Commun. Math. Phys. 338, 483–500 (2015)
Digital Object Identifier (DOI) 10.1007/s00220-015-2287-y

Communications in
**Mathematical
Physics**

Quantum Curves

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Received: 25 August 2014 / Accepted: 25 August 2014
Published online: 17 February 2015 –

Abstract: One says that a pair (P, Q) of ordinary differential operators specify a quantum curve if $[P, Q] = \hbar$. If a pair of difference operators (K, L) obey the relation $KL = qLK$, where $q = e^{\hbar}$, we say that they specify a discrete quantum curve.

This terminology is prompted by well known results about commuting differential and difference operators, relating pairs of such operators with pairs of meromorphic functions on algebraic curves obeying some conditions.

The goal of this paper is to study the moduli spaces of quantum curves. We will relate the moduli spaces for different $\hbar$. We will show how to quantize a pair of commuting differential or difference operators (i.e., to construct the corresponding quantum curve or discrete quantum curve).

1. Introduction

One says that a pair (P, Q) of ordinary differential operators specify a quantum curve if $[P, Q] = \hbar$ [4,15,18]. If a pair of difference operators (K, L) obey the relation $KL = \lambda LK$, where $\lambda = e^{\hbar}$, we say that they specify a discrete quantum curve (we can also impose an additional condition that L^{-1} is a difference operator in the definition of quantum curve).

This terminology is prompted by well known results about commuting differential and difference operators [12,17], relating pairs of such operators with pairs of meromorphic functions on algebraic curves obeying some conditions.

The goal of this paper is to study the moduli spaces of quantum curves. We will relate the moduli spaces for different $\hbar$. We will show how to quantize a pair of commuting differential or difference operators (i.e., to construct the corresponding quantum curve or discrete quantum curve). This construction generalizes the considerations of [18].

The KP-hierarchy acts on the moduli space of quantum curves; we prove that similarly the discrete KP-hierarchy acts on the moduli space of discrete quantum curves.

We consider also matrix differential and difference operators and obtain similar results. (The generalization of [18] to matrix differential operators was given in [11].)

Eynard–Orantin topological recursion [6] gives a construction of free energy and correlation functions corresponding to an algebraic curve and two meromorphic functions on it. We construct a quantum curve starting with the same data (but the conditions that we impose on meromorphic functions are different). It seems that the paper [8] can be considered as a bridge between topological recursion and our constructions. Another way to relate topological recursion to quantum curves in our sense can be based on the comparison of Virasoro constraints that can be derived in both situations. The modification of topological recursion that was required to "remodel B-model" [3] should be related to discrete quantum curves in our sense.

We will discuss the relation of our constructions to the results of [4].

2. Differential Operators: Quantum Curves

Let us define a pseudodifferential operator as a formal series

$$L = \sum a_k(x) D^k$$

where $D = \frac{d}{dx}$ and $a_k(x)$ stands for a formal power series: $a_k(x) = \sum a_{kl} x^l$. We assume that $k \in \mathbb{Z}$ and $a_k(x) = 0$ for $k >> 0$. The operator has order q if its leading term (the non-zero term with greatest k) is equal to $a_q(x) D^q$; the operator is monic if $a_q(x) = 1$, a monic operator is normalized if $a_{q-1}(x) = 0$. Monic pseudodifferential operators of order 0 form a group denoted by $\mathcal{G}$.[1]

We denote by $\mathcal{H}$ the space of Laurent series $\sum_{k<<\infty} c_k z^k$, by $\mathcal{H}_+$ its subspace consisting of polynomials and by $\mathcal{H}_-$ the subspace of Laurent series obeying $c_n = 0$ for $n \geq 0$. Pseudodifferential operators act on $\mathcal{H}$; the differentiation D acts as multiplication by z and multiplication by x acts as $-\frac{d}{dz}$. Differential operators can be characterized as pseudodifferential operators preserving $\mathcal{H}_+$. Every pseudodifferential operator L can be represented as a sum of differential operator $L_+ = \sum_{k \geq 0} a_k(x) D^k$ and "integral" operator $L_- = \sum_{k<0} a_k(x) D^k$.

