Chapter 10

Segre classes and varieties of linear spaces

Keynote Questions

(a) Let $v_1, \ldots, v_{2n}$ be general tangent vector fields on $\mathbb{P}^n$. At how many points of $\mathbb{P}^n$ is there a nonzero cotangent vector annihilated by all the v_i? (Answer on page 366.)
(b) If f is a general polynomial of degree $d = 2m - 1$ in one variable over a field of characteristic 0, then there is a unique way to write f as a sum of m d-th powers of linear forms (Proposition 10.15). If f and g are general polynomials of degree $d = 2m$ in one variable, how many linear combinations of f and g are expressible as a sum of m d-th powers of linear forms? (Answer on page 377.)
(c) If $C \subset \mathbb{P}^4$ is a general rational curve of degree d, how many 3-secant lines does C have? (Answer on page 379.)
(d) If $C \subset \mathbb{P}^3$ is a general rational curve of degree d, what is the degree of the surface swept out by the 3-secant lines to C? (Answer on page 380.)

10.1 Segre classes

Our understanding of the Chow rings of projective bundles makes accessible the computation of the classes of another natural series of loci associated to a vector bundle.

We start with a naive question. Suppose that $\mathcal{E}$ is a vector bundle on a scheme X and that $\mathcal{E}$ is generated by global sections. How many global sections does it actually take to generate $\mathcal{E}$? More generally, what sort of locus is it where a given number of general global sections fail to generate $\mathcal{E}$ locally?

We can get a feeling for these questions as follows. First, consider the case where $\mathcal{E}$ is a line bundle. In this case, each regular section corresponds to a divisor of class $c_1(\mathcal{E})$. If $\mathcal{E}$ is generated by its global sections, the linear series of these divisors is base point free, so a general collection of i of them will intersect in a codimension-i locus of class

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$c_1(\mathcal{E})^i$. That is, the locus where i general sections of $\mathcal{E}$ fail to generate $\mathcal{E}$ has "expected" codimension i and class $c_1(\mathcal{E})^i$.

Now suppose that $\mathcal{E}$ has rank $r > 1$; again, suppose that it is generated by global sections. Choose r general sections, and let X' be the codimension-1 subset of $\mathcal{E}$ consisting of points p where the sections do not generate $\mathcal{E}$. One can hope that at a general point of X' the sections have only one degeneracy relation, so that on some open set $U \subset X'$ they generate a corank-1 subbundle of $\mathcal{E}' \subset \mathcal{E}$, and the quotient $\mathcal{E}/\mathcal{E}'$ is a line bundle on U. The sections of $\mathcal{E}$ yield sections of $\mathcal{E}/\mathcal{E}'$, so if it is a line bundle they will vanish in codimension 1 in U; that is, we should expect $r+1$ general sections of $\mathcal{E}$ to generate $\mathcal{E}$ away from a codimension-2 subset of X. Continuing in this way (and assuming that $r \geq i$), it seems that $r+i-1$ sections of $\mathcal{E}$ might generate $\mathcal{E}$ away from a codimension-i locus. In particular, $r + \dim X$ sections might generate $\mathcal{E}$ locally everywhere.

A case beloved by algebraists is that of $\mathcal{E} = \mathcal{O}_{\mathbb{P}V}(1)^r$. Here a collection of $r+i-1$ general sections is a general map

$$\mathcal{O}_{\mathbb{P}V}^{r+i-1} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}V}(1)^r,$$

that is, a general $r \times (r+i-1)$ matrix of linear forms. The locus where the sections fail to generate is the support of the cokernel, which is defined by the $r \times r$ minors of the matrix. By the generalized principal ideal theorem (Theorem 0.2), the codimension of this locus is at most i, and in fact equality holds (as we shall soon see) whenever $r+1 \geq i$. In fact, the support of the cokernel is exactly the scheme defined by the ideal of minors in this case (see Buchsbaum and Eisenbud [1977]).

It turns out that the construction of projective bundles gives us an effective way of reducing this question (and many others) about vector bundles to the case of line bundles, passing from $\mathcal{E}$ to the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on $\mathbb{P}\mathcal{E}$. To relate this line bundle to classes on X, we push forward its self-intersections:

Definition 10.1. Let X be a smooth projective variety, let $\mathcal{E}$ be a vector bundle of rank r on X and $\pi : \mathbb{P}\mathcal{E} \to X$ its projectivization, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. The *$i$-th Segre class* of $\mathcal{E}$ is the class

$$s_i(\mathcal{E}) = \pi_*(\zeta^{r-1+i}) \in A^i(X),$$

and the (total) *Segre class* of $\mathcal{E}$ is the sum

$$s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \cdots .$$

(For a more general definition of the Segre classes, see Fulton [1984, Chapter 4].)

The Segre classes give the answer to our question about generating vector bundles:

Proposition 10.2. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth variety X that is generated by global sections, and let $\tau_1, \ldots, \tau_{r+i-1}$ be general sections. If X_i is the scheme where $\tau_1, \ldots, \tau_{r+i-1}$ fail to generate $\mathcal{E}$, then X_i has pure codimension i and the class $[X_i]$ is equal to $(-1)^i s_i(\mathcal{E})$.*

We will prove here only the weaker statement that $(-1)^i s_i(\mathcal{E})$ is represented by a positive linear combination of the components of X_i; the stronger version is a special case of *Porteous' formula* (Theorem 12.4), which will be proved in full in Chapter 12.

The proposition shows an interesting parallel between the Chern classes and the Segre classes of a bundle:

- The i-th Chern class $c_i(\mathcal{E})$ is the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r-i+1} \to \mathcal{E}$$
fails to be injective.
- The i-th Segre class $s_i(\mathcal{E})$ is $(-1)^i$ times the locus of fibers where a suitably general bundle map
$$\mathcal{O}_X^{\oplus r+i-1} \to \mathcal{E}$$
fails to be surjective.

The Segre classes may seem to give a way of defining new cycle class invariants of a vector bundle, but in fact they are essentially a different way of packaging the information contained in the Chern classes. Postponing the proof of Proposition 10.2 for a moment, we explain the remarkable relationship:

Proposition 10.3. *The Segre and Chern classes of a bundle $\mathcal{E}$ on X are reciprocals of one another in the Chow ring of X:*
$$s(\mathcal{E})c(\mathcal{E}) = 1 \in A(X).$$

Using the formula $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$, we deduce that
$$s_i(\mathcal{E}^*) = (-1)^i s_i(\mathcal{E}).$$
Also, for any exact sequence $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ of vector bundles, the Whitney formula gives $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$, whence
$$s(\mathcal{F}) = s(\mathcal{E})s(\mathcal{G}).$$

Proof of Proposition 10.3: If $\mathcal{S}$ and $\mathcal{Q}$ are the tautological sub- and quotient bundles on $\mathbb{P}\mathcal{E}$ and $\zeta = c_1(\mathcal{S}^*)$ is the tautological class, then $c(\mathcal{S}) = 1 - \zeta$, so by the Whitney formula
$$c(\mathcal{Q}) = \frac{c(\pi^*\mathcal{E})}{c(\mathcal{S})} = c(\pi^*\mathcal{E})(1 + \zeta + \zeta^2 + \cdots) \in A(\mathbb{P}\mathcal{E}).$$

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We now push this equation forward to X. Considering first the left-hand side, we see that for $i < r-1$ the Chern class $c_i(\mathcal{Q})$ is represented by a cycle of dimension $> \dim X$, so it maps to 0, while the top Chern class $c_{r-1}(\mathcal{Q})$ maps to a multiple of the fundamental class of X — in fact, we saw in Lemma 9.7 that the multiple is 1. Thus $\pi_*(c(\mathcal{Q})) = 1 \in A(X)$. On the other hand, the push-pull formula tells us that

$$\begin{aligned}\pi_*(c(\pi^*\mathcal{E})(1+\zeta+\zeta^2+\cdots)) &= c(\mathcal{E})\cdot\pi_*(1+\zeta+\zeta^2+\cdots)\\ &= c(\mathcal{E})s(\mathcal{E}),\end{aligned}$$

completing the argument. □

For example, if $X = \mathbb{P}^n$ and $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^n}(1))^r$, then

$$s(\mathcal{E}) = \frac{1}{c(\mathcal{E})} = \frac{1}{(1+\zeta)^r} = 1 - r\zeta + \binom{r+1}{2}\zeta^2 - \binom{r+2}{3}\zeta^3 + \cdots.$$

Proof of Proposition 10.2: Let $V = H^0(\mathcal{E})$; suppose that $\dim V = n$. Since $\mathcal{E}$ is generated by global sections, we have a natural map $\varphi : X \to G(n-r, n)$ sending each point $p \in X$ to the kernel of the evaluation map $V \to \mathcal{E}_p$, that is, the subspace of sections of $\mathcal{E}$ vanishing at p. Via this map, $\mathcal{E}$ is the pullback $\varphi^*\mathcal{Q}$ of the universal quotient bundle on $G(n-r, V)$, and by Section 5.6.2 we have correspondingly

$$c_i(\mathcal{E}) = \varphi^*(c(\mathcal{Q})) = \varphi^*(\sigma_i).$$

In fact, we can see this directly: If $\tau_1, \ldots, \tau_{r-i+1} \in V$ are general sections of $\mathcal{E}$, the locus where they fail to be independent will be the preimage of the Schubert cycle $\Sigma_i(W)$, where $W \subset V$ is the span of $\tau_1, \ldots, \tau_{r-i+1}$, and, since the plane $W \subset V$ is general, by Kleiman transversality the class $[\varphi^{-1}(\Sigma_i(W))]$ of the preimage is the pullback of the class $[\Sigma_i(W)] = \sigma_i$.

