THE BERNSTEIN-SATO POLYNOMIAL

Contents

The aim of this chapter is to introduce and study the Bernstein-Sato polynomial of a regular function, and explain its connection to invariants in birational geometry.

1. Definition, existence, and examples. In this section X is a smooth variety over $\mathbf{C}$ (or more generally over a field of characteristic 0), and f is a non-invertible regular function on X. The following theorem was proved by Bernstein when f is a polynomial, and by Björk and Kashiwara in general.[1]

Theorem 1.1. *There exists a polynomial $b(s) \in \mathbf{C}[s]$, and a polynomial $P(s) \in \mathscr{D}_X[s]$ whose coefficients are differential operators on X, such that the relation*

$$P(s)f^{s+1} = b(s) \cdot f^s$$

holds formally in the $\mathscr{D}_X$-module $\mathscr{O}_X[\frac{1}{f}, s] \cdot f^s$. (Here f^{s+1} stands for $f \cdot f^s$.)

Remark 1.2. Recall that we have discussed this $\mathscr{D}$-module in ??; the action of derivations on X on its elements is given by

$$D(wf^s) = \big(D(w) + sw\frac{D(f)}{f}\big)f^s.$$

It carries in fact a $\mathscr{D}_X[s]$-module structure via the obvious action of s.

[1] They also proved it in the case of germs of holomorphic functions on complex manifolds.

In the case of polynomials $f \in \mathbf{C}[X_1, \ldots, X_n]$, the proof of this theorem[2] is a simple application of the fact that $\mathscr{O}_{\mathbf{C}^n}[\frac{1}{f}](s) \cdot f^s$ is holonomic as a module over the Weyl algebra $A_n(\mathbf{C}(s)) \simeq A_n(\mathbf{C})(s)$ associated to the field $\mathbf{C}(s)$. This is proved using the Bernstein filtration on A_n; in the general case this is not available, but we follow a similar approach replacing it by general properties of holonomic $\mathscr{D}$-modules.

Proof of Theorem 1.1. We denote by $\mathbf{C}(s)$ the field of rational functions in the variable s. We also denote by $j\colon U \hookrightarrow X$ the natural inclusion of $U = X \smallsetminus Z(f)$. Via base field extension we can consider X and U as being defined over $\mathbf{C}(s)$; we use the notation

$$X_s := X \times_{\mathrm{Spec}\mathbf{C}} \mathrm{Spec}\mathbf{C}(s) \quad \text{and} \quad U_s := U \times_{\mathrm{Spec}\mathbf{C}} \mathrm{Spec}\mathbf{C}(s),$$

with the corresponding inclusion $j_s\colon U_s \hookrightarrow X_s$. We then have

$$\mathscr{D}_{U_s} \simeq \mathscr{D}_U \times_{\mathbf{C}} \mathbf{C}(s) \quad \text{and} \quad \mathscr{D}_{X_s} \simeq \mathscr{D}_X \times_{\mathbf{C}} \mathbf{C}(s).$$

Thinking of f^s as a formal symbol as before, we now consider the $\mathscr{D}_{U_s}$-module $\mathcal{M} := \mathscr{O}_{U_s} \cdot f^s$, where the action of a derivation on X is given by

$$D(gf^s) = \big(D(g) + sg\frac{D(f)}{f}\big)f^s.$$

(Note that f is invertible on U_s.) Note that $\mathcal{M}$ is a holonomic $\mathscr{D}_{U_s}$-module. Indeed, by analogy with the trivial filtration on $\mathscr{O}_{U_s}$, on $\mathcal{M}$ we can consider the filtration given by

$$F_k\mathcal{M} = \mathscr{O}_{U_s} \cdot f^s \quad \text{for} \quad k \geq 0 \quad \text{and} \quad F_k\mathcal{M} = 0 \quad \text{for} \quad k < 0.$$

This is a good filtration such that

$$\mathrm{Ch}(\mathcal{M}) = \mathrm{Ch}(\mathscr{O}_U)_s = (T^*_U U)s,$$

the scalar extension of the zero section of T^*U, hence holonomicity is clear. Now the main claim is:

Claim. The $\mathscr{D}_{X_s}$-submodule

$$\mathcal{N} := \mathscr{D}_{X_s} f^s \subseteq j_{s+}\mathcal{M}$$

is *holonomic*.

In order to show this, we note that the construction of a maximal holonomic submodule of a finitely generated $\mathscr{D}$-module works over arbitrary base fields, and is functorial (give reference). Hence there exists a maximal holonomic submodule $\mathcal{N}' \subseteq \mathcal{N}$, compatible with restriction to open sets. Note now that

$$\mathcal{N}_{|U_s} \subseteq \mathcal{M} = \mathscr{O}_{U_s} \cdot f^s.$$

By the observation above, we conclude that $\mathcal{N}_{|U_s}$ is holonomic, hence

$$\mathcal{N}'_{|U_s} = \mathcal{N}_{|U_s}.$$

In other words, in the short exact sequence of $\mathscr{D}_{X_s}$-modules

$$0 \longrightarrow \mathcal{N}' \longrightarrow \mathcal{N} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

[2] which we have seen last quarter; see also [Co, Ch.10, §3]

the quotient $\mathcal{Q}$ is supported on the zero locus of f in X_s. Consequently, if we look at the section f^s of $\mathcal{N}$, there exists an integer $k_0 \geq 0$ such that $f^{k_0} f^s \in \mathcal{N}'$. Hence

$$\mathscr{D}_{X_s} f^{k_0} f^s \subseteq \mathcal{N}'$$

and so $\mathscr{D}_{X_s} f^{k_0} f^s$ is holonomic. Finally note that we have an isomorphism of $\mathscr{D}_{X_s}$-modules

$$\mathscr{D}_{X_s} f^s \xrightarrow{\simeq} \mathscr{D}_{X_s} f^{k_0} f^s, \quad P(s) f^s \mapsto P(s+k_0) f^{k_0} f^s$$

induced by the automorphism $s \mapsto s+k$ of $\mathscr{D}_{X_s}$.

Having establish the claim, let us conclude the proof of the main statement. Consider the chain of submodules

$$\cdots \subseteq \mathscr{D}_{X_s} f^2 f^s \subseteq \mathscr{D}_{X_s} f f^s \subseteq \mathscr{D}_{X_s} f^s.$$

Since $\mathscr{D}_{X_s} f^s$ is holonomic, hence of finite length, this chain stabilizes. There exists therefore an integer $m \geq 0$ such that

$$f^m f^s \in \mathscr{D}_{X_s} f^{m+1} f^s.$$

Applying now the similar automorphism $s \mapsto s - m$, we conclude that $f^s \in \mathscr{D}_{X_s} f f^s$, and so there exists $Q(s) \in \mathscr{D}_{X_s} = \mathscr{D}_X \otimes_{\mathbf{C}} \mathbf{C}(s)$ such that

$$f^s = Q(s) f f^s.$$

Clearing denominators, this operator can be rewritten as $Q(s) = P(s)/b(s)$, where $P(s) \in \mathscr{D}_X[s]$ and $b(s) \in \mathbf{C}(s)$, so the identity becomes

$$b(s) \cdot f^s = P(s) f f^s,$$

which is what we were after. □

Definition 1.3. The set of all polynomials $b(s)$ satisfying an identity as in Theorem 1.1 clearly forms an ideal in the polynomial ring $\mathbf{C}[s]$. The monic generator of this ideal is called the *Bernstein-Sato polynomial* of f, and is denoted $b_f(s)$.

There is also a local version of the Bernstein-Sato polynomial. We discuss this next, together with its relationship with the global version above.

Lemma 1.4. *If x is a point in X, then there exists an open neighborhood U of x such that for any other open neighborhood V we have*

$$b_{f|U}(s) \mid b_{f|V}(s).$$

Proof. For simplicity, let's denote $b_U(s) = b_{f|U}(s)$. Start with any open neighborhood $x \in U_0$ and assume that it does not satisfy the property we want. There exists then another neighborhood $x \in U_1$ such that b_{U_0} does not divide b_{U_1}. As $b_{U_0 \cap U_1} | b_{U_1}$ since we can restrict the Bernstein-Sato identity on U_1 to $U_0 \cap U_1$, and similarly for U_0, it follows that $b_{U_0 \cap U_1}$ is a proper factor of b_{U_0}.

If the neighborhood $U_0 \cap U_1$ again does not satisfy the property in the statement, than by a similar argument there exists a neighborhood $x \in U_2$ such that $b_{U_0 \cap U_1 \cap U_2}$ is a proper factor of $b_{U_0 \cap U_1}$. Continuing this way, since b_{U_0} has finitely many factors at some point the polynomial has to stabilize, and we obtain an open set $U_0 \cap U_1 \cap \ldots \cap U_r$ satisfying the assertion. $\square$

Definition 1.5. The *local Bernstein-Sato polynomial* of f at x is

$$b_{f,x}(s) := b_{f|U}(s),$$

where U is an open neighborhood of x as in Lemma 1.4.

Proposition 1.6. *If X is affine, then the global and local Bernstein-Sato polynomials are related by the formula*

$$b_f(s) = \operatorname*{lcm}_{x \in X} b_{f,x}(s).$$

In fact, let $\{U_i\}_{i \in I}$ be any open cover of X. Then

$$b_f(s) = \operatorname*{lcm}_{i \in I} b_{f|U_i}(s).$$

Proof. Denoting as in the previous proof $b_{U_i} = b_{f|U_i}$, since clearly $b_{U_i} \mid b_f$ for all i we have

$$b'(s) := \operatorname*{lcm}_{i \in I} b_{U_i}(s) \mid b_f(s).$$

Consider now the Bernstein-Sato identity on each U_i, namely

$$b_{U_i}(s) f^s_{|U_i} = P_i(s) f^{s+1}_{|U_i} \quad \text{with} \quad P_i(s) \in \mathscr{D}_{U_i}(s).$$

Since U_i is an open set in the affine variety X, there exist for each i and operator $Q_i(s) \in \mathscr{D}_X[s]$, and $g_i \in \mathscr{O}_X(X)$, such that $g_i P_i(s) = Q_i(s)$. We then have the identity

$$g_i b_{U_i}(s) f^s = Q_i(s) f^{s+1}$$

on X, which implies that $g_i b'(s) f^s \in \mathscr{D}_X[s] f^{s+1}$. Define now

$$I = \{ g \in \mathscr{O}_X(X) \mid g b'(s) f^s \in \mathscr{D}_X[s] f^{s+1} \}.$$

This is clearly an ideal in $\mathscr{O}_X(X)$, and for each $i \in I$ we have that $g_i \in I \smallsetminus \mathfrak{m}_x$ for every $x \in U_i$. It follows that $I = (1)$, hence $b_f(s) \mid b'(s)$. $\square$

Remark 1.7. On an arbitrary X, the result of Proposition 1.6 continues to hold if we think of the Bernstein-Sato polynomial as being the monic polynomial of minimal degree such that

$$b(s) f^s \in \mathscr{D}_X[s] f^{s+1}$$

in a sheaf-theoretic sense.

Exercise .1. If g is an invertible function on X, then $b_{gf}(s) = b_f(s)$.

Remark 1.8 (Bernstein-Sato polynomials of divisors)**.** Let D be an arbitrary effective divisor on X. For any two functions f_1, f_2 defining D on an open set $U \subset X$, there exists $g \in \mathscr{O}_X^*(U)$ such that $f_1 = f_2 g$. The results and exercise above imply then that it makes sense to define a Bernstein-Sato polynomial $b_D(s)$ associated to D, and

$$b_D(s) = \operatorname*{lcm}_{x \in D} b_{f,x}(s),$$

where f is any locally defining equation for D in a neighborhood of X.

We next list a few basic facts regarding Bernstein-Sato polynomials.

Remark 1.9. (1) For f invertible we could simply take $b_f(s) = 1$. This is why we restrict to f non-invertible.

(2) If f is arbitrary, then we have

$$(s+1)|b_f(s).$$

Indeed, take $s = -1$ in the identity in Theorem 1.1, to obtain

$$b_f(-1) = P(-1)\frac{1}{f}.$$

Since f is not constant, this is only possible when $b_f(-1) = 0$.

(3) If f is smooth, by a simple reduction we can assume that $f = x_1$ in local algebraic coordinates $x_1, \ldots, x_n$. It follows that $b_f(s) = s + 1$, due to (2) and the formula

$$\partial_{x_1} x_1^{s+1} = (s+1)x_1^s.$$

The converse is also true, meaning that if $b_f(s) = (s + 1)$, then f is smooth; see ??.

In the rest of the section we discuss a few rather well-known examples.

Example 1.10. (1) Let $f = x_1^2 + \cdots + x_n^2$, and consider the Laplace operator $\Delta = \partial_1^2 + \cdots + \partial_n^2$. A simple calculation gives

$$\Delta f^{s+1} = 4(s+1)(s+\frac{n}{2})f^s.$$

According to the last comment in the Remark above, we obtain that

$$b_f(s) = (s+1)(s+\frac{n}{2}).$$

We will also obtain this as a special case of the general calculation for all weighted homogeneous singularities, Theorem 2.4 below.

(2) Let $f = x^2 + y^3$ be a cusp in the plane. A well-known, though tedious and not easily motivated, calculation is that

$$= \big(s+1\big)\big(s+\frac{5}{6}\big)\big(s+\frac{7}{6}\big)f^s,$$

and in fact

$$b_f(s) = \big(s+1\big)\big(s+\frac{5}{6}\big)\big(s+\frac{7}{6}\big).$$

This again will be implied by the general result for weighted homogeneous singularities.

(3) Let $f = \det(x_{ij})$ be the determinant of a generic matrix in $n \times n$ variables. A formula attributed to Cayley is

$$\det\big(\partial/\partial x_{ij}\big)f^{s+1} = (s+1)(s+2)\cdots(s+n)f^s.$$

We actually have $b_f(s) = (s+1)(s+2)\cdots(s+n)$, and this will also be a special case of the general result below.

(4) The case of a divisor with SNC support can also be computed explicitly. Let

$$f = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}, \quad a_i \in \mathbf{N}.$$

Note that in one variable x, and $a \ge 1$, we have the formula

$$\partial_x^a x^{as+a} = a^a \big(s+1\big)\big(s+1-\frac{1}{a}\big)\cdots\big(s+1-\frac{a-1}{a}\big)x^{as}.$$

Therefore using the operator

$$P = \frac{1}{\prod_{i=1}^n a_i^{a_i}} \cdot \partial_1^{a_1} \cdots \partial_n^{a_n},$$

a straightforward calculation gives

$$Pf^{s+1} = b_{a_1,\ldots,a_n}(s) f^s, \quad b_{a_1,\ldots,a_n}(s) := \prod_{i=1}^n \left(\prod_{k=0}^{a_i-1} \big(s+1-\frac{k}{a_i}\big) \right)$$

and so $b_{a_1,\ldots,a_n} \mid b_f$. We can in fact see that $b_f = b_{a_1,\ldots,a_n}$ as follows. Starting with an expression

$$\big(\sum_{\alpha,\beta,j} a^j_{\underline{\alpha}\underline{\beta}} s^j \underline{x}^{\underline{\alpha}} \underline{\partial}^{\underline{\beta}}\big) \underline{x}^{\underline{a}(s+1)} = c(s) \cdot \underline{x}^{\underline{a}s},$$

where we use the notation $\underline{x}^{\underline{a}} = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$, etc., by comparing the terms containing $\underline{x}^{\underline{a}s}$ on both sides we see that the only contribution coming from the left hand side is from terms of the form $a^j_{\underline{\alpha}\underline{\beta}} \underline{x}^{\underline{\alpha}} \underline{\partial}^{\underline{\beta}}$ with $a^j_{\underline{\alpha}\underline{\beta}} \neq 0$ and $\beta_i = \alpha_i + a_i$.[3] We then have that $\beta_i \ge a_i$ for all i, hence employing the one variable formula above repeatedly we see that each $\underline{\partial}^{\underline{\beta}} \underline{x}^{\underline{a}(s+1)}$ contributes a polynomial (in s) term divisible by $b_{a_1,\ldots,a_n}$. It follows that $b_{a_1,\ldots,a_n} \mid c(s)$.

