Lecture 7

1 §Extensions of complete fields (cont.)

Proof of Theorem 6.1 cont:

(Recall $(K, |\cdot|)$ complete disc valued. L/K finite de

$$|x|_L := |N_{L/K}(x)|^{\frac{1}{n}}$$

Set $\mathcal{O}_L = \{ y \in L \mid |y|_L \le 1 \}$

Claim: $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ inside L.

Proof of claim: Let $0 \ne y \in \mathcal{O}_L$ and let

$$f(X) = X^d + a_{d-1}X^{d-1} + \dots + a_0 \in K[X]$$

be minimal polynomial of y.

By property of $N_{L/K}$, $\exists\, m \geq 1$ s.t. $N_{L/K}(y) = \pm a_0^m$

By Corollary 4.5, we have

$$|a_i| \le \max(|\underbrace{N_{L/K}(y)^{1/m}}_{a_0}|, 1) = 1,$$

since $|N_{L/K}(y)|^{\frac{1}{n}} \le 1$.

Thus $a_i \in \mathcal{O}_K \ \forall i \Rightarrow f \in \mathcal{O}_K[X]$.

$\Rightarrow y$ integral over $\mathcal{O}_K$.

2 Conversely let $y \in L$ be integral over K.

Let $f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0 \in K[x]$ min poly of y and

$0 \neq g(x) \in \mathcal{O}_K[X]$ monic s.t. $g(y) = 0$.

Then $f \mid g$, and hence every root of f is a root of g.

$\Rightarrow$ every root of f in $\bar{K}$ is integral over $\mathcal{O}_K$

$\Rightarrow$ a_i integral over $\mathcal{O}_K$, $i = 0, \ldots, d-1$

$\Rightarrow$ $a_i \in \mathcal{O}_K$ (Lemma 6.8)

Again by Property of $N_{L/K}$, we have

$$N_{L/K}(y) = \pm a_0^m \in \mathcal{O}_K$$

$\Rightarrow |N_{L/K}(y)| \leq 1$.

Thus $\mathcal{O}_K^{\text{int}(L)} = \mathcal{O}_L$ and proves the claim.

Last time $|x|_L := N_{L/K}(x)$ defines an abs. value on L.

Since $N_{L/K}(x) = x^n$ for $x \in K$, $|x|_L$ extends $|\cdot|$ on K

If $|\cdot|_L'$ is another abs. value on L extending $|\cdot|$, then note that $|\cdot|_L, |\cdot|_L'$ are norms on L.

Theorem 6.5 $\Rightarrow |\cdot|_L', |\cdot|_L$ induce same topology on L

$\Rightarrow |\cdot|_L' = |\cdot|_L^c$ some $c > 0$ (Proposition 1.3)

Since $|\cdot|_L'$ extends $|\cdot|$, we have $c = 1$.

(ii) Since $|\cdot|_L$ defines a norm on L, Theorem 6.5 implies L is complete w.r.t. $|\cdot|_L$. $\square$

Corollary 6.9: Let $(K, |\cdot|)$ be a complete non-arch. discretely valued field and L/K a finite extension. Then

(i) L is discretely valued w.r.t. $|\cdot|_L$

(ii) $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.

Proof: (i) v valution on K, v_L valuation on L s.t. v_L extends v. $n = [L:K]$

$y \in L^\times$ $\quad |y|_L = |N_{L/K}(y)|^{1/n}$

$$\Rightarrow v_L(y) = \frac{1}{n} v(N_{L/K}(y))$$

$$\Rightarrow v_L(L^\times) \subseteq \frac{1}{n} v(K^\times)$$

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$$\Rightarrow v_L \text{ is discrete.}$$

(ii) Proved earlier. $\square$

Corollary 6.10: Let $(K, |\cdot|)$ complete non-arch. discretely valued field and $\bar{K}/K$ an algebraic closure of K. Then $|\cdot|$ extends to a unique abs. value $|\cdot|_{\bar{K}}$ on $\bar{K}$.

