of integers  $m_i \geq 1$ , and  $\lambda_* \in T_n(k)$ , let  $\Lambda_I(\lambda_*, m_*)$  be the closed subscheme of  $T_n$  with ideal generated by elements  $t_i^{m_i} - \lambda_i$ ,  $i \in I$ , and  $t_i - 1$ ,  $i \notin I$ . Let  $\mu_{m_*,I} = \Lambda_I(1, m_*) = \prod_{i \in I} \iota_i(\mu_{m_i})$ , so  $\Lambda_I(\lambda_*, m_*)$  is a  $\mu_{m_*,I}$ -torsor. Translating by  $\sigma_I$  gives us the closed subscheme  $\Lambda_I(\lambda_*, m_*) \cdot \sigma_I$  with  $(\Lambda_I(\lambda_*, m_*) \cdot \sigma_I)^2 = (-1)_I$ , so  $\{(-1)_I, \pm \Lambda_I(\lambda_*, m_*) \cdot \sigma_I\}$  is a subgroup scheme of  $N^n$ , denoted  $\Sigma_I(\lambda_*, m_*)$ .

Let  $\pi: N^n \to N^n/T_n = (\mathbb{Z}/2)^n$  be the projection. We identify  $(\mathbb{Z}/2)^n$  with the set of subsets  $I \subset \{1, \ldots, n\}$ , sending I to the tuple  $\sum_{i \in I} e_i$ , with  $e_i$  the i standard

basis vector in  $(\mathbb{Z}/2)^n$ . For a subgroup scheme H of  $N^n$ , we write  $\pi(H)$  for  $\pi(H(\bar{k})$  and set  $\bar{\sigma}_i := \pi(\sigma_i)$ .

**Definition 7.1.** Let  $F \supset k$  be a field, let  $\chi : T_n \to \mathbb{G}_m$  be a character with  $\chi(-\mathrm{Id}) = +1$  and take  $\lambda \in F^{\times}$ . Define the subgroup scheme  $\Lambda(\chi, \lambda) \subset N_F^n$  by

$$\Lambda(\chi,\lambda) := \chi^{-1}(1) \coprod \chi^{-1}(\lambda^{-1}) \cdot \sigma_{1,\dots,1}$$

Note that  $\pm Id$  is in  $\Lambda(\chi, \lambda)$  since  $\chi(-Id) = 1$ .

**Definition 7.2.** Let  $\chi: T_n \to \mathbb{G}_m$  be a character and let  $F \supset k$  be a field We construct four types of  $N^n$  homogeneous spaces X of dimension one over F; let  $F_X$  denote the  $N^n$ -invariant subring of  $\mathcal{O}_X(X)$ .

Type a.  $X = (N^n/\chi^{-1}(1))_F$ .

be the F' automorphism

Type b.  $\chi(-\mathrm{Id}) = 1$  and  $X = N_F^n/\Lambda(\chi, \lambda)$  for some  $\lambda \in F^{\times}$ .

Type c±. Let  $F' \supset F$  be a degree two extension.

Type c+. Suppose  $\chi(-\mathrm{Id})=1$ . Choose  $\lambda\in F^{\times}$ . Let  $\tau$  denote the conjugation of F' over F. Choose an isomorphism of  $T_n$ -homogeneous spaces  $T_n/\chi^{-1}(1)_{F'}\cong \mathbb{G}_{m\,F'}$ , giving the isomorphism

$$(N^n/\chi^{-1}(1))_{F'} = \coprod_{I \in (\mathbb{Z}/2)^n} \sigma_I \cdot \mathbb{G}_{m F'}$$

and let  $\rho: N^n/\chi^{-1}(1))_{F'} \to N^n/\chi^{-1}(1))_{F'}$  be the F' automorphism

$$\rho(\sigma_I \cdot x) = \sigma_{1,\dots,1} \sigma_I \cdot \lambda / x$$

This gives us the F-automorphism  $\tau \circ \rho$ , defining a  $\mathbb{Z}/2$ -action on  $(N^n/\chi^{-1}(1))_{F'}$  over F. One can easily check that the left  $N^n$ -action on  $N^n$  descends to a left action on the quotient, giving the  $N^n$ -homogeneous space  $X := [N^n/\chi^{-1}(1))_{F'}]/\langle \tau \circ \rho \rangle$ . Type c-. Suppose  $\chi(-\mathrm{Id}) = -1$ . Choose  $\lambda_0 \in F^\times$  and choose a generator  $\sqrt{a}$  for F' over F, for some  $a \in F^\times$ . Let  $\lambda = \lambda_0 \sqrt{a}$  and let  $\rho : N^n/\chi^{-1}(1))_{F'} \to N^n/\chi^{-1}(1))_{F'}$ 

$$\rho(\sigma_I \cdot x) = \sigma_{1,\dots,1} \sigma_I \cdot \lambda / x$$

This gives us the F-automorphism  $\tau \circ \rho$ , defining a  $\mathbb{Z}/2$ -action on  $(N^n/\chi^{-1}(1))_{F'}$  over F, giving as in case (c+) the  $N^n$ -homogeneous space

$$X := [N^n/\chi^{-1}(1))_{F'}]/\langle \tau \circ \rho \rangle.$$

Remark 7.3. Case (b) can also be described as a  $\mathbb{Z}/2$ -quotient of  $N^n/\chi^{-1}(1)$ . Indeed, writing  $N^n/\chi^{-1}(1)_F = \coprod_{I \in (\mathbb{Z}/2)^n} \sigma_I \cdot \mathbb{G}_{mF}$  we have the automorphism  $\rho: N^n/\chi^{-1}(1)_F \to N^n/\chi^{-1}(1)_F$  over F

$$\rho(\sigma_I \cdot x) = \sigma_{1,\dots,1} \sigma_I \cdot \lambda / x.$$

Since  $\chi(-\mathrm{Id}) = +1$  and  $\sigma_{1,\dots,1}^2 = (-1)$ , this defines a  $\mathbb{Z}/2$ -action on  $N^n/\chi^{-1}(1)_F$ . One can easily check that the left  $N^n$ -action on descends to a left action on the quotient.

For  $I \in (\mathbb{Z}/2)^n$ , and  $x \in \mathbb{G}_{m,F}$ , and  $\mu \in \chi^{-1}(\lambda^{-1})$ , we have in  $N_F^n/\Lambda(\chi, \lambda^{-1})$  the relation

$$\sigma_I \cdot x = \sigma_I \cdot x \cdot (\mu \cdot \sigma_{1,\dots,1}) = \sigma_{1,\dots,1} \sigma_I \cdot \chi(\mu^{-1}) \cdot x^{-1} = \sigma_{1,\dots,1} \cdot \sigma_I \lambda / x$$

Thus the quotient map

$$N^n/\chi^{-1}(1)_F \to N^n/\Lambda(\chi,\lambda^{-1})$$

defines a commutative diagram

$$N^n/\chi^{-1}(1)_F \\ \downarrow \\ N_F^n/\Lambda(\chi,\lambda^{-1}) \xrightarrow{\phi} N^n/\chi^{-1}(1)_F/\langle \sigma \rangle$$

and it is easy to see that  $\phi$  is an isomorphism of  $N^n$ -homogeneous spaces.

**Lemma 7.4.** Let X be a homogeneous space for  $N^n$ , smooth and of dimension one over Spec  $k_X := N^n \backslash X$  and let  $Y = T_n \backslash X$ , with induced structure of a  $N^n / T_n = (\mathbb{Z}/2)^n$ -homogeneous space over  $k_X$ . Suppose that  $T_1$  acts non-trivially on X via each inclusion  $\iota_i$ . Then

- 1. As Y-scheme,  $X \cong \mathbb{G}_{mY}$ .
- 2. Fix a closed point  $y_0$  of Y, let  $A \subset (\mathbb{Z}/2)^n$  be isotropy group of  $y_0$  and let  $B \subset A$  be the kernel of the action map  $A \to \operatorname{Aut}_{k_X}(k_X(y_0))$ . There are three cases

Case a. 
$$B = A = \{0\}$$

Case b. 
$$A = \langle (1, \ldots, 1) \rangle = B$$

Case c. 
$$A = \langle (1, ..., 1) \rangle, B = \{0\}.$$

3. In case (a) X is of type (a), in case (b) X is of type (b), and in both cases (a) and (b) we have  $k_X(y_0) = k_X$ . In case (c), X is of type (c±), depending on  $\chi(-\mathrm{Id}) = \pm 1$ , and  $k_X(y_0)$  is the degree two extension  $k_X'$  of  $k_X$ .

*Proof.* By construction,  $T_n$  acts on  $X_y$  for each point  $y \in Y$ , with  $y = T_n \backslash X_y$ . Thus the image of  $T_n$  in  $\operatorname{Aut}_y(X_y)$  is a quotient torus acting freely on  $X_y$ , hence is isomorphic to  $\mathbb{G}_{m\,y}$  as group-scheme over y. This defines the character  $\chi_y: T_n \to \mathbb{G}_m$  via which  $T_n$  maps to  $\mathbb{G}_{m\,y} \subset \operatorname{Aut}_y(X_y)$ . By Hilbert's theorem 90,  $X_y \cong \mathbb{G}_{m\,y}$  as  $\mathbb{G}_{m\,y}$ -torsor, with  $T_n$  acting via  $\chi_y$ . This proves (1).

as  $\mathbb{G}_{m\,y}$ -torsor, with  $T_n$  acting via  $\chi_y$ . This proves (1). Let  $\chi = \chi_{y_0}$ . For  $I = (\epsilon_1, \ldots, \epsilon_n) \in (\mathbb{Z}/2)^n = \{0, 1\}^n$ , and for  $t = (t_1, \ldots, t_n) \in T_n$ , let  $t^I = (t_1^{(-1)^{\epsilon_1}}, \ldots, t_n^{(-1)^{\epsilon_n}})$  and let  $\chi_I$  be the character

$$\chi_I(t) = \chi(t^I).$$

Then for  $y = \sigma_I \cdot y_0$ , acting by  $\sigma_I$  defines the isomorphism  $\sigma_I \times : X_{y_0} \to X_y$  and we have

$$t * \sigma_I(x) = \sigma_I(\chi_I(t)x).$$

where \* denotes the  $N^n$ -action. By our assumption that  $\iota_j(T_1)$  acts non-trivially on X for each j, we see that  $\chi((t_1,\ldots,t_n)=\prod_{i=1}^n t_i^{m_i}$  with no  $m_i=0$ .

Now take  $I \in A$ , giving the  $k(y_0)$  automorphism  $\sigma_I : \mathbb{G}_{m,y_0} \to \mathbb{G}_{m,y_0}$ . Suppose  $I \notin \langle (1,\ldots,1) \rangle$ , so there is an  $\epsilon_i = 0$  and an  $\epsilon_j = 1$ . Then

$$\sigma_I(t^{m_i}) = \sigma_I(\iota_i(t) \cdot 1) = t^{m_i} \cdot \sigma_I(1)$$

and similarly

$$\sigma_I(s^{m_j}) = s^{-m_j} \cdot \sigma_I(1)$$

for all  $t, s \in \mathbb{G}_m$ . Since both  $m_i$  and  $m_j$  are non-zero, this is impossible: take  $t = u^{m_j}$  and  $s = u^{m_i}$  for u arbitrary. Thus  $A \subset \langle (1, \ldots, 1) \rangle$ .

If  $A = \{0\}$ , then  $Y = (N^n/T_n)_{k_X}$  and thus X has the  $k_X$ -rational point  $1 \in X_{y_0} = \mathbb{G}_{m y_0}$ . The isotropy group of  $1 \in X_{y_0}$  is clearly  $\chi^{-1}(1)$ , giving the

isomorphism of  $N^n$ -homogeneous spaces.

$$X \cong N^n/\chi^{-1}(1)$$

If  $A=B=\langle (1,\ldots,1)\rangle$ , then  $k_X(y_0)=k_X$  and  $Y=\operatorname{Spec} k_X\times (\mathbb{Z}/2)^n/\langle (1,\ldots,1)\rangle$ .  $\sigma_{1,\ldots,1}$  acts as  $k_X$ -automorphism of  $X_{y_0}=\mathbb{G}_{m,k_X}$  and since  $\sigma_{1,\ldots,1}(t_1,\ldots,t_n)=(t_1^{-1},\ldots,t_n^{-1})\sigma_{1,\ldots,1}$ , we must have  $\sigma_{1,\ldots,1}(x)=\lambda/x$  for some  $\lambda\in k_X^\times$ . In particular,  $-\operatorname{Id}=\sigma_{1,\ldots,1}^2$  acts as the identity on  $X_{y_0}$ , so  $\chi(-\operatorname{Id})=1$ . Writing

$$N^n/\chi^{-1}(1) = \coprod_I \sigma_I \cdot T_n/\chi^{-1}(1)$$

and using  $1 \in X_{y_0}$  to define the action map  $N^n \to X$ , we see that  $\Lambda(\chi, \lambda^{-1})$  acts trivially, giving the map of  $N^n$ -homogeneous space

$$N^n/\Lambda(\chi,\lambda^{-1}) \to X$$

which is easily seen to be an isomorphism.

Now suppose  $A = \langle (1, \ldots, 1) \rangle$ ,  $B = \{0\}$ , so  $\sigma_{1,\ldots,1}$  acts by an involution  $\tau$  on  $k_X(y_0)$  and thus  $k_X(y_0)$  is a degree two Galois extension of  $k_X$  (we have assumed that  $\operatorname{char} k \neq 2$ ). As in the previous case,  $\sigma_{1,\ldots,1}$  acts on  $X_{y_0} = \mathbb{G}_{m,k_X(y_0)}$ , but this time as an automorphism over  $k_X$ , and acting by  $\tau$  on  $k_X(y_0)$ . As above, we have

$$\sigma_{1,\dots,1}(t^{m_i} \cdot x) = \sigma_{1,\dots,1}(\iota_i(t)(x)) = t^{-m_i}\sigma_{1,\dots,1}(x),$$

so the  $k_X$ -automorphism  $\sigma_{1,\dots,1}^*$  of  $k_X(y_0)[x,x^{-1}]$  induced by  $\sigma_{1,\dots,1}$  is of the form

$$\sigma_{1,\dots,1}^*(x) = \lambda/x$$

for some  $\lambda \in k_X(y_0)^{\times}$ , and by  $\tau$  on  $k_X(y_0)$ 

Suppose  $\chi(-\mathrm{Id}) = +1$ . Then  $\sigma_{1,\ldots,1}^2 = -\mathrm{Id}$  acts as the identity on  $X_{y_0}$ , so  $\tau(\lambda) = \lambda$  and thus  $\lambda$  is in  $k_X^{\times}$ . Similarly, if  $\chi(-\mathrm{Id}) = -1$ , then  $\sigma_{1,\ldots,1}^2(x) = -x$ , so  $\tau(\lambda) = -\lambda$ , and  $\lambda = \lambda_0 \cdot \sqrt{a}$ , where  $k_X(y_0) = k_X(\sqrt{a})$ , and  $\lambda_0$ , a are in  $k_X^{\times}$ . In both cases, this action defines the automorphism  $\sigma$  of  $N^n/\chi^{-1}(1)_{k_X(\sqrt{a})}$  by setting

$$\sigma(\sigma_I \cdot x) = \sigma_{1,\ldots,1} \sigma_I \cdot \lambda / x$$

and gives a  $\mathbb{Z}/2$ -action over  $k_X$  by acting by  $\tau \circ \sigma$ . As in cases (a), (b), the action map  $N^n/\chi^{-1}(1) \to X$  defined by the point  $1 \in \mathbb{G}_{mk_X(y_0)} = X_{y_0}$  descends to an isomorphism of  $\langle \tau \sigma \rangle \backslash N^n/\chi^{-1}(1)_{k_X(\sqrt{a})}$  with X.

Remark 7.5. If X is an arbitrary homogeneous space for  $N^n$  of positive dimension, then X admits a morphism  $X \to \bar{X}$  of  $N^n$ -homogeneous spaces with  $\dim \bar{X} = 1$  and with  $N^n \backslash X = N^n \backslash \bar{X}$ . Indeed,  $\iota_j(T_1)$  will act nontrivially on X for some j and since  $\iota_j(N)$  is a normal subgroup of  $N^n$ , we have the  $N^n$ -homogeneous space  $\iota_j(N) \backslash X$  of dimension  $\dim X - 1$  and with the same quotient by  $N^n$  as for X.

**Lemma 7.6.** Let X,  $\chi$  be as in Lemma 7.4, write  $\chi(t_1,\ldots,t_n)=\prod_{i=1}^n t_i^{m_i}$ .

In case (a),  $e_i$  goes to zero in  $H_{N^n}^*(X, \mathcal{W})$  for all i.

In case (b)  $e(\otimes_i \pi_i^* \tilde{\mathcal{O}}(m_i))$  goes to zero in  $H_{Nn}^*(X, \mathcal{W})$ .

In case (c+) (i.e.,  $\sum_i m_i$  even),  $e(\otimes_i \pi_i^* \tilde{\mathcal{O}}(m_i))$  goes to zero in  $H_{N^n}^*(X, \mathcal{W})$ .

In case (c-) (i.e.,  $\sum_i m_i$  odd), choose j with  $m_j$  is odd. Then  $e(\otimes_{i\neq j} \pi_i^* \tilde{\mathcal{O}}(m_i)) \otimes \tilde{\mathcal{O}}(2m_j)$  goes to zero in  $H_{N^n}^*(X, \mathcal{W})$ .

*Proof.* We use the fact that the  $H^*(-, \mathcal{W}(\det))$ -valued Euler characteristic is multiplicative in short exact sequences and vanishes on odd rank bundles.

We write X as a right  $N^n$ -homogeneous space. If we are in case (a), then

$$X \times^{N^n} E \operatorname{GL}_2 \cong \ker \chi \backslash N_{k_X}^n \times^{N^n} E \operatorname{GL}_2 \cong B \ker \chi_{k_X}$$

The pullback of  $\pi_i^* \tilde{O}(1)$  to  $B \ker \chi$  is a direct sum  $\mathcal{O}(\bar{\pi}_i) \oplus \mathcal{O}(-\bar{\pi}_i)$ , where  $\bar{\pi}_i$ :  $\ker \chi \to \mathbb{G}_m$  is the restriction of the projection. Thus  $\pi_i^*(e_i) = 0$ .

In case (b) there is a  $\mathbb{Z}/2$ -action on  $N^n/\chi^{-1}(1)_{k_X}$  with X the quotient. This gives the identity

$$X \times^{N^n} E \operatorname{GL}_2 \cong (\operatorname{Spec} k_X)^{\times \mathbb{Z}/2} B \chi^{-1}(1).$$

and the  $\mathbb{Z}/2$ -acting via  $\bar{\sigma}_{1,\dots,1}$  (the image of  $\sigma_{1,\dots,1}$  in  $N^n/T_n$ ). Since  $\chi(t_1,\dots,t_n)=\prod_i t_i^{m_i}$ , the tensor product  $\otimes \rho_{m_i} \circ \pi_i$  contains  $\rho_\chi \oplus \rho_{\chi^{-1}}$  a summand. This pulls back to  $B\chi^{-1}(1)$  to the trivial rank two bundle, with  $\bar{\sigma}_{1,\dots,1}$  acting as with

$$\chi^{-1}(\lambda^{-1})\sigma_{1,\dots,1} = \begin{pmatrix} 0 & \lambda^{-1} \\ (-1)^{\sum_i m_i} \lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}$$

since  $\sum_i m_i$  is even. This has the two invariant subspaces spanned by the  $\pm 1$  eigenvectors

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \begin{pmatrix} -1 \\ \lambda \end{pmatrix}$$

Thus  $\otimes_i \tilde{\mathcal{O}}(m_i)$  pulls back to a bundle which contains two rank 1 summands, hence  $e(\otimes_i \tilde{\mathcal{O}}(m_i))$  goes to zero.

In the cases  $(c\pm)$ , we have

$$X \times^{N^n} E \operatorname{GL}_2 \cong (\operatorname{Spec} k_X(\sqrt{a}))^{\times \mathbb{Z}/2} B \chi^{-1}(1).$$

with  $\mathbb{Z}/2$  acting on  $k_X(\sqrt{a})$  by conjugation over  $k_X$ . In case (c+), we have essentially the same computation as is the same as case (b), noting that in case of even  $\sum_i m_i$ ,  $\lambda$  is in  $k_X$ , so the eigenvectors we chose define subspaces that are invariant under the conjugation action of  $k(\sqrt{a})$  over  $k_X$ . In case (c-),  $\sum_i m_i$  is odd, so there is an odd  $m_j$ . We replace X with the quotient  $X' := \iota_j(\{\pm 1\}) \backslash X$ , giving the character  $\chi'(t_1, \ldots, t_m) = \prod_{i \neq j} t_i^{m_i} \cdot t_j^{2m_j}$  with  $\sum_{i \neq j} m_i + 2m_j$  even, and we are back in case (c+) with  $e(\otimes_{i \neq j} \pi_i^* \tilde{\mathcal{O}}(m_i) \otimes \tilde{\mathcal{O}}(2m_j))$  going to zero in  $H_{N^n}^*(X', \mathcal{W})$ . Pulling back further to  $H_{N^n}^*(X, \mathcal{W})$ , this class still goes to zero.

We have the canonical map  $\mathbb{Z}[e_1,\ldots,e_n]\to W(k)[e_1,\ldots,e_n]=H^*(\mathrm{BSL}_2^n,\mathcal{W}),$  giving  $H^*_{N^n}(Y,\mathcal{W})$  the structure of a  $\mathbb{Z}[e_1,\ldots,e_n]$ -module for Y a k-scheme with  $N^n$ -action.

**Proposition 7.7.** Let X be a homogeneous space for  $N^n$  of positive dimension. Then there is a homogeneous polynomial  $P \in \mathbb{Z}[e_1, \ldots, e_n]$  of positive degree such that  $H_{N^n}^*(X, \mathcal{W})[P^{-1}] = 0$ .

Proof. By repeatedly taking the quotient of X by a normal subgroups of the form  $\iota_j(N)$ , we have a map of  $N^n$ -homogeneous spaces  $X \to \bar{X}$  with  $\dim \bar{X} = 1$ . If  $\iota_j(T_1)$  acts trivially on  $\bar{X}$ , we may take a further quotient by  $\iota_j(N)$  without changing the dimension of  $\bar{X}$ , so we may assume that if  $\iota_j(T_1)$  acts trivially on  $\bar{X}$ , then so does  $\iota_j(N)$ . Letting  $\pi: N^n \to N^r$  be the quotient of  $N^n$  by all subgroups  $\iota_j(N)$  that act trivially on  $\bar{X}$ , we have the induced action of  $N^r$  on  $\bar{X}$ , with  $\iota_j(T_1)$  acting non-trivially on  $\bar{X}$  for each inclusion  $\iota_j: N \to N^r$ .

Reordering the factors in  $N^n$ , we may assume that  $\pi$  is the projection on the first r factors. This gives us the isomorphism

$$H_{N^n}^*(\bar{X}, \mathcal{W}) \cong H_{N^r}^*(\bar{X}, \mathcal{W}) \otimes_{W(k)} H^*(BN^{n-r}, \mathcal{W})$$

and with the  $H^*(\mathrm{BSL}_2^n,\mathcal{W})=H^*(\mathrm{BSL}_2^r,\mathcal{W})\otimes_{W(k)}H^*(\mathrm{BSL}_2^{n-r},\mathcal{W})$  module structure being the evident one. Since the  $H^*(\mathrm{BSL}_2^n,\mathcal{W})$ -action on  $H^*_{N^n}(X,\mathcal{W})$  factors through its action on  $H^*_{N^n}(\bar{X},\mathcal{W})$ , we thus reduce to the case  $X=\bar{X}$  of dimension one, with  $\iota_j(T_1)$  acting non-trivially on X for each  $j=1,\ldots,n$ .