(5) If $f(x_1,\ldots,x_n) = g(x_1,\ldots,x_m) \cdot h(x_{m+1},\ldots,x_n)$, it is immediate to see that

$$b_f \mid b_g \cdot b_h.$$

Whether equality holds seems to be an open problem. Note that for arbitrary $f = g \cdot h$ it is very easy to produce examples where this divisibility does not hold (and there is no reason for it to do so). Consider for instance the triple point $f = xy(x+y)$ in $\mathbf{C}^2$, and take $g = xy$ and $h = x+y$. Then

$$b_g(s) \cdot b_h(s) = (s+1)^3 \quad \text{while} \quad \big(s+\frac{2}{3}\big) \mid b_f(s).$$

For the last statement one can for instance use the general formula for weighted homogeneous singularities in Theorem 2.4, or the fact that the log canonical threshold of f is $2/3$ combined with Theorem 7.2 below.

(6) Hyperplane arrangements. TO ADD.

[3] A priori there may be terms of other type in the differential operator, but after differentiation their contributions must cancel each other.

2. Quasi-homogeneous singularities. In this section we study an extended example, proving a general formula for the Bernstein-Sato polynomial of a quasi-homogeneous isolated singularity.[4]

We start by recalling a few basic notions from singularity theory. For a polynomial $f \in \mathbf{C}[X] = \mathbf{C}[X_1, \ldots, X_n]$, inside the ring of convergent power series $\mathbf{C}\{X\}$ we consider the associated *Jacobian ideal*

$$J(f) := \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big) \subseteq \mathbf{C}\{X\}.$$

Similarly, the *Tjurina ideal* is

$$(f, J(f)) = \Big(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big) \subseteq \mathbf{C}\{X\}.$$

The *Milnor* and *Tjurina algebras* associated to f are

$$M_f := \mathbf{C}\{X\}/J(f) \quad \text{and} \quad T_f := \mathbf{C}\{X\}/(f, J(f)),$$

while the corresponding Milnor and Tjurina numbers are

$$\mu_f := \dim_{\mathbf{C}} M_f \quad \text{and} \quad \tau_f := \dim_{\mathbf{C}} T_f.$$

If $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n_{>0}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we denote

$$|w| = w_1 + \cdots + w_n \quad \text{and} \quad \langle w, \alpha \rangle = w_1\alpha_1 + \cdots + w_n\alpha_n \in \mathbb{Q}_{>0}.$$

Definition 2.1. We say that a polynomial $f = \sum_{\alpha \in \mathbf{N}^n} a_\alpha X^\alpha \in \mathbf{C}[X]$ is *quasi-homogeneous* (or *weighted homogeneous*) of type $(w; \rho)$ if for every $\alpha \in \mathbf{N}^n$ such that $a_\alpha \neq 0$ we have

$$\rho(X^\alpha) := \langle w, \alpha \rangle = \rho.$$

We simply say that f is quasi-homogeneous or weighted-homogeneous (with respect to the weights $w_1, \ldots, w_n$) if $\rho = 1$.

Remark 2.2. Note that the weights are not invariant under linear change of coordinates.

In what follows we assume that $f(0) = 0$, and 0 is an isolated singular point of $Z(f)$. It is a well-known fact that this condition is equivalent to $\mu_f < \infty$, and also to $\tau_f < \infty$; see e.g. [GLS, Lemma 2.3]. Under this assumption, a theorem of K. Saito [KSa] states that a polynomial f is quasi-homogeneous (of any weight) after some biholomorphic change of coordinates if and only if the following equivalent conditions hold:

$$f \in J(f) \iff \mu_f = \tau_f.$$

Example 2.3. (1) The most common examples of quasi-homogeneous polynomials are the diagonal hypersurfaces

$$f = X_1^{a_1} + \cdots + X_n^{a_n} \in \mathbf{C}[X],$$

with weights $w_i = \frac{1}{a_i}$ for $i = 1, \ldots, n$.

[4] I thank Mingyi Zhang for giving lectures on this topic, which I am following here.

(2) Let $f = x^3 + xy^3 \in \mathbf{C}[X,Y]$. This is quasi-homogeneous, with weights $w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{9}$. We have $\tau_f = \mu_f = 7$, and a monomial basis of the Milnor algebra seen as a $\mathbf{C}$-vector space is $1, x, x^2, y, xy, x^2y, y^2$.

(3) Let $f = x^5 + y^5 + x^2y^2 \in \mathbf{C}[X,Y]$. Then f is not a quasi-homogeneous polynomial. Indeed, it is not hard to compute that

$$10 = \tau_f < \mu_f = 11.$$

Let's now fix a quasi-homogeneous polynomial $f \in \mathbf{C}[X]$ with an isolated singularity, of weights $w_1, \ldots, w_n$, and denote $\mu = \mu_f$. We write

$$M_f = \bigoplus_{i=1}^{\mu} \mathbf{C} \cdot e_i,$$

where e_i are a monomial basis for the Milnor algebra as a $\mathbf{C}$-vector space. We also consider the set of rational numbers

$$\Sigma = \{\rho(e_1), \ldots, \rho(e_n)\},$$

where each number appears without repetitions. With this notation, the main result is the following theorem obtained in [BGM]; see also [BGMM] for an extension to the case of polynomials which are non-degenerate with respect to their Newton polygon.

Theorem 2.4. *The Bernstein-Sato polynomial of f is*

$$b_f(s) = (s+1) \cdot \prod_{\rho \in \Sigma} \big(s + |w| + \rho\big).$$

Remark 2.5. The statement of the theorem implies that all of the roots of $b_f(s)$ different from -1 are simple, while -1 appears with multiplicity 1 or 2.

Before proving the theorem, we discuss some preliminaries. First, with respect to this set of weights, we define the *Euler vector field* as

$$\chi := \sum_{i=1}^{n} w_i x_i \partial_{x_i}.$$

Quasi-homogeneity immediately implies that we have the identity

$$\chi(f) = f.$$

Lemma 2.6. *For every $u \in \mathbf{C}\{X\}$ and every $\rho \in \mathbb{Q}$ we have*

$$(s + |w| + \rho) u f^s = \Big(\sum_{i=1}^{n} w_i \partial_{x_i}(x_i u) + \rho u - \chi(u)\Big) f^s.$$

Proof. This is a simple exercise, using the identities:

$$\sum_{i=1}^{n} w_i \partial_{x_i} x_i = \chi + |w| \qquad \text{(as operators)}$$

and

$$\chi(u f^s) = s u f^s + \chi(u) f^s.$$

□

Proof of Theorem 2.4. Step 1. In this first step we show that

$$b_f(s) \mid (s+1) \cdot \prod_{\rho \in \Sigma} \big(s + |w| + \rho\big).$$

By definition, this follows if we show

$$(s+1) \cdot \prod_{\rho \in \Sigma} \big(s + |w| + \rho\big) \in \mathscr{D}_{\mathbf{C}^n}[s] f^s. \tag{2.1}$$

This in turn follows by setting $u = 1$ in the following more general:

Claim: For a quasi-homogeneous representative u of an element in M_f, we have

$$(s+1) \prod_{\rho \in \Sigma, \rho \geq \rho(u)} \big(s + |w| + \rho\big) u f^s = \sum_{i=1}^{n} A_i \partial_{x_i} \cdot f^{s+1}, \quad A_i \in A_n.$$

This can be proven by descending induction on the weight $\rho(u)$. First, if $\rho(u)$ is the maximal value in Σ, then $x_i u \in J(f)$ for all i. Since $\chi(u) = \rho(u)u$, by Lemma 2.6 we have

$$(s + |w| + \rho) u f^s = \big(\sum_{i=1}^{n} w_i \partial_{x_i}(x_i u)\big) f^s.$$

Modulo $A_n \cdot J(f) \cdot f^s$, this last term is equal to

$$\sum_{i=1}^{n} \partial_{x_i} \cdot v_i,$$

with v_i a quasi-homogeneous element in M_f of weight $\rho(u) + w_i$.

We now assume that the Claim is true for any u as above with $\rho(u) > \nu \in \Sigma$, and take a quasi-homogeneous representative u' of an element in M_f, such that $\rho(u') = \nu$. By the inductive assumption, for each $1 \leq i \leq n$ we have

$$\prod_{\rho \geq \rho(x_i u')} (s + |w| + \rho)(x_i u') f^s \in A_n \cdot J(f) \cdot f^s.$$

Notice that the action of any polynomial in s on $A_n \cdot J(f) \cdot f^s$ stays in $A_n \cdot J(f) \cdot f^s$, because

$$s \cdot \frac{\partial f}{\partial x_i} \cdot f^s = -(1 - w_i) \partial_i f f^s + \chi \cdot \frac{\partial f}{\partial x_i} \cdot f^s.$$

We can then multiply the products above by suitable factors of the form $(s + |w| + \rho')$, to get

$$\prod_{\rho \in \Sigma, \rho > \nu} \big(s + |w| + \rho\big)(x_i u') f^s \in A_n \cdot J(f) \cdot f^s$$

for all i. Using now Lemma 2.6, a straightforward calculation (acting $w_i \partial_i$ on these products and taking the sum) leads to

$$\prod_{\rho \in \Sigma, \rho \geq \nu} \big(s + |w| + \rho\big) u' f^s = \sum_{i=1}^{n} A_i \partial_{x_i} \cdot f^s,$$

with $A_i \in A_n$. Finally, multiplying both sides by $(s+1)$ and applying the formula

$$(s+1)\partial_{x_i} f^s = \partial_{x_i} f^{s+1},$$

we obtain the Claim.

Step 2. According to Step 1, we are left with proving that for every $\rho \in \Sigma$ we have

$$\widetilde{b}_f(-|w| - \rho) = 0,$$

where $\widetilde{b}_f = b_f(s)/(s+1)$ is the reduced Bernstein-Sato polynomial of f.

Note first that for the operator P in the functional equation

$$P(s) f^{s+1} = b_f(s) f^s$$

we have $P(-1) \cdot 1 = 0$, hence it is easy to check that we can write

$$P(s) = (s+1)Q(s) + \sum_{i=1}^n A_i \partial_{x_i}, \quad \text{with } \; Q(s) \in \mathbf{C}[s] \; \text{ and } \; A_i \in A_n[s].$$

Consequently, we obtain

$$\widetilde{b}_f(s) f^s = \big(Q(s) f + \sum_{i=1}^n A_i \partial_{x_i}\big) \cdot f^s.$$

Let now u be a monomial representing an element in M_f, of weight $\rho = \rho(u) \in \Sigma$. Multiplying the identity above on the left by u, we obtain

$$\widetilde{b}_f(s) \cdot u f^s = \big(Q'(s) f + \sum_{i=1}^n A_i' \partial_{x_i}\big) \cdot f^s, \tag{2.2}$$

where $Q' = uQ$, and similarly for A_i. On the other hand, we use Lemma 2.6 to get

$$(s + |w| + \rho) \cdot u f^s = \big(\sum_{i=1}^n w_i \partial_{x_i}(x_i u)\big) f^s. \tag{2.3}$$

Let's assume now that the conclusion is false, so that the polynomials $(s + |w| + \rho)$ and $\widetilde{b}_f(s)$ are coprime. In this case there exist polynomials $p(s), q(s) \in \mathbf{C}[s]$ such that

$$p(s)(s + |w| + \rho) + q(s)\widetilde{b}_f(s) = 1.$$

Using (2.2) and (2.3), we then obtain

$$u f^s = \left(p(s)\big(\sum_{i=1}^n w_i \partial_{x_i}(x_i u)\big) + q(s)\big(Q'(s) f + \sum_{i=1}^n A_i' \partial_{x_i}\big) \right) f^s.$$

Recalling that $\chi(f^s) = s f^s$ (see the proof of Lemma 2.6) and $f \in J(f)$, a straightforward calculation shows that an operator of the form

$$R = u - \sum_{i=1}^n B_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^n \partial_{x_i} C_i, \qquad B_i, C_i \in A_n$$

is in the annihilator of f^s in A_n. However the Lemma below says that R belongs to $\sum_{\alpha \in \mathbf{N}} \partial^\alpha J(f)$, and so it follows that $u \in J(f)$, which is a contradiction. $\square$

Lemma 2.7. *For any polynomial f with an isolated singularity at the origin, the annihilator of f^s in the Weil algebra A_n is generated by the operators*

$$\frac{\partial f}{\partial x_i} \cdot \partial_{x_j} - \frac{\partial f}{\partial x_j} \cdot \partial_{x_i}, \quad 1 \leq i < j \leq n.$$

Proof. This is [Ya, Theorem 2.19]; see also [Gr, Appendix B]. It is a consequence of the fact that $\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1}\}$ form a regular sequence in the case of an isolated singularity. □

3. Kashiwara's rationality theorem. In this section we prove the main theorem in [Ka1]. Note that Kashiwara's proof also works when f is the germ of an analytic function on a complex manifold X.

Theorem 3.1. *For every non-invertible regular function f on X, the roots of $b_f(s)$ are negative rational numbers.*

Main construction. We start by describing the main technical construction of Kashiwara. Let X be as always a smooth complex variety, and $f \in \mathscr{O}_X$ a regular function on X. We denote by D the effective divisor corresponding to f. We consider a log resolution $\mu\colon Y \to X$ of D, which is an isomorphism away from D, and we denote $f' = f \circ \mu$.

As in §B in the notes on the V-filtration, we denote

$$\mathcal{N}_f := \mathscr{D}_X[s] \cdot f^s \quad \text{and} \quad \mathcal{N}_{f'} := \mathscr{D}_{X'}[s] \cdot f'^s.$$

We also define

$$\mathcal{N} := \mathcal{H}^0 \mu_+ \mathcal{N}_{f'}.$$

We will see below that we can attach to $\mathcal{N}$ a Bernstein-Sato polynomial $b_{\mathcal{N}}(s)$, and that $b_{\mathcal{N}}(s) \mid b_{f'}(s)$. Hence one of the key points in the proof of Kashiwara's theorem will be to compare $b_f(s) = b_{\mathcal{N}_f}(s)$ with $b_{\mathcal{N}}(s)$. This is however not immediate; it turns out that $\mathcal{N}_f$ is a special type of sub-quotient of $\mathcal{N}$, but this requires a bit of work.

To this end, we construct a distinguished section $u \in \Gamma(X, \mathcal{N})$. First, since μ is birational, there is a natural nontrivial (hence injective) map $p^*\omega_X \to \omega_Y$, hence a natural section $s\colon \mathscr{O}_Y \to \omega_{Y/X}$. On the other hand, recall that by definition

$$\mathcal{N} = R^0\mu_*\big(\mathscr{D}_{X \leftarrow Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathcal{N}_{f'}\big),$$

where

$$\mathscr{D}_{X \leftarrow Y} = \mu^{-1}\mathscr{D}_X \otimes_{\mu^{-1}\mathscr{O}_X} \omega_{Y/X}$$

is the transfer $(\mu^{-1}\mathscr{D}_X, \mathscr{D}_Y)$-bimodule. Note that $1 \in \mathscr{D}_X$ and s give rise to a homomorphism

$$\mathscr{O}_Y \to \mathscr{D}_{X \leftarrow Y}$$

of $(\mu^{-1}\mathscr{O}_X, \mathscr{O}_Y)$-bimodules. On the other hand the section f'^s of $\mathcal{N}_{f'}$ gives rise to a $\mathscr{O}_Y$-module homomorphism $\mathscr{O}_Y \to \mathcal{N}_{f'}$. Taking the tensor product of these two we obtain a morphism

$$\mathscr{O}_Y \to \mathscr{D}_{X \leftarrow Y} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathcal{N}_{f'}$$

in the derived category of $\mu^{-1}\mathscr{O}_X$-modules, and applying $R^0\mu_*$ to this we obtain our desired section $u \in \Gamma(X, \mathcal{N})$. We define

$$\mathcal{N}' := \mathscr{D}_X[s] \cdot u \subseteq \mathcal{N},$$

a $\mathscr{D}_X[s]$-submodule of $\mathcal{N}$.

Lemma 3.2. *The module $\mathcal{N}'$ satisfies the following properties:*

(1) It sits in a digram of $\mathscr{D}_X\langle s, t\rangle$-modules

$$\begin{array}{ccc} \mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\ \big\downarrow{\scriptstyle g} & & \\ \mathcal{N}_f & & \end{array}$$

where i is the inclusion map, and g is surjective.

(2) It is a coherent $\mathscr{D}_X$-module.

Proof. To see that $\mathcal{N}'$ is a $\mathscr{D}_X\langle s, t\rangle$-module, we only need to check that it is preserved by the action of t. Denoting by $1_{X\leftarrow Y}$ the section of $\mathscr{D}_{X\leftarrow Y}$ considered above, we have

$$t \cdot (1_{X\leftarrow Y} \otimes f'^s) = 1_{X\leftarrow Y} \otimes tf'^s = 1_{X\leftarrow Y} \otimes ff'^s = f(1_{X\leftarrow Y} \otimes f'^s),$$

which indeed shows that $t \cdot \mathcal{N}' \subseteq \mathcal{N}'$.

