Chapter 1

Manifolds

1.1 Preliminaries

1.1.1 *Space and Coordinatization*

Mathematics is a natural science with a special modus operandi. It replaces concrete natural objects with mental abstractions which serve as intermediaries. One studies the properties of these abstractions in the hope they reflect facts of life. So far, this approach proved to be very productive.

The most visible natural object is the Space, the place where all things happen. The first and most important mathematical abstraction is the notion of number. Loosely speaking, the aim of this book is to illustrate how these two concepts, Space and Number, fit together.

It is safe to say that geometry as a rigorous science is a creation of ancient Greeks. Euclid proposed a method of research that was later adopted by the entire mathematics. We refer of course to the axiomatic method. He viewed the Space as a collection of points, and he distinguished some basic objects in the space such as lines, planes etc. He then postulated certain (natural) relations between them. All the other properties were derived from these simple axioms.

Euclid's work is a masterpiece of mathematics, and it has produced many interesting results, but it has its own limitations. For example, the most complicated shapes one could reasonably study using this method are the conics and/or quadrics, and the Greeks certainly did this. A major breakthrough in geometry was the discovery of *coordinates* by René Descartes in the 17th century. Numbers were put to work in the study of Space.

Descartes' idea of producing what is now commonly referred to as Cartesian coordinates is familiar to any undergraduate. These coordinates are obtained using a very special method (in this case using three concurrent, pairwise perpendicular lines, each one endowed with an orientation and a unit length standard. What is important here is that they produced a one-to-one mapping

$$\text{Euclidian Space} \to \mathbb{R}^3, \quad P \longmapsto (x(P), y(P), z(P)).$$

We call such a process *coordinatization.* The corresponding map is called (in this case) *Cartesian system of coordinates.* A line or a plane becomes via coordinatization an algebraic object, more precisely, an equation.

In general, any coordinatization replaces geometry by algebra and we get a two-way correspondence

$$\text{Study of Space} \longleftrightarrow \text{Study of Equations.}$$

The shift from geometry to numbers is beneficial to geometry as long as one has efficient tools do deal with numbers and equations. Fortunately, about the same time with the introduction of coordinates, Isaac Newton created the differential and integral calculus and opened new horizons in the study of equations.

The Cartesian system of coordinates is by no means the unique, or the most useful coordinatization. Concrete problems dictate other choices. For example, the polar coordinates represent another coordinatization of (a piece of the plane) (see Figure 1.1).

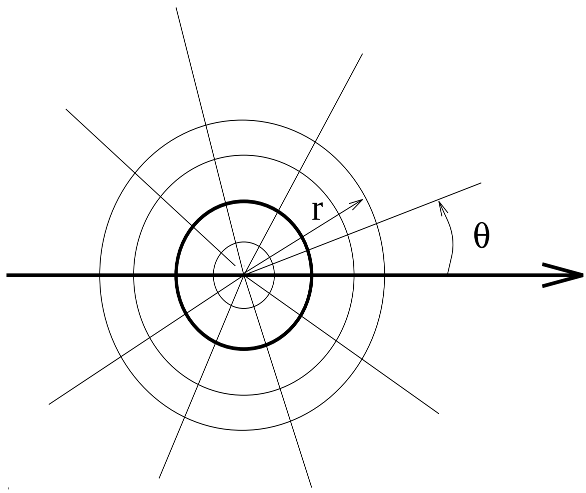

Fig. 1.1 *Polar coordinates*

$$P \mapsto (r(P), \theta(P)) \in (0, \infty) \times (-\pi, \pi).$$

This choice is related to the Cartesian choice by the well known formulae

$$x = r\cos\theta \quad y = r\sin\theta. \tag{1.1.1}$$

A remarkable feature of (1.1.1) is that $x(P)$ and $y(P)$ depend smoothly upon $r(P)$ and $\theta(P)$.

As science progressed, so did the notion of Space. One can think of Space as a *configuration set*, i.e., the collection of all possible states of a certain phenomenon. For example, we know from the principles of Newtonian mechanics that the motion of a particle in the ambient space can be completely described if we know the position and the velocity of the particle at a given moment. The space associated with this problem consists of all pairs *(position, velocity)* a particle can possibly have. We can coordinatize this space using six functions: three of them will describe the position, and the other three of them will describe the velocity. We say the configuration space is 6-dimensional. We cannot visualize this space, but it helps to think of it as an Euclidian space, only "roomier".

There are many ways to coordinatize the configuration space of a motion of a particle, and for each choice of coordinates we get a different description of the motion. Clearly, all these descriptions must "agree" in some sense, since they all reflect the same phenomenon. In other words, these descriptions should be *independent of coordinates*. Differential geometry studies the objects which are independent of coordinates.

The coordinatization process had been used by people centuries before mathematicians accepted it as a method. For example, sailors used it to travel from one point to another on Earth. Each point has a latitude and a longitude that completely determines its position on Earth. This coordinatization is not a global one. There exist four domains delimited by the Equator and the Greenwich meridian, and each of them is then naturally coordinatized. Note that the points on the Equator or the Greenwich meridian admit two different coordinatizations which are smoothly related.

The manifolds are precisely those spaces which can be piecewise coordinatized, with smooth correspondence on overlaps, and the intention of this book is to introduce the reader to the problems and the methods which arise in the study of manifolds. The next section is a technical interlude. We will review the implicit function theorem which will be one of the basic tools for detecting manifolds.

1.1.2 *The implicit function theorem*

We gather here, with only sketchy proofs, a collection of classical analytical facts. For more details one can consult [26].

Let X and Y be two Banach spaces and denote by $L(X,Y)$ the space of bounded linear operators $X \to Y$. For example, if $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, then $L(X,Y)$ can be identified with the space of $m \times n$ matrices with real entries.

Definition 1.1.1. Let $F : U \subset X \to Y$ be a continuous function (U is an open subset of X). The map F is said to be (Fréchet) differentiable at $u \in U$ if there exists $T \in L(X,Y)$ such that

$$\|F(u_0 + h) - F(u_0) - Th\|_Y = o(\|h\|_X) \ \text{ as } h \to 0. \qquad \square$$

Loosely speaking, a continuous function is differentiable at a point if, near that point, it admits a " best approximation " by a linear map.

When F is differentiable at $u_0 \in U$, the operator T in the above definition is uniquely determined by

$$Th = \frac{d}{dt}\,|_{t=0}\, F(u_0 + th) = \lim_{t \to 0} \frac{1}{t}\left(F(u_0 + th) - F(u_0)\right).$$

We will use the notation $T = D_{u_0}F$ and we will call T the *Fréchet derivative* of F at u_0.

Assume that the map $F : U \to Y$ is differentiable at each point $u \in U$. Then F is said to be of class C^1, if the map $u \mapsto D_uF \in L(X, Y)$ is continuous. F is said to be of class C^2 if $u \mapsto D_uF$ is of class C^1. One can define inductively C^k and C^∞ (or *smooth*) maps.