Referring to Lemma 7.6, in case (a), we can take  $P = e_i$  for any i between 1 and n. In cases (b), (c  $\pm$ ), we have the Euler class  $e(\tilde{\mathcal{O}}(m_1, \ldots, m_n))$ ; note that each  $m_i$  is strictly positive.

For the tensor product of two rank two bundles V, W, we have

$$e(V \otimes W) = e(V)^2 - e(W)^2$$

(see [18, Proposition 9.1]); an induction using Ananyevskiy's  $\operatorname{SL}_2$ -splitting principle implies that, for  $V_1, \ldots, V_r$  rank two bundles,  $e(V_1 \otimes \ldots \otimes V_r)$  is a non-zero homogeneous polynomial with integer coefficients and of degree  $2^{r-2}$  in  $e(V_1)^2, \ldots, e(V_r)^2$ . Also, we have  $e(\tilde{\mathcal{O}}(m)) = \pm m \cdot e(\tilde{\mathcal{O}}(1))$  if m is odd, and  $e(\tilde{\mathcal{O}}(m))^2 = m^2 e(\tilde{\mathcal{O}}(1))^2$  if m is even [18, Theorem 7.1]. Since  $e(\tilde{\mathcal{O}}(1))$  is the image of the canonical generator  $e \in H^2(\operatorname{BSL}_2, \mathcal{W})$ , it follows that for  $n \geq 2$ ,  $e(\tilde{\mathcal{O}}(m_1, \ldots, m_n)) \in H^{2^n}(BN^n, \mathcal{W})$  is the image of a non-zero polynomial  $P \in \mathbb{Z}[e_1, \ldots, e_n]$ 

If n = 1, then if m is odd, we have  $e(\mathcal{O}(m)) = \pm m \cdot e$ , and we use  $P = m \cdot e$ . If m is even then  $e(\mathcal{O}(m))^2 = m^2 \cdot e^2$  and we use  $P = m \cdot e$ .

Remark 7.8. Looking at the proof of Proposition 7.7, we have associated to each positive  $N^n$ -homogeneous space X a finite list of characters of  $T_n$  (depending on the quotients we took to reduce the dimension down to one), and for each character three different action-types, (a), (b), (c). The polynomial P depends only on the character  $\chi$  and the action-type. The polynomial  $Q := e_1 \cdot P$  only depends on  $\chi$ .

If k admits a real embedding, then as the composition  $\mathbb{Z} \to W(k) \to W(\mathbb{R}) = \mathbb{Z}$  is the identity, the polynomial P is non-zero in  $H^{*\geq 2}(\mathrm{BSL}_2^n, \mathcal{W})$ .

Remark 7.9. Take  $X \in \mathbf{Sch}^{N^n}/k$ . Since  $N^n/T_n = (\mathbb{Z}/2)^n$ , a point  $x \in X$  has 0-dimensional  $N^n$ -orbit if and only if x is a point of  $X^{T_n}$ . Thus  $X^{T_n} \subset X$  is the union of the 0-dimensional  $N^n$ -orbits.

**Theorem 7.10** (Localization for  $N^n$ ). Take X in  $\mathbf{Sch}^{N^n}/k$ , and let  $i: X^{T_n} \hookrightarrow X$  be the inclusion. Let  $\mathcal{L}$  be an invertible sheaf on X with an  $N^n$ -linearization. Then there is a homogeneous polynomial  $P \in \mathbb{Z}[e_1, \ldots, e_n]$  of positive degree such that

$$i_*: H^{\mathrm{B.M.}}_{*,N^n}(X^{T_n}/k,\mathcal{W}(i^*\mathcal{L}))[1/P] \to H^{\mathrm{B.M.}}_{*,N^n}(X/k,\mathcal{W}(\mathcal{L}))[1/P]$$

is an isomorphism. If the base-field k admits a real embedding, then P is non-zero in  $H^*(\mathrm{BSL}_2^n,\mathcal{W})$ .

*Proof.* As in the proof of Proposition 6.6, we may assume that k is perfect.

Using the localization sequence in equivariant Borel-Moore homology (Proposition 4.21), we reduce to showing, that if X has no 0-dimensional orbits, then  $H_{*,N^n}^{\mathrm{B.M.}}(X,\mathcal{W}(\mathcal{L}))=0$ ; we thus assume that  $X^{T_n}=\emptyset$ . We may assume that X is reduced

By Proposition 1.1, there is a finite stratification of X as  $X = \coprod_{\alpha} X_{\alpha}$  such that for each  $\alpha$  the quotient scheme  $X_{\alpha} \to N^n \backslash X_{\alpha}$  exists as a quasi-projective k-scheme.

Using the localization sequence again, we reduce to the case in which the quotient  $Z:=N^n\backslash X$  exists as a quasi-projective k-scheme. Localizing again, we may assume that Z is integral and that  $\pi$  is equi-dimensional; we let  $Z_{(q)}$  be the set of dimension q points of Z. It suffices to show that if  $\pi$  has strictly positive relative dimension, then  $H_{*,N^n}^{\mathrm{B,M}}(X,\mathcal{W}(\mathcal{L}))=0$ , so we assume that our  $\pi$  has strictly positive relative dimension. Using localization once more, we may replace Z with any dense open subscheme. Since  $N^n$  is smooth over k, it follows from Proposition 1.1 that we may assume that  $X\to Z$  is smooth.

We may replace  $N^n$  with the quotient of  $N^n$  by the factors  $\iota_j(N)$  for which  $\iota_j(T_1)$  acts trivially; changing notation, we may assume that  $\iota_j(T_1)$  acts non-trivially for each j. Again passing to a dense open subscheme of Z and changing notation, we may assume that for each point z of Z,  $\iota_j(T_1)$  acts non-trivially on the fiber  $X_z$  for each j.

We have the Leray spectral sequence of homological type

$$E_{a,b}^{1} = \bigoplus_{z \in Z_{b}} H_{N^{n},a+b}^{\text{B.M.}}(X_{z}/k, \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_{z}}) \Rightarrow H_{N^{n},a+b}^{\text{B.M.}}(X, \mathcal{W}(\mathcal{L})).$$

Since k is perfect, we have an isomorphism

$$H_{N^n,a}^{\mathrm{B.M.}}(X_z/k(z), \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_z}) \cong H_{N^n,a+b}^{\mathrm{B.M.}}(X_z/k, \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_z})$$

for  $z \in Z_{(b)}$ , where  $H_{N^n,a+b}^{\mathrm{B.M.}}(X_z/k,\mathcal{W}(\mathcal{L}\otimes\mathcal{O}_{X_z}))$  is defined as a colimit over models for  $X_z$  that are of finite type over k. This rewrites our spectral sequence as

$$E_{a,b}^{1} = \bigoplus_{z \in Z_{b}} H_{N^{n},a}^{\text{B.M.}}(X_{z}/k(z), \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_{z}})) \Rightarrow H_{N^{n},a+b}^{\text{B.M.}}(Y, \mathcal{W}(\mathcal{L})).$$

Referring to Remark 7.8, we have the discrete invariant  $\chi_z$  giving the character of the action of  $T_n$  on  $X_z$ ; again, we may assume that  $\chi_z = \chi_{z'}$  for all  $z, z' \in Z$  and let  $\chi$  denote this common character.

Each  $X_z$  is a homogeneous space over k(z) for  $N^n$  of positive dimension, so by Proposition 7.7, there is a non-zero polynomial  $P_z \in ]Z[e_1,\ldots,e_n]$  such that  $H_{N^n,*}^{\mathrm{B.M.}}(X_z/k(z),\mathcal{W})[1/P_z]=0$ . Following Remark 7.8, there is non-zero polynomial  $P\in\mathbb{Z}[e_1,\ldots,e_n]$  that depends only on the character  $\chi$  and is divisible by  $P_z$  for all  $z\in Z$ , thus  $H_{N^n,*}^{\mathrm{B.M.}}(X_z/k(z),\mathcal{W})[1/P]=0$  for all  $z\in Z$ .

Since  $X \to Z$  is smooth,  $X_z$  is smooth over k(z) for each  $z \in Z$ . This gives us an isomorphism  $H_{N^n,*}^{\mathrm{B.M.}}(X_z/k(z), \mathcal{W}) \cong H_{N^n}^*(X_z, \mathcal{W})$ , after a reindexing. As  $H_{N^n,*}^{\mathrm{B.M.}}(X_z/k(z), \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_z}))$  is a  $H_{N^n}^*(X_z, \mathcal{W})$  module, we have

$$H_{N^n}^*(X_z, \mathcal{W})[1/P] = 0, \ \forall z \in Z \Rightarrow H_{N^n, *}^{\mathrm{B.M.}}(X_z/k(z), \mathcal{W}(\mathcal{L} \otimes \mathcal{O}_{X_z}))[1/P] = 0, \ \forall z \in Z$$
$$\Rightarrow H_{*, N^n}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L}))[1/P] = 0,$$

the last implication following from the spectral sequence.

# 8. Homogeneous spaces and localization for N

Theorem 7.10 reduces the computation of  $H_{N^n,*}^{\mathrm{B.M.}}(Y,\mathcal{W})$ , after suitable localization, to the case of a scheme with only 0-dimensional orbits. For general n, the situation is still quite complicated, so we will restrict our discussion to the case of the group N. We will see at the end of this section that this case will be sufficient for many applications. We continue to assume that k is a field of characteristic  $\neq 2$ .

Remark 8.1. We apply Lemma 7.6 in case n=1. Let X be an N-homogeneous space of dimension one, with quotient  $N \setminus X = \operatorname{Spec} k_X$  and associated character  $\chi_m : T_1 \to \mathbb{G}_m$ ,  $\chi_m(t) = t^m$  as given by Lemma 7.4. We may assume that m > 0.

We consider the four types of homogeneous spaces:

Type a. In this case  $X \cong (N/\mu_m)_{k_X}$  and  $e := e(\tilde{\mathcal{O}}(1))$  goes to zero in  $H_N^*(X, \mathcal{W})$ , so we invert P = e to kill  $H_N^*(X, \mathcal{W})$ .

Type b. In this case, m is even and there is a  $\lambda \in k_X^\times$  with  $X \cong N_{k_X}/\Lambda(\chi, \lambda^{-1})$ .  $e(\tilde{\mathcal{O}}(m))$  goes to zero in  $H_N^*(X, \mathcal{W}(\det \tilde{\mathcal{O}}(2))$ , giving us the element  $e(\tilde{\mathcal{O}}(m))^2 = m^2 e^2$  in  $H^*(\mathrm{BSL}_2, \mathcal{W})$ . If we take  $P = m \cdot e$ , then  $H_N^*(X, \mathcal{W})[P^{-1}] = 0$ .

Type c±. In this case  $X \cong [N_{k_X(\sqrt{a})}/\mu_m]/\mathbb{Z}/2$ , for a certain  $\mathbb{Z}/2$  action on  $N_{k_X(\sqrt{a})}/\mu_m$  as  $k_X$ -scheme, with  $\mathbb{Z}/2$  acting on  $k_X(\sqrt{a})$  as conjugation over  $k_X$ . In type c+, m is even and we need to invert  $P = m \cdot e$  to kill  $H_N^*(X, \mathcal{W})$ . In type c-, m is odd and we need to invert  $2m \cdot e$  to kill  $H_N^*(X, \mathcal{W})$ .

In summary, we invert e in case (a),  $m \cdot e$  in cases (b) and (c+) and  $2m \cdot e$  in case (c-). Thus, in all cases except for (a), we are inverting 2, which unfortunately is often a zero-divisor on W(k).

Thus, the polynomial P given by Theorem 7.10 in case of n=1 is of the form  $M \cdot e$ , where M is the least common multiple of the coefficients m discussed above, as one ranges over the (finitely many) dimension one orbit types of the N-action on Y.

If  $k_X$  already contains a square root of -1, one can divide the coefficient m by 2 if m is even in the above analysis.

We let  $\bar{\sigma}$  denote the image of  $\sigma \in N$  in  $N/T_1$ , so  $N/T_1 = \langle \bar{\sigma} \rangle \cong \mathbb{Z}/2$ . We recall Definition 4 from the introduction.

**Definition 8.2.** 1. Take  $X \in \mathbf{Sch}^N/k$ . We let  $|X|^N$  denote the union of the integral components  $Z \subset X^{T_1}$  such that the generic point  $z \in Z$  is fixed by  $\bar{\sigma}$ , and let  $X_{\mathrm{ind}}^{T_1}$  be the union of the integral components  $Z \subset X^{T_1}$  such that  $\bar{\sigma} \cdot Z \cap Z = \emptyset$ .

- 2. We call the N-action semi-strict if  $X_{\mathrm{red}}^{T_1} = |X|^N \cup X_{\mathrm{ind}}^{T_1}$ .
- 3. If the N-action on X is semi-strict, we say the N-action on X is strict if  $|X|^N \cap X^{T_1}_{\mathrm{ind}} = \emptyset$  and we can decompose  $|X|^N$  as a disjoint union of two N-stable closed subschemes

$$|X|^N = X^N \coprod X_{\mathrm{fr}}^{T_1}$$

where the  $N/T_1$ -action on  $X_{\rm fr}^{T_1}$  is free.

Remark 8.3. 1. If  $X_{\mathrm{red}}^{T_1}$  is a union of closed points of X, then the N-action is strict. 2. If  $X_{\mathrm{red}}^{T_1}$  is a disjoint union of its integral components, then the N-action is semi-strict, and the N-action is strict if and only if each integral component of  $X^N$  is an integral component of  $X^{T_1}$ . In case  $X_{\mathrm{red}}^{T_1}$  is a disjoint union of its integral components, the action thus fails to be strict exactly when there is an integral component C of  $X^{T_1}$  that is  $\bar{\sigma}$  stable, but with  $C^{\bar{\sigma}}$  a proper subset of C.

In case X is smooth over k, then  $X^{T_1}$  is also smooth over k and is thus a disjoint union of its integral components.

**Lemma 8.4.** Take  $Y \in \mathbf{Sch}^N/k$  and let  $\mathcal{L}$  be an N-linearized invertible sheaf on Y. Suppose that the  $T_1$  action on Y is trivial and that the quotient map  $q: Y \to \bar{Y} := N \setminus Y$  is an étale degree two cover. Then each  $\bar{y} \in \bar{Y}$  admits an open neighborhood V such that, letting  $U = q^{-1}(V)$  with inclusion  $j: U \to Y$ ,  $j^*\mathcal{L}$  is isomorphic to  $\mathcal{O}_U$  as N-linearized sheaf, where we give  $\mathcal{O}_U$  the canonical N-linearization induced by the N-action on U.

*Proof.* This follows from Lemma 2.6 with  $\mathcal{V} = \mathcal{L}$ . Indeed, since  $\mathcal{L}$  has rank one, we have an N-stable open neighborhood  $j: U \to Y$  of  $q^{-1}(\bar{y})$  and an isomorphism of N-linearized sheaves  $j^*\mathcal{L} \cong \mathcal{O}_U \otimes_k^{\tau,\sigma} V(\rho_0) = \mathcal{O}_U$ .

We recall from Theorem 5.1 that  $H^*(\mathrm{BSL}_2, \mathcal{W}) = W(k)[e]$ , with e the Euler class of the tautological rank 2 bundle on  $\mathrm{BSL}_2$ , and that

$$H^*(BN, W) = W(k)[x, e]/((1+x) \cdot e, x^2 - 1)$$

for a certain  $x \in H^0(BN, \mathcal{W})$ . We also recall that the group scheme N has two characters, the trivial character  $\rho_0$  and the character  $\rho_0^-$ , with  $\rho_0^-(\sigma) = -1$ , and  $\rho_0^-$  restricted to  $T_1$  the trivial character.

For a character  $\chi$  of some group-scheme G over k, and F an extension field of k, we let  $F(\chi)$  denote the representation of G on F with character  $\chi$ . The character  $\chi$  determines an invertible sheaf  $\mathcal{L}_{\chi}$  on BG, and we write  $\mathcal{W}(\chi)$  for the sheaf  $\mathcal{W}(\mathcal{L}_{\chi})$  on BG.

The following result gives information on  $H_N^*(X^{T_1}, \mathcal{W}(\mathcal{L}))$ .

**Lemma 8.5.** Take  $X \in \mathbf{Sch}^N/k$  and let  $\mathcal{L}$  be an N-linearized invertible sheaf on X.

1. Let  $i_0: X^N \to X$  be inclusion. Then for each connected component  $X_i^N$  of  $X^N$  there is an invertible sheaf  $\mathcal{L}_i$  on  $X_i^N$  with trivial N-action and a character  $\chi_i$  of N with  $i_0^*\mathcal{L}_{|X_i^N|} \cong \mathcal{L}_i \otimes_k k(\chi_i)$  as N-linearized invertible sheaf. Moreover

$$H_{N,*}^{\mathrm{B.M.}}(X^N, \mathcal{W}(i_0^*\mathcal{L})) \cong \bigoplus_i H_*^{\mathrm{B.M.}}(X_i^N, \mathcal{W}(\mathcal{L}_i)) \otimes_{W(k)} H^{-*}(BN, \mathcal{W}(\chi_i)).$$

2. Let  $i_{\text{ind}}: X_{\text{ind}}^{T_1} \to X$  be the inclusion. Then

$$H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{ind}}^{T_1}, \mathcal{W}(i_{\mathrm{ind}}^*\mathcal{L}))[e^{-1}] = 0.$$

3. Let  $i_Y: Y \to |X|^N$  be a locally closed N-stable subscheme. Suppose that the  $N/T_1$ -action on Y is free; let

$$q_{\mathrm{fr}}: Y \to \bar{Y} := N \backslash Y$$

be the quotient map. Take  $\bar{z} \in \bar{Y}$  and let  $z = q_{\mathrm{fr}}^{-1}(\bar{z})$ , with inclusion  $i_z : z \to X$ , giving the degree two étale extension  $k(\bar{z}) \subset \mathcal{O}_z$ . Then the isomorphism of Lemma 8.4 induces the isomorphism of  $H^*(BN, \mathcal{W})$  modules

$$H_N^*(z, \mathcal{W}(i_z^*\mathcal{L})) \cong H_N^*(z, \mathcal{W}).$$

Moreover, writing  $\mathcal{O}_z = k(\bar{z})[X]/T^2 - a$  for suitable  $a \in k(\bar{z})^{\times}$ , then as algebra over  $H^*(BN_{k(\bar{z})}, \mathcal{W}) = W(k(\bar{z}))[x, e]/(x^2 - 1, (x + 1)e)$ , we have

$$H_N^*(z, \mathcal{W}) = H^*(BN_{k(\bar{z})}, \mathcal{W})[y]/(x - \langle a \rangle, I_a \cdot y, y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot e).$$

where  $I_a \subset W(k(\bar{z}))$  is the ideal  $\operatorname{im}(\operatorname{Tr}_{\mathcal{O}_z/k(\bar{z})}) \subset W(k(\bar{z}))$ . Finally, letting  $\bar{W}(k(\bar{z})) := W(k(\bar{z}))/\operatorname{im}(\operatorname{Tr}_{\mathcal{O}_z/k(\bar{z})})$ , we have

$$H_N^*(z, \mathcal{W})[e^{-1}] = \bar{W}(k(\bar{z}))[e, e^{-1}].$$

*Proof.* For (1), we may assume that  $X^N$  is connected. As  $i_0^*\mathcal{L}$  is an invertible sheaf on  $X^N$  with an action of N over the trivial action on  $X^N$ , N acts by a character  $\chi$  on  $i_0^*\mathcal{L}$ , and we have the isomorphism of N-linearized sheaves  $i_0^*\mathcal{L} = \mathcal{L}_0 \otimes_k k(\chi)$ , with  $\mathcal{L}_0$  having the trivial N-action. The isomorphism

$$H_{N,*}^{\mathrm{B.M.}}(X^N, \mathcal{W}(i_0^*\mathcal{L})) \cong H_*^{\mathrm{B.M.}}(X^N, \mathcal{W}(\mathcal{L}_i)) \otimes_{W(k)} H^{-*}(BN, \mathcal{W}(\chi))$$

follows from Corollary 5.3.

Let  $C_1, \ldots, C_{2r}$  be the irreducible components of  $X_{\mathrm{ind}}^{T_1}$  with  $\bar{\sigma}(C_{2i-1}) = C_{2i}$ . We prove (2) by induction on r. r = 1, then  $X_{\mathrm{ind}}^{T_1} = C_1 \coprod C_2$  with  $C_2 = \bar{\sigma} \cdot C_1$ . Thus as scheme with N-action,  $X^{T_1} \cong (N/T_1) \times_k C_1$ , with the trivial action on  $C_1$ , so

$$X_{\text{ind}}^{T_1} \times^N EN \cong C_1 \times^{T_1} EN \cong C_1 \times_k BT_1$$

There is a character  $\chi$  of  $T_1$ , an invertible sheaf  $\mathcal{L}_0$  on  $C_1$  and an isomorphism

$$H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{ind}}^{T_1}, \mathcal{W}(i_{\mathrm{ind}}^*\mathcal{L})) \cong H_{*}^{\mathrm{B.M.}}(C_1, \mathcal{L}_0 \otimes_{W(k)} H^{-*}(BT_1, \mathcal{W}(\chi)))$$

As  $H^{-*}(BT_1, \mathcal{W}(\chi))$  is a  $H^{-*}(BT_1, \mathcal{W})$ -module and  $H^{-*}(BT_1, \mathcal{W})[e^{-1}] = 0$ , this proves (2) in case r = 1.

In general, write  $X_C^{T_1} \cup C'$  with  $C = C_1 \coprod C_2$  and  $C' = C_3 \cup \ldots \cup C_{2r}$ . By induction,  $H_{N,*}^{\text{B.M.}}(C, \mathcal{W}(\mathcal{L}))[1/e] = H_{N,*}^{\text{B.M.}}(C', \mathcal{W}(\mathcal{L}))[1/e] = 0$ . Moreover,  $C \cap C' = (C_1 \cap C') \coprod \bar{\sigma} \cdot (C_1 \cap C')$ , so the argument for the case r = 1 shows that  $H_{N,*}^{\text{B.M.}}(C \cap C', \mathcal{W}(\mathcal{L}))[1/e] = 0$ . The localization sequence for  $H_{N,*}^{\text{B.M.}}(-, \mathcal{W}(\mathcal{L}))$  then shows that  $H_{N,*}^{\text{B.M.}}(X_{\text{ind}}^{T_1}, \mathcal{W}(i_{\text{ind}}^*\mathcal{L}))[1/e] = 0$ .

For (3), Lemma 8.4 gives us the isomorphism  $H_N^*(z, \mathcal{W}(i_z^*\mathcal{L})) \cong H_N^*(z, \mathcal{W})$ . Suppose first that z is an integral k-scheme, that is, a is not a square in  $k(\bar{z})^{\times}$ . Then we use Corollary 10.12 to compute  $H_N^*(z, \mathcal{W})$  and  $H_N^*(z, \mathcal{W})[e^{-1}]$ . If a is a square in  $k(\bar{z})^{\times}$ , then we use Remark 10.13.

**Theorem 8.6** (Localization for an N-action). Take  $X \in \mathbf{Sch}^N/k$  and let  $i: |X|^N \hookrightarrow X$  be the inclusion. Let  $\mathcal{L}$  be an invertible sheaf on X with an N-linearization. Suppose the N-action on X is semi-strict. Then there is an  $M \in \mathbb{Z} \setminus \{0\}$  such that

$$i_*:H^{\operatorname{B.M.}}_{N,*}(|X|^N,\mathcal{W}(i^*\mathcal{L}))[1/Me]\to H^{\operatorname{B.M.}}_{N,*}(X,\mathcal{W}(\mathcal{L}))[1/Me]$$

is an isomorphism.

