Notes for Tropical Geometry

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§ *Entry 1*

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0   Introduction/Motivation

Tropical geometry is the study of discrete structures appearing in limits of polynomial equations.
Course outline:

(1) Hypersurface amoebas, their skeleta, and tropical limits

(2)

1   Hypersurface amoebas, their skeleta, and tropical limits

1.1   Laurent polynomial ring

$\mathbb{C}[z_1^{\pm 1}, ..., z_n^{\pm}]$. Each such Laurent polynomial defines a holomorphic (algebraic) map $f : (\mathbb{C}^\times)^n \to \mathbb{C}$ whose zero locus $V(f) \subseteq (\mathbb{C}^*)^n$ $f \neq 0$ is a **complex hypersurface.** The ring $\mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}]$ is a unique factorization domain which implies $f = f_1^{\alpha_1} \cdot ... \cdot f_m^{\alpha_m}$ where the f_i are ireducible, pairwise different, and hence $Z(f) = Z(f_1) \cup ... \cup Z(f_m)$. This locus is *always* a complex submanifold, even in the case of the nodal cubic for instance, of $\dim_{\mathbb{C}} = n - 1$ outside of a real codimension 2 subset $Z(f) \cap Z(\partial_1 f) \cap ... \cap Z(\partial_n f)$.

Example 1.1.

(a) $V(z + w) \subseteq (\mathbb{C}^\times)^2$ is isomorphic as a $\mathbb{C}$-manifold or as an algebraic variety to $\mathbb{C}^\times$. The map $\mathbb{C}^\times \mapsto V(z + w)$ given $u \mapsto (u, -u)$ parameterizes this curve.

(b) $V(z + w + 1) \subseteq (\mathbb{C}^\times)^2$ is isomorphic to $\mathbb{C}^\times \setminus \{0, 1\}$ via the map $u \mapsto (u, 1 - u)$.

1.2   The Log Map

Forget phases and use logarithmic coordinates.

$$\text{Log} : (\mathbb{C}^\times)^n \xrightarrow{1.1} \mathbb{R}_{>0}^n \xrightarrow{\log} \mathbb{R}^n$$

given by

$$(z_1, ..., z_n) \mapsto (|z_1|, ..., |z_n|) \mapsto (\log |z_1|, ..., \log |z_n|).$$

Definition 1.2. The **Hypersurface amoeba** of $f \in \mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}] \setminus \{0\}$ is

$$\mathcal{A}_f = \text{Log}(V(f)) \subseteq \mathbb{R}^n$$

(Gelfand, Vapranov, Zelevabsky)

Example 1.3.

(a) $f = z + w$

(b) $f = z + w + 1$

(c) $f = 1 + 5zw + w^2 - z^2 + 3z^2w - z^2w^2$

(add pictures later) careful to draw these such that the complements of the amoeba are all convex.

<u>Observations:</u>

- connected cusps of $\mathbb{R}^n \setminus C_f$ are convex in $\dim = 2$. $\mathcal{A}_f$ looks like a thickened graph. We'll sketch a proof of a more general result.

<u>Recall:</u> $\mathcal{U} \subseteq \mathbb{C}$, $f : \mathcal{U} \setminus \{p_1, ..., p_r\} \to \mathbb{C}$ are meromorphic with mkr poles $(p_1, ..., p_r)$ and s zeros with multiplicity. This implies

$$s - r = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

This is the argument principle from complex analysis. Appears in the derivative of $\frac{1}{2\pi i} \int_{S^1} \log|f| dz$. This appears in the Jensen formula: $\mathcal{U} \subseteq \mathbb{C}$ an open subset and assume it contains a closed disk of radius r $\{z \mid |z| \le r\} = D$. Important that it includes the boundary. Then if we have a holomorphic function $f : \mathcal{U} \to \mathbb{C}$ with zeros of f in D $a_1, ..., a_k$ such that $0 < |a_1| \le |a_2| \le ... \le |a_k|$ (with multiplicity) then we have

$$\frac{1}{2\pi i} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta = \log|f(0)| + \sum_{j=1}^k \log \frac{r}{|a_j|}.$$

This is the Jensen formula.

Proof. (Rudin, "Real and complex analysis")

(1) Assume f has no zeros and hence that $\log|f|$ harmonic. Using the mean value property for harmonic functions (go review Analysis) yields the Jensen Formula.

(2) For the general case, suppose we have $|a_1|, ..., |a_n| < r$, and then that $|a_{m+1}|, ..., |a_k| = r$. Consider $g(z) = f(z) \cdot \prod_{j=1}^m \frac{r^2 - \bar{a}_j z}{r(a_j - z)} \prod_{j=m+1}^k \frac{a_j}{a_j - z}$ with no zeros in $|z| \le r$. This implies

$$g(0) = f(0) \cdot \prod_{j=1}^m \frac{r}{a_j}$$

by our first case.