Example 1.1.2. Consider $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Using Cartesian coordinates $x = (x^1, \dots, x^n)$ in $\mathbb{R}^n$ and $u = (u^1, \dots, u^m)$ in $\mathbb{R}^m$ we can think of F as a collection of m functions on U

$$u^1 = u^1(x^1, \dots, x^n), \dots, u^m = u^m(x^1, \dots, x^n).$$

The map F is differentiable at a point $p = (p^1, \dots, p^n) \in U$ if and only if the functions u^i are differentiable at p in the usual sense of calculus. The Fréchet derivative of F at p is the linear operator $D_pF : \mathbb{R}^n \to \mathbb{R}^m$ given by the *Jacobian matrix*

$$D_pF = \frac{\partial(u^1, \dots, u^m)}{\partial(x^1, \dots, x^n)} = \begin{bmatrix} \frac{\partial u^1}{\partial x^1}(p) & \frac{\partial u^1}{\partial x^2}(p) & \cdots & \frac{\partial u^1}{\partial x^n}(p) \\ \frac{\partial u^2}{\partial x^1}(p) & \frac{\partial u^2}{\partial x^2}(p) & \cdots & \frac{\partial u^2}{\partial x^n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u^m}{\partial x^1}(p) & \frac{\partial u^m}{\partial x^2}(p) & \cdots & \frac{\partial u^m}{\partial x^n}(p) \end{bmatrix}.$$

The map F is smooth if and only if the functions $u^i(x)$ are smooth. □

Exercise 1.1.3. (a) Let $\mathcal{U} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ denote the set of invertible $n \times n$ matrices. Show that $\mathcal{U}$ is an open set.
(b) Let $F : \mathcal{U} \to \mathcal{U}$ be defined as $A \to A^{-1}$. Show that $D_AF(H) = -A^{-1}HA^{-1}$ for any $n \times n$ matrix H.
(c) Show that the Fréchet derivative of the map $\det : L(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}$, $A \mapsto \det A$, at $A = \mathbb{1}_{\mathbb{R}^n} \in L(\mathbb{R}^n, \mathbb{R}^n)$ is given by $\operatorname{tr} H$, i.e.,

$$\frac{d}{dt}|_{t=0} \det(\mathbb{1}_{\mathbb{R}^n} + tH) = \operatorname{tr} H, \ \ \forall H \in L(\mathbb{R}^n, \mathbb{R}^n).$$ □

Theorem 1.1.4 (Inverse function theorem). *Let X, Y be two Banach spaces, and $F : U \subset X \to Y$ a smooth function. If at a point $u_0 \in U$ the derivative $D_{u_0}F \in L(X, Y)$ is invertible, then there exits a neighborhood U_1 of u_0 in U such that $F(U_1)$ is an open neighborhood of $v_0 = F(u_0)$ in Y and $F : U_1 \to F(U_1)$ is bijective, with smooth inverse.* □

The spirit of the theorem is very clear: the invertibility of the derivative $D_{u_0}F$ "propagates" locally to F because $D_{u_0}F$ is a very good local approximation for F.

More formally, if we set $T = D_{u_0}F$, then

$$F(u_0 + h) = F(u_0) + Th + r(h),$$

where $r(h) = o(\|h\|)$ as $h \to 0$. The theorem states that, for every v sufficiently close to v_0, the equation $F(u) = v$ has a unique solution $u = u_0 + h$, with h very small. To prove the theorem one has to show that, for $\|v - v_0\|_Y$ sufficiently small, the equation below

$$v_0 + Th + r(h) = v$$

has a unique solution. We can rewrite the above equation as $Th = v - v_0 - r(h)$ or, equivalently, as $h = T^{-1}(v - v_0 - r(h))$. This last equation is a fixed point problem that can be approached successfully via the Banach fixed point theorem.

Theorem 1.1.5 (Implicit function theorem). *Let X, Y, Z be Banach spaces, and $F : X \times Y \to Z$ a smooth map. Let $(x_0, y_0) \in X \times Y$, and set $z_0 = F(x_0, y_0)$. Set $F_2 : Y \to Z$, $F_2(y) = F(x_0, y)$. Assume that $D_{y_0}F_2 \in L(Y, Z)$ is invertible. Then there exist neighborhoods U of $x_0 \in X$, V of $y_0 \in Y$, and a smooth map $G : U \to V$ such that the set S of solution (x, y) of the equation $F(x, y) = z_0$ which lie inside $U \times V$ can be identified with the graph of G, i.e.,*

$$\big\{\,(x, y) \in U \times V\;;\; F(x, y) = z_0\,\big\} = \big\{\,(x, G(x)) \in U \times V\;;\; x \in U\,\big\}.$$

In pre-Bourbaki times, the classics regarded the coordinate y as a function of x defined implicitly by the equality $F(x, y) = z_0$.

Proof. Consider the map

$$H : X \times Y \to X \times Z,\;\; \xi = (x, y) \mapsto (x, F(x, y)).$$

The map H is a smooth map, and at $\xi_0 = (x_0, y_0)$ its derivative $D_{\xi_0}H : X \times Y \to X \times Z$ has the block decomposition

$$D_{\xi_0}H = \begin{bmatrix} \mathbb{1}_X & 0 \\ D_{\xi_0}F_1 & D_{\xi_0}F_2 \end{bmatrix}.$$

Above, DF_1 (respectively DF_2) denotes the derivative of $x \mapsto F(x, y_0)$ (respectively the derivative of $y \mapsto F(x_0, y)$). The linear operator $D_{\xi_0}H$ is invertible, and its inverse has the block decomposition

$$(D_{\xi_0}H)^{-1} = \begin{bmatrix} \mathbb{1}_X & 0 \\ & \\ -\left(D_{\xi_0}F_2\right)^{-1} \circ \left(D_{\xi_0}F_1\right) & \left(D_{\xi_0}F_2\right)^{-1} \end{bmatrix}.$$

Thus, by the inverse function theorem, the equation $(x, F(x, y)) = (x, z_0)$ has a unique solution $(\tilde{x}, \tilde{y}) = H^{-1}(x, z_0)$ in a neighborhood of (x_0, y_0). It obviously satisfies $\tilde{x} = x$ and $F(\tilde{x}, \tilde{y}) = z_0$. Hence, the set $\{(x, y)\;;\; F(x, y) = z_0\}$ is locally the graph of $x \mapsto H^{-1}(x, z_0)$. $\square$

1.2 Smooth manifolds

1.2.1 *Basic definitions*

We now introduce the object which will be the main focus of this book, namely, the concept of (smooth) manifold. It formalizes the general principles outlined in Subsection 1.1.1.

Definition 1.2.1. A *smooth manifold* of dimension m is a locally compact, paracompact Hausdorff space M together with the following collection of data (henceforth called *atlas* or *smooth structure*) consisting of the following.

