F-SINGULARITIES: A COMMUTATIVE ALGEBRA APPROACH (PRELIMINARY VERSION)

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Contents

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Introduction

The action of Frobenius on a ring of positive characteristic has a long history of being used to characterize the singularities of the associated varieties. Work of Kunz shows that a Noetherian ring R of characteristic $p > 0$ is regular if and only if the Frobenius map on R is flat [Kun69]. The use of Frobenius was also applied to several important questions, for example the study invariants rings under group actions [HR74] and the study of of cohomological dimension [HS77] in positive characteristic.

With the development of tight closure theory [HH90] [HH94a] [HH94b] there was an explosion in the understanding of singularities via the Frobenius map, and a number of classes of singularities were formally introduced, which include (strongly) F-regular, F-rational, F-pure and F-injective singularities. We give an introduction on these "F-singularities". Our approach is completely algebraic, and most of the results are stated and proved in the greatest general form.

Unless otherwise stated, all rings are assumed to be commutative, Noetherian, with multiplicative identity 1*, and of prime characteristic* $p > 0$*.*

1. Kunz's theorem and F-finite rings

Rings of prime characteristic $p > 0$ come equipped with a special endomorphism, namely the Frobenius endomorphism $F : R \to R$ defined by $F(r) = r^p$. For each $e \in \mathbb{N}$ we can iterate the Frobenius endomorphism e times and obtain the e-th Frobenius endomorphism $F^e : R \to R$ defined by $F^e(r) = r^{p^e}$. Roughly speaking, the study of prime characteristic rings is the study of algebraic and geometric properties of the Frobenius endomorphism.

In this note, we often need to distinguish the source and target of the Frobenius map. We adopt the commonly used notation F^e_*R to denote the target of the Frobenius as a module over the source, that is, $F^e\colon R \to F^e_*R$. Under this notation, elements in F^e_*R are denoted by F^e_*r where $r \in R$, and the R-module structure on F^e_*R is defined via $r_1 \cdot F^e_*r_2 = F^e_*(r_1^{p^e}r_2)$. On the other hand, $F^e_*R \cong R$ via $F^e_*r \leftrightarrow r$ as rings.

Suppose that R is reduced and let K be the total ring of fractions of R, thus $K = \prod K_i$ is a product of fields. Let $\overline{K} := \prod \overline{K_i}$. There are inclusions $R \subseteq K \subseteq \overline{K}$. We let

$$R^{1/p^e} := \{s \in \overline{K} \mid s^{p^e} \in R\}.$$

In other words, R^{1/p^e} is the collection of p^e-th roots of elements of R. Then R^{1/p^e} is unique up to non-unique isomorphism, and $R^{1/p^e} \cong R$ via $r^{1/p^e} \leftrightarrow r$ as rings. In this setup, we can view the Frobenius map as the natural inclusion $R \hookrightarrow R^{1/p^e}$.

As we already mentioned, the singularities of R are often studied via the behavior of the Frobenius map. A fundamental result in this direction is Kunz's theorem.

Theorem 1.1 ([Kun69])**.** *A prime characteristic ring R is regular if and only if the Frobenius map $F^e\colon R \to F^e_*R$ is flat for some (or equivalently, all) $e > 0$.*

Proof. First assume that R is regular, we want to show that F^e_*R is a flat R-module. Since flatness can be checked locally and we have $(F^e_*R)_P \cong F^e_*(R_P)$ as R_P-modules for all $P \in \operatorname{Spec}(R)$, we may assume $(R, \mathfrak{m})$ is local. We next consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & F^e_*R \arrow[d] \\ \widehat{R} \arrow[r] & F^e_*\widehat{R} \end{tikzcd}$$

Since both vertical maps are faithfully flat, if we can show the bottom map is flat, then it will imply that the top map is flat. Therefore we may replace R by $\widehat{R}$ to assume $(R, \mathfrak{m})$ is a complete regular local ring. By Cohen's structure theorem, $R \cong k[[x_1, \dots, x_d]]$. In this case, it is straightforward to check that F^e_*R is a free R-module with basis

$$\{F^e_*(\lambda x_1^{i_1} \cdots x_d^{i_d}) \mid 0 \le i_j < p^e, \text{ where } \{F^e_*\lambda\} \text{ is a free basis of } F^e_*k \text{ over } k\}.$$

Now we prove the converse. Note that if F^e_*R is flat over R for some e, then after iterating we see that F^{ne}_*R is flat over R for all n. In particular, we can assume F^e_*R is flat over R for infinitely many $e > 0$. Since regularity and flatness are local conditions, we may again assume that $(R, \mathfrak{m})$ is a local ring. Let $g = \operatorname{depth} R$. We pick a regular sequence in $\mathfrak{m}$ of maximal length: $x_1, \ldots, x_g$. It follows that $R/(x_1, \ldots, x_g)$ has depth 0 and thus $0 \neq N := \operatorname{Soc}(R/(x_1, \ldots, x_g)) \cong \operatorname{Hom}_R(R/\mathfrak{m}, R/(x_1, \ldots, x_g))$. Hence there exists n such that $N \nsubseteq \mathfrak{m}^n(R/(x_1, \ldots, x_g))$.

Claim 1.2. *For any finitely generated R-module M of infinite projective dimension with minimal free resolution*

$$\cdots \to R^{n_{g+2}} \xrightarrow{\phi_{g+2}} R^{n_{g+1}} \xrightarrow{\phi_{g+1}} R^{n_g} \to \cdots \to R^{n_1} \to R^{n_0} \to M \to 0,$$

the entries in the matrix representing ϕ_{g+2} are not all contained in $\mathfrak{m}^n$.

Proof of Claim. Since $\operatorname{pd}_R R/(x_1, \ldots, x_g) = g$, we have $\operatorname{Tor}^R_{g+1}(M, R/(x_1, \ldots, x_g)) = 0$. Therefore tensoring the above minimal free resolution with $R/(x_1, \ldots, x_g)$, we know that

$$(R/(x_1, \ldots, x_g))^{n_{g+2}} \xrightarrow{\phi_{g+2}} (R/(x_1, \ldots, x_g))^{n_{g+1}} \xrightarrow{\phi_{g+1}} (R/(x_1, \ldots, x_g))^{n_g}$$

is exact in the middle. Since the resolution is minimal, the socle $N^{n_{g+1}} \subseteq (R/(x_1, \ldots, x_g))^{n_{g+1}}$ is contained in $\operatorname{Ker} \phi_{g+1} = \operatorname{Im} \phi_{g+2}$. If all entries in the matrix representing ϕ_{g+2} are contained in $\mathfrak{m}^n$, then $N^{n_{g+1}} \subseteq \mathfrak{m}^n(R/(x_1, \ldots, x_g))^{n_{g+1}}$ and thus $N \subseteq \mathfrak{m}^n(R/(x_1, \ldots, x_g))$. This is a contradiction. $\square$

We now continue the proof of the theorem. Suppose $\operatorname{pd}_R R/\mathfrak{m} = \infty$. Since the Frobenius map is flat, tensoring a minimal free resolution of $R/\mathfrak{m}$ with F^e_*R and identifying F^e_*R with R, we obtain a minimal free resolution of $R/\mathfrak{m}^{[p^e]}$ such that the entries in the matrix representing each differential (in particular the $(g+2)$-th differential) are all contained in $\mathfrak{m}^{[p^e]}$, the ideal generated by p^e-th powers of elements of $\mathfrak{m}$. But for $e \gg 0$ this contradicts Claim 1.2 because n is independent of e. Therefore $\operatorname{pd}_R R/\mathfrak{m} < \infty$ and thus R is regular. $\square$

Remark 1.3. Our proof of the converse direction in Theorem 1.1 follows from [KL98] (which originates from ideas in [Her74]).

We next introduce a rather "mild" condition on the Frobenius map.

Definition 1.4. R is called *F-finite* if for some (or equivalently, all) $e > 0$, the Frobenius map $F^e \colon R \to R$ is a finite morphism, i.e., F^e_*R is a finitely generated R-module.

For example, a field k is F-finite if and only if $[k^{1/p} : k] < \infty$. More generally, it follows from Exercise 4 below (and Cohen's structure theorem) that rings essentially finite type over F-finite fields are F-finite, and complete local rings with F-finite residue fields are F-finite.

The F-finite property turns out to imply that the rings are not pathological. We will sometimes implicitly use the following two results throughout. We will give proofs of these results in Section 12.

Theorem 1.5 ([Gab04])**.** *If R is F-finite, then R is a homomorphic image of an F-finite regular ring. As a consequence, F-finite rings admit canonical modules.*

Recall that a ring R (not necessarily of characteristic $p > 0$) is called *excellent* if R satisfies the following:

(1) R is universally catenary.
(2) If S is an R-algebra of finite type, then the regular locus of S is open in $\operatorname{Spec}(S)$.
(3) For all $P \in \operatorname{Spec}(R)$, the map $R_P \to \widehat{R_P}$ has geometrically regular fibers. That is, for all $Q \in \operatorname{Spec}(R)$ such that $Q \subseteq P$, $\kappa(Q)' \otimes_{R_P} \widehat{R_P}$ is regular for all finite (or equivalently, finite and purely inseparable) field extensions $\kappa(Q)'$ of $\kappa(Q)$.

Excellent rings include most examples arising from algebraic geometry. For example, all rings essentially finite type over a field and all complete local rings are excellent.

Theorem 1.6 ([Kun76])**.** *If R is F-finite, then R is excellent. Moreover, if $(R, \mathfrak{m})$ is local, then R is F-finite if and only if R is excellent and $R/\mathfrak{m}$ is F-finite.*

We end this section with a few exercises.

Exercise 1. Verify that if $F^e_* R$ is finitely generated for one $e > 0$, then $F^e_* R$ is finitely generated for all $e > 0$.

Exercise 2. Prove that R is reduced if and only if $R \to F^e_* R$ is injective for one (or equivalently, all) $e > 0$.

Exercise 3. Prove that R is F-finite if and only if $R_{\mathrm{red}} := R/\sqrt{0}$ is F-finite.

Exercise 4. Let R be an F-finite ring. Prove the following:

(1) If $I \subseteq R$ an ideal then R/I is F-finite.
(2) If W a multiplicative subset of R then $W^{-1}R$ is F-finite.
(3) If x an indeterminate then $R[x]$ and $R[[x]]$ are F-finite.

As a consequence of Exercise 4, we see that all rings essentially of finite type over an F-finite ring are F-finite.

2. F-pure rings and Fedder's criterion

In this section, we introduce and study F-pure singularities.

Definition 2.1. A map of R-modules $M_1 \to M_2$ is *pure* if $M_1 \otimes_R N \to M_2 \otimes_R N$ is injective for every R-module N. The ring R is called *F-pure* if the Frobenius map $F^e\colon R \to F^e_* R$ is pure for some (or equivalently, all) $e > 0$. R is called *F-split* if the Frobenius map $F^e\colon R \to F^e_* R$ is split for some (or equivalently, all) $e > 0$.

Clearly, a split map is always pure, hence F-split implies F-pure. Moreover, if R is F-pure then in particular the Frobenius map is injective and thus R is reduced. So in this case we can always view the Frobenius map as the natural inclusion $R \hookrightarrow R^{1/p^e}$. Therefore R is F-pure if and only if R is reduced and the natural map $R \to R^{1/p^e}$ is pure for some (or equivalently, all) $e > 0$. Similarly, R is F-split if R is reduced and $R \to R^{1/p^e}$ is split for some (or equivalently, all) $e > 0$.

It turns out that F-split and F-pure are equivalent in most cases of interest. To establish this we prove a general criterion for purity of maps.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module. Then a map $R \to M$ is pure if and only if the induced map $E \to E \otimes_R M$ is injective where $E := E_R(R/\mathfrak{m})$ denotes the injective hull of the residue field.*

Proof. One direction is obvious. So suppose $R \to M$ is not pure, then there exists an R-module N such that $N \to N \otimes_R M$ is not injective. Since N is a directed union of its finitely generated submodules and injectivity is preserved under direct limit, we may assume N is finitely generated. Now we pick $u \in \operatorname{Ker}(N \to N \otimes_R M)$, there exists n such that $u \notin \mathfrak{m}^n N$. Consider the commutative diagram:

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N \otimes_R M \arrow[d] \\ N/\mathfrak{m}^n N \arrow[r] & (N/\mathfrak{m}^n N) \otimes_R M \end{tikzcd}$$

Since the image of $u \in N/\mathfrak{m}^n N$ is nonzero, the bottom map is not injective. Now $N/\mathfrak{m}^n N$ has finite length, so it embeds in $E^{\oplus r}$ for some r. The commutative diagram

$$\begin{tikzcd} N/\mathfrak{m}^n N \arrow[r] \arrow[d, hook] & (N/\mathfrak{m}^n N) \otimes_R M \arrow[d] \\ E^{\oplus r} \arrow[r] & E^{\oplus r} \otimes_R M \end{tikzcd}$$

then shows that the bottom map is not injective. Thus $E \to E \otimes_R M$ is not injective. $\square$

Corollary 2.3. *Let $(R,\mathfrak{m})$ be a local ring. Then R is F-pure if and only if $\widehat{R}$ is F-pure.*

Proof. We have canonical isomorphisms $E := E_R(R/\mathfrak{m}) \cong E_R(R/\mathfrak{m}) \otimes_R \widehat{R} \cong E_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R})$. Thus we have

$$E \to E \otimes_R F^e_* R \to E \otimes_R F^e_* \widehat{R} \cong E \otimes_{\widehat{R}} F^e_* \widehat{R}.$$

Since $F^e_* R \to F^e_* \widehat{R}$ is faithfully flat and hence pure (see Exercise 9), the second map is injective. Hence the composition is injective if and only if the first map is injective. Therefore the conclusion follows from Proposition 2.2. $\square$

Corollary 2.4. *Let $R \to M$ be a pure map. If either R is complete local or M is finitely generated, then $R \to M$ is split. In particular, if R is F-pure and is either complete local or F-finite, then R is F-split.*

Proof. If $(R,\mathfrak{m})$ is complete local, then taking the Matlis dual of the injection $E \hookrightarrow E \otimes_R M$ yields a surjection $\operatorname{Hom}_R(E \otimes_R M, E) \twoheadrightarrow \operatorname{Hom}_R(E,E) \cong R$. By adjunction we have

$$\operatorname{Hom}_R(E \otimes_R M, E) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(E,E)) \cong \operatorname{Hom}_R(M,R).$$

Thus we have a surjection $\operatorname{Hom}_R(M,R) \twoheadrightarrow R$, one can check that this is precisely the natural map induced by applying $\operatorname{Hom}_R(-,R)$ to $R \to M$. Thus $R \to M$ is split.

Next we assume M is finitely generated. We want to show that the map $\operatorname{Hom}_R(M,R) \to R$ is surjective. It is enough to check this locally on $\operatorname{Spec}(R)$. We have

$$R_P \otimes_R \operatorname{Hom}_R(M,R) \cong \operatorname{Hom}_{R_P}(M_P, R_P).$$

Since $R \to M$ is pure implies $R_P \to M_P$ is pure for all $P \in \operatorname{Spec}(R)$ (see Exercise 10), we may thus assume that R is local. But then the surjectivity of $\operatorname{Hom}_R(M,R) \to R$ can be checked after base change to $\widehat{R}$. Since M is finitely generated, we know that

$$\widehat{R} \otimes_R \operatorname{Hom}_R(M,R) \cong \operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \widehat{R}).$$

Therefore it remains to show that $\operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \widehat{R}) \to \widehat{R}$ is surjective. But since $R \to M$ is pure, $\widehat{R} \to M \otimes_R \widehat{R}$ is pure, and hence split by the first conclusion. So $\operatorname{Hom}_{\widehat{R}}(M \otimes_R \widehat{R}, \widehat{R}) \to \widehat{R}$ is surjective as wanted. $\square$

Since faithfully flat maps are always pure (see Exercise 9 below), regular rings are F-pure by Theorem 1.1. Thus by Corollary 2.4, complete regular local rings and F-finite regular rings are F-split. However, we warn the reader that there are examples of regular local rings (even DVRs) that are *not* F-split. The first such example was discovered by Datta–Smith [DS16] who constructed a non-excellent DVR that is not F-split. Very recently, Datta–Murayama [DM20] constructed an excellent, local, henselian DVR that is not F-split. Thus without the assumptions of Corollary 2.4, it frequently happens that F-pure rings fail to be F-split. We will not treat these examples in this note though: for most questions that we will study, one can first localize and then complete (or simply work with F-finite rings, which include most examples) so Corollary 2.4 can be applied to tell us that we do not need to distinguish between F-pure and F-split.

We next state and prove a fundamental result of Fedder [Fed83].

Theorem 2.5 (Fedder's criterion)**.** *Let $(S,\mathfrak{m})$ be a regular local ring and let $I \subseteq S$ be an ideal. Then $R := S/I$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ where $I^{[p]}$ is the ideal generated by p-th powers of elements of I.*

Proof. We first assume $(S,\mathfrak{m})$ is a complete regular local ring with perfect residue field. By Cohen's structure theorem, $S \cong k[[x_1,\dots,x_d]]$ and we know that F_*S is a finite free S-module with basis $\{F_*(x_1^{i_1}\cdots x_d^{i_d}) \mid 0 \le i_j < p\}$.

Claim 2.6. *For each tuple $(i_1,\dots,i_d)$ with $0 \le i_1,\dots,i_d < p$ there is an S-linear map $\varphi_{(i_1,\dots,i_d)}\colon F_*S \to S$ which is defined on basis elements as follows:*

$$\varphi_{(i_1,\dots,i_d)}(F_*(x_1^{j_1}\cdots x_d^{j_d})) = \begin{cases} 1 & (j_1,\dots,j_d) = (i_1,\dots,i_d) \\ 0 & (j_1,\dots,j_d) \neq (i_1,\dots,i_d) \end{cases}.$$

Moreover, $\operatorname{Hom}_S(F_*S,S) \cong (F_*S)\cdot\Phi$ *where* $\Phi = \varphi_{(p-1,\dots,p-1)}$.

Proof of Claim. The first assertion is clear and we only prove the second assertion. Since all the $\varphi_{(i_1,\dots,i_d)}$s generate $\operatorname{Hom}_S(F_*S,S)$ as an S-module, it is enough to observe that

$$\varphi_{(i_1,\dots,i_d)}(F_*\cdot -) = \Phi(F_*(x_1^{p-1-i_1}\cdots x_d^{p-1-i_d}\cdot -)) = F_*(x_1^{p-1-i_1}\cdots x_d^{p-1-i_d})\cdot\Phi.$$

Therefore Φ generates $\operatorname{Hom}_S(F_*S,S)$ as an F_*S-module as wanted. ☐

Since F_*S is a finite free S-module, every map $F_*(S/I) \to S/I$ can be lifted to a map $F_*S \to S$, and thus can be written as $\Phi(F_*(s\cdot -))$ for some $s \in S$ by Claim 2.6.

Claim 2.7. $\Phi(F_*(s\cdot -))$ *induces a map $F_*(S/I) \to S/I$ if and only if $s \in (I^{[p]} : I)$.*

Proof of Claim. If $s \in (I^{[p]} : I)$, then $\Phi(F_*(s\cdot -))$ sends F_*I to I hence it induces a map $F_*(S/I) \to S/I$. To prove the converse, suppose $r = sr' \in sI$ such that $r \notin I^{[p]}$. Since $\{F_*(x_1^{i_1}\cdots x_d^{i_d}) \mid 0 \le i_j < p\}$ is a free basis of F_*S over S, F_*r can be written uniquely as $\sum r_{i_1i_2\dots i_d}F_*(x_1^{i_1}\cdots x_d^{i_d})$ where $r_{i_1i_2\dots i_d} \in S$. Since $F_*r \notin F_*I^{[p]}$ by our choice, there exists $r_{i_1i_2\dots i_d} \notin I$ and by Claim 2.6 $\varphi_{(i_1,\dots,i_d)}(F_*r) \notin I$. But then $\Phi(F_*(rx_1^{p-1-x_1}\cdots x_d^{p-1-x_d})) \notin I$ and thus $\Phi(F_*(s\cdot r'x_1^{p-1-x_1}\cdots x_d^{p-1-x_d})) \notin I$. Therefore $\Phi(F_*(s\cdot -))$ does not send F_*I to I so it does not induce a map $F_*(S/I) \to S/I$. ☐

By Claim 2.7, S/I is F-pure (equivalently, F-split in this case by Corollary 2.4) if and only if there exists $s \in (I^{[p]} : I)$ such that $\Phi(F_*(s \cdot -))$ is surjective. But it is easy to see that $\Phi(F_*(s \cdot -))$ is surjective if and only if $s \notin \mathfrak{m}^{[p]}$: if $s \in \mathfrak{m}^{[p]}$ then the image of $\Phi(F_*(s \cdot -))$ is contained in $\mathfrak{m}$ so it cannot be surjective, while if $s \notin \mathfrak{m}^{[p]}$ then s contains a monomial $x_1^{i_1} \cdots x_d^{i_d}$ with nonzero coefficient for some $0 \le i_1, \ldots, i_d < p$, so $\Phi(F_*(s \cdot x_1^{p-1-i_1} \cdots x_d^{p-1-i_d}))$ is a unit and thus $\Phi(F_*(s \cdot -))$ is surjective. Putting all these together, we see that S/I is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.

We next treat the general case. Consider the following commutative diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \widehat{S} \cong k[[x_1, \ldots, x_d]] \arrow[r] \arrow[d] & \widetilde{S} := \overline{k}[[x_1, \ldots, x_d]] \arrow[d] \\ R = S/I \arrow[r] & \widehat{R} = \widehat{S}/I\widehat{S} \arrow[r] & \widetilde{R} := \widetilde{S}/I\widetilde{S} \end{tikzcd}$$

It is clear that all the maps in the horizontal rows are faithfully flat. Moreover, since $E_S(k) \cong k[x_1^{-1}, \ldots, x_d^{-1}]$ and similarly for $\widetilde{S}$, we have $E_{\widetilde{S}}(\overline{k}) \cong E_S(k) \otimes_S \widetilde{S}$. It follows that

$$E_R(k) \otimes_R \widetilde{R} \cong (\operatorname{Ann}_{E_S(k)} I) \otimes_S \widetilde{S} \cong \operatorname{Ann}_{E_{\widetilde{S}}(\overline{k})} I\widetilde{S} \cong E_{\widetilde{R}}(\overline{k}).$$

Therefore we have the following commutative diagram:

$$\begin{tikzcd} E_R(k) \arrow[r] \arrow[d] & E_R(k) \otimes_R F_*R \arrow[d] \\ E_{\widetilde{R}}(\overline{k}) \cong E_R(k) \otimes_R \widetilde{R} \arrow[r] & E_{\widetilde{R}}(\overline{k}) \otimes_{\widetilde{R}} F_*\widetilde{R} \cong E_R(k) \otimes_R F_*\widetilde{R} \end{tikzcd}$$

Note that a socle representative $u \in E_R(k)$ maps to a socle representative $u \otimes 1 \in E_{\widetilde{R}}(\overline{k})$. Thus u maps to zero in $E_R(k) \otimes_R F_*R$ if and only if $u \otimes 1$ maps to zero in $E_{\widetilde{R}}(\overline{k}) \otimes_{\widetilde{R}} F_*\widetilde{R}$ (the right vertical map is injective as $F_*R \to F_*\widetilde{R}$ is faithfully flat and hence pure, see Exercise 9). Thus the top map is injective if and only if the bottom map is injective. By Corollary 2.2, R is F-pure if and only if $\widetilde{R}$ is F-pure. Now $\widetilde{R} = \widetilde{S}/I\widetilde{S}$ and $\widetilde{S}$ is complete local with perfect residue field, so by what we have proved, $\widetilde{R}$ is F-pure if and only if $(I^{[p]}\widetilde{S} :_{\widetilde{S}} I\widetilde{S}) \not\subseteq \mathfrak{m}^{[p]}\widetilde{S}$. But since $S \to \widetilde{S}$ is faithfully flat, the latter holds if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$. □

Remark 2.8. We have the following graded version of Fedder's criterion: let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let $I \subseteq S$ be a homogeneous ideal. Then $R := S/I$ is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ where $\mathfrak{m} = (x_1, \ldots, x_d)$. The proof follows from the same line as in Theorem 2.5: the key point is that, when k is perfect, $\operatorname{Hom}_S(F_*S, S) \cong F_*S$ still holds and we have a graded version of Proposition 2.2 (with graded injective hull of k in place of the injective hull of k). We leave the details to the interested reader.

Remark 2.9. With the same setup as in Theorem 2.5 or Remark 2.8, it follows from the same argument that R is F-pure if and only if $(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$ for some (or equivalently, all) $e > 0$. We leave the details to the interested reader.

Fedder's criterion is *extremely* useful. We give some examples of F-pure rings.

Example 2.10.

(1) Let S be $k[[x_1, \ldots, x_d]]$ or $k[x_1, \ldots, x_d]$ and let $R = S/I$ be a Stanley-Reisner ring (i.e., I is generated by square free monomials). Then R is F-pure. The point is that $x_1x_2\cdots x_d$ is a multiple of every square free monomial, thus $(x_1\cdots x_d)^{p-1}\cdot f \in (f^p)$ for any square free monomial f. Hence $(x_1\cdots x_d)^{p-1} \in (I^{[p]} : I)$ since I is generated by square free monomials, but $(x_1\cdots x_d)^{p-1} \notin \mathfrak{m}^{[p]}$.

(2) Let R denote either $k[[x,y,z]]/(x^3+y^3+z^3)$ or $k[x,y,z]/(x^3+y^3+z^3)$. Then $(I^{[p]} : I) = (x^3+y^3+z^3)^{p-1}$. If $p \equiv 1 \bmod 3$, then there is a term $(xyz)^{p-1}$ in the monomial expansion of $(x^3+y^3+z^3)^{p-1}$ with nonzero coefficient thus R is F-pure. On the other hand, if $p \equiv 2 \bmod 3$, then one checks that $(x^3+y^3+z^3)^{p-1} \in \mathfrak{m}^{[p]} = (x^p, y^p, z^p)$ so R is not F-pure.

We end this section with various exercises about F-pure and F-split rings.

Exercise 5. Verify that $R \to F^e_*R$ is pure (resp. split) for one $e > 0$, then $R \to F^e_*R$ is pure (resp. split) for all $e > 0$.

Exercise 6. Suppose that R is F-finite and F^e_*R admits a free summand. Show that the Frobenius map $R \to F^e_*R$ is split. (Assuming F^e_*R admits a free summand is equivalent to assuming that there exists $F^e_*r \in F^e_*R$ and $\varphi : F^e_*R \to R$ so that $\varphi(F^e_*r) = 1$. We are asking you to show the existence of a map $\psi : F^e_*R \to R$ so that $\psi(F^e_*1) = 1$.)

Exercise 7. Use Fedder's criterion to show that $R = k[[x,y,z]]/(x^2+y^3+z^7)$ is not F-pure.

Exercise 8. Prove that if $R \to S$ is pure (resp. split) and S is F-pure (resp. F-split), then R is F-pure (resp. F-split).

Exercise 9. Prove that if $R \to S$ is faithfully flat, then $R \to S$ is pure. Give an example of a faithfully flat ring extension that is not split.

Exercise 10. Show that $R \to M$ is pure if and only if $R_P \to M_P$ is pure for all $P \in \operatorname{Spec}(R)$. In particular, R is F-pure if and only if R_P is F-pure for all $P \in \operatorname{Spec}(R)$. Also show that if R is F-split, then R_P is F-split for all $P \in \operatorname{Spec}(R)$.

3. F-regular rings: splitting finite extensions

In this section, we introduce and study the arguably most important class of F-singularities: strongly F-regular rings [HH90, HH94a].

Definition 3.1. An F-finite ring R is called *strongly F-regular* if for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules.

Clearly, strongly F-regular rings are F-split and in particular reduced. For local rings, we can say more.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a strongly F-regular local ring. Then R is a domain.*

Proof. Since R is reduced, it is enough to show that R has only one minimal prime. Let $P_1, \dots, P_n$ be the minimal primes of R. Suppose $n \geq 2$, we pick $f_i \in \cap_{j \neq i} P_j$. Then we have $\sum_{i=1}^n f_i$ is not contained in any minimal prime of R. Thus as R is strongly F-regular, there exists $e > 0$ and an R-linear map ϕ: $F^e_* R \to R$ such that $\phi(F^e_*(\sum_{i=1}^n f_i)) = 1$ and thus $\sum_{i=1}^n \phi(F^e_* f_i) = 1$. Since $(R, \mathfrak{m})$ is local, at least one of $\phi(F^e_* f_i)$ is a unit. Without loss of generality, we may assume $\phi(F^e_* f_1) = u \in R$ is a unit. But then as $f_1 f_2 = 0$ (since $f_1 f_2$ is contained in all minimal primes of R and R is reduced), we have

$$uf_2 = \phi(f_2 \cdot F^e_* f_1) = \phi(F^e_*(f_2^{p^e} f_1)) = \phi(F^e_* 0) = 0$$

which is a contradiction. $\square$

Like F-purity, strong F-regularity can be checked at local rings.

Lemma 3.3. *Let R be an F-finite ring. Then R is strongly F-regular if and only if R_P is strongly F-regular for every $P \in \operatorname{Spec}(R)$.*

Proof. First suppose R is strongly F-regular. Let $P_1, \dots, P_n$ be the minimal primes of R. It is enough to show that for any $c \in R$ whose image in R_P is not contained in any minimal prime of R_P, we can find $e > 0$ and an R_P-linear map $F^e_* R_P \to R_P$ sending $F^e_* c$ to 1. We may assume c is not in any minimal prime of R: for suppose c is contained in $P_1, \dots, P_i$ but not in the other minimal primes of R, then we can pick $c' \in \cap_{j=i+1}^n P_j - \cup_{j=1}^i P_i$ and replace c by $c + c'$ (the image of c' in R_P is 0 since $P_j \nsubseteq P$ for each $j = 1, \dots, i$). But then since R is strongly F-regular, there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules. So after localizing the splitting we get the desired R_P-linear map $F^e_* R_P \to R_P$ sending $F^e_* c$ to 1.