We will denote by Gr the space of all subspaces $V \subset \mathcal{H}$ such that the natural projection $\pi_+ : \mathcal{H} \to \mathcal{H}_+$ induces an isomorphism between V and $\mathcal{H}_+$. In other words the subspace $V \in Gr$ if it has a basis of the form $v_n = z^n + r_n$ where $n \geq 0$ and $r_n \in \mathcal{H}_-$. The space Gr is called Sato Grassmannian.[2]

The following theorems belong to Sato (see [16] for the proof):

Theorem 2.1. *There exists one-to-one correspondence between the elements of the group $\mathcal{G}$ of monic zeroth order differential operators and points of Gr. Namely, every subspace $V \in Gr$ has a unique representation in the form $V = S\mathcal{H}_+$ where $S \in \mathcal{G}$.*

The commutative Lie algebra γ_+ of polynomials $\sum_{k \geq 0} t_i z^i$ acts on Gr in natural way. (This action comes from the remark that we can multiply the elements of $\mathcal{H}$ by $g(t) = \exp(\sum_{k \geq 0} t_i z^i)$ where t_i are nilpotent parameters.)[3]

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[1] Pseudodifferential operators constitute an associative algebra $\mathcal{A}$ for appropriate definition of multiplication. We will not give this definition; see, for example, [16]. Let us notice, however, that the multiplicative structure in $\mathcal{A}$ can be recovered from the representation of $\mathcal{A}$ by the operators in $\mathcal{H}$ that is described in the next paragraph.

[2] It is the big cell of the index zero part of the infinite-dimensional Grassmannian; we do not need the description of this Grassmannian (see, for example, [16]).

[3] More generally, one can consider the action of the Lie algebra γ of polynomials $\sum_{-\infty<<k<<\infty} t_i z^i$; this action will be important in the next section.

It is clear from Theorem 2.1 that γ_+ acfs also on $\mathcal{G}$:

$$\frac{\partial S}{\partial t_n} = (SD^n S^{-1})_- S. \tag{1}$$

Theorem 2.2. *Every normalized pseudodifferential operator Q of order q can be represented in the form $S^{-1}D^q S$ where $S \in \mathcal{G}$; this representation is unique up to multiplication by an operator with constant coefficients.*

Using this statement we can construct the action of Lie algebra γ_+ on the space of normalized pseudodifferential operators of order q differentiating the relation $Q(t) = S^{-1}(t)D^q S(t)$ with respect to t_n.

The action on this space can be written in the form of differential equation

$$\frac{\partial Q}{\partial t_n} = [Q_+^{\frac{n}{q}}, Q] \tag{2}$$

Notice that this formula determines also the action of Lie algebra γ_+ on the space of normalized differential operators.[4]

All actions we described can be considered as different forms of KP-hierarchy.

We would like to solve the equation $[P, Q] = \hbar$. We assume that P is a differential operator of order p and Q is a normalized differential operator of order q. Using Theorem 2.2 we construct the operator $S \in \mathcal{G}$ such that $SQS^{-1} = D^q$. Introducing the notation $V = S\mathcal{H}_+$ we obtain a subspace $V \in Gr$ invariant with respect to multiplication by z^q and with respect to the action of the operator $\tilde{P} = \hbar\frac{d}{dz^q} + b(z)$ where $b(z)$ stands for the multiplication by a Laurent series denoted by the same letter. (We use the fact that the action of D^q can be interpreted as multiplication by z^q and the fact that $\mathcal{H}_+$ is invariant with respect to the action of differential operators. The form of the operator $\tilde{P} = SPS^{-1}$ follows from the relation $[\tilde{P}, \tilde{Q}] = \hbar$ where $\tilde{Q} = SQS^{-1} = z^q$.) We can invert this consideration to obtain the following statement [18]:

Theorem 2.3. *If $V \in Gr$ is invariant with respect to the operator of multiplication by z^q and with respect to the operator $\hbar\frac{d}{dz^q} + b(z)$ we can construct a differential operator P and a normalized differential operator Q obeying $[P, Q] = \hbar$. The leading term of the operator P is determined by the leading term of the Laurent series $b(z)$.*

The construction is based on the Sato theorem: We represent V in the form $V = S\mathcal{H}_+$ where $S \in \mathcal{G}$ and transform the operators acting on V by means of the operator S. We obtain pseudodifferential operators acting in $\mathcal{H}_+$, i.e. differential operators.

Notice that the points of Grassmannian that are invariant with respect to multiplication by z^q and with respect to the action of the operator of the form $\hbar\frac{d}{dz^q} + b(z)$ appeared for first time in the study of partition function of 2D-gravity in the paper [10]; they were studied later in numerous papers (sometimes these points are called string points of the Grassmannian; the operators preserving them are called Kac–Schwarz operators). A point $V \in Gr$ that is invariant with respect to multiplication by z^q and with respect to the action of the operator $A = \hbar\frac{d}{dz^q} + b(z)$ is invariant also with respect to the operators $z^{nq}A$ where $q = 0, 1, 2, \ldots$; it was shown in [10] that this implies Virasoro constraints on the state corresponding to V. (Recall that for every point $V \in Gr$ one can construct a state Ψ_V in fermionic Fock space and a bosonic state represented by tau-function. If V is invariant with respect to the operator C having matrix elements c_{mn} in the basis

[4] The fractional powers entering this formula can be defined using the representation $Q = S^{-1}D^q S$.

$z^n \in \mathcal{H}$) then Ψ_V is an eigenvector of the operator $\sum c_{mn} : \psi_m \psi_n^+ :$ where ψ_m are operators obeying canonical anticommutation relations; see [10]. This remark allows us to derive the Virasoro constraints for fermionic state and for tau-function.)

It follows from Theorem 2.3 that the Lie algebra γ_+ acts on the space of pairs (P, Q) of differential operators obeying $[P, Q] = \hbar$. (We assume that Q is normalized.) The proof is based on the remark that for $V \in Gr$ satisfying the conditions of the theorem the subspace $V(t) = g(t)V$ also obeys the same conditions with $\tilde{P} = \hbar \frac{d}{dz^q} + b(z)$ replaced with

$$g^{-1}(t)\tilde{P}g(t) = \hbar \frac{\mathrm{d}}{\mathrm{d}z^q} + b(z) - \sum \frac{i}{q} t_i z^{i-q}.$$

In other words, we can say that the KP-flows (2) are defined on the space of pairs we are interested in.

We say that the vectors $v_0, \ldots, v_{q-1}$ form a g-basis in V if the vectors $g^m v_i$ where $0 \le i < q, 0 \le m$, form a basis of V. We will use this definition in the case $g = z^q$.

To construct an example of z^q - basis we recall that $V \in Gr$ has a basis $v_n = z^n + r_n$ where $r_n \in \mathcal{H}_-$. The first q vectors of this basis (vectors $v_0, \ldots, v_{q-1}$) form a z^q-basis of V.

If $SQS^{-1} = D^q$, $V = S\mathcal{H}_+$, $v_0, \ldots, v_{q-1}$ is a z^q-basis of V we represent the operator $\tilde{P} = SPS^{-1}$ in this basis. We obtain