(a) An open cover $\{U_i\}_{i\in I}$ of M;
(b) A collection of continuous, injective maps $\{\, \Psi_i : U_i \to \mathbb{R}^m;\quad i \in I \,\}$ (called *charts* or *local coordinates*) such that, $\Psi_i(U_i)$ is open in $\mathbb{R}^m$, and if $U_i \cap U_j \neq \emptyset$, then the transition map

$$\Psi_j \circ \Psi_i^{-1} : \Psi_i(U_i \cap U_j) \subset \mathbb{R}^m \to \Psi_j(U_i \cap U_j) \subset \mathbb{R}^m$$

is smooth. (We say the various charts are *smoothly compatible*; see Figure 1.2).□

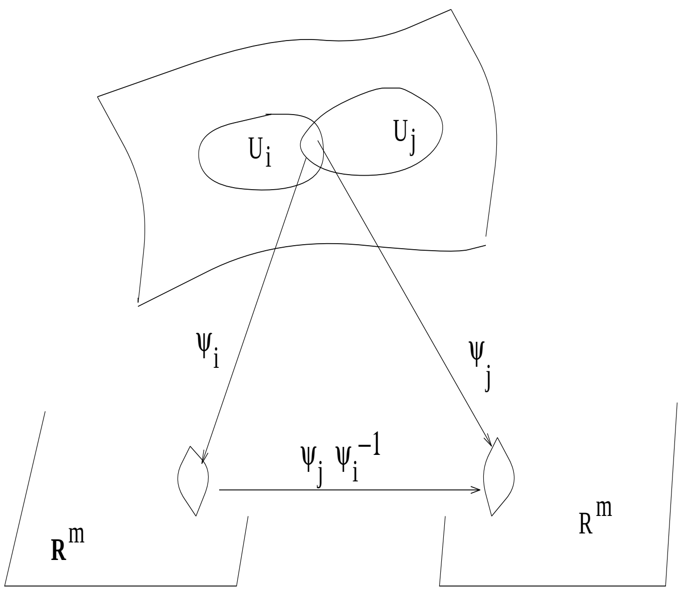

Fig. 1.2 *Transition maps*

Each chart Ψ_i can be viewed as a collection of m functions $(x^1, \dots, x^m)$ on U_i. Similarly, we can view another chart Ψ_j as another collection of functions $(y^1, \dots, y^m)$. The transition map $\Psi_j \circ \Psi_i^{-1}$ can then be interpreted as a collection of maps

$$(x^1,\dots,x^m)\mapsto \big(y^1(x^1,\dots,x^m),\dots,y^m(x^1,\dots,x^m)\big).$$

The first and the most important example of manifold is $\mathbb{R}^n$ itself. The natural smooth structure consists of an atlas with a single chart, $\mathbb{1}_{\mathbb{R}^n}:\mathbb{R}^n\to\mathbb{R}^n$. To construct more examples we will use the implicit function theorem .

Definition 1.2.2. (a) Let M, N be two smooth manifolds of dimensions m and respectively n. A continuous map $f:M\to N$ is said to be *smooth* if, for any local charts ϕ on M and ψ on N, the composition $\psi\circ f\circ\phi^{-1}$ (whenever this makes sense) is a smooth map $\mathbb{R}^m\to\mathbb{R}^n$.
(b) A smooth map $f:M\to N$ is called a *diffeomorphism* if it is invertible and its inverse is also a smooth map. □

Example 1.2.3. The map $t\mapsto e^t$ is a diffeomorphism $(-\infty,\infty)\to(0,\infty)$. The map $t\mapsto t^3$ is a homeomorphism $\mathbb{R}\to\mathbb{R}$ but it is not a diffeomorphism! □

If M is a smooth manifold we will denote by $C^\infty(M)$ the linear space of all smooth functions $M\to\mathbb{R}$.

Remark 1.2.4. Let U be an open subset of the smooth manifold M ($\dim M=m$) and $\psi:U\to\mathbb{R}^m$ a smooth, one-to one map with open image and smooth inverse. Then ψ defines local coordinates over U compatible with the existing atlas of M. Thus (U,ψ) can be added to the original atlas and the new smooth structure is diffeomorphic with the initial one. Using Zermelo's Axiom we can produce a *maximal atlas* (no more compatible local chart can be added to it). □

Our next result is a general recipe for producing manifolds. Historically, this is how manifolds entered mathematics.

Proposition 1.2.5. *Let M be a smooth manifold of dimension m and $f_1,\dots,f_k\in C^\infty(M)$. Define*

$$\mathcal{Z}=\mathcal{Z}(f_1,\dots,f_k)=\big\{\,p\in M\ ;\ f_1(p)=\cdots=f_k(p)=0\,\big\}.$$

Assume that the functions $f_1,\dots,f_k$ are functionally independent along $\mathcal{Z}$, i.e., for each $p\in\mathcal{Z}$, there exist local coordinates $(x^1,\dots,x^m)$ defined in a neighborhood of p in M such that $x^i(p)=0$, $i=1,\dots,m$, and the matrix

$$\frac{\partial\vec{f}}{\partial\vec{x}}\Big|_p:=\begin{bmatrix}\frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \cdots & \frac{\partial f_1}{\partial x^m}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f_k}{\partial x^1} & \frac{\partial f_k}{\partial x^2} & \cdots & \frac{\partial f_k}{\partial x^m}\end{bmatrix}_{x^1=\cdots=x^m=0}$$

has rank k. Then $\mathcal{Z}$ has a natural structure of smooth manifold of dimension $m-k$.

Proof. **Step 1: Constructing the charts.** Let $p_0 \in \mathcal{Z}$, and denote by $(x^1, \dots, x^m)$ local coordinates near p_0 such that $x^i(p_0) = 0$. One of the $k \times k$ minors of the matrix

$$\frac{\partial \vec{f}}{\partial \vec{x}}|_p := \begin{bmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \cdots & \frac{\partial f_1}{\partial x^m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial x^1} & \frac{\partial f_k}{\partial x^2} & \cdots & \frac{\partial f_k}{\partial x^m} \end{bmatrix}_{x^1 = \cdots = x^m = 0}$$

is nonzero. Assume this minor is determined by the last k columns (and all the k lines).

We can think of the functions $f_1, \dots, f_k$ as defined on an open subset U of $\mathbb{R}^m$. Split $\mathbb{R}^m$ as $\mathbb{R}^{m-k} \times \mathbb{R}^k$, and set

$$x' := (x^1, \dots, x^{m-k}), \;\; x'' := (x^{m-k+1}, \dots, x^m).$$

We are now in the setting of the implicit function theorem with

$$X = \mathbb{R}^{m-k}, \;\; Y = \mathbb{R}^k, \;\; Z = \mathbb{R}^k,$$

and $F : X \times Y \to Z$ given by

$$x \mapsto \begin{bmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{bmatrix} \in \mathbb{R}^k.$$

In this case, $DF_2 = \left(\frac{\partial F}{\partial x''}\right) : \mathbb{R}^k \to \mathbb{R}^k$ is invertible since its determinant corresponds to our nonzero minor. Thus, in a product neighborhood $U_{p_0} = U'_{p_0} \times U''_{p_0}$ of p_0, the set $\mathcal{Z}$ is the graph of some function

$$g : U'_{p_0} \subset \mathbb{R}^{m-k} \longrightarrow U''_{p_0} \subset \mathbb{R}^k,$$

i.e.,

$$\mathcal{Z} \cap U_{p_0} = \left\{ \, (x', g(x')\,) \in \mathbb{R}^{m-k} \times \mathbb{R}^k; \; x' \in U'_{p_0}, \;\; |x'| \text{ small} \right\}.$$

We now define $\psi_{p_0} : \mathcal{Z} \cap U_{p_0} \to \mathbb{R}^{m-k}$ by

$$(\, x', g(x')\,) \overset{\psi_{p_0}}{\longmapsto} x' \in \mathbb{R}^{m-k}.$$

The map ψ_{p_0} is a local chart of $\mathcal{Z}$ near p_0.

Step 2. The transition maps for the charts constructed above are smooth. The details are left to the reader. □

Exercise 1.2.6. Complete Step 2 in the proof of Proposition1.2.5. □

Definition 1.2.7. Let M be a m-dimensional manifold. A *codimension k submanifold* of M is a subset $N \subset M$ locally defined as the common zero locus of k functionally independent functions $f_1, \dots, f_k \in C^\infty(M)$. □

Proposition1.2.5 shows that any submanifold $N \subset M$ has a natural smooth structure so it becomes a manifold *per se*. Moreover, the inclusion map $i : N \hookrightarrow M$ is smooth.