To complete the proof of (1), it suffices to check that the mapping

$$\mathscr{D}_X[s] \cdot u \to \mathscr{D}_X[s] \cdot f^s, \quad P(s)u \mapsto P(s)f^s$$

is well defined. But assuming that $P(s)u = 0$, and recalling that u and f^s coincide on the open set U, it follows that $P(s)f^s$ is zero on U and therefore annihilated by a power of f. This means that in fact $P(s)f^s = 0$, since $\mathscr{D}_X[s] \cdot f^s$ is torsion-free as an $\mathscr{O}_X$-module; indeed, it is a submodule of $\mathscr{O}_X[\frac{1}{f}, s]f^s$, which is isomorphic to $\mathscr{O}_X[\frac{1}{f}, s]$ as an $\mathscr{O}_X$-module, and therefore torsion-free.

The assertion in (2) follows immediately from (1), since $\mathcal{N}$ is a coherent $\mathscr{D}_X$-module by the behavior of direct images under projective morphisms. □

Bernstein-sato polynomials of special $\mathscr{D}$-modules. We will make use of the following general theorem; see [Ka2, Theorem 4.45].

Theorem 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be holonomic $\mathscr{D}_X$-modules. Then $\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathcal{N})_x$ is a finite dimensional $\mathbf{C}$-vector space for every $x \in X$.*

This allows us to deduce:

Proposition 3.4. *Let $\mathcal{M}$ be a $\mathscr{D}_X[s]$-module, which is holonomic as $\mathscr{D}_X$-module. Then there exists a polynomial $b(s) \in \mathbf{C}[s]$ such that $b(s) \cdot \mathcal{M} = 0$. (Therefore the action of s on $\mathcal{M}$ has a minimal polynomial.)*

Proof. By Theorem 3.3, we know that $\mathscr{E}nd_{\mathscr{D}_X}(\mathcal{M})_x$ is a finite dimensional $\mathbf{C}$-vector space for all $x \in X$. It follows that there exists a polynomial $b_x(s)$ such that $b_x(s) \cdot \mathcal{M}_x = 0$, and this continues to hold on a neighborhood of x. We can now cover X with finitely many such neighborhoods, and take $b(s)$ to be the least common multiple of the respective $b_x(s)$. □

Lemma 3.5. *Let $\mathcal{N}$ be a $\mathscr{D}_X\langle s,t\rangle$-module which is holonomic as a $\mathscr{D}_X$-module. Then there exists an integer $N \geq 0$ such that $t^N \cdot \mathcal{N} = 0$.*

Proof. Let $b(s)$ be a polynomial as in Proposition 3.4, so that $b(s) \cdot \mathcal{N} = 0$. If

$$b(s) = \prod_{i=1}^{r} (s + \alpha_i)^{m_i},$$

then every $\mathcal{N}_x$ has a decomposition into subspaces on which some $s + \alpha_i$ acts nilpotently. Therefore for $N \gg 0$, the action of $b(s+N)$ on $\mathcal{N}$ is bijective. Recalling that $P(s)t = tP(s-1)$ for every $P \in \mathbf{C}[s]$, we have

$$b(s+N) \cdot t^N \cdot \mathcal{N} = t^N \cdot b(s) \cdot \mathcal{N} = 0,$$

hence $t^N \cdot \mathcal{N} = 0$. □

In what follows we will deal with $\mathscr{D}_X\langle s,t\rangle$-modules, as in §B of the notes on the V-filtration. Let $\mathcal{N}$ be such a module, assumed moreover to be coherent over $\mathscr{D}_X$, and such that $\mathcal{N}/t\mathcal{N}$ is holonomic. By the Proposition above, the action of s on $\mathcal{N}/t\mathcal{N}$ has a minimal polynomial, which we denote $b_{\mathcal{N}}(s)$. In other words, this is the monic polynomial of minimal degree such that

$$b_{\mathcal{N}}(s) \cdot \mathcal{N} \subseteq t\mathcal{N}.$$

Example 3.6. Recall that for $f \in \mathscr{O}_X$ we considered the module $\mathcal{N}_f = \mathscr{D}_X[s] \cdot f^s$. We have then seen in Proposition 0.17 in the notes on the V-filtration that $b_f(s) = b_{\mathcal{N}_f}(s)$. In fact Kashiwara shows in [Ka1], in the general setting of germs of holomorphic functions, that $\mathcal{N}_f/t\mathcal{N}_f$ is holonomic, which by the discussion above leads to another proof of the existence of the Bernstein-Sato polynomial.

Lemma 3.7. *Let $\mathcal{N}$ be as above, and $\mathcal{N}'$ a $\mathscr{D}_X\langle s,t\rangle$-submodule of $\mathcal{N}$. Assume that $\mathcal{N}/\mathcal{N}'$ is a holonomic $\mathscr{D}_X$-module. Then there exists an integer $N \geq 0$ such that*

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s+1)\cdots b_{\mathcal{N}}(s+N).$$

Proof. Note that $b_{\mathcal{N}'}(s)$ makes sense, as $\mathcal{N}'/t\mathcal{N}'$ is holonomic as well; see the Exercise below.

Now since $\mathcal{N}/\mathcal{N}'$ is holonomic, Lemma 3.5 implies that there exists $N \geq 0$ such that $t^N \cdot (\mathcal{N}/\mathcal{N}') = 0$, or equivalently

$$t^N \cdot \mathcal{N} \subseteq \mathcal{N}'.$$

As above, for every $j \geq 0$ we have

$$b_{\mathcal{N}}(s+j)t^j \cdot \mathcal{N} = t^j b_{\mathcal{N}}(s) \cdot \mathcal{N} \subseteq t^{j+1} \cdot \mathcal{N}.$$

Applying this repeatedly, we obtain

$$b_{\mathcal{N}}(s+N)\cdots b_{\mathcal{N}}(s)\cdot\mathcal{N}'\subseteq b_{\mathcal{N}}(s+N)\cdots b_{\mathcal{N}}(s)\cdot\mathcal{N}\subseteq t^{N+1}\cdot\mathcal{N}\subseteq t\cdot\mathcal{N}',$$

which implies the conclusion by the definition of $b_{\mathcal{N}'}(s)$. □

Exercise .2. Show that in the situation above, i.e. $\mathcal{N}'$ is a $\mathscr{D}_X\langle s,t\rangle$-submodule of a module $\mathcal{N}$ having the property that $\mathcal{N}/t\mathcal{N}$ is holonomic, we have that $\mathcal{N}'/t\mathcal{N}'$ is holonomic as well.

Proof of the Theorem. We go back to the main construction at the beginning of the section. A crucial technical result of Kashiwara, see [Ka1, Lemma 5.7] or [Ka2, p.113], is the following. Strictly speaking, in the theorem below one needs to assume that the zero locus of the Jacobian ideal $\mathrm{Jac}(f)$ (generated locally by $\partial f/\partial x_1,\ldots,\partial f/\partial_{x_n}$) is contained in the zero locus of f. This however can be accomplished after passing to an open neighborhood of the zero locus of f, which is of course all we care about if we want to study $b_f(s)$. Recall that $\mathcal{N}=\mathcal{H}^0\mu_+\mathcal{N}_{f'}$.

Theorem 3.8. *The $\mathscr{D}_X$-module $\mathcal{N}$ is subholonomic. More precisely we have*

$$\mathrm{Ch}(\mathcal{N})=W_f\cup\Lambda,$$

where Λ is a Lagrangian subvariety of T^*X, and W_f is the closure of the subset

$$\{(x,s\mathrm{dlog}f(x)\mid f(x)\neq 0,\ s\in\mathbf{C}\}\subset T^*X,$$

which is involutive and $(n+1)$-dimensional.

Corollary 3.9. *The $\mathscr{D}_X$-module $\mathcal{N}/t\mathcal{N}$ is holonomic.*

Proof. This is true for an arbitrary subholonomic $\mathscr{D}$-module preserved by the action of t. Indeed, consider the short exact sequence

$$0\longrightarrow\mathcal{N}\xrightarrow{\cdot t}\mathcal{N}\longrightarrow\mathcal{N}/t\mathcal{N}\longrightarrow 0.$$

It implies that the characteristic variety of $\mathcal{N}/t\mathcal{N}$ satisfies

$$\mathrm{Char}(\mathcal{N}/t\mathcal{N})\subseteq\mathrm{Char}(\mathcal{N}),$$

but also that its multiplicity along each irreducible component of $\mathrm{Char}(\mathcal{N})$ is equal to zero. Hence its dimension is strictly smaller than that of $\mathrm{Char}(\mathcal{N})$, so it can only be equal to $n=\dim X$. □

By the discussion above, the Corollary implies that there exists a Bernstein-Sato polynomial $b_{\mathcal{N}}(s)$; it satisfies the following:

Lemma 3.10. *We have the divisibility*

$$b_{\mathcal{N}}(s)\mid b_{f'}(s).$$

Proof. By definition we have $b_{f'}(s)\cdot\mathcal{N}_{f'}\subseteq t\mathcal{N}_{f'}$, so there exists a $\mathscr{D}_{X'}$-module endomorphism $\varphi\colon\mathcal{N}_{f'}\to\mathcal{N}_{f'}$ such that

$$b_{f'}(s)=t\varphi.$$

We can apply the functor $\mathcal{H}^0\mu_+$ to this identity; since the push-forward is in the sense of $\mathscr{D}_Y$-modules, it commutes with the s and t action, hence we obtain an induced endomorphism

$$b_{f'}(s)\colon \mathcal{N} \to \mathcal{N}, \quad b_{f'}(s) = t\psi,$$

where $\psi = \mathcal{H}^0\mu_+(\varphi)$. This immediately gives the conclusion. $\square$

We can now address the main result of the section.

Proof of Theorem 3.1. With the notation above, by Example 1.10(4) the statement holds for $b_{f'}(s)$. The result follows from the following stronger statement: there exists an integer $N \geq 0$ such that

$$b_f(s) \mid b_{f'}(s)b_{f'}(s+1)\cdots b_{f'}(s+N). \tag{3.1}$$

To show (3.1), we start by considering the $\mathscr{D}_X\langle s,t\rangle$-module

$$\mathcal{M} := \mathcal{N}/\mathcal{N}'.$$

This is a coherent $\mathscr{D}_X$-module, and we claim that it is holonomic. Indeed, on one hand we have

$$\mathrm{Ch}(\mathcal{M}) \subseteq \mathrm{Ch}(\mathcal{N}) = W_f \cup \Lambda,$$

using Theorem 3.8. On the other hand, by construction $\mathrm{Supp}(\mathcal{M}) \subseteq D = (f=0)$, since by construction $\mathcal{N}$ and $\mathcal{N}'$ coincide away from D. Hence in fact

$$\mathrm{Ch}(\mathcal{M}) \subseteq \big(W_f \cap \pi_X^{-1}(D)\big) \cup \Lambda.$$

It remains to note that by definition W_f dominates X , and therefore the right hand side has dimension n.

We can therefore apply Lemma 3.7, since by Corollary 3.9 we know that $\mathcal{N}/t\mathcal{N}$ is holonomic. We conclude that there exists an integer $N \geq 0$ such that

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s+1)\cdots b_{\mathcal{N}}(s+N).$$

By Lemma 3.10 we also know that $b_{\mathcal{N}}(s) \mid b_{f'}(s)$. It suffices then to show that $b_f(s) \mid b_{\mathcal{N}'}(s)$. But this is clear, since $\mathcal{N}_f$ is a quotient of $\mathcal{N}'$, so $b_{\mathcal{N}'}(s)\cdot\mathcal{N}_f \subseteq t\mathcal{N}_f$. $\square$

4. Lichtin's theorem and generalizations. In this section we discuss a refined version of Kashiwara's theorem due to Lichtin [Li], which relates the roots of Bernstein-Sato polynomials with invariants appearing on resolutions of singularities. We will in fact prove a recent theorem of Dirks-Mustaţă [DM], providing a further extension of Lichtin's theorem to the Bernstein-Sato polynomials of certain elements in the $\mathscr{D}$-module $\iota_+\mathscr{O}_X$, where ι is the embedding given by the graph of a regular function f.

We start by recalling a few notions and facts related to resolution of singularities. We always consider X to be a smooth complex variety of dimension n.

Definition 4.1. Let D be an effective $\mathbb{Q}$-divisor on X. A *log resolution* of the pair (X,D) is a proper birational morphism $\mu\colon Y \to X$, where Y is a smooth variety and the divisor $\mu^*D + \mathrm{Exc}(\mu)$ has simple normal crossing (SNC) support.

Log resolutions are known to exist in characteristic 0 by Hironaka's theorem. Moreover, one can always obtain such a resolution as a sequence of blow-ups along smooth centers contained in the locus where D does not have SNC support. Hence we will often assume that μ is an isomorphism away from this locus, so in particular away from $U = X \smallsetminus \mathrm{Supp}(D)$.

Remark 4.2. For concrete calculations, the existence of such a resolution can be rephrased locally as saying that for any regular function f on X, there exists a proper birational map as above, and around each point $y \in Y$ a system of algebraic coordinates $y_1, \ldots, y_n$ such that in this neighborhood

$$\mu^* f = h \cdot y_1^{a_1} \cdot \ldots \cdot y_n^{a_n},$$

with $k_i \geq 0$ and h an invertible function.

We will write

$$f^* D = \sum_{i=1}^{m} a_i \cdot E_i, \tag{4.1}$$

where $a_i \geq 0$ are rational numbers, and E_i are either components of the proper transform $\widetilde{D}$ or exceptional divisors. Another standard point to note is that while we cannot talk about canonically defined divisors K_X and K_Y, there is a canonically defined *relative* canonical divisor $K_{Y/X}$, namely the zero locus of the Jacobian

$$\mathrm{Jac}(\mu) = \det\big(\partial \mu_i / \partial y_j\big)_{i,j}$$

of the map μ. This supported on the exceptional locus of μ. We will also write

$$K_{Y/X} := \mathrm{div}(\mathrm{Jac}(\mu)) = \sum_{i=1}^{m} b_i \cdot E_i. \tag{4.2}$$

with $b_i \geq 0$. When μ is an isomorphism away from $U = X \smallsetminus \mathrm{Supp}(D)$, all exceptional divisors appear nontrivially in both sums.

Example 4.3. Let's review a few well-known examples; all the statements are left as exercises.

(1) If $\mu\colon Y = \mathrm{Bl}_W X \to X$ is the blow-up of X along a smooth subvariety of codimension c, and F is the exceptional divisor over W, then

$$K_{Y/X} = (c-1)F.$$

In particular, if $W = \{x\}$ is a point, then $K_{Y/X} = (n-1)F$.

(2) If $D = (x^2 + y^3 = 0) \subset \mathbf{C}^2$ is a cusp, then a log resolution $\mu\colon Y \to X = \mathbf{C}^2$ of (X, D) can be obtained as the composition of three succesive blow-ups at points; see the picture in [La, Example 9.1.13]. If we denote by F_1, F_2 and F_3 the exceptional divisors arising from the three blow-ups (in this order), then an easy calculation gives

$$\mu^* D = \widetilde{D} + 2F_1 + 3F_2 + 6F_3 \quad \text{and} \quad K_{Y/X} = F_1 + 2F_2 + 4F_3. \tag{4.3}$$

(3) Let $D = (f = 0)$ and assume for simplicity that $f(0) = 0$. Write $f = f_m + f_{m+1} + \cdots$, with f_i homogeneous of degree i for all i, and $f_m \neq 0$, so that $\mathrm{mult}_0 D = m$. Recall that the tangent cone of D at 0 is

$$TC_x D = (f_m = 0) \subset \mathbf{A}^n,$$

while the projectivized tangent cone is $\mathbb{P}(TC_0 D) \subset \mathbb{P}^{n-1}$. We say that D has an *ordinary singularity* at 0 if $\mathbb{P}(TC_0 D)$ is smooth.

The main example is when D is the cone in $\mathbf{A}^n$ over a smooth hypersurface in $\mathbb{P}^{n-1}$. For instance one can take $f = x_1^m + \cdots + x_n^m$, the Fermat hypersurface of degree m.