In the same way, if $\tau_1, \ldots, \tau_{r+i-1} \in V$ are general sections of $\mathcal{E}$, the scheme X_i where they fail to span will be the preimage of the Schubert cycle $\Sigma_{1^i}(W) = \Sigma_{1,1,\ldots,1}(W)$, where $W \subset V$ is the span of $\tau_1, \ldots, \tau_{r+i-1}$; again, we can invoke Kleiman to deduce that the X_i have pure codimension i and that

$$[X_i] = \varphi^*(\sigma_{1^i}).$$

Finally, we saw in Corollary 4.10 that in the Chow ring of the Grassmannian we have

$$(1+\sigma_1+\sigma_2+\cdots)(1-\sigma_1+\sigma_{1,1}-\cdots) = 1,$$

and combining these we have

$$\sum(-1)^i[X_i] = \varphi^*\Big(\sum(-1)^i\sigma_{1^i}\Big) = \varphi^*\frac{1}{\sum\sigma_i} = \frac{1}{c(\mathcal{E})} = s(\mathcal{E}),$$

as desired. □

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We can now answer Keynote Question (a). A tangent vector field on $\mathbb{P}^n$ is a section of $\mathcal{T}_{\mathbb{P}^n} = \Omega^*_{\mathbb{P}^n}$, so the question can be rephrased as: At how many points of $\mathbb{P}^n$ do $2n$ general sections of $\mathcal{T}_{\mathbb{P}^n}$ fail to generate $\mathcal{T}_{\mathbb{P}^n}$? By Proposition 10.2, this is $(-1)^n$ times the degree of the Segre class $s_n(\mathcal{T}_{\mathbb{P}^n})$. By Proposition 10.3, $s(\mathcal{T}_{\mathbb{P}^n}) = 1/c(\mathcal{T}_{\mathbb{P}^n})$. And, as we have seen (in Section 5.7.1), $c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where ζ is the hyperplane class on $\mathbb{P}^n$. Putting this together,

$$s(\mathcal{T}_{\mathbb{P}^n}) = \frac{1}{(1+\zeta)^{n+1}} = 1 - (n+1)\zeta + \binom{n+2}{2}\zeta^2 + \cdots,$$

so the answer is $\binom{2n}{n}$.

10.2 Varieties swept out by linear spaces

We can use Segre classes to calculate the degrees of some interesting varieties "swept out" by linear spaces in the following sense. Let B be a smooth variety of dimension m and $\alpha : B \to G = \mathbb{G}(k,n)$ a map to the Grassmannian of k-planes in $\mathbb{P}^n$, and let

$$X = \bigcup_{b\in B} \Lambda_{\alpha(b)} \subset \mathbb{P}^n$$

be the union of the planes in $\mathbb{P}^n$ corresponding to the points of the image of B. Let $\mathcal{S}$ be the universal subbundle on G and

$$\Phi = \mathbb{P}\mathcal{S} = \{(\Lambda, p) \in G \times \mathbb{P}^n \mid p \in \Lambda\}$$

the universal k-plane. Form the fiber product

$$\Phi_B = B \times_G \Phi = \{(b, p) \in B \times \mathbb{P}^n \mid p \in \Lambda_{\alpha(b)}\},$$

with projection maps

$$B \xleftarrow{\pi} \Phi_B \xrightarrow{\eta} \mathbb{P}^n,$$

so that we can write

$$X = \eta(\Phi_B).$$

Since Φ_B is necessarily a variety of dimension $m+k$, we see from this that X will be a subvariety of $\mathbb{P}^n$ of dimension at most $m+k$. In case it has dimension equal to $m+k$ — that is, the map η is generically finite of some degree d, or in other words a general point of X lies on d of the planes $\Lambda_{\alpha(b)}$ — we will say that X is *swept out d times* by the planes $\Lambda_{\alpha(b)}$.

Assuming now that X has the "expected" dimension $m+k$, we can ask for its degree in $\mathbb{P}^n$. This can be conveniently expressed as the degree of a Segre class:

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Proposition 10.4. *Let $B \subset \mathbb{G}(k,n)$ be a smooth projective variety of dimension m, $\alpha : B \to G = \mathbb{G}(k,n)$ any morphism and $\mathcal{E} = \alpha^*\mathcal{S}$ the pullback of the universal subbundle on G. If*

$$X = \bigcup_{b \in B} \Lambda_{\alpha(b)} \subset \mathbb{P}^n$$

is swept out d times by the planes corresponding to points of B, then

$$\deg(X) = \deg(s_m(\mathcal{E}))/d.$$

Proof: If $L \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ is a homogeneous linear form on $\mathbb{P}^n$, then L defines a section of $\mathcal{E}^*$ by restriction to each fiber of $\mathcal{E} = \mathcal{S}_B$, and hence a section σ_L of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. The preimage $\eta_B^{-1}(H)$ of the hyperplane $H = V(L) \subset \mathbb{P}^n$ given by L is the zero locus of σ_L. Thus the pullback of the hyperplane class on $\mathbb{P}^n$ under the map η_B is the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ on $\mathbb{P}\mathcal{E}$, and it follows that $d \cdot \deg(X) = \deg \zeta^{m+k} = \deg s_m(\mathcal{E})$, as required. □

Alternatively, we could argue that the degree of X is the number of points of its intersection with a general $(n-m-k)$-plane $\Gamma \subset \mathbb{P}^n$; since the class of the cycle $\Sigma_m(\Gamma) \subset G$ of k-planes meeting Γ is the Schubert class $\sigma_m \in A^m(G)$, this is $1/d$ times the degree of the pullback $\alpha^*\sigma_m$. Thus we have

$$\begin{aligned} d \cdot \deg(X) &= \deg \alpha^*\sigma_m \\ &= \deg c_m(\alpha^*\mathcal{Q}) \\ &= \deg s_m(\mathcal{E}) \end{aligned}$$

since $s(\mathcal{S}) = 1/c(\mathcal{S}) = c(\mathcal{Q})$.

10.3 Secant varieties

The study of secant varieties to projective varieties $X \subset \mathbb{P}^n$ is a rich one, with a substantial history and many fundamental open problems. In this section, we will discuss some of the basic questions. In the following sections we will use Segre classes to compute the degrees of secant varieties to rational curves.

10.3.1 Symmetric powers

A k-secant m-plane to a variety X in $\mathbb{P}^n$ is a linear space $\Lambda \cong \mathbb{P}^m \subset \mathbb{P}^n$ of dimension m that meets X in k points, so it will be useful to introduce a classical construction of a variety whose points are (unordered) k-tuples of points of X: the *k-th symmetric power* $X^{(k)}$ of X.

Formally, we define $X^{(k)}$ to be the quotient of the ordinary k-fold product X^k by the action of the symmetric group on k letters $\mathfrak{S}_k$, acting on X^k by permuting the factors. If $X = \operatorname{Spec} A$ is any affine scheme, this means that

$$X^{(k)} := \operatorname{Spec}((A \otimes A \otimes \cdots \otimes A)^{\mathfrak{S}_k}).$$

When X is quasi-projective, $X^{(k)}$ is defined by patching together symmetric powers of affine open subsets of X. The main theorem of Galois theory shows that when X is a variety the extension of rational function fields $\Bbbk(X^k)/\Bbbk(X^{(k)})$ is Galois, and of degree $k!$.

One can show that such quotients are *categorical*: Any morphism $X^{(k)} \to Y$ determines an $\mathfrak{S}_k$-invariant morphism $X^k \to Y$, and this is a one-to-one correspondence. Further, the closed points of X correspond naturally to the effective 0-cycles on X: they are usually denoted additively by $p_1 + \cdots + p_k$, where the $p_i \in X$ need not be distinct. For these results, see Mumford [2008, Chapter 12].

Since the natural map $X^k \to X^{(k)}$ is finite, $X^{(k)}$ is affine or projective if and only if X is.

A familiar example is the case $X = \mathbb{A}^1$: Here, $X = \operatorname{Spec} \Bbbk[t]$, so

$$(\mathbb{A}^1)^{(k)} = \operatorname{Spec}(\Bbbk[t_1, \dots, t_k]^{\mathfrak{S}_k}).$$

This ring of invariants is a polynomial ring on the k elementary symmetric functions (see for example Eisenbud [1995, Section 1.3] for an algebraic proof), so $(\mathbb{A}^1)^{(k)} = \mathbb{A}^k$. Set-theoretically, this is the statement that a monic polynomial is determined by the set of its roots, counting multiplicity.

A similar result holds for $\mathbb{P}^1$. We could deduce it from the case of $\mathbb{A}^1$, but instead we give a geometric proof:

Proposition 10.5. $$(\mathbb{P}^1)^{(k)} \cong \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(k)) = \mathbb{P}^k.$$

Proof: We think of $\mathbb{P}^1$ as $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, the space of linear forms in 2 variables modulo scalars. The product of k linear forms is a form of degree k, which is independent of the order in which the product is taken. Thus multiplication defines a morphism $\varphi : (\mathbb{P}^1)^k \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(k))$ that is invariant under the group $\mathfrak{S}_k$. The morphism φ is finite and generically $k!$-to-one, so it has degree $k!$.

Since φ is invariant, it factors through a morphism $\psi : (\mathbb{P}^1)^{(k)} \to \mathbb{P}^k$, and, since the degree of the quotient map $(\mathbb{P}^1)^k \to (\mathbb{P}^1)^{(k)}$ is $k!$, we see that ψ is birational. Since $\mathbb{P}^k$ is normal and ψ is finite and birational, ψ is an isomorphism. $\square$

The construction of $X^{(k)}$ is most useful when X is a smooth curve. One reason is given by the following result:

Proposition 10.6. *If X is a variety and $k > 1$, then $X^{(k)}$ is smooth if and only if X is smooth and* $\dim X \leq 1$.

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Proof: If $\dim X = 0$, then X consists of a single reduced point, and $X^{(k)}$ is also a single reduced point. Thus we may assume that $\dim X > 0$.

Away from the subsets where at least two factors are equal, the quotient map $X^k \to X^{(k)}$ is an unramified covering. Thus if X is singular at a point p, and $p, q_1, \dots, q_{k-1} \in X$ are distinct points, then near $p + q_1 + \cdots + q_{k-1}$ the variety $X^{(k)}$ looks like the product X^k near $(p, q_1, \dots, q_{k-1})$; in particular, it is singular. Thus if X is singular then $X^{(k)}$ is singular.

Now suppose that X is smooth and of dimension ≥ 2. If $X^{(k)}$ were smooth as well, the quotient map $\pi : X^k \to X^{(k)}$ would be étale away from the diagonal in X^k, a locus of codimension at least 2. But the differential $d\pi : \mathcal{T}_{X^k} \to \pi^*\mathcal{T}_{X^{(k)}}$, being a map between vector bundles of equal rank, would necessarily be singular in codimension 1, a contradiction.