1.2.2 *Partitions of unity*

This is a very brief technical subsection describing a trick we will extensively use in this book.

Definition 1.2.8. Let M be a smooth manifold and $(U_\alpha)_{\alpha\in\mathcal{A}}$ an open cover of M. A (smooth) *partition of unity* subordinated to this cover is a family $(f_\beta)_{\beta\in\mathcal{B}} \subset C^\infty(M)$ satisfying the following conditions.

(i) $0 \leq f_\beta \leq 1$.
(ii) $\exists\phi : \mathcal{B} \to \mathcal{A}$ such that $\operatorname{supp} f_\beta \subset U_{\phi(\beta)}$.
(iii) The family $(\operatorname{supp} f_\beta)$ is locally finite, i.e., any point $x \in M$ admits an open neighborhood intersecting only finitely many of the supports $\operatorname{supp} f_\beta$.
(iv) $\sum_\beta f_\beta(x) = 1$ for all $x \in M$. □

We include here for the reader's convenience the basic existence result concerning partitions of unity. For a proof we refer to [95].

Proposition 1.2.9. *(a) For any open cover* $\mathcal{U} = (U_\alpha)_{\alpha\in\mathcal{A}}$ *of a smooth manifold* M *there exists at least one smooth partition of unity* $(f_\beta)_{\beta\in\mathcal{B}}$ *subordinated to* $\mathcal{U}$ *such that* $\operatorname{supp} f_\beta$ *is compact for any* β.
(b) If we do not require compact supports, then we can find a partition of unity in which $\mathcal{B} = \mathcal{A}$ *and* $\phi = \mathbb{1}_\mathcal{A}$. □

Exercise 1.2.10. Let M be a smooth manifold and $S \subset M$ a *closed* submanifold. Prove that the restriction map

$$r : C^\infty(M) \to C^\infty(S) \quad f \mapsto f\,|_S$$

is surjective. □

1.2.3 *Examples*

Manifolds are everywhere, and in fact, to many physical phenomena which can be modelled mathematically one can naturally associate a manifold. On the other hand, many problems in mathematics find their most natural presentation using the language of manifolds. To give the reader an idea of the scope and extent of modern geometry, we present here a short list of examples of manifolds. This list will be enlarged as we enter deeper into the study of manifolds.

Example 1.2.11. (The n-dimensional sphere). This is the codimension 1 submanifold of $\mathbb{R}^{n+1}$ given by the equation

$$|x|^2 = \sum_{i=0}^{n}(x^i)^2 = r^2, \;\; x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}.$$

One checks that, along the sphere, the differential of $|x|^2$ is nowhere zero, so by Proposition 1.2.5, S^n is indeed a smooth manifold. In this case one can explicitly construct an atlas (consisting of two charts) which is useful in many applications. The construction relies on *stereographic projections*.

Let N and S denote the North and resp. South pole of S^n ($N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, $S = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$). Consider the open sets $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$. They form an open cover of S^n. The stereographic projection from the North pole is the map $\sigma_N : U_N \to \mathbb{R}^n$ such that, for any $P \in U_N$, the point $\sigma_N(P)$ is the intersection of the line NP with the hyperplane $\{x^n = 0\} \cong \mathbb{R}^n$.

The stereographic projection from the South pole is defined similarly. For $P \in U_N$ we denote by $(y^1(P), \dots, y^n(P))$ the coordinates of $\sigma_N(P)$, and for $Q \in U_S$, we denote by $(z^1(Q), \dots, z^n(Q))$ the coordinates of $\sigma_S(Q)$. A simple argument shows the map

$$\big(\, y^1(P), \dots, y^n(P)\,\big) \mapsto \big(\, z^1(P), \dots, z^n(P)\,\big), \quad P \in U_N \cap U_S,$$

is smooth (see the exercise below). Hence $\{(U_N, \sigma_N),\ (U_S, \sigma_S)\}$ defines a smooth structure on S^n. □

Exercise 1.2.12. Show that the functions y^i, z^j constructed in the above example satisfy

$$z^i = \frac{y^i}{\left(\sum_{j=1}^n (y^j)^2\right)}, \quad \forall i = 1, \dots, n. \qquad \square$$

Example 1.2.13. (The n-dimensional torus). This is the codimension n submanifold of $\mathbb{R}^{2n}(x_1, y_1; \dots; x_n, y_n)$ defined as the zero locus

$$x_1^2 + y_1^2 = \cdots = x_n^2 + y_n^2 = 1.$$

Note that T^1 is diffeomorphic with the 1-dimensional sphere S^1 (unit circle). As a set T^n is a direct product of n circles $T^n = S^1 \times \cdots \times S^1$ (see Figure 1.3). □

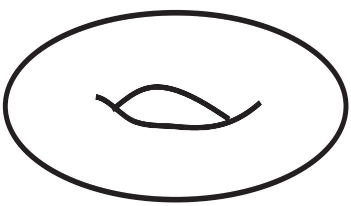

Fig. 1.3 *The 2-dimensional torus*

The above example suggests the following general construction.

Example 1.2.14. Let M and N be smooth manifolds of dimension m and respectively n. Then their topological direct product has a natural structure of smooth manifold of dimension $m + n$. □

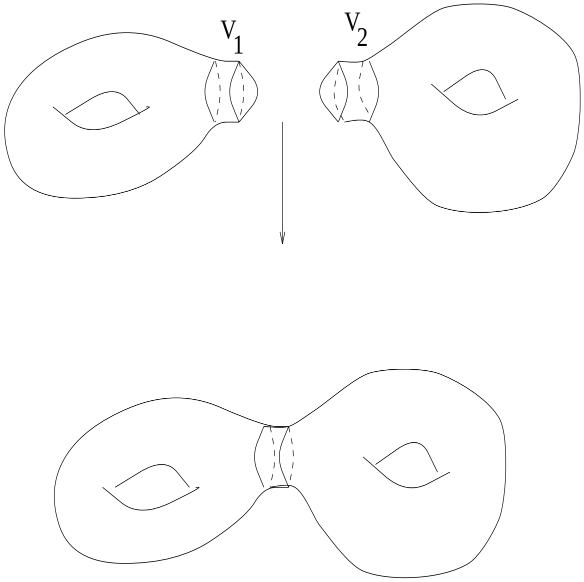

Fig. 1.4 *Connected sum of tori*

Example 1.2.15. (The connected sum of two manifolds). Let M_1 and M_2 be two manifolds of the same dimension m. Pick $p_i \in M_i$ $(i = 1, 2)$, choose small open neighborhoods U_i of p_i in M_i and then local charts ψ_i identifying each of these neighborhoods with $B_2(0)$, the ball of radius 2 in $\mathbb{R}^m$.

Let $V_i \subset U_i$ correspond (via ψ_i) to the annulus $\{1/2 < |x| < 2\} \subset \mathbb{R}^m$. Consider

$$\phi : \{\, 1/2 < |x| < 2 \,\} \to \{\, 1/2 < |x| < 2 \,\}, \quad \phi(x) = \frac{x}{|x|^2}.$$

The action of ϕ is clear: it switches the two boundary components of $\{1/2 < |x| < 2\}$, and reverses the orientation of the radial directions.