We next prove the converse. We fix $c \in R$ not in any minimal prime of R. We know that for every $P \in \operatorname{Spec}(R)$, there exists e (which may depend on P) such that $R_P \to F^e_* R_P$ sending

1 to F^e_*c splits. Since R is F-finite, $\operatorname{Hom}_{R_P}(F^e_*R_P, R_P) \cong R_P \otimes_R \operatorname{Hom}_R(F^e_*R, R)$ hence there exists a map $\phi \in \operatorname{Hom}_R(F^e_*R, R)$ sending F^e_*c to $f \notin P$. But then $R_f \to F^e_*R_f$ sending 1 to F^e_*c splits. Now for every $P \in \operatorname{Spec} R$ we can find such f thus $\cup D(f) = \operatorname{Spec}(R)$. Hence there exists $f_1, \dots, f_n$ such that $\cup_{i=1}^n D(f_i) = \operatorname{Spec}(R)$ and for each f_i there exists $e_i > 0$ such that $R_{f_i} \to F^{e_i}_*R_{f_i}$ sending 1 to $F^{e_i}_*c$ splits. It is then easy to check that, for $e_0 = \max\{e_1, \dots, e_n\}$, the map $R \to F^{e_0}_*R$ sending 1 to $F^{e_0}_*c$ splits. □

The following is a consequence of Kunz's theorem.

Theorem 3.4. *An F-finite regular ring is strongly F-regular.*

Proof. By Lemma 3.3, we may assume that $(R, \mathfrak{m})$ is an F-finite regular local ring. By Theorem 1.1, F^e_*R is a finite free R-module. For any $0 \neq c \in R$, there exists $e > 0$ such that $F^e_*c \in F^e_*R$ is part of a minimal basis of F^e_*R over R: otherwise $F^e_*c \in \mathfrak{m} \cdot F^e_*R = F^e_*(\mathfrak{m}^{[p^e]})$ for all e which implies that $c \in \cap_e \mathfrak{m}^{[p^e]} = 0$ which is a contradiction. Since $F^e_*c \in F^e_*R$ is part of a minimal basis of F^e_*R over R, the map $R \to F^e_*R$ sending 1 to F^e_*c splits as a map of R-modules. □

We next prove a crucial property of strongly F-regular rings.

Theorem 3.5. *Let R be a strongly F-regular ring. Then $R \to S$ splits for any module-finite extension S of R.*

Proof. Since S is module-finite over R, it is enough to show $R_P \to (R-P)^{-1}S$ is split for every prime $P \in \operatorname{Spec}(R)$. Thus by Lemma 3.3, we may assume $(R, \mathfrak{m})$ is a strongly F-regular local ring and hence a domain by Lemma 3.2. By killing a minimal prime of S, we may further assume that S is also a domain. Now S is a torsion-free R-module, thus there exists an R-linear map θ: $S \to R$ such that $\theta(1) = c \neq 0$. Since R is strongly F-regular, we can find e such that $R \to F^e_*R$ sending 1 to F^e_*c splits, call the splitting ϕ. Now we consider the following commutative diagram with natural maps:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ F^e_*R \arrow[r] & F^e_*S. \end{tikzcd}$$

We know that $F^e_*\theta$: $F^e_*S \to F^e_*R$ sends F^e_*1 to F^e_*c, thus $\phi \circ F^e_*\theta$ sends $F^e_*1 \in F^e_*S$ to $1 \in R$. Therefore, $R \to F^e_*S$ splits, this clearly implies $R \to S$ splits by the commutative diagram. □

Combining the results we have so far, we obtain:

Corollary 3.6. *If R is regular (of characteristic $p > 0$), then $R \to S$ splits for any module-finite extension S of R.*

Proof. Since S is module-finite over R, it is enough to show $R_P \to (R-P)^{-1}S$ is split for every prime $P \in \operatorname{Spec}(R)$ thus we may assume R is a regular local ring. We then consider the faithfully flat extensions $R \to \widehat{R} \cong k[[x_1, \dots, x_d]] \to \widetilde{R} \cong \overline{k}[[x_1, \dots, x_d]]$. Again since S is module-finite over R, it is enough to show $\widetilde{R} \to \widetilde{R} \otimes_R S$ is split. Now $\widetilde{R}$ is F-finite and regular thus strongly F-regular by Theorem 3.4. So $\widetilde{R} \to \widetilde{R} \otimes_R S$ splits by Theorem 3.5. ☐

Remark 3.7. Corollary 3.6 holds without assuming R has characteristic $p > 0$, see [And18]. We will not discuss this result beyond the characteristic $p > 0$ setting.

Another consequence of Theorem 3.5 is the following:

Corollary 3.8. *Let R be a strongly F-regular ring. Then R is normal. In particular, one-dimensional strongly F-regular rings are regular.*

Proof. Let $R \to R'$ be the normalization map. Suppose $R \neq R'$, we can pick $\frac{a}{b} \in R'$ (with b a nonzerodivisor in R) such that $\frac{a}{b} \notin R$. By Theorem 3.5, there exists an R-linear map θ: $R' \to R$ such that $\theta(1) = 1$. Thus

$$b \cdot \theta(\frac{a}{b}) = \theta(a) = a.$$

But then $\frac{a}{b} = \theta(\frac{a}{b}) \in R$, which is a contradiction. ☐

Another important property of strongly F-regular rings is the following:

Theorem 3.9. *Let R and S be F-finite rings. If R is a direct summand of S and S is strongly F-regular (e.g., S is regular), then R is strongly F-regular.*

Proof. By Lemma 3.3, it is enough to show R_P is strongly F-regular for each $P \in \operatorname{Spec}(R)$. Now R_P is a direct summand of $(R-P)^{-1}S$ and the latter is strongly F-regular by Lemma 3.3 again. Thus we may assume $(R, \mathfrak{m})$ is local. Since S is strongly F-regular, it is normal by Corollary 3.8 and hence a product of normal domains $S \cong S_1 \times S_2 \times \cdots \times S_n = \prod S_i e_i$ where e_i is the i-th idempotent corresponding to S_i (e.g., $e_1 = (1, 0, \dots, 0)$). Now a splitting ϕ: $S \to R$ sends $1 = (1, \dots, 1) = \sum e_i$ to 1. Since $(R, \mathfrak{m})$ is local, there exists i such that $\phi(e_i)$ is a unit in R. But then the induced map $\widetilde{\phi}$: $S_i \to R$ defined via $\widetilde{\phi}(s_i) := \phi(s_i e_i)$ for all $s_i \in S_i$ is an R-linear surjection $S_i \twoheadrightarrow R$. Therefore $R \to S_i$ is split (i.e., R is a direct summand of S_i). Note that, as S_i can be viewed as a localization of S, S_i is still strongly F-regular by Lemma 3.3.

Thus replacing S by S_i, we may assume that both R and S are domains. Let $0 \neq c \in R$ be given. Since S is strongly F-regular, there exists $e > 0$ and an S-linear map $\phi\colon F^e_* S \to S$ such that $\phi(F^e_* c) = 1$. Let $\theta\colon S \to R$ be a splitting. Then $\theta \circ \phi\colon F^e_* S \to R$ is an R-linear map sending $F^e_* c$ to 1. Restricting this map to $F^e_* R$ then yields an R-linear map $F^e_* R \to R$ sending $F^e_* c$ to 1. □

Theorem 3.9 allows us to write many examples of strongly F-regular rings:

Example 3.10. Let k be an F-finite field.

(1) Let $R = k[x, y, z]/(xy - z^2)$. Then $R \cong k[s^2, st, t^2]$ is a direct summand of $S = k[s, t]$. Hence R is strongly F-regular. More generally, Veronese subrings of polynomial rings (over F-finite fields) are strongly F-regular.

(2) Let $R = k[x, y, u, v]/(xy - uv)$. Then $R \cong k[a, b]\#k[c, d] \cong k[ac, ad, bc, bd]$ is a direct summand of $S = k[a, b, c, d]$. Hence R is strongly F-regular. More generally, Segre product of polynomial rings (over F-finite fields) are strongly F-regular.

Finally, we point out that to check strong F-regularity, one actually only needs to check the splitting condition in the definition for one single c. This will be very useful in later sections.

Theorem 3.11. *Let R be an F-finite ring. Suppose there exists c not in any minimal prime of R such that R_c is strongly F-regular (e.g., R_c is regular). Then R is strongly F-regular if and only if there exists $e > 0$ such that the map $R \to F^e_* R$ sending 1 to $F^e_* c$ splits as a map of R-modules.*

Proof. Given any $d \in R$ that is not in any minimal prime of R, the image of d is not in any minimal prime of R_c. Therefore, since R_c is strongly F-regular, there exists $e_0 > 0$ and a map $\phi \in \mathrm{Hom}_{R_c}(F^{e_0}_* R_c, R_c)$ such that $\phi(F^{e_0}_* d) = 1$. Since R is F-finite, we have $\mathrm{Hom}_{R_c}(F^{e_0}_* R_c, R_c) \cong R_c \otimes_R \mathrm{Hom}_R(F^{e_0}_* R, R)$ and thus $\phi = \frac{\varphi}{c^n}$ for some $n > 0$ and some $\varphi \in \mathrm{Hom}_R(F^{e_0}_* R, R)$. It follows that $\varphi(F^{e_0}_* d) = c^n$. Next we pick $e_1 > 0$ such that $n < p^{e_1 - e}$, so (the image of) $F^e_* c$ in $F^{e_1}_* R$ is a multiple of $F^{e_1}_* c^n$. Since $R \to F^e_* R$ sending 1 to $F^e_* c$ splits, it follows that $R \to F^{e_1}_* R$ sending 1 to $F^{e_1}_* c^n$ splits. We pick such a splitting θ and consider the map $\theta \circ (F^{e_1}_* \varphi)\colon F^{e_1+e_0}_* R \to R$. It is straighforward to check that this map sends $F^{e_1+e_0}_* d$ to 1. □

Corollary 3.12. *An F-finite local ring $(R, \mathfrak{m})$ is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular.*

Proof. We may assume R is a domain by Lemma 3.2. Since R is excellent, there exists $0 \neq c \in R$ such that R_c is regular and then $\widehat{R}_c$ is also regular. Consider the following commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\cong"] & E \otimes_R F^e_* R \arrow[d] \\ E \arrow[r] & E \otimes_{\widehat{R}} F^e_* \widehat{R} \end{tikzcd}$$

where $E = E_R(R/\mathfrak{m}) = E_{\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R})$ and the horizontal maps are induced by $R \to F^e_* R$ (resp. $\widehat{R} \to F^e_* \widehat{R}$) sending 1 to $F^e_* c$. It is easy to see that the first row is injective if and only if the second row is injective. Since R and $\widehat{R}$ are F-finite, by Corollary 2.4 and Proposition 2.2, $R \to F^e_* R$ sending $1 \to F^e_* c$ splits if and only if $\widehat{R} \to F^e_* \widehat{R}$ sending $1 \to F^e_* c$ splits. By Theorem 3.11, R is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular. $\square$

Similar to Fedder's criterion for F-purity, we have an analogous criterion for strong F-regularity. The following result was obtained by Glassbrenner [Gla96]. We leave this as an exercise–the strategy is similar to the proof of Fedder's criterion.

Exercise 11. Let $(S, \mathfrak{m})$ be an F-finite regular local ring (resp. a polynomial ring over an F-finite field) and let $I \subseteq S$ be an ideal (resp. a homogeneous ideal). Then the following are equivalent for $R = S/I$:

(1) R is strongly F-regular.
(2) For every $c \in S$ not in any minimal prime of I, there exists $e > 0$ such that $c(I^{[p^e]} : I) \nsubseteq \mathfrak{m}^{[p^e]}$.
(3) For some $c \in S$ not in any minimal prime of I such that R_c is strongly F-regular, there exists $e > 0$ such that $c(I^{[p^e]} : I) \nsubseteq \mathfrak{m}^{[p^e]}$.

Exercise 12. Let $R = k[x_1, \dots, x_d]/(x_1^n + \cdots + x_d^n)$. Use Exercise 11 to show that R is strongly F-regular if $n < d$ and $p \gg 0$, and R is not strongly F-regular if $n \geq d$.

Exercise 13. Let R be an $\mathbb{N}$-graded ring over a field k with homogenous maximal ideal $\mathfrak{m}$. Use Theorem 2.5 and Exercise 11 to prove that R is F-pure (resp. F-finite and strongly F-regular) if and only if so is $R_{\mathfrak{m}}$.

Exercise 14. Let $R \to S$ be a faithfully flat extension of F-finite rings. Prove that if S is strongly F-regular, then R is strongly F-regular.

A *very big* open question in tight closure and F-singularity theory is that whether the converse of Theorem 3.5 holds.

Open Problem 1. *Let R be an F-finite domain. If $R \to S$ splits for any module-finite extension S of R, then is R strongly F-regular?*

This has an affirmative answer in the following cases:

(1) If R is Gorenstein by [HH94b].
(2) If R is $\mathbb{Q}$-Gorenstein by [Sin99a].
(3) If the anti-canonical cover of R is a Noetherian ring by [CEMS18].

For the readers not familiar with the terminology $\mathbb{Q}$-Gorenstein or anti-canonical cover, we just point out that there are implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) since every Gorenstein ring is $\mathbb{Q}$-Gorenstein and every $\mathbb{Q}$-Gorenstein ring has Noetherian anti-canonical cover.

In Hochster–Huneke's foundational work [HH90, HH94a], there are three notions of F-regularity: weakly F-regular, F-regular, and strongly F-regular. The former two are defined using tight closure. Conjecturally all these notions are equivalent (for F-finite rings), but to this date this is still not clear. It turns out that even weakly F-regular rings split from all their module-finite extensions [HH90]. Thus an affirmative answer to Open Problem 1 will imply that all these notions are equivalent. For related results on the equivalence of different notions of F-regularity, see [HH94a, Wil95, Mac96, LS99, LS01, AP19].

On the other hand, it becomes apparent in recent years that strong F-regularity is the most useful concept and has most applications to algebraic geometry.

Discussion 3.13. We can define strongly F-regular rings beyond the F-finite setting, there are actually several ways to extend the definition, for example see [HH94a] or [DS16]. For technical reasons, and also because it will be quite technical to define F-signature without F-finite assumptions, we decide to keep the F-finite assumption in the definition of strong F-regularity.

4. F-rational and F-injective rings

In this section we discuss F-rational and F-injective rings [Fed83, HH94a, HH94b]. We begin by collecting some basic facts about Frobenius structure on local cohomology modules. Let $I = (f_1, \dots, f_n)$ be an ideal of R, then we have the Čech complex:

$$C^\bullet(f_1, \dots, f_n; R) := 0 \to R \to \oplus_i R_{f_i} \to \cdots \to R_{f_1 f_2 \cdots f_n} \to 0.$$

The i-th local cohomology module $H^i_I(R)$ is the i-th cohomology of $C^\bullet(f_1, \dots, f_n; R)$. It turns out that $H^i_I(R)$ only depends on the radical of I. Since the Frobenius endomorphism on R naturally induces the Frobenius endomorphism on all localizations of R, it induces a natural Frobenius action on $C^\bullet(f_1, \dots, f_n; R)$, and hence it induces a natural Frobenius action on each $H^i_I(R)$.

We know from the definition that a ring homomorphism $R \to S$ induces a map $H^i_I(R) \to H^i_{IS}(S)$. The natural Frobenius action on $H^i_I(R)$ discussed above can be alternatively described as $H^i_I(R) \to H^i_{I \cdot F_* R}(F_* R) = H^i_{F_* I^{[p]}}(F_* R)$ and then identify $H^i_{F_* I^{[p]}}(F_* R)$ with $H^i_I(R)$, where the last identification is induced by $F_* R \cong R$ as rings (note that $H^i_{I^{[p]}}(R) = H^i_I(R)$). We will be mostly interested in the case that $(R, \mathfrak{m})$ is local and $I = \mathfrak{m}$. In this case, we can compute $H^i_\mathfrak{m}(R)$ using the Čech complex on a system of parameters $x_1, \dots, x_d$ of R. For example, the top local cohomology module $H^d_\mathfrak{m}(R)$ is isomorphic to

$$\frac{R_{x_1 \cdots x_d}}{\sum_i \operatorname{Im}(R_{x_1 \cdots \widehat{x_i} \cdots x_d})},$$

and with this description, the natural Frobenius action on $H^d_\mathfrak{m}(R)$ is given by

$$\frac{r}{x_1^n \cdots x_d^n} \to \frac{r^p}{x_1^{np} \cdots x_d^{np}}.$$

Definition 4.1. A local ring $(R, \mathfrak{m})$ of dimension d is called *F-rational* if R is Cohen–Macaulay and for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the composition

$$H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_\mathfrak{m}(F^e_* R)$$

is injective. Equivalently, using $F^e \colon H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ to denote the e-th Frobenius action, this is saying that $c \cdot F^e(-)$ is injective on $H^d_\mathfrak{m}(R)$. An arbitrary ring R is called F-rational if and only if $R_\mathfrak{m}$ is F-rational for all maximal ideals $\mathfrak{m} \subseteq R$.

Remark 4.2. Our definition of F-rational rings is not the original one as in [HH94a, HH94b], but it is an equivalent definition for all rings that are homomorphic images of Cohen–Macaulay rings. This is a very mild assumption: for example, all excellent local rings satisfy this condition [Kaw02]. In fact, in Hochster's more recent course notes on tight closure

[Hoc07], being a homomorphic image of a Cohen–Macaulay ring is built into the definition of F-rationality, thus almost nothing is lost.

Example 4.3. As a first example, all regular rings are F-rational. For suppose $(R, \mathfrak{m})$ is a regular local ring of dimension d. Then a socle representative of $H^d_{\mathfrak{m}}(R)$ is $\eta = \frac{1}{x_1 \cdots x_d}$ where $x_1, \ldots, x_d$ is a regular system of parameters of R. If $c \neq 0$ such that $c \cdot F^e(\eta) = 0$ for all $e > 0$, then $\frac{c}{x_1^{p^e} \cdots x_d^{p^e}} = 0$ in $H^d_{\mathfrak{m}}(R)$ for all $e > 0$. But then $c \in \cap_e (x_1^{p^e}, \ldots, x_d^{p^e}) = 0$, a contradiction.

Proposition 4.4. *Suppose R is F-rational, then R is normal. In particular, one-dimensional F-rational rings are regular.*

Proof. We may assume $(R, \mathfrak{m})$ is local. In order to show R is normal, it is enough to prove that every principal ideal of height one is integrally closed by [HS06, Proposition 1.5.2] (if $\dim(R) = 0$, then the condition implies R is a field so R is trivially normal). Suppose $y \in \overline{(x)}$ where x is not in any minimal prime of R, then there exists $m > 0$ such that $(y, x)^n = (y, x)^m (x)^{n-m}$ for all $n > m$. Thus $x^m y^n \in (x)^n$ for every n. We can extend x to a full system of parameters $x, x_2^t, \ldots, x_d^t$ of R. Then the Čech class $\eta = \frac{y}{x x_2^t \cdots x_d^t}$ satisfies

$$x^m \cdot F^e(\eta) = x^m \cdot \frac{y^{p^e}}{x^{p^e} x_2^{tp^e} \cdots x_d^{tp^e}} = 0$$

for all $e > 0$ since $x^m y^{p^e} \in (x^{p^e})$ by construction. So by the definition of F-rationality, $\eta = 0$ in $H^d_{\mathfrak{m}}(R)$. But since R is Cohen–Macaulay, we know that $y \in (x, x_2^t, \ldots, x_d^t)$. As this is true for every $t > 0$, $y \in \cap_t (x, x_2^t, \ldots, x_d^t) = (x)$. Thus (x) is integrally closed. $\square$

An important result we want to prove next is that strongly F-regular rings are F-rational. We need a well-known lemma.

Lemma 4.5. *Let $(R, \mathfrak{m})$ be a complete and equidimensional local ring of dimension d. Suppose R_P is Cohen–Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Then $H^i_{\mathfrak{m}}(R)$ has finite length for all $i < d$.*

Proof. By Cohen's structure theorem, we can write $R = S/I$ where S is a complete regular local ring. By local duality, $H^i_{\mathfrak{m}}(R)^\vee \cong \operatorname{Ext}^{n-i}_S(R, S)$ where $n = \dim S$. It follows that

$$\operatorname{Ext}^{n-i}_S(R, S)_P \cong \operatorname{Ext}^{n-i}_{S_P}(S_P, R_P) = \operatorname{Ext}^{\dim(S_P)-(i-\dim(R/P))}_{S_P}(S_P, R_P),$$

where we abuse notation and also use P to denote the pre-image of P in S. Now by local duality over S_P,

$$\operatorname{Ext}^{\dim(S_P)-(i-\dim(R/P))}_{S_P}(S_P, R_P)^\vee \cong H^{i-\dim(R/P)}_{PR_P}(R_P).$$

Since R is equidimensional, $\dim(R/P) + \dim(R_P) = d$ hence if $i < d$ then $i - \dim(R/P) < \dim(R_P)$. Thus if $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $i < d$, then $H^{i-\dim(R/P)}_{PR_P}(R_P) = 0$ since R_P is Cohen–Macaulay, which gives $\operatorname{Ext}^{n-i}_S(R,S)_P = 0$. Thus $\operatorname{Ext}^{n-i}_S(R,S)$ is supported only at $\{\mathfrak{m}\}$ when $i < d$. By Matlis duality, $H^i_{\mathfrak{m}}(R)$ has finite length whenever $i < d$. $\square$

We can now prove the following result.

Theorem 4.6. *Let $(R,\mathfrak{m})$ be a strongly F-regular ring. Then R is F-rational (and hence Cohen–Macaulay).*

Proof. Note that $H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(\widehat{R})$ and if $c \in R$ is not in any minimal prime of R, then c is not in any minimal prime of $\widehat{R}$. Thus it is clear that $\widehat{R}$ is F-rational implies R is F-rational. Therefore we may assume R is a complete local domain by Corollary 3.12 and Lemma 3.2. Since strong F-regularity is preserved under localization by Lemma 3.3, by induction on $\dim(R)$ we may further assume R_P is Cohen–Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Thus by Lemma 4.5, $H^i_{\mathfrak{m}}(R)$ has finite length whenever $i < d = \dim(R)$.

Let $0 \neq c \in \mathfrak{m}$. Since $H^i_{\mathfrak{m}}(R)$ has finite length for $i < d$, there exists n such that $c^n H^i_{\mathfrak{m}}(R) = 0$. Replacing c with c^n we may assume $cH^i_{\mathfrak{m}}(R) = 0$. Thus $(F^e_*c) \cdot H^i_{\mathfrak{m}}(F^e_*R) = 0$. Since R is strongly F-regular, there exists $e > 0$ and an R-linear map $F^e_*R \to R$ such that the composition of the following maps is the identity map on R:

$$R \to F^e_*R \xrightarrow{\cdot F^e_*c} F^e_*R \to R.$$

Applying the i-th local cohomology functor $H^i_{\mathfrak{m}}(-)$ to the above composition of maps we see that the identity map on $H^i_{\mathfrak{m}}(R)$ factors through the zero map on $H^i_{\mathfrak{m}}(F^e_*R)$ and thus $H^i_{\mathfrak{m}}(R) = 0$ whenever $i < d$. This proves that R is Cohen–Macaulay. Finally, applying the d-th local cohomology functor $H^d_{\mathfrak{m}}(-)$ to the same composition of maps, we see that the identity map on $H^d_{\mathfrak{m}}(R)$ factors through

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_*R) \xrightarrow{\cdot F^e_*c} H^d_{\mathfrak{m}}(F^e_*R).$$

In particular, the above map is injective and thus R is F-rational. $\square$

As a consequence of the results we proved so far, we can prove the following.

Corollary 4.7. *Let $R \to S$ be a pure map of rings (of characteristic $p > 0$). If S is regular, then R is Cohen–Macaulay.*

Proof. We first observe that if R and S are both F-finite and the map $R \to S$ is split (this includes most cases of interest). Then the conclusion follows by combining Theorem 3.4, Theorem 3.9, and Theorem 4.6.

But with a careful examination of the methods we used in proving these results, we can prove the general case of the corollary. We now give the details. First of all we may assume $(R,\mathfrak{m})$ is a local ring. Since $R \to S$ is pure, $E \to E \otimes_R S$ is injective so $u \otimes 1 \neq 0$ in $E \otimes_R S$ where u is a socle representative of E. But then $u \otimes 1 \neq 0$ in $E \otimes_R S_Q$ for some $Q \in \operatorname{Spec}(S)$, and thus $E \to E \otimes_R S_Q$ is injective. This implies $R \to S_Q$ is pure by Proposition 2.2. So we may assume S is also a local ring. We may then replace R by $\widehat{R}$ and S by $\widehat{S} \cong k[[x_1, \dots, x_n]]$ and further replace $\widehat{S}$ by $\widetilde{S} := \overline{k}[[x_1, \dots, x_n]]$. Therefore we may assume $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is pure where S is a complete and F-finite regular local ring and $(R,\mathfrak{m})$ is a complete local domain. Furthemore, by induction on $\dim(R)$ we may assume R_P is Cohen–Macaulay for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. By Lemma 4.5, $H^i_\mathfrak{m}(R)$ has finite length for all $i < \dim(R)$.

For each $i < \dim(R)$, let $0 \neq c \in R$ that annihilates $H^i_\mathfrak{m}(R)$. By Theorem 3.4, S is strongly F-regular so there exists $e > 0$ such that $S \to F^e_*S$ sending 1 to F^e_*c splits. We consider the following commutative diagram:

$$\begin{tikzcd}
H^i_\mathfrak{m}(R) \arrow[r] \arrow[d] & H^i_\mathfrak{m}(F^e_*R) \arrow[r, "\cdot F^e_*c"] \arrow[d] & H^i_\mathfrak{m}(F^e_*R) \arrow[d] \\
H^i_\mathfrak{m}(S) \arrow[r] & H^i_\mathfrak{m}(F^e_*S) \arrow[r, "\cdot F^e_*c"] & H^i_\mathfrak{m}(F^e_*S)
\end{tikzcd}$$

From the bottom row, we see that the map from top left $H^i_\mathfrak{m}(R)$ to bottom right $H^i_\mathfrak{m}(F^e_*S)$ is injective, while from the first row, we see that the same map is the zero map from $H^i_\mathfrak{m}(R)$ to $H^i_\mathfrak{m}(F^e_*S)$ as c annihilates $H^i_\mathfrak{m}(R)$. This shows that $H^i_\mathfrak{m}(R) = 0$ and hence R is Cohen–Macaulay. $\square$

Remark 4.8. Corollary 4.7 holds without assuming the rings have characteristic $p > 0$, see [HH95] and [HM18]. We will not discuss the result beyond the characteristic $p > 0$ setting.

The converse of Theorem 4.6 holds if R is Gorenstein.

Proposition 4.9. *Suppose R is F-finite, Gorenstein and F-rational, then R is strongly F-regular.*

Proof. By Lemma 3.3, we may assume $(R,\mathfrak{m})$ is local. It is enough to show that for any $c \in R$ not in any minimal prime of R, there exists $e > 0$ such that the map $E \to E \otimes_R F^e_*R$ induced by sending 1 to F^e_*c is injective (see Proposition 2.2 and Corollary 2.4), where $E = E_R(R/\mathfrak{m})$ denotes the injective hull of the residue field as usual. Since R is Gorenstein, $E \cong H^d_\mathfrak{m}(R)$. Thus the map $E \to E \otimes_R F^e_*R$ can be identified with the map $H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_*R) \xrightarrow{\cdot F^e_*c} H^d_\mathfrak{m}(F^e_*R)$, which is injective by the F-rationality of R. $\square$

We next give an alternative but important characterization of F-rationality (up to completion), see [Smi97] for more details. We need a definition.

Definition 4.10. Let M be an R-module with a Frobenius action F (i.e., $F(rm) = r^pF(m)$ for all $r \in R$ and $m \in M$). An R-submodule $N \subseteq M$ is called *F-stable* if $F(N) \subseteq N$.

Proposition 4.11. *Let $(R, \mathfrak{m})$ be a local ring. Then the following are equivalent:*

(1) $\widehat{R}$ is F-rational.

(2) R is Cohen–Macaulay and the only F-stable submodules of $H^d_\mathfrak{m}(R)$ are 0 and $H^d_\mathfrak{m}(R)$, i.e., $H^d_\mathfrak{m}(R)$ is a simple object in the category of R-modules with a Frobenius action.

Proof. Since $H^d_\mathfrak{m}(R)$ is Artinian, any R-submodule of $H^d_\mathfrak{m}(R)$ carries a canonical $\widehat{R}$-module structure, and the Frobenius structure on $H^d_\mathfrak{m}(R)$ is unaffected by considering it as a module over $\widehat{R}$. Thus all conditions in (2) are unaffected by replacing R by $\widehat{R}$ and so we may assume $(R, \mathfrak{m})$ is complete.

Suppose (1) holds. By Proposition 4.4, we may assume $(R, \mathfrak{m})$ is a complete normal local domain. Let $N \subsetneq H^d_\mathfrak{m}(R)$ be a proper F-stable submodule. By Matlis duality, $H^d_\mathfrak{m}(R)^\vee \cong \omega_R \twoheadrightarrow N^\vee$ is a proper quotient. Since R is a domain, ω_R is torsion-free. $N^\vee$ and hence N is annihilated by some $c \neq 0$. If $N \neq 0$, then any $0 \neq \eta \in N$ satisfies $c \cdot F^e(\eta) = 0$ for all e, which contradicts that $c \cdot F^e(-)$ is injective for some e.

Suppose (2) holds. First notice that the Frobenius is injective on $H^d_\mathfrak{m}(R)$: otherwise the kernel is a nonzero proper submodule of $H^d_\mathfrak{m}(R)$. Now for any $c \in R$ not in any minimal prime of R, it is easy to check that

$$\{\eta \in H^d_\mathfrak{m}(R) \mid c \cdot F^e(\eta) = 0 \text{ for all } e \geq 0\}$$

is an F-stable submodule of $H^d_\mathfrak{m}(R)$. Since it is annihilated by c, it cannot be $H^d_\mathfrak{m}(R)$ so it must be 0 by the conditions of (2). But this is saying that for any $\eta \in H^d_\mathfrak{m}(R)$, there exists $e > 0$ such that $c \cdot F^e(\eta) \neq 0$. Let $N_e := \{\eta \in H^d_\mathfrak{m}(R) \mid c \cdot F^e(\eta) = 0\}$. Since the Frobenius is injective on $H^d_\mathfrak{m}(R)$, it is easy to check that $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$. Since $H^d_\mathfrak{m}(R)$ is Artinian and $\cap_e N_e = 0$, there exists e such that $N_e = 0$, which is precisely saying that $c \cdot F^e(-)$ is injective on $H^d_\mathfrak{m}(R)$. ☐

A natural question one might ask at this point is that whether R is F-rational implies $\widehat{R}$ is F-rational (it is straightforward from the definition that if $\widehat{R}$ is F-rational then R is F-rational). It turns out that this is not always true, but it holds if R is excellent. We will come back to this question in Section 6.