It remains to see that if X is a smooth curve then $X^{(k)}$ will be smooth. This in fact follows from the special case $X = \mathbb{P}^1$ described in Proposition 10.5: in the analytic topology, any collection of points p_i on any smooth curve X have neighborhoods isomorphic to open subsets of $\mathbb{P}^1$, and it follows that any point of $X^{(k)}$ has a neighborhood isomorphic to an open subset of $\mathbb{P}^k$. $\square$

The symmetric powers of a smooth curve C are central to the analysis of the geometry of C, as we will see illustrated in Appendix D. We can think of a point of $C^{(k)}$ as a subscheme $D \subset C$, and use notation such as $D \cup D'$ and $D \cap D'$ accordingly. In fact, $C^{(k)}$ is isomorphic to the Hilbert scheme of subschemes of C with constant Hilbert polynomial k — that is, zero-dimensional subschemes of degree k (see Arbarello et al. [1985] for a proof). When $\dim X > 1$ or X is singular, a point on $X^{(k)}$ does *not* in general determine a subscheme of X, and the Hilbert schemes $\mathcal{H}_k(X)$ are often more useful.

10.3.2 Secant varieties in general

In this subsection we will prove a basic result related to the dimension of secant varieties. Then we will state without proof some general results that may help to orient the reader. In the following two sections we will prove a number of results about the secant varieties of rational curves.

Let $X \subset \mathbb{P}^r$ be a projective variety of dimension n not contained in a hyperplane. Since $m \leq r + 1$ general points of X are linearly independent, for any $m \leq r$ we have a rational map

$$\tau : X^{(m)} \dashrightarrow \mathbb{G}(m-1, r),$$

called the *secant plane map*, sending a general m-tuple $p_1 + \cdots + p_m$ to the span $\overline{p_1, \dots, p_m} \cong \mathbb{P}^{m-1} \subset \mathbb{P}^r$. (In coordinates: If $p_i = (x_{i,0}, \dots, x_{i,r})$, then τ is given by the maximal minors of the matrix $(x_{i,j})$.) We define the *locus of secant* $(m-1)$*-planes* to X to be the image $\Psi_m(X) \subset \mathbb{G}(m-1, r)$ of the rational map τ — that is, the closure

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in $\mathbb{G}(m-1, r)$ of the locus of $(m-1)$-planes spanned by m linearly independent points of X. Finally, the variety

$$\operatorname{Sec}_m(X) = \bigcup_{\Lambda \in \Psi_m(X)} \Lambda \subset \mathbb{P}^r$$

is called the *m-th secant variety of X*.

Caution: If $\Lambda \in \Psi_m$ and $\Lambda \cap X$ is finite, then $\deg(\Lambda \cap X) \geq m$, but the converse is false; Exercise 10.24 suggests an example of this.

If $n > 1$ and $m > 1$ then the secant plane map $\tau : X^{(m)} \dashrightarrow \mathbb{G}(m-1, r)$ is never regular: When a point $p \in X$ on a variety of dimension 2 or more approaches another point $q \in X$, the limiting position of the secant line $\overline{p, q}$ necessarily depends on the direction of approach. (When X is a curve and q a smooth point of X, the limit is always the tangent line $\mathbb{T}_q X$.) This illustrates the point that — in this context, at least — the Hilbert scheme $\mathcal{H}_m(X)$ may be a better compactification of the space of unordered m-tuples of points on X than the symmetric power: When $m = 2$, for example, the map $\tilde{\tau} : \mathcal{H}_2(X) \to \mathbb{G}(1, r)$ sending a subscheme of length 2 to its span is always regular. Further, if we fix m and replace the embedding $X \subset \mathbb{P}^r$ by a sufficiently high Veronese re-embedding, then every length-m subscheme of X will span an $m-1$ plane, so the map $\mathcal{H}_m(X) \to \mathbb{G}(m-1, r)$ will be regular. In this chapter, we will care only about the image of τ, so it does not matter which we use.

We begin with the dimension of $\operatorname{Sec}_m(X)$:

Proposition 10.7. *If $m \leq r - n$, then the map τ is birational onto its image; in particular, $\Psi_m(X)$ has dimension* $\dim X^{(m)} = mn$.

This is slightly more subtle than it might at first appear. The first case would be the statement that if $C \subset \mathbb{P}^3$ is a nondegenerate curve then the line joining two general points of C does not meet C a third time. Though intuitively plausible, this is tricky to prove, and requires the hypothesis of characteristic 0. For the proof we will use the following general position result:

Lemma 10.8 (General position lemma)**.** *If $X \subset \mathbb{P}^r$ is a nondegenerate variety of dimension n and $\Gamma \cong \mathbb{P}^{r-n} \subset \mathbb{P}^r$ a general linear subspace of complementary dimension, then the points of $\Gamma \cap X$ are in linear general position; that is, any $r-n+1$ of them span Γ.*

We will not prove this here; a good reference is the discussion of the *uniform position lemma* in Arbarello et al. [1985, Section III.1]).

Proof of Proposition 10.7: The proposition amounts to the claim that if $p_1, \ldots, p_m \in X$ are general points, then the plane $\overline{p_1, \ldots, p_m}$ they span contains no other points of X.

To prove this, let $U \subset X^{(m)}$ be the open subset of m-tuples of distinct, linearly independent points, and consider the incidence correspondence

$$\Psi = \{(p_1 + \cdots + p_m, \Gamma) \in U \times \mathbb{G}(r-n, r) \mid p_1, \ldots, p_m \in \Gamma\}.$$

Via projection on the first factor, we see that Ψ is irreducible, and by Lemma 10.8 it dominates $\mathbb{G}(r-n, r)$; it follows that *a general $(r-n)$-plane Γ containing m general points $p_1, \ldots, p_m \in X$ is a general $(r-n)$-plane in $\mathbb{P}^r$*, and applying Lemma 10.8 again we deduce that the $(m-1)$-plane $\overline{p_1, \ldots, p_m}$ contains no other points of X. □

Let

$$\Phi = \{(\Lambda, p) \in \mathbb{G}(m-1, r) \times \mathbb{P}^r \mid p \in \Lambda\}$$

be the universal $(m-1)$-plane in $\mathbb{P}^r$, with projection maps

$$\begin{tikzcd}
\Phi \arrow[r, "\eta"] \arrow[d, "\pi"'] & \mathbb{P}^r \\
\mathbb{G}(m-1, r).
\end{tikzcd}$$

Set

$$\Phi_m(X) = \pi^{-1}(\Psi_m(X)) \quad \text{and} \quad \eta_X = \eta|_{\Phi_m(X)},$$

so that the m-th secant variety $\mathrm{Sec}_m(X)$ is the image of η_X. We will call $\Phi_m(X)$ the *abstract secant variety*.

Projection on the first factor shows $\Phi_m(X)$ is irreducible of dimension $mn + m - 1$, so that $\dim \mathrm{Sec}_m(X) \le mn + m - 1$, with equality holding when a general point on $\mathrm{Sec}_m(X)$ lies on only finitely many m-secant $(m-1)$-planes to X. By way of language, if $X \subset \mathbb{P}^r$ has dimension n we will call $\min(mn + m - 1, r)$ the *expected dimension* of the secant variety $\mathrm{Sec}_m(X)$; we will say that X is *m-defective* if $\dim \mathrm{Sec}_m(X) < \min(mn + m - 1, r)$, and *defective* if it is m-defective for some m.

Everyone's favorite example of a defective variety is the Veronese surface in $\mathbb{P}^5$:

Proposition 10.9. *The Veronese surface $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is 2-defective.*

In fact, the Veronese surface is the *only* 2-defective smooth projective surface! This much more difficult theorem was asserted, and partially proven, by Severi. The proof was completed by Moishezon in characteristic 0; see Dale [1985] for a modern treatment that works in all characteristics.

Proof: The Veronese surface may be realized as the locus where a symmetric 3×3 matrix

$$M = \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_3 & z_4 \\ z_2 & z_4 & z_5 \end{pmatrix}$$

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has rank 1. But if M has rank 1 at two points $p, q \in \mathbb{P}^5$, then M has rank at most 2 at any point of the form $\lambda p + \mu q$. Thus the determinant of M vanishes on the whole line spanned by p and q, so the cubic form $\det M$ vanishes on the secant locus $\mathrm{Sec}_2(X)$. Thus $\dim \mathrm{Sec}_2(X) \leq 5 - 1 = 4$, not $2 \times 2 + 1 = 5$. □

We can give a more geometric proof using a basic result introduced by Terracini [1911]:

Proposition 10.10 (Terracini's lemma)**.** *Let $X \subset \mathbb{P}^r$ be a variety and $p_1, \dots, p_m \in X$ linearly independent smooth points of X. If $p \in \Gamma = \overline{p_1, \dots, p_m}$ is any point in their span not in the span of any proper subset, then the image of the differential $d\eta_X$ at the point $(\Gamma, p) \in \Phi_m(X)$ is the span*

$$\operatorname{Im} d\eta_X = \overline{\mathbb{T}_{p_1} X, \dots, \mathbb{T}_{p_m} X}$$

of the tangent planes to X at the points p_i. In particular, if X has dimension n and $r \geq mn + m - 1$, then X is m-defective if and only if its tangent spaces at m general points are dependent.

For a proof, see Landsberg [2012].

We can use Terracini's lemma to see that the Veronese surface $X \subset \mathbb{P}^5$ is 2-defective as follows: A hyperplane $H \subset \mathbb{P}^5$ contains the tangent plane to X at a point p if and only if the curve $H \cap X$ is singular at p. Of course we can consider $H \cap X$ as a conic in $\mathbb{P}^2 \cong X$, and, from the definition of the Veronese surface, we see that every conic appears in this way. Now, two planes in $\mathbb{P}^5$ are dependent if and only if they are both contained in a hyperplane. Putting this together with Terracini's lemma, we see that to show that X is 2-defective we must show that given any two points in $\mathbb{P}^2$ there is a conic in $\mathbb{P}^2$ that is singular at both these points; of course, the double line passing through the points is such a conic.