Proof. Since the action is semi-strict, we have the closed immersion  $X_{\mathrm{ind}}^{T_1} \cup |X|^N \hookrightarrow X$  with open complement  $X \setminus X^{T_1}$ . By Lemma 8.5,  $H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{ind}}^{T_1}, \mathcal{W}(i^*\mathcal{L}))[1/e] = 0$ . If we apply Lemma 8.5 to the scheme  $|X|^N \in \mathbf{Sch}^N/k$ , we see that  $H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{ind}}^{T_1} \cap |X|^N, \mathcal{W}(i^*\mathcal{L}))[1/e] = 0$ , so by localization,  $H_{N,*}^{\mathrm{B.M.}}(X_{\mathrm{ind}}^{T_1} \setminus |X|^N, \mathcal{W}(i^*\mathcal{L}))[1/e] = 0$  as well, and thus the inclusion  $|X|^N \to X^{T_1}$  induces an isomorphism on  $H_{N,*}^{\mathrm{B.M.}}(-, \mathcal{W}(\mathcal{L}))$  and the result follows from Theorem 7.10.

Remark 8.7. For n > 1, it is not in general the case that for  $N^n \cdot y \subset Y$  a 0-dimensional orbit consisting of more than one point, that  $H^*_{N^n}(N^n \cdot y, \mathcal{W}) = 0$  after inverting some non-zero P, so one cannot in general localize  $H^{\mathrm{B.M.}}_{*,N^n}(Y,\mathcal{W})$  to the fixed points. We have not made a computation of the  $N^n$ -cohomology of a 0-dimension orbit in all cases.

We conclude this section by showing how to reduce an  $\mathbb{N}^n$ -action to an action of  $\mathbb{N}$ .

**Proposition 8.8.** Given odd integers  $a_1, \ldots, a_n$ , there is a unique homomorphism of group schemes  $\rho_{a_1,\ldots,a_n}: N \to N^n$  with

$$\rho_{a_1,...,a_n}(\iota(t)) = \prod_{i=1}^n \iota_i(t^{a_i}), \ \rho(\sigma) = \prod_{i=1}^n \iota_i(\sigma)$$

*Proof.* Uniqueness is clear; for existence, we note that  $\rho_{a_1,...,a_n}$  restricted to  $T_1$  is a homomorphism  $T_1 \to T_n$ . We have

$$\rho_{a_1,\dots,a_n}(\iota(t)) \circ \sigma) = \prod_{i=1}^n \iota_i(\iota(t)^{a_i} \cdot \sigma)$$
$$= \prod_{i=1}^n \iota_i(\sigma\iota(t)^{-a_i})$$
$$= \rho_{a_1,\dots,a_n}(\sigma\iota(t)^{-1})$$

and

$$\rho_{a_1,\dots,a_n}(\sigma^2) = \rho_{a_1,\dots,a_n}(\prod_{i=1}^n \iota_i(-\operatorname{Id}_N))$$

$$= \prod_{i=1}^n \iota_i((-1)^{a_i}\operatorname{Id}_N)$$

$$= -\operatorname{Id}_{N^n} = (\prod_{i=1}^n \iota_i(\sigma))^2$$

$$= \rho_{a_1,\dots,a_n}(\sigma)^2.$$

The relations defining N are thus respected by  $\rho_{a_1,...,a_n}$ , so  $\rho_{a_1,...,a_n}$  yields a well-defined homomorphism.

Remark 8.9. Suppose we have a k-scheme Y with an  $N^n$  action. By a general choice of positive odd integers  $a_1, \ldots, a_n$ , the  $T_1$ -action on Y via the co-character  $t \mapsto (t^{a_1}, \ldots, t^{a_n})$  will have the same fixed point set as for  $T_n$ ; this follows from the fact that for  $t_{\text{gen}} \in T_n$  a geometric generic point over k,  $Y^{t_{\text{gen}}} = Y^{T_n}$ , and so the co-characters  $t \mapsto (t^{a_1}, \ldots, t^{a_n})$  for which  $Y^{T_1} \neq Y^{T_n}$  all have image in a fixed proper closed subscheme of  $T_n$ .

Thus, if Y has an  $N^n$ -action with 0-dimensional  $T_n$ -fixed point locus, for a general choice of positive odd integers  $a_1, \ldots, a_n$ , the N-action on Y induced by  $\rho_{a_1,\ldots,a_n}$  will also have 0-dimensional  $T_1$ -fixed point locus, and the N-action with thus be strict.

## 9. A Bott residue theorem for Borel-Moore Witt homology

We consider a codimension r regular embedding  $i: Y \to X$  in  $\mathbf{Sch}^G/k$ . Let  $\mathcal{N}_i$  be the conormal sheaf of i, giving the normal bundle  $N_i \to Y$ ,  $N_i := \mathbb{V}(\mathcal{N}_i)$ .  $N_i$  carries a canonical G-linearization, so we have the Euler class  $e_G(N_i) \in H_G^r(Y, \mathcal{W}(\det^{-1} N_i))$ .

**Lemma 9.1.** Let  $\mathcal{L}$  be an invertible sheaf on X. The map

$$i^!i_*: H^{\mathrm{B.M.}}_{G*}(Y, \mathcal{W}(i^*\mathcal{L})) \to H^{\mathrm{B.M.}}_{G*-r}(Y, \mathcal{W}(i^*\mathcal{L} \otimes \det^{-1} N_i))$$

is cap product with  $e_G(N_i) \in H_G^*(Y, \mathcal{W}(\det^{-1} N_i))$ .

*Proof.* We have the corresponding regular embedding  $i_j: Y \times^G E_j G \to X \times^G E_j G$ . Then the sheaf  $p_1^* \mathcal{N}_i$  on  $Y \times E_j G$  is the conormal sheaf of  $i \times \mathrm{Id}: Y \times E_j G \to X \times E_j G$  and the conormal sheaf  $\mathcal{N}_{i_j}$  of  $i_j$  is the sheaf constructed from  $p_1^* \mathcal{N}_i$  by descent via the G-action. Thus the result in the G-equivariant setting follows from the case  $G = \{\mathrm{Id}\}$ .

In this case, the result is a consequence of the self-intersection formula of Déglise-Jin-Khan [6, Corollary 4.2.3], a special case of the excess intersection formula [6, Proposition 4.2.2], see also the statement in Proposition 3.2.  $\Box$ 

Remark 9.2 (Invertible elements in Witt-cohomology). We briefly return to the general setting. Let  $G \subset \operatorname{GL}_n$  be a group-scheme over k, take  $X \in \operatorname{\mathbf{Sch}}^G/k$  and let  $\mathcal L$  be a G-linearized invertible sheaf on X. Products in equivariant Witt cohomology gives  $H_G^{2*}(X,\mathcal W):=\oplus_m H_G^{2m}(X,\mathcal W)$  the structure of a commutative, graded ring, and makes  $H_G^{2*}(X,\mathcal W(\mathcal L))$  a graded  $H_G^{2*}(X,\mathcal W)$ -module. Using the canonical isomorphism  $H_G^*(X,\mathcal W(\mathcal L^{\otimes 2}))\cong H_G^*(X,\mathcal W)$ , we also have a commutative graded product

$$H_G^{2*}(X, \mathcal{W}(\mathcal{L})) \times H_G^{2*}(X, \mathcal{W}(\mathcal{L})) \to H_G^{2*}(X, \mathcal{W})$$

For  $s \in H_G^{2m}(X, \mathcal{W})$ , we say a homogeneous element  $x \in H_G^{2*}(X, \mathcal{W}(\mathcal{L}))[s^{-1}]$  is invertible if there is a homogeneous element  $y \in H_G^{2*}(X, \mathcal{W}(\mathcal{L})[s^{-1}])$  with  $xy = 1 \in H^{2*}(X, \mathcal{W})[s^{-1}]$ . Clearly, a homogeneous element  $x \in H_G^{2*}(X, \mathcal{W}(\mathcal{L})[s^{-1}])$  is invertible if and only if  $x^2 \in H_G^{2*}(X, \mathcal{W})[s^{-1}]$  is invertible in the commutative ring  $H_G^{2*}(X, \mathcal{W})[s^{-1}]$ , in the usual sense.

**Lemma 9.3.** We take  $G = \operatorname{SL}_2^n$  or G = N. Let V be a G-linearized vector bundle of rank 2r on some connected  $Y \in \operatorname{\mathbf{Sch}}^G/k$ ; we suppose that the assumptions of Construction 2.7, Case 1, Case 2 or Case 3, hold, giving us the generic Euler class  $[e_G^{gen}(V)] \subset H^{4r}(BG, \mathcal{W})$ . Choose an element  $e_G^{gen}(V) \in [e_G^{gen}(V)]$ . Then  $e_G(V) \in H_G^{2r}(Y, \mathcal{W}(\det^{-1}V))$  is invertible in  $H_G^{2*}(Y, \mathcal{W}(\det^{-1}V))[e_G^{gen}(V)^{-1}]$ .

Proof. Since  $e_G(V)$  is invertible in  $H_G^{2*}(Y, \mathcal{W}(\det^{-1} \mathcal{V}))[e_G^{gen}(V)^{-1}]$  if and only if  $e_G(V \oplus V) = e_G(V)^2$  is invertible in  $H_G^{2*}(Y, \mathcal{W})[e_G^{gen}(V \oplus V)]$ , we may assume that  $\det V \cong \mathcal{L}^{\otimes 2}$  for some G-linearized invertible sheaf  $\mathcal{L}$ , and  $\det V^{gen} \cong k(\chi)^{\otimes 2}$  for some character  $\chi$  of G. It thus suffices to show that  $e_G(V)$  is invertible in  $H_G^{2*}(Y, \mathcal{W})[e_G(V^{gen})^{-1}]$ , for some choice of representative G-representation  $V^{gen}$  for the isomorphism class  $[\mathcal{V}^{gen}]$ , in case  $\det V$  and  $\det V^{gen}$  are trivial.

It follows from Construction 2.7 that, for each G-stable trivializing open subscheme  $j_{U_1}: U_1 \hookrightarrow Y$ , we have  $j_{U_1}^*e_G(V) - e_G(V^{gen}) = 0$  in  $H_G^*(U_1, \mathcal{W})$ , so  $j_{U_1}^*e_G(V)/e_G(V^{gen}) = 1$  in  $H_G^*(U_1, \mathcal{W})[e_G(V^{gen})^{-1}]$ . Let  $i: Z_1 \to Y$  be the closed complement of  $U_1$ . The long exact sequence for cohomology with support gives us a (non-unique) element  $\alpha_1 \in H_{G,Z_1}^*(Y,\mathcal{W})[e_G(V^{gen})^{-1}]$  with  $i_*(\alpha_1) = e_G(V)/e_G(V^{gen}) - 1$ . Given a second G-stable trivializing open subscheme  $j_{U_2}: U_2 \to Y$ , we have

$$\alpha_1 \cdot (j_{U_2}^*[e_G(V)/e_G(V^{gen}) - 1]) = 0 \in H_{G,Z \cap U_2}^*(U_2, \mathcal{W})[e_G(V^{gen})^{-1}]$$

so letting  $Z_2 = Y \setminus U_2$ , there is an element  $\alpha_2 \in H^*_{G,Z_1 \cap Z_2}(Y, \mathcal{W})[e_G(V^{gen})^{-1}]$  mapping to  $\alpha_1 \cdot [e_G(V)/e_G(V^{gen}) - 1]$  in  $H^*_{G,Z_1}(Y, \mathcal{W})[e_G(V^{gen})^{-1}]$ , so  $\alpha_2$  maps to  $[e_G(V)/e_G(V^{gen}) - 1]^2$  in  $H^*_G(Y, \mathcal{W})[e_G(V^{gen})^{-1}]$ . Taking  $U_1, \ldots, U_r$  a Zariski open cover of Y, we see that  $[e_G(V)/e_G(V^{gen}) - 1]^r = 0$  in  $H^*_G(Y, \mathcal{W})[e_G^{gen}(V)^{-1}]$ , so  $e_G(V)$  is invertible in  $H^*_G(Y, \mathcal{W})[e_G(V^{gen})^{-1}]$ .

Remark 9.4. If  $\mathcal{V}$  has odd rank, then  $[e_G^{gen}(V)] = 0$ , so there is nothing to prove.

For  $X \in \mathbf{Sch}^N/k$ , we have defined the closed subscheme  $|X|^N$  of X; for  $X \in \mathbf{Sch}^{\mathrm{SL}_2^n}/k$ , we set  $|X|^{\mathrm{SL}_2^n} := X^{\mathrm{SL}_2^n}$ .

**Theorem 9.5.** We take  $G = \operatorname{SL}_2^n$  or G = N. Take  $X \in \operatorname{\mathbf{Sch}}^G/k$  and let  $\mathcal{L}$  be a G-linearized invertible sheaf on X. We assume that for each connected component  $i_j : |X|_j^G \to X$  of  $|X|^G$ ,  $i_j$  is a regular embedding. Let  $N_{i_j}$  be the normal bundle of  $i_j$ ; If G = N, we assume that for each j, the hypotheses of Case 1, Case 2, or Case 3 of Construction 2.7 for  $V = N_{i_j}$  holds. We let  $e_G^{gen}(N_i)$  be the product of the Euler classes  $e_G^{gen}(N_{i_j})$ . We also assume that the N-action on X is semi-strict.

If  $G = \operatorname{SL}_2^n$ , we suppose that the G-action on  $X \setminus |X|^G$  is localizing. We also assume that either k has characteristic zero, or  $\dim_k |X|^G = 0$ .

In case  $G = SL_2^n$ , let

$$P = \prod_{i=1}^{n} e_i \dot{\prod}_{1 \le i < j \le n} e_i - e_j \in H^{* \ge 2}(\mathrm{BSL}_2^n, \mathcal{W})$$

and in case G = N, take  $P = p \cdot M \cdot e \in H^{* \geq 2}(BSL_2, W)$ , with M > 0 the integer defined in Remark 8.1 and p the exponential characteristic.

Decomposing  $H_{G*}^{\mathrm{B.M.}}(|X|^G, \mathcal{W}(i^*\mathcal{L}))$  according to the connected components  $|X|_j^G$  gives the isomorphism

$$H^{\mathrm{B.M.}}_{G*}(|X|^G, \mathcal{W}(i^*\mathcal{L}))[P^{-1}e^{gen}(N_i)^{-1}] \cong \prod_i H^{\mathrm{B.M.}}_{G*}(|X|_j^G, \mathcal{W}(i_j^*\mathcal{L}))[P^{-1}e^{gen}(N_{i_j})^{-1}]$$

 $Then \ the \ inverse \ of \ the \ isomorphism$ 

$$i_*: H^{\mathrm{B.M.}}_{G*}(|X|^G, \mathcal{W}(i^*\mathcal{L}))[P^{-1}e^{gen}(N_i)^{-1}] \xrightarrow{\sim} H^{\mathrm{B.M.}}_{G*}(X, \mathcal{W}(\mathcal{L})[P^{-1}e^{gen}_G(N_i)^{-1}]$$
 is the map

$$x \mapsto \prod_{i} i_j!(x) \cap e_G(N_{i_j})^{-1}$$

Remark 9.6. If X is in  $\mathbf{Sm}^N/k$ , then  $X^{T_1}$  is smooth, the N-action is (semi-)strict and each  $i_j:|X|_j^N\to X$  of  $|X|^N$  is a regular embedding. Moreover,  $N_{i_j}=N_{i_j}^{\mathfrak{m}}$  for each j, so we are in Case 2 of Construction 2.7, and we may apply Theorem 9.5.

Proof of Theorem 9.5. By Lemma 9.3,  $e_G(N_{i_j})$  is invertible in the localization  $H_G^*(|X|_j^G, \mathcal{W}(\det^{-1} N_{i_j}))[P^{-1}e_G^{gen}(f_G)]$  for each j. Moreover the map  $i_j^!$  is of the form

$$\begin{split} H^{\mathrm{B.M.}}_{G,*}(X,\mathcal{W}(\mathcal{L}))[P^{-1}e_G^{gen}(N_i)^{-1}] \\ \xrightarrow{i^!_j} H^{\mathrm{B.M.}}_{G,*}(|X|^G_j,\mathcal{W}(i^*_j\mathcal{L}\otimes \det N_{i_j}^{-1}))[P^{-1}e_G^{gen}(N_{i_j})^{-1}] \end{split}$$

and we have the cap product

$$\begin{split} H^{\mathrm{B.M.}}_{G,*}(|X|_{j}^{G}, \mathcal{W}(i_{j}^{*}\mathcal{L} \otimes \det N_{i_{j}}^{-1}))[P^{-1}e_{G}^{gen}(N_{i_{j}})^{-1}] \\ & \times H^{-*}_{G}(|X|_{j}^{G}, \mathcal{W}(\det^{-1}N_{ij}))[P^{-1}e_{G}^{gen}(N_{i_{j}})^{-1}] \\ & \stackrel{\cap}{\to} H^{\mathrm{B.M.}}_{G,*}(|X|_{j}^{G}, \mathcal{W}(i_{j}^{*}\mathcal{L}))[P^{-1}e_{G}^{gen}(N_{i_{j}})^{-1}] \end{split}$$

so the formula in the statement of the Theorem makes sense.

Given  $x \in H_{G,*}^{\mathrm{B.M.}}(X, \mathcal{W}(\mathcal{L})[P^{-1}e^{gen}(N_i)^{-1}]$ , it follows from Theorem 6.7 (for  $G = \mathrm{SL}_2^n$ ) or Theorem 8.6 (for G = N), that there are

$$y_j \in H_{G,*}^{\mathrm{B.M.}}(|X|_j^G, \mathcal{W}(i_j^*\mathcal{L}))[P^{-1}e_G^{gen}(N_{i_j})^{-1}]$$

with

$$x = \sum_{j} i_{j*}(y_j)$$

But then by Lemma 9.1,

$$i_j'(x) = i_j' i_{j*}(y_j) = y_j \cap e_G(N_{i_j}).$$
 so  $y_j = i_j'(x) \cap e_G(N_{i_j})^{-1}$ .  $\square$ 

Remark 9.7. The main difference between this localization theorem and the classical one for a torus action is that the product  $P \cdot e(\mathcal{N}_i)_0$  might be zero, and in that case, the conclusion tells us nothing.

Remark 9.8. Suppose that  $|X|^G$  is a finite set of closed points of X, and that  $\pi: X \to \operatorname{Spec} k$  is smooth and proper over k. Then all the hypotheses of Theorem 9.5 are satisfied. Suppose in addition that we have a G-vector bundle V of rank  $d:=\dim_k X$  on X and a G-equivariant isomorphism  $\rho: \det^{-1} V \xrightarrow{\sim} \omega_{X/k} \otimes \mathcal{M}^{\otimes 2}$  for some invertible sheaf  $\mathcal{M}$  with G-linearization on X. Then via  $\pi_*$ , we have the equivariant degree

$$\pi_*(e_G(V)) \in H^0(BG, \mathcal{W}),$$

where

$$e_G(V) \in H_{G,0}^{\mathrm{B.M.}}(|X|^G, \mathcal{W}) = \bigoplus_{x \in |X|^G} H_{G,0}^{\mathrm{B.M.}}(x, \mathcal{W})$$

is the G-equivariant Euler class. We also have

$$\pi_*(e(V)) \in H^0(\operatorname{Spec} k, \mathcal{W}) = W(k),$$

where e(V) is the usual Euler class; these both depend on the choice of  $\rho$ .

Using our version of the Bott residue theorem, Theorem 9.5, we can compute  $\pi_*(e_G(V))$  in a suitable localization of  $H^0(BG, \mathcal{W})$  by taking the appropriate trace map applied to  $e_G(i^*V)/e(N_i)$ ). In the case G=N, for the points  $x\in X^{T_1}_{\mathrm{fr}}\subset |X|^N$ , we will need to use the computation of the push-forward given in Corollary 10.14.

Example 9.9 (Some trivial examples). 1. Let  $SL_2^n$  act on  $\mathbb{P}^{2n-1}$  by

$$(g_1, \dots, g_n)(x_0, x_1, \dots, x_{2n-2}, x_{2n-1})$$

$$= ((g_1^t \cdot (x_0, x_1)^t)^t, (g_2^t \cdot (x_2, x_3)^t)^t, \dots, (g_n^t \cdot (x_{2n-2}, x_{2n-1})^t)^t)$$

or on  $\mathbb{P}^{2n}$  by the same formula with no action on the last coordinate  $x_{2n}$ . For  $X:=\mathbb{P}^{2n-1}$ , we have  $X^G=\emptyset$  and for  $X=\mathbb{P}^{2n}$  with have  $X^G=\{(0,\ldots,0,1)\}$ , with normal bundle the fundamental representation  $F_1\oplus\ldots\oplus F_n$ . We give the tangent bundles  $T_{\mathbb{P}^{2n-1}}$  and  $T_{\mathbb{P}^{2n}}$  the induced G action and apply the Theorem to compute

$$\pi_*(e_G(T_{\mathbb{P}^{2n-1}})) = 0 \in H^0(BG, \mathcal{W})[e^{-1}]$$
  
 $\pi_*(e_G(T_{\mathbb{P}^{2n}})) = 1 \in H^0(BG, \mathcal{W})[e^{-1}]$ 

Since the localization map  $H^0(BG, \mathcal{W}) \to H^0(BG, \mathcal{W})[e^{-1}]$  is injective, this implies

$$\pi_*(e_G(T_{\mathbb{P}^{2n-1}})) = 0 \in H^0(BG, \mathcal{W})$$

$$\pi_*(e_G(T_{\mathbb{P}^{2n}})) = 1 \in H^0(BG, \mathcal{W})$$

and restricting to the fiber over the base-point of BG recovers the known results

$$\pi_*(e(T_{\mathbb{P}^{2n-1}})) = 0 \in W(k)$$
$$\pi_*(e(T_{\mathbb{P}^{2n}})) = 1 \in W(k).$$

2. We let G act on  $\mathbb{P}^{2n-1}$  or  $\mathbb{P}^{2n}$  as in (1) and consider the induced action on a Grassmann variety Gr(m,2n) or Gr(m,2n+1). Let  $e_{j-1}$  denotes the jth standard basis vector of  $k^{2n}$  or  $k^{2n+1}$  and let  $F_j$  denote the span of  $e_{2j-2}, e_{2j-1}$ , considered

as a representation of G via the jth factor  $\mathrm{SL}_2^{(j)}$ . It is clear that the G-invariant linear subspaces of  $\mathbb{P}^{2n-1}$  are exactly those of the form  $\mathbb{P}(F_{i_1} \oplus \ldots F_{i_r}) \cong \mathbb{P}^{2r-1}$ ; for  $\mathbb{P}^{2n}$ , we have all these together with those of the form  $\mathbb{P}(F_{i_1} \oplus \ldots F_{i_r} \oplus k \cdot e_{2n})$ , where  $e_j$  denotes the jth standard basis vector of  $k^{2n}$ .