(3) $|z| = r$, so on the boundary, we have

$$\left| \frac{r^2 - a_j z}{r(a_j - z)} \right| = \frac{1}{r} \left| \frac{r^2 \bar{z} - a_j |z|^2}{r(a_j - z)} \right| = \frac{r}{r} = 1$$

$$\implies \log|g(re^{i\theta})| = \log|f(re^{i\theta})| - \sum_{j=m+1}^k \log|1 - \overbrace{e^{i(\theta - \theta_j)}}^{a_j = re^{i\theta_j}}|$$

(4) Lemma: $\int_0^{2\pi} \log(1 - e^{i\theta})d\theta = 0$. These four things together prove the Jensen formula.

$\square$

For $n > 1$ we define something called the Ronkin function. We have $f \in \mathcal{O}(\mathrm{Log}^{-1}(\Omega))$, $\Omega \subseteq \mathbb{R}^n$ a (convex) open set. Then the **Ronkin Function** is defined

$$N_f(x) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathrm{Log}^{-1}(x)} \mathrm{Log}\,|f(z_1, ..., z_n)| \frac{dz_1}{z_1} \vee \cdots \vee \frac{dz_n}{z_n}$$

Theorem 1.4. *(a) N_f is a convex $\mathcal{C}^0$-function*

(b) $\mathcal{A}_f = \mathrm{Log}(V(f)) \subseteq \Omega$ an Amoeba. For all $\mathcal{U} \subseteq \Omega$ open, connected $\mathcal{U} \cap \mathcal{A}_f = \emptyset \iff N_f|_{\mathcal{U}}$ affine linear.

(c) $x \in \Omega \setminus \mathcal{A}_f \implies \mathrm{grad}\, N_f(x) = (v_1, ..., v_n)$,

$$v_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \vee \cdots \vee \frac{dz_n}{z_n}.$$

Picture: $N_f(x) = \langle \alpha_1, x \rangle + c_1$

Proof. (sketch)

(a) $\log |f|$ is plurisubharmonic (i.e. is subharmonic (i.e. somehow less than harmonic functions on a circle) on each each holomorphic image of a disk). We have the following fact: if $h : \mathcal{U} \to \mathbb{R}$ is subharmonic, $\mathcal{U} \subseteq \mathbb{C}$ a domain containing $\{|z| \leq R\}$, then $\varphi(r) = \int_{|z|=r=\exp(s)} h(x)dz$ is a convex function in $\log r = s$. Found this proof in a book of Runkin called "Introduction to the theory of entire functions," page 84.

(b) Prove this next time

(c) $x \in \Omega \setminus \mathcal{A}_f$. Note:

$$\frac{\partial}{\partial x_j} \log |f| = \frac{1}{2} \frac{\partial}{\partial x_j} \log(f\overline{f}) = \mathrm{Re}\left(z_j \frac{\partial}{\partial z_j} \log f\overline{f}\right) = \mathrm{Re}\left(\frac{z_j \partial_j f}{f}\right).$$

$x \in \Omega \setminus \mathcal{A}_f$ implies that

$$\frac{\partial}{\partial x_j} N_f(x) = \mathrm{Re}\left(\frac{1}{2\pi i}^n \int_{\mathrm{Log}^{-1}} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}\right).$$

Note: for all j, we have

$$\gamma_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

This is a locally constant n-form on $\mathcal{U} \setminus \mathcal{A}_f$ and is not defined on $\mathcal{A}_f$ since f is zero on $\mathcal{A}_f$. In fact, $\gamma_j \in \mathbb{Z} : \frac{1}{2\pi i} \int_{|z_j|=e^{x_j}} \frac{\partial_j f(z)}{f(z)} dz_j \in \mathbb{Z}$ by the argument principle.

Look at Passare, Rullgard "Amoebas, Monge – Ampere, measures and triangulations DMJ 2004" $\square$

§ *Entry 2*

Recall that last time we had $V(f) \subseteq (\mathbb{C}^\times)^n \xrightarrow{\text{Log}} \mathbb{R}^n$, and we took $f \in \mathbb{C}[z_1^\pm, ..., z_n^\pm]$. This map has image in $\mathcal{A}_f \subseteq \mathbb{R}^n$. Recall also that the complement of the amoeba decomposes as the following union of connected components.

$$\mathbb{R}^n \setminus \mathcal{A}_f = \Omega_1 \cup ... \cup \Omega_k.$$

These connected components correspond to integral points of the Newton polyhedron $\mathrm{conv}\{I \mid a_I \neq 0\}$ where $f = \sum_{\text{finite}} a_I z^I$. Ronkin function is

$$N_f(x) = \frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(x)} \text{Log}\,|f(x)| \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}$$

is convex on $\mathbb{R}^n$ and is **affine linear on each** Ω_i which then implies that each Ω_i is convex.

Note: $\mathcal{U} = \text{Log}^{-1}(\Omega)$, where Ω is open, connected is a **circular domain**, i.e. change the argument of an element in the set and you're still in the set. These are called **Reinhardt domains**.