We can also use Terracini's lemma to show that there are no defective curves:

Proposition 10.11. *If $C \subset \mathbb{P}^r$ is a nondegenerate reduced irreducible curve, then* $\dim \mathrm{Sec}_m(X) = \min(2m - 1, r)$ *for every m.*

Proof: By Terracini's lemma, it suffices to show that if $p_1, \dots, p_m \in C$ are general points then the tangent lines $\mathbb{T}_{p_i} C$ are linearly independent when $2m - 1 \leq r$ and span $\mathbb{P}^r$ when $2m - 1 \geq r$.

We have already seen in the proof of Theorem 7.13 that for a general point $p \in C$ the divisor $(r + 1)p$ spans $\mathbb{P}^r$ (that is, a general point $p \in C$ is not inflectionary); it follows that the divisor $2m \cdot p$ spans a $\mathbb{P}^{2m-1}$ when $2m \leq r + 1$ and spans $\mathbb{P}^r$ when $2m \geq r + 1$. By lower-semicontinuity of rank, it follows that for general $p_1, \dots, p_m$ the divisor $2p_1 + \cdots + 2p_m$ has span of the same dimension. □

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The general question of which nondegenerate varieties are defective is a fascinating one, with a long history. Perhaps because of Terracini's lemma, which relates the issue to the question of when multiples of general points impose independent conditions on polynomials (the *interpolation problem*), the case of Veronese embeddings of projective spaces has attracted a great deal of attention. The following is a result of Alexander and Hirschowitz [1995]. The proof was later simplified by Karen Chandler, and an exposition of this version, with a further simplification, can be found in Brambilla and Ottaviani [2008].

Theorem 10.12. *The defective Veronese varieties are the following:*

- $\nu_2(\mathbb{P}^n)$ *is 2-defective for any* n.
- $\nu_4(\mathbb{P}^2)$ *is 5-defective.*
- $\nu_4(\mathbb{P}^3)$ *is 9-defective.*
- $\nu_3(\mathbb{P}^4)$ *is 7-defective.*
- $\nu_4(\mathbb{P}^4)$ *is 14-defective.*

We will see in Exercises 10.26–10.29 that the Veronese varieties listed in the theorem are indeed defective (the hard part is the converse!). Note that by Terracini's lemma Theorem 10.12 implies (and indeed is equivalent to) the following corollary:

Corollary 10.13. *Let* $p_1, \dots, p_m$ *be general points in* $\mathbb{A}^n$, *and let* d *be any positive integer such that* $\binom{d+n}{n} \geq m(n+1)$. *There exists a polynomial* f *of degree* d *on* $\mathbb{A}^n$ *with specified values and derivatives at the points* p_i, *except in the cases* $d = 2$ *and* $(n, d, m) = (2, 4, 5), (3, 4, 9), (4, 3, 7)$ *and* $(4, 4, 14)$.

10.4 Secant varieties of rational normal curves

We turn now from secant varieties in general to the special case of rational curves. Every rational curve is the projection of a rational normal curve, and its secant varieties are correspondingly projections of the secant varieties of the rational normal curve, so we will focus initially on that case.

10.4.1 Secants to rational normal curves

We begin with the observation that finite sets of points on a rational normal curve are always "as independent as possible." (This property actually characterizes rational normal curves, as we invite the reader to show in Exercise 10.31.)

Lemma 10.14. *Let* $C \subset \mathbb{P}^d$ *be a rational normal curve. If* $D \subset C$ *is a divisor of degree* $m \leq d + 1$, *then* D *is not contained in any linear subspace of* $\mathbb{P}^d$ *of dimension* $< m - 1$.

Informally: Any finite subscheme $D \subset C$ of length $\leq d+1$ is linearly independent, in the sense that the map $H^0(\mathcal{O}_{\mathbb{P}^d}(1)) \to H^0(\mathcal{O}_D(1))$ is surjective. On an affine subset of $\mathbb{P}^1$ the parametrization of the rational normal curve looks like $t \mapsto (1, t, t^2, \ldots, t^d)$, so the independence of the images of any $d+1$ points $a_0, \ldots, a_d$ is given by the nonvanishing of the Vandermonde determinant

$$\det \begin{pmatrix} 1 & a_0 & \cdots & a_0^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_d & \cdots & a_d^d \end{pmatrix} = \prod_{i<j} (a_i - a_j).$$

Proof: If D were contained in a linear subspace L of dimension $n < m-1$, then adding $d-n-1$ general points to D we would arrive at a divisor $D' \subset C$ of degree $m+d-n-1$ contained in a hyperplane H. Since C is not contained in a hyperplane, the intersection $H \cap C$ is finite, and we deduce the contradiction $\deg C > d$. □

In Section 10.3.2 we described the secant map as a regular map on an open set, that is, as a rational map

$$\tau : C^{(m)} \dashrightarrow \mathbb{G}(m-1, d).$$

One consequence of Lemma 10.14 is that the secant plane map has a natural extension to an injective map of sets. It is not hard to show that τ is actually a morphism, and in fact an embedding. We can thus regard the restriction Φ_C to the image of τ of the universal $\mathbb{P}^{m-1}$-bundle over $\mathbb{G}(m-1, d)$ as a $\mathbb{P}^{m-1}$-bundle over $(\mathbb{P}^1)^{(m)} = \mathbb{P}^m$, and $\mathrm{Sec}_m(C)$ is the image of this bundle.

Proposition 10.15. *Let $C \subset \mathbb{P}^d$ be a rational normal curve. When $2m-1 \leq d$, the map $\eta_C : \Phi_m(C) \to \mathbb{P}^d$ is birational onto its image $\mathrm{Sec}_m(C)$; more precisely, it is one-to-one over the complement of $\mathrm{Sec}_{m-1}(C)$ in $\mathrm{Sec}_m(C)$.*

Proof: Suppose a point $p \in \mathbb{P}^d$ is the image of two different points of $\Phi_m(C)$, say $(\overline{D}, p)$ and $(\overline{D}', p)$. Let $k = \dim(\overline{D} \cap \overline{D}')$; note that $0 < k < m-1$. Since the span of $\overline{D}$ and $\overline{D}'$ has dimension $2m-2-k$, by Lemma 10.14 the union (as subschemes of C) of D and D' can have degree at most $2m-k-1$. It follows that the intersection $D \cap D'$ (again, as subschemes of C) has degree at least $k+1$. Thus $\overline{D} \cap \overline{D}'$ is a secant k-plane, and $p \in S_k(C) \subset S_{m-1}(C)$. □

Proposition 10.15 is not particularly remarkable in case $2m-1 < d$: all irreducible, nondegenerate curves $C \subset \mathbb{P}^d$ have the property that when $2m-1 < d$ a general point on the m-secant variety $\mathrm{Sec}_m(C)$ lies on a unique m-secant $(m-1)$-plane to C (see Exercises 10.33–10.35). In case $2m-1 = d$, however, it is striking. For example, the twisted cubic curve $C \subset \mathbb{P}^3$ is the *unique* nondegenerate space curve whose secant lines sweep out $\mathbb{P}^3$ only once (see Exercise 10.30).

10.4.2 Degrees of the secant varieties

Let $C \subset \mathbb{P}^d$ be a rational normal curve, and m any integer with $2m - 1 \le d$. Since the secant plane map $\tau : C^{(m)} \to \mathbb{G}(m-1, d)$ is regular, and $\Phi_C \to \mathrm{Sec}_m(C)$ is birational, it is reasonable to hope that we can answer enumerative questions about the geometry of the varieties $\mathrm{Sec}_m(C)$. We will do this in the remainder of this section and the next, starting with the calculation of the degree of $\mathrm{Sec}_m(C)$.

There are a few cases that we can do without any machinery; for example:

(a) $\mathrm{Sec}_1(C) = C$, so $\deg \mathrm{Sec}_1(C) = d$.

(b) The case $m = 2$ is not quite as trivial, but is readily done: The variety $\mathrm{Sec}_2(C)$ has dimension 3, so its degree is the number of points in which it intersects a general $(d-3)$-plane Λ. As we saw in Exercise 3.34, the projection of C from Λ to $\mathbb{P}^2$ will map C birationally onto a plane curve C_0 with nodes, and the points of $\Lambda \cap S_2(C)$ correspond to these nodes. Since C_0 has arithmetic genus $\binom{d-1}{2}$ and geometric genus 0, we conclude that $\deg \mathrm{Sec}_2(C) = \binom{d-1}{2}$.

(c) Finally, if d is odd and $m = (d+1)/2$, then the secant locus is all of $\mathbb{P}^d$, so $\deg \mathrm{Sec}_m(C) = 1$.

In order to go further, we use the Segre class technique of Proposition 10.4. To begin with, the map

$$\Phi_m(C) \xrightarrow{\pi} \Psi_m(C) = \tau((\mathbb{P}^1)^{(m)}) \cong (\mathbb{P}^1)^{(m)} \cong \mathbb{P}^m$$

has the form $\mathbb{P}\mathcal{H} \to \mathbb{P}^m$, where $\mathcal{H}$ is the pullback $\tau^*\mathcal{S}$ of the tautological subbundle on $\mathbb{G}(m-1, d)$ to $\mathbb{P}^m$.

In fact, we have already seen this bundle before, in Section 9.3.3! To see this, let $V = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. The rational normal curve lives naturally in $\mathbb{P}V^*$, as the locus C of linear functionals on V given by evaluation at a point $p \in \mathbb{P}^1$. The span of a divisor D of degree m on C is the space of linear functionals vanishing on those points, that is, the annihilator in V^* of the subspace $V_D = H^0(\mathcal{I}_{D,\mathbb{P}^1}(d))$. Thus the map $\tau : \mathbb{P}^m \to \mathbb{G}(m-1, d)$ sends $D \in \mathbb{P}^m$ to the subspace $\mathrm{Ann}(V_D) \subset V^*$, and it follows that the pullback $\tau^*\mathcal{S}$ is the dual $\mathcal{F}^*$ of the bundle $\mathcal{F}$ introduced in Section 9.3.3. We have from the results of that section that

$$c(\mathcal{F}) = \frac{1}{(1-\sigma)^{d-m+1}},$$

where $\sigma \in A^1(\mathbb{P}^m)$ is the hyperplane class. The Segre class is the inverse, and taking the dual we have

$$s(\mathcal{F}^*) = (1+\sigma)^{d-m+1}.$$

Finally, we can deduce:

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Theorem 10.16. *If $C \subset \mathbb{P}^d$ is a rational normal curve, then, for $2m-1 \leq d$,*

$$\deg \operatorname{Sec}_m(C) = \binom{d-m+1}{m}.$$

Note that in the case $d = 2m-1$, the calculation reaffirms the conclusion of Proposition 10.15 that the m-secant planes to C sweep out $\mathbb{P}^d$ exactly once.