Thus  $Gr(m, 2n)^G = \emptyset$  for m odd. Letting  $x_{i_1, \dots, i_r} = F_{i_1} \oplus \dots \oplus F_{i_r} \in Gr(2r, 2n)^G$ , the normal bundle  $N_{x_i, \dots, i_r}$  is

$$N_{x_{i_1,\ldots,i_r}} = (F_{i_1} \oplus \ldots \oplus F_{i_r})^{\vee} \otimes F_{j_1} \oplus \ldots \oplus F_{j_{n-r}}$$

where  $\{j_1,\ldots,j_{n-r}\}$  is the complement of  $\{i_1,\ldots,i_r\}$  in  $\{1,\ldots,n\}$ . For  $i\neq j$ ,  $F_i^\vee\otimes F_j\cong \operatorname{Sym}^{\epsilon_1,\ldots,\epsilon_n}$  with  $\epsilon_\ell=1$  for  $\ell=i,j,\ \epsilon_\ell=0$  otherwise, the Euler class  $e(N_{x_{i_1,\ldots,i_r}})$  is non-zero, giving

$$\pi_*(e(T_{Gr(2r,2n)})) = \binom{n}{r} \cdot 1 \in W(k)[e_1, \dots, e_n]$$
$$\pi_*(e(T_{Gr(2r+1,2n)})) = 0 \in W(k)[e_1, \dots, e_n]$$

For Gr(2r, 2n+1), we have the same description of the G-invariant even-dimensional subspaces. The normal bundle is

$$N_{x_{i_1,\ldots,i_r}} = (F_{i_1} \oplus \ldots \oplus F_{i_r})^{\vee} \otimes (F_{j_1} \oplus \ldots \oplus F_{j_{n-r}} \oplus k \cdot e_{2n})$$

giving terms of the form  $\operatorname{Sym}^{\epsilon_1,\dots,\epsilon_n}$  with  $0 \leq \epsilon_j \leq 1$  and with  $1 \leq \sum_{\ell} \epsilon_{\ell} 2$ . Thus  $e(N_{x_{i_1,\dots,i_r}}) \neq 0$  and

$$\pi_*(e(T_{\operatorname{Gr}(2r,2n+1)})) = \binom{n}{r} cdot 1 \in W(k)[e_1,\ldots,e_n]$$

In the case of odd dimensional subspaces, the G-invariant ones are all of the form  $F_{i_1} \oplus \ldots \oplus F_{i_r}) \oplus k \cdot e_{2n}$ , giving a normal bundle of the form

$$N_{x_{i_1,\ldots,i_r}} = (F_{i_1} \oplus \ldots \oplus F_{i_r} \oplus k \cdot e_{2n})^{\vee} \otimes (F_{j_1} \oplus \ldots \oplus F_{j_{n-r}})$$

Thus  $e(N_{x_{i_1,\ldots,i_r}}) \neq 0$  and

$$\pi_*(e(T_{Gr(2r+1,2n+1)})) = \binom{n}{r} \cdot 1 \in W(k)[e_1, \dots, e_n]$$

## 10. Equivariant cohomology of twisted quadrics

We turn to the study of the case of a 0-dimensional orbit for N. Other than the clear case of a fixed point with trivial action on the residue field, or the case of an  $N/T_1$ -induced action of a trivial action, the basic example is the case of an N-action is a finite integral k-scheme z with residue field k(z) a degree two extension of k, and with  $\sigma \in N$  acting as Galois conjugation. Since the characteristic is assumed to be different from 2, we can write  $k(z) = k(\sqrt{a})$  for some  $a \in k^{\times} \setminus k^{\times 2}$ , with  $\sigma^*(\sqrt{a}) = -\sqrt{a}$ . We let  $X_a := \operatorname{Spec} k(\sqrt{a}) \times^N \operatorname{SL}_2$ . Thus

(10.1) 
$$H_N^*(z, \mathcal{W}) \cong H_{\operatorname{SL}_2}^*(X_a, \mathcal{W})$$

Let F be the representation of  $\operatorname{SL}_2$  corresponding to the inclusion  $\operatorname{SL}_2 \subset \operatorname{GL}_2$ , giving (right)  $\operatorname{SL}_2$ -action on the projective line  $\mathbb{P}^1 := \mathbb{P}(F)$  and the projective plane  $\mathbb{P}^2 := \mathbb{P}(\operatorname{Sym}^2 F)$ . To fix the notation, we give F the basis  $e_1, e_2$ , and  $\operatorname{Sym}^2 F$  the basis  $e_1^2, e_1 e_2, e_2^2$ . We let  $X_0, X_1$  be the dual basis to  $e_1, e_2$  and identify the dual of  $\operatorname{Sym}^2 F$  with the space of quadratic forms on F, with corresponding basis  $T_0, T_1, T_2$ , via

$$q(x_0, x_1) = t_0 x_0^2 + t_1 x_0 x_1 + t_2 x_1^2.$$

In other words, we have

The squaring map  $F \to \operatorname{Sym}^2 F$  gives the  $\operatorname{SL}_2$ -equivariant closed immersion  $\mathbb{P}^1 \to \mathbb{P}^2$ ,  $f(x_0, x_1) = (x_0^2, 2x_0x_1, x_1^2)$  with image the conic D defined by  $Q := T_1^2 - 4T_0T_2$ . We have as well the quadratic form on  $(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F)) \times F$  over  $\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F)$ , defined by the matrix

$$Q' := \begin{pmatrix} -2T_0 & -T_1 \\ -T_1 & -2T_2 \end{pmatrix},$$

with discriminant Q. Following Lemma 11.6, Q' defines an  $SL_2$ -invariant section  $[Q'] \in H^0(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(\operatorname{Sym}^2 F)}).$ 

We have the 2-1 map  $p: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$  defined by  $p((x_0, x_1), (y_0, y_1)) = (x_0y_0, x_0y_1 + x_1y_0, x_1y_1)$ . Via the (modified) Segre embedding  $i: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ ,  $i((x_0, x_1), (y_0, y_1)) = (x_0y_0, x_0y_1 + x_1y_0, x_1y_1, x_0y_1 - x_1y_0)$ ,  $i(\mathbb{P}^1 \times \mathbb{P}^1)$  is the quadric defined by  $T_3^2 - T_1^2 + 4T_0T_2$ , the map p is given by the projection  $\pi: \mathbb{P}^3 \setminus \{(0, 0, 0, 1)\} \to \mathbb{P}^2$ ,  $(T_0, T_1, T_2, T_3) \mapsto (T_0, T_1, T_2)$ , and defines a 2-1 cover, ramified over D.

We embed  $\mathbb{P}^2 := \mathbb{P}(\operatorname{Sym}^2 F)$  in  $\mathbb{P}^3 := \mathbb{P}(\operatorname{Sym}^2 F \oplus k)$  by  $(t_0, t_1, t_2) \mapsto (t_0, t_1, t_2, 0)$ . and let  $\operatorname{SL}_2$  act on  $\mathbb{P}^3$  via its action on  $\mathbb{P}^2$ , with trivial action on the last coordinate  $T_3$ . Via this embedding, we consider D as a curve in  $\mathbb{P}^3$ . We let  $\bar{X}_a$  denote the twisted form of  $\mathbb{P}^1 \times \mathbb{P}^1$  defined by the  $\mathbb{Z}/2$ -action on

We let  $X_a$  denote the twisted form of  $\mathbb{P}^1 \times \mathbb{P}^1$  defined by the  $\mathbb{Z}/2$ -action on  $\operatorname{Spec} k(\sqrt{a}) \times \mathbb{P}^1 \times \mathbb{P}^1$  with generator acting by conjugation on  $k(\sqrt{a})$  and by exchange of factors on  $\mathbb{P}^1 \times \mathbb{P}^1$ . The diagonal (right) action of  $\operatorname{SL}_2$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  gives an  $\operatorname{SL}_2$ -action on  $\bar{X}_a$ . The map  $p_{k(\sqrt{a})}$  defines by descent the finite morphism  $p_a: \bar{X}_a \to \mathbb{P}^2$  over k.

**Lemma 10.1.** 1. The k-scheme  $\bar{X}_a$  is  $SL_2$ -equivariantly isomorphic to the quadric in  $\mathbb{P}^3$  defined by  $Q_a := T_3^2 - a(T_1^2 - 4T_0T_2)$ , with  $p_a$  equal to the restriction of  $\pi$  to  $V(Q_a)$ .

2. The k-scheme  $X_a$  is  $\operatorname{SL}_2$ -equivariantly isomorphic to the complement of  $D \subset \mathbb{P}^2 \subset \mathbb{P}^3$  in  $V(Q_a)$ .

Proof. (1) Let  $k(\sqrt{a})[\mathbb{P}^1 \times \mathbb{P}^1]$  denote the bi-homogeneous coordinate ring of  $\mathbb{P}^1 \times \mathbb{P}^1$ . The invariants of bi-degree  $\{(n,n)\}_{n\geq 0}$  of the  $\mathbb{Z}/2$ -action on  $k(\sqrt{a})[\mathbb{P}^1 \times \mathbb{P}^1]$  is the k-sub-algebra of  $k(\sqrt{a})[\mathbb{P}^1 \times \mathbb{P}^1]$  generated by  $t_0 := x_0y_0, t_2 := x_1y_1, t_1 := x_0y_1 + x_1y_0$  and  $t_3 := \sqrt{a}(x_0y_1 - x_1y_0)$ . This defines the  $\mathrm{SL}_2$ -equivariant embedding of  $\bar{X}_a$  in  $\mathbb{P}^3$  with image the quadric defined by  $Q_a$ . One easily sees that  $p_a : \bar{X}_a \to \mathbb{P}^2$  goes over to the restriction of  $\pi$  to  $V(Q_a)$ .

Let  $T \subset GL_2$  be the diagonal torus. Then  $\mathbb{G}_m \backslash SL_2 \cong T \backslash GL_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$  by

$$\begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \mapsto ((x_0, x_1), (y_0, y_1))$$

The action by  $\sigma \in \operatorname{SL}_2$  becomes the exchange of factors. Thus  $X_a$  is isomorphic to the quotient of  $\operatorname{Spec} k(\sqrt{a}) \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$  by the  $\mathbb{Z}_2$ -action used in the definition of  $\bar{X}_a$ . Sending  $((x_0, x_1), (y_0, y_1))$  to  $(t_0, t_1, t_2, t_3)$  sends  $\Delta$  to  $D \subset \mathbb{P}^2 \subset \mathbb{P}^3$  and gives the desired isomorphism of  $X_a$  with  $\bar{X}_a \setminus D$ .

We henceforth identify  $\bar{X}_a$  with  $V(Q_a)$  and  $X_a$  with  $V(Q_a) \setminus D$ .

As a quadric in  $\mathbb{P}^3$  with rational point  $p:=(1,2,1,0), \bar{X}_a$  is birationally isomorphic to  $\mathbb{P}^2$  via projection from p. Using the explicit rational map  $\pi:\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ ,  $(t_0,\ldots,t_3)\mapsto (t_0+t_2-t_1,t_2-t_0,t_3)=(x_0,x_1,x_2)$ , the point p gets blown up to the line  $X_0=0$ , and the two tangent lines to  $\bar{X}_a$  at p, defined by  $T_3^2-a(T_2-T_0)^2=$ 

 $T_0+T_2-T_1=0$ , get blown down to the point  $y\in\mathbb{P}^2$  defined by  $X_0=0$ ,  $X_2^2=aX_1^2$ . We write  $\sqrt{a}$  for the restriction to y of  $X_2/X_1$ , giving the canonical identification  $k(y)=k(\sqrt{a})$ . Similarly, the curve  $D=\bar{X}_a\cap(T_3=0)$  gets sent isomorphically to the line  $X_2=0$ 

Let  $C_p \subset \bar{X}_a$  be the closed subscheme corresponding to  $y \in \mathbb{P}^2$ , that is,  $C_p = \bar{X}_a \cap (T_0 + T_2 - T_1 = 0)$ . The normalization of  $C_p$  is thus isomorphic to  $\mathbb{P}^1_{k(y)}$ ; if we take coordinates with  $\infty$  lying over p, then  $\mathbb{P}^1_{k(y)} \to \ell_y$  is an isomorphism over  $C_p \setminus \{p\}$ , and with  $k(\infty) \supset k(p)$  the field extension  $k(y) \supset k$ .

For an arbitrary point  $q \in D \subset \bar{X}_a$ , let  $T_q \bar{X}_a \subset \mathbb{P}^3$  be the (projective) tangent plane to  $\bar{X}_a$  at q and let  $C_q = T_q \bar{X}_a \cap \bar{X}_a$ . Noting that  $T_p \bar{X}_a$  is the plane  $T_0 + T_2 - T_1 = 0$ , this agrees with our previous notation in case q = p.

Take  $q \neq p$ . The birational map  $g: \bar{X}_a \dashrightarrow \mathbb{P}^2$  is defined on all of  $C_q$ , q is sent to a point q' := g(q) on the line  $X_2 = 0$  and  $g(C_q)$  is the cone  $C_{q'}(y)$  over y with vertex q'. For  $q \in D(k) \setminus \{p\}$ , we henceforth identify  $C_q$  with the cone  $C_{q'}(y)$  via g.

Thus, for  $q \neq p$  a k-point of C the normalization of  $C_q$  is isomorphic to  $\mathbb{P}^1_{k(\sqrt{a})}$ , the canonical map  $\rho_q : \mathbb{P}^1_{k(\sqrt{a})} \to C_q = C_{q'}(y)$  is an isomorphism over  $C_q \setminus \{q\}$  and the inverse image of q is a single closed point  $0 \in \mathbb{P}^1_{k(\sqrt{a})}(k(\sqrt{a}))$  with  $k(0) = k(\sqrt{a})$ .

As a preliminary, we recall the well-known computation of the Witt cohomology for projective spaces.

**Lemma 10.2.** For n > 0, we have

$$H^{i}(\mathbb{P}^{n}_{k}, \mathcal{W}) = \begin{cases} W(k) & \text{for } i = 0\\ 0 & \text{for } 0 < i < n\\ 0 & \text{for } i = n \text{ even}\\ W(k) & \text{for } i = n \text{ odd.} \end{cases}$$

and

$$H^{i}(\mathbb{P}^{n}_{k}, \mathcal{W}(-1)) = \begin{cases} 0 & \text{for } 0 \leq i < n \\ 0 & \text{for } i = n \text{ odd} \\ W(k) & \text{for } i = n \text{ even.} \end{cases}$$

The identity  $W(k) = H^0(\mathbb{P}^n, \mathcal{W})$  is induced by pull-back from Spec k, and in cases  $H^n(\mathbb{P}^n, \mathcal{W}) = W(k)$  or  $H^n(\mathbb{P}^n, \mathcal{W}(-1)) = W(k)$ , the identity is induced by pushforward for the inclusion of a k-point.

*Proof.* Use the localization sequence for  $H^*(-, \mathcal{W}(m))$  for the closed immersion  $i: \mathbb{P}^{n-1} \to \mathbb{P}^n$  with open complement  $j: \mathbb{A}^n \to \mathbb{P}^n$ , and show that the boundary map

$$W(k) = H^0(\mathbb{A}^n, \mathcal{W}) \to H^0(\mathbb{P}^{n-1}, \mathcal{W}(m-1))$$

is an isomorphism for m odd and the zero map for m even and n=1. This proves the result for n=1 and the result in general follows by induction.

We begin by computing the Witt sheaf cohomology of  $\bar{X}_a$ .

**Lemma 10.3.** 1. Pullback by the projection  $\pi_{\bar{X}_a}: \bar{X}_a \to \operatorname{Spec} k$  defines an isomorphism  $H^0(\bar{X}_a, \mathcal{W}) \cong W(k)$ .

2. Let  $q \in D(k)$  be a k-point and let  $p_q : \mathbb{P}^1_{k(\sqrt{a})} \to \bar{X}_a$  denote the composition  $\mathbb{P}^1_{k(\sqrt{a})} \to C_q \hookrightarrow \bar{X}_a$ . Then there is a well defined map

$$p_{q*}: H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^1(\bar{X}_a, \mathcal{W})$$

that induces an isomorphism  $W(k(\sqrt{a})) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W}) \cong H^1(\bar{X}_a, \mathcal{W}).$ 

3. The inclusion  $i_p: p \to \bar{X}_a$  induces an isomorphism

$$i_{p*}: W(k) = W(k(p)) \xrightarrow{\sim} H^2(\bar{X}_a, \mathcal{W})$$

4. For 
$$i \geq 3$$
,  $H^{i}(\bar{X}_{a}, \mathcal{W}) = 0$ .

*Proof.* The assertion (4) follows from the fact that the Rost-Schmid complex for W on  $\bar{X}_a$ , whose *i*th cohomology computes  $H^i(\bar{X}_a, W)$ , is supported in degrees [0, 2].

Suppose we have a smooth surface Y over k, a closed point  $w \in Y$  and let  $Y \to Y$  be the blow-up of Y at w, with exceptional divisor  $E \cong \mathbb{P}^1_w$ . Let  $V = Y \setminus \{w\}$ . We have the exact localization sequences

$$\dots \to H^{i-2}(w, \mathcal{W}) \xrightarrow{i_{w*}} H^i(Y, \mathcal{W}) \to H^i(V, \mathcal{W}) \xrightarrow{\partial} H^{i-1}(w, \mathcal{W}) \to \dots$$

and

$$\ldots \to H^{i-1}(\mathbb{P}^1_w, \mathcal{W}(-1)) \xrightarrow{i_{\mathbb{P}^1_w*}} H^i(\tilde{Y}, \mathcal{W}) \to H^i(V, \mathcal{W}) \xrightarrow{\partial} H^i(\mathbb{P}^1_w, \mathcal{W}(-1)) \to \ldots$$

The definition of  $i_{w*}$  requires a choice of isomorphism  $\det^{-1} \mathfrak{m}/\mathfrak{m}^2 \cong k(w)$  and the definition of  $i_{\mathbb{P}^1_w}$  similarly relies on a choice of isomorphism  $N_{\mathbb{P}^1_w} \tilde{Y} \cong \mathcal{O}_{\mathbb{P}^1_w}(-1)$ ; changing one of these choices alters the corresponding map in cohomology by multiplication by an element  $\langle \lambda \rangle \in W(k(w))$ , for some  $\lambda \in k(w)^{\times}$ .

Since  $H^j(\mathbb{P}^1_w, \mathcal{W}(-1)) = 0$  for all j, we have  $H^i(\tilde{Y}, \mathcal{W}) \cong H^i(V, \mathcal{W})$  for all i. From the first sequence, we have  $H^0(Y, \mathcal{W}) \cong H^0(V, \mathcal{W})$ , so the pullback  $H^0(Y, \mathcal{W}) \to H^0(\tilde{Y}, \mathcal{W})$  is an isomorphism.

We recall that  $H^i(\mathbb{P}^2, \mathcal{W}) = 0$  for i > 0 and that the pullback  $\pi_{\mathbb{P}^2}^* : W(k) = H^0(\operatorname{Spec} k, \mathcal{W}) \to H^0(\mathbb{P}^2, \mathcal{W})$  is an isomorphism. The comments in the previous paragraph thus prove (1).

For (3), we know that  $H^2(\mathbb{P}^2, \mathcal{W}) = 0$ . From the above, this implies that  $H^2(\mathbb{P}^2 \setminus \{y\}, \mathcal{W}) = 0$ , that  $H^2(\mathrm{Bl}_y\mathbb{P}^2, \mathcal{W}) = 0$ ; since  $\mathrm{Bl}_p\bar{X}_a$  is isomorphic to  $\mathrm{Bl}_y\mathbb{P}^2$  for a certain closed point y, we see that  $H^2(\bar{X}_a \setminus \{p\}, \mathcal{W}) = 0$ . Since  $\bar{X}_a$  is a quadric in  $\mathbb{P}^3$ , the canonical sheaf  $\omega_{\bar{X}_a}$  is isomorphic to  $\mathcal{O}_{\bar{X}_a}(-2)$ , hence a square. This gives us a well-defined push-forward map  $i_{p*}: W(k) = H^0(p, \mathcal{W}) \to H^2(\bar{X}_a, \mathcal{W})$ , and the corresponding localization sequence shows that  $i_{p*}$  is surjective. Again, since  $\omega_{\bar{X}_a}$  is canonically isomorphic to a square, the structure map  $\pi_{\bar{X}_a}: \bar{X} \to \mathrm{Spec}\,k$  induces a push-forward  $\pi_{\bar{X}_{a*}}: H^2(\bar{X}_a, \mathcal{W}) \to H^0(\mathrm{Spec}\,k, \mathcal{W}) = W(k)$ , with  $\pi_{\bar{X}_{a*}} \circ i_{p*}$  an isomorphism. Thus  $i_{p*}$  is an isomorphism.

For (2), we may take  $q \in D(k) \setminus \{p\}$ ; the result for q = p will then follow by applying a suitable translation by the  $\operatorname{SL}_2$ -action. Identifying the normalization of  $C_q$  with  $\mathbb{P}^1_{k(\sqrt{a})}$ , we have the birational map  $\rho_q : \mathbb{P}^1_{k(\sqrt{a})} \to C_{q'}(y)$  which is an isomorphism over  $C_{q'}(y) \setminus \{q'\}$ . Taking  $\infty \in \mathbb{P}^1_{k(\sqrt{a})}$  to be  $\rho^{-1}(y)$ , the inclusion  $C_{q'}(y) \subset \mathbb{P}^2$  gives the finite morphism  $\rho_0 : \mathbb{A}^1_{k(\sqrt{a})} = \mathbb{P}^1_{k(\sqrt{a})} \setminus \{\infty\} \to \mathbb{P}^2 \setminus \{y\}$ . As  $\rho_0^{-1}(X_0 = 0) = \emptyset$ , we have the isomorphism  $\phi_1 : \rho_0^*(\omega_{\mathbb{P}^2}) \cong \mathcal{O}_{\mathbb{A}^1_{k(\sqrt{a})}}$ , giving us the push-forward map

$$\rho_{0*}: W(k(\sqrt{a})) = H^0(\mathbb{A}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^1(\mathbb{P}^2 \setminus \{y\}, \mathcal{W}).$$

As above, this push-forward depends on the choice of the isomorphism  $\phi_1$ , that is, up to multiplication by a quadratic form  $\langle \lambda \rangle$ ,  $\lambda \in k(\sqrt{a})$ .

We have the boundary map in the Rost-Schmid complex

$$\partial_y: H^0(C_{q'}(y)\setminus\{q',y\}, \mathcal{W}(\det N_{C_{q'}(y)\setminus\{q',y\}})) \to H^0(y, \mathcal{W}(\det N_y))$$

which gives the map

$$\partial_y \circ \rho_{0*}: W(k(\sqrt{a})) = H^0(\mathbb{A}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^0(y, \mathcal{W}(\det N_y)) = W(k(\sqrt{a})).$$

Choosing a (quadratic) defining equation g for  $C_{q'}(y)$  and using g and  $X_2/X_1$  for the defining equations for y, we find that  $\partial_y \circ \rho_{0*}$  is an isomorphism. On the other hand, the localization sequence for  $\mathbb{P}^2\setminus\{y\}\subset\mathbb{P}^2$  gives the isomorphism

$$H^1(\mathbb{P}^2 \setminus \{y\}, \mathcal{W}) \cong H^0(y, \mathcal{W}(\det N_y))$$

again, via a boundary map. This shows that the map  $\rho_{0*}: H^0(\mathbb{A}^1_{k(\sqrt{\rho_0})}, \mathcal{W}) \to$  $H^1(\mathbb{P}^2 \setminus \{y\}, \mathcal{W})$  is an isomorphism.

Next, we look on the blow-up  $\mathrm{Bl}_{\nu}\mathbb{P}^2\to\mathbb{P}^2$ . As we have seen above, the restriction

map  $H^1(\mathrm{Bl}_y\mathbb{P}^2,\mathcal{W}) \to H^1(\mathbb{P}^2\setminus\{y\},\mathcal{W})$  is an isomorphism. The map  $\rho:\mathbb{P}^1_{k(\sqrt{a})}\to\mathbb{P}^2$  lifts uniquely to a map  $\tilde{\rho}:\mathbb{P}^1_{k(\sqrt{a})}\to\mathrm{Bl}_y\mathbb{P}^2$ . In this case, we have an isomorphism

(10.2) 
$$\phi_2: \tilde{\rho}^*(\omega_{\mathrm{Bl}_y\mathbb{P}^2}) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1_{k(\sqrt{a})}}(-2) \cong \omega_{\mathbb{P}^1_{k(\sqrt{a})}},$$

so we have a push-forward

$$\tilde{\rho}_*: W(k(\sqrt{a})) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^1(\mathrm{Bl}_y\mathbb{P}^2, \mathcal{W})$$

Since the restriction of  $\tilde{\rho}_*$  over  $\mathbb{P}^2 \setminus y \subset \mathrm{Bl}_u \mathbb{P}^2$  is our isomorphism  $\rho_{0*}$ , we see that  $\tilde{\rho}_*$  is an isomorphism as well, depending as before on the choice of  $\phi_2$ .