If $x \in D$ is an ordinary singularity of multiplicity m, then it is an isolated singularity, and a log resolution is given by $\mu\colon Y = \mathrm{Bl}_x X \to X$. We then have

$$\mu^* D = \widetilde{D} + mF \quad \text{and} \quad K_{Y/X} = (n-1)F.$$

We now state Lichtin's theorem. We fix a nonzero regular function f on X, and set $D = (f = 0)$. In the rest of the section we assume that the log resolutions $\mu\colon Y \to X$ are chosen to be isomorphisms away from $\mathrm{Supp}(D)$. In this case we can write $K_{Y/X}$ and $\mu^* D$ as in (4.1) and (4.2), with all $a_i \neq 0$. The following is [Li, Theorem 5].[5]

Theorem 4.4. *With the notation above, all the roots of the Bernstein-Sato polynomial $b_f(s)$ are of the form*

$$-\frac{b_i + 1 + \ell}{a_i}$$

for some $1 \leq i \leq m$ and some integer $\ell \geq 0$.

The proof is a refinement of Kashiwara's arguments described in the previous section. A recent result of Dirks-Mustaţă uses arguments similar to Lichtin's (and Kashiwara's) to extend this further to other Bernstein-Sato polynomials related to f. The following is [DM, Theorem 1.2].

Theorem 4.5. *Let $g \in \mathscr{O}_X(X)$, and let $u = g\partial_t^p f^s \in \iota_+ \mathscr{O}_X$. With the same notation as in Theorem 4.4, we set $k_i = \mathrm{ord}_{E_i}(g)$. Then the following hold:*

(1) The greatest root of b_u is at most $\max\{-1, p - \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}\}$.

(2) If $p = 0$, then the greatest root of b_u is at most $-\min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}$.

(3) If $g = 1$, then every root of b_u is either a negative integer or of the form

$$p - \frac{b_i + 1 + \ell}{a_i} \quad \text{for some integers } \; 1 \leq i \leq m \; \text{ and } \; \ell \geq 0.$$

If we assume in addition that $D = (f = 0)$ is reduced and the proper transform $\widetilde{D}$ is smooth, then we may consider only those i such that E_i is exceptional.

[5] Note that the proof of Kashiwara's result, Theorem 3.1, only shows that the roots of $b_f(s)$ are all of the form $-\frac{1+\ell}{a_i}$ for some $1 \leq i \leq m$ and some integer $\ell \geq 0$.

Lichtin's theorem above is then the special case $g = 1$ and $p = 0$ of this result. (Note that in this case all negative integers are also of the second type described in (3).) The extra ingredient introduced by Lichtin compared to the proof of Kashiwara's theorem is to pass from left to right $\mathscr{D}$-modules, and work with a slightly modified $\mathscr{D}$-module on the resolution in order to absorb the relative canonical divisor $K_{Y/X}$ in the calculations.

We now introduce the necessary ingredients for the proof. We use the notation introduced in Sections B and C of the notes on the V-filtration (see also the previous section).

Generalities of Bernstein-Sato polynomials. A useful point for later is the following:

Lemma 4.6. *Let u be a section of $\iota_+\mathscr{O}_X$, and $h \in \mathscr{O}_X$. Then the greatest root of $b_{hu}(s)$ is at most equal to the greatest root of $b_u(s)$.*

Proof. Let $-\alpha$ be the greatest root of u. By Sabbah's description of the V-filtration (Proposition 0.28 in the notes on the V-filtration) we then have $u \in V^\gamma\iota_+\mathscr{O}_X$. But V^γ is an $\mathscr{O}_X$-module, and therefore $hu \in V^\gamma\iota_+\mathscr{O}_X$ as well. Applying Sabbah's result again, we obtain that the greatest root of $b_{hu}(s)$ is at most $-\alpha$. □

The following lemma is a generalization of Exercise .1:

Lemma 4.7. *Let p and q be invertible functions on X, and let $u = g\partial_t^p f^s$ for some regular functions f and g. If $v = (qg)\partial_t^p(pf)^s$, then $b_u(s) = b_v(s)$.*

Proof. We consider the sheaf

$$\iota_+\mathscr{O}_X(*D) \simeq \mathscr{O}_X[s, \frac{1}{f}]f^s$$

as an $\mathscr{O}_X$-module; it already has the standard $\mathscr{D}_X\langle t, \partial_t\rangle$-action, but we endow it with a new one, denoted $\star$ and given by:

- $D \star w = Dw + swD(p)p^{-1}$, for all $D \in \mathrm{Der}_{\mathbf{C}}(\mathscr{O}_X)$.
- $t \star w = (pt)w$.
- $\partial_t \star w = (p^{-1}\partial_t)w$.

We denote this $\mathscr{D}_X\langle t, \partial_t\rangle$-module by $\iota_+\mathscr{O}_X(*D)^\star$. Note that the new action of $s = -\partial_t t$ coincides with the old one.

A simple calculation shows that the map

$$\nu\colon \iota_+\mathscr{O}_X(*D) \to \iota_+\mathscr{O}_X(*D)^\star, \quad P(s)(pf)^s \mapsto P(s)f^s$$

is an isomorphism of $\mathscr{D}_X\langle t, \partial_t\rangle$-modules, mapping v to $qp^{-m}g\partial_t^p f^s$. Recalling from the notes on the V-filtration (see the proof of Proposition 0.25) that $b_v(s)$ is the monic polynomial of minimal degree such that $b_v(s)v \in \mathscr{D}_X\langle t, s\rangle tv$, it follows that it is also the monic polynomial of minimal degree satisfying

$$b_v(s)qp^{-m}g\partial_t^p f^s \in \mathscr{D}_X\langle t, s\rangle t \star qp^{-m}g\partial_t^p f^s.$$

On the other hand, for every section w of $\iota_+\mathscr{O}_X(*D)$ and every invertible function $\varphi \in \mathscr{O}_X$ we have

$$\mathscr{D}_X\langle t,s\rangle t \star w = \mathscr{D}_X\langle t,s\rangle tw \quad \text{and} \quad \mathscr{D}_X\langle t,s\rangle \varphi w = \mathscr{D}_X\langle t,s\rangle w.$$

Hence we deduce that $b_v(s) = b_u(s)$. □

Remark 4.8. If $g \in \mathscr{O}_X$ is such that g/f is not a regular function, then

$$(s+1) \mid b_{gf^s}(s).$$

Indeed, we have that

$$b_{gf^s}(s)gf^s = P(s)gf^{s+1} \quad \text{with} \quad P \in \mathscr{D}_X[s],$$

and taking $s = -1$ we obtain that $b_{gf^s}(-1) \cdot (g/f)$ is a regular function, which cannot happen unless $b_{gf^s}(-1) = 0$.

Lemma 4.9. *In local algebraic coordinates $x_1, \ldots, x_n$, let $u = g\partial_t^p f^s$ be the section of $\iota_+\mathscr{O}_X$ given by $f = x_1^{c_1} \cdots x_n^{c_n}$ and $g = x_1^{d_1} \cdots x_n^{d_n}$, for some non-negative integers c_i, d_i and p. Then the following hold:*

(1) $b_u(s)$ divides $(s+1)\prod_{i=1}^n \prod_{j=1}^{c_i}\left(s - p + \frac{d_i+j}{c_i}\right)$. (Here the second product is taken to be 1 if $c_i = 0$.)

(2) If $p = 0$, then $b_u(s)$ divides $\prod_{i=1}^n \prod_{j=1}^{c_i}\left(s + \frac{d_i+j}{c_i}\right)$.

(3) If $c_1 = 1$ and $d_1 = 0$, then $b_u(s)$ divides $(s+1)\prod_{i=2}^n \prod_{j=1}^{c_i}\left(s - p + \frac{d_i+j}{c_i}\right)$.

Proof. We use the notation

$$gf^s = x^{cs+d} := \prod_{i=1}^n x_i^{c_i s + d_i} \quad \text{and} \quad tgf^s = x^{c(s+1)+d} := \prod_{i=1}^n x_i^{c_i(s+1)+d_i}.$$

Note that we can rewrite $u = \partial_t^p x^{cs+d}$.

Consider now the polynomial

$$c(s) := \prod_{i=1}^n \prod_{j=1}^{c_i} \big(c_i(s-p) + d_i + j\big).$$

Using the identity $\partial_t P(s) = P(s-1)\partial_t$ for all $P \in \mathbf{C}[s]$, we derive

$$\partial_{x_1}^{c_1} \cdots \partial_{x_n}^{c_n} \partial_t^p x^{c(s+1)+d} = \partial_t^p \prod_{i=1}^n \prod_{j=1}^{c_i} (c_i s + d_i + j) x^{cs+d} = c(s)\partial_t^p x^{cs+d} = c(s)u. \tag{4.4}$$

This immediately implies (2), since for $p = 0$ it gives

$$c(s)u \in \mathscr{D}_X[s] \cdot tu.$$

By repeatedly applying the formula $\partial_t t = t\partial_t + 1$, and noting that $s = -\partial_t t$, we obtain

$$\partial_t^p t = -(s - p + 1)\partial_t^{p-1},$$

and therefore we have

$$(s+1)\partial_t^p t = (s-p+1)t\partial_t^p.$$

We obtain

$$(s+1)\partial_t^p x^{c(s+1)+d} = (s-p+1)tu.$$

Hence for arbitrary $p \geq 1$, going back to (4.4), this gives

$$(s+1)c(s)u = (s-p+1)\partial_{x_1}^{c_1}\cdots\partial_{x_n}^{c_n}tu,$$

which implies (1).

For (3), note first that under the assumption $c_1 = 1$ and $d_1 = 0$, g/f cannot be a regular function. By Remark 4.8 we then know that $(s+1)$ divides b_{gf^s}, so we can talk about the reduced version $\widetilde{b}_{gf^s}$. Moreover, exactly as in Example 0.33 in the notes on the V-filtration, we have that

$$b_u(s) \mid (s+1)\widetilde{b}_{gf^s}(s-p).$$

It therefore suffices to show that

$$\widetilde{b}_{gf^s}(s) \mid \prod_{i=2}^{n}\prod_{j=1}^{c_i}\left(s+\frac{d_i+j}{c_i}\right).$$

But by the assumption on c_1 and d_1, this is the same as what we saw in (2). □

Left to right correspondence. Recall that there is an equivalence of categories between left and right $\mathscr{D}_X$-modules, taking a left $\mathscr{D}_X$-module $\mathcal{M}$ to the right $\mathscr{D}_X$-module $\omega_X \otimes_{\mathscr{O}_X} \mathcal{M}$. In local coordinates $x_1, \ldots, x_n$, this is given by an involution

$$\mathscr{D}_X \to \mathscr{D}_X, \quad P \mapsto P^*,$$

where P^* is the adjoint of P, determined uniquely by the rules: $(PQ)^* = Q^*P^*$, $f^* = f$ for $f \in \mathscr{O}_X$, and $\partial_{x_i}^* = -\partial_{x_i}$. For a section u of $\mathcal{M}$, we define the section

$$u^* = dx \otimes u$$

of $\omega_X \otimes_{\mathscr{O}_X} \mathcal{M}$, where $dx := dx_1 \wedge \cdots \wedge d_{x_n}$. We then have

$$(Pu)^* = u^*P^* \quad \text{for all} \quad P \in \mathscr{D}_X.$$

We now extend this to $\mathscr{D}_X\langle s,t\rangle$-modules. The same rule

$$\mathcal{M} \mapsto \omega_X \otimes_{\mathscr{O}_X} \mathcal{M}$$

takes a left $\mathscr{D}_X\langle s,t\rangle$-module to a right one as follows. The involution of $\mathscr{D}_X$ we just described extends to one of $\mathscr{D}_X\langle t,\partial_t\rangle$ by mapping $t \mapsto t$ and $\partial_t \mapsto -\partial_t$, hence mapping

$$s = -\partial_t t \mapsto t\partial_t = -\partial_t t - 1 = -s-1.$$

We again have $(Pu)^* = u^*P^*$ for all sections u of $\mathcal{M}$ and P of $\mathscr{D}_X\langle s,t\rangle$.

Just as with $\iota_+\mathscr{O}_X(*D)$, we can also consider Bernstein-Sato polynomials for sections of the right $\mathscr{D}_X\langle s,t\rangle$-module $\omega_X \otimes_{\mathscr{O}_X} \iota_+\mathscr{O}_X(*D)$. More precisely, for a section u of $\iota_+\mathscr{O}_X(*D)$, a Bernstein-Sato relation $b_u(s)u = P(tu)$ becomes, after passing to adjoints,

$$u^*b_u(-s-1) = (u^*t)P^*,$$

hence we have

$$b_{u^*}(s) = b_u(-s-1).$$

Main construction, and proof of the theorem. Fix $g \in \mathscr{O}_X(X)$ and $p \geq 0$, and consider the $\mathscr{D}_X\langle t, s\rangle$-module

$$\mathcal{N}_{f,p}(g) := \mathscr{D}_X\langle t, s\rangle \cdot g\partial_t^p f^s \subseteq \iota_+\mathscr{O}_X.$$

where $\iota \colon X \hookrightarrow Y = X \times \mathbf{C}$ is the graph embedding induced by f. Note that when $g = 1$ and $p = 0$ we have

$$\mathcal{N}_{f,0}(1) = \mathcal{N}_f,$$

the $\mathscr{D}$-module used in the proof of Kashiwara's theorem. Just as in that case, we have

Exercise .3. The action of t preserves $\mathcal{N}_{f,p}(g)$, so that $t\mathcal{N}_{f,p}(g)$ is a $\mathscr{D}_X\langle t, s\rangle$-submodule of $\mathcal{N}_{f,p}(g)$. Furthermore, if $u = g\partial_t^p f^s$, then the Bernstein-Sato polynomial $b_u(s)$ is the minimal polynomial of the action of s on the quotient $\mathcal{N}_{f,p}(g)/t\mathcal{N}_{f,p}(g)$.

As in the proof of Theorem 3.1, by restricting to an open neighborhood of the zero locus $Z(f)$ we can make the harmless extra assumption that

$$Z(\mathrm{Jac(f)}) \subseteq Z(f), \tag{4.5}$$

where $\mathrm{Jac}(f)$ is the Jacobian ideal of f. Recall that we denote by W_f the closure of the subset

$$\{(x, sdf(x)) \mid f(x) \neq 0,\ s \in \mathbf{C}\} \subseteq T^*X.$$

This is an irreducible subvariety of T^*X, of dimension $n+1$, which dominates X. A result that Kashiwara proves at the same time as Theorem 3.8 is the following:

Theorem 4.10 ([Ka1, Theorem 5.3])**.** *The $\mathscr{D}_X$-module $\mathcal{N}_f$ is coherent, and* $\mathrm{Ch}(\mathcal{N}_f) = W_f$. *In particular $\mathcal{N}_f$ is subholonomic.*

Based on this, one can show the following:

Proposition 4.11. *If f defines a divisor with SNC support and satisfies (4.5), denoting $\mathcal{N}_{f,p} = \mathcal{N}_{f,p}(1)$, for every $p \geq 0$ we have:*

(1) As a $\mathscr{D}_X$-module, $\mathcal{N}_{f,p}$ is generated by $\partial_t^j f^s$, with $0 \leq j \leq p$.

(2) $\mathrm{Ch}(\mathcal{N}_{f,p}) = W_f$.

Proof. Recall that for every $j \geq 0$ we have

$$t\partial_t^j f^s = f\partial_t^j f^s - j\partial_t^{j-1} f^s,$$

from which descending induction on j shows that $\partial_t^j f^s \in \mathcal{N}_{f,p}$ for $0 \leq j \leq p$. The formula also shows that

$$\mathcal{N}'_{f,p} := \sum_{j=0}^{p} \mathscr{D}_X \cdot \partial_t^j f^s$$

is a $\mathscr{D}_X[t]$-submodule of $\mathcal{N}_{f,p}$. Thus by the definition of $\mathcal{N}_{f,p}$, to deduce (1) it suffices to show that

$$s\partial_t^j f^s \in \mathcal{N}'_{f,p}, \quad \text{for all} \quad 0 \leq j \leq p.$$

Recall now that $\partial_t s = (s-1)\partial_t$, and so

$$s\partial_t^j f^s = \partial_t^j (s+j) f^s.$$

It suffices thus to show that $sf^s \in \mathscr{D}_X f^s$, which is left as Exercise .4 below.