In the proof of Proposition 4.11, we crucially used the fact that the Frobenius action is injective on $H^d_\mathfrak{m}(R)$. We now formally introduce F-injective singularities.

Definition 4.12. A local ring $(R,\mathfrak{m})$ is called *F-injective* if the natural Frobenius action on $H^i_\mathfrak{m}(R)$ is injective for all i. An arbitrary ring R is called F-injective if $R_\mathfrak{m}$ is F-injective for all maximal ideals $\mathfrak{m}\subseteq R$.

It is straightforward from the definition that if R is F-rational, then R is F-injective. Since the Frobenius structure on $H^i_\mathfrak{m}(R)$ is the same when we consider it as a module over $\widehat{R}$, we also know that a local ring $(R,\mathfrak{m})$ is F-injective if and only if $\widehat{R}$ is F-injective.

We next show that F-injectivity and F-rationality are preserved under localization. For F-injectivity, the strategy is taken from [DM19], where the result is proved in its most general form.

Theorem 4.13. *Suppose R is F-injective, then R_P is F-injective for all $P\in \operatorname{Spec}(R)$*

Proof. We may assume $(R,\mathfrak{m})$ is local with $\dim(R)=d$. First we claim that we may assume R is complete. Let $P\in\operatorname{Spec}(R)$, pick a minimal prime Q of $P\widehat{R}$, then $R_P\to\widehat{R}_Q$ is faithfully flat with $\dim(R_P)=\dim(\widehat{R}_Q)$. Thus $H^i_Q(\widehat{R}_Q)\cong H^i_P(R_P)\otimes_{R_P}\widehat{R}_Q$ for all i and it is easy to see that the isomorphism is compatible with the Frobenius actions. Hence if we can show $\widehat{R}_Q$ is F-injective, then R_P is F-injective.

Now we assume $(R,\mathfrak{m})$ is complete, by Cohen's structure theorem we can write $R=S/I$ where $(S,\mathfrak{n})$ is a complete regular local ring of dimension n. We can write $F_*R=\varinjlim_j R_j$ such that each R_j is module-finite (and purely inseparable) over R, thus $F_*(R_P)=\varinjlim_j (R_j)_P$. We have the following (abusing notations a bit, we still use P to denote the corresponding prime ideal in S):

$$\begin{aligned}
R \text{ is } F\text{-injective} &\Rightarrow H^i_\mathfrak{m}(R)\to H^i_\mathfrak{m}(F_*R) \text{ is injective for all } i\\
&\Rightarrow H^i_\mathfrak{m}(R)\to H^i_\mathfrak{m}(R_j) \text{ is injective for all } i,\ j\\
&\Rightarrow \operatorname{Ext}^{n-i}_S(R_j,S)\to \operatorname{Ext}^{n-i}_S(R,S) \text{ is surjective for all } i,\ j\\
&\Rightarrow \operatorname{Ext}^{n-i}_{S_P}((R_j)_P,S_P)\to \operatorname{Ext}^{n-i}_{S_P}(R_P,S_P) \text{ is surjective for all } i,\ j\\
&\Rightarrow H_P^{\dim(S_P)-n+i}(R_P)\to H_P^{\dim(S_P)-n+i}((R_j)_P) \text{ is injective for all } i,\ j\\
&\Rightarrow H_P^{\dim(S_P)-n+i}(R_P)\to H_P^{\dim(S_P)-n+i}(F_*(R_P)) \text{ is injective for all } i\\
&\Rightarrow R_P \text{ is } F\text{-injective.}
\end{aligned}$$

where the third and fifth implications are due to local duality over S and S_P respectively. ☐

Finally, we show that F-rationality localizes.

Theorem 4.14. *Suppose R is F-rational, then R_P is F-rational for all $P\in\operatorname{Spec}(R)$.*

Proof. We may assume $(R, \mathfrak{m})$ is local with $\dim(R) = d$. By Proposition 4.4, R is a Cohen–Macaulay normal domain, and hence so is R_P. Suppose P has height h, it is then enough to show that for any $0 \neq c \in R$, there exists $e > 0$ such that $cF^e(-)$ is injective on $H^h_P(R_P)$. Suppose on the contrary, there exists $0 \neq c \in R$ such that $cF^e(-)$ is not injective for all $e > 0$. Then for all $e > 0$, we have

$$0 \neq K_e := \operatorname{Ker}(H^h_P(R_P) \xrightarrow{cF^e(-)} H^h_P(R_P)).$$

We claim that $K_{e+1} \subseteq K_e$: if $cF^{e+1}(\eta) = 0$, then $F(cF^e(\eta)) = c^p F^{e+1}(\eta) = 0$, but we know that R_P is F-injective by Theorem 4.13, thus $cF^e(\eta) = 0$. Therefore we have a descending chain of R_P-modules:

$$K_1 \supseteq \cdots \supseteq K_e \supseteq K_{e+1} \supseteq \cdots .$$

Since $H^h_P(R_P)$ is an Artinian R_P-module, this chain stabilizes and so there exists $0 \neq \eta \in \cap_e K_e$. Next we pick a system of parameters $x_1, \ldots, x_h, x_{h+1}, \ldots, x_d$ of R such that the image of $x_1, \ldots, x_h$ is a system of parameters on R_P. Note that

$$H^h_P(R_P) = \varinjlim_e \frac{R_P}{(x_1^{p^e}, \ldots, x_h^{p^e})R_P},$$

where the connection maps are multiplication by $(x_1 \cdots x_h)^{p^{e+1}-p^e}$. By replacing $x_1, \ldots, x_h$ by their powers if necessary, we may assume that $\eta \neq 0$ is the image of $\overline{y} \in R_P/(x_1, \ldots, x_h)R_P$ in $H^h_P(R_P)$. Multiplying η and y by elements in $R - P$ (which are units in R_P), we may assume that $y \in R$. We consider the following commutative diagram

$$\begin{tikzcd} H^h_P(R_P) \arrow[r, "cF^e(-)"] & H^h_P(R_P) \\ \frac{R_P}{(x_1,\ldots,x_h)R_P} \arrow[u, hook] \arrow[r, "\overline{x}\mapsto c\overline{x}^{p^e}"] & \frac{R_P}{(x_1^{p^e},\ldots,x_h^{p^e})R_P} \arrow[u, hook] \end{tikzcd}$$

where the vertical maps are injections since R_P is Cohen–Macaulay. Chasing the image of $\overline{y} \in R_P/(x_1, \ldots, x_h)R_P$, we find that for all $e > 0$, $c\overline{y}^{p^e} = 0$ in $R_P/(x_1^{p^e}, \ldots, x_h^{p^e})R_P$. That is, for every $e > 0$, there exists $z_e \notin P$ such that $cz_e y^{p^e} \in (x_1^{p^e}, \ldots, x_h^{p^e})$.

Let $(x_1, \ldots, x_h) = Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition of $(x_1, \ldots, x_h)$, with $P_i = \sqrt{Q_i}$ the corresponding associated primes. We may assume $P = P_1$. Since R is Cohen–Macaulay and $x_1, \ldots, x_h$ is a regular sequence, each P_i is a minimal prime of $(x_1, \ldots, x_h)$ and we have $\operatorname{Ass}(R/(x_1, \ldots, x_h)) = \operatorname{Ass}(R/(x_1^{p^e}, \cdots, x_h^{p^e}))$ for all $e > 0$. Let $(x_1^{p^e}, \ldots, x_h^{p^e}) = Q_{1,e} \cap \cdots Q_{s,e}$ be the irredundant primary decomposition with $P_i = \sqrt{Q_{i,e}}$. We know that $Q_{i,e}$ is the contraction of $(x_1^{p^e}, \ldots, x_h^{p^e})R_{P_i}$ to R. Since $(x_1, \ldots, x_h)^{hp^e} \subseteq$

$(x_1^{p^e},\dots,x_h^{p^e})$, we have $Q_i^{(hp^e)} \subseteq Q_{i,e}$. Now we fix $z \in (Q_2 \cap \cdots \cap Q_s)^h - P_1$, it follows that $z^{p^e} \in Q_i^{hp^e} \subseteq Q_{i,e}$ for all $i \geq 2$. Since $cz_e y^{p^e} \in (x_1^{p^e},\dots,x_h^{p^e}) \subseteq Q_{1,e}$ and $z_e \notin P = P_1$, we know that $cy^{p^e} \in Q_{1,e}$. Thus we have $z \in R - P$ such that for all $e > 0$,

$$cy^{p^e} z^{p^e} \in Q_{1,e} \cap Q_{2,e} \cap \cdots \cap Q_{s,e} = (x_1^{p^e},\dots,x_h^{p^e}).$$

Therefore for all $e > 0$ and all $n > 0$, we have

$$c(zy)^{p^e} \in (x_1^{p^e},\dots,x_h^{p^e}) \subseteq (x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e}).$$

Since R is F-rational, there exists $e > 0$ such that $cF^e(-)$ is injective on $H^d_{\mathfrak{m}}(R)$. Fix this e, we consider the following commutative diagram

$$\begin{tikzcd} H^d_{\mathfrak{m}}(R) \arrow[r, hook, "cF^e(-)"] & H^d_{\mathfrak{m}}(R) \\ \frac{R}{(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)} \arrow[u, hook] \arrow[r, "\overline{x}\mapsto c\overline{x}^{p^e}"] & \frac{R}{(x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e})} \arrow[u, hook] \end{tikzcd}$$

where the vertical maps are injective since R is Cohen–Macaulay. Chasing the diagram we find that the bottom map is injective. Since $\overline{zy} \in R/(x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)$ maps to zero in $R/(x_1^{p^e},\dots,x_h^{p^e},x_{h+1}^{np^e},\dots,x_d^{np^e})$, we obtain that $zy \in (x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n)$ for all $n > 0$. Thus

$$zy \in \cap_n (x_1,\dots,x_h,x_{h+1}^n,\dots,x_d^n) = (x_1,\dots,x_h),$$

which implies $y \in (x_1,\dots,x_h)R_P$. Therefore $0 = \overline{y} \in R_P/(x_1,\dots,x_h)R_P$ and thus $\eta = 0$, which is a contradiction. ☐

Exercise 15. Prove that if R is F-injective, then R is reduced.

Exercise 16. Let $R \to S$ be a faithfully flat extension of rings. Prove that if S is F-rational (resp. F-injective), then R is F-rational (resp. F-injective).

Exercise 17. Show that if R is F-pure, then R is F-injective. Conversely, show that if R is quasi-Gorenstein and F-injective, then R is F-pure.

Exercise 18. Let R be an $\mathbb{N}$-graded ring over a field with homogeneous maximal ideal $\mathfrak{m}$. Show that

(1) If R is F-injective, then $[H^i_{\mathfrak{m}}(R)]_{>0} = 0$ for each i.
(2) If R is F-rational, then $[H^d_{\mathfrak{m}}(R)]_{\geq 0} = 0$.

One can ask that, similar to strong F-regularity and F-purity, whether a direct summand of F-rational or F-injective ring remains F-rational or F-injective. Unfortunately, Watanabe

[Wat97] constructed an example of a direct summand of an F-rational ring that is not even F-injective. The example will be examined in Section 9. But the following question seems open.[1]

Open Problem 2. *Are direct summands of F-rational rings Cohen–Macaulay?*

We warn the reader that, one cannot expect that direct summands of Cohen–Macaulay rings are Cohen–Macaulay in general:

Exercise 19. Let R be the Segre product $(k[x,y,z]/(x^3+y^3+z^3))\#k[s,t]$, which is the subring of the hypersurface $S = k[x,y,z,s,t]/(x^3+y^3+z^3)$ generated by xs, ys, zs, xt, yt, zt. Then R is a direct summand of S. Show that R is not Cohen–Macaulay, and show that S is not F-rational.

[1] Though we suspect it is false in general.

5. The deformation problem

An interesting question in the study of singularities is how they behave under deformation. Roughly speaking, if $\mathrm{Spec}(R)$ is the total space of a fibration over a curve, then the special fiber of this fibration is a variety with coordinate ring R/xR for a nonzerodivisor x of R. The question is whether the singularity type of the total space $\mathrm{Spec}(R)$ is no worse than the singularity type as the special fiber $\mathrm{Spec}(R/xR)$.

This deformation question has been studied in details for F-singularities: it is proved that F-rationality always deforms [HH94a], and that both F-pure and strongly F-regular singularities fail to deform in general [Fed83], [Sin99b], [Sin99c]. The examples will be examined in Section 9. Here we focus on deformation of F-rational and F-injective singularities.

Theorem 5.1. *Let $(R,\mathfrak{m})$ be a local ring and x a nonzerodivisor on R. Then*

(1) If R/xR is Cohen–Macaulay and F-injective, then R is Cohen–Macaulay and F-injective.

(2) If R/xR is F-rational, then R is F-rational.

Proof. We first prove (1). It is clear that R is Cohen–Macaulay. It is enough to show that the natural Frobenius action on $H^d_\mathfrak{m}(R)$ is injective. The commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^{d-1}_\mathfrak{m}(R/xR) \arrow[r] \arrow[d, "F^e"] & H^d_\mathfrak{m}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^d_\mathfrak{m}(R) \arrow[r] \arrow[d, "F^e"] & 0 \\ 0 \arrow[r] & H^{d-1}_\mathfrak{m}(R/xR) \arrow[r] & H^d_\mathfrak{m}(R) \arrow[r, "\cdot x"] & H^d_\mathfrak{m}(R) \arrow[r] & 0 \end{tikzcd}$$

If the middle map is not injective, then we pick $\eta \in \mathrm{Soc}(H^i_\mathfrak{m}(R)) \cap \mathrm{Ker}(x^{p^e-1}F^e)$ and it is easy to see that η comes from $H^{d-1}_\mathfrak{m}(R/xR)$. But this contradicts the injectivity of F^e on $H^{d-1}_\mathfrak{m}(R/xR)$. Thus $x^{p^e-1}F^e$ and hence F^e is injective on $H^d_\mathfrak{m}(R)$.

We next prove (2). Suppose we have $c \in R$ not in any minimal prime of R. It is enough to show that the F-stable submodule $\{\eta \in H^d_\mathfrak{m}(R) \mid c \cdot F^e(\eta) = 0 \text{ for all } e > 0\}$ is 0 (see the proof of Proposition 4.11). If this submodule is nonzero, then it intersect $\mathrm{Soc}(H^d_\mathfrak{m}(R))$ nontrivially so we may assume there exists $0 \neq \eta \in H^d_\mathfrak{m}(R)$ such that $c \cdot F^e(\eta) = 0$ for all $e > 0$ and $x\eta = 0$. We can write $c = x^n c'$ where $c' \notin (x)$ and pick any $e > 0$ such that

$p^e - 1 \geq n$. Since $c \cdot F^e(\eta) = 0$, $c'x^{p^e-1}F^e(\eta) = 0$. Since $x\eta = 0$ we know that η comes from $H^{d-1}_{\mathfrak{m}}(R/xR)$ and chasing the diagram we find that $c'F^e(\eta) = 0$ in $H^{d-1}_{\mathfrak{m}}(R/xR)$. But since R/xR is F-rational, it is a normal domain by Proposition 4.4 and hence the image of c' is nonzero in R/xR. So the F-rationality of R/xR implies that $c'F^e$ is injective on $H^{d-1}_{\mathfrak{m}}(R/xR)$ for some $e > 0$. Thus $\eta = 0$, a contradiction. $\square$

Corollary 5.2. *Let $(R, \mathfrak{m})$ be an F-finite local ring and x a nonzerodivisor on R. If R/xR is Gorenstein and strongly F-regular, then R is Gorenstein and strongly F-regular.*

Proof. Simply combining Theorem 5.1 and Proposition 4.9. $\square$

The deformation question for F-injectivity is not solved completely. To this date, the best partial result towards this question is obtained in [HMS14], where it is shown that F-purity deforms to F-injectivity (note that F-purity itself does not deform in general by Example 9.5). To prove this result, we need a result from [Ma14].

Theorem 5.3 ([Ma14])**.** *If $(R, \mathfrak{m})$ is F-pure, then for all i and all F-stable submodules $N \subseteq H^i_{\mathfrak{m}}(R)$, the natural Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is injective.*

Proof. We may replace R by $\widehat{R}$ to assume R is F-split (see Corollary 2.4). We then observe the following quite general claim:

Claim 5.4. *If $R \to S$ is split, η is an element of $H^i_{\mathfrak{m}}(R)$, and N is a submodule of $H^i_{\mathfrak{m}}(R)$, then $\eta \in N$ provided that the image of η in $H^i_{\mathfrak{m}S}(S)$ is contained in the S-span of the image of N in $H^i_{\mathfrak{m}S}(S)$.*

Proof. Let $\phi\colon S \to R$ be a splitting. It is easy to check that we have the following commutative diagram

$$\begin{tikzcd}
S \otimes_R H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[dr, "\phi\otimes \mathrm{id}"'] & H^i_{\mathfrak{m}S}(S) \arrow[d, "H^i_{\mathfrak{m}}(\phi)"] \\
 & H^i_{\mathfrak{m}}(R).
\end{tikzcd}$$

Thus if the image of η is in the S-span of the image of N, say $\mathrm{Im}(1 \otimes \eta) = \sum s_i \cdot \mathrm{Im}(1 \otimes \eta_i)$ where $\eta_i \in N$. Then by the above commutative diagram, $\eta = \sum \phi(s_i)\eta_i \in N$. $\square$

We now continue the proof of the theorem. Suppose N is an F-stable submodule such that the Frobenius action on $H^i_{\mathfrak{m}}(R)/N$ is not injective, then there exists $\eta \notin N$ such that $F(\eta) \in N$ for all $e > 0$. Let N_e be the R-span of $F^e(N)$. Since N is F-stable, we have a descending chain $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$. This chain stabilizes since $H^i_{\mathfrak{m}}(R)$ is Artinian. Therefore, as $F(\eta) \in N$, $F^{e+1}(\eta) \in N_e = N_{e+1}$ for $e \gg 0$. Finally we apply Claim 5.4 to the

$(e+1)$-th Frobenius map $F^{e+1}\colon R \to R$ (which is split by assumption) and note that the R-span of the image of N is precisely N_{e+1}, hence we know that $y \in N$, a contradiction. $\square$

Theorem 5.5 ([HMS14]). *Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R. If R/xR is F-pure, then R is F-injective.*

Proof. The strategy is similar to the Cohen–Macaulay case. The commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"'] & R \arrow[r] \arrow[d, "F^e"'] & R/xR \arrow[r] \arrow[d, "F^e"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "\cdot x"] & R \arrow[r] & R/xR \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] \arrow[d, "F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] \arrow[d, "x^{p^e-1}F^e"] & H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "F^e"] & \cdots \\ 0 \arrow[r] & H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) \arrow[r] & H^i_{\mathfrak{m}}(R) \arrow[r, "\cdot x"] & H^i_{\mathfrak{m}}(R) \arrow[r] & \cdots \end{tikzcd}$$

Note that $\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$ is an F-stable submodule of $H^{i-1}_{\mathfrak{m}}(R/xR)$. So by Theorem 5.3, F^e is injective on $H^{i-1}_{\mathfrak{m}}(R/xR)/\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R))$. Now by the same argument as in Theorem 5.1, this implies that $x^{p^e-1}F^e$ and hence F^e is injective on $H^i_{\mathfrak{m}}(R)$. $\square$

In fact, it can be shown that $\operatorname{Im}(H^{i-1}_{\mathfrak{m}}(R)) = 0$ in the proof of Theorem 5.5. This was observed in [MQ18], and we leave it as an exercise below.

Exercise 20. Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R such that R/xR is F-pure. Use Theorem 5.3 to show that the natural map $H^i_{\mathfrak{m}}(R/x^nR) \to H^i_{\mathfrak{m}}(R/xR)$ is surjective for all $n \geq 1$ and all i. Then use this to show that multiplication by x on $H^i_{\mathfrak{m}}(R)$ is surjective for all i.

Exercise 21. Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R such that R/xR is quasi-Gorenstein and F-pure. Use Exercise 20 to prove that R is quasi-Gorenstein and F-pure.

We caution the reader that, in general, the quasi-Gorenstein property does not deform [STT20, Theorem 4.2].

As we already mentioned, whether F-injectivity deforms in general remains an open question. We refer the reader to [MSS17, MQ18, DSM20] for some recent progress.

Open Problem 3. *Let $(R, \mathfrak{m})$ be a local ring and x a nonzerodivisor on R. If R/xR is F-injective, then is R also F-injective?*

6. The Γ-construction and completion of F-rationality

Our goal in this section is to show that completion excellent local F-rational rings are F-rational. To establish this, we need to show that in the definition of F-rationality, we actually only need to consider one (special) c. This is not difficult to prove if R is F-finite. To reduce the general case to the F-finite case, we need a powerful tool introduced by Hochster–Huneke [HH94a]: the Γ-construction.

Discussion 6.1 (Trace map). Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a module-finite extension of local rings of dimension d. Suppose ω_R is a canonical module of R (recall that this means $\omega_R^\vee \cong H^d_\mathfrak{m}(R)$). Then the canonical map $R \to S$ induces a trace map:

$$\omega_S \cong \operatorname{Hom}_R(S,\omega_R) \xrightarrow{\mathrm{Tr}} \omega_R.$$

The Matlis dual of this map

$$H^d_\mathfrak{m}(R) \to \operatorname{Hom}_R(\omega_S, E_R) \cong \operatorname{Hom}_S(\omega_S, \operatorname{Hom}_R(S,E_R)) \cong \operatorname{Hom}_S(\omega_S, E_S) \cong H^d_\mathfrak{n}(S)$$

is precisely the natural map on top local cohomology modules induced by $R \to S$. In particular, if R is F-finite, then the natural e-th Frobenius action $H^d_\mathfrak{m}(R) \to F^e_* H^d_\mathfrak{m}(R)$ corresponds to the trace map $F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$, and it can be checked that $\mathrm{Tr}^{e_1+e_2} = \mathrm{Tr}^{e_1} \circ F^{e_1}_*(\mathrm{Tr}^{e_2})$ and $(\mathrm{Tr}^e)_P$ is the corresponding trace map for R_P. Note that here we are implicitly using that F-finite rings admit canonical modules (see Theorem 1.5).

Proposition 6.2. *Let $(R,\mathfrak{m})$ be an F-finite Cohen–Macaulay local ring. Then R is F-rational if and only if for every $c \in R$ that is not in any minimal prime of R, there exists $e > 0$ such that the composition $F^e_*\omega_R \xrightarrow{\cdot F^e_* c} F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$ is surjective (i.e., $\mathrm{Tr}^e : F^e_*(c\omega_R) \to \omega_R$ is surjective).*

Proof. This follows immediately from Discussion 6.1 and the definition of F-rationality. ☐

Proposition 6.2 implies that if R is F-finite and F-rational, then R_P is F-rational for all $P \in \operatorname{Spec}(R)$. Of course, we have already proved a more general Theorem 4.14 without assuming R is F-finite.

The next result is an analog of Theorem 3.11 for F-rationality. We will eventually extend this result to excellent Cohen–Macaulay local rings in Section 7. But at this point, we only prove it when R is F-finite.

Proposition 6.3. *Let $(R,\mathfrak{m})$ be an F-finite Cohen–Macaulay local ring of dimension d. Suppose there exists c not in any minimal prime of R such that R_c is F-rational (e.g., R_c is* *regular). Then R is F-rational if and only if there exists $e > 0$ such that the composition*

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$$

is injective, or equivalently,

$$F^e_*\omega_R \xrightarrow{\cdot F^e_* c} F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$$

is surjective.

Proof. Suppose z is not in any minimal prime of R. Then z is not in any minimal prime of R_c and thus by Proposition 6.2, there exists e_0 such that $\mathrm{Tr}^{e_0}\colon F^{e_0}_*(z\omega_{R_c}) \to \omega_{R_c}$ is surjective.[2] Since R is F-finite, we know that

$$\mathrm{Hom}_{R_c}(F^{e_0}_*(z\omega_{R_c}), \omega_{R_c}) \cong \mathrm{Hom}_R(F^{e_0}_*(z\omega_R), \omega_R)_c.$$

Therefore we know that there exists $n > 0$ such that the image of $\mathrm{Tr}^{e_0}\colon F^{e_0}_*(z\omega_R) \to \omega_R$ contains $c^n\omega_R$.

Our assumption says that there exists $e > 0$ such that $c \cdot F^e$ is injective on $H^d_{\mathfrak{m}}(R)$. If we compose this map n times we get that $c^{1+p^e+\cdots+p^{ne}} \cdot F^{ne}$ is injective on $H^d_{\mathfrak{m}}(R)$, in particular, $c^n \cdot F^{ne}$ is injective on $H^d_{\mathfrak{m}}(R)$. That is, the composition

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^{ne}_* R) \xrightarrow{\cdot F^{ne}_* c^n} H^d_{\mathfrak{m}}(F^{ne}_* R)$$

is injective. But then by Discussion 6.1, we see that $\mathrm{Tr}^{ne}\colon F^{ne}_*(c^n\omega_R) \to \omega_R$ is surjective. Now the composition

$$\mathrm{Tr}^{ne+e_0} : F^{ne+e_0}_*(z\omega_R) \xrightarrow{F^{ne}_* \mathrm{Tr}^{e_0}} F^{ne}_*\omega_R \xrightarrow{\mathrm{Tr}^{ne}} \omega_R$$

is surjective and so by Proposition 6.2, R is F-rational. $\square$

As a consequence, we prove the following result on openness of F-rational locus for F-finite local rings. We will eventually extend this result to excellent local rings.

Proposition 6.4. *Let $(R, \mathfrak{m})$ be an F-finite local ring. Then the F-rational locus of* $\mathrm{Spec}(R)$ *is open.*

Proof. Suppose R_P is F-rational, it is enough to show that there exists $f \notin P$ such that R_f is F-rational. Since R_P is a domain by Proposition 4.4, there is a unique minimal prime P' of R that is contained in P and $R_{P'}$ is a field. Since R is excellent, there exists $c \notin P'$ such that R_c is regular. Since R_P is F-finite and F-rational, by Proposition 6.2 we know that there exists $e > 0$ such that $\mathrm{Tr}^e\colon F^e_*(c\omega_{R_P}) \to \omega_{R_P}$ is surjective (note that $c \neq 0$ in R_P). It

[2]Here we are using Proposition 6.2 for an F-finite but not necessarily local ring R_c, we leave it as an exercise to check that the proposition is still valid in this case.

follows that there exists $f \notin P$ such that $\mathrm{Tr}^e\colon F^e_*(c\omega_{R_f}) \to \omega_{R_f}$ is surjective. Since R_P is Cohen–Macaulay, we can replace f by a multiple to assume that R_f is Cohen–Macaulay,[3] and further replace f by a multiple to assume f is contained in all minimal primes of R except P' while keeping $f \notin P$ (this is possible since P does not contain the other minimal primes of R). Now the only minimal prime of R_f is $P'R_f$, and we have $c \notin P'$ and R_c is regular, thus by Proposition 6.3 (applied to each R_Q such that $Q \in D(f)$), we see that R_f is F-rational. □

Remark 6.5. In the proof of Proposition 6.3, we are implicitly using R is local since we need a trace map $\mathrm{Tr}\colon F^e_*\omega_R \to \omega_R$ that localizes to the corresponding trace map of R_P for all $P \in \mathrm{Spec}(R)$. It is well-known that this holds as long as R is F-finite and "sufficiently affine" (see [BB11], we will not make this precise here).[4] Now for any F-finite ring R, we can find a finite cover of $\mathrm{Spec}(R)$ by sufficiently affine open subsets $\cup D(f_i)$, the proof of Proposition 6.3 then works for each R_{f_i}. But a subset of $\mathrm{Spec}(R)$ is open if and only if its intersection with each $D(f_i)$ is open. Therefore for any F-finite (not necessarily local) ring R, the F-rational locus of $\mathrm{Spec}(R)$ is open.

We next introduce the Γ-construction of Hochster–Huneke [HH94a] – a very useful technique to reduce questions from complete local rings to the case of F-finite local rings. The results presented here: Lemma 6.9 – Lemma 6.14, originate from [HH94a] and [EH08].

Let K be a field of positive characteristic $p > 0$ with a p-base Λ. Let Γ be a fixed cofinite subset of Λ. For $e \in \mathbb{N}$ we denote by $K^{\Gamma,e}$ the purely inseparable field extension of K that is the result of adjoining p^e-th roots of all elements in Γ to K.

Discussion 6.6 (The Γ-construction). Let $(R, \mathfrak{m})$ be a complete local ring with K a coefficient field of R. Let $x_1, \dots, x_d$ be a system of parameters for R. By Cohen's structure theorem we know that R is module-finite over $A = K[[x_1, \dots, x_d]] \subseteq R$. We define

$$A^\Gamma := \bigcup_{e\in\mathbb{N}} K^{\Gamma,e}[[x_1, \dots, x_d]],$$

which is a regular local ring faithfully flat and purely inseparable over A (note that A^Γ is Noetherian, we leave this as Exercise 23). The maximal ideal of A expands to that of A^Γ. Set $R^\Gamma := A^\Gamma \otimes_A R$. Then R^Γ is module-finite over the regular local ring A^Γ, and R^Γ is faithfully

[3] It is well-known that the Cohen–Macaulay locus is open for excellent rings. In our context, we can argue as follows: $(R, \mathfrak{m})$ is a homomorphic image of a regular local ring $(S, \mathfrak{n})$ by Theorem 1.5, it is easy to check that R_P is Cohen–Macaulay if and only if $\mathrm{Ext}^j_S(R, S)_P = 0$ for all $j \neq n - d$ where $n = \dim(S)$ and $d = \dim(R)$, but if these Ext groups vanish when localized at P, then they vanish when inverting f for some $f \notin P$.

[4] In fact, Karl Schwede communicated to us that there exists a global trace map $\mathrm{Tr}\colon F^e_*\omega_R \to \omega_R$ for all F-finite rings, but this result has not been written down.

flat and purely inseparable over R. The maximal ideal of R expands to the maximal ideal of R^Γ and the residue field of R^Γ is $K^\Gamma := \bigcup_{e\in\mathbb{N}} K^{\Gamma,e}$. Note that, since $R \to R^\Gamma$ is purely inseparable, $\operatorname{Spec}(R^\Gamma)$ can be identified with $\operatorname{Spec}(R)$. For every $Q \in \operatorname{Spec}(R)$, we use Q^Γ to denote the unique prime ideal in R^Γ corresponds to Q, i.e., $Q^\Gamma = \sqrt{QR^\Gamma}$.

Remark 6.7. With notation as in Discussion 6.6, it is easy to see that $R^\Gamma = \bigcup_{e\in\mathbb{N}} R\widehat{\otimes}_K K^{\Gamma,e}$. In particular, the definition of R^Γ depends only on K (and the choice of p-base of K), but not on the choice of $x_1,\dots,x_d$.

