D-Modules, Unit F-Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

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1 Questions

Question 1.1. What is the point of D-modules? Why do people care about them? What sort of questions do they answer? What insights do they provide?

Question 1.2. Given a scheme X over a field k, we define the sheaf $\mathcal{E}nd_k(\mathcal{O}_X) = \mathcal{H}om_k(\mathcal{O}_X, \mathcal{O}_X)$ to be the sheaf of k-linear endomorphisms on $\mathcal{O}_X$ as in [Gie75, Definition 0.3]. What I think this means is this: we have a structure morphism $X \longrightarrow \operatorname{Spec} k$ and we can therefore think of $\mathcal{O}_X$ as a $\mathcal{O}_{\operatorname{Spec} k}$-module via the map $f^\sharp : \mathcal{O}_{\operatorname{Spec} k} \longrightarrow f_*\mathcal{O}_X$. However, this is rather stupid, since this means $\operatorname{End}_k(\mathcal{O}_X)$ is a $\mathcal{O}_{\operatorname{Spec} k}$-module, which should mean it takes open sets from $\operatorname{Spec} k$. In the book [HTT08, Example A.4.2.] however, this sheaf takes open sets from X. What's going on here?

Question 1.3. How can one "naturally" make A_m a subalgebra of A_n when $m \leq n$? It seems like there are n many subalgebras of A_n isomorphic to A_{n-1}, for example.

2 Some Non-Commutative Algebra

$\mathcal{D}$-modules requires non-commutative algebra. Necessary facts are found here.

2.1 Filtered rings and modules

This subsection follows Ginzburg's notes quite closely, see [BIBTEX SETUP, GINZBURG D-MODULES Page 3].

Definition 2.1 (*Filtered Ring*)**.** Let A be an associative ring with unit. We call A a *filtered ring* if an increasing filtration $\ldots \subset A_i \subset A_{i+1} \subset \ldots$ by additive subgroups is given such that

(i) $A_i A_j \subset A_{ij}$

(ii) $1 \in A_0$,

(iii) $\bigcup A_i = A$, i.e. the filtration is *exhausting*.

Typically, either (a) $\mathbb{N}$ or (b) $\mathbb{Z}$ is chosen for the index set. In the former case A is said to be *positively filtered*. Note that (a) can be viewed as a special case of (b) by setting $A_{-1} = 0$. In the latter case we will consider the topology induced by the filtration by taking $\{A_i\}_{i\in\mathbb{Z}}$ to be the base of open sets, and we then impose two additional conditions:

(iv) $\bigcap A_i = \{0\}$, i.e. the topology defined by $\{A_i\}$ is *separating*

1. A is complete with respect to this topology.

Finally, we denote by $\operatorname{gr} A$ the associated graded ring $\bigoplus A_i/A_{i-1}$.

3 Differential Operators and D-Modules

Definition 3.1 (Quasi-coherent #1)**.** Fix X a scheme over k, $\mathcal{O}_X$ the structure sheaf, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We call $\mathcal{F}$ a *quasi-coherent* sheaf of $\mathcal{O}_X$-modules (or simply an $\mathcal{O}_X$-modules) if it satisfies the condition

$$\text{If } U \subseteq X \text{ an open affine, } f \in \mathcal{O}_X(U), \text{ and } U_f = \{u \in U \mid f(u) \neq 0\},$$

then $\mathcal{F}(U_f) = \mathcal{F}(U)_f = \mathcal{O}_X(U_f) \otimes_{\mathcal{O}_X(U)} \mathcal{F}$.

Definition 3.2 (Quasi-coherent #2)**.** Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if X can be covered by affine opens $U_i = \operatorname{Spec} A_i$ such that for each i there exists an A_i module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$. We say $\mathcal{F}_i$ is coherent if each M_i can be taken to be finitely generated.