We show (2) by induction on p; the base case $p = 0$ is Theorem 4.10. Assuming that $p \geq 1$, part (1) implies that $\mathcal{N}_{f,p-1} \subset \mathcal{N}_{p,f}$, and the quotient $\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}$ is generated over $\mathscr{D}_X$ by the class of $\partial_t^p f^s$. In particular, there is a surjective $\mathscr{D}_X$-module homomorphism

$$\mathcal{N}_{f,0} = \mathscr{D}_X f^s \to \mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}, \quad Pf^s \mapsto P\widehat{\partial_t^p f^s},$$

which implies that $\mathrm{Ch}(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}) \subseteq W_f$. The fact that $\mathrm{Ch}(\mathcal{N}_{f,p} = W_f$ follows then from the chain of inclusions

$$W_f = \mathrm{Ch}(\mathcal{N}_{f,p-1}) \subseteq \mathrm{Ch}(\mathcal{N}_{f,p}) \subseteq \mathrm{Ch}(\mathcal{N}_{f,p-1}) \cup \mathrm{Ch}\big(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}\big) \subseteq W_f.$$

$\square$

Exercise .4. Show that if f defines a divisor with SNC support and satisfies (4.5), then $sf^s \in \mathscr{D}_X f^s$.

Proof of Theorem 4.5. Step 1. Using Proposition 1.6 and Remark 1.7, we see that the statement of the theorem is local on X. We may therefore assume that X is affine, with algebraic coordinates $x_1, \ldots, x_n$. We may also assume that f is not an invertible function on X, hence after passing to an open neighborhood of the zero locus of f, that condition (4.5) is satisfied.

For each u as in the statement, we consider the section

$$u^* = dx \otimes u$$

of the right $\mathscr{D}_X\langle s,t\rangle$-module $\mathcal{N}_u := \omega_X \otimes_{\mathscr{O}_X} \mathcal{N}_{f,p}(g)$. The right $\mathscr{D}$-module version of Exercise .3 tells us that $b_{u^*}(s)$ is equal to $b_{\mathcal{N}_u}(s)$, the minimal polynomial of the action of s on $\mathcal{N}_u/\mathcal{N}_u t$. By the discussion above we also know that $b_{u^*}(s) = b_u(-s-1)$.

Recalling that $\mu\colon Y \to X$ is the fixed log resolution, we denote

$$f' := f \circ \mu \quad \text{and} \quad g' = g \circ \mu.$$

By analogy with the construction on X, on Y we consider the $\mathscr{D}_Y\langle s,t\rangle$-module

$$\mathcal{N}_{u'} := \omega_Y \otimes_{\mathscr{O}_Y} \mathcal{N}_{f',p}(g'),$$

where $u' = g'\partial_t^p f'^s \in \iota_+ \mathscr{O}_Y$ and $\mathcal{N}_{f',p}(g') = \mathscr{D}_Y\langle s,t\rangle \cdot u'$. We also consider its submodule

$$\mathcal{N}_v := v \cdot \mathscr{D}_Y\langle s,t\rangle, \quad \text{with} \quad v := \mu^* dx \otimes u'.^{6}$$

We fix an open set $V \subseteq Y$ on which we have algebraic coordinates $y_1, \ldots, y_n$ in which

$$f' = h_1 y_1^{a_1} \cdots y_n^{a_n}, \quad \mu^* dx = h_2 y_1^{b_1} \cdots y_n^{b_n} dy, \quad \text{and} \quad g' = h y_1^{k_1} \cdots y_n^{k_n},$$

according to our usual notation (the only a_i which are non-zero correspond to $1 \leq i \leq m$), where h_1 and h_2 are invertible functions on V, and h is a regular function on V. Note that

[6] Note that this is where things diverge from (the analogue of) Kashiwara's argument, which would proceed using $\mathcal{N}_{u'}$, while here this is replaced by $\mathcal{N}_v$.

$b_v(s)$ (in the sense of right $\mathscr{D}$-modules) is the same as $b_{\mathcal{N}_v}(s)$, the minimal polynomial of the action of s on $\mathcal{N}_v/\mathcal{N}_v t$. According to the right $\mathscr{D}$-module version of Lemma 4.7, we then have that

$$b_{\mathcal{N}_v}(s) = b_{w'^*}(s) = b_{w'}(-s-1),$$

where $w' = hw$, with

$$w := y_1^{b_1+k_1} \cdots y_n^{b_n+k_n} \partial_t^p (y_1^{a_1} \cdots y_n^{a_n})^s \in \iota_+ \mathscr{O}_Y[\frac{1}{f'}].$$

Using Lemma 4.6, we see that the greatest root of $b_{w'}(s)$ is at most equal to the greatest root of $b_w(s)$. On the other hand, w is an element to which we can apply Lemma 4.9. Using it, and recalling that we have $a_i \neq 0$ only for $1 \leq i \leq m$, we obtain the following:

- The greatest root of $b_w(s)$ is at most $\max\{-1, p - \min\limits_{1\leq i\leq m} \frac{b_i+1+k_i}{a_i}\}$.
- If $p = 0$, then the greatest root of $b_w(s)$ is at most $-\min\limits_{1\leq i\leq m} \frac{b_i+1+k_i}{a_i}$.
- If $g = 1$ (so that $h = 1$ and we don't need to use Lemma 4.6) every root of $b_w(s)$ is either equal to -1 or to some $p - \frac{b_i+\ell}{a_i}$, with $1 \leq i \leq m$ and $1 \leq \ell \leq a_i$. If in addition D is reduced and its proper transform $\widetilde{D}$ is smooth, then we may assume that the divisor given by $(y_i = 0)$ on Y is exceptional. Indeed, note that in this case at most one y_i satisfies $b_i = 0$, i.e. it is not exceptional, and $a_i > 0$; in this case in fact $a_i = 1$.

Covering Y by open sets V on which we have such local coordinates, the global polynomial $b_{\mathcal{N}_v}(s)$ is the least common multiple of the respective polynomials on each V, described above. Hence to conclude the proof, it suffices to show that each root of $b_{u^*}(s)$ is of the form $\alpha + \ell$, where α is a root of $b_{\mathcal{N}_v}(s)$ and ℓ is a non-negative integer.

Step 2. According to the paragraph above, it suffices to show that there exists an integer $N \geq 0$ such that

$$b_{u^*}(s) \mid b_{\mathcal{N}_v}(s) b_{\mathcal{N}_v}(s-1) \cdots b_{\mathcal{N}_v}(s-N). \tag{4.6}$$

This is now very similar to the proof of Kashiwara's theorem, only in the setting of right $\mathscr{D}$-modules, and with $\mathcal{N}_v$ playing the role of $\mathcal{N}_{f'}$ there. We define

$$\mathcal{N} := \mathcal{H}^0 \mu_+ \mathcal{N}_v = R^0 \mu_* (\mathcal{N}_v \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}).$$

Exactly as in Lemma 3.10, we have the divisibility

$$b_{\mathcal{N}}(s) \mid b_{\mathcal{N}_v}(s). \tag{4.7}$$

We now perform Kashiwara's main construction in this setting, by constructing a distinguished section $r \in \Gamma(X, \mathcal{N})$. Note first that on Y we have a morphism of $\mathscr{D}_Y$-modules

$$\mathscr{D}_Y \to \mathcal{N}_v, \quad 1 \mapsto v.$$

Taking the derived tensor product with $\mathscr{D}_{Y\to X}$, we obtain a homomorphism

$$\mathscr{D}_{Y \to X} \to \mathcal{N}_v \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}.$$

On the other hand, the section $1 \in \mathscr{D}_X$ induces an $\mathscr{O}_Y$-module homomorphism $\mathscr{O}_Y \to \mathscr{D}_{Y\to X}$, since $\mathscr{D}_{Y\to X} \simeq \mu^*\mathscr{D}_X$ as an $\mathscr{O}_Y$-module. By composition we obtain a section

$$\mathscr{O}_Y \to \mathcal{N}_v \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X},$$

and applying $R^0\mu_*$ we finally obtain a global section $r \in \Gamma(X, \mathcal{N})$. It is immediate from the construction that $r = u^*$ on U, the complement of the zero locus of f. We define

$$\mathcal{N}' := r \cdot \mathscr{D}_X\langle s,t\rangle \subseteq \mathcal{N},$$

a right $\mathscr{D}_X\langle s,t\rangle$-submodule of $\mathcal{N}$. We can also define a $\mathscr{D}_X\langle s,t\rangle$-module homomorphism

$$\mathcal{N}' \to \mathcal{N}_u, \quad r \mapsto u^*,$$

and precisely as in the proof of Lemma 3.2, this is well defined, and obviously surjective. In conclusion, we have a digram of $\mathscr{D}_X\langle s,t\rangle$-modules

$$\begin{array}{ccc} \mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\ \big\downarrow{\scriptstyle g} & & \\ \mathcal{N}_u & & \end{array}$$

where i is the inclusion map, and g is surjective.

As in the proof of Theorem 3.1, we see that the $\mathscr{D}_X\langle s,t\rangle$-module $\mathcal{M} := \mathcal{N}/\mathcal{N}'$ is holonomic as a $\mathscr{D}_X$-module. Indeed, as in the paragraph after (3.1), it suffices to have the analogue of Theorem 3.8, namely

$$\mathrm{Ch}(\mathcal{N}) = W_f \cup \Lambda,$$

where Λ is a Lagrangian subvariety of T^*X. As $\mathcal{N}_v \subseteq \mathcal{N}_{f',p}(g')$, this in turn follows by using Proposition 4.11(2) and the obvious fact that $\mathcal{N}_{f',p}(g') \subseteq \mathcal{N}_{f',p}$ (together with a standard results on the characteristic variety of a direct image, that I have not explained yet; this is also used for proving Theorem 3.8, and it will be added eventually).

The right $\mathscr{D}$-module analogue of Lemma 3.7 then implies that

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s-1)\cdots b_{\mathcal{N}}(s-N).$$

for some integer $N \geq 0$. (Indeed the signs change, since now we are using the identity $b(s)t = tb(s-1)$ multiplying from the right.) In view of (4.7), it suffices then to show that

$$b_{u^*}(s) \mid b_{\mathcal{N}'}(s).$$

But this follows immediately from the surjection $\mathcal{N}' \to \mathcal{N}_u$, since $b_{u^*}(s) = b_{\mathcal{N}_u}(s)$. $\square$

5. Analytic continuation of the archimedean zeta function. We present here one of the original motivations for the the introduction of Bernstein-Sato polynomials, namely Bernstein's approach to Gel'fand's problem on the analytic continuation of complex powers of polynomials. The original problem was stated in the context of real polynomials; we will discuss its complex version.

For an introduction to what comes next, we recall the following statement from complex analysis:

Exercise .5. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in \mathbb{R}^n$, and let $\varphi \in C_c^0(\mathbb{R}^n)$ be a continuous complex $\mathbf{C}$-valued function with compact support. Then the function

$$Z_\varphi\colon \mathbf{C} \to \mathbf{C}, \quad s \mapsto \int_{\mathbf{R}^n} f(x)^s \varphi(x) dx$$

is an analytic function, where the meaning of $f(x)^s$ is $e^{s\log f}$.

Let now $f \in \mathbf{C}[x_1, \ldots, x_n]$ be a nonconstant polynomial, and $s \in \mathbf{C}$. We will consider $|f|^{2s}$ as a distribution depending on the complex parameter s, in the sense described below. Note first that this time f has zeros, and therefore the function $|f|^2$ does not satisfy the strict positivity condition in Exercise .5 any more.

On the other hand, it is easy to see that the function $|f(x)|^{2s}$ is continuous in x if we restrict to the case when

$$s \in \mathbb{H}_0 := \{s \in \mathbf{C} \mid \mathrm{Re}(s) > 0\}.$$

For any continuous $\mathbf{C}$-valued function with compact support $\varphi \in C_c^0(\mathbf{C}^n)$, we then still have a well-defined function

$$Z_\varphi\colon \mathbb{H}_0 \to \mathbf{C}, \quad s \mapsto \int_{\mathbf{C}^n} |f(x)|^{2s} \varphi(x) dx.$$

Proposition 5.1. *Z_φ is an analytic function on $\mathbb{H}$.*

Proof. First proof. The statement is an immediate application of the complex version of the theorem on differentiation under the integral sign; see e.g. [?]. Indeed, the function $|f(x)|^{2s}\varphi(x)$ is integrable for each $s \in \mathbb{H}_0$, and is analytic as a function of s, hence with bounded partial derivatives on the support of φ. Therefore applying $\partial/\partial \bar{z}$ commutes with the integral sign, and the statement follows.

Second proof. Let me also include a slightly more tedious second proof, which has the potentially useful advantage of providing the coefficients of a power series expansion in a neighborhood of a point in $\mathbb{H}_0$.

Fix a point $s_0 \in \mathbb{H}_0$. We show thart there exists $\epsilon > 0$ such that for all $s \in \mathbb{H}$ with $|s - s_0| < \epsilon$ we have

$$Z_\varphi(s) = \sum_{k=0}^{\infty} a_k (s - s_0)^k,$$

a convergent power series with $a_k \in \mathbf{C}$.

For $s \in \mathbb{H}_0$ close enough to s_0, using the usual expansion of the exponential function we have

$$|f(x)|^{2s}\varphi(x) = \sum_{k=0}^{\infty} \frac{(2\log|f(x)|)^k |f(x)|^{2s_0}\varphi(x)}{k!} \cdot (s - s_0)^k. \tag{5.1}$$

Since φ has compact support, we can find $C > 0$ such that

$$\mathrm{Supp}(\varphi) \subseteq K := [-C, C]^n.$$

Claim. There exist real numbers $\epsilon > 0$ and $M > 0$ such that

$$a_k := \sup_{x\in K} \left| \frac{(2\log|f(x)|)^k |f(x)|^{2s_0}}{k!} \right| \leq M/\epsilon^k, \quad \forall k \in \mathbf{N},$$

with the convention that the quantity we are measuring is 0 at $x \in K$ such that $f(x) = 0$. Assuming the Claim, we obtain that the series in (5.1) converges absolutely and uniformly on $K \times \{|s - s_0| < \delta\}$ for any $0 < \delta < \epsilon$, hence

$$Z_\varphi(s) = \sum_{k=0}^{\infty} \left(\int_{\mathbf{C}^n} \frac{(2\log|f|)^k |f|^{2s_0}\varphi}{k!} \right) (s - s_0)^k \quad \text{for} \quad |s - s_0| < \epsilon,$$

which concludes the proof.

We are left with proving the Claim. Denoting $r_0 = \mathrm{Re}(s_0) \in \mathbb{R}_{>0}$, we note that

$$k! \cdot a_k = \sup_{x\in K} |(2\log|f(x)|)^k |f(x)|^{2s_0}| = |(2\log|f(x)|)^k |f(x)|^{r_0}| = \frac{2^k}{r_0^k} |(\log|f(x)|^{r_0})^k |f(x)|^{r_0}|.$$

We consider $M \in \mathbb{R}_{>0}$ such that $|f(x)|^{r_0} \leq M$ for all $x \in K$, so that

$$a_k \leq \frac{\sup_{x\in(0,M]} 2^k |(\log x)^k x|}{k! r_0^k} \leq \frac{2^k \cdot \max\{M(\log M)^k, k^k e^{-k}\}}{k! r_0^k},$$

where the second inequality uses the easily checked fact that the $x(\log x)^k$ takes its minimum on $(0, 1]$ at $x = e^{-k}$. Finally, Stirling's formula says that

$$k! \sim \sqrt{2\pi k} \cdot k^k e^{-k},$$

which implies that, choosing a suitable $M > 0$, any $0 < \epsilon < r_0$ will do for the Claim. □

The problem proposed by Gel'fand was whether one can analytically continue Z_φ to a meromorphic function on $\mathbf{C}$; he also asked whether its poles lie in a finite number of arithmetic progressions. The existence of the Bernstein-Sato polynomial leads to a positive answer.

Theorem 5.2. *With the notation above, for every smooth complex-valued function with compact support $\varphi \in C_c^\infty(\mathbf{C}^n)$, Z_φ admits an analytic continuation to $\mathbf{C}$ as a meromorphic function whose poles are of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_f(s)$ and $m \in \mathbf{N}$. In particular all the poles are negative rational numbers.*

Before giving the proof of the theorem, we record a technical point that will be useful later on as well. We start with the defining Bernstein-Sato formula

$$b(s)f^s = P(s)f^{s+1}, \quad \text{with} \quad P \in A_n[s],$$

where A_n is the Weyl algebra, and for simplicity we denote $b(s) = b_f(s)$. We also consider the conjugate $\bar{P}(s)$, where we replace x_i by $\overline{x_i}$ and ∂_{x_i} by $\partial_{\overline{x_i}}$.

