Chapter 11

Contact problems

Keynote Questions

(a) Given a general quintic surface $S \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ meet S in only one point? (Answer on page 391.)
(b) If $\{C_t = V(t_0F + t_1G + t_2H) \subset \mathbb{P}^2\}$ is a general net of cubic plane curves, how many of the curves C_t will have cusps? (Answer on page 416.)
(c) If $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}$ is a general pencil of quartic plane curves, how many of the curves C_t will have hyperflexes? (Answer on page 405.)
(d) If $\{C_t\}$ is again a general pencil of quartic plane curves, what are the degree and genus of the curve traced out by flexes of members of the pencil? (Answer in Section 11.3.2.)

Problems such as these, dealing with orders of contact of varieties with linear spaces, are known as *contact problems*. Their solution can often be reduced to the computation of the Chern classes of associated bundles. The most important of the bundles involved is a relative version of the bundle of principal parts introduced in Chapter 7 and described by Theorem 7.2. We will begin with an illustration showing how these arise.

One point of terminology: We define the *order of contact* of a curve C on a smooth variety X with a Cartier divisor $D \subset X$ at $p \in C$ to be the length of the component of the scheme of intersection $C \cap D$ supported at p, or (equivalently) if $\nu : \widetilde{C} \to C$ is the normalization, the sum of the orders of vanishing of the defining equation of D at points of $\widetilde{C}$ lying over p. If p is an isolated point of $C \cap D$, this is the same as the intersection multiplicity $m_p(C \cdot D)$, and we will use this to denote the order of contact; however, we will also adopt the convention that if $C \subset D$ then the order of contact is ∞, so that the condition $m_p(C \cdot D) \geq m$ is a closed condition on C, D and p.

Finally, we reiterate our standing hypothesis that our ground field has characteristic 0. As with most questions involving derivatives, the content of this chapter is much more complicated in characteristic p, and many of the results derived here are false in that setting.

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

11.1 Lines meeting a surface to high order

Consider a general quintic surface $S \subset \mathbb{P}^3$. A general line meets S in five points; to require them all to coincide is four conditions, and there is a four-dimensional family of lines in $\mathbb{P}^3$. Thus we would "expect" there to be only finitely many lines meeting S in just one point. On this basis we would expect, more generally, that for a general surface $S \subset \mathbb{P}^3$ of any degree $d \geq 5$ there will be a finite number of lines having a point of contact of order 5 with S.

As we shall show, this expectation is fulfilled, and we can compute the number. To verify the dimension statement, we introduce the flag variety

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\},$$

which we think of as the universal line over $\mathbb{G}(1,3)$; we can also realize Φ as the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(1,3)$. Next, we fix $d \geq 4$ and look at pairs consisting of a point $(L, p) \in \Phi$ and a surface $S \subset \mathbb{P}^3$ of degree d such that the line L has contact of order at least 5 with S at p (or is contained in S):

$$\Gamma = \{(L, p, S) \in \Phi \times \mathbb{P}^N \mid m_p(S \cdot L) \geq 5\},$$

where $\mathbb{P}^N$ is the space of surfaces of degree d in $\mathbb{P}^3$.

Assuming $d \geq 4$, the fiber of Γ over any point $(L, p) \in \Phi$ is a linear subspace of codimension 5 in the space $\mathbb{P}^N$ of surfaces of degree d. Since Φ is irreducible of dimension 5, it follows that Γ is irreducible of dimension N, and hence that the fiber of Γ over a general point $[S] \in \mathbb{P}^N$ is finite. Note that this includes the possibility that the fiber over a general point is empty, as in fact will be the case when $d = 4$: any line with a point of contact of order 5 with a quartic surface S must lie in S, but, as we saw in Chapter 6, a general quartic surface contains no lines. In the case $d = 4$, correspondingly, Γ projects with one-dimensional fibers to the hypersurface $\Sigma \subset \mathbb{P}^{34}$ of quartics that do contain a line. By contrast, we will see (as a consequence of Theorem 11.1) that if $d \geq 5$ then the projection $\Gamma \to \mathbb{P}^N$ is generically finite and surjective.

To linearize the problem, we consider for each pair $(L, p) \in \Phi$ the five-dimensional vector space

$$E_{(L,p)} = \frac{\{\text{germs of sections of } \mathcal{O}_L(d) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 5 \text{ at } p\}}.$$

To say that $m_p(S \cdot L) \geq 5$ means exactly that the defining equation F of S is in the kernel of the map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to E_{(L,p)}$$

given by restriction of F to a neighborhood of p in L.

To compute the number of lines with five-fold contact, we will define a vector bundle $\mathcal{E}$ on Φ whose fiber at a point $(L, p) \in \Phi$ is the vector space $E_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathfrak{m}_p^5(d))$ defined above, so that a polynomial F of degree d on $\mathbb{P}^3$ will give a global section σ_F of $\mathcal{E}$ by restriction in turn to each pointed line (L, p). The zeros of the section σ_F will then be the points $(L, p) \in \Phi$ such that $m_p(S \cdot L) \geq 5$, and — assuming that there are no unforeseen multiplicities — the answer to our enumerative problem will be the degree of the top Chern class $c_5(\mathcal{E}) \in A^5(\Phi)$. The necessary theory and computation will occupy the next two sections, and will prove:

Theorem 11.1. *If S is a general quintic surface, then there are exactly 575 lines meeting S in only one point. More generally, if $S \subset \mathbb{P}^3$ is a general surface of degree $d \geq 4$, then there are exactly $35d^3 - 200d^2 + 240d$ lines having a point of contact of order 5 with S.*

Note that this does return the correct answer 0 in the case $d = 4$! (In case $d \leq 3$, the number is meaningless, since the locus of such pairs (L, p) is positive-dimensional.)

11.1.1 Bundles of relative principal parts

The desired bundle $\mathcal{E}$ on Φ is a *bundle of relative principal parts* associated to the map

$$\pi : \Phi \to \mathbb{G}(1, 3), \quad (L, p) \mapsto [L].$$

The construction is a relative version of that of Section 7.2; the reader may wish to review that section before proceeding. The facts we need are the analogs of some of the properties spelled out in Theorem 7.2.

Suppose more generally that $\pi : Y \to X$ is a proper smooth map of schemes, and let $\mathcal{L}$ be a vector bundle on Y. Set $Z = Y \times_X Y$, the fiber product of Y with itself over X, with projection maps $\pi_1, \pi_2 : Z \to Y$, and let $\Delta \subset Z$ be the diagonal, so that we have a diagram

$$\begin{tikzcd}
\Delta \arrow[r] & Z = Y \times_X Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "\pi"] \\
& Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

The *bundle of relative m-th order principal parts* $\mathcal{P}^m_{Y/X}(\mathcal{L})$ is by definition

$$\mathcal{P}^m_{Y/X}(\mathcal{L}) = \pi_{2*}(\pi_1^*\mathcal{L} \otimes \mathcal{O}_Z/\mathcal{I}_\Delta^{m+1}).$$

Theorem 11.2. *With $\pi : Y \to X$ and projections $\pi_i : Y \times_X Y \to Y$ as above:*

(a) The sheaf $\mathcal{P}^m_{Y/X}(\mathcal{L})$ is a vector bundle on Y, and its fiber at a point $y \in Y$ is the vector space

$$\mathcal{P}^m_{Y/X}(\mathcal{L})_y = \frac{\{\text{germs of sections of } \mathcal{L}|_{F_y} \text{ at } y\}}{\{\text{germs vanishing to order} \geq m+1 \text{ at } y\}},$$

where $F_y = \pi^{-1}(\pi(y)) \subset Y$ is the fiber of π through y.

(b) We have an isomorphism $\pi^*\pi_*\mathcal{L} \cong \pi_{2*}\pi_1^*\mathcal{L}$.

(c) The quotient map $\pi_1^*\mathcal{L} \to \pi_1^*\mathcal{L} \otimes \mathcal{O}_Z/\mathcal{I}_\Delta^{m+1}$ pushes forward to give a map

$$\pi^*\pi_*\mathcal{L} \cong \pi_{2*}\pi_1^*\mathcal{L} \to \mathcal{P}^m_{Y/X}(\mathcal{L}),$$

and the image of a global section $G \in H^0(\mathcal{L})$ is the section σ_G of $\mathcal{P}^m_{Y/X}(\mathcal{L})$ whose value at a point $y \in Y$ is the restriction of G to a neighborhood of y in F_y.

(d) $\mathcal{P}^0_{Y/X}\mathcal{L} = \mathcal{L}$. For $m > 1$, the filtration of the fiber $\mathcal{P}^m_{Y/X}(\mathcal{L})_y$ by order of vanishing at y corresponds to a filtration of $\mathcal{P}^m_{Y/X}(\mathcal{L})$ by subbundles that are the kernels of surjections $\mathcal{P}^m_{Y/X}(\mathcal{L}) \to \mathcal{P}^k_{Y/X}(\mathcal{L})$ for $k < m$. The graded pieces of this filtration are identified by the exact sequences

$$0 \longrightarrow \mathcal{L} \otimes \mathrm{Sym}^m(\Omega_{Y/X}) \longrightarrow \mathcal{P}^m_{Y/X}(\mathcal{L}) \longrightarrow \mathcal{P}^{m-1}_{Y/X}(\mathcal{L}) \longrightarrow 0. \qquad (11.1)$$

The exact sequences in (11.1) allow us to express the Chern classes of the bundles $\mathcal{P}^m_{Y/X}(\mathcal{L})$ in terms of the Chern classes of $\mathcal{L}$ and those of $\Omega_{Y/X}$. We will compute the latter in the case where Y is a projectivized vector bundle over X in the next section, and this will allow us to complete the answer to Keynote Question (a).

Proof: Just as in the absolute case (Theorem 7.2), part (a) is an application of the theorem on cohomology and base change (Theorem B.5). Similarly, part (b) follows from statement (2) on page 525 in the appendix on cohomology and base change (Section B.2).

Part (c) is also a direct analog of the absolute case. For part (d), consider the diagonal $\Delta := \Delta_{Y/X} \subset Y \times_X Y$ and its ideal sheaf $\mathcal{I}_\Delta$. As in the absolute case, we have $\pi_{1*}(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) = \Omega_{Y/X}$ (see Eisenbud [1995, Theorem 16.24] for the affine case, to which the problem reduces). The sheaf $\Omega_{Y/X}$ is a vector bundle on Y because π is smooth. Since Δ is locally a complete intersection in $Y \times_X Y$, it follows (see, for example, Eisenbud [1995, Exercise 17.14]) that

$$\mathcal{I}_\Delta^m/\mathcal{I}_\Delta^{m+1} = \mathrm{Sym}^m(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2).$$

The desired exact sequences are derived from this exactly as in the absolute case. □

11.1.2 Relative tangent bundles of projective bundles

To use the sequences (11.1) to calculate the Chern class of $\mathcal{P}^m_{Y/X}(\mathcal{L})$, we need to understand the relative tangent bundle $\mathcal{T}_{Y/X}$. Recall first the definition: If $\pi : Y \to X$ is a smooth map, then the differential $d\pi : \mathcal{T}_Y \to \pi^*\mathcal{T}_X$ is surjective. Its kernel is called the *relative tangent bundle* of π, and denoted by $\mathcal{T}_\pi$ or, when there is no ambiguity, by $\mathcal{T}_{Y/X}$; its local sections are the vector fields on Y that are everywhere tangent to a fiber. Thus, for example, if $x \in X$ then the restriction $\mathcal{T}_{Y/X}|_{\pi^{-1}(x)}$ is the tangent bundle to the smooth variety $\pi^{-1}(x)$.

One special case in which we can describe the relative tangent bundle explicitly is when $\pi : Y = \mathbb{P}\mathcal{E} \to X$ is a projective bundle (as was the case in the example of Keynote Question (a), discussed in Section 11.1 above); in this section we will show how.

Recall from Section 3.2.4 that if $\xi \in \mathbb{P}V$ is a point in the projectivization $\mathbb{P}V$ of a vector space V, then we can identify the tangent space $T_\xi\mathbb{P}V$ with the vector space $\mathrm{Hom}(\xi, V/\xi)$. As we showed, these identifications fit together to give an identification of bundles

$$\mathcal{T}_{\mathbb{P}V} = \mathcal{H}om(\mathcal{S}, \mathcal{Q}),$$

where $\mathcal{S} = \mathcal{O}_{\mathbb{P}V}(-1)$ and $\mathcal{Q}$ are the universal sub- and quotient bundles.