10.4.3 Expression of a form as a sum of powers

We can now answer Keynote Question (b): If f and g are general polynomials of degree $d = 2m$ in one variable, how many linear combinations of f and g are expressible as a sum of m d-th powers of linear forms?

This question is related to secants of rational normal curves, because if we realize $\mathbb{P}^d$ as the projective space of forms of degree d on $\mathbb{P}^1$ then the curve of pure d-th powers is a rational normal curve — it is the image of the morphism

$$\mu : \mathbb{P}^1 \ni (s,t) \mapsto \left(s^d, ds^{d-1}t, \binom{d}{2}s^{d-2}t, \dots, t^d\right) \in \mathbb{P}^d.$$

(Note that we are relying here on the hypothesis of characteristic 0: If, for example, d is equal to the characteristic, then μ is a purely inseparable map whose image is a line!)

A point $p \in \mathbb{P}^d$ lies on the plane spanned by distinct points $q_1, \dots, q_m \in C$ if and only if the homogeneous coordinates of p can be expressed as a linear combination of the homogeneous coordinates of $q_1, \dots, q_m$. Thus a form of degree d is a linear combination of m d-th powers of linear forms if and only if the corresponding point in $\mathbb{P}^d$ lies in the union of the m-secant $(m-1)$-planes to $\mu(\mathbb{P}^1)$, and questions about the expression of a polynomial as a sum of powers become questions about the secants.

There is an important subtlety: It is *not* the case that every point of $\operatorname{Sec}_m(C)$ corresponds to a polynomial that is expressible as a sum of m d-th powers! For example, the tangent lines to C are contained in $\operatorname{Sec}_2(C)$. If $d \geq 3$, then no 2-plane in $\mathbb{P}^d$ meets the rational normal curve in four points, so a tangent line to C cannot meet any other secant line at a point off C. Thus the points on the tangent lines away from C are points of $\operatorname{Sec}_2(C)$ that cannot be expressed as the sum of two pure d-th powers.

(The points on the tangent lines do have an interesting characterization, however: At the point corresponding to the polynomial $f(t) = (t-\lambda)^d$, the tangent line is the set of linear combinations of f and $\partial f/\partial t$, or equivalently the set of polynomials that have $d-1$ roots equal to λ.)

By definition, $\operatorname{Sec}_m(C)$ contains an open set consisting of points on secant $(m-1)$-planes spanned by m distinct points of C. Further, by Proposition 10.15 a point in the open subset $\operatorname{Sec}_m(C) \setminus \operatorname{Sec}_{m-1}(C)$ of $\operatorname{Sec}_m(C)$ lies on the span of a unique divisor of degree m. Thus Theorem 10.16 yields the answer to Keynote Question (b), and even a generalization:

Corollary 10.17. *If $d \geq 2m-1$, then the number of linear combinations of $d-2m+2$ general forms of degree d that can be expressed as the sum of m pure d-th powers is* $\deg \operatorname{Sec}_m(C) = \binom{d-m+1}{m}$.

10.5 Special secant planes

For a curve $C \subset \mathbb{P}^r$ other than a rational normal curve, it is interesting to consider the subspaces that meet C in a dependent set of points; these are called *special secant planes*. Examples of this that we will investigate below include trisecant and quadrisecant lines to a curve $C \subset \mathbb{P}^3$, and trisecant lines to a curve $C \subset \mathbb{P}^4$.

We start, as usual, with the question of dimension: When would we expect a curve $C \subset \mathbb{P}^r$ to contain m points lying in a $\mathbb{P}^{m-1-k}$-plane? What would be the expected dimension of the locus $C_k^{(m)} \subset C^{(m)}$ of such m-tuples?

There are many ways to set this up. One would be to express the locus of such m-tuples as a determinantal variety: If the coordinates of points $p_1, \dots, p_m \in C$ are the rows of the matrix

$$M = \begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,0} & x_{m,1} & \dots & x_{m,r} \end{pmatrix},$$

then $C_k^{(m)}$ is just the locus where this matrix has rank $m-k$ or less. Now, in the space of $m \times (r+1)$ matrices, those of rank $m-k$ or less have codimension $k(r+1-m+k)$, so we would expect the locus $C_k^{(m)}$ of m-tuples spanning only a $\mathbb{P}^{m-1-k}$ to have dimension

$$m - k(r+1-m+k) = (k+1)(m-r-k) + r.$$

An alternative in the case C is a rational curve would be to express C as the projection $\pi_\Lambda : \widetilde{C} \to C$ of a rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ from a plane $\Lambda \cong \mathbb{P}^{d-r-1}$. The m-secant $(m-1-k)$-planes to C then correspond to the m-secant $(m-1)$-planes to $\widetilde{C}$ that intersect Λ in a $(k-1)$-plane, that is, the preimage under the secant plane map $\tau : C^{(m)} \to \mathbb{G}(m-1,d)$ of the Schubert cycle

$$\Sigma_{(r+1-m+k)^k}(\Lambda) = \{\Gamma \in \mathbb{G}(m-1,d) \mid \dim(\Gamma \cap \Lambda) \geq k-1\}.$$

This Schubert cycle has codimension $k(r+1-m+k)$, so again we would expect the preimage to have dimension $m - k(r+1-m+k)$.

The first three cases (with $k > 0$) are:

(a) trisecants to a curve $C \subset \mathbb{P}^3$ (that is, $r = 3$, $m = 3$ and $k = 1$), where we expect a one-parameter family;
(b) quadrisecants to a curve $C \subset \mathbb{P}^3$ (that is, $r = 3$, $m = 4$ and $k = 2$), where we expect finitely many; and

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(c) trisecants to a curve $C \subset \mathbb{P}^4$ (that is, $r = 4$, $m = 3$ and $k = 1$), where, again, we expect finitely many.

That our expectations for the dimensions of these loci are indeed the case for general rational curves is shown in Exercise 10.36, though it is *not* necessarily true of a general point on any component of the Hilbert scheme of curves of higher genus, as shown in Exercise 10.37.

In this section, we will show how to count the trisecants to a general rational curve in $\mathbb{P}^4$, answering Keynote Question (c), and we will determine the degree of the trisecant surface of a general rational curve in $\mathbb{P}^3$, answering Keynote Question (d). We leave the case of quadrisecants to a rational curve in $\mathbb{P}^3$ to Exercise 10.38 for now; it will also be a direct application of Porteous' formula in Section 12.4.4.

10.5.1 The class of the locus of secant planes

In case the curve C is rational, the answers to all of the above questions come directly from the answer to a question we have not yet addressed directly: If $C \subset \mathbb{P}^d$ is a rational normal curve, $\tau : C^{(m)} \cong \mathbb{P}^m \to \mathbb{G}(m-1, d)$ the secant plane map and $\Psi_m(C) \subset \mathbb{G}(m-1, d)$ the image of τ, *what is the class* $[\Psi_m(C)] \in A_m(\mathbb{G}(m-1, d))$*?*

We have all the tools to answer this question at hand: We know that the pullback $\tau^*\mathcal{S}^*$ of the dual of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(m-1, d)$ is the bundle $\mathcal{E}^*$ whose Chern classes we gave in Section 10.4.2. We know that $c_i(\mathcal{S}^*) = \sigma_{1^i}$, so this says that

$$\tau^*\sigma_{1^i} = \binom{d-m+i}{i}\zeta^i \in A^i(\mathbb{P}^m),$$

where ζ as usual is the hyperplane class in $C^{(m)} \cong \mathbb{P}^m$. Equivalently, since we also know that the Segre class of $\mathcal{S}$ is $s(\mathcal{S}) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{d-m}$, we have

$$\tau^*\sigma_i = \binom{d-m+1}{i}\zeta^i \in A^i(\mathbb{P}^m). \tag{10.1}$$

Since the classes σ_i generate the Chow ring of $\mathbb{G}(m-1, d)$ (and τ is an embedding), this determines the class of the image. We will use this idea to compute $[\Psi_m(C)]$ explicitly in the cases below. It is an interesting fact that the map $\tau : C^{(m)} \cong \mathbb{P}^m \to \mathbb{G}(m-1, d)$ composed with the Plücker embedding of the Grassmannian is the d-th Veronese map on $\mathbb{P}^m$; see Exercise 10.32

Trisecants to a rational curve in $\mathbb{P}^4$

How many trisecant lines does a general rational curve of degree d in $\mathbb{P}^4$ possess? We already know the answer in at least two cases. First, there are no trisecant lines to a rational normal curve in the case $d = 4$. If $d = 5$, Proposition 10.15 says that a general point $p \in \mathbb{P}^5$ lies on a unique 3-secant 2-plane to a rational normal curve $\widetilde{C} \subset \mathbb{P}^5$, and thus the projection of that curve from a general point has just one trisecant line.

The general case may be handled similarly to the case $d = 5$ above: We use the fact that a rational curve $C \subset \mathbb{P}^4$ of degree d is the projection of a rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ from a $(d-5)$-plane $\Lambda \subset \mathbb{P}^d$. The trisecant lines to C then correspond to 3-secant 2-planes to $\widetilde{C}$ of degree d that meet Λ. The trisecant lines to C correspond to the intersection of the Schubert cycle $\Sigma_3(\Lambda) \subset \mathbb{G}(2,d)$ of 2-planes meeting Λ with the cycle $\Psi_3(\widetilde{C}) \subset \mathbb{G}(2,d)$ of 3-secant 2-planes to $\widetilde{C}$.