Finally, we map down to  $\bar{X}_a$  by the blow-down map  $\psi: \mathrm{Bl}_v \mathbb{P}^2 \to \bar{X}_a$ . As the exceptional divisor E of  $\psi$  is the proper transform of the line  $X_0 = 0$ , we have  $\tilde{\rho}^{-1}(E) = \emptyset$ . The map  $p_q : \mathbb{P}^1_{k(\sqrt{a})} \to \bar{X}_a$  is just the map  $\psi \circ \tilde{\rho}$ ; since  $\psi$  is an isomorphism in a neighborhood of the image of  $\tilde{\rho}$ , we have a well-defined pushforward map

$$p_{q*}: W(k(\sqrt{a})) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^1(\bar{X}_a, \mathcal{W}).$$

Using our two exact sequences above, the fact that  $\tilde{\rho}_*$  is an isomorphism implies that  $p_{q*}$  defines an isomorphism after composing with the restriction map  $j^*$ :  $H^1(\bar{X}_a, \mathcal{W}) \to H^1(\bar{X}_a \setminus \{p\}, \mathcal{W})$  for the inclusion  $j : \bar{X}_a \setminus \{p\} \to \bar{X}_a$ . The proof will be finished once we show that  $j^*$  an isomorphism.

For this, we have the exact sequence

$$0 \to H^1(\bar{X}_a, \mathcal{W}) \xrightarrow{j^*} H^1(\bar{X}_a \setminus \{p\}, \mathcal{W}) \xrightarrow{\partial} H^0(p, \mathcal{W}) \xrightarrow{i_*} H^2(\bar{X}_a, \mathcal{W}) \to \dots$$

We have already shown that  $i_*$  is an isomorphism, so  $j^*$  is an isomorphism, as desired.

Next, we use the Leray spectral sequence to compute the SL<sub>2</sub>-equivariant cohomology of  $X_a$ . As a preliminary to the computation, we recall the computation of  $H^*_{\mathrm{SL}_2}(\mathbb{P}(F),\mathcal{W})$  discussed in [18, Lemma 5.1] and describe the push-forward and pullback maps in Witt cohomology for the map  $i:D\to \bar{X}_a$ . Let  $\iota:W(k)\to$  $W(k(\sqrt{a}))$  be the base-change map and let  $\operatorname{Tr} = \operatorname{Tr}_{k(\sqrt{a})/k} : W(k(\sqrt{a})) \to W(k)$  be the trace map for the finite separable extension  $k \subset k(\sqrt{a})$ .

To make the computation of the Leray spectral sequence more concrete, we use the identification  $\bar{X}_a = V(Q_a) \subset \mathbb{P}^3(T_0, T_1, T_2, T_3)$  given by Lemma 10.1, with  $D = \bar{X}_a \cap (T_3 = 0)$  and we fix our choice of point  $q \in D$  by taking q = (1, 0, 0, 0). We fix our choice of the isomorphism  $\phi_2$  (10.2) as follows. The tangent plane to  $\bar{X}_a$  at q is then the plane  $T_2 = 0$ . We have the relative dualizing sheaf  $\omega_{p_q}$  of the map  $p_q : \mathbb{P}^1_{k(\sqrt{a})} \to \bar{X}_a$ .

Let g be the rational function on  $\bar{X}_a$ ,  $g := T_2/T_3$ . Since  $q = \bar{X}_a \cap (T_2 = T_3 = 0)$ , g is a defining equation for  $C_q \setminus \{q\}$  in  $\bar{X}_a \setminus D$ , and thus  $p_q^*(dg)$  gives a generating section of  $\omega_{p_q}$  over  $\mathbb{P}^1_{k(\sqrt{a})} \setminus \{0\}$ . We claim that  $p_q^*(dg)$  extends to a generating section of  $\omega_{p_q}$  over  $\mathbb{P}^1_{k(\sqrt{a})}$ , which will then give us our isomorphism  $\phi_2$ .

To verify our claim, we may take the base-extension over  $k(\sqrt{a})$ , and consider the induced map  $\tilde{p}_q: \mathbb{P}^1_{k(\sqrt{a})} \otimes_k k(\sqrt{a}) \to \bar{X}_a \otimes_k k(\sqrt{a}) = \mathbb{P}^1 \times \mathbb{P}^1_{k(\sqrt{a})}.$   $C_q \otimes_k k(\sqrt{a}) \subset \mathbb{P}^1 \times \mathbb{P}^1_{k(\sqrt{a})}$  is the curved defined by  $T_2 = x_1y_1$ , and thus the map  $\tilde{p}_q$  becomes the disjoint union of the two inclusions  $i_1: (1,0) \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ ,  $i_2: \mathbb{P}^1 \times (1,0) \to \mathbb{P}^1 \times \mathbb{P}^1$ . The rational function g pulls back to the rational function  $h:=x_1y_1/\sqrt{a}(x_0y_1-x_1y_0)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ , with

$$dh = \frac{1}{\sqrt{a(x_0y_1 - x_1y_0)}} \cdot (x_1dy_1 + y_1dx_1) + (x_1y_1) \cdot d(\frac{1}{\sqrt{a(x_0y_1 - x_1y_0)}})$$

Thus

$$i_1^*dh = \frac{-1}{\sqrt{a}}d(x_1/x_0), \ i_2^*dh = \frac{1}{\sqrt{a}}d(y_1/y_0)$$

which verifies our claim.

Remark 10.4. With q = (1, 0, 0, 0) as above, we note that the isotropy group of q in  $SL_2$  is the Borel subgroup of lower triangular matrices, and that the matrix

$$\alpha := \begin{pmatrix} t & 0 \\ b & t^{-1} \end{pmatrix}$$

acts on  $T_2$  by multiplication by  $t^{-2}$ . Since a square factor does not affect the pushforward map, we have a well-defined,  $\operatorname{SL}_2$  invariant isomorphism  $\rho: W(k(\sqrt{a}) \xrightarrow{\sim} H^1(\bar{X}_a, \mathcal{W}))$ , defined for the choice of point q = (1, 0, 0, 0) using the generating section  $p_q^*dg$  as described above, and for a general  $q \in D(k)$ , translating everything by an element  $\alpha \in \operatorname{SL}_2(k)$  sending (1, 0, 0, 0) to q. For instance, taking q = (1, 2, 1, 0), we use the function  $g := (T_0 + T_2 - T_1)/T_3$  to define the generating section for  $\omega_{p_q}$ .

We need similar choices to give a uniquely defined and compatible push-forward maps for the inclusions  $i_p: p \to \bar{X}_a, i_p^D: p \to D, i_D: D \to \bar{X}_a$  and the projections  $\pi_{\bar{X}_a}: \bar{X}_a \to \operatorname{Spec} k, \ \pi_D: D \to \operatorname{Spec} k$ . These will all follow by constructing isomorphisms of the corresponding relative dualizing sheaves with the square of some line bundle on the appropriate scheme. To allow these morphisms to extend to equivariant cohomology, we will make these isomorphisms  $\operatorname{SL}_2$ -equivariant.

Recall that  $D = \mathbb{P}(F)$ . The Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(F)}(-1) \to \mathcal{O}^2_{\mathbb{P}(F)} \to T_{\mathbb{P}(F)}(-1) \to 0$$

is an exact sequence of  $SL_2$ -equivariant maps, and thus gives the canonical  $SL_2$ -equivariant isomorphism

$$\omega_{\mathbb{P}(F)} \cong \mathcal{O}_{\mathbb{P}(F)}(-1)^{\otimes 2}$$

The embedding  $i_{\mathbb{P}(F)}: \mathbb{P}(F) \to \mathbb{P}((\mathrm{Sym}^2 F \oplus k))$  induces the canonical  $\mathrm{SL}_2$ -equivariant isomorphism

$$i_{\mathbb{P}(F)}^* \mathcal{O}_{\mathbb{P}((\mathrm{Sym}^2 F \oplus k)}(1) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}(F)}(1)^{\otimes 2}$$

Similarly, we have the SL<sub>2</sub>-equivariant isomorphism

$$\omega_{\mathbb{P}(\operatorname{Sym}^2 F \oplus k)} \cong \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^2 F \oplus k)}(-4).$$

The embedding  $i_{\bar{X}_a}: \bar{X}_a \to \mathcal{O}_{\mathbb{P}(\operatorname{Sym}^2 F \oplus k)}$  is  $\operatorname{SL}_2$ -equivariant and the generating section  $Q_a = T_3^2 - a(T_1^2 - 4T_0T_1)$  of  $\mathcal{I}_{\mathbb{P}(\operatorname{Sym}^2 F \oplus k)}$  gives an  $\operatorname{SL}_2$ -equivariant isomorphism

$$\times Q_a: \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^2 F \oplus k)}(-2) \xrightarrow{\sim} \mathcal{I}_{\bar{X}_a}.$$

Together with the adjunction formula, we arrive at the SL<sub>2</sub>-equivariant isomorphism

$$\omega_{\bar{X}_a} \cong \mathcal{O}_{\bar{X}_a}(-2) = \mathcal{O}_{\bar{X}_a}(-1)^{\otimes 2}$$

Altogether, these induce the SL<sub>2</sub>-equivariant isomorphism

$$\omega_{i_D} = \omega_{\mathbb{P}(F)} \otimes i_D^* \omega_{\bar{X}_a}^{-1} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}(F)}(-2+4) = \mathcal{O}_{\mathbb{P}(F)}(1)^{\otimes 2}.$$

In local coordinates, this can be described as follows. Letting q=(1,0,0,0) as before, we use the rational function  $1/g=T_3/T_2$  to give a section of  $\mathcal{I}_D$  on the affine open subset  $\bar{X}_a \setminus (T_2=0)$ ,  $T_3/T_2$  is clearly a defining equation for  $D \setminus \{q\} \subset \bar{X}_a \setminus (T_2=0)$ . Similarly,  $T_3/T_0$  is a defining equation for  $D \setminus \{q'\} \subset \bar{X}_a \setminus (T_1=0)$ , with q'=(0,0,1,0). We have the isomorphism  $\phi: \mathbb{P}^1 \to D$ , with  $\phi(t_0,t_1)=(t_0^2,2t_0t_1,t_1^2,0)$ , via which  $\phi^*(T_2/T_0)=(t_1/t_0)^2$ , so we may use  $i_D^*d(1/g)$  as a generating section of  $\omega_{i_D}$  over  $D \setminus \{q\}$ , which agrees, modulo squares (i.e., modulo squares of sections of  $\mathcal{O}_{\mathbb{P}(F)}(1)$ ), with the generating section  $i_D^*d(T_3/T_0)$  over  $D \setminus \{q'\}$ .

Recall that we have a fixed element  $\sqrt{a} \in k(y)$  generating k(y) over k. We use our choice of generating sections for  $\omega_{p_q}$  and  $\omega_{i_D}$  to give our isomorphism  $H^1(\bar{X}_a, \mathcal{W}) \cong W(k(\sqrt{a}))$  and to define the push-forward map  $i_{D*}$ .

**Lemma 10.5.** With respect to the choices made above, the following holds.

- 1. The map  $i_{D*}: W(k) = H^0(D, \mathcal{W}) \to H^1(\bar{X}_a, \mathcal{W}) \cong W(k(\sqrt{a}))$  is given by  $i_{D*}(\alpha) = \iota(\alpha)$
- 2. The map  $i_{D*}: W(k) = H^1(D, \mathcal{W}) \to H^2(\bar{X}_a, \mathcal{W}) = W(k)$  is the identity map. 3. The map  $i_D^*: H^1(\bar{X}_a, \mathcal{W}) \cong W(k(\sqrt{a})) \to H^1(D, \mathcal{W}) \cong W(k)$  is given by a modified trace map  $i_D^*(\alpha) = Tr_{k(\sqrt{a})/k}(\sqrt{a} \cdot \alpha)$ .

*Proof.* We use the Rost-Schmid complex to prove (1). For  $\alpha \in W(k) = H^0(D, \mathcal{W})$ ,  $i_{D*}(\alpha)$  is represented by  $\alpha \otimes i_D^* d(1/g)$  in  $W(k(D)) \otimes \omega_{i_D}$ . We use  $\langle g \rangle \cdot \alpha \in W(k(\bar{X}_a))$  to give the relation  $\partial \langle g \rangle = 0$ , that is

$$0 = \partial \langle g \rangle = (\alpha \otimes p_q^* dg \text{ on } C_q) + (\alpha \otimes i_D^* dg \text{ on } D)$$

Since  $d(1/g) = -1/g^2 \cdot dg$ , this gives

$$(\alpha \otimes p_q^* dg \text{ on } C_q) + (\langle -1 \rangle \alpha \otimes i_D^* dg \text{ on } D) = 0$$

Since  $\langle -1 \rangle = -1$  in W(k), it follows from our choices that

$$i_{D*}(\alpha) = p_{q*}(\alpha)$$

in  $H^1(\bar{X}_a, \mathcal{W})$ .

For (2), we use our canonical isomorphisms of the various dualizing sheaves with squares to define the push-forward maps. As in the proof of Lemma 10.3, the canonical isomorphisms  $\omega_{\mathbb{P}(F)} \cong \mathcal{O}_{\mathbb{P}(F)}(-2)$ ,  $\omega_{\bar{X}_a} \cong \mathcal{O}_{\bar{X}_a}(-2)$  gives the push-forward maps

$$i_{p*}^D: W(k(p)) \to H^1(D, \mathcal{W}), \ i_{p*}: W(k(p)) \to H^2(\bar{X}_a, \mathcal{W})$$

with  $i_{p*} = i_{D*} \circ i_{p*}^D$ ; both of these maps are isomorphisms. Since we have used  $i_{p*}$  to identify  $H^2(\bar{X}_a, \mathcal{W})$  with W(k) = W(k(p)), and  $i_{p*}^D$  to identify  $H^1(D, \mathcal{W})$  with W(k) = W(k(p)), this proves (2).

For (3), consider the cartesian diagram

$$0_{k(\sqrt{a})} \xrightarrow{i_0} \mathbb{P}^1_{k(\sqrt{a})}$$

$$p_{q_0} \downarrow \qquad p_q \downarrow$$

$$D \xrightarrow{i_D} \overline{X}_a$$

defining the morphism  $p_{q0}$  as the projection with image  $q \in D(k)$ . We have the base-change identity  $i_D^* \circ p_{q*} = p_{q0*} \circ i_0^*$ ; since  $p_{q*} : W(k(\sqrt{a}) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W}) \to H^1(\bar{X}_a, \mathcal{W})$  is an isomorphism, we can use  $p_{q0*} \circ i_0^*$  to compute  $i_D^*$ . Note that the base-change identity requires a compatible choice of generating section (mod squares) for  $\omega_{p_q}$  and  $\omega_{p_{q0}}$ , so we need to compute  $i_0^*$  of our chosen generating section  $p_q^*(dg)$  for  $\omega_{p_q}$ .

For our choice of q,  $p_{q0}(0_{k(\sqrt{a})})$  is the point  $q=(1,0,0,0)\in D$ ; via our isomorphism  $\psi:(x_0,x_1)\mapsto (x_0^2,2x_0x_1,x_1^2)$  of  $\mathbb{P}^1_k$  with D, q corresponds to the point  $(1,0)\in \mathbb{P}^1_k$ . We use the standard generating section  $d(x_1/x_0)$  for  $\omega_{\mathbb{P}^1}$  near (1,0), giving the generating section  $p_{q0}^*(d(x_1/x_0))$  for  $\omega_{p_{q0}}$ .

We thus need to compute the unique  $\lambda \in k(\sqrt{a})$  with  $i_0^*p_q^*(dg) = \lambda \cdot p_{q_0}^*(d(x_1/x_0))$ . We can do this after taking the base-change by  $k \subset k(\sqrt{a})$ , which transforms our cartesian diagram to the diagram

$$0_1 \coprod 0_2 \xrightarrow{i_{01} \coprod i_{02}} \{(1,0)\} \times \mathbb{P}^1_{k(\sqrt{a})} \coprod \mathbb{P}^1_{k(\sqrt{a})} \times \{(1,0)\}$$

$$\downarrow^{p_0} \qquad \qquad \downarrow^{i_1 \coprod i_2}$$

$$\mathbb{P}^1_{k(\sqrt{a})} \xrightarrow{\delta} \mathbb{P}^1_{k(\sqrt{a})} \times_{k(\sqrt{a})} \mathbb{P}^1_{k(\sqrt{a})}$$

Here  $0_1 = 0_2 = ((1,0),(1,0))$  and  $i_{01}$ ,  $i_{02}$ ,  $i_1$ ,  $i_2$  are the evident inclusions,  $p_0 := (p_{01},p_{02})$  is the evident projection to  $(1,0) \in \mathbb{P}^1$ , and  $\delta$  is the diagonal map. On  $\mathbb{P}^1_{k(\sqrt{a})} \times \mathbb{P}^1_{k(\sqrt{a})}$  we have  $g = x_1y_1/\sqrt{a}(x_0y_1 - x_1y_0)$  and

$$i_1^*(dg) = \frac{-1}{\sqrt{a}}d(x_1/x_0), \ i_2^*(dg) = \frac{1}{\sqrt{a}}d(y_1/y_0)$$

so

$$i_{01}^*i_1^*(dg) = \frac{-1}{\sqrt{a}} p_{01}^*(d(x_1/x_0)), \ i_{02}^*i_2^*(dg) = \frac{1}{\sqrt{a}} p_{02}^*(d(x_1/x_0))$$

By our conventions, the element  $\sqrt{a}$  has value  $+\sqrt{a}$  along  $(1,1)\times \mathbb{P}^1_{k(\sqrt{a})}\subset \mathbb{P}^1_{k(\sqrt{a})}\times \mathbb{P}^1_{k(\sqrt{a})}$  and  $-\sqrt{a}$  along  $\mathbb{P}^1_{k(\sqrt{a})}\times (1,1)\subset \mathbb{P}^1_{k(\sqrt{a})}\times \mathbb{P}^1_{k(\sqrt{a})}$ . Under the inclusion

$$k(\sqrt{a}) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{O}) \xrightarrow{p^*} H^0(\mathbb{P}^1_{k(\sqrt{a})} \otimes_k k(\sqrt{a}), \mathcal{O}) = k(\sqrt{a}) \otimes_k k(\sqrt{a}) \cong k(\sqrt{a}) \times k(\sqrt{a})$$

induced by the projection  $\mathbb{P}^1_{k(\sqrt{a})} \otimes_k k(\sqrt{a}) \to \mathbb{P}^1_{k(\sqrt{a})}$ , an element  $\lambda \in k(\sqrt{a})$  goes to the pair  $(\lambda, \bar{\lambda})$ , so under our conventions, the pair consisting of the constant

functions  $\sqrt{a}$  on  $\mathbb{P}^1 \times (1,0)$  and  $-\sqrt{a}$  on  $(1,0) \times \mathbb{P}^1$  comes from the constant function  $\sqrt{a}$  on  $\mathbb{P}^1_{k(\sqrt{a})}$ . Thus we have

(10.3) 
$$i_0^* p_q^*(dg) = \frac{1}{\sqrt{a}} \cdot p_{q_0}^*(d(x_1/x_0)).$$

We may factor  $p_{q0}: 0_{k(\sqrt{a})} \to D$  as the projection  $\bar{p}: 0_{k(\sqrt{a})} \to q$  followed by the inclusion  $i_q: q \to D$ . We use the canonical choice  $d(x_1/x_0)$  to define  $i_{q*}$  and we modify the usual trace map defining  $\bar{p}_*$  by defining

$$\bar{p}_*(\alpha) := \operatorname{Tr}_{k(\sqrt{a})/k}(\langle \sqrt{a} \rangle \cdot \alpha)$$

By (10.3), we then have

$$p_{q0*} = i_{q*} \circ \bar{p}_*.$$

Since  $i_{q*}$  is used to define our isomorphism  $W(k) \cong H^1(D, \mathcal{W})$ , putting everything together gives us the identity for  $\alpha \in W(k(\sqrt{a})) = H^0(\mathbb{P}^1_{k(\sqrt{a})}, \mathcal{W})$ ,

$$i_D^* \circ p_{q*}(\alpha) = p_{q0*} \circ i_0^*(\alpha) = \operatorname{Tr}_{k(\sqrt{a})/k}(\langle \sqrt{a} \rangle \cdot \alpha) \in W(k) = H^1(D, \mathcal{W}).$$

This completes the proof of (2).

We let

$$\operatorname{Tr}^{\langle \sqrt{a} \rangle} : W(k(\sqrt{a})) \to W(k)$$

denote the map  $\alpha \mapsto \operatorname{Tr}_{k(\sqrt{a})/k}(\langle \sqrt{a} \rangle \cdot \alpha)$ .

**Proposition 10.6.** Let  $\pi: \bar{X}_a \to \operatorname{Spec} k$  be the structure map. As an  $H^*(B\operatorname{SL}_2, \mathcal{W}) = W(k)[e]$ -module,  $H^*_{\operatorname{SL}_2}(\bar{X}_a, \mathcal{W})$  has the following description.

- 1. The canonical map  $W(k) = H^0(BSL_2, \mathcal{W}) \to H^0_{SL_2}(\bar{X}_a, \mathcal{W})$  is an isomorphism.
- 2.  $H^2_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W})$  consists of two cyclic W(k)-summands, with respective generators the equivariant Euler classes  $\pi^*e$  and  $e_{\mathrm{SL}_2}(T_{\bar{X}_a})$  and is isomorphic to

$$(W(k)/\mathrm{im}(\mathit{Tr}^{\langle\sqrt{a}\rangle})) \cdot p^*e \oplus (W(k)/\mathrm{im}(\mathit{Tr}^{\langle\sqrt{a}\rangle})) \cdot e_{\mathrm{SL}_2}(T_{\bar{X}_a})$$

In particular, the natural map

$$H^2(\mathrm{BSL}_2, \mathcal{W}) = W(k) \cdot e \to H^2_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W})$$

has kernel the image of the map  $\operatorname{Tr}^{\langle \sqrt{a} \rangle}$ .

- 3.  $H^1_{\operatorname{SL}_2}(\bar{X}_a, \mathcal{W}) \cong \operatorname{im}(i_* : W(k) \to W(k(\sqrt{a})).$
- 4.  $H^{2n+1}_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}) = 0$  for  $n \geq 1$  and the multiplication map

$$\times e^n: H^2_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}) \to H^{2n+2}_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W})$$

is an isomorphism for all  $n \geq 1$ .

In addition, the kernel of  $Tr^{\langle\sqrt{a}\rangle}$  is equal to the image of the base-change map  $\iota_*:W(k)\to W(k(\sqrt{a}))$ , the kernel of  $\iota_*$  is the ideal  $(1-\langle a\rangle)\subset W(k)$ . The image of  $Tr^{\langle\sqrt{a}\rangle}$  is the kernel of the map  $W(k)\to W(k)$  given by multiplication by  $1-\langle a\rangle$ , and is the same as the image of the usual trace map  $Tr_{k(\sqrt{a})/k}$ .

