Example Sheet 4

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EXERCISE 1. Let D be a degree 0 Weil divisor on $\mathbb{P}^1_k$. Construct a rational function f, i.e. a section of the structure sheaf over the generic point, such that the divisor associated to f is D. Deduce that every degree 0 divisor on $\mathbb{P}^1_k$ is principal.

> *Proof:* Let $D = D_1 - D_2$, where D_1 and D_2 are two effective Weil divisors.
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> Recall first that any codimension one subvariety $Y = V(\mathfrak{a}) =\subset \mathbb{P}^1_k$ is determined by a single homogeneous equation. To see this, suppose $f \in \mathfrak{a}$ is some nonzero homogeneous equation. Then $V(\mathfrak{a}) \subseteq V(f) \subsetneq \mathbb{P}^1_k$, hence $Y = V(\mathfrak{a}) = V(f)$.
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> We now suppose that D is a Weil divisor on $\mathbb{P}^1_k$ as in the question. Write $D = D_1 - D_2$ for some effective divisors D_1 and D_2, where
> $$D_1 = \sum n_i B_i \quad \text{and} \quad D_2 = \sum m_j C_j$$
> for prime divisors B_i and C_j. For each i, $B_i = V(f_i)$ for some homogeneous irreducible polynomial by our earlier comment, and likewise $C_j = V(g_j)$ with g_j an irreducible homogeneous polynomial for each j. Define $f = \prod_i f_i^{n_i}$ and $g = \prod_j g_j^{m_j}$ and let $h = \frac{f}{g}$. Because $\deg(D) = 0$ we have by design that $\deg(f) = \deg(D_1) = \deg(D_2) = \deg(g)$, and hence $h \in K^*$ as it is a quotient of two homogeneous function of equal degree. We have that
> $$(h) = (f) - (g) = \sum_i n_i B_i \; - \; \sum_j m_j C_j = D_1 - D_2 = D,$$
> and therefore D is principal.
>
> By starting with a rational function $\frac{f}{g}$ in K^* and factoring both the numerator and denominator into irreducible components, we similarly see that $(f) = 0$ and can then conclude that a Weil divisor D has degree 0 if and only if it is principal. $\square$