This identification extends to families of projective spaces. Explicitly, suppose $\mathcal{E}$ is a vector bundle on X and $\mathbb{P}\mathcal{E}$ its projectivization, with universal sub- and quotient bundles $\mathcal{S} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ and $\mathcal{Q}$. At every point $(x, \xi) \in \mathbb{P}\mathcal{E}$, with $x \in X$ and $\xi \subset \mathcal{E}_x$, we have an identification $T_\xi\mathbb{P}\mathcal{E}_x = \mathrm{Hom}(\xi, \mathcal{E}_x/\xi) = \mathrm{Hom}(\mathcal{S}_{(x,\xi)}, \mathcal{Q}_{(x,\xi)})$, and these agree on overlaps of such open sets to give a global isomorphism:

Proposition 11.3. $$\mathcal{T}_{\mathbb{P}\mathcal{E}/X} \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q}).$$

Proof: This is a special case of the statement that with notation as in the proposition the relative tangent bundle of the Grassmannian bundle $G(k, \mathcal{E}) \to X$ is

$$\mathcal{T}_{\mathrm{Gr}(k,\mathcal{E})/X} = \mathcal{H}om(\mathcal{S}, \mathcal{Q}).$$

Over an open subset where $\mathcal{E}$ is trivial this is an immediate consequence of the isomorphism described in Section 3.2.4 between the tangent bundle of a Grassmannian and the bundle $\mathcal{H}om(\mathcal{S}, \mathcal{Q})$, and as in that setting the fact that these isomorphisms are independent of choices says they fit together to give the desired isomorphism $\mathcal{T}_{\mathrm{Gr}(k,\mathcal{E})/X} = \mathcal{H}om(\mathcal{S}, \mathcal{Q})$. □

Using the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

Proposition 11.3 yields an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}\mathcal{E}} \longrightarrow \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \longrightarrow \mathcal{T}_{\mathbb{P}\mathcal{E}/X} \longrightarrow 0,$$

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

the *relative Euler sequence*. Applying the formula for the Chern classes of the tensor product of a vector bundle with a line bundle (Proposition 5.17), we arrive at the following theorem:

Theorem 11.4. *If $\mathcal{E}$ is a vector bundle of rank $r+1$ on the smooth variety X, then the Chern classes of the relative tangent bundle $\mathcal{T}_{\mathbb{P}\mathcal{E}/X}$ are*

$$c_k(\mathcal{T}_{\mathbb{P}\mathcal{E}/X}) = \sum_{i=0}^{k} \binom{r+1-i}{k-i} c_i(\mathcal{E})\zeta^{k-i},$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\mathbb{P}\mathcal{E})$ and we identify $A(X)$ with its image in $A(\mathbb{P}\mathcal{E})$ via the pullback map.

11.1.3 Chern classes of contact bundles

Returning to Keynote Question (a), we again let

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\}$$

be the universal line over $\mathbb{G}(1,3)$. Via the projection $\pi : \Phi \to \mathbb{G}(1,3)$, this is the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(1,3)$. Let $\mathcal{E}$ be the bundle on Φ given by

$$\mathcal{E} = \mathcal{P}^4_{\Phi/\mathbb{G}(1,3)}(\beta^*\mathcal{O}_{\mathbb{P}^3}(d)),$$

where $\beta : \Phi \to \mathbb{P}^3$ is the projection $(L, p) \mapsto p$ on the second factor. By Theorem 11.2, this has fiber $\mathcal{E}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathfrak{m}_p^5(d))$ at a point $(L, p) \in \Phi$. Thus, counting multiplicities, the number of lines having a point of contact of order at least 5 with a general surface of degree d is the degree of the Chern class $c_5(\mathcal{E})$.

To find the degree of $c_5(\mathcal{E})$, we recall first the description of the Chow ring of Φ given in Section 9.3.1: Since

$$\Phi = \mathbb{P}\mathcal{S} \to \mathbb{G}(1,3)$$

is the projectivization of the universal subbundle on $\mathbb{G}(1,3)$, and

$$c_1(\mathcal{S}) = -\sigma_1 \quad \text{and} \quad c_2(\mathcal{S}) = \sigma_{11},$$

Theorem 9.6 yields

$$A(\Phi) = A(\mathbb{G}(1,3))[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{11}),$$

where $\zeta \in A^1(\Phi)$ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$. Recall, moreover, that the class ζ can also be realized as the pullback $\zeta = \beta^*\omega$, where $\beta : \Phi \to \mathbb{P}^3$ is the projection $(L, p) \mapsto p$ on the second factor and $\omega \in A^1(\mathbb{P}^3)$ is the hyperplane class.

The relative tangent bundle $\mathcal{T}_{\Phi/\mathbb{G}(1,3)}$ is a line bundle on Φ. By Theorem 11.4, its first Chern class is

$$c_1(\mathcal{T}_{\Phi/\mathbb{G}(1,3)}) = 2\zeta - \sigma_1.$$

By Theorem 11.2, the bundle $\mathcal{E} = \mathcal{P}^4_{\Phi/\mathbb{G}(1,3)}(\beta^*\mathcal{O}_{\mathbb{P}^3}(d))$ has a filtration with successive quotients

$$\beta^*\mathcal{O}_{\mathbb{P}^3}(d),\ \beta^*\mathcal{O}_{\mathbb{P}^3}(d)\otimes\Omega_{\Phi/\mathbb{G}(1,3)},\ldots,\ \beta^*\mathcal{O}_{\mathbb{P}^3}(d)\otimes\operatorname{Sym}^4\Omega_{\Phi/\mathbb{G}(1,3)}.$$

The bundle $\Omega_{\Phi/\mathbb{G}(1,3)}$ is dual to the relative tangent bundle $\mathcal{T}_{\Phi/\mathbb{G}(1,3)}$, so its m-th symmetric power has Chern class

$$c(\operatorname{Sym}^m\Omega_{\Phi/\mathbb{G}(1,3)}) = 1 + m(\sigma_1 - 2\zeta).$$

With the formula $c_1(\beta^*\mathcal{O}_{\mathbb{P}^3}(d)) = d\zeta$, this gives

$$c(\beta^*\mathcal{O}_{\mathbb{P}^3}(d)\otimes\operatorname{Sym}^m\Omega_{\Phi/\mathbb{G}(1,3)}) = 1 + (d-2m)\zeta + m\sigma_1,$$

and altogether

$$c(\mathcal{E}) = \prod_{m=0}^{4}(1 + (d-2m)\zeta + m\sigma_1).$$

In particular,

$$c_5(\mathcal{E}) = d\zeta\cdot((d-2)\zeta+\sigma_1)\cdot((d-4)\zeta+2\sigma_1)\cdot((d-6)\zeta+3\sigma_1)\cdot((d-8)\zeta+4\sigma_1).$$

We can evaluate the degrees of monomials of degree 5 in ζ and σ_1 by using the Segre classes introduced in Section 10.1, and in particular Proposition 10.3: We have

$$\deg(\zeta^a\sigma_1^b) = \deg\pi_*(\zeta^a\sigma_1^b) = \deg(s_{a-1}(\mathcal{S})\sigma_1^b)),$$

where $s_k(\mathcal{S})$ is the k-th Segre class of $\mathcal{S}$. Combining Proposition 10.3 and the Whitney formula, we have

$$s(\mathcal{S}) = \frac{1}{c(\mathcal{S})} = c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2,$$

and so we have

$$\begin{aligned}\deg(\zeta\sigma_1^4)_\Phi &= \deg(\sigma_1^4)_{\mathbb{G}(1,3)} = 2,\\ \deg(\zeta^2\sigma_1^3)_\Phi &= \deg(\sigma_1^4)_{\mathbb{G}(1,3)} = 2,\\ \deg(\zeta^3\sigma_1^2)_\Phi &= \deg(\sigma_2\sigma_1^2)_{\mathbb{G}(1,3)} = 1.\end{aligned}$$

The remaining monomials of degree 5 in ζ and σ_1 are all zero: $\sigma_1^5 = 0$ since the Grassmannian $\mathbb{G}(1,3)$ is four-dimensional, while $\zeta^4\sigma_1 = \zeta^5 = 0$ because the Segre classes of $\mathcal{S}$ vanish above degree 2 (alternatively, since $\zeta = \beta^*\omega$ is the pullback of a class on $\mathbb{P}^3$ we see immediately that $\zeta^4 = 0$).

Putting this together with the formula above for $c_5(\mathcal{E})$, we get

$$\begin{aligned}\deg c_5(\mathcal{E})&\\ &= \deg\big(d\zeta((d-2)\zeta+\sigma_1)((d-4)\zeta+2\sigma_1)((d-6)\zeta+3\sigma_1)((d-8)\zeta+4\sigma_1)\big)\\ &= \deg\big(24d\zeta\sigma_1^4 + d(50d-192)\zeta^2\sigma_1^3 + d(35d^2-200d+240)\zeta^3\sigma_1^2\big)\\ &= 35d^3 - 200d^2 + 240d.\end{aligned}$$

Assuming there are only finitely many and counting multiplicities, this is the number of lines having a point of contact of order at least 5 with S.

To answer the keynote question, we need to know in addition that for a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 5$ all the lines having a point of contact of order 5 with S "count with multiplicity 1" — that is, all the zeros of the corresponding section of the bundle $\mathcal{E}$ on Φ are simple zeros. To do this, we invoke the irreducibility of the incidence correspondence

$$\Gamma = \{(L, p, S) \mid m_p(S \cdot L) \geq 5\} \subset \Phi \times \mathbb{P}^N,$$

introduced in Section 11.1. By virtue of the irreducibility of Γ, it is enough to show that at just one point $(L, p, S) \in \Gamma$ the section of $\mathcal{E}$ corresponding to S has a simple zero at $(L, p) \in \Phi$: Given this, the locus of (L, p, S) for which this is not the case, being a proper closed subvariety of Γ, will have strictly smaller dimension, and so cannot dominate $\mathbb{P}^N$. As for locating such a triple (L, p, S), Exercise 11.17 suggests one. We should also check that for S general, no line has a point of contact of order at least 6 with S, or more than one point of contact of order at least 5; this is implied by Exercise 11.18. This completes the proof of Theorem 11.1.

11.2 The case of negative expected dimension

In this section, we will describe a rather different application of the contact calculus developed so far: We will use it to bound the maximum number of occurrences of some phenomena that occur in negative "expected dimension."

We begin by explaining an example. We do not expect a surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ to contain any lines. But some smooth quartics do contain a line and some contain several. Thus we can ask: How many lines can a smooth surface of degree d contain?

We observe to begin with that the number of lines a smooth surface of degree d can contain is certainly bounded: If we let $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$, and write

$$\Sigma = \{(S, L) \in \mathbb{P}^N \times \mathbb{G}(1, 3) \mid L \subset S\}$$

for the incidence correspondence, then the set of points of $\mathbb{P}^N$ over which the fiber of the map $\pi : \Sigma \to \mathbb{P}^N$ is finite of degree $\geq m$ is a locally closed subset of $\mathbb{P}^N$ for any m. Since, as we saw in Section 2.4.2, a smooth surface in $\mathbb{P}^3$ of degree > 2 cannot contain a positive-dimensional family lines, by the Noetherian property the degrees of the fibers over the open set $U \subset \mathbb{P}^N$ of smooth surfaces are bounded. We can thus ask in particular:

Question 11.5. *What is the largest number $M(d)$ of lines that a smooth surface $S \subset \mathbb{P}^3$ of degree d can have?*

Remarkably, we do not know the answer to this in general!

The situation here is typical: there is a large range of quasi-enumerative problems where the actual number is indeterminate because the expected dimension of the solution set is negative. In general, almost every time we have an enumerative problem there are analogous "negative expected dimension" variants. For example, we can ask:

Question 11.6. *(a) How many isolated singular points can a hypersurface $X \subset \mathbb{P}^n$ of degree d have?*

(b) How many tritangents can a plane curve $C \subset \mathbb{P}^2$ of degree d have? How many hyperflexes?

(c) How many cuspidal curves can a pencil of plane curves of degree d have? How many reducible ones? How many totally reducible ones (that is, unions of lines)?

We can even go all the way back to Bézout, and ask:

Question 11.7. *How many isolated points of intersection can $n+k$ linearly independent hypersurfaces of degree d in $\mathbb{P}^n$ have?*

Here there is at least a conjecture, described in Eisenbud et al. [1996] and proved in the case $k = 1$ for reduced sets of points by Lazarsfeld [1994, Exercise 4.12]. For a general discussion of these questions (and a more general conjecture), see Eisenbud et al. [1996].

All of these problems are attractive (especially Question 11.7). But we will not pursue them here; rather, we will focus on the original problem of bounding the number of lines on a smooth surface in $\mathbb{P}^3$, in order to illustrate how we can use enumerative methods to find such a bound.

11.2.1 Lines on smooth surfaces in $\mathbb{P}^3$

Since the number of lines on a smooth surface S of degree $d \geq 4$ is variable, it cannot be the solution to an enumerative problem of the sort we have been considering. But we can still use enumerative geometry to bound the number. What we will do is to find a curve F on S whose degree is determined enumeratively and such that F contains all the lines on S. In this we follow a line of argument proposed in Clebsch [1861, p. 106].

A natural approach is to relax the condition that a line L be contained in S to the condition that L meets S with multiplicity $\geq m$ at some point $p \in S$. We can adjust m so that the set of points p for which some line satisfies this condition has expected dimension 1, defining a curve on the surface (as we will see, the right multiplicity is 4). Since this curve must contain all the lines lying on the surface, its degree — which we can compute by enumerative means — is a bound for the number of such lines.

Figure 11.1 A flecnode is a node in which one branch has a flex at the node.

First of all, we say that a point $p \in S$ is *flecnodal* if there exists a line $L \subset \mathbb{P}^3$ having contact of order 4 or more with S at p; let $F \subset S$ be the locus of such points. (The reason for the name comes from another characterization of such points: for a general surface S, a general flecnodal point $p \in S$ will be one such that the intersection $S \cap \mathbb{T}_p S$ has a *flecnode* at p, that is, a node such that one branch has contact of order at least 3 with its tangent line; see Figure 11.1.) As we will show in Proposition 11.8, the flecnodal locus $F \subset S$ of a smooth surface of degree $d \geq 3$ will always have dimension 1.