Now "glue" V_1 to V_2 using the "prescription" given by $\psi_2^{-1} \circ \phi \circ \psi_1 : V_1 \to V_2$. In this way we obtain a new topological space with a natural smooth structure induced by the smooth structures on M_i. Up to a diffeomeorphism, the new manifold thus obtained is independent of the choices of local coordinates ([19]), and it is called *the connected sum of* M_1 *and* M_2 and is denoted by $M_1 \# M_2$ (see Figure 1.4). □

Example 1.2.16. (The real projective space $\mathbb{RP}^n$). As a topological space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1}$ modulo the equivalence relation

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{R}^* : \;\; x = \lambda y.$$

The equivalence class of $x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \setminus \{0\}$ is usually denoted by $[x^0, \dots, x^n]$. Alternatively, $\mathbb{RP}^n$ is the set of all lines (directions) in $\mathbb{R}^{n+1}$. Traditionally, one attaches a point to each direction in $\mathbb{R}^{n+1}$, the so-called "point at infinity" along that direction, so that $\mathbb{RP}^n$ can be thought as the collection of all points at infinity along all directions.

The space $\mathbb{RP}^{n+1}$ has a natural structure of smooth manifold. To describe it consider the sets

$$U_k = \left\{ [x^0, \dots, x^n] \in \mathbb{RP}^n \; ; \; x^k \neq 0 \right\}, \quad k = 0, \dots, n.$$

Now define

$$\psi_k : U_k \to \mathbb{R}^n \qquad [x^0, \dots, x^n] \mapsto (x^0/x^k, \dots, x^{k-1}/x^k, x^{k+1}/x^k, \dots x^n).$$

The maps ψ_k define local coordinates on the projective space. The transition map on the overlap region $U_k \cap U_m = \{[x^0, \dots, x^n] \; ; \; x^k x^m \neq 0\}$ can be easily described. Set

$$\psi_k([x^0, \dots, x^n]) = (\xi_1, \dots, \xi_n), \;\; \psi_m([x^0, \dots, x^n]) = (\eta_1, \dots, \eta_n).$$

The equality

$$[x^0, \dots, x^n] = [\xi_1, \dots, \xi_{k-1}, 1, \xi_k, \dots, \xi_n] = [\eta_1, \dots, \eta_{m-1}, 1, \eta_m, \dots, \eta_n]$$

immediately implies (assume $k < m$)

$$\begin{cases} \xi_1 = \eta_1/\eta_k, \;\; \dots, \; \xi_{k-1} = \eta_{k-1}/\eta_k, \quad \xi_{k+1} = \eta_k \\ \xi_k = \eta_{k+1}/\eta_k, \dots, \xi_{m-2} = \eta_{m-1}/\eta_k, \, \xi_{m-1} = 1/\eta_k \\ \xi_m = \eta_m \eta_k, \;\; \dots, \qquad \xi_n = \eta_n/\eta_k \end{cases} \tag{1.2.1}$$

This shows the map $\psi_k \circ \psi_m^{-1}$ is smooth and proves that $\mathbb{RP}^n$ is a smooth manifold. Note that when $n = 1$, $\mathbb{RP}^1$ is diffeomorphic with S^1. One way to see this is to observe that the projective space can be alternatively described as the quotient space of S^n modulo the equivalence relation which identifies antipodal points. □

Example 1.2.17. (The complex projective space $\mathbb{CP}^n$). The definition is formally identical to that of $\mathbb{RP}^n$. $\mathbb{CP}^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ modulo the equivalence relation

$$x \sim y \stackrel{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{C}^* : \;\; x = \lambda y.$$

The open sets U_k are defined similarly and so are the local charts $\psi_k : U_k \to \mathbb{C}^n$. They satisfy transition rules similar to (1.2.1) so that $\mathbb{CP}^n$ is a smooth manifold of dimension $2n$. □

Exercise 1.2.18. Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2. □

In the above example we encountered a special (and very pleasant) situation: the gluing maps not only are smooth, they are also *holomorphic* as maps $\psi_k \circ \psi_m^{-1} : U \to V$ where U and V are open sets in $\mathbb{C}^n$. This type of gluing induces a "rigidity" in the underlying manifold and it is worth distinguishing this situation.

Definition 1.2.19. (Complex manifolds). A complex manifold is a smooth, $2n$-dimensional manifold M which admits an atlas $\{(U_i, \psi_i) : U_i \to \mathbb{C}^n\}$ such that all transition maps are holomorphic. □

The complex projective space is a complex manifold. Our next example naturally generalizes the projective spaces described above.

Example 1.2.20. (The real and complex Grassmannians $\mathbf{Gr}_k(\mathbb{R}^n)$, $\mathbf{Gr}_k(\mathbb{C}^n)$).

Suppose V is a real vector space of dimension n. For every $0 \leq k \leq n$ we denote by $\mathbf{Gr}_k(V)$ the set of k-dimensional vector subspaces of V. We will say that $\mathbf{Gr}_k(V)$ is the *linear Grassmannian of k-planes in E*. When $V = \mathbb{R}^n$ we will write $\mathbf{Gr}_{k,n}(\mathbb{R})$ instead of $\mathbf{Gr}_k(\mathbb{R}^n)$.

We would like to give several equivalent descriptions of the natural structure of smooth manifold on $\mathbf{Gr}_k(V)$. To do this it is very convenient to fix a Euclidean metric on V.

Any k-dimensional subspace $L \subset V$ is uniquely determined by the orthogonal projection onto L which we will denote by P_L. Thus we can identify $\mathbf{Gr}_k(V)$ with the set of rank k projectors

$$\mathbf{Proj}_k(V) := \bigl\{ P : V \to V;\ \ P^* = P = P^2,\ \ \operatorname{rank} P = k \bigr\}.$$

We have a natural map

$$P : \mathbf{Gr}_k(V) \to \mathbf{Proj}_k(V),\ \ L \mapsto P_L$$

with inverse $P \mapsto \operatorname{Range}(P)$.

The set $\mathbf{Proj}_k(V)$ is a subset of the vector space of symmetric endomorphisms

$$\operatorname{End}^+(V) := \bigl\{ A \in \operatorname{End}(V),\ \ A^* = A \bigr\}.$$

The sapce $\operatorname{End}^+(V)$ is equipped with a natural inner product

$$(A, B) := \frac{1}{2} \operatorname{tr}(AB),\ \ \forall A, B \in \operatorname{End}^+(V). \tag{1.2.2}$$

The set $\mathbf{Proj}_k(V)$ is a closed and bounded subset of $\operatorname{End}^+(V)$. The bijection

$$P : \mathbf{Gr}_k(V) \to \mathbf{Proj}_k(V)$$

induces a topology on $\mathbf{Gr}_k(V)$. We want to show that $\mathbf{Gr}_k(V)$ has a natural structure of smooth manifold compatible with this topology. To see this, we define for every $L \subset \mathbf{Gr}_k(V)$ the set

$$\mathbf{Gr}_k(V, L) := \bigl\{ U \in \mathbf{Gr}_k(V);\ \ U \cap L^\perp = 0 \bigr\}.$$

Lemma 1.2.21. *(a) Let $L \in \mathbf{Gr}_k(V)$. Then*

$$U \cap L^\perp = 0 \Longleftrightarrow \mathbb{1} - P_L + P_U : V \to V \ \ \textit{is an isomorphism}. \tag{1.2.3}$$

(b) The set $\mathbf{Gr}_k(V, L)$ is an open subset of $\mathbf{Gr}_k(V)$.