Proof: Let $x \in \bar{K}$, then $x \in L$ some L/K finite. Define $|x|_{\bar{K}} = |x|_L$. Well-defined, i.e.

independent of L by the uniqueness in Theorem 6.1. The axioms for $| \ |_{\bar{K}}$ to be an abs. value can be checked over finite extensions. Uniqueness: clear. $\square$

Remark: $|\cdot|_{\bar{K}}$ on $\bar{K}$ is never discrete.

Eg. $K = \mathbb{Q}_p$, $\sqrt[n]{p} \in \bar{\mathbb{Q}}_p$ $\forall n \in \mathbb{N}_{>0}$, $v_p(\sqrt[n]{p}) = \frac{1}{n} v_p(p)$
$= \frac{1}{n}$.

$\bar{\mathbb{Q}}_p$ is not complete w.r.t. $|\cdot|_{\bar{\mathbb{Q}}_p}$.

Ex. Sheet 2: $\mathbb{C}_p :=$ completion of $\bar{\mathbb{Q}}_p$ w.r.t. $| \ |_{\bar{\mathbb{Q}}_p}$, then $\mathbb{C}_p$ is algebraically closed

Proposition 6.11: Let L/K finite extension of complete discretely valued fields. Assume the residue extension k_L/k is separable and $\mathcal{O}_K$ is compact.

Then $\exists \alpha \in \mathcal{O}_L$ s.t. $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

Proof: (Later we'll see that k_L/k is finite).

Then k_L/k separable $\Rightarrow \exists \bar{\alpha} \in k_L$ s.t. $k(\bar{\alpha}) = k_L$.

Let $\alpha \in \mathcal{O}_L$ a lift of $\bar{\alpha}$, and $g(x) \in \mathcal{O}_K[x]$ a monic lift of the min. polynomial of $\bar{\alpha}$.

Fix $\pi_L \in \mathcal{O}_L$ a uniformizer.

Then $\bar{g}(X) \in k[X]$ irreducible and separable $\Rightarrow g(\alpha) \equiv 0 \bmod \pi_L$ and $g'(\alpha) \not\equiv 0 \bmod \pi_L$.

But $g(\alpha + \pi_L) \equiv g(\alpha) + \pi_L g'(\alpha) \mod \pi_L^2$.

Let v_L normalized valuation for L,

so that $v_L(\pi_L g'(\alpha)) = 1$

It follows that either

$$v_L(g(\alpha)) = 1 \quad \text{or} \quad v_L(g(\alpha + \pi_L)) = 1.$$

Upon possibly replacing α by $\alpha + \pi_L$, WMA $v_L(g(\alpha)) = 1$.

Set $\beta := g(\alpha) \in \mathcal{O}_K[\alpha]$

$\mathcal{O}_K[\alpha] \subseteq L$ is image of continuous map.

$\mathcal{O}_K^n \to L : (x_0, \ldots, x_{n-1}) \mapsto \sum_{i=0}^{n-1} x_i \alpha^i$.

where $n := [K(\alpha) : K]$.

$\mathcal{O}_K$ compact $\Rightarrow \mathcal{O}_K[\alpha] \subseteq L$ compact, hence closed.

Since $k_L = k(\bar{\alpha})$, $\mathcal{O}_K[\alpha]$ contains coset reps for

$k_L = \mathcal{O}_L / \pi_L \mathcal{O}_L = \frac{\mathcal{O}_L}{\beta \mathcal{O}_L}$.

Let $y \in \mathcal{O}_L$. Prop. 3.5. $\Rightarrow y = \sum_{i=0}^{\infty} \lambda_i \beta^i$,

$\lambda_i \in \mathcal{O}_K[\alpha]$.

Then $y_n := \sum_{i=0}^{n} \lambda_i \beta^i \in \mathcal{O}_K[\alpha]$

$\Rightarrow y \in \mathcal{O}_K[\alpha]$ since $\mathcal{O}_K[\alpha]$ closed $\square$