Solutions to Example Sheet 1.

We describe the Zariski spectra of the rings $\mathbb{R}[x]$, $\mathbb{C}[x, y]$, $\mathbb{Z}[x]$, and $\mathbb{C}[\![x]\!]$.

1. $\mathbb{R}[x]$ is a PID, so any prime is generated by an irreducible polynomial. We have the zero ideal (0) and maximal ideals generated by a linear polynonomial $x - a$, $a \in \mathbb{R}$ or an irreducible quadratic polynomial $(x - c)(x - \bar{c})$ for $c \in \mathbb{C} \setminus \mathbb{R}$. Thus the closed points are in one-to-one correspondence with orbits of Galois conjugation acting on $\mathbb{C}$. The closed sets are either finite or everything.

To fully understand $\mathbb{C}[x, y]$, we need to use some dimension theory from commutative algebra (most necessary results can be found in Chapter 11 of Atiyah and MacDonald). First, $\mathbb{C}[x, y]$ is a ring of Krull dimension 2. Prime ideals of height 2 are of the form $(x - a, y - b)$ by Hilbert's Nullstellensatz, so these are in one-to-one correspondence with the points of $\mathbb{C}^2$. Primes of height 1 are principal as $\mathbb{C}[x, y]$ is a UFD. Thus they are generated by a single irreducible polynomial. Note that this is not a hard fact: if x is an element in the prime, write out an irreducible factorization of it. One such irreducible must be contained in the prime, and that has to be a generator for the whole ideal because of the height 1 condition.

If $f \in \mathbb{C}[x, y]$ is irreducible, then the closure of the point (f) is $V((f))$, and this consists of (f) and all maximal ideals $(x-a, y-b)$ with $f(a, b) = 0$. Finally there is the zero ideal (0). The closure of this point is all of $\operatorname{Spec} \mathbb{C}[x, y]$. By primary decomposition (Atiyah and MacDonald Chapter 4) it follows that all closed sets are finite unions of the closed sets described above.

As $\dim \mathbb{Z} = 1$, $\dim \mathbb{Z}[x] = 2$. First note that if $\mathfrak{p} \subseteq \mathbb{Z}[x]$ is prime, then so is $\mathfrak{p} \cap \mathbb{Z}$, and we split into cases:

Case I. $\mathfrak{p} \cap \mathbb{Z} = (p)$ for a prime number p. Then as $(p) \subseteq \mathfrak{p}$, $\mathfrak{p}$ is the inverse image of $\mathfrak{p}' = \mathfrak{p}/(p)$ in $\mathbb{F}_p[x]$. As $\mathbb{F}_p[x]$ is a UFD of dimension one, the prime ideals are either (0) or $(\bar{f})$ for an irreducible polynomial $\bar{f} \in \mathbb{F}_p[x]$. In these two cases, $\mathfrak{p} = (p)$ or, lifting $\bar{f}$ to $f \in \mathbb{Z}[x]$, $\mathfrak{p} = (p, f)$.

Case II. $\mathfrak{p} \cap \mathbb{Z} = (0)$. In this case, localizing $\mathbb{Z}[x]$ at the set $S = \mathbb{Z} \setminus \{0\}$ yields $\mathbb{Q}[x]$, and there is a one-to-one correspondence between prime ideals of $\mathbb{Z}[x]$ disjoint from S and prime ideals of $\mathbb{Q}[x]$. The latter consists of the ideals (0) or (f) for f an irreducible polynomial over $\mathbb{Q}$. We can clear denominators and assume $f \in \mathbb{Z}[x]$, in which case $\mathfrak{p} = (0)$ or (f) in the two cases.

Note then that a maximal ideal in $\mathbb{Z}[x]$ corresponds to a Galois orbit of solutions to a polynomial equation with coefficients in $\mathbb{F}_p$ for some prime p, while a prime ideal of the form (f) with f irreducible in $\mathbb{Q}[x]$ corresponds to a Galois orbit of solutions to the equation over $\mathbb{Q}$.

In terms of the topology, the closure of the set $\{(p)\}$ consists of all points "over p", i.e., (p) plus the maximal ideals (p, f), and the closure of (f) consists of ideals of the form (p, f') where f' is a lift of an irreducible factor of f modulo p. Again, by primary decomposition, any closed set is a finite union of closed sets of the above form.