It is a fact that $\mathcal{U}$ is a domain of holomorphy if and only if Ω is convex. Laurent series converge on $\text{Log}^{-1}(\Omega)$ since Ω is convex.

Corollary 1.5. $\text{Log}^{-1}(\Omega_i)$ are the domains of convergence of the Laurent series expansions of f.

1.3 The spine of a hypersurface amoeba

Let $\varphi_i = N_f|_{\text{Log}^{-1}(\Omega_i)} = \langle \alpha_i, \cdot \rangle + c_i$ with $\alpha_i \in (\mathbb{R}^n)^*$ and $c_i \in \mathbb{R}$ be the piecewise affine approximation of N_f. Define

$$\varphi = \max\{\varphi_i\}.$$

Note that whenever N_f is convex we get that $\varphi \leq N_f$. **CHECK THIS, SWAPPED FROM MIN TO MAX, CHECK THIS INEQUALITY REMAINS SAME**

Definition 1.6.

$$
\begin{aligned}
\varphi_f &:= \{x \in \mathbb{R}^n \mid \varphi \text{ not affine linear near } x\} \\
&= \{x \in \mathbb{R}^n \mid \varphi \text{ not differentiable at } x\} \\
&= \{x \in \mathbb{R}^n \mid \exists i \neq j \text{ s.t. } \varphi_i(x) = \varphi_j(x = \max_k\{\varphi_k(x)\})\}
\end{aligned}
$$

is called the **spine** of $\mathcal{A}_f$.

Theorem 1.7. *[(Passare, Rullgard)]*

  (a) φ_f *is the* $(n-1)$*-skeleton of a face-fitting decomposition of* $\mathbb{R}^n$ *into convex (with integrally defined facets) polyhedra.*

  (b) $\mathcal{A}_f$ *deformation retracts onto* φ_f.

This notation is slightly confusing to me – φ_f is a subset of the graph of φ_f, it is not itself a function.

1.4 Tropical Limits and Maslov "dequantization"

$(\mathbb{R}_{>0}, +, \cdot) \xrightarrow{h \cdot \log = \log_t} (\mathbb{R}, \oplus_h, \odot_h)$ is a semiring isomorphism. The inverse is $(\mathbb{R}_{>0}, +, \cdot) \xleftarrow{\exp(x/h) \leftarrow x}$
$(\mathbb{R}, \oplus_h, \odot)$ with

$$x \oplus_h y = h \cdot \log \left(\exp \left(\frac{x}{h} \right) + \exp \left(\frac{y}{h} \right) \right) \xrightarrow{h \to 0} \max\{x, y\}$$

$$x \odot y = h \cdot \log \left(\exp \left(\frac{x}{h} \right) \cdot \exp \left(\frac{y}{h} \right) \right) = x + y.$$

Now consider $f_h \in \mathbb{C}(h)[z_1^{\pm}, ..., z_n^{\pm}]$ e.g. $\frac{h^2+1}{h} z_1^2 + (h^3 - h^2) z_1 z_2^{-1}$. For all h we have that

$$\mathcal{A}_n(f_n) = \mathrm{Log}_t \left(V(f_h) \right) = h \cdot \mathcal{A}(f_h) \subseteq \mathbb{R}^n$$

are the amoeba for the rescaled Log-map $\mathrm{Log}_t = h \, \mathrm{Log}$. Here's a theorem from a paper prior to tropical geometry truly kicking off.

Theorem 1.8. $\mathcal{A}_h(f_h)$ *converges for $h \to 0$ in the Hausdorff distance to the tropical hypersurface $V(\mathrm{trop}(f_h))$.*

$$f_h = \alpha_1 z^{\underline{u}_1} + ... + a_r z^{\underline{u}_r}, \ a_i \in \alpha_i \in \mathbb{C}(h)$$

then

$$\mathrm{trop} \, f_h = \max\{\langle \underline{u}_1, - \rangle + c_1, ..., \langle \underline{u}_r, - \rangle + c_r\}$$

where $c_i = \mathrm{val}_0(\alpha_i)$, order of $\alpha_i(h)$ at $h = 0$.

$$\mathrm{val}_0(\frac{h^2 + 1}{h}) = -1, \mathrm{val}_0(h^3 - h^2)) = 2.$$

INCLUDE BOARD WITH HAUSDORFF DISTANCE

2 Tropical Arithmetic

2.1 Tropical semiring

Definition 2.1. $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is the tropical semiring or the min-plus algebra. We set

- $x \oplus y := \min\{x, y\}$

- $x \odot y := x + y.$

Both operations are commutative, associative, and are together distributive.

We have the following identities:

- $x \odot (y \oplus z) = x \odot y \oplus x \odot z$

- $x \oplus \infty = x$

- $x \oplus 0 = \begin{cases} 0 & x \geq 0 \\ x & x < 0 \end{cases}$

- $x \odot 0 = x$

- $x \odot \infty := \infty$

Explanation:

$$
\begin{aligned}
(x \oplus y)^3 &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\
&= 3 \min\{x, y\} \\
&= \min\{3x, 3y\} = x^3 \oplus y^3 \\
&= \min\{3x, 2x + y, x + 2y, 3y\} = x^3 \oplus x^2 y \oplus x y^2 \oplus y^3
\end{aligned}
$$

Noting that $x^3 = 0 \odot x^3$, $x^2 y = 0 \odot x^2 y$, etc. we see that these are the coefficients of Pascal's triangle in tropical land, and that the coefficients are all 0. Hence the tropical Pascal triangle is just a bunch of 0's.