Lemma 5.3. *For every $s \in \mathbf{C}$ we have*

$$b(s)^2|f|^{2s} = P(s)\bar{P}(s)|f|^{2(s+1)}.$$

Proof. Conjugating the Bernstein-Sato formula above gives
$$b(s)\bar{f}^s = \bar{P}(s)\bar{f}^{s+1}.$$
(Note that by Kashiwara's theorem, $b(s)$ has rational coefficients.) Hence multiplying the two formulas leads to
$$b(s)^2|f|^{2s} = (P(s)f^{s+1}) \cdot (\bar{P}(s)\bar{f}^{s+1}).$$
The fact that the coefficients of $\bar{P}(s)$ are antiholomorphic differential operators implies that
$$\bar{P}(s)\big(f^{s+1} \cdot \bar{f}^{s+1}\big) = f^{s+1} \cdot \bar{P}(s)\bar{f}^{s+1},[7]$$
and therefore
$$P(s)\bar{P}(s)|f|^{2(s+1)} = P(s)\bar{P}(s)\big(f^{s+1} \cdot \bar{f}^{s+1}\big) = (P(s)f^{s+1}) \cdot (\bar{P}(s)\bar{f}^{s+1})$$
as similarly since P is holomorphic we have $P(s)(\bar{P}(s)\bar{f}^{s+1}) = 0$. □

Proof. (*of Theorem 5.2*). For every $m \in \mathbf{N}$, denote
$$\mathbb{H}_m := \{s \in \mathbf{C} \mid \mathrm{Re}(s) > -m\}.$$
We have seen that all such Z_φ are holomorphic on $\mathbb{H}_0$, and we show by induction on m that they can be extended to meromorphic functions on $\mathbb{H}_m$ with poles as stated.

Going back to the identity
$$b(s) \cdot f^s = P(s) \cdot f^{s+1},$$
we write
$$P(s) = \sum_{i=0}^{p} Q_i \cdot s^i, \quad Q_i \in A_n.$$
We then have
$$P(s)\bar{P}(s) = \sum_{i=0}^{2p} R_i \cdot s^i,$$
for operators $R_i = \overline{R_i}$. For $s \in \mathbb{H}_0$ we can then write
$$b(s)^2 \int_{\mathbf{C}^n} |f(x)|^{2s}\varphi(x) = \sum_{i=0}^{2p} s^i \int_{\mathbf{C}^n} (R_i|f(x)|^{2(s+1)}) \cdot \varphi(x).$$
Now a simple application of integration by parts (see Exercise .6 below) shows that
$$\int_{\mathbf{C}^n} (R_i|f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbf{C}^n} |f(x)|^{2(s+1)} \cdot (R_i\varphi)(x).$$
As a consequence, for $s \in \mathbb{H}_0$ we obtain
$$Z_\varphi(s) = \frac{\sum_{i=0}^{2p} s^i Z_{R_i\varphi}(s+1)}{b(s)^2}.$$
But we are assuming by induction that $Z_{R_i\varphi}(s)$ can be extended to a meromorphic function with poles as prescribed on $\mathbb{H}_m$. Therefore the identity shows that $Z_\varphi(s)$ can be extended to a meromorphic function $\mathbb{H}_{m+1}$, whose poles are among the roots of $b(s)$ and the poles

[7] Apply one $\partial/\partial\bar{z}_i$ at a time, and use the product rule.

of the extension of $Z_{R_i\varphi}(s+1)$ to $\mathbb{H}_{m+1}$, again of the prescribed form. The last statement follows from Kashiwara's result, Theorem 3.1. $\square$

Exercise .6. Let $Q \in A_n$, and define Q' to be its image under the (anti-)automorphism of A_n sending x_i to x_i and ∂_i to $-\partial_i$. For every complex-valued $\varphi \in C_c^\infty(\mathbf{C}^n)$ we have

$$\int_{\mathbf{C}^n} (Q|f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbf{C}^n} |f(x)|^{2(s+1)} \cdot (Q'\varphi)(x),$$

and similarly for $\overline{Q}$. (Hint: apply the change of variables formula when $Q = \partial_i$, for each i, and then use induction on the degree of Q.)

Remark 5.4 (The distribution $|f|^{2s}$)**.** Usually the quantity $|f|^{2s}$ is thought of as a distribution, which on any $\varphi \in C_c^\infty(\mathbf{C}^n)$ is defined by

$$(|f|^{2s}, \varphi) := \int_{\mathbf{C}^n} |f(x)|^{2s}\varphi(x).$$

This depends holomorphically on s for $\mathrm{Re}(s) > 0$, and the theorem above says that it can be continued meromorphically to $\mathbf{C}$, with poles of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_f(s)$ and $m \in \mathbf{N}$. Here if $P(\cdot)$ denotes the set of poles, by definition we have

$$P(|f|^{2s}) := \bigcup_{\varphi} P\big(Z_\varphi(s)\big).$$

Note. Bernstein-Gel'fand and Atyiah originally answered Gel'fand's question making use of resolution of singularities. The proof presented here, making use instead of the existence of the Bernstein-Sato polynomial, was given later by Bernstein in [Be]. The exposition draws also on the lecture notes [Ay], [Gr].

6. Log resolutions, log canonical thresholds, multiplier ideals. In this section we review a few basic concepts from birational geometry. Most of the material below is covered in great detail in [La, Ch.9]. For general singularities of pairs, a great introduction is [KM].

Definition 6.1. (1) Let $\mu\colon Y \to X$ be a log resolution of (X, D) (see §4), and fix a prime divisor E on Y. The *discrepancy* of E (with respect to D) is

$$a(E) = a(E; X, D) := \mathrm{ord}_E(K_{Y/X} - \mu^*D) \in \mathbb{Q}.$$

Consequently we have

$$K_Y - \mu^*(K_X + D) = \sum_E a(E) \cdot E,$$

where the sum is taken over all prime divisors E in X, or equivalently

$$K_Y + \widetilde{D} - \mu^*(K_X + D) = \sum_{E \text{ exceptional}} a(E) \cdot E.$$

(2) The pair (X, D) is called *log-canonical* if $a(E) \geq -1$ for all exceptional divisors E on any log resolution. It is called *klt (Kawamata log terminal)* if $a(E) > -1$ for all prime divisors E on any log resolution.

(3) The *log canonical threshold* of D is

$$\mathrm{lct}(D) := \inf\{c \in \mathbb{Q} \mid (X, cD) \text{ is not log canonical}\}.$$

In particular, if D is an integral divisor, then (X, D) is log canonical if and only if $\mathrm{lct}(D) = 1$. There is also a more refined local invariant, defined for any $x \in X$ to be

$$\mathrm{lct}_x(D) := \inf\{c \in \mathbb{Q} \mid (X, cD) \text{ is not log canonical around } x\}.$$

Exercise .7. We have $\mathrm{lct}(D) = \min_{x\in X}\mathrm{lct}_x(D)$.

Lemma 6.2. *We have $\mathrm{lct}_x(D) \in \mathbb{Q}$, and the infimum in the definition is in fact a minimum that can be computed on any resolution of singularities.*

Proof. We fix a log resolution $\mu\colon Y \to X$, and use the notation introduced in §4. We then have, for $c \in \mathbb{Q}$,

$$K_Y - \mu^*(K_X + cD) = \sum_{i=1}^{m}(b_i - ca_i)E_i,$$

and so with respect to the pair (X, cD) we get, for every exceptional divisor E_i, that

$$a(E_i) \geq -1 \iff b_i - ca_i \geq -1 \iff \frac{b_i + 1}{a_i} \geq c.$$

Hence we obtain that

$$\mathrm{lct}_x(D) = \min_{1\leq i\leq m} \frac{b_i + 1}{a_i}, \tag{6.1}$$

where the minimum is taken over all E_i such that $x \in \mu(E_i)$, at least after showing that this quantity is independent of the choice of log resolution. This is left as an exercise, but note that it will also follow from the analytic interpretation given below. $\square$

Exercise .8. Show that the function $x \mapsto \mathrm{lct}_x(D)$ is lower semicontinuous on X.

Remark 6.3. When D is a reduced integral divisor, the proof above shows that

$$\mathrm{lct}(D) = \min\{1, \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}\}.$$

Example 6.4. Going back to Example 4.3, we see that:

(1) The log canonical threshold of the cusp $(x^2 + y^3 = 0)$ is $5/6$.

(2) If D has an ordinary singularity of multiplicity m at x, then $\mathrm{lct}_x(D) = \min\{1, \frac{n}{m}\}$.

We now come to a concept of great importance in modern birational geometry.

Definition 6.5. The *multiplier ideal* of the $\mathbb{Q}$-divisor D on X is defined as

$$\mathcal{J}(D) := \mu_*\mathscr{O}_Y\big(K_{Y/X} - [\mu^*D]\big) \subseteq \mathscr{O}_X,$$

where $\mu\colon Y \to X$ is any log resolution of (X, D). The fact that it is an ideal sheaf follows since clearly $\mathcal{J}(D) \subseteq \mu_*\omega_{Y/X} \simeq \mathscr{O}_X$.

It is well known, and originally due to Esnault-Viehweg, that this definition is independent of the choice of resolution of singularities. By dominating any two log resolutions by a third, this reduces to the following statement, which can be found in [La, Lemma 9.2.19].

Lemma 6.6. *Let D be an effective $\mathbb{Q}$-divisor on X with SNC support, and $\mu\colon Y \to X$ a log resolution of (X, D). Then*

$$\mu_*\mathscr{O}_Y\big(K_{Y/X} - [\mu^*D]\big) \simeq \mathscr{O}_X(-[D]).$$

The behavior of multiplier ideals under birational maps is not hard to describe:

Exercise .9 (Birational transformation rule)**.** Show that if $\mu\colon Y \to X$ is a proper birational map of smooth varieties, and D an effective $\mathbb{Q}$-divisor on X, then

$$\mathcal{J}(D) \otimes_{\mathscr{O}_X} \omega_X \simeq \mu_*\mathscr{O}_Y\big(\mathcal{J}(\mu^*D) \otimes_{\mathscr{O}_Y} \omega_Y\big).$$

(Hint: take a log resolution of the pair $(Y, \mu^*D + \mathrm{Exc}(\mu))$, and use the projection formula.)

Exercise .10. The notions of singularities of pairs defined above can be interpreted in terms of multiplier ideals as follows:

$$(X, D) \text{ is klt} \iff \mathcal{J}(D) = \mathscr{O}_X$$

and

$$(X, D) \text{ is log canonical} \iff \mathcal{J}\big((1-\epsilon)D\big) = \mathscr{O}_X, \quad \text{for all } \ 0 < \epsilon < 1.$$

In fact we have

$$\mathrm{lct}(D) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(cD) \neq \mathscr{O}_X\}$$

and more precisely

$$\mathrm{lct}_x(D) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(cD)_x \subseteq \mathfrak{m}_x\}.$$

Example 6.7. (1) Using the definition and the projection formula, if D is an integral effective divisor we have $\mathcal{J}(D) = \mathscr{O}_X(-D)$. For the same reason, if D is an arbitrary effective $\mathbb{Q}$-divisor and E is an integral divisor, then

$$\mathcal{J}(D + E) = \mathcal{J}(D) \otimes \mathscr{O}_X(-E).$$

(2) If D has SNC support, then $\mathcal{J}(D) = \mathscr{O}_X(-[D])$.

(3) If $D = (x^2 + y^3 = 0)$ is a cusp in $\mathbf{C}^2$, using the resolution and calculations in Example 4.3(2) we have

$$\mathcal{J}(cD) = \mu_*\mathscr{O}_Y\big(K_{Y/X} - [c\mu^*D]\big) =$$

$$\mu_*\mathscr{O}_Y\big((1-[2c])E_1 + (2-[3c])E_2 + (4-[6c])E_3 - [c]\widetilde{D}\big).$$

We can focus on the case $0 < c < 1$ (see (1)), and so we obtain $\mathcal{J}(cD) = \mathscr{O}_X$ as long as all the coefficients in the parenthesis are nonnegative, i.e.

$$\mathcal{J}(cD) = \mathscr{O}_X \iff 0 < c < 5/6$$

and

$$\mathcal{J}(cD) = \mu_*\mathscr{O}_Y(-E_3) = \mathfrak{m}_0 \iff 5/6 \leq c < 1.$$

(4) A more general example that will be discussed later in the analytic setting is that of a general $D = (x^a + y^b = 0) \subset \mathbf{C}^2$, with $a, b \geq 2$. We will see in Example 6.18 that

$$\mathcal{J}(cD) = \mathcal{O}_X \iff c < \frac{1}{a} + \frac{1}{b},$$

so in particular $\mathrm{lct}(D) = 1/a + 1/b$.

The log canonical thrreshold is the first in a sequence of rational numbers describing the "jumps" of the multiplier ideals associated to multiples of a fixed divisor.

Proposition 6.8. *If D is an effective $\mathbb{Q}$-divisor and $x \in X$ is a point, then there exists an increasing sequence of rational numbers*

$$0 = c_0 < c_1 < c_2 < \cdots$$

such that:

- $\mathcal{J}(cD)_x = \mathcal{J}(c_i D)_x$ *for* $c \in [c_i, c_{i+1})$.
- $\mathcal{J}(c_{i+1}D)_x \neq \mathcal{J}(c_i D)_x$ *for all* i.

(Here by convention $\mathcal{J}(0 \cdot D) = \mathcal{O}_X$.) In particular $c_1 = \mathrm{lct}_x(D)$.

Proof. Just as in the proof of Lemma 6.2, we can write

$$\mathcal{J}(cD) = \mu_* \mathcal{O}_Y\big(\sum_{i=1}^{m}(b_i - [ca_i])E_i\big),$$

and clearly the coefficients $b_i - [ca_i]$ are constant on intervals as indicated. Moreover, the endpoints of these intervals belong to the set

$$\{\frac{b_i + m}{a_i} \mid \text{some } i \text{ and some } m \geq 1\} \subset \mathbb{Q}.$$

$\square$

Definition 6.9. The numbers c_i in Proposition 6.8 are called the *jumping numbers* (or *jumping coefficients*) of D at x.

Example 6.10. (1) If D is an integral divisor, we have noted earlier that $\mathcal{J}\big((c+1)D\big) = \mathcal{J}(cD) \otimes \mathcal{O}_X(-D)$, and therefore c is a jumping number for D if and only if $c+1$ is one. So all the jumping numbers are determined by those in the interval $[0, 1]$, which form a finite set according to the formula at the end of the proof of Proposition 6.8.

(2) We will see later that if $f = x_1^{d_1} + \cdots + x_n^{d_n}$, then the jumping numbers of f are all the rational numbers of the form

$$\frac{e_1 + 1}{d_1} + \cdots + \frac{e_n + 1}{d_n}, \quad \text{for all } \; e_1 \ldots, e_n \in \mathbf{N}.$$

We record the following for later use:

Lemma 6.11. *The jumping numbers of D at x satisfy the inequalities*

$$c_{i+1} \leq c_1 + c_i.$$

Proof. This is a consequence of the Subadditivity Theorem for multiplier ideals, for which I refer to [La, Theorem 9.5.20]. It says that for any $\mathbb{Q}$-divisors D_1 and D_2 on X we have

$$\mathcal{J}(D_1 + D_2) \subseteq \mathcal{J}(D_2) \cdot \mathcal{J}(D_2).$$

In our case, we then have

$$\mathcal{J}\big((c_i + c_1)D\big)_x \subseteq \mathcal{J}(c_i D)_x \cdot \mathcal{J}(c_1 D)_x \subsetneq \mathcal{J}(c_i D)_x,$$

and therefore by definition $c_{i+1} \leq c_1 + c_i$. □

Analytic interpretation. An excellent source for this material is [De, §5.B].

Let $D = \sum_{i=1}^k a_i D_i$ be an effective divisor on X, with D_i prime divisors and $a_i \in \mathbb{Q}$. We fix an open set U on which we have $D_i = (f_i = 0)$ for a regular function f_i. We next view X as a complex manifold.

Definition 6.12. The *analytic multiplier ideal* of D (on U) is

$$\mathcal{J}_{\mathrm{an}}(D) := \{g \in \mathscr{O}_X(U) \mid \frac{|g|^2}{\prod_{i=1}^k |f_i|^{2a_i}} \text{ is locally integrable}^{8}\}.$$

In particular, if D is an integral divisor given by $f \in \mathscr{O}_X(U)$, then for every $c \in \mathbb{Q}_{>0}$ we have

$$\mathcal{J}_{\mathrm{an}}(cD) := \{g \in \mathscr{O}_X(U) \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally integrable}\}.$$

It is not hard to see that the local definitions glue to give a global sheaf of ideals on X^{an}.