As we have observed, any line lying in S is contained in the flecnodal locus F. Of course when $d = 3$ any line meeting S with multiplicity ≥ 4 must lie in S, so the flecnodal locus is exactly the union of the 27 lines in S. To describe the locus of flecnodes on S more generally, we again write Φ for the incidence correspondence

$$\Phi = \{(L, p) \in \mathbb{G}(1,3) \times \mathbb{P}^3 \mid p \in L\},$$

and we let $\zeta, \sigma_1 \in A^1(\Phi)$ be the pullbacks of the corresponding classes on $\mathbb{P}^3$ and $\mathbb{G}(1,3)$. Given a surface S, we wish to find the class of the locus

$$\Gamma = \{(L, p) \in \Phi \mid m_p(L \cdot S) \geq 4\}.$$

Since the flecnodal locus $F \subset S$ is the image of Γ under the projection of Φ to $\mathbb{P}^3$, knowledge of this class will determine in particular the degree of F.

To compute the class of Γ, consider the bundle $\mathcal{F} = \mathcal{P}^3_{\Phi/\mathbb{G}(1,3)}(\pi_2^*\mathcal{O}_{\mathbb{P}^3}(d))$ of third-order relative principal parts of $\pi_2^*\mathcal{O}_{\mathbb{P}^3}(d)$. It is a bundle of rank 4 on Φ whose fiber at a point (L, p) is the vector space of germs of sections of $\mathcal{O}_L(d)$ at p, modulo those vanishing to order at least 4 at p:

$$\mathcal{F}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}_p^4(d)).$$

If $A \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is a homogeneous polynomial of degree d defining a surface S, then the restrictions of A to each line $L \subset \mathbb{P}^3$ yield a global section σ_A of the bundle $\mathcal{F}$, whose zeros are the pairs (L, p) such that L meets S with multiplicity ≥ 4 at p.

Since $\dim \Phi = 5$ and $\mathcal{F}$ has rank 4, the locus Γ (if not empty) is at least one-dimensional; if it has dimension exactly 1 then its class is the top Chern class

$$[\Gamma] = c_4(\mathcal{F}) \in A^4(\Phi).$$

We can calculate this class as before: We can filter the bundle $\mathcal{F}$ by order of vanishing — that is, invoke the exact sequences (11.1) — and apply the Whitney formula to arrive at

$$c_4(\mathcal{F}) = d\zeta \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1).$$

Of course, none of this will help us bound the number of lines on S if every point of S is a flecnode! The following result, which was assumed by Clebsch, is thus crucial for this approach. A proof can be found in McCrory and Shifrin [1984, Lemma 2.10]. For partial results in finite characteristic see Voloch [2003].

Proposition 11.8. *If $S \subset \mathbb{P}^3$ is a smooth surface of degree $d \geq 3$ over a field of characteristic 0, the locus*

$$\Gamma = \{(L, p) \in \Phi \mid m_p(L \cdot S) \geq 4\}$$

has dimension 1. In particular, the general point of S is not flecnodal.

We will defer the proof of this proposition to the next section, and continue to derive our bound on the number of lines. By the proposition, the flecnodal locus $F \subset S$ of S is a curve, whose degree is the degree of the intersection of Γ with the class ζ. We can evaluate this as before:

$$\begin{aligned}\deg(F) &= d\zeta^2 \cdot ((d-2)\zeta + \sigma_1) \cdot ((d-4)\zeta + 2\sigma_1) \cdot ((d-6)\zeta + 3\sigma_1)\\ &= d(11d - 24).\end{aligned}$$

Putting this together, we have proven a bound on the number of lines in S:

Proposition 11.9. *The maximum number $M(d)$ of lines lying on a smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$ is at most $d(11d - 24)$.*

In the case $d = 3$ this gives the exact answer since $d(11d - 24) = 27$. But for $d \geq 4$ the bound is not sharp: Segre [1943] proved the slightly better bound $M(d) \leq d(11d - 28) + 12$. Even this is not sharp; for example with $d = 4$ we have $d(11d - 28) + 12 = 76$, but Segre also showed that the maximum number of lines on a smooth quartic surface is exactly $M(4) = 64$.

Of course, we can give a lower bound for $M(d)$ simply by exhibiting a surface with a large number of lines. The Fermat surface $V(x^d + y^d + z^d + w^d)$, for example, has exactly $3d^2$ lines (Exercise 11.25), whence $M(d) \geq 3d^2$. This is still the record-holder for general d. More is known for some particular values of d; Exercises 11.26 and 11.27 exhibit surfaces with more lines in the cases $d = 4, 6, 8, 12$ and 20 (respectively, 64, 180, 256, 864, and 1600 lines), and Boissière and Sarti [2007] find an octic with 352 (the current champion!).

11.2.2 The flecnodal locus

It remains to prove that for any smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$ the locus $\Gamma \subset \Phi$ of pairs (L, p) with $m_p(L \cdot S) \geq 4$ has dimension 1. The following proof was shown us by Francesco Cavazzani:

Proof of Proposition 11.8: Suppose on the contrary that the locus $\Gamma \subset \Phi$ has a component Γ_0 of dimension 2 or more, and let (L_0, p_0) be a general point of Γ_0. Since the fiber of Γ over a point $p \in S$ consists of lines through p in $\mathbb{T}_p S$, it has dimension at most 1. By Theorem 7.11(a), there are only finitely many points p over which the fiber has positive dimension. Thus Γ_0 must dominate S, so p_0 is a general point of S. By Theorem 7.11(b), the intersection $S \cap \mathbb{T}_{p_0} S$ has a node at p_0.

We will proceed by introducing local coordinates on Φ and writing down the defining equations of the subset Γ. To start with, we can find an affine open $\mathbb{A}^3 \subset \mathbb{P}^3$ and choose coordinates (x, y, z) on $\mathbb{A}^3$ so that the point p_0 is the origin $(0,0,0) \in \mathbb{A}^3$ and the line $L_0 = \{(x, 0, 0)\}$ is the x-axis; we can also take the tangent plane $\mathbb{T}_{p_0} S$ to be the plane $z = 0$, and, given that the tangent plane section $S \cap \mathbb{T}_{p_0} S$ has a node at p_0, we can take the tangent cone at p to the intersection $S \cap \mathbb{T}_{p_0} S$ to be the union $V(z, xy)$ of the x- and y-axes.

We can take coordinates (a, b, c, d, e) in a neighborhood U of $(L_0, p_0) \in \Phi$ so that if (L, p) is the pair corresponding to (a, b, c, d, e) then

$$p = (a, b, c) \quad \text{and} \quad L = \{(a + t, b + dt, c + et)\}.$$

Let $f(x, y, z)$ be the defining equation of S in $\mathbb{A}^3$. If we write the restriction of f to L as

$$f|_L = f(a + t, b + dt, c + et) = \sum_{i \geq 0} \alpha_i(a, b, c, d, e) t^i,$$

the four functions $\alpha_0, \alpha_1, \alpha_2$ and α_3 will be the defining equations of Γ in U. We want to show their common zero locus has codimension 4 in Φ; we will actually prove the stronger fact that their differentials at (L_0, p_0) are independent.

By the specifications above of p_0, L_0, $\mathbb{T}_{p_0} S$ and $\mathbb{T}C_{p_0}(S \cap \mathbb{T}_{p_0} S)$, we can write

$$f(x, y, z) = z \cdot u(x, y, z) + xy \cdot v(x, y) + y^3 \cdot l(y) + x^4 \cdot m(x).$$

Note that since S is smooth at p_0 we have $u(0,0,0) \neq 0$, and since the tangent plane section $S \cap \mathbb{T}_{p_0} S$ has multiplicity 2 at p_0 we have $v(0,0) \neq 0$; rescaling the coordinates, we can assume $u(0,0,0) = v(0,0) = 1$. Note by contrast that we may have $m(0) = 0$; this will be the case exactly when $m_{p_0}(L_0 \cdot S) \geq 5$.

Now, we can just plug $(a + t, b + dt, c + et)$ in for (x, y, z) in this expression to write out $f|_L$, and hence the coefficients $\alpha_i(a, b, c, d, e)$. This is potentially messy, but in fact it will be enough to evaluate the differentials of the α_i at $(L_0.p_0)$ — that is, at

$(a,b,c,d,e) = (0,0,0,0,0)$ — and so we can work modulo the ideal $(a,b,c,d,e)^2$. That said, we have

$$f|_L = f(a+t, b+dt, c+et) = \\ (c+et)u + (a+t)(b+dt)v + (b+dt)^3 l(b+dt) + (a+t)^4 m(a+t),$$

and thus

$$\alpha_0 \equiv c \mod (a,b,c,d,e)^2,$$
$$\alpha_1 \equiv e + b \mod (c) + (a,b,c,d,e)^2,$$
$$\alpha_2 \equiv d \mod (b,c,e) + (a,b,c,d,e)^2,$$

and finally

$$\alpha_3 \equiv 4a \cdot m(0) \mod (b,c,d,e) + (a,b,c,d,e)^2.$$

What we see from this is that *the differentials of* $\alpha_0, \dots, \alpha_3$ *at* (L_0, p_0) *are linearly independent, unless* $m(0) = 0$; or in other words, *if there is a two-dimensional locus* $\Sigma \subset \Phi$ *of pairs* (L, p) *such that* $m_p(L \cdot S) \geq 4$, *then in fact we must have* $m_p(L \cdot S) \geq 5$ *for all* $(L, p) \in \Sigma$. But we can carry out exactly the same argument again to show that if there is a two-dimensional family of lines having contact of order 5 or more with S, then all these lines in fact have contact of order 6 or more with S, and so on. We conclude that if Γ has dimension 2 or more, then S must be ruled by lines; in other words, S must be a plane or a quadric. □

11.3 Flexes via defining equations

In our initial discussion of flexes in Section 7.5, we gave the curve $C \subset \mathbb{P}^2$ in question *parametrically* — that is, as the image of a map $\nu : \tilde{C} \to \mathbb{P}^2$ from a smooth curve $\tilde{C}$, the normalization of C, to $\mathbb{P}^2$. We defined flexes as the points $p \in \tilde{C}$ such that for some line $L \subset \mathbb{P}^2$ the multiplicity $m_p(\nu^* L)$ is at least 3.

This definition does not work well in families of curves. As we shall see, when a smooth plane curve degenerates to one with a node, a certain number of the flexes approach the node; but, according to the definition in Section 7.5, the nodal point will generally not be a flex, since in general neither branch of the node will have contact of order 3 or more with its tangent line. Thus to track the behavior of flexes in families we need a different way of describing them, related to the notion of Cartesian flexes described in Section 7.5.2. We will call the objects described below “flex lines” rather than flexes.

We define a *flex line* of $C \subset \mathbb{P}^2$ to be a pair (L, p) with $L \subset \mathbb{P}^2$ a line and $p \in L$ a point such that C and L intersect at p with multiplicity ≥ 3; that is, the set Γ of flex lines is the locus

$$\Gamma = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid m_p(C \cdot L) \geq 3\}.$$

Thus if C is the vanishing locus of a polynomial F, then (L, p) is a flex line if and only if the restriction of F to L vanishes to order at least 3 at p; in other words, instead of taking the defining equation of L and restricting to C (or, more precisely, pulling back to the normalization of C), we are restricting the defining equation of C to L. The two are the same when C is smooth, but different in general: For example, if C is a general curve with a node at p, the tangent lines to the two branches are flex lines at the node.

To compute the number of flexes on a curve defined by a homogeneous form F, we define Ψ to be the incidence correspondence

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

thought of as the universal line over $\mathbb{P}^{2*}$, and consider

$$\mathcal{E} = \mathcal{P}^2_{\Psi/\mathbb{P}^{2*}}(\pi_2^* \mathcal{O}_{\mathbb{P}^2}(d)),$$

a rank-3 vector bundle on the three-dimensional variety Ψ, whose fiber at a point (L, p) is

$$\mathcal{E}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}_p^3(d)).$$

The homogeneous polynomial F gives rise to a section σ_F of $\mathcal{E}$, and the zeros of this section correspond to the flex lines of the corresponding plane curve $C = V(F)$. Thus the number of flex lines — when this number is finite, of course, and counting multiplicity — is the degree of $c_3(\mathcal{E}) \in A^3(\Psi)$.

Since the projection on the first factor expresses Ψ as the projectivization

$$\Psi = \mathbb{P}\mathcal{S} \to \mathbb{P}^{2*}$$

of the universal subbundle $\mathcal{S}$ on $\mathbb{P}^{2*}$, we can give a presentation of the Chow ring exactly as in the case of the universal line Φ over $\mathbb{G}(1, 3)$ in Section 11.1.3. Letting $\sigma \in A^1(\mathbb{P}^{2*})$ be the hyperplane class, we have

$$A(\Psi) = A(\mathbb{P}^{2*})[\zeta]/(\zeta^2 - \sigma\zeta + \sigma^2),$$

where $\zeta \in A^1(\Phi)$ is the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$. Recall from Section 9.3.1, moreover, that the class ζ can also be realized as the pullback $\zeta = \beta^*\omega$, where $\beta : \Phi \to \mathbb{P}^2$ is the projection $(L, p) \mapsto p$ on the second factor and $\omega \in A^1(\mathbb{P}^2)$ is the hyperplane class.

We can also evaluate the degrees of monomials of degree 3 in ζ and σ as before by using the Segre classes introduced in Section 10.1, and in particular Proposition 10.3: We have

$$\deg(\zeta\sigma^2) = \deg(\zeta^2\sigma) = 1 \quad \text{and} \quad \deg(\zeta^3) = \deg(\sigma^3) = 0.$$

(We could also see these directly by observing that $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ is a hypersurface of bidegree $(1, 1)$, and the classes σ and ζ are the pullbacks of the hyperplane classes in the two factors.)