2.2 Linear algebra

The usual operations (formally) make sense over $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, e.g.

$$
\begin{aligned}
(u_1, u_2, u_3) \cdot (v_1, v_2, v_3)^T &= u_1 \odot v_1 \oplus u_2 \odot v_2 \oplus u_3 \odot v_3 \\
&= \min\{u_1 + v_1, u_2 + v_2, u_3 + v_3\}.
\end{aligned}
$$

$$
(u_1, u_2, u_3)^T \odot (v_1, v_2, v_3) = \begin{pmatrix} u_1 \odot v_1 & u_1 \odot v_2 & \cdots \\ u_2 \odot v_2 & \cdots & \\ & & u_3 \ddots v_3 \end{pmatrix}
$$

Definition 2.2. Matrices that can be written as $u^t \odot v$ have **tropical rank** 1.

Definition 2.3. The Barvihok rank of $A \in M(m \times n, \mathbb{R})$ is $\min\{k \mid \exists u_1, ..., u_k, v_1, ..., v_k, A = u_1^T \odot v_1 \oplus ... \oplus u_k^T \odot v_k\}$.

There are other notions of rank: Kapronov rank, tropical rank [MLS, S.5.3].

Looking at **tropical linear systems** $A \odot x = b$ has applications in engineering, dynamic programming (optimization via recursive structures, e.g. Find a shortest (weighted) path through a directed graph) etc. More on this in section 3.

2.3 Tropical Polynomials

Definition 2.4. A **Tropical polynomial** is a Laurent polynomial over $x_1, ..., x_n$, i.e. is a function on $\mathbb{R}, \oplus, \odot)^n$. A monomial is

$$
x_1^{u_1} \odot x_2^{u_2} \cdot ... \cdot x_n^{u_n}
$$

§ *Entry 3*

Recall that a tropical polynomial $f = a_1 \odot x^{\underline{u}_1} \oplus ... \oplus a_n \odot x^{\underline{u}_n}$ is a concave piecewise affine function

$$p(x) = \min\{\langle \underline{u_1} \rangle + a_1, ..., \langle \underline{u_n}, - \rangle + a_n\}.$$

Example 2.5. $p = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d = \min\{3x + a, 2x + b, x + c, d\}$. We say that the linear breaks of this graph are the vanishing points of p.

Lemma 2.6. For any concave, piecewise affine function with $\mathbb{Z}$-derivatives $p : \mathbb{R}^n \to \mathbb{R}$ there exists a tropical polynomial f with $p(x) = (x \mapsto f(x)$ in $(\mathbb{R}, \oplus, \odot))$.

Note: $f = \oplus a_I \odot x^I$ is only unique if we assume that for each I with $a_I \neq \infty$ we have that the map $x \mapsto \langle I, x \rangle + a_I$ agrees with h in a neighborhood of $x \in \mathbb{R}^n$.

Exercise 2.7 (Tropical Fundamental Theorem of Algebra). Every PA function $p : \mathbb{R} \to \mathbb{R}$ with integral derivatives (constant derivatives which are integers) can be written uniquely as a minimal product of tropical linear functions $a \odot x$.

Example 2.8 (Example of Tropical FTA Decomposition). Take $f = x^2 \oplus 17 \odot x \oplus 2$. We then have

$$\begin{aligned}
f &= x^2 \oplus 17 \odot \oplus x \oplus 2 \\
&= \min\{2x, x + 17, 2\} \\
&= \min\{2x, x + 1, 2\} \\
&= (x \oplus 1) \odot (x \oplus 1)
\end{aligned}$$

Unique factorization fails for $n > 1$.

Example 2.9. Take $f(x, y) = (x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0) = (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0)$. The Newton polygon of $p = a_1 x^{\underline{u}_1} \oplus ... \oplus a_r x^{\underline{u}_r}$ gives us

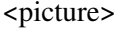$$\text{Newt}(p) = \text{conv}\{\underline{u}_i \mid a_i \neq \infty\}$$

Here: $f = x^2 y^2 \oplus x^2 y \oplus xy^2 \oplus xy \oplus x \oplus y \oplus 0.$

3 Dynamic Programming

3.1 Shortest paths in graphs

If G is a directed graph with n nodes $1, 2, ..., n$ and directed edges (i, j) have a weight $d_{ij} \in \mathbb{R}_{\geq 0}$ with $d_{ii} = 0$. We say $d_{ij} = \infty$ if there is no edge from i to j. We can conveniently present these distances in an $n \times n$ **adjacency matrix** in the extended reals, i.e

$$D_G = (d_{ij})_{i,j=1,...,n} \in M(n \times n, \mathbb{R} \cup \{\infty\}).$$

Example 3.1. <picture>

$$D_G = \begin{pmatrix} 0 & 3 & 1 & \infty \\ 1 & 0 & \infty & 3 \\ 1 & 2 & 0 & 0 \\ \infty & 1 & 1 & 0 \end{pmatrix}.$$