This gives the answer to Keynote Question (c):

Proposition 10.18. *If $C \subset \mathbb{P}^4$ is a general rational curve of degree d, then C has $\binom{d-2}{3}$ trisecant lines.*

Proof: Since C is general, it is the projection of the rational normal curve $\widetilde{C}$ from a general $(d-5)$-plane Λ. The number of trisecant lines is the number of points in which $\Sigma_3(\Lambda)$ meets $\Psi_3(\widetilde{C})$. By Kleiman transversality, this is the degree of the intersection class $[\Psi_3(C)]\sigma_3$, or equivalently the degree of the pullback $\tau^*\sigma_3$; by the above, this is $\binom{d-2}{3}$. □

Trisecants to a rational curve in $\mathbb{P}^3$

We next turn to Keynote Question (d): If $C \subset \mathbb{P}^3$ is a general rational curve of degree d, what is the degree of the surface $S \subset \mathbb{P}^3$ swept out by the 3-secant lines to C?

Again we already know the answer in the simplest cases: 0 in the case $d = 3$ (a rational normal curve has no trisecants); and 2 in the case $d = 4$, since a smooth rational quartic is a curve of type $(1,3)$ on a quadric surface $Q \subset \mathbb{P}^3$, and the trisecants of C comprise one ruling of Q.

To set up the general case, let C be the projection $\pi_\Lambda(\widetilde{C})$ of a rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ from a general plane $\Lambda \cong \mathbb{P}^{d-4}$; let $L \subset \mathbb{P}^3$ be a general line, and let $\Gamma = \pi_\Lambda^{-1}(L) \subset \mathbb{P}^d$ be the corresponding $(d-2)$-plane containing Λ. The points of intersection of L with S correspond to 3-secant 2-planes to $\widetilde{C} \subset \mathbb{P}^d$ that

(a) meet Λ, and
(b) intersect Γ in a line.

These are the points of intersection of $\Psi_3(\widetilde{C})$ with the Schubert cycle $\Sigma_{2,1}(\Lambda, \Gamma)$. Kleiman transversality shows that the cardinality of this intersection is the degree of the pullback $\tau^*(\sigma_{2,1})$.

To evaluate this we express $\sigma_{2,1}$ as a polynomial in $\sigma_1, \sigma_2, \ldots$ and evaluate each term using (10.1). Giambelli's formula (Proposition 4.16) tells us that

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_1\sigma_2 - \sigma_3$$

(an equality we could readily derive by hand). By (10.1),

$$\deg \tau^*(\sigma_1\sigma_2) = \binom{d-2}{1}\binom{d-2}{2} \quad \text{and} \quad \deg \tau^*(\sigma_3) = \binom{d-2}{3}.$$

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Putting these things together, we have the answer to the question:

Proposition 10.19. *If $C \subset \mathbb{P}^3$ is a general rational curve of degree d, then the degree of the surface swept out by trisecant lines to C is*

$$\binom{d-2}{1}\binom{d-2}{2} - \binom{d-2}{3} = 2\binom{d-1}{3}.$$

10.5.2 Secants to curves of positive genus

It is instructive to ask whether we could extend the computations of Sections 10.4 and 10.5 to curves other than rational ones. There is one key problem: in treating rational curves, we made essential use of the fact that the space of effective divisors of degree m on $\mathbb{P}^1$ is the variety $\mathbb{P}^m$, whose Chow ring we know. But when C has positive genus, the Chow rings $A(C^{(k)})$ of symmetric powers of C are unknown.

In the case of $g = 1$ this is not an insurmountable problem; it is the content of Exercises 10.48 and 10.49. For genera $g \geq 2$, however, it typically necessitates the use of a coarser equivalence relation on cycles, such as homology rather than rational equivalence. Given this framework, however, it is indeed possible to extend the results of this chapter to curves of arbitrary genus; see (Arbarello et al. [1985, Chapter 8]), where there are explicit formulas generalizing all the above formulas to arbitrary genus.

10.6 Dual varieties and conormal varieties

We next turn to a remarkable property of projective varieties called *reflexivity*. A corollary of reflexivity is the deep fact that the dual of the dual of a variety is the variety itself. See Kleiman [1986] for a comprehensive account of the history of these matters (our account is based on that in Kleiman [1984]). We emphasize that the statements below are very much dependent on the hypothesis of characteristic 0; see the references above for the characteristic p case.

Let $X \subset \mathbb{P}^n$ be a subvariety of dimension k. If X is smooth, we define the *conormal variety* $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ to be the incidence correspondence

$$CX = \{(p, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} \mid p \in X \text{ and } \mathbb{T}_p X \subset H\}.$$

If X is singular, we define CX to be the closure in $\mathbb{P}^n \times \mathbb{P}^{n*}$ of the locus CX° of such pairs (p, H), where p is a smooth point of X. Whatever the dimension of X, the conormal variety CX will have dimension $n-1$: it is the closure of the locus CX°, which maps onto the smooth locus of X with fibers of dimension $n-k-1$. The *dual variety* $X^* \subset \mathbb{P}^{n*}$ of X is the image of CX under projection on the second factor.

In these terms, we can state the main theorem of this section:

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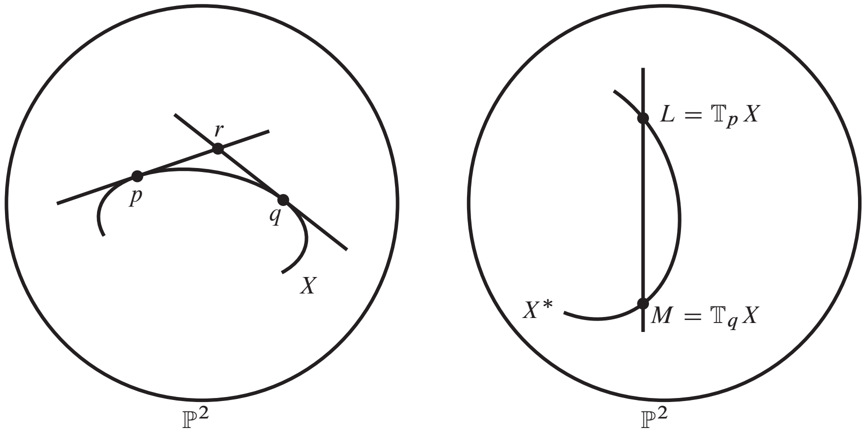

Figure 10.1 The tangent line to $X^* \subset \mathbb{P}^{2*}$ at $L = \mathbb{T}_p X$ is the line dual to p.

Theorem 10.20 (Reflexivity). *If $X \subset \mathbb{P}^n$ is any variety and $X^* \subset \mathbb{P}^{n*}$ its dual, then the conormal variety $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ is equal to $C(X^*) \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ with the factors reversed. It follows that $(X^*)^* = X$ — that is, the dual of the dual of X is X.*

For example, if X is a plane curve, then the statement $X^{**} = X$ says that if $p \in X$ is a smooth point then the tangent line to X^* at the point $L = \mathbb{T}_p X$ is the line $p^* \subset \mathbb{P}^{2*}$ of lines through p. More picturesquely put, the tangent lines to points near $x \in X$ "roll" on the point x. It is true more generally that the osculating k-planes to a smooth curve $X \subset \mathbb{P}^n$ at points near $p \in X$ move, to first order, by rotating around the osculating $(k-1)$-plane to X at p while staying in the osculating $(k+1)$-plane to X at p (see Exercise 10.47).

This picture, for plane curves, can be made precise as follows. Observe that if $p \in X$ is a smooth point, then the tangent line $\mathbb{T}_p X \subset \mathbb{P}^2$ is the limit of the secant lines $\overline{p, q}$ as $q \in X$ approaches p. Applied to the dual curve $X^* \subset \mathbb{P}^{2*}$, this says that the tangent line $\mathbb{T}_L X^* \subset \mathbb{P}^{2*}$ to the curve X^* at a point L is the limit of the secant lines $\overline{L, M}$ as $M \in X^*$ approaches L. But the line $\overline{L, M} \subset \mathbb{P}^{2*}$ joining two points $L, M \in \mathbb{P}^{2*}$ corresponding to lines $L, M \subset \mathbb{P}^2$ is the line in $\mathbb{P}^{2*}$ dual to the point $L \cap M$ in $\mathbb{P}^2$.

Now, the equality $X^{**} = X$ means that the tangent line to X^* at the point $L = \mathbb{T}_p X$ corresponding to $p \in X$ is p itself; this amounts to saying that the limit as $q \in X$ approaches p of the point of intersection $r = \mathbb{T}_p X \cap \mathbb{T}_q X$ is just the point p itself, which is clear from Figure 10.1.

Combining Theorem 10.20 with the argument at the beginning of Section 2.1.3, we see that the Gauss map of a smooth hypersurface is birational as well as finite:

Corollary 10.21. *If X is a hypersurface whose dual is also a hypersurface, then the Gauss map $\mathcal{G}_X : X \to X^*$ is birational, with inverse $\mathcal{G}_{X^*} : X^* \to X$. Thus if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d then the Gauss map is finite and birational, and X^* is a hypersurface of degree $d(d-1)^{n-1}$.*

This allows us, finally, to complete the proof of Proposition 2.9.

Proof of Corollary 10.21: By Section 2.1.3, the dual of a smooth hypersurface is always a hypersurface.

If X and X^* are both hypersurfaces then both $\mathcal{G}_X$ and $\mathcal{G}_{X^*}$ are well-defined rational maps. Since the graphs of $\mathcal{G}_X$ and $\mathcal{G}_{X^*}$ are equal after exchanging factors, the two rational maps are inverse to each other, and are thus birational. As already noted in Section 2.1.3, the degree computation follows from the birationality of $\mathcal{G}_X$. □

One aspect of Theorem 10.20 may seem puzzling. The only way the dual of a variety $X \subset \mathbb{P}^n$ can fail to be a hypersurface is if the map $CX \to X^*$ has positive-dimensional fibers — in other words, if every singular hyperplane section of X has positive-dimensional singular locus. This is a rare circumstance; as we will see in Exercise 10.42, it can never be the case for a smooth complete intersection, and, as we will see in Exercise 10.44, it can only happen for X swept out by positive-dimensional linear spaces. But if we have a one-to-one correspondence between varieties $X \subset \mathbb{P}^n$ and their dual varieties, we seem to be suggesting that there are as many hypersurfaces as varieties of arbitrary dimension in $\mathbb{P}^n$! The discrepancy is due to the fact that the duals of smooth varieties tend to be highly singular — see, for example, Exercise 10.45.