Proof. (a) Note first that a dimension count implies that

$$U \cap L^\perp = 0 \Longleftrightarrow U + L^\perp = V \Longleftrightarrow U^\perp \cap L = 0.$$

Let us show that $U\cap L^{\perp}=0$ implies that $\mathbb{1}-P_L+P_L$ is an isomorphism. It suffices to show that

$$\ker(\mathbb{1}-P_L+P_U)=0.$$

Suppose $v\in\ker(\mathbb{1}-P_L+P_U)$. Then

$$0=P_L(\mathbb{1}-P_L+P_U)v=P_LP_Uv=0\Longrightarrow P_Uv\in U\cap\ker P_L=U\cap L^{\perp}=0.$$

Hence $P_Uv=0$, so that $v\in U^{\perp}$. From the equality $(\mathbb{1}-P_L-P_U)v=0$ we also deduce $(\mathbb{1}-P_L)v=0$ so that $v\in L$. Hence

$$v\in U^{\perp}\cap L=0.$$

Conversely, we will show that if $\mathbb{1}-P_L+P_U=P_{L^{\perp}}+P_U$ is onto, then $U+L^{\perp}=V$.

Indeed, let $v\in V$. Then there exists $x\in V$ such that

$$v=P_{L^{\perp}}x+P_Ux\in L^{\perp}+U.$$

(b) We have to show that, for every $K\in\mathbf{Gr}_k(V,L)$, there exists $\varepsilon>0$ such that any U satisfying

$$\|P_U-P_K\|<\varepsilon$$

intersects $L^{\perp}$ trivially. Since $K\in\mathbf{Gr}_k(V,L)$ we deduce from (a) that the map $\mathbb{1}-P_L-P_K:V\to V$ is an isomorphism. Note that

$$\|(\mathbb{1}-P_L-P_K)-(\mathbb{1}-P_L-P_U)\|=\|P_K-P_U\|.$$

The space of isomorphisms of V is an open subset of $\operatorname{End}(V)$. Hence there exists $\varepsilon>0$ such that, for any subspace U satisfying $\|P_U-P_K\|<\varepsilon$, the endomorphism $(\mathbb{1}-P_L-P_U)$ is an isomorphism. We now conclude using part (a). □

Since $L\in\mathbf{Gr}_k(V,L)$, $\forall L\in\mathbf{Gr}_k(V)$, we have an open cover of $\mathbf{Gr}_k(V)$

$$\mathbf{Gr}_k(V)=\bigcup_{L\in\mathbf{Gr}_k(V)}\mathbf{Gr}_k(V,L).$$

Note that for every $L\in\mathbf{Gr}_k(V)$ we have a natural map

$$\Gamma:\operatorname{Hom}(L,L^{\perp})\to\mathbf{Gr}_k(V,L),$$

which associates to each linear map $S:L\to L^{\perp}$ its graph (see Figure 1.5)

$$\Gamma_S=\{x+Sx\in L+L^{\perp}=V;\ \ x\in L\}.$$

We will show that this is a homeomorphism by providing an explicit description of the orthogonal projection P_{Γ_S}

Observe first that the orthogonal complement of Γ_S is the graph of $-S^*:L^{\perp}\to L$. More precisely,

$$\Gamma_S^{\perp}=\Gamma_{-S^*}=\big\{y-S^*y\in L^{\perp}+L=V;\ \ y\in L^{\perp}\big\}.$$

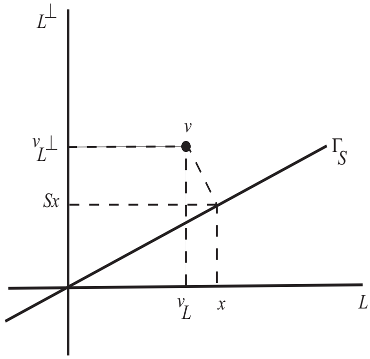

Fig. 1.5 *Subspaces as graphs of linear operators.*

Let $v = P_L v + P_{L^\perp} v = v_L + v_{L^+} \in V$ (see Figure 1.5). Then

$$P_{\Gamma_S} v = x + Sx, \;\; x \in L \Longleftrightarrow v - (x + Sx) \in \Gamma_S^{\perp}$$

$$\Longleftrightarrow \exists x \in L, \;\; y \in L^{\perp} \;\text{ such that } \begin{cases} x + S^* y = v_L \\ Sx - y = v_{L^\perp} \end{cases}.$$

Consider the operator $\mathcal{S} : L \oplus L^{\perp} \to L \oplus L^{\perp}$ which has the block decomposition

$$\mathcal{S} = \begin{bmatrix} \mathbb{1}_L & S^* \\ S & -\mathbb{1}_L^{\perp} \end{bmatrix}.$$

Then the above linear system can be rewritten as

$$\mathcal{S} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_L \\ v_{L^\perp} \end{bmatrix}.$$

Now observe that

$$\mathcal{S}^2 = \begin{bmatrix} \mathbb{1}_L + S^*S & 0 \\ 0 & \mathbb{1}_{L^\perp} + SS^* \end{bmatrix}.$$

Hence $\mathcal{S}$ is invertible, and

$$\mathcal{S}^{-1} = \begin{bmatrix} (\mathbb{1}_L + S^*S)^{-1} & 0 \\ 0 & (\mathbb{1}_{L^\perp} + SS^*)^{-1} \end{bmatrix} \cdot \mathcal{S}$$

$$= \begin{bmatrix} (\mathbb{1}_L + S^*S)^{-1} & (\mathbb{1}_L + S^*S)^{-1} S^* \\ (\mathbb{1}_{L^\perp} + SS^*)^{-1} S & -(\mathbb{1}_{L^\perp} + SS^*)^{-1} \end{bmatrix}.$$

We deduce

$$x = (\mathbb{1}_L + S^*S)^{-1} v_L + (\mathbb{1}_L + S^*S)^{-1} S^* v_{L^\perp}$$

and

$$P_{\Gamma_S} v = \begin{bmatrix} x \\ Sx \end{bmatrix}.$$

Hence P_{Γ_S} has the block decomposition

$$P_{\Gamma_S} = \begin{bmatrix} \mathbb{1}_L \\ S \end{bmatrix} \cdot [(\mathbb{1}_L + S^*S)^{-1} \quad (\mathbb{1}_L + S^*S)^{-1}S^*]$$

$$= \begin{bmatrix} (\mathbb{1}_L + S^*S)^{-1} & (\mathbb{1}_L + S^*S)^{-1}S^* \\ S(\mathbb{1}_L + S^*S)^{-1} & S(\mathbb{1}_L + S^*S)^{-1}S^* \end{bmatrix}. \qquad (1.2.4)$$