Proposition 3.2. The shortest length of a path from i to j is

$$(ij)\text{-entry of } D_G^{\otimes(n-1)} = \overbrace{D_G \odot ... \odot D_G}^{(n-1) - times}.$$

Proof: We have that

$$d_{ij}^r := \min\{(\text{weighted}) \text{ length of a path from } i \text{ to } j \text{ with } \leq r \text{ edges}\}.$$

We have that $d_{ij}^{(1)} = d_{ij}$. If $d_{ij} \geq 0$, then a shortest path in the number of edges runs through each node at most once (otherwise, reverse the loop from i to i to arrive at a shorter path).

This implies that $d_{ij \cdot (n-1)} = $ length of shortest weighted path from i to j. Recursively this gives

$$d_{ij}^{(r)} = \min\{d_{ik}^{(r-1)} + d_{kj} \mid k \in \{1, ..., n\}\}$$
$$= d_{i_1}^{(r-1)} \odot d_{1j} \oplus ... \oplus d_{in}^{(r-1)} \odot d_{nj}$$
$$= \begin{pmatrix} d_{i1}^{(r-1)} & \cdots & d_{in}^{(r-n)} \end{pmatrix} \odot \begin{pmatrix} d_{1j} \\ \vdots \\ d_{nj} \end{pmatrix}$$
$$= d_{ij}^{(r)} = (i, j)\text{-th entry of } D_G^{\odot r}.$$

$\square$

This can also be viewed as a limit of a quantum computation (Maslov's dequantization). Replace D_G with a matrix $A_G(\epsilon)$ where $A_G(\epsilon)_{ij} = \epsilon^{D_G(i,j)}$.

3.2 Integer Linear Programming

Given $A = (a_{ij}) \in M(d \times n, \mathbb{N})$ with $w \in \mathbb{R}^n$ and $b \in \mathbb{N}^d$. We'd like to solve the optimization problem $w \cdot u$ for $u \in \mathbb{N}^n$ subject to $Au \leq b$ or $Au = b$.

We can simplify this in the following way. For all j, take $\sum_i a_{ij} = \alpha$. Column sums are equal. We then have $b_1 + ... + b_d = m\alpha$, for $m \in \mathbb{N}$.

<u>Then</u>: $Au = b \implies u_1 + ... + u_n = m$. Indeed, $m\alpha = b_1 + ... + b_d = \sum_{i,j} \alpha_{ij} u_j = \sum_j (\sum_i a_{ij}) u_j = \alpha(u_1 + ... + u_n)$.

Proposition 3.3.

$$\min\{w \cdot u \mid Au = b\} = \text{coeff of } x_1^{b_1} \oplus ... \oplus x_d^{b_d}$$

in $(w \odot x_1^{a_{12}} \odot)$

3.3 The assignment problem and the tropical determinant

go back and review this

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§ *Entry 4*

Written: 2022-Sept-01

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4 Plane Curves

4.1 Tropical polynomial in finitely many variables

Let p be a tropical polynomial in $x_1, ..., x_n$ or let it be the associated piecewise affine function $p : \mathbb{R}^n \to \mathbb{R}$. Let $V(p) = \{x \in \mathbb{R}^n \mid x \mapsto p(x) \text{ is not locally affine }\}$. This is a **tropical hypersurface.**

For $n = 2$: we get plane tropical curves, e.g. conics, $a \odot x^2 \oplus b \odot xy \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f$. We call the graph of p a "tent over $\mathbb{R}^2$"
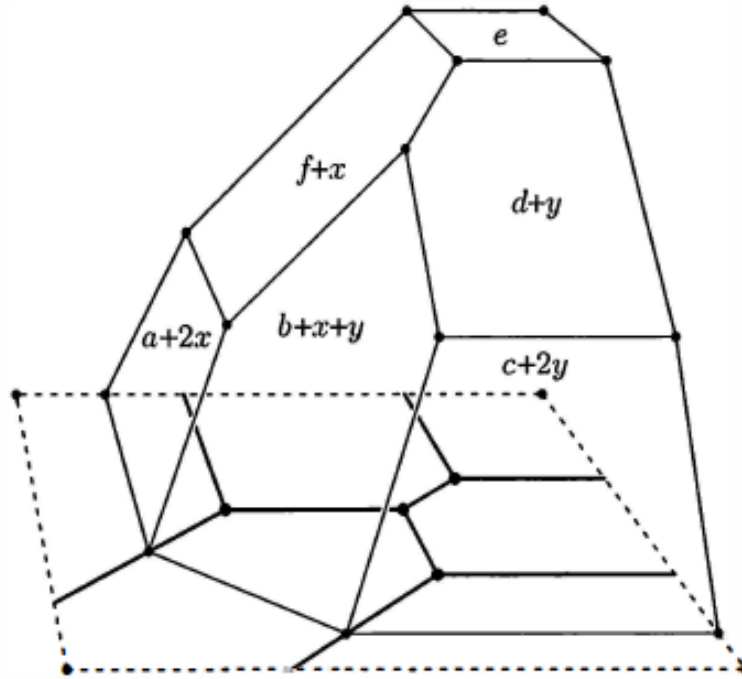

