Lecture 6

1 ## § Extension of complete valued fields

Theorem 6.1: Let $(K, |\cdot|)$ be a complete non-arch discretely valued field and L/K a finite extension of degree n

(i) $|\cdot|$ extends uniquely to an abs. value $|\cdot|_L$ on L defined by
$$|y|_L = |N_{L/K}(y)|^{1/n} \quad \forall y \in L.$$

(ii) L is complete w.r.t. $|\cdot|_L$.

Recall: If L/K finite, $N_{L/K} : L \to K$ is defined by $N_{L/K}(y) = \mathrm{Det}_K(\mathrm{mult}(y))$, where $\mathrm{mult}(y) : L \to L$ is the K-linear map induced by multiplication by y.

Fact: • $N_{L/K}(xy) = N_{L/K}(x) N_{L/K}(y)$.

• Let $X^n + a_{n-1}X^{n-1} + \dots + a_0 \in K[X]$ be minimal polynomial of $y \in L$. Then $N_{L/K}(y) = \pm a_0^m$, $m \geq 1$.

$N_{L/K}(x) = 0 \iff x = 0$.

Definition 6.2: Let $(K, |\cdot|)$ be a non-arch. valued field, V a vector space over K. A

2 norm on V is a function $\|\cdot\| : V \to \mathbb{R}$...

satisfying:

(i) $\|x\| = 0$ iff $x = 0$

(ii) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall \lambda \in K, x \in V.$

(iii) $\|x + y\| \leq \max(\|x\|, \|y\|) \ \forall x, y \in V.$

Example: If V finite dimensional and $e_1, \dots, e_n$ is a basis of V. The sup norm $\| \ \|$ on V is defined by

$$\|x\|_{sup} = \max_i |x_i|$$

where $x = \sum_{i=1}^{n} x_i e_i$

Exercise: $\| \ \|_{sup}$ is a norm.

Definition 6.3: Two norms $\| \cdot \|_1, \| \ \|_2$ on V are equivalent if $\exists C, D > 0$ s.t.

$$C\|x\|_1 \leq \|x\|_2 \leq D\|x\|_1 \quad \forall x \in V.$$

Fact: A norm defines a topology on V, and equivalent norms induce the same topology.

Proposition 6.4: Let $(K, | \ |)$ be complete non-arch., V a finite dim. vector space over K. Then V is complete w.r.t. $\| \cdot \|_{sup}$.

Proof: Let $(v_i)_{i=1}^{\infty}$ a Cauchy sequence in V, $e_1, \dots, e_n$ a basis for V.

Write $v_i = \sum_{j=1}^{n} x_j^i e_j$; then $(x_j^i)_{i=1}^{\infty}$ is a Cauchy sequence in K. Let $x_j^i \to x_j \in K$, then $v_i \to v = \sum_{j=1}^{n} x_j e_j$. □

Theorem 6.5: Let $(K, |\cdot|)$ be complete non-arch. and V a finite dim. v.s. over K. Then any two norms on V are equivalent. In particular V is complete w.r.t. any norm.

Proof: Since equivalence defines an equiv. relation on set of norms, suffices to show any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{sup}$.

Let $e_1, \dots, e_n$ basis for V,

4 Set $D := \max_i \|e_i\|$

Then for $x = \sum_{i=1}^{n} x_i e_i$, we have

$$\|x\| \leq \max_i \|x_i e_i\| = \max_i |x_i| \|e_i\|$$
$$\leq D \max_i |x_i|$$
$$= D \|x\|_{sup}$$

To find C s.t. $C\|\cdot\|_{sup} \leq \|\cdot\|$, we induct on $n = \dim V$.

$n = 1$: $\|x\| = \|x_1 e_1\| = \overset{\|x\|_{sup}}{|x_1|} \|e_1\|$

so take $C = \|e_1\|$

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$n > 1$: Set $V_i = \mathrm{Span}\langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n \rangle$

By induction, V_i is complete w.r.t. $\|\cdot\|$, hence closed.