Remark 6.13 (Plurisubharmonic functions)**.** The multiplier ideal of D defined above is a special example of a more general analytic notion. To this end, note that

$$\varphi_D := \sum_{i=1}^k a_i \log|f_i|$$

is a *plurisubharmonic* function[9] (since $\log|z|$ is so). Now for any plurisubharmonic function φ on X, one defines its multiplier ideal $\mathcal{J}(\varphi)$ via

$$\mathcal{J}(\varphi)(U) := \{g \in \mathscr{O}_X(U) \mid |g|^2 \cdot e^{-2\varphi} \text{ is locally integrable}\}.$$

It can be shown that this is a coherent sheaf of ideals; see [De, Proposition 5.7].

It turns out that analytic multiplier ideals satisfy the same birational transformation formula as the algebraic ones:

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[8] Recall that this means that its integral with respect to the Lebesgue measure is finite on every compact subset of U.

[9] This means a function $\varphi\colon X \to [-\infty, \infty)$ that is upper semicontinuous, locally integrable, and satisfies the mean-value inequality

$$(\varphi \circ \gamma)(0) \leq \frac{1}{\pi} \int_\Delta (\varphi \circ \gamma) d\mu$$

for any holomorphic map $\gamma\colon \Delta \to X$ from the open unit disk $\Delta \subset \mathbf{C}$. See [De, §1B] for basic properties.

Lemma 6.14. *Let $\mu\colon Y \to X$ be a proper bimeromorphic holomorphic map, and φ any plurisubharmonic function on X. Then*

$$\mathcal{J}(\varphi) \otimes_{\mathscr{O}_X} \omega_X \simeq \mu_* \mathscr{O}_Y\big(\mathcal{J}(\varphi \circ \mu) \otimes_{\mathscr{O}_Y} \omega_Y\big).$$

Proof. Using the definition of $\mathcal{J}(\varphi)$, we can interprete the sheaf $\mathcal{J}(\varphi) \otimes_{\mathscr{O}_X} \omega_X$ as the subsheaf of ω_X consisting, for each open set U, of n-forms ω such that

$$\omega \wedge \overline{\omega} \cdot e^{-2\varphi} \quad \text{is locally integrable on } U.$$

It is in fact enough to restrict to forms defined on the open set $V \subseteq U$ over which μ is a biholomorphism, as they automatically extend over the complement of V by virtue of being locally L^2. Thus the change of variables formula gives

$$\int_U \omega \wedge \overline{\omega} \cdot e^{-2\varphi} = \int_{\mu^{-1}(U)} \mu^*\omega \wedge \mu^*\overline{\omega} \cdot e^{-2(\varphi\circ\mu)},$$

and therefore

$$\omega \in \Gamma\big(U, \mathcal{J}(\varphi) \otimes_{\mathscr{O}_X} \omega_X\big) \iff \mu^*\omega \in \Gamma\big(\mu^{-1}(U), \mathcal{J}(\varphi \circ \mu) \otimes_{\mathscr{O}_Y} \omega_Y\big).$$

□

Proposition 6.15. *For every effective $\mathbb{Q}$-divisor D on X we have*

$$\mathcal{J}_{\text{an}}(D) = \mathcal{J}(D)^{\text{an}}.$$

Proof. Since by Exercise .9 and Lemma 6.14 the two types of multiplier ideals satisfy the same birational transformation formula, by passing to a log resolution it is enough to check the statement when D is assumed to have SNC support.

Let's assume then that $D = \sum_{i=1}^k a_i D_i$, where $a_i \in \mathbb{Q}_{>0}$ and $\sum D_i$ is SNC. We need to show that

$$\mathcal{J}_{\text{an}}(D) = \mathscr{O}_X(-[D])^{\text{an}}.$$

Choosing local coordinates x_i such that $D_i = (x_i = 0)$, this can be reinterpreted as saying that for a holomorphic function g on such a neighborhood we have

$$\frac{|g|^2}{\prod_{i=1}^k |x_i|^{2a_i}} \text{ is locally integrable } \iff x_1^{[a_1]} \cdot \ldots \cdot x_n^{[a_n]} \mid g.$$

A standard reduction allows us to assume that g is a monomial in the x_i, in which case by Fubini's theorem we can separate the variables and reduce the statement to the fact that for a single variable z, on a a ball B around the origin, say of radius ϵ, we have

$$\int_B \frac{1}{|z|^{2c}} < \infty \iff c < 1.$$

In polar coordinates $z = re^{i\theta}$ the integral on the left is $2\pi \int_0^\epsilon r^{1-2c} dr$, and this is easily checked. □

Corollary 6.16. *If D is as in Definition 6.12, then*

$$\operatorname{lct}(D) = \sup\{c > 0 \mid \frac{1}{\prod_{i=1}^k |f_i|^{2ca_i}} \text{ is locally integrable around } x\}.$$

Proof. We have seen that

$$\mathrm{lct}_x(D) = \sup\{c \in \mathbb{Q} \mid \mathcal{J}(cD)_x = \mathscr{O}_{X,x}\}.$$

Thanks to Proposition 6.15 we then have

$$\mathrm{lct}_x(D) = \sup\{c \in \mathbb{Q} \mid 1 \in \mathcal{J}_{\mathrm{an}}(cD)_x\},$$

which is equivalent by definition to the assertion in the Corollary. $\square$

Remark 6.17. Note that Corollary 6.16 (and the calculation in Proposition 6.15) shows that the invariant defined in terms of discrepancies in (6.1) is indeed independent of the choice of log resolution.

Example 6.18. Let $D = (x^a + y^b = 0) \subset \mathbf{C}^2$. We check using the interpretation in Corollary 6.16 that

$$\mathrm{lct}(D) = \frac{1}{a} + \frac{1}{b}.$$

According to its statement, we have

$$\mathrm{lct}(D) = \sup\{c > 0 \mid \int_B |x^a + y^b|^{-2c} < \infty\}$$

on any ball B in the neighborhood of the origin. Consider now the unit ball $B = B(0;1) \subset \mathbf{C}^2$, and the transformation

$$T\colon B \to B, \quad (x,y) \mapsto \left(\frac{x}{2^a}, \frac{y}{2^b}\right).$$

Denote $T^{(r)} = T \circ \ldots \circ T$, the r-fold composition, and set

$$\Omega_r := T^{(r-1)}(B) \smallsetminus T^{(r)}(B).$$

We clearly have

$$\bigcup_{r \geq 1} \Omega_r = B \smallsetminus \{0\}.$$

Denoting $I_r = \int_{\Omega_r} |x^a + y^b|^{-2c}$, and using the change of variables $x \mapsto 2^{-b}x$, $y \mapsto 2^{-a}y$ (hence $dxdy \mapsto 2^{-(a+b)}dxdy$), we have

$$I_{r+1} = \int_{\Omega_r} 2^{2abc}|x^a + y^b|^{-2c}2^{-(a+b)} = 2^{2(abc-(a+b))} \cdot I_r.$$

Hence

$$\int_{B\smallsetminus\{0\}} |x^a + y^b|^{-2c} = \sum_{r\geq 1} I_r = I_1 \cdot (1 + u + u^2 + \cdots),$$

where $u = 2^{2(abc-(a+b))}$, and so this is finite if and only if $abc < a + b$.

7. Log canonical threshold and jumping numbers as roots. We go back to the discussion of the archimedean zeta function. Let f be a regular function on X. It turns out that the greatest pole of the distribution $|f|^{2s}$ is the well-known invariant of the singularities of f discussed in the previous section.

Proposition 7.1. *The greatest pole of* $|f|^{2s}$ *is* $-\mathrm{lct}(f)$.

Proof. We use the analytic interpretation of the log-canonical threshold discussed in Corollary 6.16, namely

$$c_0 := \mathrm{lct}(f) = \sup\{c > 0 \mid |f|^{-2c} \text{ is locally integrable}\}.$$

By the same argument as in Proposition 5.1, we have that $Z_\varphi(s) = \int_{\mathbf{C}^n} |f|^{2s}\varphi$ is analytic at $s = -c$ for any $c < c_0$ and any $\varphi \in C_c^\infty(\mathbf{C}^n)$. Therefore we need to show that there exists a φ such that the continuation of $Z_\varphi(s)$ does have a pole at $s = -c_0$.

Let B be a ball in the neighborhood of a point x in the zero locus of f such that

$$\int_B |f|^{-2c_0} = \infty,$$

and let φ be a bump function in a neighborhood of x such that φ is identically 1 on B. If we assumed that the continuation of $Z_\varphi(s)$ were analytic at $s = -c_0$, and so the limit of $|Z_\varphi(-c)|$ would be finite as $c \to c_0$ from the left. On the other hand, by Fatou's Lemma we have that

$$\liminf_{c\to c_0} \int_{\mathbf{C}^n} |f|^{-2c}\varphi \geq \int_{\mathbf{C}^n} |f|^{-2c_0}\varphi \geq \int_B |f|^{-2c_0} = \infty,$$

which is a contradiction. □

According to Theorem 5.2 and Proposition 7.1, the log canonical threshold of f should then be an integral shift of a root of $b_f(s)$. The best possible scenario does in fact happen, according to the following theorem appearing in works of Yano, Lichtin, and Kollár. We follow Kollár's approach in [Ko], similar to the methods discussed above.

Theorem 7.2. *Let* $f \in \mathbf{C}[X_1, \ldots, X_n]$ *be a nontrivial polynomial (or germ of analytic function), and let* α_f *be the negative of the greatest root of the Bernstein-Sato polynomial* $b_f(s)$. *Then*

$$\alpha_f = \mathrm{lct}(f).$$

Proof. Step 1. In this step we only show that $-\mathrm{lct}(f)$ is a root of $b(s) = b_f(s)$. We use the analytic interpretation of the log canonical threshold, namely

$$c_0 := \mathrm{lct}(f) = \sup\{c > 0 \mid \frac{1}{|f|^{2c}} \text{ is locally } L^1\}.$$

Therefore for some point x in the zero locus of f and some small ball B around x, the function $\frac{1}{|f|^{2c}}$ with $c = c_0 - \epsilon$ for $0 < \epsilon \ll 1$ is integrable on B, but $\frac{1}{|f|^{2c_0}}$ is not integrable on some compact ball B' strictly contained in B.

Let us now take $s = -c$ in Lemma 5.3, so that

$$b(-c)^2|f|^{-2c} = \big(P(-c)\bar{P}(-c)\big)|f|^{2(-c+1)}.$$

We can think of this as being an equality of distributions, since both sides are integrable on B. Therefore, for any smooth positive test function φ supported on B, we have

$$\int_B b(-c)^2|f|^{-2c}\varphi = \int_B \big(P(-c)\bar{P}(-c)\big)|f|^{2(-c+1)}\varphi. \tag{7.1}$$

We can in fact take φ to be a bump function with support in B, identically equal to 1 on B', in which case we obtain that the left-hand side of (7.1) is at least

$$b(-c)^2\int_{B'}|f|^{-2c}.$$

Using integration by parts in a way similar to Exercise .6, we see that the right-hand side of (7.1) is equal to

$$\int_B |f|^{2(-c+1)}\big(P(-c)\bar{P}(-c)\varphi\big),$$

and therefore if ϵ is in a fixed interval $(0,\delta]$, then it is bounded above by some $M>0$ depending only on φ (as $c-1$ belongs to a closed interval of values for which the integral is finite and depends continuously on c). We deduce that

$$b(-c)^2\int_{B'}|f|^{-2c}\le M<\infty$$

for every such c. On the other hand, $\frac{1}{|f|^{2c_0}}$ is not integrable on B', hence by Fatou's Lemma we have

$$\int_{B'}|f|^{-2c}\to\infty \quad \text{as} \quad c\to c_0.$$

The only way this can happen is if $b(-c_0)=0$.

Step 2. To show in addition that $-c_0$ is the greatest root of $b(s)$, we need to use in addition Lichtin's refinement of Kashiwara's result on the rationality of the roots of $b(s)$, Theorem 4.4. If $f\colon Y\to \mathbf{C}^n$ is log resolution of $D=(f=0)$ with the property that it is an isomorphism away from D, then writing

$$f^*D=\widetilde{D}+\sum_{i=1}^m a_iE_i \quad \text{and} \quad K_{Y/X}=\sum_{i=1}^m b_iE_i,$$

with E_i the exceptional divisors, Lichtin's theorem tells us that all the roots of $b(s)$ are of the form

$$-\frac{b_i+1+\ell}{a_i} \quad \text{with} \quad 0\le i\le m,$$

where by convention $a_0=1$ and $b_0=0$, and $\ell\ge 0$ is an integer. Since on the other hand we know by the proof of Lemma 6.2 that

$$c_0=\min_{1\le i\le m}\frac{b_i+1}{a_i},$$

and $c_0\le 1$, it is then clear that no root can exceed $-c_0$. □

In [ELSV], Theorem 7.2 was extended along similar lines to the following statement:

Theorem 7.3. *With the same hypothesis as in Theorem 7.2, let ξ be a jumping coefficient of f in the interval $(0,1]$. Then $-\xi$ is a root of $b_f(s)$.*

Proof. Let ξ' be the previous jumping coefficient (taken by convention to be equal to 0 if ξ is the first jumping coefficient, i.e. the log canonical threshold). Recall that this means that for every $c \in [\xi', \xi)$ we have

$$\mathscr{I}(\xi' \cdot f) = \mathscr{I}(c \cdot f),$$

but there exists $x \in Z(f)$ such that

$$\mathscr{I}(\xi \cdot f)_x \subsetneq \mathscr{I}(c \cdot f)_x.$$

Recall now that the analytic interpretation of multiplier ideals gives

$$\mathscr{I}(c \cdot f) = \{g \in \mathscr{O}_{\mathbf{C}^n} \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally } L^1\}.$$

Therefore for any $c \in [\xi', \xi)$, there exists a function g and a small ball B around x such that

$$\int_B \frac{|g|^2}{|f|^{2c}} < \infty \quad \text{but} \quad \int_B \frac{|g|^2}{|f|^{2\xi}} = \infty.$$

The argument then goes through exactly as in the proof of Theorem 7.2, after multiplying the two integrands in (7.1) by $|g|^2$. □

Remark 7.4. The converse of the Theorem above is not true. For instance, Saito [Sa1, Example 3.5] shows that if $f(x,y) = x^5 + y^4 + x^3y^2$ then $b_f(-s)$ has roots in $(0,1]$ that are not jumping numbers for f; cf. also [ELSV, Example 2.5]. Despite this, we nevertheless have the following behavior, similar to that of jumping numbers:

Corollary 7.5. *Let*

$$-1 = \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 < 0$$

be the distinct roots of $b_{f,x}(s)$ in the interval $[-1,0)$, for some $x \in X$. Then

$$\alpha_i + \alpha_1 \leq \alpha_{i+1}, \quad \text{for all } \ 1 \leq i < m.$$

Proof. Set $\beta_i = -\alpha_i$, so that the inequality to be shown is

$$\beta_{i+1} \leq \beta_i + \beta_1.$$

Denote by c_j and c_{j+1} the two consecutive jumping numbers of f at x such that $c_j \leq \beta_i < c_{j+1}$. Note also that the first nontrivial jumping number is $c_1 = \mathrm{lct}_x(f) = \beta_1$, according to Theorem 7.2. Now according to Lemma 6.11, we have

$$c_{j+1} \leq c_j + c_1 \leq \beta_i + \beta_1.$$

On the other hand, clearly $c_j < 1$, hence $c_{j+1} \leq 1$. Therefore by Theorem 7.3 there exists some $k > i$ such that $c_{j+1} = \beta_k \geq \beta_{i+1}$, and the result follows. □

Note. A detailed account of the developments leading to Theorems 7.2 and 7.3 can be found in [ELSV]. These theorems are also consequences of the stronger Budur-Saito theorem [BS], comparing multiplier ideals with the V-filtration, explained in the next section. Another generalization, giving a criterion for jumping numbers of higher Hodge ideals to be roots of the Bernstein-Sato polynomial, is proved in [MP, Proposition 6.14].