Figure 1: Tent over $\mathbb{R}^2$

Proposition 4.1. The polynomial $V(p) \subseteq \mathbb{R}^2$ is a finite embedded graph (with bounded and unbounded edges) and all edge slopes are rational. Moreoever, each edge E of $V(p)$ comes with a weight $w(E) \in \mathbb{N} \setminus \{0\}$ such that at each vertex $v \in V(p)$ we have

$$\sum w(E) \cdot u_{V,E} = 0.$$

This is called the **balancing condition**. The element $u_{V,E} \in \mathbb{Z}^n$ is a primitive vector in the direction E.

Example 4.2.

4.2 Relation to subdivisions of the Newton polygon

Let $p = \bigoplus_{I \subseteq \mathbb{Z}^2} a_I \odot x^I$.

$$\text{Newt}(p) = \text{conv}\{I \in I^2 \subseteq \mathbb{R}^2 \mid a_I \neq \infty\} \subseteq \mathbb{R}^2$$

Example 4.3. Suppose $p(x) = 1 \odot x^2 \oplus 0 \odot xy \oplus 1 \odot y^2 \oplus 0 \odot y \oplus 1$. The a_I provide a function

$$a : \text{Newt}(p) \cap \mathbb{Z}^2 \to \mathbb{R} \cup \{\infty\},$$

Given by $I \mapsto a_I$. Take the "overgraph of a" $= \text{conv}\{(I, a_I) \in \mathbb{R}^3 \mid I \in \text{Newt}(p) \cap \mathbb{Z}^2\} + \mathbb{R}_{\geq 0} \cdot (0, 0, 1)$. The lower body is the union of bd. cells of boundary which is equal to the graph of a piecewise affine function $\varphi : \text{Newt}(p) \to \mathbb{R}$

Domains of affine linearity of φ define a polydral decomposition P of $\text{Newt}(p)$ into convex polyhedra with integral vertices. In the previous example, $V(p)$ is dual to $\text{Newt}(p)$. Furthermore, the edges in the interior of $\text{Newt}(p)$ correspond to bounded edges in $V(p)$ and the boundary edges of $\text{Newt}(p)$ correspond to the unbounded edges of $V(p)$.

Proposition 4.4. $V(p)$ is combinatorially the dual complex of the 1-skeleton of P. Edges in $\partial \text{Newt}(p)$ correspond to unbounded edges (tentacles) of $V(p)$. Edge directions $u_{V,E}$ are the interior normal vectors to the 2-cell dual to v at the edge dual to E.

Example 4.5. Fill in

This proposition connecting Newton polygons to $V(p)$ also explains the balancing condition. Vertices $v \in V(p)$ is dual to a convex polygon $\sigma \in P$ with integral vertices correspond precisely to slopes of the affine functions defining $V(p)$ locally at p. The balancing condition holds if and only if the sum of the edge vectors of σ are close to a polygon.

Weights = integral lengths of edge $w = \#(w \cap \mathbb{Z}^2)$.

Note: Subdivision into standard simplicies ($n = 2$: std $\iff$ Area $= \frac{1}{2}$)

4.3 Intersections of plane tropical curves

Theorem 4.6 (Bezout's Theorem). *Suppose $f_1, f_2 \in \mathbb{C}[x, y, z]$ are homogeneous of degree $d_i > 0$. Suppose also that f_i is irreducible and that for $c \in \mathbb{C}$ $f_i = cf_j \implies i = j$. Then the degree d algebraic curve*

$$C_i = V(f_i) = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} \mid f_i(x, y, z) = 0\}$$

has intersection number $d_i d_j$ with C_j, i.e. $C_i \# C_j = d_1 \cdot d_2$. More precisely,

$$C_i \# C_j = \#(C_i^c \cap C^j)$$

if $V(f_1^c) = C_i^c$ is a small perurbation of C_i.

§ *Entry 5*

We quickly finish our summary of algebraic geometry. If $Y, Z \subseteq \mathbb{A}^n/T^n/\mathbb{P}^n$ are varieties (affine, toric, projective) with $I = I_Y = \sqrt{I}, J = I_Z = \sqrt{J}$, then

$$I_{\overline{Y \setminus Z}} = (I : J) = (I : J^\infty).$$

By taking the closure naïvely In particular, $Z = V(h)$ a hypersurface, then

$$I_{Y \setminus Z} = (I : h^\infty) \text{ (saturation)}.$$

<u>Practical computation by elimination</u>: say $I \subseteq k[x_1, ..., x_n]$,

$$(I : h^\infty) = (I, hz - 1) \cap k[x_1, ..., x_n],$$

i.e. eliminate z. Note that $(I, hz - 1)$ lies in $k[z, x_1, ..., x_n]$ and $(I, hz - 1) \simeq I \cdot k[x_1, ..., x_n]_h)$.
The geometric interpretation is $V(I, hz - 1) = $ the graph of $1/h$ in $\mathbb{A}^{n+1}$ over $Y = V(I) \subseteq \mathbb{A}^n$.