There are many fascinating results about the geometry of dual varieties and conormal varieties. We recommend in particular Kleiman [1986], and the surprising and beautiful theorems of Ein and Landman (see Ein [1986]) and Zak [1991]. Ein and Landman proved, for example, that for any smooth variety $X \subset \mathbb{P}^n$ of dimension d in characteristic 0 the difference $(n-1) - \dim X^*$ is congruent to $\dim X$ modulo 2!

As we mentioned earlier, it is relatively rare for the dual X^* of a smooth variety $X \subset \mathbb{P}^n$ to not be a hypersurface. Exercises 10.43 and 10.46 give two circumstances where it does happen.

10.6.1 The universal hyperplane as projectivized cotangent bundle

The proof of the reflexivity theorem will make use of the *universal hyperplane*

$$\Psi = \{(p, H) \in \mathbb{P}^n \times \mathbb{P}^{n*} \mid p \in H\},$$

a special case of the universal k-plane introduced in Section 3.2.3 and analyzed further in Section 9.3.1. To express it another way, let V be an $(n+1)$-dimensional vector space and $W = V^*$ its dual; we can then write

$$\Psi = \{(v, w) \in \mathbb{P}V \times \mathbb{P}W \mid w(v) = 0\}.$$

Write the tautological sequence on $\mathbb{P}V$ as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}V} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Thus $\mathcal{Q}^* \subset W \otimes \mathcal{O}_{\mathbb{P}V}$. From the inclusion it follows that the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{Q}^*}(-1)$ on $\mathbb{P}\mathcal{Q}^*$ is the restriction from $\mathbb{P}^n \times \mathbb{P}^{n*}$ of the bundle $\pi_2^*\mathcal{O}_{\mathbb{P}^{n*}}(-1)$. In Section 6.1.1 we observed that $\Psi \subset \mathbb{P}V \times \mathbb{P}W$ may be regarded as the projectivization $\mathbb{P}\mathcal{Q}^*$ inside $\mathbb{P}W \otimes \mathcal{O}_{\mathbb{P}V} = \mathbb{P}V \times \mathbb{P}W$.

To simplify notation, we write π_V and π_W for the projections from $Z := \mathbb{P}V \times \mathbb{P}W$ to $\mathbb{P}V$ and $\mathbb{P}W$ respectively, and we set $\mathcal{O}_Z(a, b) := \pi_V^*\mathcal{O}_{\mathbb{P}V}(a) \otimes \pi_W^*\mathcal{O}_{\mathbb{P}W}(b)$. In this language, $\Psi = \pi_V^*(\mathcal{Q}^*)$ and $\mathcal{O}_{\mathbb{P}\mathcal{Q}^*}(-1)$ is the restriction of $\mathcal{O}_Z(0, -1)$. For our present purpose, we want to give a more symmetric description.

Proposition 10.22. *The map $\pi_V : \Psi \to \mathbb{P}V$ may be described as the projectivization of the cotangent bundle $\mathbb{P}\Omega_{\mathbb{P}V}$ of $\mathbb{P}V$. The tautological subbundle*

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}V}}(-1) \subset \pi_V^*(\Omega_{\mathbb{P}V})$$

is the restriction to $\Psi \subset \mathbb{P}V \times \mathbb{P}W = Z$ of $\mathcal{O}_Z(-1, -1)$.

Proof: The Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}V} \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}V}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}V} \longrightarrow 0$$

that may be taken as the definition of $\Omega_{\mathbb{P}V}$ is the tautological sequence twisted by $\mathcal{O}_{\mathbb{P}V}(-1)$, and in particular $\Omega_{\mathbb{P}V} = \mathcal{Q}^* \otimes \mathcal{O}_{\mathbb{P}V}(-1)$. By Corollary 9.5, we have $\mathbb{P}\mathcal{Q}^* \cong \mathbb{P}\Omega$, with

$$\mathcal{O}_{\mathbb{P}\Omega}(-1) = \mathcal{O}_{\mathbb{P}\mathcal{Q}^*} \otimes \pi_V^*\mathcal{O}_{\mathbb{P}V}(-1),$$

and this is the restriction to Ψ of $\mathcal{O}_Z(0, -1) \otimes \pi_V^*\mathcal{O}_{\mathbb{P}V}(-1) = \mathcal{O}_Z(-1, -1)$, as required. $\square$

Proof of Theorem 10.20: If $X \subset \mathbb{P}V$ is any subvariety then, over the open set where X is smooth, the conormal variety $CX \subset \Psi = \mathbb{P}\Omega_{\mathbb{P}V} = \operatorname{Proj} \operatorname{Sym} \mathcal{T}_{\mathbb{P}V}$ is the projectivized conormal bundle $\mathbb{P}K = \operatorname{Proj} \operatorname{Sym} K^*$, where

$$K := \operatorname{Ker}(\Omega_{\mathbb{P}V}|_X \to \Omega_X).$$

The conormal variety of X itself is defined as the closure of this set. Over the open set where X is smooth, K^* is the cokernel of the map of bundles

$$\mathcal{T}_X \to \mathcal{T}_{\mathbb{P}V}|_X,$$

so $\operatorname{Sym} K^*$ is equal to $\operatorname{Sym} \mathcal{T}_{\mathbb{P}V}|_X$ modulo the ideal generated by $\mathcal{T}_X$, thought of as being contained in the degree-1 part $\mathcal{T}_{\mathbb{P}V}|_X$ of the graded algebra $\operatorname{Sym} \mathcal{T}_{\mathbb{P}V}|_X$. Sheafifying, this means that the ideal sheaf of CX in $\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}V}}$ is the image of the composite map u in the diagram

$$\begin{tikzcd} \pi_V^*\mathcal{T}_X \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(-1) \arrow[r] \arrow[dr, "u"'] & \pi_V^*\mathcal{T}_{\mathbb{P}^n} \otimes \mathcal{O}_{\Omega_{\mathbb{P}^n}}(-1) \arrow[d] \\ & \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}} \end{tikzcd}$$

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where the vertical map is the dual of

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum A_i\, dZ_i} \pi_V^*\Omega_{\mathbb{P}^n},$$

the tautological inclusion, tensored with $\mathcal{O}_{\Omega\mathbb{P}^n}(-1)$.

With these equations, we can tell whether a given subvariety C of $\Psi \cap \pi_V^{-1}(X)$ is a subset of CX. Let $\iota : C \to \Psi$ be the inclusion. Set

$$v = u^* \otimes \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) : \mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \to \pi_V^*\Omega_X,$$

and consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1)|_C \arrow[r, "\sum A_i\, dZ_i"] \arrow[dr, "v|_C"'] & \pi_V^*\Omega_{\mathbb{P}^n}|_C \arrow[r, "d\pi_V"] \arrow[d] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\
& \pi_V^*\Omega_X|_C \arrow[r, "d(\pi_1|_C)"] & \Omega_C
\end{tikzcd}$$

From what we have said about the equations of the conormal variety, we see that $C \subset CX$ if and only if $v|_C = 0$; since $d\pi_V|_C$ is generically injective, we see that $C \subset CX$ if and only if the composition $d\pi_V|_C \circ v|_C$ is zero.

We will show that this condition is symmetric, so that $C \subset CX$ if and only if $C \subset C(X^*)$ (with the factors reversed). Since Ψ is defined by a hypersurface of bidegree $(1, 1)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1) \xrightarrow{\varphi} \Omega_{\mathbb{P}^n \times \mathbb{P}W}|_\Psi \longrightarrow \Omega_\Psi \longrightarrow 0.$$

In coordinates, using the decomposition

$$\Omega_{\mathbb{P}^n \times \mathbb{P}W} = \pi_V^*(\Omega_{\mathbb{P}^n}) \oplus \pi_W^*(\Omega_{\mathbb{P}W})$$

this becomes

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1)|_\Psi \xrightarrow{\sum A_i\, dZ_i, \sum Z_i\, dA_i} \pi_V^*(\Omega_{\mathbb{P}^n})|_\Psi \oplus \pi_W^*(\Omega_{\mathbb{P}W})|_\Psi$$
$$\xrightarrow{(d\pi_V, d\pi_W)} \Omega_\Psi \longrightarrow 0.$$

Noting that $\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}W}(-1,-1)|_\Psi$, we see that if $\iota : C \to \Psi$ is the inclusion of any subvariety, then the composition

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum A_i\, dZ_i} \pi_V^*(\Omega_{\mathbb{P}^n}) \xrightarrow{d\pi_V} \Omega_\Psi$$

is the negative of the composition

$$\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1) \xrightarrow{\sum Z_i\, dA_i} \pi_W^*(\Omega_{\mathbb{P}W}) \xrightarrow{d\pi_W} \Omega_\Psi.$$

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It follows that the composition

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}^n}}(-1)|_C \arrow[r, "\sum A_i\,dZ_i"] & \pi_V^*\Omega_{\mathbb{P}^n}|_C \arrow[r, "d\pi_V"] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\
& & \Omega_C
\end{tikzcd}$$

is zero if and only if the composition

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}\Omega_{\mathbb{P}W}}(-1)|_C \arrow[r, "\sum Z_i\,dA_i"] & \pi_W^*\Omega_{\mathbb{P}W}|_C \arrow[r, dashed, "d\pi_W"] & \Omega_\Psi|_C \arrow[d, "d\iota"] \\
& & \Omega_C
\end{tikzcd}$$

is zero.

If $C \subset C(X^*)$ then the composite above is zero, and it follows that $C \subset CX$. If $C \subset \Psi$, then $C \subset CX$ if and only if $C \subset CX'$. Applying this argument to $C = CX$ and $C = C(X^*)$, we obtain the desired equality. □

10.7 Exercises

Exercise 10.23. Use the result of Exercise 9.36 (describing the class of the universal k-plane in $\mathbb{G}(k,r) \times \mathbb{P}^r$) to give an alternative proof of Proposition 10.4.