Remark 6.8. With notation as in Discussion 6.6, we have $\operatorname{depth} R_Q = \operatorname{depth} R^\Gamma_{Q^\Gamma}$ since $R_Q \to R^\Gamma_{Q^\Gamma}$ is purely inseparable. In particular, R_Q is Cohen–Macaulay if and only if $R^\Gamma_{Q^\Gamma}$ is Cohen–Macaulay.

Lemma 6.9. *With notation as in Discussion 6.6, R^Γ is F-finite.*

Proof. It is enough to show that A^Γ is F-finite, that is, F_*A^Γ is finitely generated as an A^Γ-module, or equivalently, $(A^\Gamma)^{1/p}$ is finitely generated over A^Γ – it will be convenient to use this latter notation in the proof.

Let $\theta_1,\dots,\theta_n$ be the finitely many elements in $\Lambda-\Gamma$. Then the following finite set

$$\Theta := \{\theta_1^{i_1/p}\cdots\theta_n^{i_n/p}\cdot x_1^{j_1/p}\cdots x_d^{j_d/p} | 0 \le i_k, j_k \le p-1\}$$

is a generating set of $(A^\Gamma)^{1/p}$ over A^Γ. To see this, note that

$$(A^\Gamma)^{1/p} = \bigcup_{e\in\mathbb{N}} (K^{\Gamma,e}[[x_1,\dots,x_d]])^{1/p},$$

and it is easy to check that $(K^{\Gamma,e}[[x_1,\dots,x_d]])^{1/p}$ is generated over $K^{\Gamma,e+1}[[x_1,\dots,x_d]]$ by Θ. Thus after passing to the union, we see that Θ is a generating set of $(A^\Gamma)^{1/p}$ over A^Γ. $\square$

Lemma 6.10. *With notation as in Discussion 6.6, if Q is a prime ideal of R, then for all sufficiently small choices of Γ, we have $Q^\Gamma = QR^\Gamma$.*

Proof. Replacing R by R/Q, it is enough to show that if R is a complete local domain, then R^Γ is a domain for all sufficiently small choices of Γ (see Remark 6.7).

We let L, L^Γ, L_R denote the fraction field of A, A^Γ, R respectively. Since A^Γ is purely inseparable over A, we know that $L^\Gamma = L\otimes_A A^\Gamma$. Also note that L_R is a finite extension of L. We first observe that it suffices to show $L_R\otimes_L L^\Gamma$ is a field for sufficiently small choices of Γ: for if this is true, then we have

$$R^\Gamma = R\otimes_A A^\Gamma \hookrightarrow L_R\otimes_A A^\Gamma = L_R\otimes_L L\otimes_A A^\Gamma = L_R\otimes_L L^\Gamma$$

and hence R^Γ is a domain as desired (the injection above follows because A^Γ is flat over A). We next note that, since $A^\Gamma \hookrightarrow K^\Gamma[[x_1,\dots,x_d]]$, we have $L^\Gamma \subseteq \mathrm{Frac}(K^\Gamma[[x_1,\dots,x_d]])$ and thus

$$\bigcap_{\Gamma\subseteq\Lambda \text{ cofinite}} L^\Gamma \subseteq \bigcap_{\Gamma\subseteq\Lambda \text{ cofinite}} \mathrm{Frac}(K^\Gamma[[x_1,\dots,x_d]]) = \mathrm{Frac}(K[[x_1,\dots,x_d]]) = L$$

where the middle equality follows from [Mat70, 30.E] since $\bigcap K^\Gamma = K$. Thus we have $\bigcap L^\Gamma = L$. Let $\{\lambda_1,\dots,\lambda_n\}$ be a basis of L_R over L. To show $L_R\otimes_L L^\Gamma$ is a field, it is enough to show $\{\lambda_1,\dots,\lambda_n\}$ are linearly independent over L^Γ (view all fields in a fixed ambient $\overline{L}$). We pick Γ such that the number of linearly independent vectors of $\{\lambda_1,\dots,\lambda_n\}$ over L^Γ is maximum among all the L^Γ. If this number is $h<n$, then without loss of generality we can assume $\{\lambda_1,\dots,\lambda_h\}$ are linearly independent over L^Γ but $\lambda_{h+1} = \ell_1\lambda_1 + \cdots \ell_h\lambda_h$ where $\ell_i\in L^\Gamma$ and at least one of the ℓ_i, say ℓ_1, is not in L. Since $\bigcap L^\Gamma = L$, we can pick $\Gamma'\subseteq\Gamma$ such that $\ell_1\notin L^{\Gamma'}$. But then λ_{h+1} cannot be written as a linear combination of $\lambda_1,\dots,\lambda_h$ over $L^{\Gamma'}$ (if so then we have two expressions of λ_{h+1} as linear combinations of $\lambda_1,\dots,\lambda_h$ over L^Γ which contradict the linear independency of $\{\lambda_1,\dots,\lambda_h\}$ over L^Γ), it follows that $\{\lambda_1,\dots,\lambda_{h+1}\}$ are linearly independent over $L^{\Gamma'}$ contradicting our choice of Γ. Therefore, for all sufficiently small choices of Γ, $L_R\otimes_L L^\Gamma$ is a field. $\square$

Remark 6.11. With notation as in Discussion 6.6, if R is a domain, then we have $\mathrm{Frac}(R) = \bigcap \mathrm{Frac}(R^\Gamma)$ where the intersection is taken over all sufficiently small Γ such that R^Γ is a domain. In fact, following the notation as in the proof of Lemma 6.10, we have

$$\bigcap \mathrm{Frac}(R^\Gamma) = \bigcap (L_R\otimes_L L^\Gamma) = L_R\otimes_L \bigcap L^\Gamma = L_R\otimes_L L = L_R = \mathrm{Frac}(R)$$

where the second equality is because L_R is a finite field extension of L and the third equality uses $\bigcap L^\Gamma = L$ as in the proof of Lemma 6.10.

Lemma 6.12. *With notation as in Discussion 6.6, if R_Q is regular then $R^\Gamma_{Q^\Gamma}$ is regular for all sufficiently small choices of Γ. In fact, the regular locus of $\mathrm{Spec}(R)$ can be identified with the regular locus of $\mathrm{Spec}(R^\Gamma)$ for all sufficiently small choices of Γ.*

Proof. By Lemma 6.10, for sufficiently small Γ, $QR^\Gamma = Q^\Gamma$ is a prime ideal. Thus $R_Q\to R^\Gamma_{Q^\Gamma}$ is a faithfully flat extension whose closed fiber is a field, so it follows that $R^\Gamma_{Q^\Gamma}$ is regular.

We use $\mathrm{Reg}(R)$ to denote the regular locus of $\mathrm{Spec}(R)$. For any $\Gamma'\subseteq\Gamma$ two cofinite subsets of Λ, we have a faithfully flat purely inseparable extension $R^{\Gamma'}\to R^\Gamma$ which induces a faithfully flat extension $R^{\Gamma'}_{P^{\Gamma'}}\to R^\Gamma_{P^\Gamma}$. Thus if $P^\Gamma\in\mathrm{Reg}(R^\Gamma)$, then $P^{\Gamma'}\in\mathrm{Reg}(R^{\Gamma'})$. Thus after we identify $\mathrm{Spec}(R^\Gamma)$ with $\mathrm{Spec}(R)$, we have $\mathrm{Reg}(R^\Gamma)\subseteq\mathrm{Reg}(R^{\Gamma'})$ (note that these are open subsets of $\mathrm{Spec}(R)$ since all R^Γ are F-finite by Lemma 6.9 and hence excellent). Since open subsets of $\operatorname{Spec}(R)$ satisfy ascending chain condition, we know that for all sufficiently small choices of Γ, $\operatorname{Reg}(R^\Gamma) = \operatorname{Reg}(R^{\Gamma'})$ for all $\Gamma' \subseteq \Gamma$. Fix such a Γ, we will show that $\operatorname{Reg}(R) = \operatorname{Reg}(R^\Gamma)$. Clearly $\operatorname{Reg}(R^\Gamma) \subseteq \operatorname{Reg}(R)$. Suppose there exists $Q \in \operatorname{Reg}(R)$ but $Q^\Gamma \notin \operatorname{Reg}(R^\Gamma)$. Then by the first part of the lemma we can pick a sufficiently small $\Gamma' \subseteq \Gamma$ such that $Q^{\Gamma'} \in \operatorname{Reg}(R^{\Gamma'})$, but then $\operatorname{Reg}(R^{\Gamma'}) \neq \operatorname{Reg}(R^\Gamma)$ which is a contradiction. $\square$

Lemma 6.13. *With notation as in Discussion 6.6, if $Q \in \operatorname{Spec}(R)$ and W is an Artinian R_Q-module with an injective Frobenius action, then for all sufficiently small choices of Γ the induced Frobenius action is injective on $W^\Gamma := W \otimes_{R_Q} R^\Gamma_{Q^\Gamma}$.*

Proof. By Lemma 6.10, we may assume Γ is small enough such that $Q^\Gamma = QR^\Gamma$. Then we have $\kappa(Q^\Gamma) = \operatorname{Frac}(R^\Gamma/QR^\Gamma)$ and $\bigcap \kappa(Q^\Gamma) = \kappa(Q)$ (see Remark 6.11).

Let V be the socle of W. Since W is Artinian, V is a finite dimensional vector space over $\kappa(Q)$ and $V^\Gamma := V \otimes_{R_Q} R^\Gamma_{Q^\Gamma} = V \otimes_{\kappa(Q)} \kappa(Q^\Gamma)$ is the socle of W^Γ (as a module over $R^\Gamma_{Q^\Gamma}$). Let F be the given Frobenius action on W and let F^Γ be the induced Frobenius action on W^Γ. Set $U^\Gamma := V^\Gamma \cap \operatorname{Ker}(F^\Gamma)$ which is a $\kappa(Q^\Gamma)$-subspace of V^Γ.

Note that $U^{\Gamma'} \subseteq U^\Gamma$ whenever $\Gamma' \subseteq \Gamma$ and F^Γ is injective on W^Γ if and only if $U^\Gamma = 0$. We pick Γ sufficiently small such that $\dim(U^\Gamma)$ is the smallest. We next fix a basis $v_1, \dots, v_n$ of V over $\kappa(Q)$. If $\dim(U^\Gamma) > 0$, then we choose a basis of U^Γ over $\kappa(Q^\Gamma)$ and write each basis vector as $\sum a_{ij} v_j$ where $a_{ij} \in \kappa(Q^\Gamma)$. Now the reduced row echelon form of (a_{ij}) is *uniquely determined* by U^Γ, and in this reduced row echelon form, each row must contain an entry not in $\kappa(Q)$ since $U^\Gamma \cap V = 0$ (as F is injective on W). But since $\bigcap \kappa(Q^\Gamma) = \kappa(Q)$, there exists $\Gamma' \subseteq \Gamma$ such that at least one of these entries is not in $\kappa(Q^{\Gamma'})$, it follows that $U^{\Gamma'}$ must have dimension strictly smaller than $\dim(U^\Gamma)$ (choose a basis of $U^{\Gamma'}$ and look at the reduced row echelon form with respect to $v_1, \dots, v_n$ again, it must have fewer rows). This contradicts our choice of Γ. Thus for all sufficiently small Γ, $U^\Gamma = 0$ and so F^Γ is injective as desired. $\square$

Lemma 6.14. *With notation as in Discussion 6.6, if R_Q is F-rational, then $R^\Gamma_{Q^\Gamma}$ is F-rational for all sufficiently small choices of Γ. In fact, the F-rational locus of $\operatorname{Spec}(R)$ can be identified with the F-rational locus of $\operatorname{Spec}(R^\Gamma)$ for all sufficiently small choices of Γ.*

Proof. Since R_Q is an excellent local domain (by Proposition 4.4), there exists $c \in R$ whose image in R_Q is nonzero such that $(R_Q)_c$ is regular. Since $\operatorname{Reg}(R) = \operatorname{Reg}(R^\Gamma)$ for sufficiently small choices of Γ by Lemma 6.12, $\operatorname{Reg}(R_c) = \operatorname{Reg}(R^\Gamma_c)$ and thus $(R^\Gamma_{Q^\Gamma})_c$ is regular. Since $R^\Gamma_{Q^\Gamma}$ is F-finite and Cohen–Macaulay, by Proposition 6.3 it is enough to show there exists $e > 0$ such that for all sufficiently small choices of Γ,

$$H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \to F^e_* H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \xrightarrow{\cdot F^e_* c} F^e_* H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma})$$

is injective, where $h = \mathrm{ht}(Q)$. This follows from Lemma 6.13 since R_Q is F-rational and $H^h_{Q^\Gamma}(R^\Gamma_{Q^\Gamma}) \cong H^h_Q(R_Q) \otimes_{R_Q} R^\Gamma_{Q^\Gamma}$.

The rest of the proof is very similar to Lemma 6.12. We use $\mathrm{Frat}(R)$ to denote the F-rational locus of $\mathrm{Spec}(R)$. For any $\Gamma' \subseteq \Gamma$ two cofinite subsets of Λ, we have a faithfully flat extension $R^{\Gamma'} \to R^\Gamma$ which induces a faithfully flat extension $R^{\Gamma'}_{P^{\Gamma'}} \to R^\Gamma_{P^\Gamma}$. Thus if $P^\Gamma \in \mathrm{Frat}(R^\Gamma)$, then $P^{\Gamma'} \in \mathrm{Frat}(R^{\Gamma'})$ by Exercise 16. Thus after we identify $\mathrm{Spec}(R^\Gamma)$ with $\mathrm{Spec}(R)$, we have $\mathrm{Frat}(R^\Gamma) \subseteq \mathrm{Frat}(R^{\Gamma'})$ (note that these are open subsets of $\mathrm{Spec}(R)$ by Proposition 6.4). Since open subsets of $\mathrm{Spec}(R)$ satisfy ascending chain condition, we know that for all sufficiently small choices of Γ, $\mathrm{Frat}(R^\Gamma) = \mathrm{Frat}(R^{\Gamma'})$ for all $\Gamma' \subseteq \Gamma$. Fix such a Γ, we will show that $\mathrm{Frat}(R) = \mathrm{Frat}(R^\Gamma)$. Clearly $\mathrm{Frat}(R^\Gamma) \subseteq \mathrm{Frat}(R)$. Suppose there exists $Q \in \mathrm{Frat}(R)$ but $Q^\Gamma \notin \mathrm{Frat}(R^\Gamma)$. Then by the first part of the lemma we can pick a sufficiently small $\Gamma' \subseteq \Gamma$ such that $Q^{\Gamma'} \in \mathrm{Frat}(R^{\Gamma'})$, but then $\mathrm{Frat}(R^{\Gamma'}) \neq \mathrm{Frat}(R^\Gamma)$ which is a contradiction. $\square$

Corollary 6.15. *Let $(R, \mathfrak{m})$ be a complete local ring. Then the F-rational locus of* $\mathrm{Spec}(R)$ *is open.*

Proof. By Lemma 6.9, for all sufficiently small choices of Γ, R^Γ is F-finite. Thus by Proposition 6.4, the F-rational locus of $\mathrm{Spec}(R^\Gamma)$ is open. Hence so is the F-rational locus of $\mathrm{Spec}(R)$ by Lemma 6.14. $\square$

We can now prove the following.

Theorem 6.16. *Let $(R, \mathfrak{m})$ be an excellent Cohen–Macaulay local ring. Suppose there exists c not in any minimal prime of R such that R_c is regular. Then $\widehat{R}$ is F-rational (and hence R is F-rational) if there exists $e > 0$ such that the composition*

$$H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_\mathfrak{m}(F^e_* R)$$

is injective. In particular, R is excellent and F-rational if and only if $\widehat{R}$ is F-rational.

Proof. We first note that $H^d_\mathfrak{m}(R) = H^d_\mathfrak{m}(\widehat{R})$ and if $c \in R$ is not in any minimal prime of R, then c is not in any minimal prime of $\widehat{R}$. Thus it is clear that $\widehat{R}$ is F-rational implies R is F-rational (this is also a special case of Exercise 16, and we do not need to assume R is excellent).

Since R is excellent, $R \to \widehat{R}$ has geometrically regular fibers and hence we know that $\widehat{R}_c$ is also regular. By Lemma 6.12, $\widehat{R}^\Gamma_c$ is regular for sufficiently small choices of Γ. Moreover, since

$$H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_\mathfrak{m}(F^e_* R)$$

is injective, By Lemma 6.13, it follows that for sufficiently small choices of Γ,

$$H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \to F^e_* H^d_{\mathfrak{m}}(\widehat{R}^\Gamma) \xrightarrow{\cdot F^e_* c} F^e_* H^d_{\mathfrak{m}}(\widehat{R}^\Gamma)$$

is injective.

Since $\widehat{R}^\Gamma$ is F-finite and Cohen–Macaulay, and $\widehat{R}^\Gamma_c$ is regular (note that c is not in any minimal prime of $\widehat{R}^\Gamma$ since $R \to \widehat{R}^\Gamma$ is flat), Proposition 6.3 shows that $\widehat{R}^\Gamma$ is F-rational. But then since $\widehat{R} \to \widehat{R}^\Gamma$ is faithfully flat, $\widehat{R}$ is F-rational by Exercise 16. The last conclusion follows since the assumptions are clearly satisfied if R is F-rational. $\square$

Remark 6.17. There are examples of non-excellent F-rational local ring $(R, \mathfrak{m})$ such that $\widehat{R}$ is not F-rational, see [LR01].

Exercise 22. Let R be an F-finite ring. Prove that the F-injective, F-pure and strongly F-regular locus of $\operatorname{Spec}(R)$ are open.

Exercise 23. With notation as in Discussion 6.6, prove that $A^\Gamma \to K^\Gamma[[x_1, \ldots, x_d]]$ is faithfully flat, use this to show that A^Γ is Noetherian.

Exercise 24. With notation as in Discussion 6.6, use Lemma 6.10 to prove that if J is a radical ideal of R, then for all sufficiently small choices of Γ, we have JR^Γ is radical (in particular if R is reduced then R^Γ is reduced for all sufficiently small Γ).

In Proposition 6.3, Remark 6.5, and Exercise 22, we have seen that for F-finite rings, the loci of $\operatorname{Spec}(R)$ such that R is F-rational (resp. F-injective, F-pure) are open. In Section 7, we will show that the same holds for excellent *local* rings. It is natural to ask the following question.

Open Problem 4. *Let R be an excellent ring. Are the F-rational (resp. F-injective, F-pure) locus of* $\operatorname{Spec}(R)$ *open?*

This has an affirmative answer when R is F-finite, or R is essentially of finite type over an excellent local ring – this is basically because the theory of Γ-construction can extended to this set up (see [HH94a] or [Mur18]). But the general case seems open.

We caution the reader that, one cannot expect the openness of loci for these F-singularities without the excellent assumption, for example see [DM19, Theorem 5.10] (which is based on [Hoc73]).

7. F-singularities under faithfully flat base change

The goal of this section is to study F-singularities under faithfully flat base change. The general question we are interested is the following : suppose $(R,\mathfrak{m})\to(S,\mathfrak{n})$ is a flat local extension such that the base ring R and the closed fiber $S/\mathfrak{m}S$ has certain type of F-singularities, then whether S has the same type of F-singularities? For example, if R is a DVR with uniformizer t, then $S/\mathfrak{m}S\cong S/tS$ where t is a nonozerodivisor of S, and this is precisely the deformation question we studied in Section 5.

Since even the deformation question has a negative answer in general (e.g., for F-pure and strongly F-regular singularities, see Section 9), one cannot expect the general question hold without additional assumptions. We will present what is known and point out some open questions in this area in the end. We first recall a well-known lemma.

Lemma 7.1 ([Mat70, Section 21]). *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local extension such that $S/\mathfrak{m}S$ is Cohen–Macaulay. Let $\underline{x}:=x_1,\dots,x_d$ be a system of parameters of $S/\mathfrak{m}S$. Then $x_1,\dots,x_d$ is a regular sequence on S and $S/(\underline{x})S$ is faithfully flat over R. In particular, $H^d_{(\underline{x})}(S)$ is faithfully flat over R.*

We also need the following result on the behavior of injective hull under faithfully flat extension with Gorenstein closed fiber, which is due to Hochster–Huneke (in this generality).

Lemma 7.2 ([HH94a, Lemma 7.10]). *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local extension such that $S/\mathfrak{m}S$ is Gorenstein. Let $\underline{x}:=x_1,\dots,x_d$ be a system of parameters of $S/\mathfrak{m}S$. Then $E_S\cong E_R\otimes_R H^d_{(\underline{x})}(S)$. Moreover, if u is a socle representative of E_R and the image of $\frac{v}{x_1\cdots x_d}\in H^d_{(\underline{x})}(S)$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ is a socle representative of $H^d_{(\underline{x})}(S/\mathfrak{m}S)$, then $u\otimes\frac{v}{x_1\cdots x_d}$ is a socle representative of $E_S\cong E_R\otimes_R H^d_{(\underline{x})}(S)$.*

We first prove general base change results on F-pure and strongly F-regular singularities. These results are due to Aberbach.

Theorem 7.3 ([Abe01]). *Let $(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local extension such that R is F-pure and $S/\mathfrak{m}S$ is Gorenstein and F-pure. Then S is F-pure.*

Proof. By Lemma 7.2 and Proposition 2.2, it is enough to show that

$$E_R\otimes_R H^d_{(\underline{x})}(S)\to E_R\otimes_R H^d_{(\underline{x})}(S)\otimes_S F^e_*S\cong E_R\otimes_R F^e_*R\otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S)$$

is injective for all $e>0$. Now the image of the socle representative $u\otimes\frac{v}{x_1\cdots x_d}$ under the map is $u\otimes F^e_*1\otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$. Thus it is enough to show this element is nonzero in $E_R\otimes_R F^e_*R\otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S)$. Since R is F-pure, $u\otimes F^e_*1\neq 0$ in $E_R\otimes_R F^e_*R$. Thus there exists a nonzero (F^e_*R)-linear map $F^e_*R \to E_R \otimes_R F^e_*R$ sending F^e_*1 to $u \otimes_R F^e_*1$, say with kernel F^e_*J. Since $F^e_*H^d_{(\underline{x})}(S)$ is faithfully flat over F^e_*R by Lemma 7.1, we have an injection:

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \hookrightarrow E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

The image of $F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$ under this map is precisely $u \otimes F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$. Thus to show the latter one is nonzero, it is enough to show $F^e_*1 \otimes F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$. But

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \twoheadrightarrow (F^e_*R/F^e_*\mathfrak{m}) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \cong F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)),$$

thus it is enough to show that $F^e_*(\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$ in $F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S))$, that is, $\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$. But $S/\mathfrak{m}S$ is F-pure, in particular F-injective, hence the Frobenius action on $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ is injective. Since $\frac{v}{x_1\cdots x_d} \neq 0$, $\frac{v^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} = F^e(\frac{v}{x_1\cdots x_d}) \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$. $\square$

Theorem 7.4 ([Abe01])**.** *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local extension of F-finite rings such that R is strongly F-regular and $S/\mathfrak{m}S$ is Gorenstein and strongly F-regular. Then S is strongly F-regular.*

Proof. Since $S/\mathfrak{m}S$ is strongly F-regular, it is a normal domain by Proposition 3.8. Thus $\mathfrak{m}S$ is a prime ideal in S. We first show that $S' = S_{\mathfrak{m}S}$ is strongly F-regular. We know that $R \to S'$ is a flat local extension such that $S'/\mathfrak{m}S'$ is a field. Moreover, by Proposition 3.12, we may replace R and S' by their completion to assume R and S' are both complete.

Suppose there exists $c \in S'$ not in any minimal prime of S' such that for all $e > 0$, the map $S' \to F^e_*S'$ sending 1 to F^e_*c is not split, then by Corollary 2.4 and Proposition 2.2, the map $E_{S'} \to E_{S'} \otimes_{S'} F^e_*S'$ induced by sending 1 to F^e_*c is not injective for all $e > 0$, thus the socle of $E_{S'}$ maps to zero under this map. By Lemma 7.2, $E_{S'} \cong E_R \otimes_R S'$ and a socle representative is $u \otimes 1$ where u is a socle representative of E_R. It follows that

$$E_{S'} \otimes_{S'} F^e_*S' \cong E_R \otimes_R S' \otimes_{S'} F^e_*S' \cong E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'$$

and that $u \otimes F^e_*1 \otimes F^e_*c = 0$ in $E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'$ for all $e > 0$. Thus

$$F^e_*c \in \operatorname{Ann}_{E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*S'}(u \otimes F^e_*1 \otimes F^e_*1) \cong (\operatorname{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)) \otimes_{F^e_*R} F^e_*S'$$

for all $e > 0$ where the isomorphism follows from that F^e_*S' is flat over F^e_*R. However, since R is strongly F-regular, we know that for all $0 \neq z \in R$, there exists $e > 0$ such that the map $R \to F^e_*R$ sending 1 to F^e_*z is split, thus $F^e_*z \notin \operatorname{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)$ (again by Corollary 2.4 and Proposition 2.2). Therefore, if we define $F^e_*I_e := \operatorname{Ann}_{E_R \otimes_R F^e_*R}(u \otimes F^e_*1)$, then $\cap_e I_e = 0$ and $0 \neq c \in \cap_e(I_e \otimes_R S')$. But by Chevalley's lemma, for all $n > 0$, there exists $e(n)$ such that $I_{e(n)} \subseteq \mathfrak{m}^n$, thus $\cap_e(I_e \otimes_R S') \subseteq \cap_n \mathfrak{m}^n S' = 0$ which is a contradiction.

So far we have proved that $S_{\mathfrak{m}S}$ is strongly F-regular. By Exercise 22, there exists $c \notin \mathfrak{m}S$ such that S_c is strongly F-regular. Note that c is a nonzerodivisor on $S/\mathfrak{m}S$ and thus it is a nonzerodivisor on S by Lemma 7.1, in particular, c is not in any minimal prime of S. By Theorem 3.11, it is enough to show that there exists $e > 0$ such that the map $S \to F^e_*S$ sending 1 to F^e_*c is split. The rest of the proof is very similar to the proof of Theorem 7.3. By Corollary 2.4 and Proposition 2.2, it is enough to show that the map $E_S \to E_S \otimes_S F^e_*S$ induced by sending 1 to F^e_*c is injective for some $e > 0$. By Lemma 7.2, this is the same as the map

$$E_R \otimes_R H^d_{(\underline{x})}(S) \to E_R \otimes_R H^d_{(\underline{x})}(S) \otimes_S F^e_*S \cong E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

Now the image of the socle representative $u \otimes \frac{v}{x_1\cdots x_d}$ under the map is $u \otimes F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$. Thus it is enough to show this element is nonzero in $E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S)$. Since R is strongly F-regular (in particular F-pure), $u \otimes F^e_*1 \neq 0$ in $E_R \otimes_R F^e_*R$. Thus there exists a nonzero (F^e_*R)-linear map $F^e_*R \to E_R \otimes_R F^e_*R$ sending F^e_*1 to $u \otimes_R F^e_*1$, say with kernel F^e_*J. Since $F^e_*H^d_{(\underline{x})}(S)$ is faithfully flat over F^e_*R by Lemma 7.1, we have an injection:

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \hookrightarrow E_R \otimes_R F^e_*R \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S).$$

The image of $F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$ under this map is precisely $u \otimes F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}})$. Thus to show the latter one is nonzero for some $e > 0$, it is enough to show $F^e_*1 \otimes F^e_*(\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$ for some $e > 0$. But we have

$$(F^e_*R/F^e_*J) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \twoheadrightarrow (F^e_*R/F^e_*\mathfrak{m}) \otimes_{F^e_*R} F^e_*H^d_{(\underline{x})}(S) \cong F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)).$$

Thus it is enough to show that $F^e_*(\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}}) \neq 0$ in $F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S))$ for some $e > 0$, that is, $\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ for some $e > 0$. But $S/\mathfrak{m}S$ is strongly F-regular, and hence F-rational by Theorem 4.6. Therefore since $\frac{v}{x_1\cdots x_d} \neq 0$, $\frac{cv^{p^e}}{x_1^{p^e}\cdots x_d^{p^e}} = cF^e(\frac{v}{x_1\cdots x_d}) \neq 0$ in $H^d_{(\underline{x})}(S/\mathfrak{m}S)$ for some $e > 0$ as desired. ☐

We next prove the general base change result on F-injectivity, due to Datta–Murayama.

Theorem 7.5 ([DM19])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension such that R is F-injective and $S/\mathfrak{m}S$ is Cohen–Macaulay and geometrically F-injective. Then S is F-injective.*

Proof. Let $\underline{x} := x_1, \dots, x_d$ be a system of parameters of $S/\mathfrak{m}S$. We first claim the following:

Claim 7.6. *For any Artinian R-module M, the map $F^e_*M \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(M \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_*m \otimes \eta \to F^e_*(m \otimes F^e(\eta))$ is injective for all $e > 0$, where $F^e(-)$ is the natural Frobenius action on $H^d_{(\underline{x})}(S)$.*

Proof. By taking a direct limit, it suffices to prove the claim for all R-modules of finite length. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence, then since $F^e_*(-)$ and $\otimes_R H^d_{(\underline{x})}(S)$ are both exact (by Lemma 7.1), we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F^e_*M_1 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & F^e_*M_2 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & F^e_*M_3 \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^e_*(M_1 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & F^e_*(M_2 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & F^e_*(M_3 \otimes_R H^d_{(\underline{x})}(S)) \arrow[r] & 0.
\end{tikzcd}$$

Thus to prove the claim for M_2, it is enough to prove it for M_1 and M_3. So by induction on the length of M, it is enough to prove the claim for $M = R/\mathfrak{m} =: k$. But we have the following commutative diagram

$$\begin{tikzcd}
F^e_*k \otimes_R H^d_{(\underline{x})}(S) \arrow[r] \arrow[d, "\cong"] & F^e_*(k \otimes_R H^d_{(\underline{x})}(S)) \arrow[d, "\cong"] & \\
H^d_{(\underline{x})}(F^e_*k \otimes_k S/\mathfrak{m}S) \arrow[r] & F^e_*(H^d_{(\underline{x})}(S/\mathfrak{m}S)) \arrow[r] & F^e_*(H^d_{(\underline{x})}(F^e_*k \otimes_k S/\mathfrak{m}S))
\end{tikzcd} .$$

The composition map in the second row is injective, because it is a direct limit of the natural Frobenius map $H^d_{(\underline{x})}(k' \otimes_k S/\mathfrak{m}S) \to F^e_*(H^d_{(\underline{x})}(k' \otimes_k S/\mathfrak{m}S))$ (where k' is a finite extension of k in F^e_*k), which is injective since $S/\mathfrak{m}S$ is geometrically F-injective. Thus the map in the first row is injective as desired. $\square$

Now Claim 7.6 implies that the natural map

$$F^e_*H^i_\mathfrak{m}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^i_\mathfrak{m}(R) \otimes_R H^d_{(\underline{x})}(S)) \cong F^e_*H^{i+d}_\mathfrak{n}(S)$$

is injective (the last isomorphism follows from Lemma 7.1 and a simple computation using the spectral sequence $H^i_\mathfrak{m}H^j_{(\underline{x})}(S) \Rightarrow H^{i+j}_\mathfrak{n}(S)$). But $H^i_\mathfrak{m}(R) \to F^e_*H^i_\mathfrak{m}(R)$ is injective since R is injective, thus as $H^d_{(\underline{x})}(S)$ is faithfully flat over R by Lemma 7.1, we know that

$$H^{i+d}_\mathfrak{n}(S) \cong H^i_\mathfrak{m}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*H^i_\mathfrak{m}(R) \otimes_R H^d_{(\underline{x})}(S)$$

is injective. Composing the two maps we find that $H^{i+d}_\mathfrak{n}(S) \to F^e_*H^{i+d}_\mathfrak{n}(S)$ is injective for all i (we leave it to the reader to check that this map is precisely the natural Frobenius action on $H^{i+d}_\mathfrak{n}(S)$). Thus S is F-injective. $\square$

It will take us considerable effort to prove the corresponding base change result for F-rationality. We first prove a special case, that is, when $S/\mathfrak{m}S$ is geometrically regular. This result was originally obtained by Vélez (which extended some results in [HH94a]).