*Proof.* Recall that we use the schemes  $U_{2,n}$  of  $2 \times n$  matrices of rank 2 to define the  $\mathrm{SL}_2$ -equivariant cohomology. Let  $\pi_n: \mathrm{SL}_2 \setminus \bar{X}_a \times U_{2,n} \to \mathrm{SL}_2 \setminus U_{2,n}$  be the projection. This is a Zariski locally trivial fiber bundle with fiber  $X_a$ , giving for each n we have the spectral sequence

$$E_{2,n}^{p,q} = H^p(\operatorname{SL}_2 \setminus U_{2,n}, R^q \pi_{n*} \mathcal{W}) \Rightarrow H^{p+q}(\operatorname{SL}_2 \setminus \bar{X}_a \times U_{2,n}, \mathcal{W}).$$

which, taking n >> 0 gives us the spectral sequence

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, R^q \pi_* \mathcal{W}) \Rightarrow H^{p+q}_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}).$$

This is justified by using the Rost-Schmid complex for W to show that the transition maps from  $U_{2,n+1}$  to  $U_{2,n}$  induce isomorphism

$$H_{\mathrm{SL}_2}^p(U_{2,n+1}, R^q \pi_{n+1*} \mathcal{W}) \to H_{\mathrm{SL}_2}^p(U_{2,n}, R^q \pi_{n*} \mathcal{W})$$

and

$$H^{p+q}(\operatorname{SL}_2 \setminus \bar{X}_a \times U_{2,n+1}, \mathcal{W}) \to H^{p+q}(\operatorname{SL}_2 \setminus \bar{X}_a \times U_{2,n}, \mathcal{W}).$$

for n >> 0

Let  $p: U_{2,n} \times_k k(\sqrt{a}) \to U_{2,n}$  be the projection and let  $\mathcal{W}_{k(\sqrt{a})} = p_*\mathcal{W}$ ; we use the same notation for the corresponding sheaves on BSL<sub>2</sub>. With our choices, the isomorphisms

$$H^0(\bar{X}_a, \mathcal{W}) \cong W(k), \ H^1(\bar{X}_a, \mathcal{W}) \cong W(k(\sqrt{a})), H^2(\bar{X}_a, \mathcal{W}) \cong W(k)$$

are all SL<sub>2</sub>-invariant, so we have canonical SL<sub>2</sub>-equivariant isomorphisms

$$R^0\pi_*(\mathcal{W}) \cong \mathcal{W} \cong R^2\pi_*(\mathcal{W}), \ R^1\pi_*(\mathcal{W}) \cong \mathcal{W}_{k(\sqrt{a})}$$

This gives the isomorphism

$$E_2^{p,q} = \begin{cases} H^p(\mathrm{BSL}_2, \mathcal{W}) & \text{for } q = 0, 2\\ H^p(\mathrm{BSL}_2, \mathcal{W}_{k(\sqrt{a})}) & \text{for } q = 1\\ 0 & \text{otherwise.} \end{cases}$$

Recall that  $H^p(\mathrm{BSL}_2,\mathcal{W})=W(k)e^n$  for p=2n, and is 0 for p odd. Also, by Gersten's conjecture for the Witt sheaves, we have  $R^np_*\mathcal{W}=0$  for n>0, so we have the isomorphism  $H^p(\mathrm{BSL}_2,\mathcal{W}_{k(\sqrt{a})})=H^p(\mathrm{BSL}_2\times_k k(\sqrt{a}),\mathcal{W})$ . This gives the isomorphism

$$E_2^{p,q}(\bar{X}_a) = \begin{cases} W(k) \cdot e^n & \text{for } q = 0, 2, p = 2n \\ W(k(\sqrt{a})) \cdot e^n & \text{for } q = 1, p = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the sequence degenerates at  $E_3$ . Using the multiplicative structure on the spectral sequence, we reduce to computing the differentials

$$d^{0,2}(\bar{X}_a): W(k) = H^0(BSL_2, H^2(\bar{X}_a, \mathcal{W})) \to H^2(BSL_2, H^1(\bar{X}_a, \mathcal{W})) = W(k(\sqrt{a}))$$

$$d^{0,1}(\bar{X}_a):W(k(\sqrt{a}))=H^0(\mathrm{BSL}_2,H^1(\bar{X}_a,\mathcal{W})\to H^2(\mathrm{BSL}_2,H^0(\bar{X}_a,\mathcal{W}))=:W(k).$$

To facilitate the computation of the differentials, we compare with the Leray spectral sequence for  $\mathbb{P}(F)$ :

$$E_2^{p,q}(\mathbb{P}(F)) = H^p(\mathrm{BSL}_2, R^q \pi_*(\mathcal{W})) \Rightarrow H^{p+q}_{\mathrm{SL}_2}(\mathbb{P}(F), \mathcal{W})$$

where now  $\pi$  is the family of projections  $\mathbb{P}(F) \times U_{2,n} \to U_{2,n}$ . Computing as above, we find

$$E_2^{p,q}(\mathbb{P}(F)) = \begin{cases} \mathcal{W} \cdot e^n & \text{ for } q = 0, 1, p = 2n \\ 0 & \text{ else.} \end{cases}$$

The fact that e maps to zero in  $H^2_{\mathrm{SL}_2}(\mathbb{P}(F),\mathcal{W})$  implies that the differential

$$d_2^{0,1}(\mathbb{P}(F)):W(k)=H^0(\mathrm{BSL}_2,H^1(\mathbb{P}(F),\mathcal{W}))\to H^2(\mathrm{BSL}_2,H^0(\mathbb{P}(F),\mathcal{W}))=W(k)\cdot e^{-\frac{1}{2}}$$

is an isomorphism.

We have the  $SL_2$ -equivariant inclusion  $i_D: D = \mathbb{P}(F) \to \bar{X}_a$  and the  $SL_2$ -equivariant isomorphism of  $SL_2$ -linearized bundles  $\omega_{i_D} \cong \mathcal{O}_{\mathbb{P}(F)}(-1)^{\otimes 2}$ . This induces a map of spectral sequences,

$$i_{D*}: E_*^{*,*}(\mathbb{P}(F)) \to E_*^{*,*+1}(\bar{X}_a)$$

with the map on the  $E_2$ -terms induced from  $i_{D*}: H^q(\mathbb{P}(F), \mathcal{W}) \to H^{q+1}(\bar{X}_a, \mathcal{W})$ . Since  $i_{D*}: H^1(D, \mathcal{W}) \to H^2(\bar{X}_a, \mathcal{W})$  is an isomorphism, and  $i_{D*}: H^0(D, \mathcal{W}) \to H^1(\bar{X}_a, \mathcal{W})$  is the base-change map  $\iota_*: W(k) \to W(k(\sqrt{a}])$ , the differential  $d_2^{0,1}(\mathbb{P}(F))$  determines  $d_2^{0,2}(\bar{X}_a)$ ; via Lemma 10.5 this gives the formula

$$d^{0,2}(\bar{X}_a))(\alpha) = \iota_*(d_2^{0,1}(D)(\bar{\alpha})) \in W(k(\sqrt{a}))$$

for  $\alpha \in W(k) = H^0(\mathrm{BSL}_2, H^2(\bar{X}_a, \mathcal{W}))$ , where  $\bar{\alpha} = i_{D_*}^{-1}(\alpha)$ . As  $d_2^{0,1}(\mathbb{P}(F))$  is an isomorphism, this gives

$$\ker d^{0,2}(\bar{X}_a) = \ker \iota_*, \ \operatorname{im} d^{0,2}(\bar{X}_a) = \operatorname{im} \iota_*$$

Similarly,  $i_D^*$  induces a map of spectral sequences

$$i_D^*: E_*^{*,*}(\bar{X}_a) \to E_*^{*,*}(\mathbb{P}(F)).$$

Since  $i_D^*: W(k) = H^0(\bar{X}_a, \mathcal{W}) \to H^0(D, \mathcal{W}) = W(k)$  is the identity map and  $i_D^*: W(k(\sqrt{a})) = H^1(\bar{X}_a, \mathcal{W}) \to H^1(D, \mathcal{W}) = W(k)$  is the map  $\operatorname{Tr}^{\langle \sqrt{a} \rangle}$  (see Lemma 10.5), we have

$$d^{0,1}(\bar{X}_a)(\alpha) = d_2^{0,1}(\mathbb{P}(F))(\operatorname{Tr}^{\langle \sqrt{a} \rangle}(\alpha)) \in W(k)$$

for  $\alpha \in W(k(\sqrt{a}) = H^0(BSL_2, H^1(\bar{X}_a, \mathcal{W}))$ . Thus

$$\ker d^{0,1}(\bar{X}_a) = \ker \operatorname{Tr}^{\langle \sqrt{a} \rangle}, \ \operatorname{im} d^{0,2}(\bar{X}_a)) = \operatorname{im} \operatorname{Tr}^{\langle \sqrt{a} \rangle}.$$

Multiplying by  $e^n$  gives us corresponding expressions for the kernel and image of  $d^{2n,q}$ , q=1,2,  $n\geq 1$ . We conclude by invoking "Lam's distinguished triangle"

$$W(k) \xrightarrow{i_*} W(k(\sqrt{a}))$$

$$\times (1 - \langle a \rangle) \qquad W(k)$$

This sequence is exact at all three points (see e.g. [10, 34.12, Prop. 34.1]). The fact that  $\operatorname{im}\operatorname{Tr}^{\langle\sqrt{a}\rangle}=\operatorname{im}\operatorname{Tr}_{k(\sqrt{a})/k}$  follows from the fact that  $\langle\sqrt{a}\rangle$  is a unit in  $W(k(\sqrt{a}))$ .

We now use our computation of the equivariant cohomology

$$H^*_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}), \ H^*_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(1))$$

given in §11. Let  $\bar{p}: \bar{X}_a \to \mathbb{P}(\mathrm{Sym}^2 F)$  be the projection and let  $j: X_a \to \bar{X}_a$  be the inclusion; let  $p: X_a \to \mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F)$  be the projection induced by  $\bar{p}$ . We have the SL<sub>2</sub>-invariant element  $q' \in H^0(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(\mathrm{Sym}^2 F)}))$  (see Lemma 11.6). We have the SL<sub>2</sub>-invariant section  $\Omega$  of  $\omega_{\mathbb{P}(\mathrm{Sym}^2 F)}(3)$  (11.1) and we use  $2\Omega/T_3^3$  to trivialize  $j^*\bar{p}^*\omega_{\mathbb{P}(\mathrm{Sym}^2 F)} = \omega_{X_a}$ .

**Proposition 10.7.** 1.  $j^*: H^p_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}) \to H^p_{\mathrm{SL}_2}(X_a, \mathcal{W})$  is an isomorphism for  $p \geq 2$ .

- 2.  $H^1_{\mathrm{SL}_2}(X_a, \mathcal{W}) = 0$
- 3. The localization sequence for j gives rise to an exact sequence

$$0 \to H^0_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}) \xrightarrow{j^*} H^0_{\mathrm{SL}_2}(X_a, \mathcal{W}) \xrightarrow{\partial} H^0_{\mathrm{SL}_2}(D, \mathcal{W}) \xrightarrow{i_{D^*}} H^1_{\mathrm{SL}_2}(\bar{X}_a, \mathcal{W}) \to 0.$$

4. Via the canonical isomorphism  $W(k) \cong H^0_{\operatorname{SL}_2}(D, \mathcal{W})$ , the image of  $\partial$  is the ideal  $(1 - \langle a \rangle)W(k)$  and the element  $[\tilde{Q}] := p^*[Q'] \in H^0_{\operatorname{SL}_2}(X_a, \mathcal{W})$  generates a W(k)-module summand of  $H^0_{\operatorname{SL}_2}(X_a, \mathcal{W})$  which determines a W(k) splitting of the surjection  $\partial: H^0_{\operatorname{SL}_2}(X_a, \mathcal{W}) \to (1 - \langle a \rangle)W(k)$ .

*Proof.* (1), (2) and the exact sequence of (3) except for the surjectivity of  $i_{D*}$  follow from the localization sequence (Proposition 4.21)

$$\ldots \to H^{p-1}_{\operatorname{SL}_2}(D,\mathcal{W}) \xrightarrow{i_{D*}} H^p_{\operatorname{SL}_2}(\bar{X}_a,\mathcal{W}) \xrightarrow{j^*} H^p_{\operatorname{SL}_2}(X_a,\mathcal{W}) \xrightarrow{\partial} H^p_{\operatorname{SL}_2}(D,\mathcal{W}) \to \ldots$$
 and the computation

$$H_{\mathrm{SL}_2}^p(D, \mathcal{W}) = \begin{cases} 0 & \text{for } p > 0 \\ W(k) & \text{for } p = 0 \end{cases}$$

The surjectivity of  $i_{D*}$  follows from Lemma 10.5(1) and Lemma 10.3(3). These also imply that, via our identification of  $H^0_{\mathrm{SL}_2}(D,\mathcal{W})$  with W(k), the image of  $\partial$  is the kernel of the base-extension map  $\iota:W(k)\to W(k(\sqrt{a}))$ , which by Lam's exact triangle is the ideal  $(\langle 1\rangle - \langle a\rangle)W(k)$ .

To complete the proof, it suffices to show

- i.  $\partial([\tilde{Q}]) = \langle a \rangle \langle 1 \rangle$
- ii. For  $\alpha \in W(k)$ , if  $\alpha \cdot (\langle 1 \rangle \langle a \rangle) = 0$  in W(k), then  $\alpha \cdot [\tilde{Q}] = 0$ .

For (i), recall from Lemma 11.6 below that for a linear form  $L(T_0, T_1, T_2)$ , [Q'] is represented on  $\mathbb{P}(\mathrm{Sym}^2 F) \setminus (\mathbb{P}(F) \cup V(L))$  by the  $\omega_{\mathbb{P}(\mathrm{Sym}^2 F)}$ -valued quadratic form

$$Q'_L(x,y) := (-T_0x^2 - T_1xy - T_2y^2) \otimes \frac{2\Omega}{L^4}$$

Taking  $L = T_0$ , we diagonalize  $Q'_L$  to give the isometric form

$$q' = [\langle -1 \rangle + \langle Q/T_0^2 \rangle] \otimes \frac{2\Omega}{T_0^3}$$

where Q is our quadratic polynomial  $T_1^2 - 4T_0T_2$  defining  $\mathbb{P}(F) \subset \mathbb{P}(\mathrm{Sym}^2 F)$ . We pull back to  $X_a$ , use  $2\Omega/T_3^3$  to trivialize  $\omega_{X_a}$  and noting the relation  $aQ/T_0^2 = (T_3/T_0)^2$  on  $X_a$ , we have

$$[\tilde{Q}] = [(\langle -1 \rangle + \langle a \rangle) \langle t_3 \rangle] \in H^0(X_a, \mathcal{W}), \ t_i := T_i/T_0.$$

For our localization sequence, we are using  $T_3/T_2$  to trivialize the normal bundle of D in  $\bar{X}_a \setminus \{T_2 = 0\}$ , but as  $T_2/T_0 = (T_1/2T_2)^2$  on D, we may instead use  $t_3$  on  $\bar{X}_a \setminus \{T_0 = 0\}$ . Thus

$$\partial_D([\tilde{Q}]) = \partial_{t_3}((\langle -1 \rangle + \langle a \rangle)\langle t_3 \rangle) = \langle a \rangle - \langle 1 \rangle,$$

which proves (i).

If  $\alpha \cdot (\langle a \rangle - \langle 1 \rangle) = 0$ , then by our computation above, we have

$$\alpha \cdot [\tilde{Q}] = [\alpha \cdot (\langle -1 \rangle + \langle a \rangle) \langle t_3 \rangle] = 0,$$

proving (ii).  $\Box$ 

**Definition 10.8.** For  $a \in k^{\times}$ , let  $I_a \subset W(k)$  denote the kernel of the multiplication map  $W(k) \to W(k)$ ,  $x \mapsto x \cdot (1 - \langle a \rangle)$ .

By Lam's exact triangle,  $I_a$  is also the image of the trace map

$$\operatorname{Tr}_{k(\sqrt{a})/k}: W(k(\sqrt{a})) \to W(k).$$

**Proposition 10.9.** Let  $p: X_a \to Y := \mathbb{P}(\operatorname{Sym}^2 F) \setminus D$  be the projection. We use the generating section  $T_3$  of  $\mathcal{O}_{X_a}(1)$  to define the  $\operatorname{SL}_2$ -equivariant isomorphisms  $\mathcal{O}_{X_a}(1) \cong \mathcal{O}_{X_a}$  and  $\mathcal{W}(\mathcal{O}_{X_a}(1)) \cong \mathcal{W}$  and use  $2\Omega/T_3^3$  to give the  $\operatorname{SL}_2$  isomorphism  $\mathcal{W}(\omega_{X_a}) \cong \mathcal{W}$ . Let  $[\tilde{Q}] = p^*([Q']) \in H^0_{\operatorname{SL}_2}(X_a, \mathcal{W})$ . Let  $\bar{W}(k) := W(k)/I_a$ . Then as an  $H^*(B\operatorname{SL}_2, \mathcal{W}) = W(k)[e]$ -algebra,

$$H_{\mathrm{SL}_2}^*(X_a, \mathcal{W}) \cong W(k)[e, y]/(y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot y, I_a \cdot e),$$

by the map sending y to  $[\tilde{Q}]$ . Moreover,  $[\tilde{Q}]e = e_{SL_2}(T_{X_a})$  and  $[\tilde{Q}]^2e = 4e$ .

*Proof.* Since p is étale, we have  $p^*T_Y \cong T_{X_a}$  and hence  $e_{SL_2}(T_{X_a}) = p^*(e_{SL_2}(T_Y))$ . With Lemma 11.7(3) and Lemma 11.8(1), this gives the relations

$$[\tilde{Q}]^2 = 2 \cdot (\langle 1 \rangle - p^*[Q]), \ [\tilde{Q}]e = e_{SL_2}(T_{X_a}).$$

Moreover, since  $X_a$  is the affine variety defined by  $T_3^2 = aQ$ ,  $T_3 \neq 0$ , we have

$$p^*[Q] = \langle a \rangle \in H^0(X_a, \mathcal{W})$$

which gives us the relation  $[\tilde{Q}]^2 = 2 \cdot (\langle 1 \rangle - \langle a \rangle)$ .

By Proposition 10.7 and Proposition 10.6, the degree  $\geq 2 \ W(k)[e]$ -submodule of  $H^*_{\mathrm{SL}_2}(X_a, \mathcal{W})$  is the free  $W(k)/I_a$ -module with basis  $e, e_{\mathrm{SL}_2}(T_{X_a})$ . The degree 0 W(k)-submodule of  $H^*_{\mathrm{SL}_2}(X_a, \mathcal{W})$  is  $W(k) \cdot 1 \oplus W(k) \cdot [\tilde{Q}]$ , which is isomorphic as W(k)-algebra to  $W(k)[y]/I_a \cdot y, y^2 - 2(\langle 1 \rangle - \langle a \rangle)$ . Thus the W(k)[e]-algebra

$$W(k)[e,y]/I_a \cdot y, y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot e)$$

is isomorphic to  $H^*_{\mathrm{SL}_2}(X_a, \mathcal{W})$  by sending y to  $[\tilde{Q}]$ .

Finally, since  $\langle 1 \rangle + \langle a \rangle = \langle 2 \rangle \cdot \text{Tr}_{k(\sqrt{a}/k)}(\langle 1 \rangle)$ , we have  $\langle a \rangle e = -e$ , so

$$[\tilde{Q}]^2 e = 2 \cdot (\langle 1 \rangle - \langle a \rangle) \cdot e = 4e.$$

Remark 10.10. We have  $\operatorname{Pic}(\operatorname{Sym}^2(F) \setminus \operatorname{Sym}(F)) \cong \mathbb{Z}/2$ , with generator  $\gamma$  induced by the  $\operatorname{SL}_2$ -linearized invertible sheaf  $\mathcal{O}(1)$  on  $\operatorname{Sym}^2(F) \setminus \operatorname{Sym}(F)$ . The étale double cover  $X_a \to \operatorname{Sym}^2(F) \setminus \operatorname{Sym}(F)$  is formed by taking a square root out of the section aQ of  $\gamma$ , so the map  $\operatorname{Pic}(\operatorname{Sym}^2(F) \setminus \operatorname{Sym}(F)) \to \operatorname{Pic}(X_a)$  is the zero map. If  $a \in k^\times$  is not a square, it follows from Hilbert's theorem 90 and the exact sequence

$$0 \to H^1(\langle \sigma \rangle, k(\sqrt{a})^{\times}) \to \operatorname{Pic}(X_a) \to H^0(\langle \sigma \rangle, \operatorname{Pic}(X_{a \ k(\sqrt{a})}) \to \dots$$

that  $\operatorname{Pic}(X_a)$  is the trivial group  $(\operatorname{Pic}(X_{a\ k(\sqrt{a})}) = \mathbb{Z}, \text{ with } \sigma \text{ acting by } -1)$ . Thus, we need not concern ourselves with computing twisted cohomology  $H^*_{\operatorname{SL}_2}(X_a, \mathcal{W}(\mathcal{L}))$ .

As a direct consequence of Proposition 10.9 and (10.1), we have achieved our goal of computing the N-equivariant Witt cohomology in the case of a non-trivial action on an integral, 0-dimensional scheme.

Remark 10.11. We have the  $SL_2$ -equivariant isomorphism

$$N \backslash \operatorname{SL}_2 \cong \mathbb{P}(\operatorname{Sym}^2 F) \backslash \mathbb{P}(F),$$

which gives the isomorphism

$$H^*(BN, \mathcal{W}) \cong H^*_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}).$$

Via this isomorphism, the element  $x \in H^0(BN, W)$  in our description (Theorem 5.1(2)) of  $H^*(BN, W)$  as  $W(k)[e, x]/(x^2 - 1, (x + 1) \cdot e)$  corresponds to the  $\mathcal{O}(2)$ -valued,  $\mathrm{SL}_2$  invariant quadratic form  $\langle Q \rangle \in H^0(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), W)$ , where  $Q = T_1^2 - 4T_0T_2$ . The element  $e \in H^2(BN, W)$  is just the pullback of the Euler class  $e \in H^2(\mathrm{BSL}_2, W)$  via the canonical projection  $BN \to \mathrm{BSL}_2$ .

**Corollary 10.12.** Let z be an integral finite k-scheme with k(z) a degree two extension of k and with N acting on z by letting  $\sigma$  act as conjugation over k. Then as an  $H^*(BN, \mathcal{W}) = W(k)[e, x]/((1+x)e, x^2-1)$ -algebra, we have

$$H_N^*(z, \mathcal{W}) \cong W(k)[e, x, y]/(x - \langle a \rangle, y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot y, I_a \cdot e)$$

After inverting e, we have

$$H_N^*(z, \mathcal{W})[e^{-1}] = \bar{W}(k)[e, y, e^{-1}]/(y^2 - 2(\langle 1 \rangle - \langle a \rangle))$$

where  $\bar{W}(k) = W(k)/I_a$ , and the map  $H_N^*(z, \mathcal{W}) \to H_N^*(z, \mathcal{W})[1/e]$  sends x to -1.

