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§ Chapter 1

EXERCISE 2.1 (Stronger separation). Let (S, \mathcal{S}, μ) be a measure space and let $f, g \in \mathcal{L}^0(S, \mathcal{S})$ satisfy $\mu(\{x \in S : f(x) < g(x)\}) > 0$. Prove or construct a counterexample for the following statement:

“There exist constants $a, b \in \mathbb{R}$ such that $\mu(\{x \in S : f(x) \leq a < b \leq g(x)\}) > 0$.”

Proof: We prove that this statement is true. Define a function $m : S \rightarrow \mathbb{R}$ by

$$m(x) = \frac{f(x) + g(x)}{2}.$$

Think of $m(x)$ as the midpoint between $f(x)$ and $g(x)$. Since \mathcal{L}^0 is a real vector space, $m(x)$ is a measurable function.

Now define for each $n \in \mathbb{N}$

$$A_n = \left\{ x \in S \mid f(x) \leq m(x) - \frac{1}{2^n} < m(x) + \frac{1}{2^n} \leq g(x) \right\}.$$

Notice that because $m(x) - \frac{1}{2^n} < m(x) + \frac{1}{2^n}$ for all $x \in S$ we can write these sets instead as

$$\begin{aligned} A_n &= \left\{ x \in S \mid f(x) \leq m(x) - \frac{1}{2^n} \text{ and } m(x) + \frac{1}{2^n} \leq g(x) \right\} \\ &= \left\{ x \in S \mid f(x) - m(x) \leq -\frac{1}{2^n} \right\} \cap \left\{ x \in S \mid \frac{1}{2^n} \leq g(x) - m(x) \right\} \\ &= (f - m)^{-1} \left(-\infty, -\frac{1}{2^n} \right] \cap (g - m)^{-1} \left[\frac{1}{2^n}, \infty \right). \end{aligned}$$

The differences $f - m$ and $g - m$ are measurable functions since $\mathcal{L}^0(S, \mathcal{S})$ is a \mathbb{R} -vector space, so A_n is an intersection of measurable sets and hence itself measurable. Furthermore, if $x \in A_n$, then we have

$$f(x) \leq m(x) - \frac{1}{2^n} < m(x) - \frac{1}{2^{n+1}} < m(x) + \frac{1}{2^{n+1}} < m(x) + \frac{1}{2^n} \leq g(x)$$

which implies $x \in A_{n+1}$ and $A_n \subseteq A_{n+1}$. Thus the sequence $\{A_n\}_{n \in \mathbb{N}}$ is nested increasing.

We now suggestively name the set in question A , i.e. set

$$A = \{x \in S \mid f(x) < g(x)\}.$$

I claim that $A = \bigcup_{n \in \mathbb{N}} A_n$. One inclusion is obvious, since $A_n \subseteq A$ for all $n \in \mathbb{N}$. For the other inclusion, suppose $x \in A$. Then $f(x) < g(x)$, and hence $f(x) < m(x) < g(x)$. Set choose N so that $\frac{1}{N} < \min\{m(x) - f(x), g(x) - m(x)\}$ (note that $m(x) - f(x) = g(x) - m(x)$, but this is faster). Then $\frac{1}{2^N} < \frac{1}{N}$, and hence

$$f(x) \leq m(x) - \frac{1}{2^N} \text{ and } m(x) + \frac{1}{2^N} \geq g(x).$$

This implies that $x \in A_n$. Since every element of A is contained in A_n for some $n \in \mathbb{N}$, we conclude that $A \subseteq \bigcup_n A_n$, and therefore have equality $A = \bigcup_n A_n$.

Now, by the continuity measures with respect to increasing sequences,

$$\mu(A) = \mu\left(\bigcup_n A_n\right) = \lim_n \mu(A_n).$$

If $\mu(A_n) = 0$ for all $n \in \mathbb{N}$, then we would have $\mu(A) = \lim_n \mu(A_n) = 0$, which is not the case. There must then be some $n \in \mathbb{N}$ such that $\mu(A_n) > 0$. Setting $a = m(x) - \frac{1}{2^n}$ and $b = m(x) + \frac{1}{2^n}$ then gives us that

$$\mu(\{x \in S \mid f(x) \leq a < b \leq g(x)\}) = \mu(A_n) > 0.$$

□

EXERCISE 2.2 (A uniform distribution on a circle.) Let S^1 be the unit circle and let $f : [0, 1) \rightarrow S^1$ be the “winding map”

$$f(x) = (\cos(2\pi x), \sin(2\pi x)), \quad x \in [0, 1).$$

- (1) Show that the map f is $(\mathcal{B}([0, 1)), \mathcal{S}^1)$ -measurable, where \mathcal{S}^1 denotes the Borel σ -algebra on S^1 (with topology inherited from \mathbb{R}^2).
- (2) For $\alpha \in (0, 2\pi)$, let R_α denote the (counter-clockwise) rotation of \mathbb{R}^2 with center $(0, 0)$ and angle α . Show that $R_\alpha(A) = \{R_\alpha(x) : x \in A\}$ is in \mathcal{S}^1 if and only if $A \in \mathcal{S}^1$.
- (3) Let μ^1 be the pushforward of the Lebesgue measure λ by the map f . Show that μ^1 is rotation-invariant, i.e. that $\mu^1(A) = \mu^1(R_\alpha(A))$. *Note:* The measure μ^1 is called the **uniform measure** (or the **uniform distribution** on S^1).