Theorem 7.7 ([Vél95])**.** *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local extension of excellent rings such that R is F-rational and $S/\mathfrak{m}S$ is geometrically regular. Then S is F-rational.*

Proof. Since $S/\mathfrak{m}S$ is geometrically regular (so clearly Cohen–Macaulay and geometrically F-injective), by Claim 7.6 we know that

$$F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S))$$

is injective for all $e > 0$, where $n = \dim(R)$ and $d = \dim(S/\mathfrak{m}S)$.

Furthermore, since R is excellent and $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ is flat local with $S/\mathfrak{m}S$ geometrically regular, $\kappa(P) \otimes_R S$ is geometrically regular over $\kappa(P)$ for all $P \in \operatorname{Spec}(R)$ by [And74, Thm on page 297]. In particular, there exists $0 \neq c \in R$ such that R_c and S_c are both regular (note that R is a domain by Proposition 4.4 and thus c is not in any minimal prime of S since $R \to S$ is flat). Now since R is F-rational, there exists $e > 0$ such that

$$H^n_{\mathfrak{m}}(R) \to F^e_* H^n_{\mathfrak{m}}(R) \xrightarrow{\cdot F^e_* c} F^e_* H^n_{\mathfrak{m}}(R)$$

is injective. This injection is preserved after tensoring with $H^d_{(\underline{x})}(S)$ since the latter is flat over R by Lemma 7.1, and thus the composition

$$H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \xrightarrow{\cdot F^e_* c} F^e_* H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)))$$

is injective. After identifying $H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)$ with $H^{n+d}_{\mathfrak{n}}(S)$ (again, this follows from Lemma 7.1 and the spectral sequence $H^i_{\mathfrak{m}} H^j_{(\underline{x})}(S) \Rightarrow H^{i+j}_{\mathfrak{n}}(S)$), the above injection is precisely

$$H^{n+d}_{\mathfrak{n}}(S) \to F^e_* H^{n+d}_{\mathfrak{n}}(S) \xrightarrow{\cdot F^e_* c} H^{n+d}_{\mathfrak{n}}(S).$$

Since S is excellent Cohen–Macaulay and S_c is regular (and c is not in any minimal prime of S), S is F-rational by Theorem 6.16. $\square$

The above theorem allows us to prove the following criterion for F-rationality. This is a full generalization of Proposition 6.3 and Theorem 6.16.

Theorem 7.8. *Let $(R,\mathfrak{m})$ be an excellent Cohen–Macaulay local ring. Suppose there exists c not in any minimal prime of R such that R_c is F-rational. Then R is F-rational if and only if there exists $e > 0$ such that the composition*

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$$

is injective.

Proof. Since $R_c \to \widehat{R}_c$ has geometrically regular fibers (as R is excellent), we know that $\widehat{R}_c$ is F-rational by Theorem 7.7. It follows that for sufficiently small choices of Γ, $\widehat{R}^{\Gamma}_c$ is F-rational by Lemma 6.14. Moreover, since

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R) \xrightarrow{\cdot F^e_* c} H^d_{\mathfrak{m}}(F^e_* R)$$

is injective, it follows that for sufficiently small choices of Γ,

$$H^d_{\mathfrak{m}}(R^\Gamma) \to F^e_* H^d_{\mathfrak{m}}(R^\Gamma) \xrightarrow{\cdot F^e_* c} F^e_* H^d_{\mathfrak{m}}(R^\Gamma)$$

is injective. Since $\widehat{R}^\Gamma$ is F-finite and $\widehat{R}^\Gamma_c$ is F-rational (and c is not in any minimal prime of $\widehat{R}^\Gamma$ since $R \to \widehat{R}^\Gamma$ is flat), Proposition 6.3 shows that $\widehat{R}^\Gamma$ is F-rational. But then since $R \to \widehat{R}^\Gamma$ is faithfully flat, R is F-rational by Exercise 16. $\square$

We can also extend Proposition 6.4 to the case of excellent local rings, this was originally proved by Vélez.

Theorem 7.9 ([Vél95])**.** *Let $(R, \mathfrak{m})$ be an excellent local ring. Then the F-rational locus of* $\operatorname{Spec}(R)$ *is open.*

Proof. By Corollary 6.15, we know that the F-rational locus of $\operatorname{Spec}(\widehat{R})$ is open. Let $V(I) \subseteq \operatorname{Spec}(\widehat{R})$ be the non-F-rational locus where $I \subseteq \widehat{R}$ is a radical ideal. We claim that the non-F-rational locus of $\operatorname{Spec}(R)$ is precisely $V(I \cap R)$.

To see this, first note that if $P \in \operatorname{Spec}(R)$ such that P does not contain $I \cap R$, then any prime $Q \in \operatorname{Spec}(\widehat{R})$ lying over P does not contain I and thus $\widehat{R}_Q$ is F-rational, which implies R_P is F-rational by Exercise 16 since $R_P \to \widehat{R}_Q$ is faithfully flat.

Now suppose $P \in \operatorname{Spec}(R)$ contains $I \cap R$, we want to show R_P is not F-rational. Write $I = Q_1 \cap \cdots \cap Q_n$ where $Q_1, \ldots, Q_n$ are minimal primes of I. Then $I \cap R = P_1 \cap \cdots \cap P_n$ where $P_i = Q_i \cap R$. Since $I \cap R \subseteq P$, we know $P_i \subseteq P$ for some i. If R_P is F-rational, then R_{P_i} is F-rational by Theorem 4.14. But then as Q_i contracts to P_i and R is excellent, $R_{P_i} \to \widehat{R}_{Q_i}$ is a faithfully flat extension of excellent local rings with geometrically regular fibers. Thus Theorem 7.7 implies that $\widehat{R}_{Q_i}$ is F-rational, which is a contradiction to $I \subseteq Q_i$ (recall that $V(I)$ is the non-F-rational locus of $\operatorname{Spec}(\widehat{R})$). $\square$

Remark 7.10. In fact, the proof of Theorem 7.9 follows from a more general result: if $R \to S$ is a faithfully flat extension and $U \subseteq \operatorname{Spec}(R)$, then U is open if and only if the pre-image of U in $\operatorname{Spec}(S)$ is open, see [sta16, Lemma 29.25.12].

Finally we can prove the following result on base change of F-rational singularities, which was originally proved by Aberbach–Enescu.

Theorem 7.11 ([AE03])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of excellent rings such that R is F-rational and $S/\mathfrak{m}S$ is geometrically F-rational. Then S is F-rational.*

Proof. Since $S/\mathfrak{m}S$ is F-rational, it is a normal domain by Proposition 4.4. Thus $\mathfrak{m}S$ is a prime ideal in S. We first show that $S' := S_{\mathfrak{m}S}$ is F-rational: since $S/\mathfrak{m}S$ is geometrically F-rational, we know that $R \to S'$ is a flat local extension such that $S'/\mathfrak{m}S'$ is geometrically regular and so by Theorem 7.7, S' is F-rational.

Since S is an excellent local domain and $S_{\mathfrak{m}S}$ is F-rational, by Theorem 7.9 we know that there exists $c \notin \mathfrak{m}S$ such that S_c is F-rational. Note that c is a nonzerodivisor on $S/\mathfrak{m}S$ and thus it is a nonzerodivisor on S by Lemma 7.1, in particular, c is not in any minimal prime of S. Let $\underline{x} := x_1, \ldots, x_d$ be a system of parameters of $S/\mathfrak{m}S$. In analogy with Claim 7.6, we have the following.

Claim 7.12. *For any Artinian R-module M, the map $F^e_*M \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(M \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_*m \otimes \eta \to F^e_*(m \otimes cF^e(\eta))$ is injective for some $e > 0$, where $F^e(-)$ is the natural Frobenius action on $H^d_{(\underline{x})}(S)$.*

Proof. This follows from the same argument as in Claim 7.6, using $S/\mathfrak{m}S$ is geometrically F-rational instead of geometrically F-injective. □

As a consequence, we see that there exists $e > 0$ such that the map $F^e_*H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*(H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S))$ sending $F^e_*\eta' \otimes \eta \to F^e_*(\eta' \otimes cF^e(\eta))$ is injective. But since R is F-injective and $H^d_{(\underline{x})}(S)$ is faithfully flat over R (see Lemma 7.1), composing this injection with the injection

$$H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \to F^e_*H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S)$$

and using that $H^n_{\mathfrak{m}}(R) \otimes_R H^d_{(\underline{x})}(S) \cong H^{n+d}_{\mathfrak{n}}(S)$, we find that $H^{n+d}_{\mathfrak{n}}(S) \to F^e_*H^{n+d}_{\mathfrak{n}}(S)$ sending η to $F^e_*(cF^e(\eta))$ is injective. Since S is excellent Cohen–Macaulay and S_c is F-rational (and c is not in any minimal prime of S), by Theorem 7.8, we see that S is F-rational. □

It is natural to ask whether we can drop "geometrically" in Theorem 7.7 or Theorem 7.11. This is unfortunately not known, in fact, the following question is open.

Open Problem 5. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of excellent rings such that R is F-rational and $S/\mathfrak{m}S$ is regular. Then is S also F-rational?*

On the other hand, it is known that we cannot drop "geometrically" in Theorem 7.5 (though there are no known examples with R normal). The following example is due to Enescu [Ene09, Proposition 4.2], which was based on [EH08, Example 2.16]. We leave the details in a few exercises.

Exercise 25. Let K be an F-finite field and let $K \to L$ be a finite field extension that is not separable such that $L^p \cap K = K^p$. Let x be an indeterminate and let $R = K + xL[[x]] \subseteq L[[x]]$. Prove the following:

(1) R is a (Noetherian) complete local domain with $\dim(R) = 1$.
(2) R is F-injective.
(3) $K^{1/p} \otimes_K R$ is not reduced, and hence not F-injective.

In particular, $(R, \mathfrak{m}) \to S := K^{1/p} \otimes_K R$ is a flat local extension such that R is F-injective and $S/\mathfrak{m}S$ a field, but S is not F-injective.

Exercise 26. Let k be a perfect field. Set $K = k(u, v)$ and $L = K[y]/(y^{2p} + uy^p + v)$. Prove that K, L satisfy the assumptions of Exercise 25.

Exercise 27. With notation as in Discussion 6.6, prove that if R_Q is F-injective (resp. F-pure), then $R^{\Gamma}_{Q^{\Gamma}}$ is F-injective (resp. F-pure) for all sufficiently small choices of Γ. Furthermore, prove that the F-injective (resp. F-pure) locus of $\mathrm{Spec}(R)$ can be identified with the F-injective (resp. F-pure) locus of $\mathrm{Spec}(R^{\Gamma})$ for all sufficiently small choices of Γ.

Exercise 28. Let $(R, \mathfrak{m})$ be an excellent local ring. Prove that the F-pure and F-injective locus of $\mathrm{Spec}(R)$ are open.

The ideal I_e that shows up in the proof of Theorem 7.4 plays an important role in the study of F-singularities (e.g., see Section 10).

Exercise 29. Let $(R, \mathfrak{m})$ be an F-finite ring. Let E_R be the injective hull of the residue field and let u be a socle representative. Recall that $F^e_* I_e := \mathrm{Ann}_{E_R \otimes_R F^e_* R}(u \otimes F^e_* 1)$. Prove that

$$I_e = \{r \in R \mid \text{for all } \phi \in \mathrm{Hom}_R(F^e_* R, R), \phi(F^e_* r) \in \mathfrak{m}\}.$$

In connection with Exercise 21, the following question seems open.

Open Problem 6. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension such that R is F-pure and $S/\mathfrak{m}S$ is quasi-Gorenstein and F-pure. Then is S also F-pure?*

Remark 7.13. Though the results in the section are stated for local rings, one can immediately deduce the corresponding global results (as these F-singularities are local properties). Namely, if $R \to S$ is a faithfully flat extension such that R is F-pure (resp. F-injective, excellent and F-rational, F-finite and strongly F-regular) and all fibers of $R \to S$ are Gorenstein and F-pure (resp. Cohen–Macaulay and geometrically F-injective, geometrically F-rational, Gorenstein and strongly F-regular), then S is F-pure (resp. F-injective, F-rational if S is excellent, strongly F-regular if S is F-finite).

8. Determinantal rings

Our goal in this section is to show that generic determinantal rings over a field k are F-rational, in fact strongly F-regular if k is F-finite. To establish this, we first prove a very powerful criterion of F-rationality for graded rings [FW89], this criterion will also be useful when we study more examples in Section 9 (the analogous criterion for rational singularities in characteristic 0 was first proved by Watanabe [Wat83]).

Theorem 8.1 (Fedder–Watanabe's criterion)**.** *Let R be an $\mathbb{N}$-graded ring over a field k with homogeneous maximal ideal $\mathfrak{m}$. Then R is F-rational if and only if*

(1) R is Cohen–Macaulay.
(2) R_P is F-rational for all homogeneous prime $P \neq \mathfrak{m}$.
(3) $a(R) := \max\{n | H^d_{\mathfrak{m}}(R)_n \neq 0\} < 0$.
(4) R is F-injective.

Proof. If R is F-rational, then (1) and (4) clearly hold, (2) holds since F-rationality localizes by Theorem 4.14, and (3) holds by Exercise 18.

Now we suppose R satisfies (1) – (4) and we want to prove R is F-rational. We first assume R is F-finite, that is, k is an F-finite field. Note that R has a global canonical module ω_R (in the sense that $(\omega_R)_P \cong \omega_{R_P}$ for all $P \in \operatorname{Spec}(R)$). Moreover we can choose ω_R such that it is graded (see [BS98, Chapter 14]). Similar to Discussion 6.1, we have a graded trace map $F^e_*\omega_R \xrightarrow{\mathrm{Tr}^e} \omega_R$, and it is easy to verify that the analogs of Proposition 6.2 and Proposition 6.3 (the statments involving ω_R) hold in this set up.

Condition (2) implies R_P is a field for all minimal primes of R, so there exists a homogeneous $c \in R$ not in any minimal prime of R such that R_c is regular. By condition (2) again, for each homogenous prime $P \neq \mathfrak{m}$, there exists $e > 0$ such that $F^e_*(c\omega_R)_P \xrightarrow{\mathrm{Tr}^e} (\omega_R)_P$ is surjective. Thus there exists a homogenous $f_P \notin P$ such that $F^e_*(c\omega_R)_{f_P} \xrightarrow{\mathrm{Tr}^e} (\omega_R)_{f_P}$ is surjective. At this point, we note that $\cup D(f_P) = \operatorname{Spec}(R) - \{\mathfrak{m}\}$ where the union is taken over all homogenous primes $P \neq \mathfrak{m}$. Since $\operatorname{Spec}(R) - \{\mathfrak{m}\}$ is quasi-compact, there exists a finite collection $\{f_1, \dots, f_n\}$ that generates $\mathfrak{m}$ up to radical such that for each f_i there is an associated e_i such that $F^{e_i}_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^{e_i}} (\omega_R)_{f_i}$ is surjective. Pick $e \gg e_i$ for all i, it follows that $F^e_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^e} (\omega_R)_{f_i}$ is surjective for all f_i.[5] But then we know that

$$\operatorname{Coker}(F^e_*(c\omega_R) \xrightarrow{\mathrm{Tr}^e} \omega_R)$$

[5]We leave this as an exercise, the point is that $\mathrm{Tr}^e : F^e_*(\omega_R)_{f_i} \to (\omega_R)_{f_i}$ is surjective for all e since R_{f_i} is F-injective, so we can enlarge the e_i while preserving the surjectivity of $F^{e_i}_*(c\omega_R)_{f_i} \xrightarrow{\mathrm{Tr}^{e_i}} (\omega_R)_{f_i}$.

is a graded finite length module supported only at $\mathfrak{m}$. It is enough to show that this cokernel is 0, since then $F^e_*(c\omega_R) \xrightarrow{\mathrm{Tr}^e} \omega_R$ is surjective, and by the analog of Proposition 6.3 we will be done.

But the graded Matlis dual of this cokernel is $N_e = \{\eta \in H^d_{\mathfrak{m}}(R) \mid cF^e(\eta) = 0\}$, and for $e \gg 0$, we know that N_e is a (graded) F-stable submodule of $H^d_{\mathfrak{m}}(R)$ (see the proof of Proposition 4.11). Now by (4), any graded F-stable submodule of finite length must concentrate in degree 0, but then it vanishes by (3). We have completed the proof when k is an F-finite field.

Finally, if k is not F-finite, we can replace k by k^Γ (and R by $R^\Gamma := R \otimes_k k^\Gamma$) for Γ sufficiently small and run the above argument for R^Γ (it can be shown, in analogy with the local case, that (1) – (4) are preserved[6]). The outcome is that R^Γ is F-rational and hence R is F-rational by Exercise 16. □

Now we explain that generic determinantal rings are F-rational [HH94b].

Example 8.2. Let $S = k[x_{ij}|1 \le i \le m, 1 \le j \le n]$ be a polynomial ring in $m \times n$ variables with $m \le n$. Let I_t be the ideal of S generated by $t \times t$ minors of the matrix $[x_{ij}]_{1 \le i \le m, 1 \le j \le n}$. Then $R = S/I_t$ is F-rational. Moreover, if k is F-finite then R is strongly F-regular.

Proof. We will use Theorem 8.1 to show R is F-rational. First of all, property (1) and (3) are well-known: for example see [HE71] or [BV88].

We now prove (2). For any homogeneous prime $P \neq \mathfrak{m}$, there exists $x_{ij} \notin P$. Without loss of generality we may assume $x_{11} \notin P$. After inverting the element x_{11}, we may perform row and column operations to transform our matrix:

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & 0 & \dots & 0 \\ 0 & x'_{22} & \dots & x'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x'_{m2} & \dots & x'_{mn} \end{bmatrix}$$

where $x'_{ij} = x_{ij} - \frac{x_{i1}x_{1j}}{x_{11}}$. The ideal $I_t S_{x_{11}}$ is generated by $(t-1) \times (t-1)$ minors of the second displayed matrix. Therefore,

$$R_{x_{11}} = S_{x_{11}}/I_t S_{x_{11}} \cong (S'/I'_{t-1})[x_{11}, \frac{1}{x_{11}}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{m1}]$$

[6]Only (2) requires some work and we omit the details, as the argument is entirely similar as in the local case we carried out in Section 6. In fact, as we already mentioned before, the theory of Γ-construction can be extended to all rings essentially finite type over a complete local ring (e.g., a field), see [HH94a] for details. In the sequel we will apply Theorem 8.1 mainly in the case that k is perfect or algebraically closed.

where $S' = k[x'_{ij} | 2 \leq i \leq m, 2 \leq j \leq n]$ and I'_{t-1} denotes the ideal generated by the $(t-1) \times (t-1)$ minors of the matrix $[x'_{ij}]$. By induction, we know that S'/I'_{t-1} is F-rational, thus so is $R_{x_{11}}$ by Theorem 7.7 and Theorem 4.14. Since R_P can be viewed as a localization of $R_{x_{11}}$, R_P is F-rational by Theorem 4.14 again.

It remains to prove (4). In fact the method below will also reprove (1) along the way we prove (4). We need the following result from combinatorial commutative algebra:

Theorem 8.3 ([Stu90]). *The $t \times t$ minors of $[x_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ form a Gröbner basis of I_t with respect to the term order $x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots\cdots > x_{mn}$.*

At this point we follow the standard construction as in [Eis95, 15.16 and 15.17]. We choose an appropriate weight function λ such that $\mathrm{in}_\lambda(I_t) = \mathrm{in}_>(I_t)$. Let $\widetilde{I}$ be the λ-homogenization of I_t in $S[z]$. We have

$$(S[z]/\widetilde{I}) \otimes_{k[z]} k(z) \cong R \otimes_k k(z) \text{ and } (S[z]/\widetilde{I})/z \cong S/\mathrm{in}_>(I_t).$$

Therefore if we can show that $S/\mathrm{in}_>(I_t)$ is Cohen–Macaulay and F-injective, then so is $S[z]/\widetilde{I}$ by Theorem 5.1.[7] But then $R \otimes_k k(z)$ is Cohen–Macaulay and F-injective by Theorem 4.13 and hence R is Cohen–Macaulay and F-injective by Exercise 16. But since the $t \times t$ minors form a Gröbner basis by Theorem 8.3,

$$\mathrm{in}_>(I_t) = (x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_t j_t} | 1 \leq i_1 < i_2 < \cdots < i_t \leq m, 1 \leq j_1 < j_2 < \cdots < j_t \leq n)$$

is a square-free monomial ideal. Thus $S/\mathrm{in}_>(I_m)$ is F-pure and hence F-injective, and one can check that $S/\mathrm{in}_>(I_m)$ is Cohen–Macaulay using Hochster's criterion [Hoc72] that a Stanley-Reisner ring is Cohen–Macaulay if the corresponding simplicial complex is shellable (the a-invariant can also be computed using the combinatorial structure [BH92]). This completes the proof that R is F-rational.

Finally we prove that S/I_t is strongly F-regular when k is F-finite. Note that we can enlarge the $m \times n$ generic matrix to an $n \times n$ generic matrix and consider the corresponding quotients S'/I_t of $t \times t$ minors in the $n \times n$ matrix. Then $S/I_t \to S'/I_t$ splits (we can map the new variables to zero to obtain a splitting), thus S/I_t is strongly F-regular provided S'/I_t is strongly F-regular by Theorem 3.9. But S'/I_t is F-finite and Gorenstein (see [BV88]) and thus F-rationality of S'/I_t implies the strong F-regularity of S'/I_t by Proposition 4.9. $\square$

[7]Even without knowing $S/\mathrm{in}_>(I_t)$ is Cohen–Macaulay, $S[z]/\widetilde{I}$ is F-injective because we showed $S/\mathrm{in}_>(I_t)$ is F-pure and so we can use Theorem 5.5 instead of Theorem 5.1.

9. Examples

We start this section with a quick summary of the relations between the F-singularities we have introduced so far (all the arrows that go to strongly F-regular also require F-finite assumption as usual):

$$
\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[l, bend right, "+\text{Gorenstein}"'] & \text{normal} \\
 & F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \arrow[r, Rightarrow] \arrow[l, bend right, "+\text{quasi-Gorenstein}"'] & \text{reduced}
\end{tikzcd}
$$

A natural question one might ask is that whether there are other implications between these F-singularities: for example, whether there are relations between F-rational and F-pure singularities. However, Watanabe [Wat91] constructed examples of F-rational rings that are not F-pure, and examples of F-rational and F-pure rings that are not strongly F-regular. To study these examples we first collect some basic facts about section rings of divisors with rational coefficients. Let X be a projective variety over an algebraically closed field $k = \overline{k}$ and let D be an effective $\mathbb{Q}$-divisor such that mD is an ample Cartier divisor on X. Then

$$R = R(X, D) := \oplus_{n\geq 0} H^0(X, O_X(\lfloor nD \rfloor)) \cdot t^n$$

is a normal $\mathbb{N}$-graded ring over k. We can explicitly describe the graded canonical module of R and its symbolic powers using sheaf cohomology as follows (see [Wat91], which follows from [Wat81] and [Dem88]):

$$
\begin{aligned}
\omega_R &= \oplus_{n\in\mathbb{Z}}\, H^0(X, O_X(\lfloor K_X + D' + nD \rfloor)) \cdot t^n. \\
\omega_R^{(q)} &= \oplus_{n\in\mathbb{Z}}\, H^0(X, O_X(\lfloor q(K_X + D') + nD \rfloor)) \cdot t^n.
\end{aligned} \tag{9.1}
$$

where D' is defined as follows: if $D = \sum \frac{a_i}{b_i} E_i$ such that $(a_i, b_i) = 1$ and E_i's are prime divisors, then $D' = \sum \frac{b_i - 1}{b_i} E_i$.

Example 9.1 ([Wat91])**.** Let $R = R(\mathbb{P}^1_k, D)$ where $k = \overline{k}$ and let $D = \frac{1}{a}P_1 + \frac{1}{b}P_2 + \frac{1}{c}P_3$ be an effective $\mathbb{Q}$-divisor where P_1, P_2, P_3 are distinct points on $\mathbb{P}^1$. Then we have

(1) R is F-rational for all $a, b, c \geq 1$.
(2) R is not F-pure if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$.
(3) If $a = b = c = 3$, then R is F-pure if $p \equiv 1 \bmod 3$ but R is not strongly F-regular.

Proof. We first prove (1). We use Theorem 8.1. R is a two-dimensional normal $\mathbb{N}$-graded ring so it is Cohen–Macaulay and R_P is regular for all $P \neq \mathfrak{m}$. To see $a(R) < 0$, it is enough to show that $[\omega_R]_{\leq 0} = 0$, which follows from (9.1) as

$$[\omega_R]_n = H^0(\mathbb{P}^1, O(-2) \otimes O(\lfloor D' + nD \rfloor)) \cdot t^n = 0$$

for $n \leq 0$. Finally we show R is F-injective. Let x be the parameter of $\mathbb{P}^1$ and let P_1, P_2, P_3 correspond to $(x-\alpha)$, $(x-\beta)$, $(x-\gamma)$. It is straightforward to check that R is generated by t, $y_1 := \frac{1}{x-\alpha}t^a$, $y_2 := \frac{1}{x-\beta}t^b$, $y_3 := \frac{1}{x-\gamma}t^c$. But then we observe that

$$R/tR \cong k[y_1, y_2, y_3]/(y_1y_2, y_1y_3, y_2y_3).$$

To see this, note that mod t, $y_1y_2 = \frac{1}{(x-\alpha)(x-\beta)} \cdot t^{a+b} = (\alpha - \beta) \cdot (\frac{1}{x-\beta}t^{a+b} - \frac{1}{x-\alpha}t^{a+b}) = 0$ and similarly $y_1y_3 = y_2y_3 = 0$. Hence R/tR is Cohen–Macaulay and F-pure and thus R is F-injective by Theorem 5.1. This completes the proof that R is F-rational.

We next prove (2) and (3). We note that the canonical map $E_R(k) \to F^e_*R \otimes E_R(k)$ can be identified with $H^2_{\mathfrak{m}}(\omega_R) \to F^e_*R \otimes_R H^2_{\mathfrak{m}}(\omega_R) \cong H^2_{\mathfrak{m}}(F^e_*\omega_R^{(p^e)})$, where the isomorphism results from the fact that the natural map $F^e_*R \otimes_R \omega_R \to F^e_*\omega_R^{(p^e)}$ is an isomorphism in codimension one (after we localize at height one primes, ω_R is a rank one free module). We then have the degree-preserving identifications:

$$\begin{array}{ccc} E_R(k) \cong H^2_{\mathfrak{m}}(\omega_R) & \longrightarrow & F^e_*R \otimes E_R(k) \cong H^2_{\mathfrak{m}}(F^e_*\omega_R^{(p^e)}) \\ \downarrow \cong & & \downarrow \cong \\ \oplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D' + nD \rfloor)) \cdot t^n & \longrightarrow & \oplus_{n\in\mathbb{Z}} H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1} + D') + nD \rfloor)) \cdot t^n. \end{array}$$

It is easy to check that the socle of $E_R(k)$ corresponds to $H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D' \rfloor)) \cong k$ (the point is that all the degree > 0 piece vanish by a simple computation). Thus by Proposition 2.2, R is F-pure if and only if the map

$$H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D' \rfloor)) \to H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D') \rfloor))$$

is injective. But if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, then a simple computation shows that

$$\deg(\lfloor p(K_{\mathbb{P}^1} + D') \rfloor) = -2p + \lfloor p \cdot \frac{a-1}{a} \rfloor + \lfloor p \cdot \frac{b-1}{b} \rfloor + \lfloor p \cdot \frac{c-1}{c} \rfloor \geq -1$$

and thus $H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D') \rfloor)) \cong H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1} + D') \rfloor)) = 0$. Hence R is not F-pure if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, which proves (2).