*Proof.* As a direct consequence of Proposition 10.9 and (10.1), we have

$$H_N^*(z, \mathcal{W}) \cong W(k)[e, y]/(y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot y, I_a \cdot e)$$

as algebra over  $H^*(\mathrm{BSL}_2,\mathcal{W})=W(k)[e]$ . The  $\mathrm{SL}_2$ -invariant section  $Q:=T_1^2-4T_0T_2$  of  $\mathcal{O}_{\mathbb{P}^2}(2)$  gives rise to the section  $x\in H^0_{\mathrm{SL}_2}(\mathbb{P}^2\setminus D,\mathcal{W})\cong H^0(BN,\mathcal{W})$  used in the presentation of  $H^0(BN,\mathcal{W})$  given in Theorem 5.1 (see [18, Proposition 5.5]). Since  $X_a\to\mathbb{P}^2\setminus D$  is the double cover defined by  $T_3^2=aQ$ , the element x maps to  $\langle a\rangle\in H^0_{\mathrm{SL}_2}(X_a,\mathcal{W})$  and we arrive at the desired presentation of  $H^*_N(z,\mathcal{W})$  as  $H^*(BN,\mathcal{W})$ -algebra. The description of  $H^*_N(z,\mathcal{W})[e^{-1}]$  follows directly from this and the presentation of  $H^0(BN,\mathcal{W})[e^{-1}]$  given in Theorem 5.1

Remark 10.13. Take  $b \in k^{\times}$  and let  $a = b^2$ ,  $\mathcal{O} = k[X]/X^2 - a$ , and let  $z = \operatorname{Spec} \mathcal{O}$ . We let N act on z through  $N/T_1 = \langle \bar{\sigma} \rangle$ , with  $\bar{\sigma}^*(X) = -X$ . Then  $\langle a \rangle = 1$ , so the ideal  $I_a := \ker((-) \cdot (1 - \langle a \rangle) : W(k) \to W(k)$  is the unit ideal. Moreover, we have

$$H_N^*(z, \mathcal{W}) = H_{T_1}^*(\operatorname{Spec} k, \mathcal{W}) \cong W(k)[e]/(e)$$

$$\cong W(k)[e, x, y]/(x - \langle a \rangle, y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot y, I_a \cdot e)$$

so the description of  $H_N^*(\operatorname{Spec} k[X]/(X^2-a), \mathcal{W})$  given by Corollary 10.12 in case of  $a \in k^{\times}$ , not a square, extends to the case of a a square.

We complete this section by computing the push-forward map  $H_N^*(z, \mathcal{W}) \to H^*(BN, \mathcal{W})$ .

Corollary 10.14. Let  $z \in \operatorname{\mathbf{Sch}}^N/k$  be as in Corollary 10.12. Let  $\pi: z \to \operatorname{Spec} k$  be the structure map, where N acts trivially on  $\operatorname{Spec} k$ . Then with respect to the presentation of  $H_N^*(z, \mathcal{W})$  given in Corollary 10.12 and  $H_N^*(\operatorname{Spec} k, \mathcal{W}) = H^*(BN, \mathcal{W})$  given in Theorem 5.1(2), the map

$$W(k)[e, x, y]/(x - \langle a \rangle, y^2 - 2(\langle 1 \rangle - \langle a \rangle), I_a \cdot y, I_a \cdot e) \xrightarrow{\pi_*} W(k)[e, x]/(x^2 - 1, (1 + x)e)$$

is the W(k)[e]-module map given by

$$\pi_*(1) = <2> + <2a> x$$
 $\pi_*(e) = (<2> - <2a>) \cdot e$ 
 $\pi_*(y) = 0$ 

After inverting e, this is the W(k)[e]-module map  $\bar{W}(k)[e,y,e^{-1}] \to W(k)[e,e^{-1}]$  given by

$$\pi_*(1) = <2 > - <2a >$$

$$\pi_*(e) = (<2 > - <2a >) \cdot e$$

$$\pi_*(y) = 0$$

*Proof.* The fact that  $\pi_*$  is a  $W(k)[e] = H^*(BSL_2, \mathcal{W})$ -module map is the projection formula; this implies that  $\pi_*$  is determined by its values on 1, e and y.

The  $\operatorname{SL}_2$ -equivariant projection  $\pi: X_a \to \operatorname{Sym}^2(F) \setminus \operatorname{Sym}(F)$  realizes  $X_a$  as the étale double cover of  $\mathbb{P}^2 \setminus D$  defined by  $T_3^2 = aQ$ . This implies that  $\pi_* \circ \pi^*$  is multiplication by the corresponding trace form, namely  $\langle 2 \rangle + \langle 2aQ \rangle$ . Since

$$1 = \pi^*(1), \ x = \pi^*([Q]), \ e = \pi^*(e), \ y = \pi^*([Q']),$$

and xe = -e, x = [Q] in  $H^*(BN, \mathcal{W})$ , this gives the first set of formulas for  $\pi_*(1)$  and  $\pi_*(e)$ . The formula for  $\pi_*(y)$  follows by diagonalizing the restriction of  $\tilde{Q}$  to  $T_0 \neq 0$ , as in proof of Proposition 10.7,

$$q' = (\langle -1 \rangle + \langle a \rangle) \langle t_3 \rangle$$

Since  $X_a \cap (T_0 \neq 0)$  is defined by  $t_3^2 = a(Q/T_0^2)$ , we have

$$\pi_*(q') = (\langle -1 \rangle + \langle a \rangle) \cdot \pi_*(\langle t_3 \rangle) = 0.$$

Inverting e gives the relation x=-1, which accounts for the second set of formulas.

#### 11. Spectral sequences and Euler classes

In this section we compute the  $SL_2$ -equivariant cohomology of  $Y := \mathbb{P}(Sym^2F) \setminus \mathbb{P}(F)$ . As a first step, we identify the Euler class of the universal rank two bundle with the value of a differential in a Leray spectral sequence.

To begin, let  $p: E \to X$  be a rank two vector bundle on some smooth k-scheme X. Suppose we have a trivializing Zariski open cover  $\mathcal{U} = \{U_i \mid i \in I\}$ , with vector bundle isomorphisms  $\psi_i: p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^2$ , and let  $\psi_{ij}:=\psi_{i|U_{ij}} \circ \psi_{j|U_{ij}}^{-1}$  be the corresponding cocycle defined on  $U_{ij}:=U_i \cap U_j$ .

We will assume that  $U_{ij} = \operatorname{Spec} A_{ij}$  is affine and that the element  $\psi_{ij} \in \operatorname{GL}_2(A_{ij})$  is a product of elementary matrices.

We use the definition of the Euler class in terms of the Rost-Schmid complex  $\mathcal{C}^*_{RS}(E,\mathcal{W})$  on E. The trivialization of  $\det E$  gives us a canonical generating section  $d\omega$  of  $\omega_{E/X}$  along the zero-section  $s:X\to E$ . This gives the element  $\langle 1\rangle_{|s(X)}\otimes d\omega\in\mathcal{C}^2_{RS}(E,\mathcal{W})$  with zero differential, giving rise to the corresponding class  $[\langle 1\rangle_{|s(X)}\otimes d\omega]\in H^2(E,\mathcal{W})$ . Via homotopy invariance, we thus have the Euler class  $e(E):=(p^*)^{-1}([\langle 1\rangle_{|s(X)}\otimes d\omega])\in H^2(X,\mathcal{W})$ . Via the isomorphism  $p^*$  all our further constructions will take place on E.

Next, we give a translation into Čech cohomology for the cover  $p^{-1}\mathcal{U} := \{p^{-1}(U_i)\}$  of E. The isomorphism  $\psi_i : p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^2$  gives us the trivialized sub-bundle

 $\mathbb{A}_i^1: \psi_i^{-1}(\mathbb{A}_{U_i}^1 \times 0) \subset p^{-1}(U_i)$ . Let  $p_1, p_2: \mathbb{A}^2 \to \mathbb{A}^1 = \operatorname{Spec} k[t]$  be the two projections. Let  $t_i$  be the regular function on  $\mathbb{A}_i^1$  corresponding via  $\psi_i$  to  $p_1^*(t)$  and let  $s_i$  be the regular function on  $p^{-1}(U_i)$  corresponding to  $p_2^*(t)$ , giving us the corresponding element  $\langle t_i \rangle_{|\mathbb{A}^1} \otimes ds_i \in \mathcal{C}^1_{RS}(p^{-1}(U_i), \mathcal{W})$ , with

$$\partial(\langle t_i\rangle_{|\mathbb{A}^1}\otimes ds_i)=\langle 1\rangle_{|p^{-1}(U_i)\rangle}\otimes d\omega.$$

If we apply the Čech coboundary  $\check{\delta}^0$  to the element  $\{\langle t_i \rangle_{|\mathbb{A}^1_i} \otimes ds_i \}_i \in \check{C}^0(\mathcal{U}, \mathcal{C}^1_{RS}(\mathcal{W}))$ , we have for each ij a Rost-Schmid cocycle  $\check{\delta}^0(\{\langle t_i \rangle_{|\mathbb{A}^1_i} \otimes ds_i \}_i)_{ij} \in \mathcal{C}^1(p^{-1}(U_{ij}), \mathcal{W})$ . By our assumption on the  $\psi_{ij}$ , each  $\check{\delta}^0(\{\langle t_i \rangle_{|\mathbb{A}^1_i} \otimes ds_i \}_i)_{ij}$  is a Rost-Schmid coboundary, so there is a section  $\lambda_{ij} \in \mathcal{W}(k(p^{-1}(U_{ij})))$  with

$$\partial_{RS}(\lambda_{ij}) = \check{\delta}^0(\{\langle t_i \rangle_{|\mathbb{A}^1_i} \otimes ds_i \}_i)_{ij}.$$

Letting  $\{\lambda_{ijk}\}_{ijk} := \check{\delta}^1(\{\partial_{RS}(\lambda_{ij})\}_{ij})$  be the Čech coboundary, we see that each  $\lambda_{ijk}$  has zero Rost-Schmid coboundary, and thus determines an element  $\lambda_{ijk} \in H^0(p^{-1}(U_{ijk}), \mathcal{W})$ , defining a Čech cocycle  $\{\lambda_{ijk}\}_{ijk} \in \check{C}^2(p^{-1}(\mathcal{U}), \mathcal{W})$ .

As before, let  $U_{2,n}$  be the scheme of  $2 \times n$  matrices of rank 2 and  $U_{2,\infty} = \operatorname{colim} U_{2,n}$ , with the evident left  $\operatorname{SL}_2$ -action. We use  $U_{2,\infty}$  as our model for  $E\operatorname{SL}_2$  and  $\operatorname{SL}_2 \setminus U_{2,\infty}$  for  $\operatorname{BSL}_2$ . The tautological vector bundle  $\pi: E_2 \to \operatorname{BSL}_2$  is the vector bundle associated to the fundamental representation  $\operatorname{SL}_2 \subset \operatorname{GL}_2$ .

We apply the above considerations to an open cover  $\mathcal{U}$  of  $X := \mathrm{SL}_2 \setminus U_{2,n}$  for  $n \geq 3$  and the tautological bundle  $E_{2,n} \to X$ . The canonical map  $H^2(\mathrm{BSL}_2, \mathcal{W}) \to H^2(X, \mathcal{W})$  is an isomorphism, so we will write  $H^2(E_2, \mathcal{W})$  for  $H^2(E_{2,n}, \mathcal{W})$  and  $H^2(\mathrm{BSL}_2, \mathcal{W})$  for  $H^2(\mathrm{SL}_2 \setminus U_{2,n}, \mathcal{W})$  in what follows.

Lemma 11.1. Under the canonical map

$$\check{H}^2(p^{-1}\mathcal{U},\mathcal{W}) \to H^2(E_2,\mathcal{W}) \xrightarrow{\sim} H^2(\mathrm{BSL}_2,\mathcal{W})$$

the cocycle  $\{\lambda_{ijk}\}_{ijk}\check{C}^2(p^{-1}(\mathcal{U}),\mathcal{W})$  maps to  $e(E_2)$ .

Proof. Consider the double complex  $\check{C}^*(p^{-1}(\mathcal{U}), \mathcal{C}_{RS}^*(\mathcal{W}))$ . Since the Rost-Schmid complex is a flasque resolution of  $\mathcal{W}$ , a spectral sequence argument implies that the augmentation  $\epsilon_1: \mathcal{C}_{RS}^*(E,\mathcal{W}) \to \operatorname{Tot}\check{C}^*(p^{-1}(\mathcal{U}), \mathcal{C}_{RS}^*(\mathcal{W}))$  induces an isomorphism on cohomology. Similarly, the map  $\check{H}^n(p^{-1}(\mathcal{U}), \mathcal{W}) \to H^n(E, \mathcal{W})$  is induced by the augmentation  $\epsilon_2: \check{C}^*(p^{-1}(\mathcal{U}), \mathcal{W}) \to \operatorname{Tot}\check{C}^*(p^{-1}(\mathcal{U}), \mathcal{C}_{RS}^*(\mathcal{W}))$ . The construction we have just performed shows that the elements  $\epsilon_2(\{\lambda_{ijk}\}_{ijk})$  and  $\epsilon_1(\langle 1\rangle_{|s(X)} \otimes d\omega)$  differ by a coboundary in  $\operatorname{Tot}\check{C}^*(p^{-1}(\mathcal{U}), \mathcal{C}_{RS}^*(\mathcal{W}))$ .

Let  $\pi^0: E_2 \setminus \{0\} \to \mathrm{BSL}_2$  be the restriction of  $\pi: E_2 \to \mathrm{BSL}_2$ . We apply similar constructions to compute the differential  $d_2^{0,1}$  in the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, \mathcal{H}^q(E_2 \setminus \{0\}, \mathcal{W})) \Rightarrow H^{p+q}_{\mathrm{SL}_2}(E_2 \setminus \{0\}, \mathcal{W}).$$

Here  $\mathcal{H}^q(E_2 \setminus \{0\}, \mathcal{W})$  is notation for the sheaf  $R^q \pi^0_* \mathcal{W}$ . For a field K, we have

$$H^{q}((\mathbb{A}^{2} \setminus \{0\})_{K}, \mathcal{W}) \cong \begin{cases} W(K) & \text{for } q = 0, 1\\ 0 & \text{else} \end{cases}$$

 $SL_2$  acts trivially on the W-cohomology, so

$$\mathcal{H}^q((E_2 \setminus \{0\}), \mathcal{W}) \cong \begin{cases} \mathcal{W} & \text{for } q = 0, 1\\ 0 & \text{else} \end{cases}$$

The differential  $d_2^{0,1}$  is thus a map

$$d_2^{1,0}: H^0(\mathrm{BSL}_2, \mathcal{W}) \to H^2(\mathrm{BSL}_2, \mathcal{W})$$

We can use the approximation  $\operatorname{SL}_2 \setminus \mathcal{U}_{2,3}$  to compute  $d_2^{1,0}$ , since the restriction map from  $\operatorname{BSL}_2$  to  $\operatorname{SL}_2 \setminus \mathcal{U}_{2,3}$  is an isomorphism on  $H^i(-,\mathcal{W})$  for  $i \leq 2$ . Let  $p: \bar{E}_2 \to \operatorname{SL}_2 \setminus \mathcal{U}_{2,3}$  be the pullback of  $E_2$  to  $\operatorname{SL}_2 \setminus \mathcal{U}_{2,3}$ .

Sending a matrix  $A \in \mathcal{U}_{2,3}$  to the triple  $(|A|_{23}, -|A|_{13}, |A|_{12})$ , where  $|A|_{ij}$  denotes the determinant of the 2 by 2 submatrix of A with columns i, j, gives an isomorphism of  $\mathrm{SL}_2 \setminus \mathcal{U}_{2,3}$  with  $\mathbb{A}^3 \setminus \{0\}$ . We use coordinates  $y_0, y_1, y_2$  on  $\mathbb{A}^3$ , and let  $U_i \subset \mathbb{A}^3 \setminus \{0\}$  be the open subscheme  $y_i \neq 0$ . This gives the affine open cover  $\mathcal{U} = \{U_i\}_{i=0,1,2}$  of  $\mathbb{A}^3 \setminus \{0\} = \mathrm{SL}_2 \setminus \mathcal{U}_{2,3}$ .

Sending  $(x, y, z) \in U_0$  to

$$s_0(x,y,z) := \begin{pmatrix} -y & x & 0 \\ -z/x & 0 & 1 \end{pmatrix}$$

gives a section  $s_0: U_0 \to \mathcal{U}_{2,3}$  of  $\mathcal{U}_{2,3} \to \mathbb{A}^3 \setminus \{0\}$ ; we have similarly defined sections over  $U_1, U_2$ ,

$$s_1(x, y, z) := \begin{pmatrix} -y & x & 0 \\ 0 & -z/y & 1 \end{pmatrix}, \ s_2(x, y, z) := \begin{pmatrix} z & 0 & -x \\ 0 & 1 & -y/z \end{pmatrix}.$$

These define trivializations  $\Psi_i$  of the  $\operatorname{SL}_2$ -principal bundle  $\mathcal{U}_{2,3} \to \mathbb{A}^3 \setminus \{0\}$  over  $U_i$ ; it is not hard to check that the resulting transition map  $\psi_{ij} \in \operatorname{SL}_2(\mathcal{O}(U_{ij}))$  is a product of elementary matrices. The  $\Psi_i$  induce trivializations  $\psi_i$  of  $E_{2,3} \to \mathbb{A}^3 \setminus \{0\}$  over  $U_i$ , giving the same cocycle  $\psi_{ij}$ .

Let Y be a smooth k-scheme with right  $\operatorname{SL}_2$ -action and let  $p_Y: Y \times^{\operatorname{SL}_2} \mathcal{U}_{2,3} \to \mathbb{A}^3 \setminus \{0\}$  be the projection induced by the structure map  $\pi_Y: Y \to \operatorname{Spec} k$ . Let L be an  $\operatorname{SL}_2$  linearized invertible sheaf on Y, giving the invertible sheaf  $\mathcal{L}$  on  $Y \times^{\operatorname{SL}_2} \mathcal{U}_{2,3}$  by descent. Let  $\mathcal{U} = \{U_i \mid i = 0, 1, 2\}$  be the affine open cover of  $\mathbb{A}^3 \setminus \{0\}$  defined above. Define the double complex  $\check{C}^*(\mathcal{U}, \mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L})))$  to be the Čech complex with values in the presheaf of Rost-Schmid complexes for  $\mathcal{W}(\mathcal{L})$ , with p, q term  $\check{C}^p(\mathcal{U}, \mathcal{C}^q_{RS}(\mathcal{W}(\mathcal{L})))$ .

### Lemma 11.2. The natural maps

$$\check{H}^p(\mathcal{U}, H^q(\mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L}))) \to H^p(\mathbb{A}^3 \setminus \{0\}, R^q p_{Y*}\mathcal{W}(\mathcal{L}))$$

$$H^{p+q}(\operatorname{Tot}\check{C}^*(\mathcal{U}, \mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L}))) \to H^{p+q}(Y \times^{\operatorname{SL}_2} \mathcal{U}_{2,3}, \mathcal{W}(\mathcal{L}))$$

induce an isomorphism of the spectral sequence

$$E_2^{p,q} := \check{H}^p(\mathcal{U}, H^q(\mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L}))) \Rightarrow H^{p+q}(\operatorname{Tot}\check{C}^*(\mathcal{U}, \mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L}))))$$

for the double complex  $\check{C}^*(\mathcal{U}, \mathcal{C}^*_{RS}(\mathcal{W}(\mathcal{L})))$  with the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathbb{A}^3 \setminus \{0\}, R^q p_{Y*} \mathcal{W}(\mathcal{L})) \to H^{p+q}(Y \times^{\mathrm{SL}_2} \mathcal{U}_{2,3}, \mathcal{W}(\mathcal{L}))$$

*Proof.* For each subset  $I \subset \{0, 1, 2\}$ , the corresponding open subset  $U_I \subset \mathbb{A}^3 \setminus \{0\}$  is isomorphic to  $\mathbb{A}^{3-|I|} \times (\mathbb{A}^1 \setminus \{0\})^{|I|}$ . Multiplication by the algebraic Hopf map  $\eta$  is invertible on  $\mathcal{W}$ , and after inverting  $\eta$ ,  $\mathbb{A}^1 \setminus \{0\}$  is isomorphic to Spec k II Spec k (in SH(k)). Thus

$$H^p(U_I, R^q p_{Y*} \mathcal{W}(\mathcal{L})) = \{0\}$$

for p > 0, that is,  $\mathcal{U}$  is a Leray cover for the sheaves  $R^q p_{Y*} \mathcal{W}(\mathcal{L})$ . This implies that the natural map of the first spectral sequence to the second one is an isomorphism

on the  $E_2$ -terms. As both spectral sequences are strongly convergent, this proves the assertion.

For q=0, the isomorphism  $W(K)\cong H^0((\mathbb{A}^2\setminus\{0\})_K,\mathcal{W})$  is given by the pullback by the structure morphism. For q=1, we choose the W(K)-generator of  $H^1((\mathbb{A}^2\setminus\{0\})_K,\mathcal{W})$  to be the class of  $\langle x\rangle_{|y=0}\otimes dy$  in  $\mathcal{C}^1_{RS}((\mathbb{A}^2\setminus\{0\})_K,\mathcal{W})$ , where x,y are the first and second projections (i.e., the standard coordinates). The isomorphism  $H^0(U_i,\mathcal{H}^1(\bar{E}_2\setminus\{0\},\mathcal{W}))\cong H^0(U_i,\mathcal{W})$  thus sends  $1\in H^0(U_i,\mathcal{W})$  to the restriction to  $\mathbb{A}^1_i\setminus\{0\}$  of the element we denoted as  $[(\langle t_i\rangle_{|\mathbb{A}^1_i}\otimes ds_i)]$  in our previous discussion.

Using the Rost-Schmid flasque resolution of  $\mathcal{W}$ , the differential  $d_2^{0,1}$  is represented in Čech cohomology by precisely the steps we used in our previous discussion of the Čech representative of the Euler class. The only difference is that we are working on  $E_{2,3} \setminus \{0\}$  rather than on  $E_{2,3}$ , we are starting with the elements  $[(\langle t_i \rangle_{|\mathbb{A}^1_i \setminus \{0\}} \otimes ds_i)]$  rather than with their Rost-Schmid coboundaries  $\langle 1 \rangle_{|p^{-1}(U_i)} \otimes d\omega$  on  $E_2$ , and we are identifying  $\mathcal{H}^0(E_{2,3} \setminus \{0\}, \mathcal{W})$  with  $\mathcal{H}^0(E_{2,3}, \mathcal{W})$  by the restriction map. This gives the following result:

Lemma 11.3. In the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, \mathcal{H}^q(E_2 \setminus \{0\}, \mathcal{W})) \Rightarrow H^{p+q}(E_2 \setminus \{0\}, \mathcal{W}).$$

and with respect to our choice of isomorphisms

$$\mathcal{W} \cong \mathcal{H}^0(E_2 \setminus \{0\}, \mathcal{W}), \ \mathcal{W} \cong \mathcal{H}^1(E_2 \setminus \{0\}, \mathcal{W})$$

described above, the differential

$$d_2^{0,1}: H^0(\mathrm{BSL}_2, \mathcal{H}^1(E_2 \setminus \{0\}, \mathcal{W})) \to H^2(\mathrm{BSL}_2, \mathcal{H}^0(E_2 \setminus \{0\}, \mathcal{W}))$$

satisfies

$$d_2^{0,1}(\langle 1 \rangle) = e(E_2).$$

We also have the Leray spectral sequence for the associated projective space bundle

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, \mathcal{H}^q(\mathbb{P}(E_2), \mathcal{W})) \Rightarrow H^{p+q}(\mathbb{P}(E_2), \mathcal{W}).$$

Recall that the algebraic Hopf map  $\eta: \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$  induces an isomorphism on Witt cohomology

$$\eta^*: H^q(\mathbb{P}^1_K, \mathcal{W}) \to H^q(\mathbb{A}^2_K \setminus \{0\}, \mathcal{W})$$

for a field K. Using our chosen isomorphisms

$$\mathcal{W} \cong \mathcal{H}^0(E_2 \setminus \{0\}, \mathcal{W}), \ \mathcal{W} \cong \mathcal{H}^1(E_2 \setminus \{0\}, \mathcal{W})$$

this gives us isomorphisms

$$\mathcal{W} \cong \mathcal{H}^0(\mathbb{P}(E_2), \mathcal{W}), \ \mathcal{W} \cong \mathcal{H}^1(\mathbb{P}(E_2), \mathcal{W})$$

and we have

**Lemma 11.4.** In the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, \mathcal{H}^q(\mathbb{P}(E_2), \mathcal{W})) \Rightarrow H^{p+q}(\mathbb{P}(E_2), \mathcal{W}).$$

and with respect to our choice of isomorphisms

$$\mathcal{W} \cong \mathcal{H}^0(\mathbb{P}(E_2), \mathcal{W}), \ \mathcal{W} \cong \mathcal{H}^1(\mathbb{P}(E_2), \mathcal{W})$$

described above, the differential

$$d^{0,1}: H^0(\mathrm{BSL}_2, \mathcal{H}^1(\mathbb{P}(E_2), \mathcal{W})) \to H^2(\mathrm{BSL}_2, \mathcal{H}^0(\mathbb{P}(E_2), \mathcal{W}))$$

satisfies

$$d^{0,1}(\langle 1 \rangle) = e(E_2).$$

*Proof.* Just apply the Hopf map isomorphism to the spectral sequence to reduce to Lemma 11.3.  $\Box$ 

Remark 11.5. The isomorphisms described above arise from the isomorphisms

$$H^0(\mathbb{P}^1_K, \mathcal{W}) \cong W(K), \ H^1(\mathbb{P}^1_K, \mathcal{W})) \cong W(K)$$

defined as follows: for  $H^0$ , this is the pullback by the structure map. For  $H^1$ , this is the W(K)-module map that sends  $\langle 1 \rangle \in W(K)$  to  $\langle 1 \rangle_{X_0=0} \otimes d(X_0/X_1)$ . Indeed, it is easy to check that  $\eta^*$  sends these isomorphisms to the ones for  $H^*(\mathbb{A}^2 \setminus \{0\}, \mathcal{W})$  we have used above.