We can now calculate the Chern classes of $\mathcal{E}$ by applying the exact sequences (11.1) and using Whitney's formula, and we get

$$c_3(\mathcal{E}) = d\zeta \cdot ((d-2)\zeta + \sigma) \cdot ((d-4)\zeta + 2\sigma).$$

Hence

$$\begin{aligned}\deg(c_3(\mathcal{E})) &= 2d(d-2) + d(d-4) + 2d \\ &= 3d(d-2).\end{aligned}$$

This shows that the number of flex lines, counted with multiplicity, is the same in the singular case as in the smooth case, whenever the number is finite. (Note that if $F = 0$ defines a nonreduced curve, or a curve containing a straight line as a component, the section defined by F vanishes in the wrong codimension.) The present derivation allows us to go further in two ways, both having to do with the behavior of flexes in families. In particular, it will permit us to solve Keynote Question (c).

11.3.1 Hyperflexes

We define a *hyperflex line* to a plane curve C similarly: It is a pair (L, p) such that L and C meet with multiplicity at least 4 at p. As with ordinary flex lines (and for the same reason), this definition is equivalent to the definition of a hyperflex given in Section 7.5 when the point p is a smooth point of C, but not in general: If a curve $C \subset \mathbb{P}^2$ has an ordinary flecnode at p (that is, two branches, one not a flex and the other a flex that is not a hyperflex), then the tangent line to the flexed branch of C at p will be a hyperflex line, though p is not a hyperflex in the sense of Section 7.5. Since a general pencil of plane curves will not include any elements possessing a flecnode (Exercise 11.29), this will not affect our answer to Keynote Question (c).

To describe the locus of hyperflex lines in a family of curves, we denote by $\mathbb{P}^N$ the space of plane curves of degree d, and consider the incidence correspondence

$$\Sigma = \{(L, p, C) \in \Psi \times \mathbb{P}^N \mid m_p(L \cdot C) \geq 4\}.$$

When $d \geq 3$, the fibers of the projection $\Sigma \to \Psi$ are linear spaces of dimension $N-4$, from which we see that Σ is irreducible of dimension $N-1$; in particular, it follows that a general curve $C \subset \mathbb{P}^2$ of degree $d > 1$ has no hyperflexes. Furthermore, since for $d \geq 4$ the general fiber of the projection $\Sigma \to \mathbb{P}^N$ is finite (see the proof of Theorem 7.13), the locus $\Xi \subset \mathbb{P}^N$ of curves that do admit a hyperflex is a hypersurface in $\mathbb{P}^N$ in this case. Keynote Question (c) is equivalent to asking for the degree of this hypersurface in the case $d = 4$. We will actually compute it for all d.

To do this, we consider the three-dimensional variety $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ as above, and introduce the rank-4 bundle

$$\mathcal{E} = \mathcal{P}^3_{\Psi/\mathbb{P}^{2*}}(\pi_2^* \mathcal{O}^2_{\mathbb{P}}(d))$$

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

whose fiber at a point (L, p) is

$$\mathcal{E}_{(L,p)} = H^0(\mathcal{O}_L(d)/\mathcal{I}_p^4(d)).$$

With this definition in hand, we consider a general pencil $\{t_0F + t_1G\}_{t\in\mathbb{P}^1}$ of homogeneous polynomials of degree d on $\mathbb{P}^2$. The polynomials F, G give rise to sections σ_F, σ_G of $\mathcal{E}$, and the set of pairs (L, p) that are hyperflexes of some element of our pencil is the locus where these sections fail to be linearly independent. Thus the number of hyperflex lines, counted with multiplicities, is the degree of $c_3(\mathcal{E}) \in A^3(\Psi)$.

We do this as before: Filtering the bundle $\mathcal{E}$ by order of vanishing, we arrive at the expression

$$c(\mathcal{E}) = (1 + d\zeta)(1 + (d-2)\zeta + \sigma)(1 + (d-4)\zeta + 2\sigma)(1 + (d-6)\zeta + 3\sigma).$$

Thus

$$\begin{aligned} c_3(\mathcal{E}) &= (18d^2 - 88d + 72)\zeta^2\sigma + (22d - 36)\zeta\sigma^2 \\ &= 18d^2 - 66d + 36 \\ &= 6(d-3)(3d-2). \end{aligned}$$

This gives zero when $d = 3$, as it should: a cubic with a hyperflex is necessarily reducible, and a general pencil of plane cubics will not include any reducible ones. We also remark that the number is meaningless in the cases $d = 1$ and $d = 2$, since every point on a line is a hyperflex and a pencil of conics will contain reducible conics equal to the union of two lines.

To show that the actual number of elements of a general pencil possessing hyperflexes is equal to the number predicted, we have to verify that for general polynomials F and G the degeneracy locus $V(\sigma_F \wedge \sigma_G) \subset \Psi$ is reduced. We do this, as in the argument carried out in Section 7.3.1, in two steps: We first use an irreducibility argument to reduce the problem to exhibiting a single pair F, G of polynomials and a point $(L, p) \in \Psi$ such that $V(\sigma_F \wedge \sigma_G)$ is reduced at (L, p), then use a local calculation to show that there do indeed exist such F, G and (L, p).

For the first, a standard incidence correspondence suffices: We let $\mathbb{P}^N$ be the space of plane curves of degree d and $\mathbb{G} = \mathbb{G}(1, N)$ the Grassmannian of pencils of such curves, and consider the locus

$$\Upsilon = \{(\mathcal{D}, L, p) \in \mathbb{G} \times \Psi \mid \text{some } C \in \mathcal{D} \text{ has a hyperflex line at } (L, p)\}.$$

The fiber of Υ over (L, p) is irreducible of dimension $2N - 5$: It is the Schubert cycle $\Sigma_3(\Lambda) \subset \mathbb{G}$, where $\Lambda = \{C \in \mathbb{P}^N \mid m_p(L \cdot C) \geq 4\}$ is the codimension-4 subspace of $\mathbb{P}^N$ consisting of curves with a hyperflex line at (L, p). It follows that Υ is irreducible of dimension $2N - 2 = \dim \mathbb{G}$. Now, if $\Upsilon' \subset \Upsilon$ is the locus of $(\mathcal{D}, L, p)$ such that $V(\sigma_F \wedge \sigma_G)$ is *not* reduced of dimension 0 at (L, p) (where $\mathcal{D}$ is the pencil spanned by F and G), then, since $\Upsilon' \subset \Upsilon$ is closed,

$$\Upsilon' \neq \Upsilon \implies \dim \Upsilon' < 2N - 2,$$

and it follows that if $\Upsilon' \neq \Upsilon$ then Υ' cannot dominate $\mathbb{G}$.

Thus we need only exhibit a single F, G and (L, p) such that $V(\sigma_F \wedge \sigma_G)$ is reduced at (L, p). We do this using local coordinates. Choose $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates x, y so that $p = (0, 0)$ is the origin and $L \subset \mathbb{A}^2$ is the line $y = 0$. Set $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$.

For local coordinates on Ψ in a neighborhood of the point (L, p), we can take the functions x, y and b, where

$$p = (x, y) \quad \text{and} \quad L = \{(x + t, y + bt)\}_{t \in \Bbbk}.$$

We can trivialize the bundle $\mathcal{E}$ in this neighborhood of (L, p), so that the section σ_F of $\mathcal{E}$ is given by the first four terms in the Taylor expansion of the polynomial $f(x + t, y + bt)$ around $t = 0$. Thus, for example, the section associated to the polynomial $f(x, y) = y + x^4$ (that is, $F(x, y, z) = yz^{d-1} + x^4 z^{d-4}$) is represented by the first four terms in the expansion of $y + bt + (x + t)^4$:

$$\sigma_F = (y + x^4, b + 4x^3, 6x^2, 4x),$$

and the general polynomial $g(x, y) = \sum a_{i,j} x^i y^j$ gives rise to the section σ_G represented by the vector

$$(a_{0,0}+a_{1,0}x+a_{0,1}y+\cdots,\ a_{1,0}+a_{0,1}b+a_{1,1}y+2a_{2,0}x+\cdots,\ a_{2,0}+\cdots,\ a_{3,0}+\cdots).$$

(Here we are omitting terms in the ideal $(x, y, b)^2$.) The section $\sigma_F \wedge \sigma_G$ is given by the 2×2 minors of the matrix

$$\begin{pmatrix} y + x^4 & b + 4x^3 & 6x^2 & 4x \\ a_{0,0} + \cdots & a_{1,0} + \cdots & a_{2,0} + \cdots & a_{3,0} + \cdots \end{pmatrix}.$$

We have minors with linear terms $a_{1,0}y - a_{0,0}b$, $a_{3,0}y - 4a_{0,0}x$ and $a_{3,0}b - 4a_{2,0}x$, and for general values of the $a_{i,j}$ these are independent. This shows that the section $\sigma_F \wedge \sigma_G$ vanishes simply at p, as required. Thus:

Proposition 11.10. *In a general pencil of degree-d plane curves, exactly $6(d-3)(3d-2)$ will have hyperflexes; in particular, in a general pencil of quartic plane curves, exactly 60 members will have hyperflexes.*

11.3.2 Flexes on families of curves

We can also use the approach via defining equations to answer another question about flexes in pencils, one that sheds some more light on how flexes behave in families. Again, suppose that $\{C_t = V(t_0 F + t_1 G)\}_{t \in \mathbb{P}^1}$ is a general pencil of plane curves of degree d. The general member C_t of the pencil will have, as we have seen, $3d(d-2)$ flex points, and as t varies these points will sweep out another curve B in the plane. We can ask: What are the degree and genus of this curve? What is the geometry of this curve

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

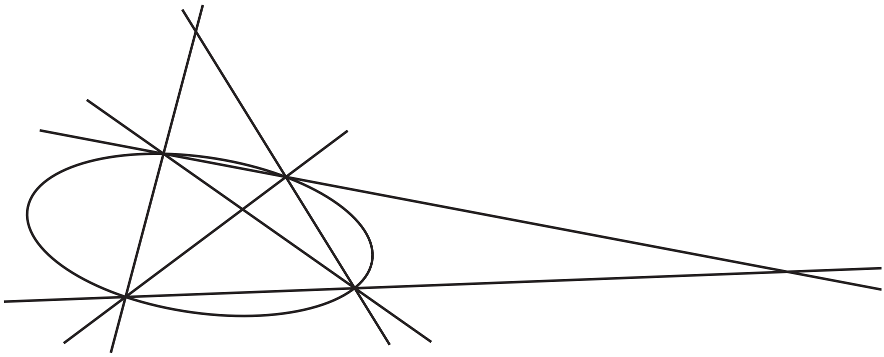

Figure 11.2 The singular elements of a pencil of conics are the pairs of lines joining the four base points.

around singular points of curves in the pencil? We will answer these questions in this section and the next.

To this end, we again write

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

and set

$$\Gamma = \{(t, L, p) \in \mathbb{P}^1 \times \Psi \mid m_p(L \cdot C_t) \geq 3\}.$$

We will describe Γ as the zero locus of a section of a rank-3 vector bundle on the four-dimensional variety $\mathbb{P}^1 \times \Psi$. For $d \geq 2$, we will show that Γ has the expected dimension 1, and we will ask the reader to show that in fact Γ is smooth by completing the sketch given in Exercise 11.33. This will allow us to determine not only the class of Γ (which will give us the degree of its image B under the projection $\mathbb{P}^1 \times \Psi \to \Psi \to \mathbb{P}^2$) but its genus as well. We will also describe the projection of Γ to $\mathbb{P}^1$, which tells how the flexes may come together as the curve moves in the pencil.

The case of a pencil of conics, $d = 2$, is easy to analyze directly, and already exhibits some of the phenomena involved. As we saw in Proposition 7.4 and the discussion immediately following, a general pencil of conics will have three singular elements, each consisting of two of the straight lines through two of the four base points of the pencil.

A smooth conic has no flexes, while the flex lines of a singular conic C are the pairs (L, p) with $p \in L \subset C$. Thus the curve B, consisting of points $p \in \mathbb{P}^2$ such that some (L, p) is a flex line of some member of the pencil, is the union of the singular members of the pencil — that is, the union of the six lines joining two of the four base points. As such it has degree 6, four triple points, and three additional double points. However, the points of the curve Γ "remember" the flex line to which they belong, so Γ is the *disjoint* union of the six lines — a smooth curve, which is the normalization of B. The singularities of B are typical of the situation of pencils of curves of higher degree, as we shall see: In general, B will have triple points at the base points of the pencil and nodes at the nodes of the singular elements of the pencil. In the case of conics, the projection map $\Gamma \to \mathbb{P}^1$ has three nonempty fibers, each consisting of one of the singular members of the pencil. For general pencils of degree > 2 we shall see that the projection is a finite cover.

Returning to the general case, we again write $\mathcal{E}$ for the rank-3 vector bundle

$$\mathcal{E} = \mathcal{P}^2_{\Psi/\mathbb{P}^{2*}}(\pi_2^* \mathcal{O}_{\mathbb{P}^2}(d)).$$

Writing V for the two-dimensional vector space spanned by F and G, the sections σ_F and σ_G define a map of bundles

$$V \otimes \mathcal{O}_\Psi \to \mathcal{E}.$$

We now pull this map back to $\mathbb{P}^1 \times \Psi$ via the projection $\nu : \mathbb{P}^1 \times \Psi \to \Psi$. If $\mathbb{P}^1 = \mathbb{P}V$ is the projective line parametrizing our pencil, we also have a natural inclusion

$$\mathcal{O}_{\mathbb{P}V}(-1) \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}V},$$

which we can pull back to the product $\mathbb{P}^1 \times \Psi$ via the projection $\mu : \mathbb{P}^1 \times \Psi \to \mathbb{P}^1$. Composing these, we arrive at a map

$$\rho : \mu^* \mathcal{O}_{\mathbb{P}^1}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1 \times \Psi} \to \nu^* \mathcal{E};$$

over the point $(t, L, p) \in \mathbb{P}^1 \times \Psi$, this is the map that takes a scalar multiple of $t_0 F + t_1 G$ to its restriction to L (modulo sections of $\mathcal{O}_L(d)$ vanishing to order 3 at p). In particular, *the zero locus of this map is the incidence correspondence* Γ.