Finally, the same analysis (via Proposition 2.2) shows that R is strongly F-regular if and only if for any $0 \neq f \in H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor nD \rfloor))$, there exists $e > 0$ such that the composition:

$$H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(\lfloor K_{\mathbb{P}^1} + D' \rfloor)) \to H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1} + D') \rfloor)) \xrightarrow{\cdot F^e_*f} H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1} + D') + nD \rfloor))$$

is injective. Now if $a=b=c=3$, then again a simple computation shows that for n large,

$$\deg(\lfloor p^e(K_{\mathbb{P}^1}+D')+nD\rfloor) = -2p^e + 3\lfloor \frac{2}{3}p^e + \frac{1}{3}n\rfloor \geq -1$$

for all $e>0$ and thus $H^1(\mathbb{P}^1, F^e_*O_{\mathbb{P}^1}(\lfloor p^e(K_{\mathbb{P}^1}+D')+nD\rfloor))=0$. Hence R is not strongly F-regular. On the other hand, if $p\equiv 1 \bmod 3$, then one checks that

$$H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1}+D')\rfloor)) \cong H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(-2)),$$

and if we use $[z_0 : z_1]$ to denote the coordinate of $\mathbb{P}^1$, then the induced map $H^1(\mathbb{P}^1, O_{\mathbb{P}^1}(-2)) \to H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(\lfloor p(K_{\mathbb{P}^1}+D')\rfloor))$ can be described as

$$\frac{1}{z_0z_1} \to \frac{(z_0-\alpha z_1)^{\lfloor p\cdot\frac{a-1}{a}\rfloor}(z_0-\beta z_1)^{\lfloor p\cdot\frac{b-1}{b}\rfloor}(z_0-\gamma z_1)^{\lfloor p\cdot\frac{c-1}{c}\rfloor}}{z_0^pz_1^p} = \frac{u}{z_0z_1} \in H^1(\mathbb{P}^1, F_*O_{\mathbb{P}^1}(-2))$$

where $0\neq u\in k$. Thus the map is injective and hence R is F-pure. □

Remark 9.2. One can write some concrete examples: for instance let $P_1=\infty$, $P_2=0$, $P_3=1$ and $a=b=c=4$, then $R\cong k[t, xt^4, x^{-1}t^4, (x-1)^{-1}t^4]$ is F-rational but not F-pure, while if we take $a=b=c=3$ and $p\equiv 1 \bmod 3$, then $R\cong k[t, xt^3, x^{-1}t^3, (x-1)^{-1}t^3]$ is F-rational and F-pure but not strongly F-regular. We can complete at the homogenous maximal ideal to obtain examples of complete local domains.

We next give Watanabe's example that direct summand of F-rational rings are not necessarily F-injective. Our construction slightly differs from [Wat97].

Example 9.3 ([Wat97])**.** Let R be an $\mathbb{N}$-graded ring over a field k with homogeneous maximal ideal $\mathfrak{m}$ such that

(1) R is Cohen–Macaulay.
(2) R_P is regular for all homogeneous prime $P\neq\mathfrak{m}$.
(3) $a(R)<0$.

Then R is a direct summand of an F-rational ring.

Proof. Let $S=k[x_1,\dots,x_n]$ be a standard graded (i.e., $\deg(x_i)=1$) polynomial ring and let $T=R\#S$ be the Segre product, that is, $T=\oplus_{j\geq 0}(R_j\otimes_k S_j)$. Then R is a direct summand of T: we can map R to T by sending $r\in[R]_j$ to $r\#x_1^j$ and the map $T\to k[x_1]$ sending x_i to 0 for all $i\geq 2$ induces a splitting $T\to R$.

We claim that T is F-rational for all $n\gg 0$ and we use Theorem 8.1. We use the following formula to compute the local cohomology of Segre product [GW78]:

$$H^{d+n}_\mathfrak{m}(T) = H^i_\mathfrak{m}(R\#S) \cong R\#H^i_\mathfrak{m}(S)\oplus H^i_\mathfrak{m}(R)\#S \oplus \left(\oplus_{a+b=i+1}H^a_\mathfrak{m}(R)\#H^b_\mathfrak{m}(S)\right)$$

where we abuse notations a bit and use $\mathfrak{m}$ to denote the corresponding homogeneous maximal ideals of R, S, and T respectively. Let $d = \dim R$. Since R is Cohen–Macaulay, we know that $H^j_{\mathfrak{m}}(R) = 0$ for all $j < d$. Thus the only possible nonzero contributions for the local cohomology modules of T are $H^d_{\mathfrak{m}}(R)\#S$, $R\#H^n_{\mathfrak{m}}(S)$, and $H^d_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S)$. But the first two terms are zero by our a-invariant assumption. Thus T is Cohen–Macaulay with top local cohomology module $H^{d+n}_{\mathfrak{m}}(T) \cong H^d_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S)$. In particular $a(T) = \min\{a(R), a(S)\} < 0$.

We next show that T is F-rational for all homogeneous primes $P \neq \mathfrak{m}_T$. If we invert a homogenous element $r\#s \in \mathfrak{m}_T$, then $T_{r\#s}$ is a direct summand of $(R\otimes_k S)_{r\otimes s}$ (since T is a direct summand of $R\otimes_k S$). But $(R\otimes_k S)_{r\otimes s} \cong R[x_1,\dots,x_n]_{rs}$ is a localization of $R_r[x_1,\dots,x_n]$, hence regular (because R_r is regular by our assumption). Therefore $T_{r\#s}$ is a direct summand of a regular ring and thus strongly F-regular by Theorem 3.9 and thus F-rational by Theorem 4.6.[8]

Finally we show that T is F-injective. Our assumptions on R implies (see the proof of Theorem 8.1) that the largest proper F-stable submodule of $H^d_{\mathfrak{m}}(R)$ has finite length. In particular, there exists $m \gg 0$ such that the Frobenius action on $[H^d_{\mathfrak{m}}(R)]_{\leq m}$ injective. Now for $n \geq m$, the Frobenius action on $H^{d+n}_{\mathfrak{m}}(T) \cong H^d_{\mathfrak{m}}(R)\#H^n_{\mathfrak{m}}(S) = [H^d_{\mathfrak{m}}(R)]_{\leq m}\#[H^n_{\mathfrak{m}}(S)]_{\leq m}$ is injective. $\square$

Remark 9.4. Perhaps the simplest concrete example is $R = \mathbb{F}_3[x,y,z]/(x^2+y^3+z^5)$ with $\deg(x) = 15$, $\deg(y) = 10$, and $\deg(z) = 6$. Since R is a two-dimensional normal domain with $a(R) = -1$, it satisfies all the conditions in Example 9.3 and thus R is a direct summand of an F-rational ring. However, it is a straightforward computation that the Čech class $[\frac{x}{yz}] \in H^2_{\mathfrak{m}}(R)$ is nonzero, but $F([\frac{x}{yz}]) = [\frac{x^3}{y^3z^3}] = 0$ since $x^3 \in (y^3, z^3)R$. Hence R is not F-injective (thus not F-rational). Again, we can complete at $\mathfrak{m}$ to obtain examples of complete local domains.

We next give Singh's example [Sin99c] showing that if we drop the Gorenstein assumption, then R/xR is strongly F-regular does not even imply R is F-pure.

Example 9.5 ([Sin99c]). Let m and n be positive integers satisfying $m - m/n > 2$. Consider the ring $R = k[a,b,c,d,t]/I$ where k is an F-finite field of characteristic $p > 2$ and I is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} a^2+t^m & b & d \\ c & a^2 & b^n - d \end{pmatrix}.$$

[8]Even without the F-finite assumptions, direct summand of regular rings are always F-rational, for example one can use the argument in Corollary 4.7 and we omit the details.

Then t is a nonzerodivisor on R and the ring R/tR is strongly F-regular. But if p and m are relatively prime, then R is not F-pure.

Proof. Let $S = k[a, b, c, d, b', c', d', t]/J$ where J is the generic 2×3 matrix

$$\begin{pmatrix} c' & b & d \\ c & b' & d' \end{pmatrix}.$$

Obviously we have $R = S/(c' - a^2 - t^m, b' - a^2, d' - b^n + d)$ and $R/(t, c, d) \cong k[a, b]/(a^4, b^{n+1})$ is Artinian. Since S is Cohen–Macaulay of dimension 6, it follows that R is Cohen–Macaulay and t, c, d is a system of parameters of R. In particular, t is a nonzerodivisor on R. We note that

$$R/tR \cong k[a, b, c, d]/(a^4 - bc, a^2(b^n - d) - cd, b(b^n - d) - a^2 d).$$

Note that we can assign weights to the variables to make R/tR $\mathbb{N}$-graded. We first claim that R/tR is normal, and hence a domain because it is $\mathbb{N}$-graded. We know that R/tR is Cohen–Macaulay of dimension 2 and c, d is a homogeneous system of parameters, thus it is enough to show that $(R/tR)_c$ and $(R/tR)_d$ are regular. These are straightforward to check: $(R/tR)_d \cong k[a, b, d][\frac{1}{d}]/(\frac{b^{n+1}}{d} - b - a^2)$ and $(R/tR)_c \cong k[a, c, d][\frac{1}{c}]/(\frac{a^{4n+2}}{c^{n+1}} - \frac{a^2 d}{c} - d)$ are both regular.

We next claim that R/tR is isomorphic to the $(2n+1)$-th Veronese subring of $k[a, x, y]/(a^2 - xy(x^n - y))$ where the variables a, x, y have weights $2n + 1, 2, 2n$ respectively. To see this, we define a map

$$R/tR \cong \frac{k[a, b, c, d]}{(a^4 - bc, a^2(b^n - d) - cd, b(b^n - d) - a^2 d)} \to (\frac{k[a, x, y]}{(a^2 - xy(x^n - y))})^{(2n+1)}$$

by sending b, c, d to $xy^2, x(x^n - y)^2, y^{2n+1}$ respectively. One easily checks that the map is well-defined and is surjective: the Veronese subring is generated over k by $a, y^2 x, x^{n+1} y, x^{2n+1}, y^{2n+1}$ and it is straightforward to check that all these generators are in the image (modulo the equation $a^2 - xy(x^n - y)$). Now both rings have dimension 2 and we know that R/tR is a domain, it follows that the map is injective and hence an isomorphism.

To prove R/tR is strongly F-regular, it is enough to show that $k[a, x, y]/(a^2 - xy(x^n - y))$ is strongly F-regular by Theorem 3.9. We now apply Exercise 11 with $c = x$ (since x is part of a system of parameters and after inverting x the ring becomes regular), it is enough to show that there exists $e > 0$ such that $x(a^2 - xy(x^n - y))^{p^e - 1} \notin (a^{p^e}, x^{p^e}, y^{p^e})$. Since $p > 2$, for $e = 1$, the term $a^{p-1} x^{\frac{p+1}{2}} y^{p-1}$ appears in $x(a^2 - xy(x^n - y))^{p-1}$ with nonzero coefficient, this term is not in (a^p, x^p, y^p).

It remains to prove that R is not F-pure if p and m are relatively prime. The key is the following elementary but tricky computation.

Claim 9.6 ([Sin99c, Lemma 4.2])**.** *If s is a positive integer such that $s(m-m/n-2)\geq 1$, then*

$$(b^n t^{m-1})^{2ms+1}\in (a^{2ms+1}, d^{2ms+1}).$$

Proof. Let $\tau = a^2+t^m$ and $\alpha = a^2$. It suffices to work in the polynomial ring $k[\tau,\alpha,b,c,d]$ and establish that

$$b^{n(2ms+1)}(\tau-\alpha)^{2s(m-1)}\in (\alpha^{ms+1}, d^{2ms+1})+I'$$

where I' is the ideal generated by 2×2 minors of the matrix

$$\begin{pmatrix}\tau & b & d\\ c & \alpha & b^n-d\end{pmatrix}.$$

Taking the binomial expansion of $(\tau-\alpha)^{2s(m-1)}$, it is enough to show that for all $1\leq i\leq ms+1$, we have

$$b^{n(2ms+1)}\alpha^{ms+1-i}\tau^{ms-2s+i-1}\in (\alpha^{ms+1}, d^{2ms+1})+I'.$$

Thus it is enough to show that

$$b^{n(2ms+1)}\tau^{ms-2s+i-1}\in (\alpha^{i}, d^{2ms+1})+I'.$$

Since $\alpha d - b(b^n-d)$ and $b^n\tau - d(c+\tau)$ belongs to I', it suffices to establish that

$$b^{n(2ms+1)}\tau^{ms-2s+i-1}\in (b^i(b^n-d)^i, d^{2ms+1}, b^n\tau-d(c+\tau)).$$

Now we work modulo the element $b^i(b^n-d)^i$, we may reduce $b^{n(2ms+1)}$ to a polynomial in b and d such that the highest power of b that occurs is less than $i(n+1)$. Thus it suffices to show that

$$b^{n(2ms+1-j)}\tau^{ms-2s+i-1}d^j\in (d^{2ms+1}, b^n\tau-d(c+\tau))$$

where $n(2ms+1-j)<i(n+1)$, i.e., $j\geq 2ms+(1-i)(1+1/n)$. So it is enough to check

$$b^{n(2ms+1-j)}\tau^{ms-2s+i-1}\in (d^{2ms+1-j}, b^n\tau-d(c+\tau)).$$

At this point, it only needs to check that $ms-2s+i-1\geq 2ms+1-j$, since modulo $b^n\tau-d(c+\tau)$, we can then express $b^{n(2ms+1-j)}\tau^{ms-2s+i-1}$ as a multiple of $d^{2ms+1-j}$. But

$$\begin{aligned}
ms-2s+i-1-(2ms+1-j) &= j-ms-2s+i-2\\
&\geq ms+(1-i)(1+1/n)-2s+i-2\\
&= ms-2s+(1-i)/n-1\\
&\geq ms-2s-(ms)/n-1\\
&= s(m-m/n-2)-1\geq 0
\end{aligned}$$

where the second $\geq$ is because $i \leq ms+1$ and the last $\geq$ follows from our assumption that $s(m-m/n-2) \geq 1$. $\square$

Finally, since p and m are relatively prime, $p>2$, and $m-m/n>2$ by our assumptions, there exists $e \gg 0$ and $s>0$ such that $p^e = 2ms+1$ and $s(m-m/n-2) \geq 1$. Claim 9.6 then shows that $(b^n t^{m-1})^{p^e} \in (a^{p^e}, d^{p^e})$. If R is F-pure, then $R \to F^e_* R$ is pure and hence the induced Frobenius map $R/(a,d) \to F^e_*(R/(a^{p^e}, d^{p^e}))$ is injective. Thus $(b^n t^{m-1})^{p^e} \in (a^{p^e}, d^{p^e})$ implies $b^n t^{m-1} \in (a,d)$. But $R/(a,d) \cong k[b,c,t]/(b^{n+1}, b^n t^m, bc)$ and it is clear that $b^n t^{m-1} \neq 0$ in this ring, which is a contradiction. $\square$

Remark 9.7. Take $m=5$, $n=2$, $k=\mathbb{F}_3$ in Example 9.5, we have

$$R = \frac{\mathbb{F}_3[a,b,c,d,t]}{((a^2+t^5)a^2 - bc, (a^2+t^5)(b^2-d) - cd, b(b^2-d) - a^2 d)}$$

is not F-pure, but R/tR is strongly F-regular. We leave the interested reader to check these using Theorem 2.5 and Exercise 11 and Macaulay 2.

Exercise 30. With notation as in Example 9.1, prove that R is not F-pure if $a=b=c=3$ and $p \equiv 2 \bmod 3$.

Exercise 31. Let $R = k[[x,y,z,w]]/(xy, xz, y(z-w^2))$. Prove the following:

(1) R is Cohen–Macaulay and w is a nonzerodivisor on R.
(2) R/wR is F-pure but R is not F-pure.

Exercise 32. Let R be as in Example 9.5. Prove that R is not strongly F-regular without assuming p and m are relatively prime.

10. F-signature: measuring Frobenius splittings

Let $(R, \mathfrak{m})$ be an F-finite local ring. Kunz's Theorem, Theorem 1.1, tells us R is regular if and only if F^e_*R is a free R-module for some (or equivalently, all) $e \in \mathbb{N}$. It is therefore natural to consider the number of free summands the R-modules F^e_*R admit as e ranges through the natural numbers. In doing so, we develop the theory of F-signature to numerically measure the severity of a strongly F-regular singularity.

Suppose M is a finitely generated R-module. We let $\operatorname{frk}_R(M)$ denote the largest number of free summands appearing in all various direct sum decompositions of M into irreducible R-modules. Equivalently, $\operatorname{frk}_R(M)$ is the largest rank of a free module F so that there exists a surjective R-linear map $M \to F$. The free ranks of F^e_*R as e varies through the natural numbers are called the *Frobenius splitting numbers of* R, denoted by $a_e(R) := \operatorname{frk}_R(F^e_*R)$. Observe that if R is a domain then $a_e(R) \leq \operatorname{rank}_R(F^e_*R)$. The F-signature of R, $s(R)$, is defined to be

$$s(R) := \lim_{e\to\infty} \frac{a_e(R)}{\operatorname{rank}_R(F^e_*R)}.$$

We will discuss more precise information of $\operatorname{rank}_R(F^e_*R)$ below. We point out that, since $0 \leq \frac{a_e(R)}{\operatorname{rank}_R(F^e_*R)} \leq 1$ for all $e \in \mathbb{N}$, we have $0 \leq s(R) \leq 1$ provided $s(R)$ exists as a limit.

The purpose of this chapter is to cover the three fundamental theorems on F-signature:

(1) [Tuc12, Main Result]: F-signature exists, i.e., the sequence of numbers $\left\{\frac{a_e(R)}{\operatorname{rank}(F^e_*R)}\right\}_{e\in\mathbb{N}}$ is a Cauchy sequence and $s(R)$ is well-defined.
(2) [HL02, Corollary 16]: F-signature detects regularity, i.e., $s(R) = 1$ if and only if R is a regular local ring.
(3) [AL03, Main Result] F-signature detects strong F-regularity, i.e., $s(R) > 0$ if and only if R is strongly F-regular.

The origins of F-signature theory can be found in [SVdB97] and was formally developed by Huneke and Leuschke in [HL02]. Researchers understood that F-signature served as a numerical measurement of singularities long before it was shown to exist in full generality. Under the assumption of existence, it was first shown in the early 2000's that $s(R) = 1$ if and only if R is regular by Huneke and Leuschke, and that $s(R) > 0$ if and only if R is strongly F-regular by Aberbach and Leuschke. Tucker's proof of the existence of F-signature came nearly 10 years later.

Our presentation of F-signature theory will significantly deviate from the historical development of the theory. We will not present the fundamental theorems of F-signature in the order they were discovered nor we will follow the original techniques. We will utilize modern techniques developed in [PT18, PS19, Pol20] to present streamlined and elementary proofs of (1), (2), and (3) respectively.

Before continuing with the theory of F-signature the reader should first observe that computing the Frobenius splitting numbers of R does not require looking at all possible choices of direct sum decompositions of F^e_*R into irreducibles and then counting free summands. More specifically, we have the following lemma.

Lemma 10.1. *Let $(R,\mathfrak{m},k)$ be a local ring, not necessarily of prime characteristic. Suppose that M is a finitely generated R-module and $M\cong R^{\oplus t_1}\oplus N_1\cong R^{\oplus t_2}\oplus N_2$ are choices of direct sum decompositions of M so that N_1,N_2 do not admit a free summand. Then $t_1=t_2$.*

Proof. There exists onto map $\varphi: R^{\oplus t_1}\oplus N_1\to R^{\oplus t_2}$. Because we are assuming that N_1 does not admit a free summand we must have that $\varphi(0\oplus N_1)\subseteq \mathfrak{m}R^{\oplus t_2}$. In particular, if we base change to the residue field $k=R/\mathfrak{m}$ we find that there is an onto map $k^{\oplus t_1}\to k^{\oplus t_2}$. Therefore $t_1\geq t_2$. By symmetry we conclude that $t_1=t_2$. ☐

Corollary 10.2. *Let $(R,\mathfrak{m},k)$ be an F-finite local ring. If $F^e_*R\cong R^{\oplus t}\oplus M$ is a choice of direct sum decomposition of F^e_*R so that M does not admit a free summand then $t=a_e(R)$, i.e., t is the maximal number of free summands appearing in any choice of direct sum decomposition of F^e_*R into irreducible modules.*

Proof. This is immediate by applying Lemma 10.1 to the finitely generated R-modules F^e_*R. ☐

10.1. **The rank of F^e_*R.** Suppose that K is an F-finite field. Consider the Frobenius map $F: K\to F_*K$; an element $F_*r\in F_*K$ satisfies the monic polynomial equation $x^p-r=0$. Therefore the degree of the minimal polynomial of every element of F_*K divides p. It follows that $[F_*K:K]=p^\gamma$ for some $\gamma\in\mathbb{N}$ and $[F^e_*K:K]=p^{e\gamma}$ for every $e\in\mathbb{N}$. If R is an F-finite domain with fraction field K then we define $\gamma(R)$ to be the unique integer such that $[F^e_*K:K]=p^{e\gamma(R)}$ for all $e\in\mathbb{N}$, i.e., $\gamma(R)$ is unique integer such that $\operatorname{rank}_R(F^e_*R)=p^{e\gamma(R)}$ for all $e\in\mathbb{N}$.

Lemma 10.3. *Let R be an F-finite ring. Suppose that $P\subseteq Q$ are prime ideals of R. Then $\gamma(R/P)=\gamma(R_Q/PR_Q)$.*

Proof. This is immediate by the observation that

$$\operatorname{rank}_{R/P}(F^e_*R/P)=\operatorname{rank}_{R_Q/PR_Q}(F^e_*R_Q/PR_Q).$$ ☐

Lemma 10.4. *Let $(R,\mathfrak{m},k)$ be an F-finite local ring. Then $F^e_*\widehat{R}\cong(F^e_*R)\otimes_R\widehat{R}$ for all $e>0$. As a consequence, $\widehat{R}$ is reduced if R is reduced.*

Proof. Since F^e_*R is a finitely generated R-module, we have $(F^e_*R)\otimes_R \widehat{R} \cong \widehat{(F^e_*R)}$. But $\widehat{(F^e_*R)} \cong F^e_*\widehat{R}$: if we identify F^e_*R with R, then $\widehat{(F^e_*R)}$ is the completion of R with respect to the ideal $\mathfrak{m}^{[p^e]}$ while $F^e_*\widehat{R}$ is the completion of R with respect to $\mathfrak{m}$, so they are the same since $\sqrt{\mathfrak{m}^{[p^e]}} = \mathfrak{m}$. If R is reduced, then $R \hookrightarrow F^e_*R$ and thus $\widehat{R} \hookrightarrow F^e_*R \otimes \widehat{R} \cong F^e_*\widehat{R}$, which implies $\widehat{R}$ is reduced by Exercise 2. $\square$

Lemma 10.5. *Let $(R,\mathfrak{m},k)$ be an F-finite local domain and let K be the fraction field of R. Let P be a minimal prime of $\widehat{R}$ and let $L = \widehat{R}_P$. Then L is a field and $F^e_*L \cong F^e_*K \otimes_K L$. In particular, $[F^e_*L : L] = [F^e_*K : K]$.*

Proof. By Lemma 10.4, $\widehat{R}$ is reduced so L is a field. Now we have

$$F^e_*L \cong (F^e_*\widehat{R})_P \cong F^e_*\widehat{R} \otimes_{\widehat{R}} \widehat{R}_P \cong F^e_*R \otimes_R \widehat{R} \otimes_{\widehat{R}} \widehat{R}_P \cong F^e_*R \otimes_R \widehat{R}_P \cong F^e_*K \otimes_K L$$

where the third isomorphism follows from Lemma 10.4. $\square$

Theorem 10.6. *Let $(R,\mathfrak{m},k)$ be an F-finite local domain of dimension d. Then for each $e \in \mathbb{N}$ we have that $\operatorname{rank}_R(F^e_*R) = [F^e_*k : k]p^{ed}$.*

Proof. We first suppose that R is complete. By Cohen's structure theorem, R module-finite over $A = k[[x_1, x_2, \ldots, x_d]]$. Consider the following commutative diagram of local domains:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \\ F^e_*A \arrow[r] & F^e_*R \end{tikzcd} .$$

Since rank is multiplicative across compositions, we have

$$\operatorname{rank}_A(F^e_*R) = \operatorname{rank}_R(F^e_*R)\operatorname{rank}_A(R) = \operatorname{rank}_{F^e_*A}(F^e_*R)\operatorname{rank}_A(F^e_*A).$$

The extension of local domains $A \to R$ is isomorphic to $F^e_*A \to F^e_*R$. Therefore $\operatorname{rank}_A(R) = \operatorname{rank}_{F^e_*A}(F^e_*R)$ and hence $\operatorname{rank}_R(F^e_*R) = \operatorname{rank}_A(F^e_*A)$. As mentioned in the proof of Theorem 1.1 it is straightforward to check that F^e_*A is a free A-module with basis

$$\{F^e_*(\lambda x_1^{i_1}\cdots x_d^{i_d}) \mid 0 \le i_j < p^e, \text{ where } \{F^e_*\lambda\} \text{ is a free basis of } F^e_*k \text{ over } k\}.$$

Therefore $\operatorname{rank}_A(F^e_*A) = [F^e_*k : k]p^{ed}$ as claimed.

Now we suppose that R is not necessarily complete. Let P be a minimal prime of $\widehat{R}$ such that $d = \dim(R) = \dim(\widehat{R}/P)$. Let K be the fraction field of R and L the fraction field of $\widehat{R}/P$. By Lemma 10.5 we have that $[F^e_*K : K] = [F^e_*L : L]$, i.e., $\operatorname{rank}_R(F^e_*R) =$

$\operatorname{rank}_{\widehat{R}/P}(F^e_*(\widehat{R}/P))$. This completes the proof as we already showed that for the complete local domain $\widehat{R}/P$ that $\operatorname{rank}_{\widehat{R}/P}(F^e_*\widehat{R}/P) = [F^e_*k : k]p^{ed}$. ☐

Remark 10.7. The proof of Theorem 10.6 shows something more. It shows that if $(R, \mathfrak{m}, k)$ is an F-finite local domain of dimension d with fraction field K then $\widehat{R}$ is (reduced and) equidimensional. That is, for each minimal prime $Q \in \operatorname{Spec}(\widehat{R})$ we have that $\dim(\widehat{R}/Q) = d$. Indeed, if Q is a minimal prime of $\widehat{R}$ and L_Q is the fraction field of $\widehat{R}/Q$ then $[F^e_*K : K] = [F^e_*L_Q : L_Q]$ by Lemma 10.5. But by Theorem 10.6, $[F^e_*K : K] = [F^e_*k : k]p^{ed}$ and $[F^e_*L_Q : L_Q] = [F^e_*k : k]p^{e\dim(\widehat{R}/Q)}$. Therefore $d = \dim(\widehat{R}/Q)$.

This observation that the completion of an F-finite local domain is reduced and equidimensional is not surprising. Indeed, by Theorem 1.6 every F-finite ring is excellent (we will prove this in Section 12), and the completion of any excellent local domain is known to be reduced and equidimensional.

Corollary 10.8. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and $P \subsetneq Q$ be prime ideals. Then $\gamma(R/Q) < \gamma(R/P)$.*

Proof. Since $\dim(R/Q) < \dim(R/Q)$, $\gamma(R/Q) < \gamma(R/P)$ by Theorem 10.6. ☐

10.2. **F-signature exists.** Let R be an F-finite ring, not necessarily a domain. We set $\gamma(R) = \max\{\gamma(R/P) \mid P \in \operatorname{Spec}(R)\}$. Corollary 10.8 implies that $\gamma(R) = \max\{\gamma(R/P) \mid P \in \operatorname{Min}(R)\}$. If R is not necessarily a domain, so that the notion of generic rank is not necessarily well-defined, then in the spirit of Theorem 10.6 we set $\operatorname{rank}_R(F^e_*R) = p^{e\gamma(R)}$. Equivalently, we set $\operatorname{rank}_R(F^e_*R)$ to be the maximal generic rank of $F^e_*(R/P)$ over R/P as P varies through the (minimal) prime ideals of R.

Lemma 10.9. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and M a finitely generated R-module. There exists a constant $C \in \mathbb{R}$ so that for all $e \in \mathbb{N}$,*

$$\mu_R(F^e_*R) \le C \operatorname{rank}_R(F^e_*R).$$

Proof. Begin by considering a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ so that $M_i/M_{i-1} \cong R/P_i$ for some prime $P_i \in \operatorname{Spec}(R)$. Counting minimal generators is subadditive on short exact sequences, see Exercise 33, therefore $\mu_R(F^e_*R) \le \sum_{i=1}^t \mu_R(F^e_*R/P_i)$. Thus we may assume $M = R$ is an F-finite local domain.

We induct on $\gamma(R)$, the unique integer so that $\operatorname{rank}_R(F^e_*R) = p^{e\gamma(R)}$ for all $e \in \mathbb{N}$. If $\gamma(R) = 0$ is minimal then $R = k$ is a perfect field by Theorem 10.6 and there is nothing to show. Suppose that $\gamma(R) > 0$. Because we may assume that R is a domain we have that

F_*R is generically free of rank $p^{\gamma(R)}$ and hence there exists a short exact sequence

$$0 \to R^{\oplus p^{\gamma(R)}} \to F_*R \to T \to 0$$

where T is a finitely generated torsion R-module. In particular, T is a module over $R/(c)$ for some $c \neq 0$. Since $\gamma(R/(c)) < \gamma(R)$ by Corollary 10.8, we may assume by induction that there exists a constant C so that $\mu(F^e_*T) \leq Cp^{e(\gamma(R)-1)}$ for all $e \in \mathbb{N}$.

Applying $F^{e-1}_*(-)$ to the above short exact sequence we find new short exact sequences

$$0 \to F^{e-1}_*R^{\oplus p^{\gamma(R)}} \to F^e_*R \to F^{e-1}_*T \to 0.$$

Counting minimal number of generators is sub-additive on short exact sequences hence

$$\begin{aligned}\mu(F^e_*R) &\leq \mu(F^{e-1}_*R^{\oplus p^{\gamma(R)}}) + \mu(F^{e-1}_*T)\\ &= p^{\gamma(R)}\mu(F^{e-1}_*R) + \mu(F^{e-1}_*T)\\ &\leq p^{\gamma(R)}\mu(F^{e-1}_*R) + Cp^{e(\gamma(R)-1)}.\end{aligned}$$

Dividing by $\operatorname{rank}_R(F^e_*R) = p^{e\gamma(R)}$ we find that

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq \frac{\mu(F^{e-1}_*R)}{\operatorname{rank}_R(F^{e-1}_*R)} + \frac{C}{p^e}. \tag{10.1}$$

Similarly, there is an inequality

$$\frac{\mu(F^{e-1}_*R)}{\operatorname{rank}_R(F^{e-1}_*R)} \leq \frac{\mu(F^{e-2}_*R)}{\operatorname{rank}_R(F^{e-2}_*R)} + \frac{C}{p^{e-1}}. \tag{10.2}$$

Applying the inequality of (10.2) to (10.1) we find that

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq \frac{\mu(F^{e-2}_*R)}{\operatorname{rank}_R(F^{e-2}_*R)} + \frac{C}{p^{e-1}} + \frac{C}{p^e}.$$

Inductively, we derive the inequality

$$\frac{\mu(F^e_*R)}{\operatorname{rank}_R(F^e_*R)} \leq 1 + \frac{C}{p} + \cdots + \frac{C}{p^{e-1}} + \frac{C}{p^e} \leq C\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e-1}} + \frac{1}{p^e}\right) \leq \frac{C}{1-\frac{1}{p}} \leq 2C.$$

Therefore $\mu(F^e_*R) \leq 2C \operatorname{rank}_R(F^e_*R)$ for all $e \in \mathbb{N}$. $\square$

Corollary 10.10. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and let T be a finitely generated R-module not supported at any minimal prime of R. Then there exists a constant C so that*

$$\mu_R(F^e_*T) \leq Cp^{e(\gamma(R)-1)}.$$

Proof. Let $I = \operatorname{Ann}_R(T)$ and consider a surjection of the form $(R/I)^{\oplus N} \to T$. Then $\mu_R(F^e_*T) = \mu_{R/I}(F^e_*T)$. By Lemma 10.9 there exists an constant C so that $\mu_{R/I}(F^e_*T) \leq Cp^{e\gamma(R/I)}$. But $\gamma(R/I) < \gamma(R)$ by Corollary 10.8. $\square$

Lemma 10.11. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. If R is not strongly F-regular then $s(R) = 0$.*

Proof. If $F^e_* R \cong R^{\oplus a_e(R)} \oplus M_e$ is a choice of direct sum decomposition of $F^e_* R$ so that M_e does not have a free summand, see Corollary 10.2, then $N_e = \mathfrak{m}^{\oplus a_e(R)} \oplus M_e$. In particular, $N_e \subseteq F^e_* R$ is an R-submodule, $F^e_* R/N_e \cong k^{\oplus a_e(R)}$, and $a_e(R) = \ell_R(F^e_* R/N_e)$.