We now use the results discussed above to compute  $H^*_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \backslash \mathbb{P}(F), \mathcal{W}(\omega))$ . Here  $\omega$  is the canonically  $\mathrm{SL}_2$ -linearized sheaf  $\omega_{\mathbb{P}(\mathrm{Sym}^2 F) \backslash \mathbb{P}(F)/k}$ .

We recall some facts from [18]. We identify  $\operatorname{Sym}^2 F$  with the space of quadratic forms, with coordinates  $(t_0, t_1, t_2)$  corresponding to the form  $q(X, Y) = t_0 X^2 + t_1 XY + t_2 Y^2$ . Under this identification,  $\mathbb{P}(F)$  is the curve  $D \subset \mathbb{P}^2$  and we have an isomorphism of  $\operatorname{SL}_2$ -homogeneous spaces

$$N \backslash \operatorname{SL}_2 \cong \mathbb{P}(\operatorname{Sym}^2 F) \backslash \mathbb{P}(F).$$

In [18, Proposition 5.5], we have computed  $H_{\mathrm{SL}_2}^*(N\backslash \mathrm{SL}_2, \mathcal{W})$  as the  $H^*(\mathrm{BSL}_2, \mathcal{W}) = W(k)[e]$ -algebra W(k)[e,x]/(1+x)e = 0, with  $x := [Q] \in H_{\mathrm{SL}_2}^0(N\backslash \mathrm{SL}_2, \mathcal{W})$  corresponding to the  $\mathrm{SL}_2$ -invariant quadratic form (with values in  $\mathcal{O}(2)$ )  $\langle T_1^2 - 4T_0T_2 \rangle$ .

We first compute  $H^*(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$  via the localization sequence

$$\dots \to H^{p-1}(D, \mathcal{W}(\omega_D)) \xrightarrow{i_{D*}} H^p(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$$

$$\xrightarrow{j^*} H^p(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$$

$$\xrightarrow{\partial} H^p(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}) \to \dots$$

We have  $H^p(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))) = 0$  for  $q \neq 2$ . Letting  $e' = e(T_{\mathbb{P}(\mathrm{Sym}^2 F)}) \in H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$ ,  $H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  is the free W(k)-module with basis e'.

The Leray spectral sequence for  $H^*_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  thus degenerates at  $E_2$ , giving the isomorphism of  $H^*_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  with the free  $H^*(\mathrm{BSL}_2, \mathcal{W}) = W(k)$  module with basis  $e_{\mathrm{SL}_2}(T_{\mathbb{P}(\mathrm{Sym}^2 F)})$  (in degree 2).

We use the generator  $d(x_1/x_0)$  for  $\omega_{\mathbb{P}(F)}$  on  $\mathbb{P}(F) \setminus \{(0,1)\}$ , and  $-d(x_0/x_1) = (x_0/x_1)^2 d(x_1/x_0)$  on  $\mathbb{P}(F) \setminus \{(1,0)\}$ . Sending  $\alpha \in W(k)$  to  $\alpha \otimes d(x_1/x_0)$  gives our chosen isomorphism  $H^0(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)})) = H^0_{\mathrm{SL}_2}(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)})) \cong W(k)$  and  $i_{(1,0)*}: W(k) \to H^1(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}))$  gives the canonical isomorphism  $W(k) = H^1(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)})) = H^1_{\mathrm{SL}_2}(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}))$ . We use the same isomorphisms on D, via our chosen  $\mathrm{SL}_2$ -equivariant isomorphism  $\mathbb{P}(F) \cong D$ .

The localization sequence for  $\mathbb{P}(F) \subset \mathbb{P}(\mathrm{Sym}^2 F)$  and our description of

$$H^*_{\operatorname{SL}_2}(\mathbb{P}(F), \mathcal{W})(\omega_{\mathbb{P}(F)})) = W(k)$$

gives the isomorphism

$$j^*: H^p_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F), \mathcal{W}(\omega)) \xrightarrow{\sim} H^p_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$$

for  $p \ge 1$  and the isomorphism

$$H^0_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega)) \xrightarrow{\partial} H^0_{\operatorname{SL}_2}(\mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)})) \cong W(k)$$

for p = 0.

We have the SL<sub>3</sub>-invariant generating section  $\Omega$  of  $\omega(3)$ ,

(11.1) 
$$\Omega = T_0 dT_1 dT_2 - T_1 dT_0 dT_2 + T_2 dT_1 dT_0 dT_1.$$

**Lemma 11.6.** Let  $[Q'] \in H^0(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$  be the unique element with  $\partial([Q']) = \langle 1 \rangle \otimes d(x_1/x_0)$ . For  $L = L(T_0, T_1, T_2)$  a linear form, [Q'] on is represented on  $\mathbb{P}(\operatorname{Sym}^2 F) \setminus (\mathbb{P}(F) \cup L)$  by the  $\omega$ -valued quadratic form

$$Q'_L(x,y) := -T_0 x^2 - T_1 xy - T_2 y^2 \otimes \frac{2\Omega}{L^4}$$

Proof. The form  $-(T_0/L)x^2-(T_1/L)xy-(T_2/L)y^2$  has discriminant  $(T_0T_2-(1/4)T_1^2)/L^2$ , and  $Q'_L = (L'/L)^4Q'_{L'}$ , hence the  $Q'_L$  defines a section [Q'] of  $\mathcal{W}(\omega)$  over  $\mathbb{P}^2 \setminus D$ ; since the  $\mathrm{SL}_2$  action on  $H^0(\mathcal{W}(\omega))$  is trivial, [Q'] is  $\mathrm{SL}_2$ -invariant.

To compute  $\partial_D([Q'])$ , we may restrict to any open subset of  $\mathbb{P}^2$  that intersects D non-trivially. Take  $L = T_0$ . We can diagonalize  $Q'_{T_0}$ :

$$Q'_{T_0} \sim (\langle -1 \rangle + \langle Q/T_0^2 \rangle) \otimes \frac{2\Omega}{T_0^3}$$

Let  $t_i = T_i/T_0$ ,  $f = t_1^2 - 4t_2 = Q/T_0^2$  and  $\Omega/T_0^3 = dt_1 dt_2$ . Then  $dt_1 dt_2 = (1/4) df dt_1$  and thus

$$\partial_D([Q']) = \partial_f(\langle f \rangle \otimes \partial/\partial f \otimes 2dfdt_1) = \langle 2 \rangle \otimes dt_1 \text{ on } D$$

Since  $t_1 = T_1/T_0 = 2x_1/x_0$ , this gives us

$$\partial_D([Q']) = \langle 2 \rangle \otimes dt_1 \text{ on } D = \langle 1 \rangle \otimes d(x_1/x_0)$$

**Lemma 11.7.** 1. With respect to the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathrm{BSL}_2, \mathcal{H}^q(\mathcal{W}(\omega))) \Rightarrow H^{p+q}_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$$

the edge homomorphism

$$H^2(\mathrm{BSL}_2, \mathcal{H}^0(\mathcal{W}(\omega))) \to H^2_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$$

is an isomorphism.

2. Using  $[Q'] \in H^0(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$  as  $\mathcal{W}$ -generator for the sheaf  $\mathcal{H}^0(\mathcal{W}(\omega))) \cong \mathcal{W}$  on  $\operatorname{BSL}_2$ , we have the isomorphism

$$H^2(\mathrm{BSL}_2, \mathcal{H}^0_{\mathrm{SL}_2}(\mathcal{W}(\omega))) = [Q'] \cdot H^2(\mathrm{BSL}_2, \mathcal{W}) = W(k) \cdot ([Q'] \cdot e)$$

3. Under the isomorphisms of (1) and (2), the Euler class  $e_{SL_2}(T_{\mathbb{P}(Sym^2F)\setminus\mathbb{P}(F)})$  gets sent to  $[Q']\cdot e$ .

*Proof.* Let  $j: \mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F) \to \mathbb{P}(\operatorname{Sym}^2 F)$  be the inclusion, so we have

$$e_{\mathrm{SL}_2}(T_{\mathbb{P}(\mathrm{Sym}^2F)\backslash\mathbb{P}(F)}) = j^*e_{\mathrm{SL}_2}(T_{\mathbb{P}(\mathrm{Sym}^2F)}).$$

Since the inclusion  $\mathbb{A}^3 \setminus \{0\} \to \mathrm{BSL}_2$  induces an isomorphism on  $H^p(-, \mathcal{W}(1))$  for  $p \leq 2$ , we may restrict to the corresponding bundles over  $\mathbb{A}^3 \setminus \{0\}$ .

Under the isomorphism

$$\check{H}^2(\mathcal{U}, H^q(\mathcal{C}^*_{RS, \mathbb{P}(\operatorname{Sym}^2F)}(\mathcal{W}(\omega)))) \to H^2(\mathbb{A}^3 \setminus \{0\}, R^q p_{\mathbb{P}(\operatorname{Sym}^2F)*}\mathcal{W}(\omega))$$

of Lemma 11.2, the Euler class  $e_{SL_2}(T_{\mathbb{P}(Sym^2F)})$  is represented by a sum of elements

$$e_{p,q}^{\mathrm{SL}_2}(T_{\mathbb{P}(\mathrm{Sym}^2F)}) \in \check{C}^p(\mathcal{U}, \mathcal{C}_{RS, \mathbb{P}(\mathrm{Sym}^2F)}^q(\mathcal{W}(\omega))), \ p+q=2.$$

Via the isomorphisms

$$H^{0}(\mathbb{A}^{3} \setminus \{0\}, R^{2} p_{\mathbb{P}(\operatorname{Sym}^{2} F)*} \mathcal{W}(\omega)) \cong H^{0}(\mathbb{A}^{3} \setminus \{0\}, \mathcal{H}^{2}(\mathcal{W}(\omega)))$$
  
$$\cong H^{2}(\mathbb{P}(\operatorname{Sym}^{2} F), \mathcal{W}(\omega_{\mathbb{P}(\operatorname{Sym}^{2} F)})) \cong W(k)$$

the image  $[e_{0,2}] \in H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega)))$  of the term  $e_{0,2}$  determines the class of  $e_{\mathrm{SL}_2}(T_{\mathbb{P}(\mathrm{Sym}^2 F)})$ , and is equal to the (non-equivariant) Euler class  $e(T_{\mathbb{P}(\mathrm{Sym}^2 F)})$ . This gives the element  $e_{0,2}^i \in C_{RS}^2(U_i \times \mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  with  $e_{0,2}^i = p_2^*([e_{0,2}])$ , and yields our representative  $e_{0,2} := \{e_{0,2}^i\}_i \in \check{C}^0(\mathcal{U}, \mathcal{H}^2(\mathcal{W}(\omega)))$ .

By the Gauß-Bonnet theorem for the W(k)-valued Euler characteristic  $\chi^{\mathcal{W}}(-/k)$  [19, Theorem 1.5], and the computation of  $\chi(\mathbb{P}^2/k) = \langle 1 \rangle + H \in \mathrm{GW}(k)$  by Hoyois [15, Example 1.7], we have

$$\pi_{\mathbb{P}(\mathrm{Sym}^2 F)*}(e(T_{\mathbb{P}(\mathrm{Sym}^2 F)})) = \chi^{\mathcal{W}}(\mathbb{P}^2/k) = \langle 1 \rangle$$

in W(k). In addition,

$$\pi_{\mathbb{P}(\mathrm{Sym}^2 F)*}: H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))) \to W(k)$$

is the isomorphism we use to identify  $H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega)))$  with W(k). Moreover,  $\pi_{\mathbb{P}(\mathrm{Sym}^2 F)*}$  is inverse to the isomorphism

$$i_{p*}: W(k) \to H^2(\mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega)))$$

for the inclusion  $i_p : \operatorname{Spec} k \to \mathbb{P}(\operatorname{Sym}^2 F)$  for an arbitrary k-point p of  $\mathbb{P}(\operatorname{Sym}^2 F)$ ; these are canonical push-forwards, so do not require any choices.

Thus, we may take  $p = (1, 0, 0) \in \mathbb{P}(F) \subset \mathbb{P}(\mathrm{Sym}^2 F)$ , giving the representative

$$\langle 1 \rangle \otimes \partial / \partial t_1 \wedge \partial / \partial t_2 \otimes dt_1 dt_2$$
 on  $p$ 

for  $e(T_{\mathbb{P}(\mathrm{Sym}^2 F)})$  in the Rost-Schmid complex, where  $t_i := T_i/T_0$ . Under our isomorphism of  $\mathbb{P}(F)$  with  $\mathbb{P}^1$ , we have

 $\langle 1 \rangle \otimes \partial / \partial t_1 \wedge \partial / \partial t_2 \otimes dt_1 dt_2$  on  $p = i_{\mathbb{P}(F)_*}(\langle 1 \rangle \otimes \partial / \partial (x_1/x_0) \otimes d(x_1/x_0)$  on (1,0)).

We define  $e^i_{\mathbb{P}(F),0,1} \in C^1_{RS}(U_i \times \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}))$  to be the uniquely determined elements with  $i_{\mathbb{P}(F)*}(e^i_{\mathbb{P}(F),0,1}) = e^i_{0,2}$ . Explicitly, this is

$$e_{\mathbb{P}(F),0,1}^i = \langle 1 \rangle \otimes d(x_1/x_0)$$
 on  $U_i \times (1,0)$ .

The term  $e_{1,1}$  is a collection of elements  $\{e_{1,1}^{ij} \in C_{RS}^1(U_{ij} \times \mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))\}$  with

$$\partial_{RS}^{1}(\{e_{1,1}^{ij}\}_{ij}) = \check{\delta}^{0}(\{e_{0,2}^{i}\}_{i})$$

where  $\check{\delta}^0, \partial_{RS}^1$  are the Čech, respectively, Rost-Schmid, coboundaries. Since the canonical push-forward  $i_{\mathbb{P}(F)*}$  gives a map of the Čech-Rost-Schmid double complex, we can solve this equation by taking  $e_{1,1}^{ij}=i_{\mathbb{P}(F)*}(e_{\mathbb{P}(F),1,0}^{i,j})$ , where the  $e_{\mathbb{P}(F),1,0}^{i,j}\in C_{RS}^0(U_{ij}\times\mathbb{P}(F),\mathcal{W})$  solve the corresponding equation for  $\{e_{\mathbb{P}(F),0,1}^i\}_i$ , that is

$$\partial_{RS}^{0}(\{e_{\mathbb{P}(F),1,0}^{ij}\}_{ij}) = \check{\delta}^{0}(\{e_{\mathbb{P}(F),0,1}^{i}\}_{i})$$

The  $e^{ij}_{\mathbb{P}(F),1,1}$  exist since  $\mathrm{SL}_2$  acts trivially on  $H^1(\mathbb{P}(F),\mathcal{W})$ , so for each ij,  $\check{\delta}^0(\{e^i_{\mathbb{P}(F),0,1}\}_i)_{ij}$  goes to zero in  $H^1(U_{ij}\times\mathbb{P}(F))=H^1(C^*_{RS}(U_{ij}\times\mathbb{P}(F),\mathcal{W}).$ 

Finally,  $e_{2,0}$  is an element  $e_{2,0}^{012} \in C_{RS}^0(U_{012} \times \mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  with

$$\partial_{RS}^{0}(\{e_{2,0}^{012}\}_{012}) = \check{\delta}^{1}(\{e_{1,1}^{ij}\}_{ij})$$

Letting  $e^{012}_{\mathbb{P}(F),2,0} \in H^0(U_{012} \times \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}))$  be the Čech coboundary  $\check{\delta}^1(\{e^{ij}_{\mathbb{P}(F),1,0}\}_{ij})$ , we have

$$\check{\delta}^1(\{e_{1,1}^{i,j}\}_{ij}) = i_{\mathbb{P}(F)*}(e_{\mathbb{P}(F),2,0}^{012}).$$

so

(11.2) 
$$\partial_{RS}^{0}(e_{2,0}^{012}) = i_{\mathbb{P}(F)*}(e_{\mathbb{P}(F),2,0}^{012}).$$

Letting K be the function field of  $U_{012} \times \mathbb{P}(\mathrm{Sym}^2 F)$ ,  $C_{RS}^0(U_{012} \times \mathbb{P}(\mathrm{Sym}^2 F), \mathcal{W}(\omega))$  is by definition  $W(K; \omega)$ , and the Rost-Schmid boundary map is

$$W(K;\omega) \xrightarrow{\partial_{RS}^0 = \oplus_x \partial_{RS,x}^0} \oplus_{x \in (U_{012} \times \mathbb{P}(\mathrm{Sym}^2 F))^{(1)}} W(k(x);\omega \otimes \mathfrak{m}_x/\mathfrak{m}_x^2)$$

The identity (11.2) says that  $\partial_{RS,x}^0 = 0$  except for x the generic point  $\eta$  of  $U_{012} \times \mathbb{P}(F)$  and

$$\partial_{RS,\eta}^0(e_{2,0}^{012}) = e_{\mathbb{P}(F),2,0}^{012} \in W(k(U_{012} \times \mathbb{P}(F)); \omega_{\mathbb{P}(F)})$$

This implies that  $e_{2,0}^{012}$  extends uniquely to an element  $e_{012} \in H^0(U_{012} \times (\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$  and

$$\partial_{\mathbb{P}(F)}(e_{012}) = e^{012}_{\mathbb{P}(F),2,0} \in H^0(U_{012} \times \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}),$$

where

$$\partial_{\mathbb{P}(F)}: H^0(U_{012} \times (\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F)) \to H^0(U_{012} \times \mathbb{P}(F), \mathcal{W}(\omega_{\mathbb{P}(F)}))$$

is the coboundary in the localization sequence for  $U_{012} \times \mathbb{P}(F) \subset U_{012} \times \mathbb{P}(\mathrm{Sym}^2 F)$ .

From our construction, the steps we used to construct  $e_{\mathbb{P}(F),2,0}$  show that the class  $[e_{\mathbb{P}(F),2,0}] \in \check{H}^2(\mathbb{A}^3 \setminus \{0\}, \mathcal{H}^0(\mathbb{P}(F), \mathcal{W}(\omega)))$  represented by  $e_{\mathbb{P}(F),2,0}^{012}$  is exactly  $d_2^{0,1}(i_{(1,0)*}(\langle 1 \rangle))$ . By Lemma 11.3, we have

$$[d_2^{0,1}(i_{(1,0)*}(\langle 1 \rangle))] = e \otimes d(x_1/x_0),$$

and by Lemma 11.6 (or really, the definition of [Q']), we may take

$$e_{2,0}^{012} := [Q'] \cdot \tilde{e}$$

where  $\tilde{e} \in \check{C}^2(\mathbb{A}^3 \setminus \{0\}, \mathcal{W})$  represents e.

Now we apply  $j^*$ . From the Leray spectral sequence, we see that the edge homomorphism

$$H^2(\mathrm{BSL}_2, \mathcal{H}^0(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))) \to H^2_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$$

is an isomorphism. Since we can compute  $j^*e_{\operatorname{SL}_2}(T_{\mathbb{P}(\operatorname{Sym}^2F)})$  by restricting the terms in our double complex to  $\mathcal{U} \times \mathbb{P}(\operatorname{Sym}^2F) \setminus \mathbb{P}(F)$ , the term  $e_{2,0}^{012}$  represents a class in  $H^2(\operatorname{BSL}_2, \mathcal{H}^0(\mathbb{P}(\operatorname{Sym}^2F) \setminus \mathbb{P}(F), \mathcal{W}(\omega)) = H^2(\mathbb{A}^3 \setminus \{0\}, \mathcal{H}^0(\mathbb{P}(\operatorname{Sym}^2F) \setminus \mathbb{P}(F), \mathcal{W}(\omega))$  that maps to  $j^*e_{\operatorname{SL}_2}(T_{\mathbb{P}(\operatorname{Sym}^2F)}) = e_{\operatorname{SL}_2}(T_{\mathbb{P}(\operatorname{Sym}^2F) \setminus \mathbb{P}(F)})$ , which completes the proof.

**Lemma 11.8.** With respect to the canonical isomorphism  $W(\omega^{\otimes 2}) \cong W$ , we have the following identities.

1. 
$$[Q']^2 = 2 \cdot (1 - [Q])$$
 in  $H^0_{\mathrm{SL}_2}(\mathbb{P}(\mathrm{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W})$ .

2. 
$$[Q'] \cdot e_{\operatorname{SL}_2}(T_{X_a}) = 4e \text{ in } H^2_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W}).$$

3.  $(e_{\operatorname{SL}_2}(T_{X_a}))^2 = 4e^2$  in  $H^2_{\operatorname{SL}_2}(\mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F), \mathcal{W})$ .

*Proof.* Let  $Y = \mathbb{P}(\operatorname{Sym}^2 F) \setminus \mathbb{P}(F)$ .

(1) Since  $H^0_{\mathrm{SL}_2}(Y, \mathcal{W}) = H^0(Y, \mathcal{W})$ , we can pass to any open subset of Y to compute  $[Q']^2$ . After inverting  $T_0$ , we diagonalize Q' as

$$q' = (\langle -1 \rangle + \langle Q/T_0^2 \rangle) \otimes \frac{2\Omega}{T_0^3} = (\langle -1 \rangle + \langle Q \rangle) \otimes \frac{2\Omega}{T_0^3},$$

so after applying the isomorphism  $\mathcal{W}(\omega^{\otimes 2}) \cong \mathcal{W}$ , we have

$$[Q']^2 = (-1 + [Q])^2 = 2 \cdot (1 - [Q])$$

For (2) we have already shown that  $e_{SL_2}(T_Y) = [Q'] \cdot e$ , so

$$[Q'] \cdot e_{\mathrm{SL}_2}(T_Y) = [Q']^2 \cdot e = 2 \cdot (1 - [Q])e^{-\frac{1}{2}}$$

Since  $[Q] \cdot e = -\langle 1 \rangle \cdot e$ , we have  $[Q] \cdot e_{SL_2}(T_{X_a}) = 4e$ .

(3) is similar:  $e_{SL_2}(T_Y) = [Q'] \cdot e \Rightarrow$ 

$$(e_{\mathrm{SL}_2}(T_Y))^2 = [Q']^2 \cdot e \cdot e = 4e^2$$

by (2).

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