The power series ring $\mathbb{C}[\![x]\!]$ is an integral domain so (0) is prime. The quotient by (x) is $\mathbb{C}$ so (x) is prime and maximal. There are no other primes. To see this, use the fact that a power series that is not in (x) has non-zero constant term and therefore is invertible.

2. Proving that the preimage of a prime is prime is a formal exercise; given $\varphi : A \to B$ and elements x, y in A such that $xy \in \varphi^{-1}(\mathfrak{p})$, $f(x)f(y)$ must lie in the prime $\mathfrak{p}$, so by primality of p either x or y lies in $f^{-1}(\mathfrak{p})$.

The key example to keep in mind for homomorphisms $A \to B$ where the preimage of a maximal is not maximal are localizations at (the complement) a prime ideal. Localization at a prime turns that prime into a maximal ideal. The silliest first example of this is $\mathbb{Z} \hookrightarrow \mathbb{Q}$, but you can try to localize $\mathbb{C}[x, y]$ at (y) to see it in a more geometric fashion.

4. We have already seen in the first question that $\mathbb{C}[\![x]\!]$ has spectrum equal to the connected doubleton. The $\star$-question is significantly harder. A first guess is to take $\mathbb{C}[\![x, y]\!]$ but this has one too many points, and one of the height 1 primes has to be removed. The correct answer is to first take the monoid $\mathbb{Z}^2_{\mathrm{lex},\geq 0}$ of lexicographically positive elements in $\mathbb{Z}^2$ (draw a grid and write these out!) and then examine the power series ring with exponents in this monoid, denoted $\mathbb{C}[\![\mathbb{Z}^2_{\mathrm{lex},\geq 0}]\!]$.

Care is required to define this power series ring. One can take the set of power series

$$f = \sum_{u \in \mathbb{Z}^2_{\mathrm{lex},\geq 0}} a_u \chi^u,$$

where a_u is any complex number, with the addition property that the set of non-zero u appearing in each sum is well-ordered – "well-ordered support".

One easily checks that the arithmetic operations are well-defined because of the support condition. There are two lessons to learn from this question, and it has exactly these prime ideals (you can do this very explicitly). First, removing sets that are neither open nor closed from a scheme can have unexpected outcomes. Second, there are more exotic rings in the world that the ones you know. The Hahn series rings are what you want to Google.

7. A non-zero presheaf with stalks equal to 0 can be defined on the discrete space D with 2 elements. Specify the value on the two points to be 0 and the value on the D to be non-zero. The stalks are all zero, and so is the sheafification.

8 and 9. I will first first show 9, which states that given a morphism of sheaves the image sheaf is a subsheaf of the codomain sheaf. The key is really to show that if $f : \mathcal{F} \to \mathcal{G}$ is an injective map of presheaves, the corresponding map $f^+ : \mathcal{F}^+ \to \mathcal{G}^+$ of sheaves is injective. But $f : \mathcal{F} \to \mathcal{G}$ injective implies $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. By 4, we know this is the same as $f_x^+ : \mathcal{F}_x^+ \to \mathcal{G}_x^+$, and thus f^+ is injective, as we showed in lecture that a morphism of sheaves was injective if it was injective on stalks.

Now suppose $\mathcal{F}$, $\mathcal{G}$ are sheaves. Clearly the presheaf image $im(f)$ of f is a sub-presheaf of $\mathcal{G}$, and thus $im(f)^+$ is a subsheaf of $\mathcal{G}^+ = \mathcal{G}$.

Ok now back to 8. We will show that exactness is stalk local. Write $f_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ for each i. Exactness of the sequence is precisely the statement that $\operatorname{im} f_{i-1} = \ker f_i$. Suppose the sequence is exact. We wish to show it is exact after taking stalks at x. We asserted in lecture (and I told you to become happy with the fact) that kernels and images commute with taking stalks, so

$$\operatorname{im} f_{i-1,x} = (\operatorname{im} f_{i-1})_x = (\ker f_i)_x = \ker f_{i,x},$$

so the sequence is exact at the level of stalks.