Example 4.7. Set $Y = V(I) \subseteq T^3, I = \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - 1, \frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} \right)$. Then

$$I_{\text{aff}} = I \cap K[x_1, x_2, x_3]$$

(moved on too quickly)

4.4   Morphisms

Suppose X, Y are both affine varieties, then $\varphi : X \to Y$ is given by a homomorphism $\varphi^* : K(Y) \to K(X)$.
If K is algebraically closed, then $K(Y) \subseteq \text{Map}(Y, K)$. Image: algebraic functions. A continuous map $\varphi : X \to Y$ is a morphism if and only if $\varphi^*(\text{alg. func})$ is an algebraic function. That is, you can check whether a continuous map is indeed a morphism by checking it on the regular functions.

Example 4.8. $\varphi : T_K^n \to T_K^m \leftrightarrow \varphi^* : K[x_1^\pm, ..., x_n^\pm] \to K[y_1^\pm, ..., y_m^\pm]$. Furthermore, φ is a homomorphism of algebraic groups if and only if φ^* is a monomial, i.e. induced by homomorphisms $A : \mathbb{Z}^m \to \mathbb{Z}^n$.

A very important example is that of **characters**. A character is an algebraic group homomorphism $\varphi : T_K^n \to T_K^1 = \mathbb{G}_m \leftrightarrow K[x^\pm] \to K[y_1^\pm, ..., y_n^\pm]$. The invertible elements of $K[y_1^\pm, ..., y_n^\pm]$ correspond precisely to $K^\times \times \mathbb{Z}^n \ni a \cdot y_1^{i_1}...y_n^{i_n}$.

Now, φ is a homomorphism of algebraic groups implies that

$$(1, ..., 1) \mapsto 1 \implies x - 1 \mapsto f - 1 = \lambda y_1^{i_1}...y_n^{i_n} - 1 \in (y_1 - 1, ..., y_n - 1)$$
$$= \lambda = 1.$$

Conclusion: $\{\text{characters } T_K^n \to T_K^1\} = \mathbb{Z}^n$. To make this more intrinsic, $T_K^n = K[M]$ for some $M \simeq \mathbb{Z}^n$, then the characters of T_K^n form the dual lattice $N = \text{Hom}_{grps}(T_K^n, \mathbb{Z})$.

Corollary 4.9. $\text{Aut}_{\text{grp}}(T_K^n) = \text{GL}(n, \mathbb{Z})$.

4.5 The Grassmannian

Suppose $r, m \in \mathbb{N}$, $0 < r < m$, $V = K^m$. As a set: $\mathrm{Gr}(r, m) = \{W \subseteq V$ a linear subspace , $\dim W = r\}$. Representatin of W by row space of $A \in M(r \times m, K)$...

non-uniqueness: A and $G \cdot A$ span the same W $\forall G \in \mathrm{GL}(r, K)$. instead: Take the $\binom{m}{r}$ $r \times r$-minors of A, yields $Plcker$ vector. We then obtain a map $\Psi : G(r, m) \to \mathbb{P}^{\binom{m}{r}-1}$.

intrinsically: $W = \mathrm{span}\{w_1, ..., w_k\} \mapsto [w_1 \wedge ... \wedge w_k] \in \mathbb{P}(\bigwedge^r V)$. The inverse map/reconstruction of W: $W = \{v \in V \mid \Psi(W) \wedge v = 0$, lift of $\Psi(W)$ to $\bigwedge^r V\}$.

Plücker ideal: $I_{r,m} = I_{\Psi(G(r,m))} \subseteq K[p_I]_{I \subseteq \{1,...,m\}, |I|=r}$

Proposition 4.10. $I_{r,m}$ is generated by the Plücker relations

$$P_{I,Y} := \sum_{j \in J \setminus I} \mathrm{sgn}(j; I, Y) \cdot p_{I \cup \{j\}} \cdot p_{J \setminus \{j\}}.$$

5 Polyhedral Geometry

For good literature: see Ziegler.

First, a number of notions/reminders:

- $X \subseteq \mathbb{R}^n$ is **convex** if for all $u, v \in X$ and all $\lambda \in [0, 1]$ we have that $\lambda u + (1 - \lambda)v \in X$. Equivalently, can say X is convex if and only if it is an intersection of half spaces, but this intersection might be infinite (think of a sphere).