Then $e_i + V_i$ is closed $\forall i$, and hence $S := \bigcup_{i=1}^{n} e_i + V_i$ is a closed subset not containing 0.

Thus $\exists\, C > 0$ s.t.

$$B(0, C) \cap S = \emptyset$$

where $B(0, C) = \{ x \in V \mid \|x\| < C \}$

Let $x = \sum_{i=1}^{n} x_i e_i$ and suppose $|x_j| = \max_i |x_i|$

Then $\|x\|_{sup} = |x_j|$, and $\frac{1}{x_j} x \in S$

Thus $\| \frac{1}{x_j} \cdot x \| \geq C$

$\Rightarrow \|x\| \geq C |x_j| = C \|x_j\|_{sup}$.

The completeness of V follows since V is complete w.r.t. $\|\cdot\|_{sup}$. $\square$

Proof of Theorem 6.1: We show $|\cdot|_L = |N_{L/K}(-)|^{1/n}$ satisfies the three axioms in definition of abs. values.

(i) $|y|_L = 0 \Leftrightarrow |N_{L/K}(y)|^{1/n} = 0$

$\Leftrightarrow N_{L/K}(y) = 0$

$\iff N_{L/K}(y) = 0$

$\iff y = 0$ (Property of $N_{L/K}$)

(ii) $|y_1 y_2|_L = |N_{L/K}(y_1 y_2)|^{1/n}$

$= |N_{L/K}(y_1) N_{L/K}(y_2)|^{1/n}$

$= |N_{L/K}(y_1)|^{1/n} |N_{L/K}(y_2)|^{1/n}$

6 $= |y_1|_L |y_2|_L$.

For (iii) need preparation.

Definition 6.6: • Let $R \subseteq S$ be rings. We say $s \in S$ is integral over R if there exists a monic polynomial $f(X) \in R[X]$ s.t. $f(s) = 0$.

• The integral closure $R^{int(S)}$ of R inside S is defined to be

$$R^{int(S)} = \{ s \in S \mid s \text{ integral over } R \}$$

• We say R is integrally closed in S if $R^{int(S)} = R$

Proposition 6.7: $R^{int(S)}$ is a subring of S. Moreover $R^{int(S)}$ is integrally closed in S.

Proof: Example sheet 2.

Lemma 6.8: Let $(K, |\cdot|)$ non-arch. valued field. Then $\mathcal{O}_K$ is integrally closed in K.

Proof: Let $x \in K$ be integral over $\mathcal{O}_K$, w.l.o.g. $x \neq 0$. Let $f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in \mathcal{O}_K[X]$

s.t. $f(x) = 0$. Then

$$x = -a_{n-1}\frac{1}{x} - \dots - a_0\frac{1}{x^{n-1}}.$$

If $|x| > 1$, we have $\left|-a_{n-1}\frac{1}{x} - \dots - a_0\frac{1}{x^{n-1}}\right| < 1$;

Thus $|x| \leq 1 \Rightarrow x \in \mathcal{O}_K$ $\square$

(iii) Set $\mathcal{O}_L = \{y \in L \mid |y|_L \leq 1\}$

Claim: $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ inside L.

Assuming this we prove (iii)

, let $x, y \in L$.

w.l.o.g. assume $|x|_L \leq |y|_L$,

then $\left|\frac{x}{y}\right|_L \leq 1. \Rightarrow \frac{x}{y} \in \mathcal{O}_L$.

Since $1 \in \mathcal{O}_L$ and $\mathcal{O}_K^{\mathrm{int}(L)}$, we have

$1 + \frac{x}{y} \in \mathcal{O}_L$ and hence $\left|1 + \frac{x}{y}\right|_L \leq 1$.

$\Rightarrow |x+y|_L \leq |y|_L = \max(|y|_L, |x|_L)$

Thus (iii) is satisfied.