We are assuming R is not strongly F-regular. So there exists an element $c \in R$ not in any minimal primes of R such that $R \xrightarrow{\cdot F^e_* c} F^e_* R$ does not split for all $e \in \mathbb{N}$. Observe then that $\mathfrak{m}F^e_* R + \operatorname{span}_{F^e_* R}\{F^e_* c\} \subseteq N_e$ for all $e \in \mathbb{N}$. Therefore we can estimate

$$\begin{aligned} a_e(R) = \ell(F^e_* R/N_e) \leq & \ell_R(F^e_* R/(\mathfrak{m}F^e_* R + \operatorname{span}_{F^e_* R}\{F^e_* c\})) \\ = & \ell_R(F^e_*(R/(c)) \otimes_R R/\mathfrak{m}) = \mu_R(F^e_*(R/(c))). \end{aligned}$$

By Corollary 10.10 there is a constant C such that

$$\mu(F^e_*(R/(c))) \leq C p^{e(\gamma(R)-1)}.$$

Dividing by $p^{e\gamma(R)}$ and taking a limit as $e \to \infty$ shows that

$$0 \leq s(R) = \lim_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}} \leq \lim_{e\to\infty} \frac{C}{p^e} = 0. \qquad \square$$

Lemma 10.12. *Let $(R, \mathfrak{m}, k)$ be a local ring, not necessarily of prime characteristic, and let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of finitely generated R-modules. Then

$$\operatorname{frk}_R(M_2) \leq \operatorname{frk}_R(M_1) + \mu_R(M_3).$$

Proof. Begin by choosing direct sum decompositions $M_1 \cong R^{\oplus \operatorname{frk}_R(M_1)} \oplus \overline{M_1}$ and $M_2 \cong R^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}$ where $\overline{M_1}$ and $\overline{M_2}$ are R-modules without a free summand. Because $\overline{M_1}$ is a module without a free summand we have that $0 \oplus \overline{M_1} \subseteq \mathfrak{m}^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}$. In particular, there is an induced map

$$\frac{M_1}{0 \oplus \overline{M_1}} \to \frac{M_2}{\mathfrak{m}^{\oplus \operatorname{frk}_R(M_2)} \oplus \overline{M_2}}.$$

Equivalently, there is a right exact sequence

$$R^{\oplus \operatorname{frk}_R(M_1)} \to k^{\oplus \operatorname{frk}_R(M_2)} \to M_3' \to 0$$

and the cokernel M_3' is a homomorphic image of M_3. Counting minimal generators is subadditive on right exact sequences and therefore

$$\operatorname{frk}_R(M_2) \leq \operatorname{frk}_R(M_1) + \mu_R(M_3') \leq \operatorname{frk}_R(M_1) + \mu_R(M_3). \qquad \square$$

Now we can prove the first main result of this section.

Theorem 10.13. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Then the F-signature of R exists, i.e., the sequence of numbers $\left\{\frac{a_e(R)}{p^{e\gamma(R)}}\right\}_{e\in\mathbb{N}}$ defines a Cauchy sequence.*

Proof. By Lemma 10.11 we are reduced to the scenario that R is strongly F-regular. By Lemma 3.2 the ring R is a domain. Let

$$s_+(R) = \limsup_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}}, \text{ and } s_-(R) = \liminf_{e\to\infty} \frac{a_e(R)}{p^{e\gamma(R)}}.$$

We aim to show $s_+(R) \leq s_-(R)$.

Since $\operatorname{rank}_R(F_*R) = p^{\gamma(R)}$, we have a short exact sequence

$$0 \to F_*R \to R^{\oplus p^{\gamma(R)}} \to T \to 0$$

where T is a finitely generated torsion R-module. Applyig $F^e_*(-)$ gives us short exact sequences

$$0 \to F^{e+1}_* R \to F^e_* R^{\oplus p^{\gamma(R)}} \to F^e_* T \to 0.$$

By Lemma 10.12 we have that for each $e \in \mathbb{N}$ the inequality

$$\operatorname{frk}_R(F^e_* R^{\oplus p^{\gamma(R)}}) \leq \operatorname{frk}_R(F^{e+1}_* R) + \mu_R(F^e_* T),$$

that is,

$$p^{\gamma(R)} a_e(R) \leq a_{e+1}(R) + \mu_R(F^e_* T).$$

By Corollary 10.10 there exists a constant C so that $\mu_R(F^e_*T) \leq Cp^{e(\gamma(R)-1)}$. Dividing by $p^{(e+1)\gamma(R)}$ yields that

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq \frac{a_{e+1}(R)}{p^{(e+1)\gamma(R)}} + \frac{C}{p^e}.$$

We can similarly bound the ratio $\frac{a_{e+1}(R)}{p^{(e+1)\gamma(R)}}$ from above by $\frac{a_{e+2}(R)}{p^{(e+2)\gamma(R)}} + \frac{C}{p^{e+1}}$ and therefore

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq \frac{a_{e+2}(R)}{p^{(e+2)\gamma(R)}} + \frac{C}{p^e} + \frac{C}{p^{e+1}}.$$

Inductively, we find that for all $e, e_0 \in \mathbb{N}$ that

$$\begin{aligned}\frac{a_e(R)}{p^{e\gamma(R)}} &\leq \frac{a_{e+e_0}(R)}{p^{(e+e_0)\gamma(R)}} + \frac{C}{p^e} + \frac{C}{p^{e+1}} + \cdots + \frac{C}{p^{e+e_0-1}} \\ &= \frac{a_{e+e_0}(R)}{p^{(e+e_0)}} + \frac{C}{p^e}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{e_0-1}}\right) \leq \frac{a_{e+e_0}(R)}{p^{(e+e_0)}} + \frac{2C}{p^e}.\end{aligned}$$

Taking a limit infimum as $e_0 \to \infty$ shows that for all $e \in \mathbb{N}$ that

$$\frac{a_e(R)}{p^{e\gamma(R)}} \leq s_-(R) + \frac{2C}{p^e}.$$

Taking a limit supremum as $e \to \infty$ then shows that

$$s_+(R) \le s_-(R),$$

i.e., the F-signature of R exists. $\square$

10.3. **F-signature and strong F-regularity.** We aim to prove that an F-finite local ring $(R, \mathfrak{m}, k)$ is strongly F-regular if and only if $s(R) > 0$. Lemma 10.11 provides the "easy" implication that $s(R) > 0$ implies that R is strongly F-regular. It remains to show that if R is strongly F-regular then $s(R) > 0$. This was first proved by Aberbach–Leuschke in [AL03]. Their approach relies on a "valuative criterion" for tight closure given by Hochster–Huneke in [HH91] (generalized by Aberbach in [Abe01]) and the Izumi–Rees theorem [Ree89] which linearly bounds any two Rees valuations centered on the maximal ideal of an analytically irreducible local domain. We shall not resort to these techniques and will instead follow a novel and more elementary path laid out in [Pol20]. We begin with the module version of the celebrated Chevalley's lemma.

Lemma 10.14 ([Che43])**.** *Let $(R, \mathfrak{m}, k)$ be a complete local ring, not necessarily of prime characteristic, and M a finitely generated R-module. Suppose that $I \subseteq R$ is an $\mathfrak{m}$-primary ideal and $\{M_n\}_{n\in\mathbb{N}}$ is a descending sequence of submodules of M so that $\bigcap_{n\in\mathbb{N}} M_n = 0$. Then there exists an $n > 0$ such that $M_n \subseteq IM$.*

Proof. Since M/IM us Artinian, the descending chain of submodules $\{(M_n + IM)/IM\}_{n\in\mathbb{N}}$ eventually stabilizes. Thus there exists n_1 so that for all $n \ge n_1$ we have that $(M_n + IM)/IM = (M_{n_1} + IM)/IM$. Similarly, there exists $n_2 > n_1$ so that $(M_n + I^2M)/I^2M = (M_{n_2} + I^2M)/I^2M$ for all $n \ge n_2$. Inductively choose n_t so that $n_{t+1} > n_t$ and $(M_n + I^tM)/I^tM = (M_{n_t} + I^tM)/I^tM$ for all $n \ge n_t$. Replacing M_t by M_{n_t}, we may assume that the sequence of modules $\{M_n\}_{n\in\mathbb{N}}$ is such that $(M_n + I^tM)/IM = (M_t + I^tM)/I^tM$ for all $n \ge t$.

We claim that $M_1 \subseteq IM$. Choose an element $\eta_1 \in M_1$. Because $(M_2 + IM)/IM = (M_1 + IM)/IM$ we can choose $\eta_2 \in M_2$ so that $\eta_2 \equiv \eta_1 \mod IM$. Inductively, we choose elements $\eta_t \in M_t$ so that $\eta_{t+1} \equiv \eta_t \mod I^tM$. The sequence of elements $\{\eta_t\}$ forms a Cauchy sequence. Let $\tilde{\eta} \in M$ denote its limit (which exists since M is complete: it is a finitely generated module over a complete local ring). Because each $\eta_t \in M_t$ and $\bigcap M_t = 0$ we must have that $\tilde{\eta} = 0$. In particular, there exists a t such that $\eta_t \in IM$. Recall that $\eta_t \equiv \eta_{t-1} \mod I^{t-1}M$. Hence $\eta_t - \eta_{t-1} \in I^tM \subseteq IM$ and therefore $\eta_{t-1} \in IM$. By induction $\eta_1 \in IM$ and hence $M_1 \subseteq IM$ as claimed. $\square$

Lemma 10.15. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring and M a finitely generated R-module. For each $e \in \mathbb{N}$ let*

$$I_e(M) = \{\eta \in M \mid R \xrightarrow{\cdot F^e_* \eta} F^e_* M \text{ does not split}\}.$$

1. *For each $e \in \mathbb{N}$ the set $I_e(M)$ is a submodule of M containing $\mathfrak{m}^{[p^e]}M$.*
2. *For each $e \in \mathbb{N}$ we have that $a_e(M) = \ell(M/I_e(M))[F^e_* k : k]$.*
3. *$\{I_e(M)\}_{e\in\mathbb{N}}$ is a descending chain of submodules of M.*
4. *If R is strongly F-regular and M is torsion-free then $\bigcap_{e\in\mathbb{N}} I_e(M) = 0$.*

Proof. (1) Suppose that $\eta_1, \eta_2 \in I_e(M)$ and $r \in R$ we aim to show that $r\eta_1 + \eta_2 \in I_e(M)$. Suppose by way of contradiction that there exists $\varphi \in \operatorname{Hom}_R(F^e_* M, R)$ so that $\varphi(F^e_*(r\eta_1 + \eta_2)) = \varphi(F^e_* r\eta_1) + \varphi(F^e_* \eta_2) = 1$. Because R is local we must have that either $\varphi(F^e_* r\eta_1)$ is a unit of R or $\varphi(F^e_*\eta_2)$ is a unit of R. If $\varphi(F^e_* r\eta_1)$ is a unit then $\eta_1 \notin I_e(M)$ if $\varphi(F^e_*\eta_2)$ is a unit then $\eta_2 \notin I_e(M)$.

(2) Suppose that $F^e_* M \cong R^{\oplus a_e(M)} \oplus N$ is a choice of direct sum decomposition of $F^e_* M$ so that N does not admit a free summand. Under this choice of direct sum decomposition we have that $F^e_* I_e(M) = \mathfrak{m}^{\oplus a_e(M)} \oplus N$. Therefore

$$\ell_R(M/I_e(M)) = \ell_{F^e_* R}(F^e_*(M/I_e(M))) = \frac{\ell_R(F^e_* M/F^e_* I_e(M))}{[F^e_* k : k]} = \frac{a_e(M)}{[F^e_* k : k]}.$$

(3) We want to show $I_e(M) \supseteq I_{e+1}(M)$, this is clear if $I_e(M) = R$ and so we assume $I_e(M) \neq R$. Suppose $\eta \notin I_e(M)$ and choose splitting $\varphi : F^e_* M \to R$ so that $\varphi(F^e_*\eta) = 1$. We will show that $\eta \notin I_{e+1}(M)$. Observe that R is F-pure: consider $R \xrightarrow{\cdot\eta} M$, then $F^e_* R \xrightarrow{\cdot F^e_*\eta} F^e_* M \xrightarrow{\varphi} R$ is a splitting of $F^e_* R$. In particular, we can choose a splitting $F_* R \xrightarrow{\psi} R$ so that $\psi(F_* 1) = 1$, see Exercise 6. Then $\psi(F_*\varphi(F^{e+1}_*\eta)) = 1$ and $\eta \notin I_{e+1}(M)$ as claimed.

(4) Because M is torsion-free and finitely generated there exists an inclusion of M into a free module $R^{\oplus N}$. Let $m \in M$ be a non-zero element. By mapping onto an appropriate summand of $R^{\oplus N}$ we find that there exists a map $\varphi : M \to R$ so that $\varphi(m) = r \neq 0$. We are assuming R is strongly F-regular. So there exists an $e \in \mathbb{N}$ and $\psi : F^e_* R \to R$ so that $\psi(F^e_* r) = 1$. Therefore $\psi(F^e_*\varphi(m)) = 1$ and hence $m \notin I_e(M)$. $\square$

Theorem 10.16. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring. Then there exists an $e_0 \in \mathbb{N}$ so that if M is a finitely generated maximal Cohen–Macaulay R-module and $\eta \in M \setminus \mathfrak{m}M$ then there exists $\varphi \in \operatorname{Hom}_R(F^{e_0}_* M, R)$ so that $\varphi(F^{e_0}_*\eta) = 1$.*

Proof. First of all we observe that the finitely generated R-module $F^{e_0}_* M$ has a free R-summand if and only if $F^{e_0}_* \widehat{M}$ has a free $\widehat{R}$-summand (see Exercise 34). Therefore one can replace R by $\widehat{R}$ to assume that R is complete (note that strong F-regularity is preserved under completion by Corollary 3.12). In particular, R admits a canonical module ω_R. Given a finitely generated R-module N, we use N^* to denote the ω_R-dual $\operatorname{Hom}_R(N, \omega_R)$.

Surject a free module $R^{\oplus N}$ onto M^*, let K denote the kernel, and consider the short exact sequence

$$0 \to K \to R^{\oplus N} \to M^* \to 0.$$

The module $R^{\oplus N}$ is Cohen–Macaulay by Theorem 4.6, M^* is Cohen–Macaulay by [BH93, Theorem 3.3.10], and therefore K is seen to be Cohen–Macaulay by examining the induced long exact sequence of local cohomology modules with support in the maximal ideal $\mathfrak{m}$. If we apply $\operatorname{Hom}_R(-, \omega_R)$ to the above short exact sequence and utilize [BH93, Theorem 3.3.10] a second time we find that there is a short exact sequence of Cohen–Macaulay R-modules

$$0 \to M \to \omega_R^{\oplus N} \to K^* \to 0. \tag{10.3}$$

Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R and let $I = (\underline{x})$. Then $\operatorname{Tor}_1^R(R/I, K^*)$ agrees with the first Koszul homology module $H_1(\underline{x}; K^*)$ and $H_1(\underline{x}; K^*) = 0$ as $\underline{x}$ is a regular sequence on K^*. Therefore if we apply $- \otimes_R R/I$ to the short exact sequence in (10.3) we produce a new short exact sequence

$$0 \to \frac{M}{IM} \to \frac{\omega_R^{\oplus N}}{I\omega_R^{\oplus N}} \to \frac{K^*}{IK^*} \to 0.$$

Consequently, if $\eta \in M \backslash IM$ then under the inclusion $M \subseteq \omega_R^{\oplus N}$ we find that $\eta \in \omega_R^{\oplus N} \backslash I\omega_R^{\oplus N}$.

For each natural number $e \in \mathbb{N}$ let

$$I_e(\omega_R) = \{m \in \omega_R \mid R \xrightarrow{\cdot F^e_* m} F^e_* \omega_R \text{ does not split}\}.$$

By Lemma 10.15, $\bigcap_{e \in \mathbb{N}} I_e(\omega_R) = 0$. By Lemma 10.14 there exists an integer e_0, depending only on ω_R and I, so that $I_{e_0}(\omega_R) \subseteq I\omega_R$. Thus if $\eta \in M \setminus IM$ then under the inclusion $M \subseteq \omega_R^{\oplus N}$ we must have that $\eta \in \omega_R^{\oplus N} \setminus I_{e_0}(\omega_R)^{\oplus N}$. In particular, there exists an R-linear map $\varphi : F^{e_0}_* \omega_R^{\oplus N} \to R$ so that $\varphi(F^{e_0}_* \eta) = 1$. Restricting the domain of φ to $F^{e_0}_* M$ then shows that $F^{e_0}_* M$ admits a free summand. $\square$

Theorem 10.17. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Then $s(R) > 0$ if and only if R is strongly F-regular.*

Proof. By Lemma 10.11, if $s(R) > 0$ then R is strongly F-regular. So we assume R is strongly F-regular and our goal is to show that $s(R) > 0$. Let e_0 be as in Theorem 10.16 and for each $e \in \mathbb{N}$ consider the Frobenius non-splitting ideals

$$I_e := I_e(R) = \{r \in R \mid R \xrightarrow{\cdot F^e_* r} F^e_* R \text{ does not split}\},$$

as in Lemma 10.15. We claim that $I_{e+e_0} \subseteq \mathfrak{m}^{[p^e]}$ for all $e \in \mathbb{N}$. Indeed, $r \in R \setminus \mathfrak{m}^{[p^e]}$ if and only if $F^e_* r \in F^e_* R \setminus \mathfrak{m} F^e_* R$. The modules $F^e_* R$ are maximal Cohen–Macaulay and so by Theorem 10.16 there exists $\varphi : F^{e+e_0}_* R \to R$ so that $\varphi(F^{e+e_0}_* r) = 1$, i.e., $r \in R \setminus I_{e+e_0}$. Therefore

$$\frac{a_{e+e_0}(R)}{[F^{e+e_0}_* k : k]} = \ell(R/I_{e+e_0}) \geq \ell(R/\mathfrak{m}^{[p^e]}) = \frac{\mu(F^e_* R)}{[F^e_* k : k]} \geq \frac{p^{e\gamma(R)}}{[F^e_* k : k]} = p^{ed}.$$

Dividing by $p^{(e+e_0)d}$ and taking a limit as $e \to \infty$ shows that

$$s(R) \geq \frac{1}{p^{e_0 d}} > 0. \qquad \square$$

10.4. **F-signature and regularity.** Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Huneke–Leuschke were the first to prove in [HL02] that R is a regular local ring if and only if $s(R) = 1$. They showed that $s(R) = 1$ implies that a related numerical invariant called the Hilbert–Kunz multiplicity of R, $\mathrm{e}_{\mathrm{HK}}(R)$, must also be equal to 1. Then they appeal to a result of Watanabe–Yoshida [WY00] that analytically irreducible local rings with Hilbert–Kunz multiplicity equal to 1 must be regular. The proof of Huneke–Leuschke's theorem presented here follows the methodology of [PS19] and allows us to bypass Hilbert–Kunz theory.

Our proof that $s(R) = 1$ if and only if R is regular is a consequence of developing an equimultiplicity theory of F-signature in strongly F-regular rings. More specifically, we need to study the behavior of F-signature and Frobenius splitting numbers under localization.

Suppose that $F^e_* R \cong R^{\oplus a_e(R)} \oplus M_e$ and the module M_e does not admit a free summand. If $P \in \operatorname{Spec}(R)$ then

$$F^e_* R \otimes_R R_P \cong F^e_* R_P \cong R_P^{\oplus a_e(R)} \oplus (M_e)_P.$$

By Lemma 10.1 we find that $a_e(R_P) \geq a_e(R)$ and equality holds if and only if $(M_e)_P$ does not admit a free R_P-summand. Therefore to keep track of the differences of the Frobenius splitting numbers of R and a localization of R at a prime ideal P it is beneficial to keep track of the number of summands $F^e_* R$ isomorphic to a particular module. To this end, if M is a finitely generated R-module we let

$$a^M_e(R) = \max\{n \mid M^{\oplus n} \text{ is a direct summand of } F^e_* R\}.$$

Observe that if M does not admit a free summand and M is also a direct summand of $F^e_* R$ so that M_P has at least one free R_P-summand then $a_e(R_P) \geq a_e(R) + a^M_e(R)$.

The following lemma is an elementary observation that for a strongly F-regular local ring R, if a finitely generated R-module M is a direct summand of $F^{e_0}_* R$ for some e_0, then the numbers $a^M_e(R)$ are asymptotically comparable to the numbers $\operatorname{rank}_R(F^e_* R) = p^{e\gamma(R)}$.

Lemma 10.18. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring and let M be a finitely generated R-module. If $a_{e_0}^M(R) \geq 1$ for some $e_0 \in \mathbb{N}$ then*

$$\liminf_{e\to\infty} \frac{a_e^M(R)}{p^{e\gamma(R)}} > 0.$$

Proof. Suppose that $F_*^{e_0} R \cong M \oplus N$ and then consider a direct sum decomposition of $F_*^e R$ as $F_*^e R \cong R^{\oplus a_e(R)} \oplus P$. Then

$$F_*^{e+e_0} R \cong F_*^{e_0} R^{\oplus a_e(R)} \oplus F_*^{e_0} P \cong (M \oplus N)^{\oplus a_e(R)} \oplus F_*^{e_0} P.$$

In particular,

$$a_{e+e_0}^M(R) \geq a_e(R).$$

Dividing by $p^{(e+e_0)\gamma(R)}$ and taking a limit infimum as $e \to \infty$ reveals that

$$\liminf_{e\to\infty} \frac{a_e^M(R)}{p^{e\gamma(R)}} \geq \frac{s(R)}{p^{e_0\gamma(R)}},$$

a quantity that is positive by Theorem 10.17. $\square$

A consequence of Lemma 10.18 is an equimultiplicity theory of F-signature. The following corollary gives us that F-signature is unchanged under localization at a prime ideal if and only if each of the Frobenius splitting numbers too are unchanged under localization.

Corollary 10.19. *Let $(R, \mathfrak{m}, k)$ be an F-finite and strongly F-regular local ring. Suppose that $P \in \operatorname{Spec}(R)$. Then the following are equivalent:*

(1) $a_e(R) = a_e(R_P)$ for all $e \in \mathbb{N}$;
(2) $s(R) = s(R_P)$.

Proof. If $a_e(R) = a_e(R_P)$ for all $e \in \mathbb{N}$ then $s(R) = s(R_P)$: The sequences of numbers $\{\frac{a_e(R)}{p^{e\gamma(R)}}\}$ and $\{\frac{a_e(R_P)}{p^{e\gamma(R_P)}}\}$ defining the F-signature of R and R_P respectively are identical sequences, see Lemma 10.3.

Suppose that $a_{e_0}(R) \neq a_e(R_P)$, or equivalently, $F_*^{e_0} R \cong R^{\oplus a_{e_0}(R)} \oplus M_{e_0}$ where M_{e_0} does not admit a free summand but $(M_{e_0})_P$ has a free R_P-summand (see Lemma 10.1). By Lemma 10.18 we have that

$$\liminf_{e\to\infty} \frac{a_e^{M_{e_0}}(R)}{p^{e\gamma(R)}} > 0.$$

For each $e \in \mathbb{N}$ consider a direct sum decomposition of the form

$$F_*^e R \cong R^{\oplus a_e(R)} \oplus M_{e_0}^{\oplus a_e^{M_{e_0}}(R)} \oplus N_e.$$

Localizing at P and counting free summands gives us

$$a_e(R_P) \geq a_e(R) + a_e^{M_{e_0}}(R).$$

Diving by $p^{e\gamma(R)} = p^{e\gamma(R_P)}$ and taking a limit infimum as $e \to \infty$ shows that

$$s(R_P) \geq s(R) + \liminf_{e\to\infty} \frac{a_e^{M_{e_0}}(R)}{p^{e\gamma(R)}} > s(R). \qquad \square$$

Now we can prove the following.

Theorem 10.20. *Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Then $s(R) = 1$ if and only if R is a regular local ring.*

Proof. If R is regular then $F^e_* R$ is a free R-module for all $e \in \mathbb{N}$ by Theorem 1.1. Hence $\frac{a_e(R)}{p^{e\gamma(R)}} = 1$ for all $e \in \mathbb{N}$ and so $s(R) = 1$.

Conversely, if $s(R) = 1$ then R is strongly F-regular by Theorem 10.17 and hence a domain by Lemma 3.2. Consider the localization of R at the prime ideal 0 and observe then that $1 = s(R) = s(R_0)$. By Corollary 10.19 we must have that $a_e(R) = a_e(R_0) = \operatorname{rank}_R(F^e_* R)$ for all $e \in \mathbb{N}$. Therefore $F^e_* R$ is a free R-module for all $e \in \mathbb{N}$ and therefore R is a regular local ring by Theorem 1.1. $\square$

Exercise 33. Let $(R, \mathfrak{m}, k)$ be a local ring, not necessarily of prime characteristic, and $M' \to M \to M'' \to 0$ a right exact sequence of finitely generated R-modules. Show that if M', M, M'' are of finite length then

$$\ell_R(M) \leq \ell_R(M') + \ell_R(M'').$$

Conclude that if M, M', M'' are not necessarily of finite length then $\mu_R(M) \leq \mu_R(M') + \mu_R(M'')$ where $\mu_R(N)$ counts the minimal number of elements needed to generate a finitely generated R-module N.

Exercise 34. Let $(R, \mathfrak{m}, k)$ be a local ring, not necessarily of prime characteristic, and M a finitely generated R-module. Show that $\operatorname{frk}_R(M) = \operatorname{frk}_{\widehat{R}}(\widehat{M})$.

Exercise 35. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring. Prove that $s(R) \leq s(R_P)$ for all $P \in \operatorname{Spec}(R)$.

11. The Radu–André theorem

The goal of this section is to explain the following theorem that is a relative version of Kunz's theorem. Recall that a map $R \to S$ of (Noetherian) rings is called *regular* if it is flat and all fibers are geometrically regular, i.e., $\kappa(P) \otimes_R S$ is geometrically regular over $\kappa(P)$ for all $P \in \operatorname{Spec}(R)$.

Theorem 11.1 (Radu–André [Rad92, And93])**.** *A homomorphism $R \to S$ of (Noetherian) rings is regular if and only if $F^e_* R \otimes_R S \to F^e_* S$ is flat for some (equivalently, all) $e > 0$.*

The difficulty of the theorem is that it is not clear in priori that $F^e_* R \otimes_R S$ is a Noetherian ring (though it will follow from the conclusion of the theorem that $F^e_* R \otimes_R S$ is in fact Noetherian, see Exercise 37).[9] Therefore we proceed carefully. We first record some criteria for flatness, see [sta16, Tag 00MD] for more details.

Lemma 11.2 ([sta16, Lemma 10.98.11])**.** *Let $R \to S$ be a map of (Noetherian) rings. Let $I \subseteq R$ be an ideal and let M be a finitely generated S-module. Suppose for each $n \geq 1$, $M/I^n M$ is flat over R/I^n. Then for each prime $Q \in \operatorname{Spec}(S)$ such that $I \subseteq Q$, M_Q is flat over R. In particular, if $(S, \mathfrak{n})$ is local and $IS \subseteq \mathfrak{n}$, then M is flat over R.*

Lemma 11.3 ([sta16, Lemma 10.98.8])**.** *Let A be a ring that is not necessarily Noetherian, $I \subseteq A$ an ideal, and M an A-module. If M/IM is flat over A/I and $\operatorname{Tor}^A_1(A/I, M) = 0$, then*

(1) $M/I^n M$ is flat over A/I^n for all $n \geq 1$.
(2) For any A-module N that is annihilated by I^m for some $m \geq 0$, $\operatorname{Tor}^A_1(N, M) = 0$.

In particular, if I is nilpotent, then M is flat over R.

The next criteria is well-known to experts, as we cannot find a good reference beyond the Noetherian set up, we deduce it from Lemma 11.3.

Lemma 11.4 (Fiberwise criteria for flatness)**.** *Let A be a ring that is not necessarily Noetherian and let M be an A-module. Let $t \in A$ such that t is a nonzerodivisor on both A and M. If M/tM is flat over A/tA and M_t is flat over A_t, then M is flat over A.*

Proof. By Lemma 11.3 applied to $I = (t)$, we know that $\operatorname{Tor}^A_1(N, M) = 0$ for all t^∞-torsion A-module (by taking direct limit). For any t^m-torsion A-module N, we have $0 \to K \to F \to N \to 0$ where F is a free $A/t^m A$-module and K is t^m-torsion. Since t is a nonzerodivisor on A and M, $\operatorname{Tor}^A_j(F, M)$=0 for all $j > 0$. The long exact sequence of Tor then shows that

[9] If we know $F^e_* R \otimes_R S$ is Noetherian in priori (e.g., if R is F-finite), then at least one direction of the theorem follows quite easily from Kunz's theorem and the local criterion for flatness [sta16, Lemma 10.98.10].

$\operatorname{Tor}_j^A(N, M) = 0$ for all $j > 0$. By taking direct limit we know that $\operatorname{Tor}_j^A(N, M) = 0$ for all $j > 0$ and all t^∞-torsion A-module N.