- the **convex hull** of $U \subseteq \mathbb{R}^n$ is the smallest convex set containing U:K

$$\mathrm{conv}(U) = \bigcap_{\substack{X \subseteq \mathbb{R}, \text{ convex} \\ U \subseteq X}} X.$$

When $U = \{u_1, ..., u_n\}$ is a finite set, we say that $\mathrm{conv}(U)$ is a polytope, and

$$\mathrm{conv}(U) = \left\{ \sum_{i=1}^r \lambda_i - u_i \;\middle|\; \sum \lambda_i = 1 \right\}.$$

- The **polyhedral cone**

- A **face of a cone** C determined by $w \in (\mathbb{R}^n)^\times$ is

$$\mathrm{face}_w(C) = \{x \in C \mid \forall y \in C, w \cdot x \leq w \cdot y\}.$$

The face $\mathrm{face}_w(C)$ may be empty,

- **convex polyhedra** are intersections of finitely many half-spaces in some $\mathbb{R}^n$:

$$P = \{x \in \mathbb{R}^n \mid A \cdot x \leq b\}$$

It is a fact that polyhedra are exactly the bounded polytopes.

- a **facet** is a maximal face which is not equal to the entire polyhedra.

- A **polyhedral complex** in $\mathbb{R}^n$ is a collection Σ of polyhedra (cells of Σ) such that

  (i) $P \in \Sigma$, $F \subseteq P$ a face implies $F \in \Sigma$
  (ii) for all $P, P' \in \Sigma$, $P \cap P' \neq \emptyset \implies P \cap P'$ is a face of P, P'.

- **faces of** Σ are the maximal cells of Σ (confusingly)

- **ridges of** Σ are the facets of facts

- the **support of** Σ:

- **lineality space**

- **affine span**

- **dimension**

- **polyhedral complex** Σ is of **pure dimension** d if all facets are d-dimensional

- **relative interior** of P

- Γ-rational polyhedra, $\Gamma \subseteq (\mathbb{R}^n, +)$ a subgroup

- Γ-rational polyhedral complex Σ

- **normal fan** of P: fan $\mathcal{N}_P$ with cones

§ *Entry 6*

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More definitions:

- **star of** $\sigma \in \Sigma$, Σ a cell complex, is the fan $\star_\Sigma(\sigma)$ with cones generated by the tangent wedges of the cells $\tau \in \Sigma$ containing σ:

$$\overline{\tau} = \{\lambda \cdot (x - y) \mid x \in \tau, y \in \sigma, \lambda \in \mathbb{R}_{\geq 0}\}, \ \tau \in \Sigma, \sigma \leq \tau$$

  <u>note</u>: $\overline{\tau}$ has lineality space T_σ = lineality space of σ.

- **regular subdivisions of cone:** $v_1, ..., v_j \in \mathbb{R}^n$, $w = (w_1, ..., w_\gamma) \in \mathbb{R}^r$ weight vector induces a subdivision of the con $\text{pos}(v_1, ..., v_r)$ by the cones

$$(v_i, i \in \sigma) \ \text{for all} \ \sigma \subseteq \{1, ..., r\} \ \text{such that} \ \exists c \in \mathbb{R}^n$$

$c \cdot v_i = w_i, i \in \sigma, c \cdot v_j < w_i, i \notin \sigma$

6 Gröbner Bases (over a field with a valuation)

This book really represents Sturmfel's point of view of tropical geometry. He realized a special case of Gröbner bases can be used to read off info in tropical world.

In our setup, K is a field with valuation val, not necessarily algebraically closed, but we assume Γ_{val} is the dense image of $a \in \mathcal{O}_K$ in $\underline{k} = \mathcal{O}_K/\mathfrak{m}_K : \overline{a}$. $\varphi : \Gamma_{\text{val}} \to K^\times$ splitting $w \mapsto t^w$, if $f \in R[x_0^{\pm 1}, ..., x_n^{\pm 1}]$ then $\overline{f} \in \underline{k}[x_0^{\pm 1}, ..., x_n^{\pm 1}]$. We mostly will work homogeneously.

6.1 Initial Ideals

The tropicalization of a polynomial $f = \sum_{u \in \mathbb{N}^{u+1}} c_u x^u \in K[x_0, ..., x_n]$:

$$\text{trop}(f) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}, \ \ \text{trop}(f)(w) = \min\{\text{val}(c_u) + w \cdot u \mid u \in \mathbb{N}^{n+1}, c_u \neq 0\}$$

is a concave piecewise linear function.

Thus, $\text{trop}(f)$ is the tropical polynomial induced by the classical polynomial f. Fix a *weight vector* $w \in \mathbb{R}^{n+1}$, and let $W = \text{trop}(f)(w) = \text{val}(c_u) + w \cdot u \mid c_u \neq 0$. the *inital form* of f with respect to w is

$$\text{in}_w(f) = \sum_{\substack{u \in \mathbb{N}^{n+1} \\ \text{val}(c_u) + w \cdot u = W}} \overline{c_u t^{-\text{val}(c_u)}} x^u \in \underline{k}[x_0, ..., x_n]$$