Tensoring with the line bundle $\mu^* \mathcal{O}_{\mathbb{P}^1}(1)$, we can think of ρ as a section of the bundle $\mu^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^* \mathcal{E}$; the class of Γ is thus given by the Chern class

$$[\Gamma] = c_3(\mu^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^* \mathcal{E}) \in A^3(\mathbb{P}^1 \times \Psi).$$

We denote by η the class of a point in $A^1(\mathbb{P}^1)$, or its pullback to $\mathbb{P}^1 \times \Psi$. Similarly, we use the notation ζ and σ, introduced as classes in $A(\Psi)$ above, to denote the pullbacks of these classes to $\mathbb{P}^1 \times \Psi$. With this notation we have

$$c(\mu^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^* \mathcal{E}) = (1 + \eta + d\zeta)(1 + \eta + (d-2)\zeta + \sigma)(1 + \eta + (d-4)\zeta + 2\sigma).$$

Collecting the terms of degree 3, we get

$$\begin{aligned} &c_3(\mu^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^* \mathcal{E}) \\ &\qquad = (3d^2 - 8d)\zeta^2\sigma + 2d\zeta\sigma^2 + \eta((3d^2 - 12d + 8)\zeta^2 + (6d-8)\zeta\sigma + 2\sigma^2). \end{aligned}$$

To find the degree of the curve $B \subset \mathbb{P}^2$ swept out by the flex points of members of the family, we intersect with the (pullback of the) class ζ of a line $L \subset \mathbb{P}^2$; we get

$$\deg(B) = \zeta \cdot c_3(\mu^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^* \mathcal{E}) = 6d - 6.$$

Note that this yields the answer 6 in the case $d = 2$, consistent with our previous analysis.

We can use the same constructions to find the geometric genus of the curve Γ. As we observed in Proposition 6.15, the normal bundle to Γ in the product $\mathbb{P}^1 \times \Psi$ is the restriction to Γ of the bundle $\mu^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \nu^*\mathcal{E}$. Since $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ is a hypersurface of bidegree $(1, 1)$, its canonical class is

$$-c_1(\mathcal{T}_\Psi) = K_\Psi = K_{\mathbb{P}^{2*}\times\mathbb{P}^2} + \zeta + \sigma = -2\zeta - 2\sigma;$$

it follows that

$$K_{\mathbb{P}^1\times\Psi} = -2\eta - 2\zeta - 2\sigma.$$

By the calculation above,

$$c_1(\mathcal{E}) = 3\eta + (3d - 6)\zeta + 3\sigma,$$

and so we have

$$K_\Gamma = (\eta + (3d - 8)\zeta + \sigma)|_\Gamma.$$

We have seen that the degree of $\eta|_\Gamma$ is $3d(d-2)$, and $\deg(\zeta|_\Gamma) = 6d - 6$; similarly, we can calculate

$$\deg(\sigma|_\Gamma) = (3d^2 - 12d + 8) + (6d - 8) = 3d^2 - 6d.$$

Altogether, we have

$$2g(\Gamma) - 2 = \deg(K_\Gamma) = 24d^2 - 78d + 48,$$

and so

$$g(\Gamma) = 12d^2 - 39d + 25.$$

Note that when $d = 2$ this yields $g(\Gamma) = -5$, as it should: As we saw, in this case Γ consists of the disjoint union of six copies of $\mathbb{P}^1$.

11.3.3 Geometry of the curve of flex lines

We will leave the proofs of most of the assertions in this section to Exercises 11.34–11.38; here, we simply outline the main points of the analysis.

We begin with the geometry of the plane curve B traced out by the flex points of the curves C_t — that is, the image of the curve Γ under projection to $\mathbb{P}^2$. We have already seen that the degree of B is $6d - 6$.

The singularities of B can be located as follows: At each base point p of the pencil, all members of the pencil are smooth. We will see in Exercise 11.34 that three members of the pencil have flexes at p, so that B has a triple point at each base point of the pencil. The only other singularities of B occur at points $p \in \mathbb{P}^2$ that are nodes of the curve C_t containing them. As we have seen, at such a point the tangent lines to the two branches are each flex lines to C_t, so that map $\Gamma \to B$ is two-to-one there; as we will verify in Exercise 11.35, the curve B will have a node there.

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

Since the projection $\Gamma \to B$ is the normalization, these observations give another way to derive the formula for the genus of Γ: There are in general d^2 base points of the pencil, and as we saw in Chapter 7 there will be $3(d-1)^2$ nodes of elements C_t of our pencil, so that the genus of Γ is

$$\begin{aligned} g(\Gamma) &= p_a(B) - 3d^2 - 3(d-1)^2 \\ &= \tfrac{1}{2}(6d-7)(6d-8) - 3d^2 - 3(d-1)^2 \\ &= 12d^2 - 39d + 25. \end{aligned}$$

We can study the geometry of the curve Γ in another way as well: via the projection $\Gamma \to \mathbb{P}^1$ on the first factor. Since a general member of our pencil has $3d(d-2)$ flexes, Γ is a degree $3d(d-2)$ cover of the line $\mathbb{P}^1$ parametrizing our pencil. Where is this cover branched? The Plücker formula of Section 7.5.2 shows that if C_t is smooth it can fail to have exactly $3d(d-2)$ flexes only if it has a hyperflex, in which case the hyperflex counts as two ordinary flexes. Such hyperflexes are thus ordinary ramification points of the cover $\Gamma \to \mathbb{P}^1$.

That leaves only the singular elements of the pencil to consider, and this is where it gets interesting. By the formula of Section 7.5.2, a curve of degree d with a node has genus one lower, and hence six fewer flexes (in the sense of that section), than a smooth curve of the same degree. If C_{t_0} is a singular element of a general pencil of plane curves, then as $t \to t_0$ three of the flex lines of the curves C_t approach each of the tangent lines to the branches of C_{t_0} at the node (Exercise 11.38). Thus each of the tangent lines to the branches of C_{t_0} at the node is a ramification point of index 2 of the cover $\Gamma \to \mathbb{P}^1$.

We can put this all together with the Riemann–Hurwitz formula to compute the genus of Γ yet again: Since there are $6(d-3)(3d-2)$ hyperflexes in the pencil, and $3(d-1)^2$ singular elements,

$$2g(\Gamma) - 2 = -2 \cdot 3d(d-2) + 6(d-3)(3d-2) + 4 \cdot 3(d-1)^2,$$

and so

$$\begin{aligned} g(\Gamma) &= -3d(d-2) + 3(d-3)(3d-2) + 2 \cdot 3(d-1)^2 + 1 \\ &= 12d^2 - 39d + 25. \end{aligned}$$

11.4 Cusps of plane curves

As a final application we will answer the second keynote question of this chapter: How many curves in a general net of cubics in $\mathbb{P}^2$ have cusps? This will finally complete our calculation, begun in Section 2.2, of the degrees of loci in the space $\mathbb{P}^9$ of plane cubics corresponding to isomorphism classes of cubic curves. Solving this problem requires the introduction of a new class of vector bundles that further generalize the idea of the bundles of principal parts.

We start by saying what we mean by a cusp. An *ordinary cusp* of a plane curve C over the complex numbers is a point p such that, in an analytic neighborhood of p in the plane, the equation of C can be written as $y^2 - x^3 = 0$ in suitable (analytic) coordinates. If we were working over an algebraically closed field $\Bbbk$ other than the complex numbers, we could say instead that the completion of the local ring of C at p is isomorphic to $\Bbbk[\![x, y]\!]/(y^2 - x^3)$, and this is equivalent when $\Bbbk = \mathbb{C}$. Similar generalizations can be made for many of the remarks below.

It is inconvenient to do enumerative geometry with ordinary cusps directly, because the locus of ordinary cusps in a family of curves is not in general closed: ordinary cusps can degenerate to various other sorts of singularities (as in the family $y^2 - tx^3 + x^n$ when $t \to 0$). For this reason we will define a *cusp* of a plane curve to be point where the Taylor expansion of the equation of the curve has no constant or linear terms, and where the quadratic term is a square (possibly zero). As will become clearer in the next section, this means that a cusp is a point at which the completion of the local ring of the curve, in some local analytic coordinate system, has the form

$$\widehat{\mathcal{O}}_{C,p} = \Bbbk[\![x, y]\!]/(ay^2 + \text{terms of degree at least 3}),$$

where a is a constant that may be equal to 0. From the point of view of a general net of curves of degree at least 3, the difference between an ordinary cusp and a general cusp, in our sense, is immaterial: Proposition 11.13 will show that no cusps other than ordinary ones appear.

It is interesting to ask questions about curves on other smooth surfaces besides $\mathbb{P}^2$. Most of the results of this section can be carried over to general nets of curves in any sufficiently ample linear series on any smooth surface, but we will not pursue this generalization.

11.4.1 Plane curve singularities

Before plunging into the enumerative geometry of cusps, we pause to explain a little of the general picture of curve singularities.

Let $p \in C$ be a point on a reduced curve. In an analytic neighborhood of the point, C looks like the union of finitely many branches, each parametrized by a one-to-one map from a disc. Over the complex numbers these maps can be taken to be parametrizations by holomorphic functions of one variable; in general, this statement should be interpreted to mean simply that the completion $\widehat{\mathcal{O}}_{C,p}$ of the local ring $\mathcal{O}_{C,p}$ of C at p is reduced and the normalization of each of its irreducible components (the branches) has the form $\Bbbk[\![t]\!]$, where $\Bbbk$ is the ground field (if our ground field were not algebraically closed then the coefficient field might be a finite extension of the ground field). These statements are part of the theory of completions; see Eisenbud [1995, Chapter 7].

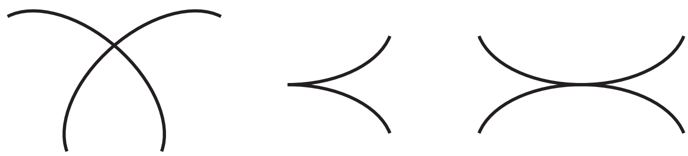

Figure 11.3 The simplest double points.

It is a consequence of the Weierstrass preparation theorem that over the complex numbers two reduced germs of analytic curves are isomorphic if and only if the completions of their local rings are isomorphic, so we will use the analytic language although we will work with the completions. See Greuel et al. [2007] for more details.

The theory of singularities in general is vast. But what will concern us here are double points of curves, and in that very limited setting we can actually give a classification, which we will do now.

To begin with, we have already defined the notion of the *multiplicity* of a variety X at a point $p \in X$ in Section 1.3.8. One consequence is that if X has multiplicity 2 at p ("p is a *double point* of X"), then the Zariski tangent space $T_p X$ has dimension $\dim X + 1$. In particular, if p is a double point of a curve C then $\dim T_p C = 2$, so that an analytic neighborhood of p in C is embeddable in the plane, and hence the completion $\widehat{\mathcal{O}}_{C,p}$ of the local ring $\mathcal{O}_{C,p}$ of C at p has the form $\Bbbk[\![x, y]\!]/(g(x, y))$, where g has leading term of degree exactly 2.

Definition 11.11. Let $p \in C$ be a double point of a reduced curve C. We say that p is *an A_n-singularity* of C (or that C *has an A_n-singularity at* p) if $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x, y]\!]/(y^2 - x^{n+1})$, that is, if in suitable (analytic) coordinates, C has equation $y^2 - x^{n+1} = 0$ around p.

For example, a double point $p \in C$ is a *node* (C has two smooth branches meeting transversely at p) if and only if C has, in suitable analytic coordinates, equation $y^2 - x^2 = 0$, and is thus an A_1-singularity. Similarly, an *ordinary cusp* is a point $p \in C$ with local analytic equation $y^2 - x^3 = 0$ (A_2-singularity), and an *ordinary tacnode* is a point with local analytic equation $y^2 - x^4 = 0$ (A_3-singularity); this looks like two smooth curves simply tangent to one another at p. In general, if $p \in C$ is an A_n-singularity and $n = 2m + 1$ is odd, then a neighborhood of p in C consists of two smooth branches meeting with multiplicity $m + 1$ at p (we can write $y^2 - x^{2m+2} = (y - x^{m+1})(y + x^{m+1})$), while if n is even then C is analytically irreducible at p.