Exercise 10.24. (Improper secants) Let $X \subset \mathbb{P}^r$ be a variety, and $\Psi_m(X) \subset \mathbb{G}(m-1,r)$ the image of the secant plane map $\tau : X^{(m)} \dashrightarrow \mathbb{G}(m-1,r)$. Show by example that not every $(m-1)$-plane Λ such that $\deg(\Lambda \cap X) \geq m$ lies in $\Psi_m(X)$. (For example, try X a curve in $\mathbb{P}^5$ with a trisecant line, with $m = 3$.)

Exercise 10.25. Prove Proposition 10.7 in the case of a nondegenerate space curve $C \subset \mathbb{P}^3$ — that is, that the line joining two general points of C does not meet the curve a third time — without using the general position lemma (Lemma 10.8).

Exercises 10.26–10.29 verify that the Veronese varieties listed in Theorem 10.12 are indeed defective.

Exercise 10.26. Show that for $p, q \in \mathbb{P}^n$ the subspace $H^0(\mathcal{I}_p^2\mathcal{I}_q^2(2)) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(2))$ of quadrics singular at p and q has codimension $2n+1$ (rather than the expected $2n+2$). Deduce that any two tangent planes to the quadratic Veronese variety $\nu_2(\mathbb{P}^n)$ meet, and thus that $\nu_2(\mathbb{P}^n)$ is 2-defective for any n.

Exercise 10.27. Show that for any five points $p_1, \dots, p_5 \in \mathbb{P}^2$ there exists a quartic curve double at all five; deduce that the tangent planes $\mathbb{T}_{p_i} S$ to the quartic Veronese surface $S = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ are dependent (equivalently, fail to span $\mathbb{P}^{14}$), and hence that S is 5-defective.

Exercise 10.28. Show that for any nine points $p_1, \dots, p_9 \in \mathbb{P}^3$ there exists a quartic surface double at all nine; deduce that the tangent planes $\mathbb{T}_{p_i} X$ to the quartic Veronese threefold $X = \nu_4(\mathbb{P}^3) \subset \mathbb{P}^{34}$ are dependent (equivalently, fail to span $\mathbb{P}^{34}$), and hence that X is 9-defective.

Exercise 10.29. Finally, show that for any seven points $p_1, \dots, p_7 \in \mathbb{P}^4$ there exists a cubic threefold double at all seven; deduce that the tangent planes $\mathbb{T}_{p_i} X$ to the cubic Veronese fourfold $X = \nu_3(\mathbb{P}^4) \subset \mathbb{P}^{34}$ are dependent (equivalently, fail to span $\mathbb{P}^{34}$), and hence that X is 7-defective.
Hint: This problem is harder than the preceding three; you have to use the fact that through seven general points in $\mathbb{P}^4$ there passes a rational normal quartic curve.

The following exercises can be solved using the following fact, the *completeness of the adjoint series* for plane curves: if C is a nodal curve of degree d in $\mathbb{P}^2$, and $\widetilde{C}$ its normalization, then we obtain the entire canonical series $H^0(K_{\widetilde{C}})$ by pulling back polynomials of degree $d-3$ on $\mathbb{P}^2$ vanishing on the nodes of C (see Arbarello et al. [1985, Appendix A]).

Exercise 10.30. Show that the twisted cubic curve is the unique nondegenerate curve $C \subset \mathbb{P}^3$ such that a general point $p \in \mathbb{P}^3$ lies on a unique secant line to C. (Note: This can be done without it, but it is easy if you apply the *Castelnuovo bound* on the genus of a curve in $\mathbb{P}^3$; see Chapter 3 of Arbarello et al. [1985] for a statement and proof.)

Exercise 10.31. Show that the rational normal curve and the elliptic normal curve of degree $d+1$ are the only nondegenerate curves $C \subset \mathbb{P}^d$ with the property that every divisor of degree d on C spans a hyperplane.

Exercise 10.32. Let $C \subset \mathbb{P}^d$ be a rational normal curve. Show that the map $\tau : C^{(m)} \cong \mathbb{P}^m \to \mathbb{G}(m-1, d)$ sending a divisor of degree m on C to its span composed with the Plücker embedding of the Grassmannian is the d-th Veronese map on $\mathbb{P}^m$.
Hint: Show that the hypersurface in $\mathbb{P}^m$ associated to any monomial of degree d is the preimage of a hyperplane section of $\mathbb{G}(m-1, d)$

For the following three exercises, $C \subset \mathbb{P}^d$ will be an irreducible, nondegenerate curve and $2m - 1 < d$. The exercises will prove the assertion made in the text that a general point on the m-secant variety $\mathrm{Sec}_m(C)$ lies on a *unique* m-secant $(m-1)$-plane to C.

Exercise 10.33. Show by a dimension count that a general point of $\mathrm{Sec}_m(C)$ lies on only *proper* secants, that is, $m-1$ planes spanned by m distinct points of C.

Exercise 10.34. Using Lemma 10.8, show that the variety of $2m$-secant $(2m-2)$-planes to C (equivalently, the locus $C_1^{(2m)}$ of divisors of degree $2m$ on C contained in a $(2m-2)$-plane) has dimension at most $2m-2$.

Exercise 10.35. Now suppose that a general point of $\mathrm{Sec}_m(C)$ lay on two or more m-secant planes. Show that the dimension of the variety of $2m$-secant $(2m-2)$-planes to C would be at least $2m-1$.

Exercise 10.36. Show that if $C \subset \mathbb{P}^r$ is a general rational curve of degree d, and k is a number such that $d \geq r+k$ and $m-1 \geq k$, then the locus of m-secant $(m-k-1)$-planes has the expected dimension $m-k(r+1-m+k)$.

Exercise 10.37. By contrast with the last exercise, show that there exist components $\mathcal{H}$ of the Hilbert scheme of curves in $\mathbb{P}^3$ whose general point corresponds to a smooth, nondegenerate curve $C \subset \mathbb{P}^3$ with a positive-dimensional family of quadrisecant lines, or with a quintisecant line.

Exercise 10.38. Compute the number of quadrisecant lines to a general rational curve $C \subset \mathbb{P}^3$ of degree d.
Hint: In the notation of Section 10.5, the answer is the degree of the class $\deg \tau^*(\sigma_{2,2}) \in A^4(\mathbb{P}^4)$. Express the class $\sigma_{2,2}$ in terms of the special Schubert classes σ_i and use (10.1) to evaluate it.

Exercise 10.39. Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d, and let g be the genus of a general hyperplane section of S; let e and f be the degrees of the classes $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S) \in A^2(S)$. Find the class of the cycle $T_1(S) \subset \mathbb{G}(1,n)$ of lines tangent to S in terms of d, e, f and g. (Note: From Exercise 4.21, we need only the intersection number $\deg([T_1(S)] \cdot \sigma_3)$; do this using Segre classes.)

Exercise 10.40. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree n and genus g, and S and $T \subset \mathbb{P}^3$ two smooth surfaces containing C, of degrees d and e. At how many points of C are S and T tangent?

Exercise 10.41. Show the conclusion of Corollary 10.21 fails in characteristic $p > 0$:

(a) Let $\mathbb{k}$ be a field of characteristic 2, and consider the plane curve
$$C = V(X^2 - YZ) \subset \mathbb{P}^2.$$
Show that C is smooth, but the dual curve $C^* \subset \mathbb{P}^{2*}$ is a line, so that $C^{**} \neq C$.

(b) Now suppose that the ground field $\mathbb{k}$ has characteristic $p > 0$, set $q = p^e$ and consider the plane curve
$$C = V(YZ^q + Y^qZ - X^{q+1}) \subset \mathbb{P}^2.$$
Show that C is smooth, and that the double dual curve C^{**} is equal to C, but that $\mathcal{G}_C : C \to C^*$ is not birational!

Exercise 10.42. We saw in Section 2.1.3 that if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$ then the dual variety $X^* \subset \mathbb{P}^{n*}$ must again be a hypersurface. Show more generally that if $X \subset \mathbb{P}^n$ is any smooth complete intersection of hypersurfaces of degrees $d_i > 1$ then X^* will be a hypersurface.

Exercise 10.43. Let $X \subset \mathbb{P}^n$ be a k-dimensional *scroll*, that is, a variety given as the union

$$X = \bigcup \Lambda_b$$

of a one-parameter family of $(k-1)$-planes $\{\Lambda_b \cong \mathbb{P}^{k-1} \subset \mathbb{P}^n\}$; suppose that $k \geq 2$ (see Section 9.1.1).

(a) Show that if $H \subset \mathbb{P}^n$ is a general hyperplane containing the tangent plane $\mathbb{T}_p X$ to X at a smooth point p then the hyperplane section $H \cap X$ is reducible.
(b) Deduce that $\dim X^* \leq n - k + 2$ when $k \geq 3$.

Exercise 10.44. This is a sort of partial converse to Exercise 10.43 above. Let $X \subset \mathbb{P}^n$ be any variety. Use Theorem 10.20 to deduce that if the dual X^* is not a hypersurface, then X must be swept out by positive-dimensional linear spaces.

Exercise 10.45. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that the dual variety X^* is necessarily singular.

Exercise 10.46. Let $X = \mathbb{G}(1, 4) \subset \mathbb{P}^9$ be the Grassmannian of lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the dual of X is projectively equivalent to X itself!

Exercise 10.47. Let $X \subset \mathbb{P}^n$ be a smooth curve, and for any $k = 1, \ldots, n-1$ let

$$\nu_k : X \to \mathbb{G}(k, n)$$

be the map sending a point $p \in X$ to its osculating k-plane. Show that the tangent line to the curve $\nu_k(X) \subset \mathbb{G}(k, n)$ at $\nu_k(p)$ is the (tangent line to the) Schubert cycle of k-planes containing the osculating $(k-1)$-plane to X at p and contained in the osculating $(k+1)$-plane to X at p; in other words, to first order the osculating k-planes move by rotating around the osculating $(k-1)$-plane to X at p while staying in the osculating $(k+1)$-plane to X at p.

Exercise 10.48. If E is a smooth elliptic curve (over an algebraically closed field this means a curve of genus 1 with a chosen point), the addition law on E expresses the k-th symmetric power E_k as a $\mathbb{P}^{k-1}$-bundle over E. Verify this, and use it to give a description of $A(E_k)$.

Exercise 10.49. Using the preceding exercise, find the degrees of the secant varieties of an elliptic normal curve $E \subset \mathbb{P}^d$.