$$\tilde{P} v_i = M_i^j (z^q) v_j \tag{3}$$

where the entries of the matrix M are polynomials with respect to z^q. We say that the matrix M is the companion matrix of the pair (P, Q). (Alternatively one can define the companion matrix as a matrix of P in a Q-basis of $\mathcal{H}_+$.) The companion matrix depends on the choice of z^q-basis. Starting with a z^q-basis v_i we can construct a new z^q-basis by the formula $v_i' = g_i^j(z^q)v_j$ where $g_i^j(z^q)$ is an invertible polynomial matrix (its entries are polynomials with respect to z^q). The transformation rule for the matrix M by the change of z^q-basis has the form of gauge transformation : $M' = gMg^{-1} + \hbar \frac{dg}{dz^q} g^{-1}$. This means that M specifies a connection. *We will choose the basis v_i in such a way that $v_i = c_i z^i$+lower order terms*, $c_i \neq 0$. (Notice that this condition does not specify the vectors v_i uniquely; we can replace v_i with $\tilde{v}_i = t_i^j v_j$ where t is a constant invertible triangular matrix: $t_i^j = 0$ for $i \le j$.)

Let us prove the following theorem:

Theorem 2.4. *If the leading coefficient (the coefficient of the leading term) of the matrix $z^{j-i} M_i^j(z^q)$ has q distinct eigenvalues then there exist q pairs of differential operators obeying $[P, Q] = \hbar$ and having M as the companion matrix.*

To prove this theorem we should find q Laurent series $b(z)$ and corresponding $v_0, \ldots, v_{q-1}$ in such a way that

$$(\hbar \frac{d}{dz^q} + b(z))v_i = M_i^j(z^q)v_j \tag{4}$$

and $z^{mq} v_i$ where $0 \le i < q, 0 \le m$ form a basis of a subspace $V \in Gr$. Then we can apply Theorem 2.3.

We change the variables in the Eq. (4) substituting $v_i = z^i u_i$ where $u_i = c_i$+lower order terms. We obtain the equation

$$(\hbar \frac{d}{dz^q} + b(z))u_i = B_i^j(z)u_j \tag{5}$$

where

$$B(z) = \left(B_i^j(z)\right) = (M_i^j(z^q)z^{j-i} - \frac{i\hbar}{qz^q}\delta_i^j\right)$$

Let us consider first the case when $\hbar = 0$. Then $b(z)$ is one of eigenvalues $\lambda_k(z)$ of the matrix $B_i^j(z)$ and u_i are components of the eigenvector. The existence of b and of the vector $u(z) = (u_0, \ldots, u_{q-1})$ obeying the conditions we need follows immediately from the perturbation theory. (If B is replaced by its leading term this statement follows from our assumptions. All other terms of B can be considered as a perturbation of the leading term. The leading term of the matrix $B_i^j(z)$ coincides with the leading term of the matrix $z^{j-i}M_i^j(z^q)$, therefore its eigenvalues are distinct. This allows us to construct b as a Laurent series and $u(z)$ as a power series with respect to z^{-1}.)

If $\hbar \neq 0$ we consider the auxiliary equation

$$\hbar\frac{d}{dz^q}w_i = B_i^j(z)w_j. \tag{6}$$

or, equivalently,

$$\hbar\frac{d}{dz}w_i = qz^{q-1}B_i^j(z)w_j. \tag{7}$$

As we have noticed the eigenvalues of the leading term of $B(z)$ are distinct. This allows us to say that the equation can be diagonalized by means of the formal change of variables $w(z) = R(z)t(z)$ where $R(z) = 1 + \sum_{k\geq 1} R_k z^{-k}$; see [19]. This means that the equation for the components of the vector $t(z)$ looks as follows:

$$\hbar\frac{d}{dz^q}t_i = \Lambda_i t_i. \tag{8}$$

Let us consider q solutions of the Eq. (11) having the form

$$t_k = \exp(\int \hbar^{-1}\Lambda_k(z)qz^{q-1}dz),$$

$t_i = 0$ for $i \neq k$.