Proposition 11.12. *Over an algebraically closed field any double point of a plane curve C is an A_n-singularity for some $n \geq 1$.*

Proof: We work in the power series ring $\mathbb{C}[\![x, y]\!]$, and we must show that if a power series $f(x, y)$ has nonzero leading term of degree 2 then, after multiplication by a unit of $\mathbb{C}[\![x, y]\!]$ and a change of variables, it can be written in the form $y^2 - x^{n+1}$. Since any nonzero quadratic form over $\mathbb{C}$ may be written (modulo scalars) as $y^2 + ax^2$ with $a \in \mathbb{C}$, we may assume that f has the form $f = y^2 + g(x) + yg_1(x) + y^2 g_2(x, y)$ for some g, g_1 and g_2 with $g_2(0, 0) = 0$. Multiplying f by the unit $1 - g_2(x, y)$, we reduce to the case $g_2 = 0$. Making a change of variable of the form $y' = y - g_1(x)$ (called a *Tschirnhausen transformation*), we can raise the order of vanishing of g_1; repeating this operation and taking the limit we may assume that $g_1 = 0$ as well. But if g has order $n + 1$, then, by Hensel's lemma (Eisenbud [1995, Theorem 7.3]), g has an $(n + 1)$-st root of the form $x + ax^2 + \cdots$. We may take this power series to be a new variable, and after this change of variables we get $f(x, y) = y^2 - x^{n+1}$, as required. □

In the space $\mathbb{P}^N$ parametrizing all plane curves of given degree d, we can estimate the dimension of the locus of curves having certain types of singularities, at least when the degree of the curves is large compared with the complexity of the singularity (this is an open problem when the degree is small; see Greuel et al. [2007] for more information):

Proposition 11.13. *Let $\mathbb{P}^N$ be the space of plane curves of degree $d \geq k$, and let $\Delta_k \subset \mathbb{P}^N \times \mathbb{P}^2$ be the set of pairs (C, p) such that C has an A_k-singularity at p. Δ_k is locally closed and has codimension $k + 2$ in $\mathbb{P}^N \times \mathbb{P}^2$. Its closure is irreducible, and contains in addition the locus $\Phi \subset \mathbb{P}^N \times \mathbb{P}^2$ of pairs (C, p) such that C has multiplicity 3 or more at p and the locus $\Xi \subset \mathbb{P}^N \times \mathbb{P}^2$ of pairs (C, p) such that p lies on a multiple component of C; in fact,*

$$\overline{\Delta_k} = \Phi \cup \Xi \cup \bigcup_{l \geq k} \Delta_l.$$

Note that since the projection $\Delta_k \to \mathbb{P}^N$ on the first factor is generically finite, the image of Δ_k will have codimension k in $\mathbb{P}^N$. Thus, among all plane curves we will see curves with a node in codimension 1, curves with a cusp in codimension 2 and curves with a tacnode in codimension 3; all other singularities should occur in codimension 4 and higher.

Finally, note that the situation is much less clear when k is large relative to d; for example, as we mentioned in Section 2.2, it is not known for all d and k whether there exists an irreducible plane curve of degree d with an A_k-singularity. In particular, it is not known for $d > 6$ whether there exists an irreducible plane curve $C \subset \mathbb{P}^2$ of degree d with an $A_{(d-1)(d-2)}$-singularity (this is the largest value allowed by the genus formula).

11.4.2 Characterizing cusps

As in the case of the simpler problem of counting singular elements of a pencil of curves, the first thing we need to do to study the cusps in a net of plane curves is to linearize the problem. The difficulty arises from the fact that even after we specify a point $p \in \mathbb{P}^2$ it is not a linear condition on the curves in our linear system to have a cusp at p. It becomes linear, though, if we specify both the point p and a line $L \subset \mathbb{P}^2$ through p with which we require our curve to have intersection multiplicity at least 3. Thus we will work on the universal line over $\mathbb{P}^{2*}$

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\},$$

which we used in Section 11.3 above. In the present circumstances, we also want to think of Ψ as a subscheme of the Hilbert scheme $\mathcal{H}_2(\mathbb{P}^2)$ parametrizing subschemes of $\mathbb{P}^2$ of degree 2. Specifically, it is the locus in $\mathcal{H}_2(\mathbb{P}^2)$ of subschemes of $\mathbb{P}^2$ supported at a single point: We associate to $(L, p) \in \Psi$ the subscheme $\Gamma = \Gamma_{L,p} \subset \mathbb{P}^2$ of degree 2 supported at p with tangent line $\mathbb{T}_p\Gamma = L \subset \mathbb{P}^2$.

For a given point $(L, p) \in \Psi$, we want to express the condition that the curve $C = V(\sigma)$ associated to a section σ of the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$ on $\mathbb{P}^2$ have a cusp at p with $m_p(C \cdot L) \geq 3$. This suggests that we introduce for each (L, p) the ideal $J_{L,p}$ of functions whose zero locus has such a cusp; that is, we set

$$J_{L,p} = \mathfrak{m}_p^3 + I_\Gamma^2,$$

where $\Gamma = \Gamma_{L,p} \subset \mathbb{P}^2$ is the subscheme of degree 2 supported at p with tangent line L.

We want to construct a vector bundle $\mathcal{E}$ on Ψ whose fiber at a point (L, p) is

$$\mathcal{E}_{(L,p)} = H^0(\mathbb{P}^2, \mathcal{L}/\mathcal{L} \otimes J_{L,p}).$$

To do this, consider the product $\Psi \times \mathbb{P}^2$, with projection maps π_1 and π_2 to Ψ and $\mathbb{P}^2$. Let $\Delta \subset \Psi \times \mathbb{P}^2$ be the graph of the projection map $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \to \mathbb{P}^2$ — in other words,

$$\Delta = \{((L, p), q) \in \Psi \times \mathbb{P}^2 \mid p = q\}.$$

Likewise, let $\Gamma \subset B \times \mathbb{P}^2$ be the universal scheme of degree 2 over $\Psi \subset \mathcal{H}_2(\mathbb{P}^2)$. We then take

$$\mathcal{E} = \pi_{1*}\big(\pi_2^*\mathcal{L}/\pi_2^*\mathcal{L} \otimes \big(\mathcal{I}^3_{\Delta/\Psi\times\mathbb{P}^2} + \mathcal{I}^2_{\Gamma/\Psi\times\mathbb{P}^2}\big)\big);$$

by the theorem on cohomology and base change (Theorem B.5), this is the bundle we want.

A global section of the line bundle $\mathcal{L}$ gives rise to a section of $\mathcal{E}$ by restriction. Given a net $\mathcal{D}$ corresponding to a three-dimensional vector space $V \subset H^0(\mathcal{L})$, we get three sections of $\mathcal{E}$, and the locus in B where they fail to be independent — that is, where some linear combination is zero — is the locus of (p, ξ) such that some element of the net has a cusp at p in the direction ξ. In sum, observing that two elements of a general net

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

cannot have cusps at the same point, and that a general cuspidal curve of degree $d > 2$ has a unique cusp (we leave the verification of this fact to the reader), the (enumerative) answer to our question is the degree of the Chern class $c_3(\mathcal{E})$. In the remainder of this section we will calculate this.

One remark before we launch into the calculation. We are using here the fact that we can characterize the condition that a curve C have a cusp at p by saying that C contains a scheme isomorphic to $\operatorname{Spec} \Bbbk[x, y]/J_{L,p}$; the parameter space Ψ can be viewed as parametrizing such subschemes of $\mathbb{P}^2$. We could use the same technique to count curves with tacnodes; this is sketched in Exercises 11.46–11.48.

If we try to apply the same techniques to count curves with other singularities, however, we run into trouble. For example, the condition that C have an A_5-singularity (an *oscnode*, in the classical terminology) is that C contain a scheme isomorphic to $\operatorname{Spec} \Bbbk[x, y]/(y, x^3)^2$. But the parameter space for such subschemes of the plane is not complete (schemes isomorphic to $\operatorname{Spec} \Bbbk[x, y]/(y, x^3)$ can specialize to the "fat point" scheme $\operatorname{Spec} \Bbbk[x, y]/(x, y)^2$), and if we try to complete it in the most natural way, by taking the closure in the Hilbert scheme, the relevant bundle $\mathcal{E}$ does not extend as a bundle to the closure. This problem is addressed and largely solved in Russell [2003].

11.4.3 Solution to the enumerative problem

We start by recalling the description of the Chow ring $A(\Psi)$ of $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2$ from Section 11.3: We have

$$A(\Psi) = A(\mathbb{P}^{2*})[\zeta]/(\zeta^2 - \sigma\zeta + \sigma^2) = \mathbb{Z}[\sigma, \zeta]/(\sigma^3, \zeta^2 - \sigma\zeta + \sigma^2),$$

where $\sigma \in \Psi$ is the pullback of the hyperplane class in $\mathbb{P}^{2*}$ and ζ is the pullback of the hyperplane class in $\mathbb{P}^2$ (equivalently, if we view Ψ as the projectivization of the universal subbundle $\mathcal{S}$ on $\mathbb{P}^{2*}$, the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$). The degrees of monomials of top degree 3 in σ and ζ are

$$\deg(\zeta\sigma^2) = \deg(\zeta^2\sigma) = 1 \quad \text{and} \quad \deg(\zeta^3) = \deg(\sigma^3) = 0.$$

Now, in order to find the Chern class of $\mathcal{E}$ we want to relate it to more familiar bundles. To this end, we observe that the inclusions

$$\mathfrak{m}_p^3 \hookrightarrow J_{L,p} \hookrightarrow \mathfrak{m}_p^2$$

and the corresponding quotients

$$\frac{\mathcal{L}_p}{\mathfrak{m}_p^3\mathcal{L}_p} \to \frac{\mathcal{L}_p}{J_{L,p}\mathcal{L}_p} \to \frac{\mathcal{L}_p}{\mathfrak{m}_p^2\mathcal{L}_p}$$

globalize to give us surjections of sheaves

$$\pi_2^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L}) \xrightarrow{\alpha} \mathcal{E} \quad \text{and} \quad \mathcal{E} \longrightarrow \pi_2^*\mathcal{P}^1_{\mathbb{P}^2}(\mathcal{L});$$

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

the composition

$$\pi_2^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L}) \xrightarrow{\beta} \pi_2^* \mathcal{P}^1_{\mathbb{P}^2}(\mathcal{L})$$

is the standard quotient map of Theorem 7.2.

Consider the corresponding inclusion

$$\mathrm{Ker}(\alpha) \hookrightarrow \mathrm{Ker}(\beta) = \pi_2^*(\mathrm{Sym}^2 \mathcal{T}^*_{\mathbb{P}^2} \otimes \mathcal{L}).$$

What is the image? It is the tensor product of $\mathcal{L}$ with the sub-line bundle of $\pi_2^* \,\mathrm{Sym}^2 \mathcal{T}^*_{\mathbb{P}^2}$ whose fiber at each point (L, p) is the subspace spanned by the square of the linear form on $T_p \mathbb{P}^2$ vanishing on $T_p L \subset T_p \mathbb{P}^2$. In other words, the inclusion $T_p L \hookrightarrow T_p \mathbb{P}^2$ at each point $(L, p) \in \Psi$ gives rise to a sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \beta^* \mathcal{T}^*_{\mathbb{P}^2} \longrightarrow \mathcal{T}^*_{\Psi/\mathbb{P}^{2*}} \longrightarrow 0, \tag{11.2}$$

where $\mathcal{N}$ is the sub-line bundle of $\beta^* \mathcal{T}^*_{\mathbb{P}^2}$ whose fiber at (L, p) is the space of linear forms on $T_p \mathbb{P}^2$ vanishing on $T_p L \subset T_p \mathbb{P}^2$ (we can think of $\mathcal{N}$ as the "relative conormal bundle" of the family $\Psi \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \to \mathbb{P}^{2*}$). Taking symmetric squares, we have an inclusion

$$\mathrm{Sym}^2 \mathcal{N} \hookrightarrow \beta^* \,\mathrm{Sym}^2 \mathcal{T}^*_{\mathbb{P}^2},$$

and tensoring with the pullback of $\mathcal{L}$ we arrive at an inclusion

$$\mathrm{Sym}^2 \mathcal{N} \otimes \beta^* \mathcal{L} \hookrightarrow \beta^*(\mathrm{Sym}^2 \mathcal{T}^*_{\mathbb{P}^2} \otimes \mathcal{L}),$$

whose image is exactly $\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})/\mathcal{E}$.

We can put this all together to calculate the Chern class $c(\mathcal{E})$. To begin with, we know the classes of the bundle $\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})$ from Proposition 7.5: We have

$$c(\beta^* \mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})) = (1 + (d-2)\zeta)^6 = 1 + 6(d-2)\zeta + 15(d-2)^2\zeta^2.$$

Next, the Chern class of the line bundle $\mathcal{N}$ can be found from the sequence (11.2): We have

$$\begin{aligned} c_1(\mathcal{N}) &= c_1\big(\beta^* \mathcal{T}^*_{\mathbb{P}^2}\big) - c_1\big(\mathcal{T}^*_{\Psi/\mathbb{P}^{2*}}\big) \\ &= -3\zeta - (-2\zeta + \sigma) \\ &= -\sigma - \zeta, \end{aligned}$$

where the equality $c_1(\mathcal{T}^*_{\Psi/\mathbb{P}^{2*}}) = -2\zeta + \sigma$ comes from Theorem 11.4. Thus

$$c(\mathrm{Sym}^2 \mathcal{N} \otimes \beta^* \mathcal{L}) = 1 + (d-2)\zeta - 2\sigma,$$

and since $\beta^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})/\mathcal{E} \cong \mathrm{Sym}^2\mathcal{N} \otimes \beta^*\mathcal{L}$ the Whitney formula gives

$$\begin{aligned} c(\mathcal{E}) &= \frac{c\big(\beta^*\mathcal{P}^2_{\mathbb{P}^2}(\mathcal{L})\big)}{c(\mathrm{Sym}^2\mathcal{N} \otimes \beta^*\mathcal{L})} \\ &= \frac{1+6(d-2)\zeta+15(d-2)^2\zeta^2}{1-(2\sigma-(d-2)\zeta)} \\ &= (1+6(d-2)\zeta+15(d-2)^2\zeta^2)\sum_{k=0}^{3}(2\sigma-(d-2)\zeta)^k. \end{aligned}$$

In particular, the third Chern class $c_3(\mathcal{E})$ is

$$c_3(\mathcal{E}) = (2\sigma-(d-2)\zeta)^3+6(d-2)\zeta(2\sigma-(d-2)\zeta)^2+15(d-2)^2\zeta^2(2\sigma-(d-2)\zeta),$$

and taking degrees we have

$$\begin{aligned} \deg c_3(\mathcal{E}) &= 30(d-2)^2+24(d-2)-24(d-2)^2-12(d-2)+6(d-2)^2 \\ &= 12d^2-36d+24. \end{aligned}$$

We have thus proven the enumerative formula:

Proposition 11.14. *The number of cuspidal elements of a net $\mathcal{D}$ of curves of degree d on $\mathbb{P}^2$, assuming there are only finitely many and counting multiplicities, is*

$$12d^2-36d+24 = 12(d-1)(d-2).$$

Of course, to answer Keynote Question (b) we have to verify that for a general net there are indeed only finitely many cusps, and that they all count with multiplicity 1. The first of these statements follows easily from the dimension count of Proposition 11.13. The second can be verified by explicit calculation in local coordinates, analogous to what we did to verify, for example, that hyperflexes in a general pencil occur with multiplicity 1; alternatively, we can use the method described in Section 11.4.4 below.