Conversely, suppose the sequence is exact at the level of stalks. The first claim is that $f_i \circ f_{i-1} = 0$. It is sufficient to show that $\ker f_i \circ f_{i-1} = \mathcal{F}_{i-1}$. But

$$\mathcal{F}_{i-1,x} = \ker(f_{i,x} \circ f_{i-1,x}) = \ker(f_i \circ f_{i-1})_x = (\ker f_i \circ f_{i-1})_x,$$

showing the natural inclusion $\ker f_i \circ f_{i-1} \hookrightarrow \mathcal{F}_{i-1}$ is an isomorphism. This shows $f_i \circ f_{i-1} = 0$. In particular, this implies that the presheaf image of f_{i-1} is contained in the kernel of f_i, which by the argument of 5 implies that $\operatorname{im} f_{i-1} \subseteq \ker f_i$. This inclusion is an isomorphism on the level of stalks, again by exactness at the level of stalks, and hence we have $\operatorname{im} f_{i-1} = \ker f_i$.

10. We have essentially already shown that a morphism of sheaves is an isomorphism if and only if it is both surjective and injective. Formally, we showed that

$f : \mathcal{F} \to \mathcal{G}$ was injective (surjective) if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ was injective (surjective) for all $x \in X$. We also showed that f was an isomorphism if f_x was an isomorphism for all $x \in X$. There's that.

11. Two comments before we get started. There is a general theory about adjunctions in category theory. I personally don't find that language all that illuminating but if you like category theory, you can go and look it up. Second, you should keep two examples in mind always when proving statements such as these. First, when the morphism $X \to Y$ is the inclusion of a point and when it is a map from a general X to a point. Run the inverse image and pushforward constructions to see how these two behave rather differently.

Onto the adjunction. Define $\Psi : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ as follows. On an open set U, a section in $f^{-1}f_*(\mathcal{F})(U)$ is represented by a pair (V, s) where V is an open neighbourhood of $f(U)$ and $s \in f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Then as $U \subseteq f^{-1}(V)$, we obtain a section $s|_U \in \mathcal{F}(U)$. It is easy to check this is well-defined and gives a morphism of sheaves.

Define $\Phi : \mathcal{G} \to f_*f^{-1}\mathcal{G}$ on an open set U as follows. Note U is an open neighbourhood of $f(f^{-1}(U))$. Thus we can map $s \in \mathcal{G}(U)$ to the representative $(U, s) \in f_*f^{-1}(\mathcal{G})(U) = f^{-1}\mathcal{G}(f^{-1}(U))$.

Now given $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$, we get functorially a morphism $f_*\varphi : f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$, defined in the obvious way, and this can be composed on the left by Φ, giving a morphism $\mathcal{G} \to f_*\mathcal{F}$. Conversely, given $\psi : \mathcal{G} \to f_*\mathcal{F}$, we obtain functorially $f^{-1}\psi : f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F}$, which we compose on the right by Ψ. This gives maps

$$F : \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}), \quad G : \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}).$$

We need to show FG and GF are the identity. Given $\psi : \mathcal{G} \to f_*\mathcal{F}$, $FG(\psi)$ is obtained as the composition of natural maps

$$\mathcal{G} \xrightarrow{\Phi} f_*f^{-1}\mathcal{G} \xrightarrow{f_*f^{-1}\psi} f_*f^{-1}f_*\mathcal{F} \xrightarrow{f_*\Psi} f_*\mathcal{F}.$$

Unwinding the definitions, for $s \in \mathcal{G}(U)$, this sequence of maps can be viewed as taking $s \mapsto (U, s) \mapsto (U, \psi(s)) \mapsto \psi(s)$, which is ψ. Here $(U, \psi(s))$ can be viewed as representing a section of $f^{-1}f_*\mathcal{F}$ over $f^{-1}(U)$. Similarly, given $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$, $GF(\varphi)$ is given as a composition

$$f^{-1}\mathcal{G} \xrightarrow{f^{-1}\Phi} f^{-1}f_*f^{-1}\mathcal{G} \xrightarrow{f^{-1}f_*\varphi} f^{-1}f_*\mathcal{F} \xrightarrow{\Psi} \mathcal{F}.$$

For an element of $f^{-1}\mathcal{G}(U)$ represented by (U, s) for $s \in \mathcal{G}(V)$, $V \supseteq f(U)$ an open set, this sequence of maps is given by $(V, s) \mapsto (V, s) \mapsto (V, \varphi(s)) \mapsto \varphi(s)$, suitably interpreted at each step.

[Part of the point of writing out this solution is that this level of detail is all that is expected, nothing more, but probably not much less.]