8. Multiplier ideals vs. V-filtration. We fix a non-invertible function f on X, and denote $D = (f = 0)$. We have seen in the previous sections that certain aspects of the behavior of the multiplier ideals $\mathcal{J}(\alpha D)$ and the filtration $V^\alpha \mathscr{O}_X$ induced on $\mathscr{O}_X$ by the V-filtration $V^\bullet \iota_+ \mathscr{O}_X$ are very similar. For instance the threshold where they both become trivial is $\mathrm{lct}(f)$, and more generally they both change at the jumping coefficients of the pair (X, D) in the interval $[0, 1]$ (as by Theorem 7.3 these are roots of $b_f(s)$), though with different semicontinuity behavior.

Budur and Saito [BS, Theorem 0.1] have noted that this is not an accident, enhancing these numerical properties to the following statement, proved using the theory of mixed Hodge modules:

Theorem 8.1. *For every $\alpha \in \mathbb{Q}_{>0}$, we have*

$$V^\alpha \mathscr{O}_X = \mathcal{J}\big((\alpha - \epsilon)D\big) \quad \text{for} \;\; 0 < \epsilon \ll 1.$$

In this section, following [DM] we give a different proof of this result, which does not make use of the Hodge filtration, but only of more elementary statements discussed in these notes.

Recall that in analytic terms we have, for every $c > 0$, that

$$\mathcal{J}(cD)^{\mathrm{an}} = \{g \in \mathscr{O}_X \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally integrable}\}.$$

For a function $g \in \mathscr{O}_X(X)$, by analogy with the usual definition we denote

$$\mathrm{lct}_g(f) := \sup\{c > 0 \mid g \in \mathcal{J}(cD)\}.$$

Fixing a log resolution $\mu \colon Y \to X$ which is an isomorphism away from the support of D, recall that we write

$$\mu^* D = \sum_{i=1}^m a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i E_i.$$

For each i, we also denote $k_i = \mathrm{ord}_{E_i}(g)$. By analogy with (6.1), we have:

Exercise .11. The threshold $\mathrm{lct}_g(f)$ is computed on Y by the formula

$$\mathrm{lct}_g(f) = \min_{1 \le i \le m} \frac{b_i + 1 + k_i}{a_i}.$$

On the other hand, as in the notes on the V-filtration we denote

$$u = g f^s \in \iota_+ \mathscr{O}_X,$$

and according to Sabbah's description of the V-filtration we have

$$g \in V^\alpha \mathscr{O}_X \iff u \in V^\alpha \iota_+ \mathscr{O}_X \iff c \le -\alpha \;\; \text{for all } c \text{ such that} \;\; b_u(c) = 0.$$

Therefore Theorem 8.1 is equivalent to the following analogue of Theorem 7.2:

Theorem 8.2. *The greatest root of $b_u(s)$ is $-\mathrm{lct}_g(f)$.*

Proof. Just as with Theorem 7.2, first we show that $-\mathrm{lct}_g(f)$ is a root of $b_u(s)$. Let us first recall that by definition we have

$$b_u(s)\cdot gf^s = P(s)t(gf^s), \quad P(s)\in \mathscr{D}_X[s].$$

Now the action of t on this element is $t(gf^s)=gf^{s+1}$ (recall that we are identifying f^s with δ, and see Exercise B.3 in the notes on the V-filtration), where as always $f^{s+1}:=f\cdot f^s$. Hence this can be rewritten as

$$b_u(s)\cdot gf^s = P(s)gf^{s+1}.$$

We now proceed precisely as in the proof of Theorem 7.2; we repeat the argument for convenience. We denote $c_0=\mathrm{lct}_g(f)$. Therefore for some point x in the zero locus of f and some small ball B around x, the function $\frac{|g|^2}{|f|^{2c}}$ with $c=c_0-\epsilon$ for $0<\epsilon\ll 1$ is integrable on B, but $\frac{|g|^2}{|f|^{2c_0}}$ is not integrable on some compact ball B' strictly contained in B.

Arguing as in Lemma 5.3, and taking $s=-c$, we obtain the identity

$$b_u(-c)^2|g|^2|f|^{-2c} = \big(P(-c)\bar{P}(-c)\big)|g|^2|f|^{2(-c+1)}.$$

Both sides are integrable on B, and so for any smooth positive test function φ supported on B we have

$$\int_B b_u(-c)^2|g|^2|f|^{-2c}\varphi = \int_B \big(P(-c)\bar{P}(-c)\big)|g|^2|f|^{2(-c+1)}\varphi. \tag{8.1}$$

We can in fact take φ to be a bump function with support in B, identically equal to 1 on B', in which case we obtain that the left-hand side of (8.1) is at least

$$b_u(-c)^2\int_{B'}|g|^2|f|^{-2c}.$$

Using integration by parts in a way similar to Exercise .6, we see that the right-hand side of (8.1) is equal to

$$\int_B |g|^2|f|^{2(-c+1)}\big(P(-c)\bar{P}(-c)\varphi\big),$$

and therefore if ϵ is in a fixed interval $(0,\delta]$, then it is bounded above by some $M>0$ depending only on φ (as $c-1$ belongs to a closed interval of values for which the integral is finite and depends continuously on c). We deduce that

$$b_u(-c)^2\int_{B'}|g|^2|f|^{-2c}\le M<\infty$$

for every such c. On the other hand, $\frac{|g|^2}{|f|^{2c_0}}$ is not integrable on B', hence by Fatou's Lemma we have

$$\int_{B'}|g|^2|f|^{-2c}\to\infty \quad \text{as} \quad c\to c_0.$$

The only way this can happen is if $b_u(-c_0)=0$.

Having established that $-\mathrm{lct}_g(f)$ is a root of $b_u(s)$, the full statement now follows from Theorem 4.5(2), which thanks to Exercise .11 can be rephrased as saying that the greatest root of $b_u(s)$ is at most equal to $-\mathrm{lct}_g(f)$. $\square$

It makes sense to wonder whether $-\mathrm{lct}_g(f)$ is also related to the poles of an archimedean zeta function as in §5, and this is indeed the case. Concretely, this time we can consider the distribution $|g|^2|f|^{2s}$, which on any $\varphi \in C_c^\infty(\mathbf{C}^n)$ is defined by

$$(|g|^2|f|^{2s}, \varphi) := \int_{\mathbf{C}^n} |g|^2|f(x)|^{2s}\varphi(x) = Z_\varphi^g(s).$$

Arguments completely analogous to those in Proposition 5.1, Theorem 5.2, and Proposition 7.1 show the following:

Theorem 8.3. *With the notation above, for every smooth complex-valued function with compact support $\varphi \in C_c^\infty(\mathbf{C}^n)$, Z_φ^g admits an analytic continuation to $\mathbf{C}$ as a meromorphic function whose poles are of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_u(s)$ and $m \in \mathbf{N}$. Moreover, the greatest pole of the distribution $|g|^2|f|^{2s}$ (meaning the maximum over the poles of Z_φ^g for all φ) is equal to $-\mathrm{lct}_g(f)$, hence to the greatest root of $b_u(s)$.*

9. Minimal exponent. Let X be a smooth variety of dimension n, and $f \in \mathscr{O}_X(X)$ a non-invertible function. We have seen that -1 is always a root of $b_f(s)$. We can therefore consider the polynomial

$$\widetilde{b}_f(s) = \frac{b_f(s)}{s+1},$$

called the *reduced Bernstein-Sato polynomial* of f.

Definition 9.1. The negative $\widetilde{\alpha}_f$ of the greatest root of the reduced of $\widetilde{b}_f(s)$ is called the *minimal exponent* of f.[10]

According to Theorem 7.2, the log canonical threshold $\mathrm{lct}(f)$ is equal to α_f, the greatest root of $b_f(s)$. Therefore if $\widetilde{\alpha}_f \le 1$, then it coincides with lct(f); more precisely

$$\alpha_f = \min\{1, \widetilde{\alpha}_f\}.$$

Thus $\widetilde{\alpha}_f$ is a refinement of the log canonical threshold, and it provides a new interesting invariant precisely when the pair (X, D) is log canonical.

Remark 9.2 (Local version)**.** Recall that we also have a local version $b_{f,x}$ of he Bernstein-Sato polynomial, around a point $x \in X$; see Definition 1.5. If f is not invertible around x, then $(s+1) \mid b_{f,x}(s)$, and we define $\widetilde{\alpha}_{f,x}$ to be the negative of the greatest root of $\widetilde{b}_{f,x}(s) = b_{f,x}(s)/(s+1)$.

Remark 9.3 (Global version)**.** We can also define a global version of the minimal exponent. For each non-trivial effective divisor D on X, there is an associated Bernstein-Sato polynomial $b_D(s)$ such that $(s+1) \mid b_D(s)$; see Remark 1.8. We have

$$b_D(s) = \underset{x \in D}{\mathrm{lcm}}\, b_{D,x}(s),$$

[10] This is also called the *microlocal log canonical threshold* of f in [Sa5].

where$b_{D,x}(s) := b_{f,x}$ for any locally defining equation f for D in a neighborhood of x. As above, the minimal exponent $\widetilde{\alpha}_D$ is the negative of the greatest root of $\widetilde{b}_D(s) = b_D(s)/(s+1)$. The description above implies that

$$\widetilde{b}_D(s) = \min_{x \in D} b_{D,x}(s).$$

Example 9.4 (Quasi-homogeneous isolated singularities)**.** Let $f \in \mathbf{C}[X_1, \ldots, X_n]$ be a quasi-homogeneous polynomial, with weights $w_1, \ldots, w_n$ (see §2), having an isolated singularity. Since $\rho(1) = 0$, Theorem 2.4 implies that

$$\widetilde{\alpha}_f = |w| := w_1 + \cdots + w_n.$$

In particular, for a diagonal hypesurface $f = X_1^{a_1} + \cdots + X_n^{a_n}$ we have the celebrated

$$\widetilde{\alpha}_f = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Just like α_f, the minimal exponent it is known to be related to standard types of singularities due the following results of Saito:

Theorem 9.5. *Assume that D is reduced around a point x. Then*

(1) [Sa1, Theorem 0.4] *D has rational singularities at x if and only if $\widetilde{\alpha}_{D,x} > 1$.*
(2) [Sa4, Theorem 0.5] *D has Du Bois singularities at x if and only if $\widetilde{\alpha}_{D,x} \geq 1$.*

Note that this implies that rational hypersurface singularities are du Bois; this is in fact known to be true for arbitrary varieties, by work of Kovács [Kov] and Saito [Sa3]. For a first look at du Bois singularities, see for instance [KS].

The proof of Theorem 9.5 requires the Hodge filtration, and therefore will be given later. For (2) we can alternatively not worry right now about what du Bois means; clearly $\widetilde{\alpha}_D \geq 1$ is equivalent to the pair (X, D) having log canonical singularities, while on the other hand using birational geometry arguments it is shown in [KS, Corollary 6.6] that

Proposition 9.6. *The pair (X, D) has log canonical singularities if and only if the divisor D has du Bois singularities.*

Let now D be a reduced effective divisor. One of the main questions about the minimal exponent of D is whether we can express it explicitly in terms of discrepancies on a log resolution, like in the case of the log canonical threshold as in Remark 6.3. We use again the notation introduced in §4, and denote

$$\gamma := \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}.$$

We have noted in Remark 6.3 that $\alpha_D = \min\{1, \gamma\}$, while on the other hand by definition $\alpha_D = \min\{1, \widetilde{\alpha}_D\}$.

It is natural then to ask whether $\widetilde{\alpha}_D = \gamma$, and Lichtin [Li, Remark 2, p.303] did indeed pose this question. This would provide a very simple description, but as noted by Kollár [Ko, Remark 10.8] in general the answer is negative, since γ usually depends on the choice of log resolution. Nevertheless, at least if we assume that the proper transform $\widetilde{D}$ is smooth,[11] one inequality does hold:

Theorem 9.7 ([MP, Corollary D]). *We always have $\widetilde{\alpha}_D \geq \gamma$.*

It is worth noting that the inequality follows easily from Lichtin's result, Theorem 4.4, if $\widetilde{\alpha}_D$ is not an integer; however, it is not clear how to use it otherwise. The original proof of the theorem in [MP] relies on the theory of Hodge ideals. However a simpler proof can be given using Theorem 4.5 above, due to Dirks-Mustaţă, and we present this next.

Proof of Theorem 9.7. Let's assume for simplicity that D is defined globally by a function f. We write $\gamma = p + \alpha$, where p is a non-negative integer, and $\alpha \in (0, 1]$. Recall now from Example 0.33 in the notes on the V-filtration (see also [MP, Proposition 6.12]) that

$$b_{\partial_t^p f^s}(s) \mid (s+1)\widetilde{b}_f(s-p) \quad \text{and} \quad \widetilde{b}_f(s-p) \mid b_{\partial_t^p f^s}(s).$$

We deduce that all roots of $\widetilde{b}_f(s)$ are $\leq -\gamma$ (which is what we want) if and only if all roots of $b_{\partial_t^p f^s}(s)$ are $\leq -\alpha$.

On the other hand, by Theorem 4.5(3) we know that for every root β of $b_{\partial_t^p f^s}(s)$ we either have that β is a negative integer, in which case we clearly have $\beta \leq -1 \leq -\alpha$, or we have

$$\beta = p - \frac{b_i + 1 + \ell}{a_i}$$

for some exceptional divisor E_i and some non-negative integer ℓ. Now by definition

$$\frac{b_i + 1 + \ell}{a_i} \geq \gamma = p + \alpha,$$

and therefore $\beta \leq -\alpha$. $\square$

Mustaţă and I expect a substantially stronger statement to hold; we have formulated the following:[12]

Conjecture 9.8. *On every log resolution of (X, D), there exists an exceptional divisor E_i for which $\widetilde{\alpha}_D = \frac{b_i+1}{a_i}$.*

The log canonical threshold is well known to satisfy a few fundamental semicontinuity and restriction properties, as well as numerical bounds in terms of the multiplicity; see for instance [Ko, §8]. It turns out that the same can be said about the minimal exponent; however the proofs are more complicated, and go beyond what we have studied up to this point (they depend on the theory of Hodge ideals). I am nevertheless including some statements below for completeness.

Theorem 9.9 ([MP, Theorem E]). *Let X be a smooth n-dimensional complex variety, and D an effective divisor on X.*

[11] This can always be achieved by performing a few more blow-ups, if needed.

[12] This conjecture is also heuristically motivated by what is called Igusa's Strong Monodromy Conjecture for the local zeta function associated to polynomials $f \in \mathbb{Z}[X_1, \ldots, X_n]$.

(1) If Y is a smooth subvariety of X such that $Y \not\subseteq D$, then for every $x \in D \cap Y$, we have

$$\widetilde{\alpha}_{D|_Y,x} \leq \widetilde{\alpha}_{D,x}.$$

(2) Consider a smooth morphism $\pi\colon X \to T$, together with a section $s\colon T \to X$ such that $s(T) \subseteq D$. If D does not contain any fiber of π, so that for every $t \in T$ the divisor $D_t = D|_{\pi^{-1}(t)}$ is defined, then the function

$$T \ni t \to \widetilde{\alpha}_{D_t,s(t)}$$

is lower semicontinuous.

(3) For every $x \in X$, if $m = \mathrm{mult}_x(D) \geq 2$, then

$$\frac{n-r-1}{m} \leq \widetilde{\alpha}_{D,x} \leq \frac{n}{m},$$

where r is the dimension of the singular locus of the projectivized tangent cone $\mathbf{P}(C_xD)$ of D at x (with the convention that $r = -1$ if $\mathbf{P}(C_xD)$ is smooth).

Example 9.10 (Ordinary singularities)**.** In the case of a Fermat hypersurface

$$f = X_1^m + \cdots + X_n^m,$$

Example 9.4 gives $\widetilde{\alpha}_f = n/m$. This is however also an example of an ordinary singular point of multiplicity m (see Example 4.3(3)), and for these we always have

$$\widetilde{\alpha}_f = \frac{n}{m}.$$

This is the highest one can go: it turns out that for any f and any singular point x of multiplicity m we have $\widetilde{\alpha}_{f,x} \leq \frac{n}{m}$; see Theorem 9.9(3). The equality above follows from work of Saito. Alternatively, one can again use Theorem 9.9(3), noting that ordinary singularities are precisely the case $r = -1$. Note that the inequality $\widetilde{\alpha}_f \geq \frac{n}{m}$ follows also from Theorem 4.4, since for an ordinary singularity $\gamma = n/m$ (just take the resolution given by blowing up the singular point).

Remark 9.11. Another general fact worth mentioning is the following result due to Saito [Sa2, Theorem 0.4]: the negative of every root of $\widetilde{b}_f$ is in the interval $[\widetilde{\alpha}_f, n - \widetilde{\alpha}_f]$.[13]

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