Note that the formula yields 0 in the cases $d = 1$ and 2, as it should. And, in the case $d = 3$, we see that a general net of plane cubics will have 24 cuspidal members, answering Keynote Question (b). Equivalently, we see that the locus of cuspidal plane cubics has degree 24 in the space $\mathbb{P}^9$ of all plane cubics, completing the analysis begun in Section 2.2.

Note that there was no need to restrict ourselves to nets of curves in $\mathbb{P}^2$; a similar analysis could be made for the number of cusps (possibly with multiplicities) in a sufficiently general net of divisors associated to a sufficiently ample line bundle $\mathcal{L}$ on any surface S. (Here the role of Ψ would be played by the projectivized tangent bundle $\mathbb{P}\mathcal{T}_S$.) We leave this version of the calculation to the reader; the answer is that the number of cuspidal elements in a net of curves $\mathcal{D} = (\mathcal{L}, V)$ on a surface S is

$$\deg(12\lambda^2-12\lambda c_1+2c_1^2+2c_2),$$

where $\lambda = c_1(\mathcal{L})$ and $c_i = c_i(\mathcal{T}_S)$. As always, this number is subject to the usual caveats: it is meaningful only if the number of cuspidal curves in the net is in fact finite; in this case, it represents the number of cuspidal curves counted with multiplicity (with multiplicity defined as the degree of the component of the zero scheme of the corresponding section of $\mathcal{E}$ supported at (p, ξ)).

11.4.4 Another approach to the cusp problem

There is another approach to the problem of counting cuspidal curves in a linear system, one that gives a beautiful picture of the geometry of nets. It is not part of the overall logical structure of this book, so we will run through the sequence of steps involved without proof; the reader who is interested can view supplying the verifications as an extended exercise.

To begin with, let S be a smooth projective surface and $\mathcal{L}$ a very ample line bundle; let $\mathcal{D} \subset |\mathcal{L}|$ be a general two-dimensional subseries, corresponding to a three-dimensional vector subspace $V \subset H^0(\mathcal{L})$. We have a natural map

$$\varphi : S \to \mathbb{P}^2 = \mathbb{P}V^*$$

to the projectivization of the dual V^*; the preimages $\varphi^{-1} \subset S$ of the lines $L \subset \mathbb{P}V^*$ are the divisors $C \subset S$ of the linear system $\mathcal{D}$. If we want, we can think of the complete linear system $|\mathcal{L}|$ as giving an embedding of S in the larger projective space $\mathbb{P}^n = \mathbb{P}H^0(\mathcal{L})^*$, and the map φ as the projection of S corresponding to a general $(n-3)$-plane.

Now, the geometry of generic projections of smooth varieties is well understood in low dimensions. Mather [1971; 1973] showed that these are the same in the algebro-geometric setting as in the differentiable; in the latter context the singularities of general projections of surfaces are described in Golubitsky and Guillemin [1973, Section 6.2]. The upshot is that if $\varphi : S \to \mathbb{P}^2$ is the projection of a smooth surface $S \subset \mathbb{P}^n$ from a general $(n-3)$-plane, then:

- The ramification divisor $R \subset S$ of the map φ is a smooth curve.
- The branch divisor $B \subset \mathbb{P}V^*$ is the birational image of R, and has only nodes and ordinary cusps as singularities.

In fact, étale locally around any point $p \in S$, one of three things is true. Either:

(i) The map is étale (if $p \notin R$).

(ii) The map is simply ramified, that is, of the form $(x, y) \mapsto (x, y^2)$ (if p is a point of R not lying over a cusp of B).

(iii) The surface S is given, in terms of local coordinates (x, y) on $\mathbb{P}V^*$ around $\varphi(p)$, by the equation

$$z^3 - xz - y = 0.$$

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

(This is the picture around a point where three sheets of the cover come together; in a neighborhood of $\varphi(p)$ the branch curve is the zero locus of the discriminant $4x^3 - 27y^2$, and in particular has a cusp at $\varphi(p)$.)

The interesting thing about this set-up is that we have two plane curves associated to it, lying in dual projective planes:

(a) The branch curve $B \subset \mathbb{P}V^*$ of the map φ.
(b) In the dual space $\mathbb{P}V$ parametrizing divisors in the net $\mathcal{D}$, we have the *discriminant curve* $\Delta \subset \mathbb{P}V$, that is, the locus of singular elements of the net.

What ties everything together is the observation that *the discriminant curve* $\Delta \subset \mathbb{P}V$ *is the dual curve of the branch curve* $B \subset \mathbb{P}V^*$. To see this, note that if $L \subset \mathbb{P}V^*$ is a line transverse to B (in particular, not passing through any of the singular points of B), then the preimage $\varphi^{-1}(L) \subset S$ will be smooth: This is certainly true away from points of B, where the map φ is étale, and at a point $p \in L \cap B$ we can take local coordinates (x, y) on $\mathbb{P}V^*$ with L given by $y = 0$ and B by $x = 0$; at a point of $\varphi^{-1}(p)$ the cover $S \to \mathbb{P}V^*$ will either be étale or given by $z^2 = x$. A similar calculation shows conversely that if L is tangent to B at a smooth point then $\pi^{-1}(L)$ will be singular.

At this point, we invoke the classical *Plücker formulas for plane curves*. These say that if $C \subset \mathbb{P}^2$ is a plane curve of degree $d > 1$ and geometric genus g having δ nodes and κ cusps as its only singularities, and the dual curve C^* has degree d^* and δ^* nodes and κ^* cusps as singularities, then

$$
\begin{aligned}
d^* &= d(d-1) - 2\delta - 3\kappa,\\
d &= d^*(d^*-1) - 2\delta^* - 3\kappa^*,\\
g &= \tfrac{1}{2}(d-1)(d-2) - \delta - \kappa = \tfrac{1}{2}(d^*-1)(d^*-2) - \delta^* - \kappa^*.
\end{aligned}
$$

See, for example, Griffiths and Harris [1994, p. 277*ff.*]. Given these, all we have to do is write down everything we know about the curves R, B and Δ. To begin with, we invoke the Riemann–Hurwitz formula for finite covers $f : X \to Y$: If η is a rational canonical form on Y with divisor D, the divisor of the pullback $f^*\eta$ will be the preimage of D plus the ramification divisor $R \subset X$; thus

$$K_X = f^*K_Y + R \in A^1(X).$$

In our present circumstances, this says that

$$K_S = \varphi^* K_{\mathbb{P}V^*} \otimes \mathcal{O}_S(R);$$

since the pullback $\varphi^*\mathcal{O}_{\mathbb{P}V^*}(1)$ is equal to $\mathcal{L}$, we can write this as

$$K_S = \mathcal{L}^{-3}(R),$$

or, in terms of the notation $c_1 = c_1(\mathcal{T}_S^*)$ and $\lambda = c_1(\mathcal{L})$, the class of R is

$$[R] = c_1 + 3\lambda \in A^1(S).$$

Among other things, this tells us the genus g of the curve R: Since R is smooth, by adjunction we have

$$\begin{aligned} g &= \tfrac{1}{2} R \cdot (R + K_S) + 1 \\ &= \tfrac{1}{2}(c_1 + 3\lambda)(2c_1 + 3\lambda) + 1 \\ &= \tfrac{1}{2}(9\lambda^2 + 9\lambda c_1 + 2c_1^2) + 1. \end{aligned}$$

It also tells us the degree d of the branch curve $B = \varphi(R) \subset \mathbb{P}V^*$: This is the intersection of R with the preimage of a line, so that

$$d = \lambda(c_1 + 3\lambda) = 3\lambda^2 + \lambda c_1.$$

Finally, we also know the degree e of the discriminant curve $\Delta \subset \mathbb{P}V$: This is the number of singular elements in a pencil, which we calculated back in Chapter 7; we have

$$e = 3\lambda^2 + 2\lambda c_1 + c_2.$$

We now have enough information to determine the number of cusps of Δ. Let δ and κ denote the number of nodes and cusps of Δ respectively. First off, the geometric genus of Δ is given by

$$g = \tfrac{1}{2}(e-1)(e-2) - \delta - \kappa,$$

and the degree d of the dual curve is

$$d = e(e-1) - 2\delta - 3\kappa.$$

Subtracting twice the first equation from the second yields

$$\begin{aligned} \kappa &= 2g - d + 2(e-1) \\ &= 9\lambda^2 + 9\lambda c_1 + 2c_1^2 + 2 - (3\lambda^2 + \lambda c_1) + 2(3\lambda^2 + 2\lambda c_1 + c_2 - 1) \\ &= 12\lambda^2 + 12\lambda c_1 + 2c_1^2 + 2c_2, \end{aligned}$$

agreeing with result stated at the end of Section 11.4.3. Note that this method also gives us a geometric sense of when a cusp "counts with multiplicity one;" in particular, if all the hypotheses above about the geometry of the map φ are satisfied, the count is exact.

This also gives us a formula for the number of curves C in the net with two nodes. This is the number δ of nodes of the curve Δ, which we get by subtracting three times the equation for g above from the equation for d: this yields

$$\delta = d - 3g - e(e-1) + \tfrac{3}{2}(e-1)(e-2),$$

where d, e and g are given in terms of the classes λ, c_1 and c_2 by the equations above.

Note that the formula returns 0 in the cases $d = 1$ and $d = 2$, as it should, and in the case $d = 3$ it gives 21 — the degree of the locus of reducible cubics in the $\mathbb{P}^9$ of all cubics, as calculated in Section 2.2.

Exercises 11.49–11.51 describe an alternative (and perhaps cleaner) way of deriving the formula for the number of binodal curves in a net, via linearization.

11.5 Exercises

Exercise 11.15. Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geq 6$. How many lines $L \subset \mathbb{P}^4$ will have a point of contact of order 7 with X?

Exercise 11.16. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 2$. Using the dimension counts of Proposition 11.13 and incidence correspondences, show that:

(a) For p in a dense open subset $U \subset S$, the intersection $S \cap \mathbb{T}_p S$ has an ordinary double point (a node) at p.
(b) There is a one-dimensional locally closed locus $Q \subset S$ such that for $p \in Q$ the intersection $S \cap \mathbb{T}_p S$ has a cusp at p.
(c) There will be a finite set Γ of points $p \in S$, lying in the closure of Q, such that the intersection $S \cap \mathbb{T}_p S$ has a tacnode at p.
(d) S is the disjoint union of U, Q and Γ; that is, no singularities other than nodes, cusps and tacnodes appear among the plane sections of S.

Exercise 11.17. Let Φ be the universal line over $\mathbb{G}(1,3)$ and $\mathcal{E}$ the bundle on Φ introduced in Section 11.1. Let $L \subset \mathbb{P}^3$ be the line $X_2 = X_3 = 0$, and let $p \in L$ be the point $[1,0,0,0]$. By trivializing the bundle $\mathcal{E}$ in a neighborhood of $(L, p) \in \Phi$ and writing everything in local coordinates, show that the section of $\mathcal{E}$ coming from the polynomial $X_1^5 + X_0^4 X_2 + X_0^2 X_1^2 X_3$ has a simple zero at (L, p).

Exercise 11.18. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 4$. Show that, for any line $L \subset \mathbb{P}^3$ and any pair of distinct points $p, q \in L$:

(a) $m_p(S \cdot L) \leq 5$.
(b) $m_p(S \cdot L) + m_q(S \cdot L) \leq 6$.

Exercise 11.19. A point p on a smooth surface $S \subset \mathbb{P}^3$ is called an *Eckhart point* of S if the intersection $S \cap \mathbb{T}_p S$ has a triple point at p. Recall that in Exercise 7.42 we saw that a general surface $S \subset \mathbb{P}^3$ of degree d has no Eckhart points.

(a) Show that the locus of smooth surfaces that do have an Eckhart point is an open subset of an irreducible hypersurface $\Psi \subset \mathbb{P}^{\binom{d+3}{3}-1}$ in the space of all surfaces.
(b) Show that a general surface $S \subset \mathbb{P}^3$ that does have an Eckhart point has only one.
(c) Find the degree of the hypersurface Ψ.

Exercise 11.20. Consider a smooth surface $S \subset \mathbb{P}^4$. Show that we would expect there to be a finite number of hyperplane sections $H \cap S$ of S with triple points, and count the number in terms of the hyperplane class $\zeta \in A^1(S)$ and the Chern classes of the tangent bundle to S.

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

Exercise 11.21. Applying your answer to the preceding exercise, find the number of hyperplane sections of $S \subset \mathbb{P}^4$ with triple points in each of the following cases:

(a) S is a complete intersection of two quadrics in $\mathbb{P}^4$.
(b) S is a cubic scroll (Section 9.1.1).
(c) S is a general projection of the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$.