For an arbitrary A-module N, we have two short exact sequences:

$$0 \to \Gamma_{(t)}N \to N \to \overline{N} \to 0, \text{ and } 0 \to \overline{N} \to \overline{N}_t \to N' \to 0$$

Now $\operatorname{Tor}_j^A(\Gamma_{(t)}N, M) = \operatorname{Tor}_j^A(N', M) = 0$ for all $j > 0$ since $\Gamma_{(t)}N, N'$ are both t^∞-torsion, and $\operatorname{Tor}_j^A(\overline{N}_t, M) \cong \operatorname{Tor}_j^{A_t}(\overline{N}_t, M_t) = 0$ for all $j > 0$ since M_t is flat over A_t. By examining the long exact sequence of Tor, it is easy to see that $\operatorname{Tor}_j^A(N, M) = 0$ for all $j > 0$. Thus M is flat over A. $\square$

Now we can prove the theorem of Radu–André.

Proof of Theorem 11.1. We first prove that if $F_*^eR \otimes_R S \to F_*^eS$ is flat for some $e > 0$, then $R \to S$ is regular. We observe that if $F_*^eR \otimes_R S \to F_*^eS$ is flat, then applying $F_*^e(-)$, we see that $F_*^{2e}R \otimes_{F_*^eR} F_*^eS \to F_*^{2e}S$ is flat, while applying $\otimes_{F_*^eR}F_*^{2e}R$, we see that $F_*^{2e}R \otimes_R S \to F_*^{2e}R \otimes_{F_*^eR} F_*^eS$ is flat. Thus composing these two maps we see that $F_*^{2e}R \otimes_R S \to F_*^{2e}S$ is flat. Thus iterating this process, we find that there are infinitely many $e > 0$ such that $F_*^eR \otimes_R S \to F_*^eS$ is flat.

We set $\kappa = \kappa(P)$. To see $\kappa \otimes_R S$ is geometrically regular. Note that for any finite and purely inseparable field extension κ' of κ, we can pick $e \gg 0$ such that $\kappa' \subseteq F_*^e\kappa$ and $F_*^eR \otimes_R S \to F_*^eS$ is flat. Base change the flat map $F_*^eR \otimes_R S \to F_*^eS$ along $F_*^eR \to F_*^e\kappa$, we know that $F_*^e\kappa \otimes_R S \to F_*^e(\kappa \otimes_R S)$ is flat. Consider the composition:

$$\kappa' \otimes_R S \to F_*^e\kappa \otimes_R S \to F_*^e(\kappa \otimes_R S) \to F_*^e(\kappa' \otimes_R S)$$

where the first and third maps are flat as they are base changed from field extensions, and the middle map is flat by previous discussion. Thus the composition is flat and so $\kappa' \otimes_R S$ is regular by Theorem 1.1. Therefore $\kappa \otimes_R S$ is geometrically regular.

To show $R \to S$ is flat, we may localize a prime ideal of S and localize the contraction of that prime ideal to R. Thus we may assume that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism. By Lemma 11.2, it is enough to show that $R/\mathfrak{m}^{[p^e]} \to S/\mathfrak{m}^{[p^e]}S$ is flat for infinitely many $e > 0$. Base change the flat map $F_*^eR \otimes_R S \to F_*^eS$ along $R \to R/\mathfrak{m}$, we see that $F_*^e(R/\mathfrak{m}^{[p^e]}) \otimes_{R/\mathfrak{m}} S/\mathfrak{m}S \to F_*^e(S/\mathfrak{m}^{[p^e]}S)$ is flat. Thus the composition:

$$F_*^e(R/\mathfrak{m}^{[p^e]}) \to F_*^e(R/\mathfrak{m}^{[p^e]}) \otimes_{R/\mathfrak{m}} S/\mathfrak{m}S \to F_*^e(S/\mathfrak{m}^{[p^e]}S)$$

is flat (the first map is flat since it is base changed over a field) as desired.

We now prove the other direction that if $R \to S$ is regular, then $F_*^eR \otimes_R S \to F_*^eS$ is flat for all $e > 0$. For any ideal $J \subseteq R$, consider the ideal $F_*^eJ(F_*^eR \otimes_R S) \cong F_*^eJ \otimes_R S \subseteq F_*^eR \otimes_R S$.

Since $R \to S$ (and hence $F^e_*R \to F^e_*R \otimes_R S$) is flat, we know that

$$\operatorname{Tor}_j^{F^e_*R\otimes_R S}(F^e_*S, (F^e_*R \otimes_R S)/F^e_*J(F^e_*R \otimes_R S)) \cong \operatorname{Tor}_j^{F^e_*R}(F^e_*S, F^e_*R/F^e_*J) = 0$$

for all $j > 0$. Apply the above discussion to the nilradical J of R, since $J^n = 0$ for $n \gg 0$ as R is Noetherian, if we can show that

$$F^e_*(R/J) \otimes_{R/J} (S/JS) \cong (F^e_*R \otimes_R S)/F^e_*J(F^e_*R \otimes_R S) \to F^e_*S/F^e_*J(F^e_*S) \cong F^e_*(S/J)$$

is flat, then by Lemma 11.3 (applied to $A = F^e_*R \otimes_R S$ and $I = F^e_*J(F^e_*R \otimes_R S)$) we will get that $F^e_*R \otimes_R S \to F^e_*S$ is flat as desired. Therefore, we may replace R by R/J to assume R is reduced.

We next note that, to show $F^e_*R \otimes_R S \to F^e_*S$ is flat, it is enough to check this at each prime ideal of S. Thus we may localize S at a prime ideal and localize R at the contraction to assume $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a regular local homomorphism.

Now we use induction on $\dim(R)$. If $\dim(R) = 0$, then since we may assume R is local and reduced, $R = k$ is a field and our hypothesis becomes that S is geometrically regular over k. Consider the composition:

$$F^e_*k \otimes_k S \to F^e_*S \to F^{2e}_*k \otimes_{F^e_*k} F^e_*S \cong F^e_*(F^e_*k \otimes_k S).$$

This composition is flat: $F^e_*k \otimes_k S = \varinjlim_{k'} k' \otimes_k S$ where k' runs over all finite field extensions of k contained in F^e_*k, since each $k' \otimes_k S$ is regular by our assumption, $k' \otimes_k S \to F^e_*(k' \otimes_k S)$ is flat by Theorem 1.1, and a direct limit of flat maps is flat. But the second map in the composition is obviously faithfully flat as it is base changed from field extensions. Thus the first map in the composition, $F^e_*k \otimes_k S \to F^e_*S$, is flat. This proves the case $\dim(R) = 0$.

Finally, we assume $\dim(R) > 0$. We may assume $(R, \mathfrak{m})$ is local and reduced. Thus there exists a nonzerodivisor $t \in \mathfrak{m}$. Since $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is flat, $F^e_*t \otimes 1$ is a nonzerodivisor on $F^e_*R \otimes_R S$ and F^e_*t is a nonzerodivisor on F^e_*S. By Lemma 11.4, to show $(F^e_*R \otimes_R S) \to F^e_*S$ is flat, it is enough to show that

(1) $(F^e_*R \otimes_R S)/F^e_*t(F^e_*R \otimes_R S) \to F^e_*S/F^e_*t(F^e_*S)$ is flat, and
(2) $(F^e_*R \otimes_R S)[\frac{1}{F^e_*t\otimes 1}] \to (F^e_*S)[\frac{1}{F^e_*t}]$ is flat.

Now the first map is the same as $F^e_*(R/tR) \otimes_{R/tR} S/tS \to F^e_*(S/tS)$, while the second map is the same as $F^e_*(R_t) \otimes_{R_t} S_t \to F^e_*(S_t)$. Since t is a nonzerodivisor, $\dim(R/tR) < \dim(R)$ and $\dim(R_t) < \dim(R)$. Thus by induction on dimension, we know both maps are flat (note that R_t is not local, but this doesn't matter, since to show $F^e_*(R_t) \otimes_{R_t} S_t \to F^e_*(S_t)$ is flat, we can localize at primes of S_t and their contractions to R_t again). This completes the proof. $\square$

Exercise 36. Let $R \to S$ be a homomorphism of (Noetherian) rings such that $F^e_* R \otimes_R S \to F^e_* S$ is pure. Prove that all fibers of $R \to S$ are F-pure.

Exercise 37. Let $R \to S$ be a regular homomorphism of (Noetherian) rings. Prove that $F^e_* R \otimes_R S$ is a Noetherian ring.

Exercise 38. Let R be a (Noetherian) complete local ring with coefficient field k. Prove that $k^{1/p^\infty} \otimes_k R$ is a Noetherian local ring.

We refer the reader to [Dum96] and [Has10] for some more results about the Radu–André theorem and F-singularities.

12. F-finite rings, excellence, and canonical module

Our first goal in this section is to prove the following result of Kunz.

Theorem 12.1 ([Kun76])**.** *If R is F-finite, then R is excellent. Moreover, if $(R,\mathfrak{m})$ is local, then R is F-finite if and only if R is excellent and $R/\mathfrak{m}$ is F-finite.*

We start with a lemma.

Lemma 12.2. *Let $(R,\mathfrak{m})$ be an F-finite local domain and let K be the fraction field of R. Then for any finite field extension L of K, $L\otimes_R \widehat{R}$ is regular.*

Proof. For all $e>0$, we have

$$F^e_*(L\otimes_R \widehat{R}) = F^e_*L\otimes_{F^e_*R} F^e_*\widehat{R} \cong F^e_*L\otimes_{F^e_*R} F^e_*R\otimes_R \widehat{R} = F^e_*L\otimes_R \widehat{R}$$

where the isomorphism in the middle follows from Lemma 10.4. Since F^e_*L is free over L, $F^e_*L\otimes_R \widehat{R}$ is free over $L\otimes_R \widehat{R}$. Thus by Theorem 1.1,[10] $L\otimes_R \widehat{R}$ is regular. □

We will also need the following fact about excellent rings, see [sta16, Tag 032E] for more details.

Lemma 12.3 ([sta16, Lemma 10.160.2])**.** *Let R be an excellent reduced ring with total quotient ring K. Then the integral closure of R in any finite reduced extension L of K is module-finite over R.*

Now we are ready to prove Kunz's theorem. Recall that R is excellent if R satisfies the following:

(1) R is universally catenary.
(2) If S is an R-algebra of finite type, then the regular locus of S is open in $\operatorname{Spec}(S)$.
(3) For all $P\in\operatorname{Spec}(R)$, the map $R_P\to\widehat{R_P}$ has geometrically regular fibers.

Proof of Theorem 12.1. We first show that if R is F-finite, then R is excellent. Since any ring finite type over an F-finite ring is still F-finite (see Exercise 4), to show R is universally catenary, it is enough to show that any F-finite ring is catenary.

Now let $P\subseteq Q$ be two prime ideals in R, we want to show any saturated chain of primes between P and Q have the same length. Suppose we have two saturated chains:

$$P=P_0\subseteq P_1\subseteq\cdots\subseteq P_n=Q, \text{ and } P=Q_0\subseteq Q_1\subseteq\cdots\subseteq Q_m=Q.$$

[10] Note that here we are implicitly using that $L\otimes_R \widehat{R}$ is Noetherian: it is module-finite over $K\otimes_R \widehat{R}$, which is a localization of $\widehat{R}$.

Applying Theorem 10.6 to $R_{P_{i+1}}/P_iR_{P_{i+1}}$, we find that

$$[F^e_*\kappa(P_i) : \kappa(P_i)] = p^e \cdot [F^e_*\kappa(P_{i+1}) : \kappa(P_{i+1})] \text{ for all } i.$$

Thus $[F^e_*\kappa(P) : \kappa(P)] = p^{en} \cdot [F^e_*\kappa(Q) : \kappa(Q)]$, but then the same argument for the other chain shows that $[F^e_*\kappa(P) : \kappa(P)] = p^{em} \cdot [F^e_*\kappa(Q) : \kappa(Q)]$. It follows that $n = m$.

We next show that for any finite type R-algebra S, the regular locus of S is an open subset of $\operatorname{Spec}(S)$. But since S is F-finite, F^e_*S is a finitely generated S-module. By Theorem 1.1, S_P is regular if and only if $(F^e_*S)_P$ is a finite free S_P-module. Since F^e_*S is finitely generated, it is easy to see that if $(F^e_*S)_P$ is finite free over S_P, then there exists $f \notin P$ such that $(F^e_*S)_f$ is finite free over S_f. Thus the regular locus of S is open in $\operatorname{Spec}(S)$ and we have completed the proof that F-finite implies excellent.

It remains to show $R_P \to \widehat{R_P}$ has geometrically regular fibers. That is, for any $Q \subseteq P$ and any finite field extension $\kappa(Q)'$ of $\kappa(Q)$, $\kappa(Q)' \otimes_{R_P} \widehat{R_P}$ is regular. This follows immediately from Lemma 12.2 applied to R_P/QR_P.

We now prove that if $(R, \mathfrak{m}, k)$ is an excellent local ring with k an F-finite field, then R is F-finite. By Exercise 3 we may assume R is reduced. Let K be the total quotient ring of R, which is a product of fields $K = K_1 \times K_2 \times \cdots \times K_s$. Since R is excellent, each $K_i \otimes_R \widehat{R}$ is regular and thus $K \otimes_R \widehat{R}$ is regular and hence reduced. But since $\widehat{R} \hookrightarrow K \otimes_R \widehat{R}$, we see that $\widehat{R}$ is reduced. By Cohen's structure theorem, $\widehat{R}$ is a homomorphic image of $k[[x_1, \ldots, x_n]]$ and so by Exercise 4, $\widehat{R}$ is F-finite since k is F-finite. We next claim the following.

Claim 12.4. *$F^e_*K \otimes_R \widehat{R}$ is finitely generated over $K \otimes_R \widehat{R}$ for all $e > 0$.*

Proof. For any $L = L_1 \times L_2 \times \cdots \times L_s$ where L_i is a finite field extension of K_i, since $R \to \widehat{R}$ has geometrically regular fibers, we know that $L \otimes_R \widehat{R}$ is regular. Thus by Theorem 1.1, $L \otimes_R \widehat{R} \to F^e_*(L \otimes_R \widehat{R})$ is faithfully flat. By considering all finite extensions L_i between K_i and $F^e_*K_i$ and taking a direct limit, we find that $F^e_*K \otimes_R \widehat{R} \to F^e_*(F^e_*K \otimes_R \widehat{R})$ is faithfully flat. But this map factors as

$$F^e_*K \otimes_R \widehat{R} \to F^e_*(K \otimes_R \widehat{R}) \to F^e_*(F^e_*K \otimes_R \widehat{R})$$

and obviously, $K \otimes_R \widehat{R} \to F^e_*K \otimes_R \widehat{R}$ is faithfully flat as K is a product of field (or one can use Theorem 1.1 since K is regular). Therefore we find that $F^e_*K \otimes_R \widehat{R} \to F^e_*(K \otimes_R \widehat{R})$ is faithfully flat, in particular it is injective. But since $\widehat{R}$ is F-finite, $K \otimes_R \widehat{R}$ is F-finite since it is a localization of $\widehat{R}$, we know that $F^e_*(K \otimes_R \widehat{R})$ is finitely generated over $K \otimes_R \widehat{R}$. Therefore $F^e_*K \otimes_R \widehat{R}$ is finitely generated over $K \otimes_R \widehat{R}$ as desired. ☐

Finally, since $\widehat{R}$ is faithfully flat over R, by Claim 12.4 we see that F^e_*K is finitely generated over K. Now we apply Lemma 12.3, we know that the integral closure of R inside F^e_*K is module-finite over R. But clearly F^e_*R is contained inside this integral closure, hence F^e_*R is module-finite over R, that is, R is F-finite. $\square$

Our second goal in this section is to explain the following result of Gabber.

Theorem 12.5 ([Gab04])**.** *If R is F-finite, then R is a homomorphic image of an F-finite regular ring. As a consequence, F-finite rings admit canonical modules.*

Proof. Let R^p be the subring of R consisting of p-th powers of elements of R. Note that R is F-finite is equivalent to saying that R is module-finite over R^p. Let $a_1, \ldots, a_s$ be generators of R as a module over R^p. Set

$$R_n := \frac{R[z_1, \ldots, z_s]}{(z_1^{p^n} - a_1, \ldots, z_s^{p^n} - a_s)}.$$

Consider the inverse system:

$$\cdots \twoheadrightarrow R_n \twoheadrightarrow R_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow R_0 = R$$

where each $R_n \to R_{n-1}$ is the Frobenius map on R and the identity map on $z_1, \ldots, z_s$, it is easy to see that the map is surjective for all n. Set $R_\infty := \varprojlim_n R_n$ and we will show R_∞ is a (Noetherian) regular ring. By Theorem 1.1, it is enough to show:

(1) R_∞ is Noetherian
(2) R_∞ is reduced
(3) R_∞ is generated over R_∞^p freely by $\{z_{1\bullet}^{i_1} \cdots z_{s\bullet}^{i_s}\}_{0 \le i_j \le p-1}$ where $z_{j\bullet}$ denotes the constant sequence $(\cdots \to z_j \to z_j \to \cdots \to z_j) \in R_\infty$.

We first prove (3). Since R is generated by $a_1, \ldots, a_s$ over R^p. By the definition of R_n, it is easy to check that R_n is generated *freely* over R_n^p by $\{z_1^{i_1} \cdots z_s^{i_s}\}_{0 \le i_j \le p-1}$ for any $n \ge 1$.[11] Thus the conclusion follows as we pass to the inverse limit.

We next prove (2). To ease the presentation we will use the following notations for the rest of the argument: $\underline{i}$ denotes an s-tuple $i_1, \ldots, i_s$, $\lambda \underline{i}$ means $\lambda i_1, \ldots, \lambda i_s$, $\underline{i} \equiv \underline{j}$ means $i_k \equiv j_k$ for each k, and $\alpha \le \underline{i} \le \beta$ means $\alpha \le i_k \le \beta$ for each k. Moreover, we set $\underline{z}^{\underline{i}} := z_1^{i_1} \cdots z_s^{i_s}$ and $\underline{a}^{\underline{i}} := a_1^{i_1} \cdots a_s^{i_s}$.

Claim 12.6. $\operatorname{Ker}(R_n \to R_{n-1}) = \{x \in R_n | x^p = 0\}$.

Proof. Suppose $x = \sum_{0 \le \underline{i} < p^n} a_{\underline{i}} \underline{z}^{\underline{i}} \in R_n$ where $a_{\underline{i}} \in R$. Then $x^p = \sum_{0 \le \underline{i} < p^n} a_{\underline{i}}^p \underline{z}^{p\underline{i}}$. Write

$$\sum_{0 \le \underline{i} < p^n} a_{\underline{i}}^p \underline{z}^{p\underline{i}} = \sum_{0 \le \underline{j} < p^{n-1}} \sum_{\substack{\underline{i} \equiv \underline{j} \\ \bmod p^{n-1}}} a_{\underline{i}}^p \underline{z}^{p(\underline{i}-\underline{j})} \underline{z}^{p\underline{j}} = \sum_{0 \le \underline{j} < p^{n-1}} \Big(\sum_{\substack{\underline{i} \equiv \underline{j} \\ \bmod p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \Big) \underline{z}^{p\underline{j}},$$

[11] We caution the reader that one cannot invoke Theorem 1.1 to say that R_n is regular, this is because R_n is not reduced so we cannot identify $R_n^p \to R_n$ with $R_n \to F_* R_n$.

we see that
$$x^p = 0 \text{ if and only if for each } \underline{j}, \quad \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} = 0.$$

But this is equivalent to saying that
$$\sum_{0\leq \underline{j}<p^{n-1}} \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \underline{z}^{\underline{j}} = 0 \text{ in } R_{n-1}$$

since R_{n-1} is finite free over R with basis $\{\underline{z}^{\underline{j}}\}_{0\leq\underline{j}<p^{n-1}}$. But note that in R_{n-1}, we have
$$\sum_{0\leq \underline{j}<p^{n-1}} \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^{n-1}}} a_{\underline{i}}^p \underline{a}^{\frac{1}{p^{n-1}}(\underline{i}-\underline{j})} \underline{z}^{\underline{j}} = \sum_{0\leq\underline{i}<p^n} a_{\underline{i}}^p \underline{z}^{\underline{i}},$$

which is precisely the image of x under the map $R_n \to R_{n-1}$ (by definition of this map). Therefore $x^p = 0$ if and only if $x \in \operatorname{Ker}(R_n \to R_{n-1})$. □

Claim 12.6 immediately implies that $R_\infty = \varprojlim_n R_n$ is reduced. We have completed the proof of (2).

Finally, we prove (1). This will take some work. We first let
$$K_{n+m,n} := \operatorname{Ker}(R_{n+m} \to R_n)$$
and we claim the following.

Claim 12.7. *For all $n \geq 0$ and $m \geq 1$, $K_{n+m,n} = (K_{n+m,0})^{[p^n]}$ as ideals in R_{n+m}.*

Proof. By Claim 12.6 (and an easy induction), we have that $(K_{n+m,0})^{[p^n]} \subseteq K_{n+m,n}$. Now let $r \in K_{n+m,n}$. We write
$$r = \sum_{0\leq\underline{i}<p^{n+m}} r_{\underline{i}} \underline{z}^{\underline{i}} = \sum_{0\leq\underline{j}<p^n} \Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} r_{\underline{i}} \underline{z}^{\underline{i}-\underline{j}} \Big) \underline{z}^{\underline{j}}$$

where $r_{\underline{i}} \in R$. Since $\underline{a}$ generates R over R^p, $\{\underline{a}^{\underline{k}}\}_{0\leq\underline{k}<p^n}$ generates R over R^{p^n}. Thus we can write
$$r_{\underline{i}} = \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^n} \underline{a}^{\underline{k}} = \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^n} \underline{z}^{p^{n+m}\underline{k}} \text{ in } R_{n+m},$$
where $b_{\underline{i},\underline{k}} \in R$. Thus we have
$$r = \sum_{0\leq\underline{j}<p^n} \Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}} \underline{z}^{p^m\underline{k}} \underline{z}^{\frac{1}{p^n}(\underline{i}-\underline{j})} \Big)^{p^n} \underline{z}^{\underline{j}}.$$

In order to show $r \in K_{n+m,0}^{[p^n]}$, it is enough to show that for each $\underline{j}$,

$$\sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}} \underline{z}^{p^m\underline{k}} \underline{z}^{\frac{1}{p^n}(\underline{i}-\underline{j})} \in K_{n+m,0}.$$

But its image in $R = R_0$ is (note that in R_0, $\underline{z} = \underline{a}$)

$$c_{\underline{j}} := \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^{n+m}} \underline{a}^{p^m\underline{k}} \underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})},$$

and our hypothesis $r \in K_{n+m,n}$ implies that

$$\sum_{0\leq\underline{j}<p^n} \Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} r_{\underline{i}}^{p^m} \underline{z}^{\underline{i}-\underline{j}} \Big) \underline{z}^{\underline{j}} = \sum_{0\leq\underline{j}<p^n} \Big(\sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} r_{\underline{i}}^{p^m} \underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})} \Big) \underline{z}^{\underline{j}} = 0 \text{ in } R_n.$$

Since R_n is finite free over R with basis $\{\underline{z}^{\underline{j}}\}_{0\leq\underline{j}<p^n}$, this implies that for every $\underline{j}$,

$$0 = \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} r_{\underline{i}}^{p^m} \underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})} = \sum_{\substack{\underline{i}\equiv\underline{j} \\ \bmod p^n}} \sum_{0\leq\underline{k}<p^n} b_{\underline{i},\underline{k}}^{p^{n+m}} \underline{a}^{p^m\underline{k}} \underline{a}^{\frac{1}{p^n}(\underline{i}-\underline{j})} = c_j,$$

which is exactly what we want. $\square$

At this point, we set $J_n := \operatorname{Ker}(R_\infty \to R_n)$. Note that we have

$$J := J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

We next claim the following

Claim 12.8. *For each $n \geq 0$, $J_n \subseteq \cap_{m\geq0}(J^{[p^n]} + J_m) \subseteq \cap_{m\geq0}(J^n + J_m)$.*

Proof. The second inclusion is trivial. We prove the first inclusion. Pick $x_\bullet \in J_n$, which can be thought of as a sequence

$$x_\bullet = \cdots \to x_{m+1} \to x_m \to \cdots \to x_n = 0 \to \cdots \to x_0 = 0.$$

In particular, $x_m \in K_{m,n} = K_{m,0}^{[p^n]}$ by Claim 12.7 and thus we can write $x_m = \sum r_{im} y_{im}^{p^n}$ where $r_{im} \in R_m$ and $y_{im} \in K_{m,0}$. Since the inverse system has surjective transition maps, r_{im}, y_{im} are images of $r_{i\bullet}, y_{i\bullet} \in R_\infty$ and $y_{i\bullet} \in J_0 = J$ by construction. Thus by looking at the m-th entry we find that $x_\bullet - \sum r_{i\bullet} y_{i\bullet}^{p^n} \in J_m$. Therefore $J_n \subseteq J^{[p^n]} + J_m$ as desired. $\square$

Next we set $I_n = \cap_m(J^n + J_m)$. It is clear that $J = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ and that $\{I_n\}_{n\geq1}$ is a graded family of ideals in R_∞ (i.e., $I_n I_m \subseteq I_{n+m}$).

Claim 12.9. *R_∞ is complete with respect to the topology defined by $\{I_n\}_{n\geq1}$.*

Proof. By Claim 12.8, $J_n \subseteq I_n$ for each $n \geq 1$. Consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & I_{n+1}/J_{n+1} \arrow[r] \arrow[d, hook] & I_n/J_n \arrow[r] \arrow[d, hook] & I_{n-1}/J_{n-1} \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r, two heads] & R_{n+1} = R_\infty/J_{n+1} \arrow[r, two heads] \arrow[d] & R_n = R_\infty/J_n \arrow[r, two heads] \arrow[d] & R_{n-1} = R_\infty/J_{n-1} \arrow[r, two heads] \arrow[d] & \cdots \\
\cdots \arrow[r, two heads] & R_\infty/I_{n+1} \arrow[r, two heads] & R_\infty/I_n \arrow[r, two heads] & R_\infty/I_{n-1} \arrow[r, two heads] & \cdots .
\end{tikzcd}$$

The inverse limit of the second row is R_∞ by definition. Thus to prove the claim it is enough to show that the first row is a null system, that is, for each $n \geq 1$ there exists $k \gg 0$ such that $I_k \subseteq J_n$.

For each $y_\bullet = (\cdots \to y_{n+1} \to y_n \to \cdots \to 0) \in I_1 = J$, we have $y_{n+1} \in K_{n+1,0}$ for all $n \geq 0$. Since $K_{n+1,n} = (K_{n,0})^{[p^n]}$ by Claim 12.7, we can pick $k \gg 0$ (depends on n) such that $(K_{n,0})^k \subseteq (K_{n,0})^{[p^n]} = K_{n+1,n}$ (this is possible since $K_{n+1,0}$ is finitely generated, as it is an ideal in a Noetherian ring R_{n+1}). Therefore for each $x_\bullet \in I_k = \cap_{m\geq 0}(J^k + J_m) \subseteq J^k + J_{n+1}$, the $(n+1)$-th entry x_{n+1} is contained in $(K_{n+1,0})^k$ as this holds for all elements in J^k and elements in J_{n+1} have $(n+1)$-th entry 0. Thus $x_{n+1} \in K_{n+1,n}$ by our choice of k and hence $x_n = 0$, which implies $x_\bullet \in J_n$. So $I_k \subseteq J_n$ as desired. □

Finally, we claim the following.

Claim 12.10. *The associated graded ring* $gr_{I_\bullet} R_\infty := (R_\infty/I_1) \oplus (I_1/I_2) \oplus \cdots$ *is Noetherian.*

Proof. Since $R_\infty/I_1 = R_\infty/J \cong R$ is Noetherian and I_1/I_2 is finitely generated (it can be viewed as an ideal in R_∞/I_2, which is Noetherian since it is a quotient of $R_\infty/J_2 \cong R_2$). Thus to show $gr_{I_\bullet} R$ is Noetherian, it is enough to show that $I_n/I_{n+1} = (I_1/I_2)^n$, that is, $I_n \subseteq I_1^n + I_{n+1}$ for all $n \geq 1$ (the other inclusion is clear). Since $I_{n+1} \supseteq J_{n+1}$ by Claim 12.8, it is enough to show $I_n \subseteq I_1^n + I_{n+1}$ modulo J_{n+1}. But recall that $I_n = \cap_{m\geq 0}(J^n + I_m)$, thus after modulo J_{n+1}, I_n is generated by $J^n = I_1^n$. □

Now the conclusion of (1) that R_∞ is Noetherian follows from Claim 12.9 and Claim 12.10. For any ideal $I \subseteq R_\infty$, its image in $gr_{I_\bullet} R_\infty$ is finitely generated, say by $\overline{f}_1, \ldots, \overline{f}_t$. We claim that I is generated by $f_1, \ldots, f_t$: given any $x \in I$, suppose $x \in I_n - I_{n+1}$, then we can find $x_1, \ldots, x_t$ such that $x' := x - (f_1 x_1 + \cdots f_t x_t) \in I_{n+1} \cap I$, now pick $n' > n$ such that $x' \in I_{n'} - I_{n'+1}$, we can find $x'_1, \ldots, x'_t$ such that $x'' := x' - (f_1 x'_1 + \cdots f_t x'_t) \in I_{n'+1} \cap I$, continuing this process and using R_∞ is complete with respect to $\{I_n\}_{n\geq 1}$, it is easy to check that eventually we can write $x = f_1 y_1 + \cdots + f_n y_n$, so I is generated by $f_1, \ldots, f_n$.

We have completed the proof that R is a homomorphic image of an F-finite regular ring, call it S. By Exercise 40, we have $\dim(R) = d < \infty$ and $\dim(S) = n < \infty$. Therefore $\operatorname{Ext}_S^{n-d}(R, S)$ is a canonical module of R. $\square$

It is worth pointing out that not all excellent local rings admit canonical modules, for example see [Nis12, Example 6.1].

Exercise 39. Let R be an F-finite ring and let $P \subseteq Q$ be two prime ideals of R. Show that $\operatorname{ht}(P) + \log_p \operatorname{rank}_{\kappa(P)}(F_*\kappa(P)) = \operatorname{ht}(Q) + \log_p \operatorname{rank}_{\kappa(Q)}(F_*\kappa(Q))$.

Exercise 40. Let R be a (not necessarily local) F-finite ring. Prove that $\dim(R) < \infty$.

It is natural to ask whether the property of being F-finite is a local property. It turns out that this is not always true! Counter-examples are constructed (quite recently) in [DI20]. On the other hand, we have the following.

Exercise 41. Let R be an excellent ring. Prove that if R_P is F-finite for all $P \in \operatorname{Spec}(R)$, then R is F-finite.

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