Remark 3.3. If A is a ring and M an A-module, the sheaf associated to M is denoted by $\tilde{M}$ and is formed as follows. For each $\mathfrak{p} \in \operatorname{Spec} A$, $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$ is the localization with respect to $\mathfrak{p}$. Given an open set $U \subseteq \operatorname{Spec} A$, define

$$\tilde{M}(U) = \left\{ s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \;\middle|\; s(\mathfrak{p}) \in M_{\mathfrak{p}}, \text{ and locally } s = \frac{m}{f}, m \in M, f \in A \right\}.$$

More verbosely, this last condition means that for each $\mathfrak{p} \in U$ there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{f} \in M_{\mathfrak{q}}$.

Alternatively, one may define

$$\tilde{M}(U_f) = M_f,$$

and then

$$\tilde{M}(U) = \varinjlim_{U_f \subseteq U} \tilde{M}(U_f).$$

Note that U_f is implied to be a distinguished open in one of the U_i, so really we need to take the limit above over all U_f in all U_i which intersect U nontrivially. This is a non-issue if U is affine.

Lemma 3.4. The following are equivalent conditions for $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules:

(a) $\mathcal{F}$ is the direct limit of its coherent subschemes

(b) For any Zariski open affine subset $U \subseteq X$ and any $f \in \mathcal{O}(U)$ one has $\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$.

A *quasi-coherent* sheaf is then one which satisfies these conditions.

Lemma 3.5 (Noether Normalization Lemma)**.** Let k be a field, A a finitely generated k-algebra. Then there exists algebraically independent elements $y_1, \ldots, y_d$ in A for some positive d such that A is finitely generated as a module over $k[y_1, \ldots, y_n]$.

Remark 3.6. The Noether normalization lemma provides a way to define differential operators using a manifold-esque coordinate approach. I prefer the following coordinate-free approach provided by Gröthendieck, however.

Definition 3.7 (Differential Operators)**.** Let A be a commutative ring. For any pair of A-modules M, N we define the module $\mathcal{D}\!\mathit{iff}^k_A(M, N)$ inductively as follows:

(i) $\mathcal{Diff}_A^0(M,N) = \mathrm{Hom}_A(M,N)$

(ii) $\mathcal{Diff}_A^{k+1}(M,N) = \left\{ \text{ additive maps } u : M \longrightarrow N \;\middle|\; \forall a \in A, (au - ua) \in \mathcal{Diff}_A^k(M,N) \right\}$

It follows from the definition that $\mathcal{Diff}_A^k(M,N) \subset \mathcal{Diff}_A^{k+1}(M,N)$. We define

$$\mathcal{Diff}_A(M,N) := \bigcup_k \mathcal{Diff}_A^k(M,N).$$

In the case that $M+N$, we write $\mathcal{Diff}_A(M)$ and note that it is a filtered almost commutative ring.

The case in which we will be most interested is when $M = N = A$, i.e. when we consider A to be a ring over itself. Let's repeat the above construction for this case.

Definition 3.8. Let A be a commutative K-algebra for K a characteristic 0 field. Let $D \in \mathrm{End}(R)$. We define the **order** of D inductively.

- $\mathrm{ord}(D) = 0$ if $[a,D] = -[D,a] = 0$ for all $a \in A$.
- $\mathrm{ord}(D) = n \in \mathbb{Z}_{\geq 0}$ if $\mathrm{ord}(D) \neq k$ for all $k < n$ and if $\mathrm{ord}([a,D]) = k_a$ for some $k_a < n$ for each $a \in A$.

The set $D^n(R)$ is defined to be the K-vector space of of all operators of order $\leq n$.

Definition 3.9. A **derivation** $D \in \mathrm{End}(R)$ is an operator which satisfies the Leibniz rule

$$D(ab) = aD(b) + D(a)b$$

for every $a,b \in A$. The set of all derivations $\mathrm{Der}_K(A) \subseteq \mathrm{End}_K(A)$ is a K-vector space and a left A-module under the action $(a \cdot D)(b) = a(D(B))$.