In each case, can you check your answer directly?

Exercise 11.22. For $S \subset \mathbb{P}^3$ a general surface of degree d, find the degree of the surface swept out by the lines in $\mathbb{P}^3$ having a point of contact of order at least 4 with S.

The following exercise describes in some more detail the geometry of the *flecnodal locus* $\Gamma \subset S$ of a smooth surface $S \subset \mathbb{P}^3$, introduced in Section 11.2.1; we will use the notation of that section.

Exercise 11.23. Let $S \subset \mathbb{P}^3$ be a general surface of degree d.

(a) Find the first Chern class of the bundle $\mathcal{F}$.
(b) Show that the curve Γ is smooth, and that the projection $\Gamma \to C$ is generically one-to-one.
(c) Using the preceding parts, find the genus of the curve Γ.
(d) Show, on the other hand, that the flecnodal curve of S is the intersection of S with a surface of degree $11d - 24$, and use this to calculate the arithmetic genus of C.
(e) Can you describe the singularities of the curve C? Do these account for the discrepancy between the genera of Γ and of C?

Exercise 11.24. Let $\mathbb{P}^N$ be the space of surfaces of degree $d \geq 4$ in $\mathbb{P}^3$ and $\Psi \subset \mathbb{P}^N$ the locus of surfaces containing a line. Show that the maximum possible number $M(d)$ of lines on a smooth surface $S \subset \mathbb{P}^3$ of degree d is at most the degree of Ψ by considering the pencil spanned by S and a general second surface T. Is this bound better or worse than the one derived in Section 11.2.1?

Exercise 11.25. Show that for $d \geq 3$ the Fermat surface $S_d = V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$ contains exactly $3d^2$ lines.

Exercise 11.26. For $F(x, y)$ any homogeneous polynomial of degree d, consider the surface $S \subset \mathbb{P}^3$ given by the equation

$$F(x, y) - F(z, w) = 0.$$

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

If α is the order of the group of automorphisms of $\mathbb{P}^1$ preserving the polynomial F (that is, carrying the set of roots of F to itself), show that S contains at least $d^2 + \alpha d$ lines. *Hint:* if L_1 and L_2 are the lines $z = w = 0$ and $x = y = 0$ respectively, and $\varphi : L_1 \to L_2$ any isomorphism carrying the zero locus $F(x, y) = 0$ to $F(z, w) = 0$, consider the intersection of S with the quadric

$$Q_\varphi = \bigcup_{p \in L_1} \overline{p, \varphi(p)}.$$

Exercise 11.27. Using the preceding exercise, exhibit smooth surfaces $S \subset \mathbb{P}^3$ of degrees 4, 6, 8, 12 and 20 having at least 64, 180, 256, 864 and 1600 lines, respectively.

Exercise 11.28. Verify that the Fermat quartic curve $C = V(x^4 + y^4 + z^4) \subset \mathbb{P}^2$ has 12 hyperflexes and no ordinary flexes.

Exercise 11.29. Recall that a node $p \in C$ of a plane curve is called a *flecnode* if one of the branches of C at p has contact of order 3 or more with its tangent line. Show that the closure, in the space $\mathbb{P}^N$ of all plane curves of degree $d \geq 4$, of the locus of curves with a flecnode is irreducible of dimension $N - 2$.

Exercise 11.30. How many elements of a general net of plane curves of degree d will have flecnodes?

Exercise 11.31. Verify that for a general pencil $\{C_t = V(t_0 F + t_1 G)\}$ of plane curves of degree d, if (L, p) is a hyperflex of some element C_t of the pencil, then:

(a) $m_p(C_t \cdot L) = 4$; that is, no line has a point of contact of order 5 or more with any element of the pencil.
(b) p is a smooth point of C_t.
(c) p is not a base point of the pencil.

Using these facts, show that the degeneracy locus of the sections σ_F and σ_G of the bundle $\mathcal{E}$ introduced in Section 11.3.1 is reduced.

Exercise 11.32. Let $\{C_t = V(t_0 F + t_1 G)\}$ be a general pencil of plane curves of degree d. If $p \in \mathbb{P}^2$ is a general point, how many flex lines to members of the pencil $\{C_t\}$ pass through p?

For Exercises 11.33–11.38, we let $\{C_t = V(t_0 F + t_1 G)\}$ be a general pencil of plane curves of degree d,

$$\Psi = \{(L, p) \in \mathbb{P}^{2*} \times \mathbb{P}^2 \mid p \in L\}$$

be the universal line and

$$\Gamma = \{(t, L, p) \in \mathbb{P}^1 \times \Phi \mid m_p(L \cdot C_t) \geq 3\}.$$

Let $B \subset \mathbb{P}^2$ be the image of Γ under the projection $\Gamma \to \Phi \to \mathbb{P}^2$; that is, the curve traced out by flex points of members of the pencil.

Exercise 11.33. First, show that Γ is indeed smooth, by showing that the "universal flex"

$$\Sigma = \{(C, L, p) \mid m_p(C \cdot L) \geq 3\} \subset \mathbb{P}^N \times \Phi$$

(where $\mathbb{P}^N$ is the space parametrizing all degree-d plane curves) is smooth and invoking Bertini. Can you give explicit conditions on the pencil equivalent to the smoothness of Γ?

Exercise 11.34. If $p \in \mathbb{P}^2$ is a base point of the pencil, show that exactly three members of the pencil have a flex point at p, and that the curve B has an ordinary triple point at p.

Exercise 11.35. If a point $p \in \mathbb{P}^2$ is a node of the curve C_t containing it, the tangent lines to the two branches are each flex lines to C_t, so that the map $\Gamma \to B$ is two-to-one there. Show that the curve B has correspondingly a node at p.

Exercise 11.36. Finally, show that the triple points and nodes of B described in the preceding two exercises are the only singularities of B.

Exercise 11.37. Let C_t be an element of our pencil with a hyperflex (L, p). Show that the map $\Gamma \to \mathbb{P}^1$ is simply ramified at (t, L, p), and simply branched at t.

Exercise 11.38. Let C_t be an element of our pencil with a node p; let L_1 and L_2 be the tangent lines to the two branches of C_t at p. Show that $(t, p, L_i) \in \Gamma$, and that these are ramification points of weight 2 of the map $\Gamma \to \mathbb{P}^1$ (that is, each of the lines L_i is a limit of three flex lines of nearby smooth curves in our pencil, and these three are cyclically permuted by the monodromy in the family). Conclude that t is a branch point of multiplicity 4 for the cover $\Gamma \to \mathbb{P}^1$.

Exercise 11.39. Let $\{C_t\}$ be a general pencil of plane curves of degree d including a cuspidal curve C_0. (That is, let $C_0 = V(F)$ be a general cuspidal curve, $C_\infty = V(G)$ a general curve and $\{C_t = V(F + tG)\}$ the pencil they span.) As $t \to 0$, how many flexes of C_t approach the cusp of C_0? How about if C_0 has a tacnode?

The following series of exercises (Exercises 11.40–11.44) sketches a proof of Proposition 11.13.

Exercise 11.40. Suppose that $p \in C$ is an A_n-singularity for $n \geq 3$. Show that the blow-up $C' = \mathrm{Bl}_p C$ of C at p has a unique point q lying over p, and that $q \in C'$ is an A_{n-2}-singularity. Conclude in particular that the normalization $\widetilde{C} \to C$ of C at p has genus

$$p_a(\widetilde{C}) = p_a(C) - \left\lfloor \tfrac{1}{2}(n+1) \right\rfloor.$$

Exercise 11.41. Let S be a smooth surface and $C \subset S$ a curve with an A_{2n-1}-singularity at p.

(a) Show that there is a unique curvilinear subscheme $\Gamma \subset S$ of degree n supported at p such that a local defining equation of $C \subset S$ at p lies in the ideal $\mathcal{I}_\Gamma^2$.

(b) If $\widetilde{S} = \mathrm{Bl}_\Gamma S$ is the blow-up of S along Γ, show that the proper transform $\widetilde{C}$ of C in $\widetilde{S}$ is smooth over p and intersects the exceptional divisor E transversely twice at smooth points of $\widetilde{S}$.

(c) Conversely, show that if $D \subset \widetilde{S}$ is any such curve then the image of D in S has an A_{2n-1}-singularity at p.

Exercise 11.42. Prove the analog of Exercise 11.41 for A_{2n}-singularities. This is the same statement, except that in the second and third parts the phrase "intersects the exceptional divisor E transversely twice at smooth points of $\widetilde{S}$" should be replaced with "is simply tangent to the exceptional divisor E at a smooth point of $\widetilde{S}$ and does not meet E otherwise."

Exercise 11.43. Let $\mathcal{L}$ be a line bundle on a smooth surface S, and assume that for any curvilinear subscheme $\Gamma \subset S$ of degree n supported at a single point we have

$$H^1(\mathcal{L} \otimes \mathcal{I}_\Gamma^2) = 0.$$

Show that the locus $\Delta_k \subset \mathbb{P}H^0(\mathcal{L})$ of curves in the linear series $|\mathcal{L}|$ with an A_k-singularity is locally closed and irreducible of codimension k in $\mathbb{P}H^0(\mathcal{L})$ for all $k \leq 2n-2$.

Exercise 11.44. Deduce from the above exercises the statement of Proposition 11.13.

Exercise 11.45. Show that if $\mathcal{L}$ is the n-th power of a very ample line bundle, then the condition $H^1(\mathcal{L} \otimes \mathcal{I}_\Gamma^2) = 0$ is satisfied for any curvilinear subscheme $\Gamma \subset S$ of degree $n/2$ or less. Conclude in particular that if $\mathcal{D} \subset |\mathcal{L}|$ is a general net in the complete linear series $|\mathcal{L}|$ associated to the fourth or higher power of a very ample bundle then no curve $C \in \mathcal{D}$ has singularities other than nodes and ordinary cusps.

The following three exercises sketch out a calculation of the number of curves $C \subset S$ with a tacnode in a suitably general three-dimensional linear system. (Here, as in the case of cusps, when we use the term "tacnode" without the adjective "ordinary" we include as well singularities that are specializations of ordinary tacnodes, that is, A_n-singularities for any $n \geq 3$, triple points or points on multiple components.)

Exercise 11.46. Let S be a smooth surface and $\mathcal{L}$ a line bundle on S. Let $B = \mathbb{P}\mathcal{T}_S$ be the projectivization of the tangent bundle of S, which we may think of as a parameter space for subschemes $\Gamma \subset S$ of degree 2 supported at a single point. Construct a vector bundle $\mathcal{E}$ on B whose fiber at a point $\Gamma \in B$ may be naturally identified with the vector space

$$\mathcal{E}_\Gamma = H^0(\mathcal{L}/\mathcal{L} \otimes \mathcal{I}_\Gamma^2)$$

Exercise 11.47. In terms of the description of the Chow ring $A(B)$ of $B = \mathbb{P}\mathcal{T}_S$ given in Section 11.4.2, calculate the top Chern class of the bundle constructed in Exercise 11.46.

https://doi.org/10.1017/CBO9781139062046.013 Published online by Cambridge University Press

Exercise 11.48. Using the preceding two exercises, find an enumerative formula for the number of curves in a three-dimensional linear series $\mathcal{D} \subset |\mathcal{L}|$ that have a tacnode. If $S \subset \mathbb{P}^3$ is a smooth surface of degree d, apply this to find the expected number of plane sections with a tacnode. Check your answer by calculating the number directly in the cases $d = 2$ and 3.

The following three exercises describe a way of deriving the formula for the number of binodal curves in a net via linearization. We begin by introducing a smooth, projective compactification of the space of unordered pairs of points $p, q \in \mathbb{P}^2$: We set

$$\tilde{\Phi} = \{(L, p, q) \mid p, q \in L\} \subset \mathbb{P}^{2*} \times \mathbb{P}^2 \times \mathbb{P}^2,$$

and let Φ be the quotient of $\tilde{\Phi}$ by the involution $(L, p, q) \mapsto (L, q, p)$. To put it differently, Φ consists of pairs (L, D) with $L \subset \mathbb{P}^2$ a line and $D \subset L$ a subscheme of degree 2; or, differently still, Φ is the Hilbert scheme of subschemes of $\mathbb{P}^2$ with Hilbert polynomial 2. (Compare this with the description in Section 9.7.4 of the Hilbert scheme of conic curves in $\mathbb{P}^3$ — this is the same thing, one dimension lower.)

Exercise 11.49. Observe that the projection $\Phi \to \mathbb{P}^{2*}$ expresses Φ as a projective bundle over $\mathbb{P}^{2*}$, and use this to calculate its Chow ring.

Exercise 11.50. Viewing Φ as the Hilbert scheme of subschemes of $\mathbb{P}^2$ of dimension 0 and degree 2, construct a vector bundle $\mathcal{E}$ on Φ whose fiber at a point D is the space

$$\mathcal{E}_{(L,p,q)} = H^0(\mathcal{O}_{\mathbb{P}^2}(d)/\mathcal{I}_D^2(d)).$$

(What would go wrong if instead of using the Hilbert scheme Φ as our parameter space we used the Chow variety — that is, the symmetric square of $\mathbb{P}^2$?) Express the condition that a curve $C = V(F) \subset \mathbb{P}^2$ be singular at p and q in terms of the vanishing of an associated section σ_F of $\mathcal{E}$ on $\mathcal{H}$ at (L, p, q)

Exercise 11.51. Calculate the Chern classes of this bundle, and derive accordingly the formula for the number of binodal curves in a net.