Note that if $U \in \mathbf{Gr}_k(V, L)$, then with respect to the decomposition $V = L + L^\perp$ the projector P_U has the block form

$$P_U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} P_L P_U I_L & P_L P_U I_{L^\perp} \\ P_{L^\perp} P_U I_L & P_L L^\perp P_U I_{L^\perp} \end{bmatrix},$$

where for every subspace $K \hookrightarrow V$ we denoted by $I_K : K \to V$ the canonical inclusion, then $U = \Gamma_S$, where $S = CA^{-1}$. This shows that the graph map

$$\operatorname{Hom}(L, L^\perp) \ni S \mapsto \Gamma_S \in \mathbf{Gr}_k(V)$$

is a homeomorphism. Moreover, the above formulæ show that if $U \in \mathbf{Gr}_k(V, L_0) \cap \mathbf{Gr}_k(V, L_1)$, then we can represent U in two ways,

$$U = \Gamma_{S_0} = \Gamma_{S_1}, \ \ S_i \in \operatorname{Hom}(L_i, L_i^\perp), \ \ i = 0, 1,$$

and the correspondence $S_0 \to S_1$ is smooth. This shows that $\mathbf{Gr}_k(V)$ has a natural structure of smooth manifold of dimension

$$\dim \mathbf{Gr}_k(V) = \dim \operatorname{Hom}(L, L^\perp) = k(n-k).$$

$\mathbf{Gr}_k(\mathbb{C}^n)$ is defined as the space of complex k-dimensional subspaces of $\mathbb{C}^n$. It can be structured as above as a smooth manifold of dimension $2k(n-k)$. Note that $\mathbf{Gr}_1(\mathbb{R}^n) \cong \mathbb{RP}^{n-1}$, and $\mathbf{Gr}_1(\mathbb{C}^n) \cong \mathbb{CP}^{n-1}$. The Grassmannians have important applications in many classification problems. □

Exercise 1.2.22. Show that $\mathbf{Gr}_k(\mathbb{C}^n)$ is a *complex* manifold of complex dimension $k(n-k)$. □

Example 1.2.23. (Lie groups). A *Lie group* is a smooth manifold G together with a group structure on it such that the map

$$G \times G \to G \qquad (g, h) \mapsto g \cdot h^{-1}$$

is smooth. These structures provide an excellent way to formalize the notion of symmetry.
(a) $(\mathbb{R}^n, +)$ is a commutative Lie group.

(b) The unit circle S^1 can be alternatively described as the set of complex numbers of norm one and the complex multiplication defines a Lie group structure on it. This is a commutative group. More generally, the torus T^n is a Lie group as a direct product of n circles[1].

[1] One can show that any connected commutative Lie group has the form $T^n \times \mathbb{R}^m$.

(c) The general linear group $\mathrm{GL}(n, \mathbb{K})$ defined as the group of invertible $n \times n$ matrices with entries in the field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ is a Lie group. Indeed, $\mathrm{GL}(n, \mathbb{K})$ is an open subset (see Exercise 1.1.3) in the linear space of $n \times n$ matrices with entries in $\mathbb{K}$. It has dimension $d_{\mathbb{K}} n^2$, where $d_{\mathbb{K}}$ is the dimension of $\mathbb{K}$ as a linear space over $\mathbb{R}$. We will often use the alternate notation $\mathrm{GL}(\mathbb{K}^n)$ when referring to $\mathrm{GL}(n, \mathbb{K})$.

(d) The orthogonal group $O(n)$ is the group of real $n \times n$ matrices satisfying

$$T \cdot T^t = \mathbb{1}.$$

To describe its smooth structure we will use the Cayley transform trick as in [84] (see also the classical [100]). Set

$$M_n(\mathbb{R})^\# := \big\{ T \in M_n(\mathbb{R}) \,;\, \det(\mathbb{1} + T) \neq 0 \big\}.$$

The matrices in $M_n(\mathbb{R})^\#$ are called non exceptional. Clearly $\mathbb{1} \in O(n)^\# = O(n) \cap M_n(\mathbb{R})^\#$ so that $O(n)^\#$ is a *nonempty* open subset of $O(n)$. The *Cayley transform* is the map $\# : M_n(\mathbb{R})^\# \to M_n(\mathbb{R})$ defined by

$$A \mapsto A^\# = (\mathbb{1} - A)(\mathbb{1} + A)^{-1}.$$

The Cayley transform has some very nice properties.
(i) $A^\# \in M_n(\mathbb{R})^\#$ for every $A \in M_n(\mathbb{R})^\#$.
(ii) $\#$ is involutive, i.e., $(A^\#)^\# = A$ for any $A \in M_n(\mathbb{R})^\#$.
(iii) For every $T \in O(n)^\#$ the matrix $T^\#$ is skew-symmetric, and conversely, if $A \in M_n(\mathbb{R})^\#$ is skew-symmetric then $A^\# \in O(n)$.

Thus the Cayley transform is a homeomorphism from $O(n)^\#$ to the space of non-exceptional, skew-symmetric, matrices. The latter space is an open subset in the linear space of real $n \times n$ skew-symmetric matrices, $\underline{o}(n)$.

Any $T \in O(n)$ defines a self-homeomorphism of $O(n)$ by *left translation in the group*

$$L_T : O(n) \to O(n) \quad S \mapsto L_T(S) = T \cdot S.$$

We obtain an open cover of $O(n)$:

$$O(n) = \bigcup_{T \in O(n)} T \cdot O(n)^\#.$$

Define $\Psi_T : T \cdot O(n)^\# \to \underline{o}(n)$ by $S \mapsto (T^{-1} \cdot S)^\#$. One can show that the collection

$$\Big(T \cdot O(n)^\#, \Psi_T \Big)_{T \in O(n)}$$

defines a smooth structure on $O(n)$. In particular, we deduce

$$\dim O(n) = n(n-1)/2.$$

Inside $O(n)$ lies a normal subgroup (the *special orthogonal group*)

$$SO(n) = \{ T \in O(n) \,;\, \det T = 1 \}.$$

The group $SO(n)$ is a Lie group as well and $\dim SO(n) = \dim O(n)$.

(e) The unitary group $U(n)$ is defined as

$$U(n) = \{T \in \mathrm{GL}(n, \mathbb{C}) \; ; \; T \cdot T^* = \mathbb{1}\},$$

where T^* denotes the conjugate transpose (adjoint) of T. To prove that $U(n)$ is a manifold one uses again the Cayley transform trick. This time, we coordinatize the group using the space $\underline{\mathrm{u}}(n)$ of skew-adjoint (skew-Hermitian) $n \times n$ complex matrices ($A = -A^*$). Thus $U(n)$ is a smooth manifold of dimension

$$\dim U(n) = \dim \underline{\mathrm{u}}(n) = n^2.$$

Inside $U(n)$ sits the normal subgroup $SU(n)$, the kernel of the group homomorphism $\det : U(n) \to S^1$. $SU(n)$ is also called the *special unitary group*. This a smooth manifold of dimension $n^2 - 1$. In fact the Cayley transform trick allows one to coordinatize $SU(n)$ using the space

$$\underline{\mathrm{su}}(n) = \{A \in \underline{\mathrm{u}}(n) \; ; \; \mathrm{tr}\, A = 0\}. \qquad \square$$

Exercise 1.2.24. (a) Prove the properties (i)-(iii) of the Cayley transform, and then show that $\big(T \cdot O(n)^{\#}, \Psi_T\big)_{T \in O(n)}$ defines a smooth structure on $O(n)$.
(b) Prove that $U(n)$ and $SU(n)$ are manifolds.
(c) Show that $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact spaces.
(d) Prove that $SU(2)$ is diffeomorphic with S^3 (Hint: think of S^3 as the group of unit quaternions.) $\square$

Exercise 1.2.25. Let $SL(n; \mathbb{K})$ denote the group of $n \times n$ matrices of determinant 1 with entries in the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Using the Cayley trick show that $SL(n; \mathbb{K})$ is a smooth manifold modeled on the linear space

$$\underline{\mathrm{sl}}(n, \mathbb{K}) = \{A \in M_{n \times n}(\mathbb{K}) \; ; \; \mathrm{tr}\, A = 0\}.$$