All derivations are order 1 operators. As one might hope, they're actually *all* order 1 operators.

Lemma 3.10. $D^1(A) = \mathrm{Der}_K(R) + R$. (See proof in *A Primer on D-modules* page 21.)

We can now define the ring of differential operators on A.

Definition 3.11. Let A be a K-algebra with K a characteristic zero field. The set of all finite order operators on A forms a noncommutative ring with pointwise addition and composition as multiplication. We denote this ring by $D(A)$ and we call it the **ring of differential operators:**

$$D(A) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} D^n(A) \subseteq \mathrm{End}_K(A).$$

It's obvious that the addition of two finite order operators yields a differential operator with order equal to the maximum order of the two summands, it's *not* obvious that the composition of two finite order operators yields a finite order operator. We therefore require the following proposition for this definition to work:

Proposition 3.12. If $D \in D^n(A)$ and $D' \in D^m(A)$ then $D \circ D', D' \circ D \in D^{n+m}(A)$.

3.1 First examples of Differential Operators

Example 3.13. Let K be a field of characteristic zero and recall that $K[x_1,...,x_n]$ is an infinite dimensional K vector space. We define $\hat{x}_i, \partial_i \in \text{End}_K(K[x_1,...,x_n])$ by $\hat{x}_i f \mapsto x_i \cdot f$ and $\partial_i f \mapsto \frac{\partial f}{\partial x_i}$. One can then check that $[\partial_j, \hat{x}_j] = \partial_i \hat{x}_j - \hat{x}_j \partial_i = \delta_{ij}\,\text{id}$ where id is the identity operator on $K[x_1,...,x_n]$ and δ_{ij} is the Kronecker delta. In other words,

$$[\partial_i, \hat{x}_i](f) = f \quad \text{and} \quad [\partial_i, \hat{x}_j](f) = 0$$

when $i \neq j$. This is quite easy to check for an arbitrary polynomial but is nonetheless quite magical:

$$\begin{aligned} \partial_x \left(x \cdot (3x^2+2y) \right) &= 9x^2+2y, \\ x \cdot \left(\partial_x (3x^2+2y) \right) &= 6x^2, \\ (\partial_x \cdot \hat{x} - \hat{x} \cdot \partial_x)(3x^2+2y) &= 3x^2+2y, \end{aligned}$$

but

$$\partial_x \left(y \cdot (3x^2+2y) \right) - y \cdot \left(\partial_x (3x^2+2y) \right) = 6xy - 6xy = 0.$$

Definition 3.14. The nth Weyl algebra of K is the $2n$-dimensional K-subalgebra of $\text{End}_K(K[x_1,...,x_n])$ generated by $\hat{x_1},...,\hat{x_n}, \partial_1,...,\partial_n$, and is denoted by $A_n(K)$ or A_n when the field is known. We let $A_0(K) = K$.

Note also that for $m \leq n$, we can make A_m a subalgebra of A_n in a "natural way".

3.2 D-Modules

Definition 3.15. A $\mathcal{D}$-module is a sheaf over the sheaf $\mathcal{D}_X$ of regular differential operators over a variety (scheme, manifold, analytic complex manifold) which is quasi-coherent as an $\mathcal{O}_X$-module.

4 Berstein-Sato Polynomial

Theorem 4.1 (Björk, Kashiwara)**.** *Let X be a smooth variety over the complex numbers and let f be a non-invertible regular function on X (i.e. a locally rational function whose numerator is non-invertible). There exists a polynomial $b(s) \in \mathbb{C}[s]$ and a polynomial $P(s) \in \mathcal{D}_X[s]$ whose coefficients are differential operators on X, such that the relation*

$$P(s) f^{s+1} = b(s) \cdot f^s$$

holds formally in the $\mathcal{D}_X$-module $\mathcal{O}_X[\frac{1}{f}, s] \cdot f^s$. Here, f^{s+1} stands for $f \cdot f^s$.

References

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