Proof:

(1): If this were a topology class, we’d simply state that “it is clear that f is continuous,” as it is a continuous map in each component. Instead, we will prove that it is continuous, and hence Borel measurable. We take for granted the continuity of \sin and \cos as functions on \mathbb{R} .

Suppose $x, a \in [0, 1)$, and consider $\|f(x) - f(a)\|^2$. With the help of trig identities, we have the following:

$$\begin{aligned} \|f(x) - f(a)\|^2 &= |(\cos(2\pi x) - \cos(2\pi a))^2 + (\sin(2\pi x) - \sin(2\pi a))^2| \\ &= |\cos^2(2\pi x) - 2\cos(2\pi x)\cos(2\pi a) + \cos^2(2\pi a) + \sin^2(2\pi x) \\ &\quad - 2\sin(2\pi x)\sin(2\pi a) + \sin^2(2\pi a)| \\ &= |2 - \cos(2\pi x - 2\pi a) - \cos(2\pi x + 2\pi a) - \cos(2\pi x - 2\pi a) + \cos(2\pi x + 2\pi a)| \\ &= 2 - 2\cos(2\pi x - 2\pi a). \end{aligned}$$

Note that we may drop the absolute value in the final equality since $2\cos(2\pi x - 2\pi a) \leq 2$ for all $x, a \in [0, 1)$. Thus, as x approaches a in $[0, 1)$, we have that

$$\lim_{x \rightarrow a} \|f(x) - f(a)\| = \lim_{x \rightarrow a} (2 - 2\cos(2\pi x - 2\pi a)) = 2 - 2\cos(0) = 0,$$

and hence f is continuous and therefore Borel measurable.

(2): I claim that R_α is a homeomorphism on \mathbb{R}^2 , from which it will follow immediately that it induces a bijection on S^1 . First, notice that rotation any point $x \in \mathbb{R}^2$ first by $\alpha \in (0, 2\pi)$ and then by $2\pi - \alpha$ gives back x , i.e. $R_{2\pi-\alpha} \circ R_\alpha = \text{id}_{\mathbb{R}^2}$. To see this more rigorously, we can realize $R_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as the \mathbb{R} -linear map given by left multiplication by

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

in which case the composition R_α with $R_{2\pi-\alpha}$ is the matrix product

$$\begin{aligned} \begin{pmatrix} \cos(2\pi - \alpha) & -\sin(2\pi - \alpha) \\ \sin(2\pi - \alpha) & \cos(2\pi - \alpha) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} &= \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\alpha) + \sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) \\ -\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha) & \sin^2(\alpha) + \cos^2(\alpha) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

We have a similar result for the composition $R_\alpha \circ R_{2\pi-\alpha}$. Since linear maps are continuous on \mathbb{R}^2 (this is a fact from undergraduate analysis that I feel doesn't warrant proof) R_α is a continuous map with continuous inverse, and is hence a homeomorphism.

Finally, note that R_α fixes S^1 , which was implicitly assumed by the problem statement.

Now suppose that $A \subseteq S^1$ is an open set. This means there must be some open set $U \subseteq \mathbb{R}^2$ such that $A = S^1 \cap U$. Since R_α is a homeomorphism on \mathbb{R}^2 , $R_\alpha(U) = R_{2\pi-\alpha}^{-1}(U)$, which is open by the continuity of $R_{2\pi-\alpha}$. Since R_α fixes S^1 ,

$$R_\alpha(A) = R_\alpha(U \cap S^1) = R_\alpha(U) \cap S^1 = R_{2\pi-\alpha}^{-1}(U) \cap S^1,$$

which is open in the subspace topology on S^1 . Likewise, if $R_\alpha(A)$ is open, then $R_{2\pi-\alpha}^{-1}(R_\alpha(A)) = A$ is open.

The Borel algebra on S^1 is generated by open sets, and since the maps $A \mapsto R_\alpha(A)$ and $R_\alpha(A) \mapsto A$ send open sets to open (and hence measurable) sets, by Proposition 1.10 in the notes we conclude that R_α induces a bijection on S^1 .

(3):

□

EXERCISE 2.3

EXERCISE 2.4

EXERCISE 2.5