In particular, it has dimension $d_{\mathbb{K}}(n^2 - 1)$, where $d_{\mathbb{K}} = \dim_{\mathbb{R}} \mathbb{K}$. $\square$

Exercise 1.2.26. (Quillen). Suppose V_0, V_1 are two real, finite dimensional Euclidean space, and $T : V_0 \to V_1$ is a linear map. We denote by T^* is adjoint, $T^* : V_1 \to V_0$, and by Γ_T the graph of T,

$$\Gamma_T = \big\{(v_0, v_1) \in V_0 \oplus V_1; \;\; v_1 = T v_0\, l\big\}.$$

We form the skew-symmetric operator

$$X : V_0 \oplus V_1 \to V_0 \oplus V_1, \;\; X \cdot \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & T^* \\ -T & 0 \end{bmatrix} \cdot \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}.$$

We denote by C_T the Cayley transform of X,

$$C_T = (\mathbb{1} - X)(\mathbb{1} + X)^{-1},$$

and by $R_0 : V_0 \oplus V_1 \to V_0 \oplus V_1$ the reflection

$$R_0 = \begin{bmatrix} \mathbb{1}_{V_0} & 0 \\ 0 & -\mathbb{1}_{V_1} \end{bmatrix}.$$

Show that $R_T = C_T R_0$ is an orthogonal involution, i.e.,

$$R_T^2 = \mathbb{1}, \;\; R_T^* = R_T,$$

and $\ker(\mathbb{1} - R_T) = \Gamma_T$. In other words, R_T is the orthogonal reflection in the subspace Γ_T,

$$R_T = 2P_{\Gamma_T} - \mathbb{1},$$

where P_{Γ_T} denotes the orthogonal projection onto Γ_T. □

Exercise 1.2.27. Suppose G is a Lie group, and H is an abstract subgroup of G. Prove that the closure of H is also a subgroup of G.

Exercise 1.2.28. (a) Let G be a *connected* Lie group and denote by U a neighborhood of $1 \in G$. If H is the subgroup algebraically generated by U show that H is dense in G.
(b) Let G be a *compact* Lie group and $g \in G$. Show that $1 \in G$ lies in the closure of $\{g^n;\;\; n \in \mathbb{Z} \setminus \{0\}\}$. □

Remark 1.2.29. If G is a Lie group, and H is a *closed* subgroup of G, then H is in fact a smooth submanifold of G, and with respect to this smooth structure H is a Lie group. For a proof we refer to [44, 95]. In view of Exercise 1.2.27, this fact allows us to produce many examples of Lie groups. □

1.2.4 *How many manifolds are there?*

The list of examples in the previous subsection can go on forever, so one may ask whether there is any coherent way to organize the collection of all possible manifolds. This is too general a question to expect a clear cut answer. We have to be more specific. For example, we can ask

Question 1: *Which are the* compact, *connected manifolds of a given dimension d?*

For $d = 1$ the answer is very simple: the only compact connected 1-dimensional manifold is the circle S^1. (Can you prove this?)

We can raise the stakes and try the same problem for $d = 2$. Already the situation is more elaborate. We know at least two surfaces: the sphere S^2 and the torus T^2. They clearly look different but we have not yet proved rigorously that they are indeed not diffeomorphic. This is not the end of the story. We can connect sum two tori, three tori or any number g of tori. We obtain doughnut-shaped surface as in Figure 1.6.

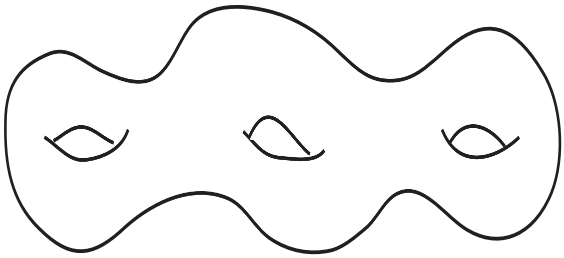

Fig. 1.6 *Connected sum of 3 tori*

Again we face the same question: do we get non-diffeomorphic surfaces for different choices of g? Figure 1.6 suggests that this may be the case but this is no rigorous argument.

We know another example of compact surface, the projective plane $\mathbb{RP}^2$, and we naturally ask whether it looks like one of the surfaces constructed above. Unfortunately, we cannot visualize the real projective plane (one can prove rigorously it does not have enough room to exist inside our 3-dimensional Universe). We have to decide this question using a little more than the raw geometric intuition provided by a picture. To kill the suspense, we mention that $\mathbb{RP}^2$ does not belong to the family of donuts. One reason is that, for example, a torus has two faces: an inside face and an outside face (think of a car rubber tube). $\mathbb{RP}^2$ has a weird behavior: it has "no inside" and "no outside". It has only one side! One says the torus is *orientable* while the projective plane is not.

We can now connect sum any numbers of $\mathbb{RP}^2$'s to any donut an thus obtain more and more surfaces, which we cannot visualize and we have yet no idea if they are pairwise distinct. A classical result in topology says that all compact surfaces can be obtained in this way (see [68]), but in the above list some manifolds are diffeomorphic, and we have to describe which. In dimension 3 things are not yet settled[2] and, to make things look hopeless, in dimension ≥ 4 Question 1 is algorithmically undecidable.

We can reconsider our goals, and look for all the manifolds with a given property X. In many instances one can give fairly accurate answers. Property X may refer to more than the (differential) topology of a manifold. Real life situations suggest the study of manifolds with additional structure. The following problem may give the reader a taste of the types of problems we will be concerned with in this book.

Question 2 *Can we wrap a planar piece of canvas around a metal sphere in a one-to-one fashion? (The canvas is flexible but not elastic.)*

A simple do-it-yourself experiment is enough to convince anyone that this is not possible. Naturally, one asks for a rigorous explanation of what goes wrong.

[2] Things are still not settled in 2007, but there has been considerable progress due to G. Perelman's proof of the Poincaré conjecture.

The best explanation of this phenomenon is contained in the celebrated Theorema Egregium (Golden Theorem) of Gauss. Canvas surfaces have additional structure (they are made of a special material), and for such objects there is a rigorous way to measure "how curved" are they. One then realizes that the problem in Question 2 is impossible, since a (canvas) sphere is curved in a different way than a plane canvas.

There are many other structures Nature forced us into studying them, but they may not be so easily described in elementary terms.

A word to the reader. The next two chapters are probably the most arid in geometry but, keep in mind that, behind each construction lies a natural motivation and, even if we do not always have the time to show it to the reader, it is there, and it may take a while to reveal itself. Most of the constructions the reader will have to "endure" in the next two chapters constitute not just some difficult to "swallow" formalism, but the basic language of geometry. It might comfort the reader during this less than glamorous journey to carry in the back of his mind Hermann Weyl's elegantly phrased advise.

> "It is certainly regrettable that we have to enter into the purely formal aspect in such detail and to give it so much space but, nevertheless, it cannot be avoided. Just as anyone who wishes to give expressions to his thoughts with ease must spend laborious hours learning language and writing, so here too the only way we can lessen the burden of formulæ is to master the technique of tensor analysis to such a degree that we can turn to real problems that concern us without feeling any encumbrance, our object being to get an insight into the nature of space [...]. Whoever sets out in quest of these goals must possess a perfect mathematical equipment from the outset."
>
> *H. Weyl: Space, Time, Matter.*

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