Notes for Probability Prelim

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Contents

Professor: Gordan Z Email: gordanz@math.utexas.edu Administrative stuff only today.

- Exams will be take-home, there will be midterms. Dates will be decided upon later.

- Professor will suggestion books to supplement notes. Many books available. You can get by with the lecture notes, but not a recommended strategy.

- Real analysis is the big hurdle for the course.

§ *Entry 2*

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1 Measurable Spaces

The **sample space** of a probability problem is the set of all possible outcomes of an experiment, typically denotes Ω.

Definition 1.1. Suppose we have a nonempty set S and a set $\mathcal{S} \subseteq 2^S$. We say that $\mathcal{S}$ is an **algebra** if

(1) $\emptyset, S \in \mathcal{S}$

(2) $A \in \mathcal{S} \implies A^c \in \mathcal{S}$

(3) $\mathcal{S}$ is closed under finite unions, i.e. $A, B \in \mathcal{S} \implies A \cup B \in \mathcal{S}$.

Definition 1.2. Let S be a set. A **sigma algebra** is a collection $\mathcal{S}$ of subsets of S which is an algebra and is additionally closed under countable unions: $A_n \in \mathcal{S}$ for $n \in \mathbb{N}$ implies $\bigcup A_n \in \mathcal{S}$.

Example 1.3. Let $S = \mathbb{Z}$ and set $\mathcal{S}$ to be the collection of open and closed sets under the Zariski topology. Then $\mathcal{S}$ is an algebra but not a sigma algebra.

Definition 1.4. $\mathcal{S}$ is a π-system if $A, B \in \mathcal{S}$ implies that $A \cap B \in \mathcal{S}$.

Definition 1.5. $\mathcal{S}$ is a lambda system if

(1) $S \in \mathcal{S}$

(2) if $A, B \in S$ and $A \subseteq B$ then $B \setminus A \in \mathcal{S}$

(3) if $(A_n)_{n \in \mathbb{N}} \in \mathcal{S}$ and $A_n \nearrow A$ then $A \in \mathcal{S}$

Notice that every sigma algebra is a lambda set but not every lambda set is a sigma algebra. Simply any nested sequence of sets and their set differences inside an ambient space (NOT VERIFIED but I think this works).

§ *Entry 3*

We would like to generate a σ-algebra. This is the approach to building a measure space, we start by defining a collection $\mathcal{A} \subseteq 2^S$ and we'd like to generate a σ-algebra containing $\mathcal{A}$. Morally, we're specifying a collection of sets which we know how to measure and then adding in everything else we should be able to measure given our methods for measuring simple sets.

Definition 1.6. Let S be a set and $\mathcal{A} \subseteq 2^S$. The **sigma algebra generated by** $\mathcal{A}$ is denoted $\sigma(\mathcal{A})$ and is the smallest σ-algebra containing $\mathcal{A}$

$$\sigma(\mathcal{A}) = \bigcap_{\substack{\mathcal{A} \subseteq \mathcal{B} \\ \mathcal{B} \text{ is a sigma algebra}}} \mathcal{B}.$$

Remark 1.7. One cannot simply take all unions and complements to generate a σ-algebra. To do such a thing algorithmically, one needs transfinite induction. Instead, simply define the σ-algebra generated by $\mathcal{A}$ to be the intersection, and accept that we don't really understand the collection of everything in there.

Definition 1.8. Let X be a topological space with topology $\mathcal{T}$. The **Borel sigma algebra** of X is denoted $\mathcal{B}(X)$ or $\mathcal{B}(\mathcal{T})$ and is the sigma algebra generated by the open sets of X. We say that $B \subseteq X$ is a **Borel set** if $B \in \mathcal{B}$.

Remark 1.9. No set you can construct in finite words in $\mathbb{R}^n$ is not going to be a Borel set. To give an example of a set which is not Borel, one needs serious set theoretic machinery, which is scary and foreign to the author of these notes. However, $|\mathcal{B}(\mathbb{R}^n)| < |2^{\mathbb{R}^n}|$. So it is that in one sense, $\mathcal{B}(\mathbb{R}^n)$ is almost all of $2^{\mathbb{R}^n}$, and in another, it is almost none of it.

Example 1.10. The following are all valid generating sets of $\mathcal{B}(\mathbb{R})$:

- $\mathcal{A}_1 = \{\text{open subsets of} \mathbb{R}\}$

- $\mathcal{A}_2 = \{\text{open intervals}\}$

- $\mathcal{A}_3 = \{\text{closed intervals}\}$

- $\mathcal{A}_4 = \{(a, b) \mid a, b \in \mathbb{Q}\}.$

All of these $\mathcal{A}$ satisfy $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$.

Analysts define sigma algebras in order to solve the Banach-Tarski paradox, in some sense. The goal is to strictly define which sets are measurable and which are not. In probability, we have another use for sigma algebras, we think of them abstractly as ways of organizing data/information, very much like topologies.

Definition 1.11. Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be σ-algebras. A function $f : S \to T$ is said to be **measurable** if $A \in \mathcal{T} \implies f^{-1}(A) \in \mathcal{S}$, if pullbacks of measurable sets are measurable.

GO BACK AND THINK ABOUT NATURALITY OF THIS DEFINITION Measurable functions are the morphisms in the category of σ-algebras. asdf