The corresponding solutions of the Eq. (6) have the form

$$w_i = \exp(\int \hbar^{-1}\Lambda_k(z)qz^{q-1}dz)r_{ik}(z),$$

where $r_{ik}(z) = c_{ik}+$ lower order terms. Now for every k it is easy to find $b(z)$ in such a way that $r_{ik}(z)$ becomes a solution to the Eq. (5). Namely, we should take

$$b_k(z) = \Lambda_k(z).$$

(We use the fact that the Eq. (5) can be reduced to the Eq. (6) by means of the substitution $w = \rho u$.) This gives the proof of the theorem.

A shorter proof can be given in the following way. Notice that the formal change of variables $w(z) = R(z)t(z)$ transforms (6) into equation

$$\hbar\frac{\mathrm{d}}{\mathrm{d}z^q}t_i = \Lambda_i^j t_j. \tag{9}$$

where

$$\Lambda_i^j(z) = S_i^r B_r^m R_m^j - \hbar S_i^m \frac{dR_m^J}{dz^q}, \tag{10}$$

and S denotes the matrix inverse to the matrix R.

We choose R in such a way that the matrix Λ is a diagonal matrix with entries Λ_i. Then it follows from (10) that for every r the series $u_i(z) = R_i^r(z)$ is a solution of (5) with $b(z) = \Lambda_r(z)$.

Notice that $\Lambda_k(z) = \lambda_k(z) + O(\hbar)$ where $\lambda_k(z)$ stands for the eigenvalue of the matrix $B_i^j(z)$. (This follows, for example, from the comparison with the case $\hbar = 0$).

Let us give another proof of the theorem that can be applied in more general situations. We would like to find solutions of the Eq. (5) as power series:

$$u(z) = \sum_{k \geq 0} {}^k u z^{-k},$$
$$b(z) = \sum_{k \geq 0} {}^k b z^{p-k}.$$

We introduce the notation

$$B = \sum_{k \geq 0} {}^k B z^{p-k}.$$

Here $u(z)$ and the corresponding coefficients ${}^k u$ are considered as k-dimensional vectors; $B(z)$ and ${}^k B$ are $q \times q$ dimensional matrices. We can solve the Eq. (5). The recursion formula looks as follows:

$$({}^0 b - {}^0 B)({}^k u) = ({}^k B - {}^k b)({}^0 u) + \text{known terms}. \tag{11}$$

In particular,

$$({}^0 b - {}^0 B)({}^0 u) = 0,$$

i.e. ${}^0 b$ is an eigenvalue of ${}^0 B$. It follows from our assumptions that all eigenvalues of ${}^0 B$ (that coincides with the leading coefficient of $z^{j-i} M_i^j(z^q)$) are simple. This means that the image of the operator ${}^0 b -{}^0 B$ has codimension 1. We denote by ρ a non-zero linear functional vanishing on this image. (It can be interpreted as an eigenvector the matrix transposed to ${}^0 B$ with eigenvalue ${}^0 b$.) Applying ρ to both parts of recursion formula and noticing that $\rho({}^0 u) \neq 0$ we can calculate ${}^k b$. Then the recursion formula gives us ${}^k u$. Noticing that we can take any eigenvalue of ${}^0 B$ as ${}^0 b$ we obtain the proof of the theorem.

The group C_q of qth roots of unity acts on solutions of the Eq. (6) (if $(u_0(z), \ldots, u_{q-1}(z))$ is a solution and $\epsilon^q = 1$ then $(u_0(\epsilon z), \ldots, \epsilon^{-i} u_i(\epsilon z), \ldots)$ is again a solution). One can use this fact to check that the group C_q acts also on the set of solutions constructed above. If p and q are coprime then this action is transitive therefore for appropriate labeling $\Lambda_{k+1}(z) = \Lambda_k(\epsilon z)$. (Here ϵ stands for a primitive root of unity.) Similarly, we can assume that $\lambda_{k+1}(z) = \lambda_k(\epsilon z)$. From this equation one can derive the properties of eigenvalues of the leading term of the matrix $B_i^j(z)$. If this leading term has degree p then the leading terms of eigenvalues have the same degree: $\lambda_k(z) = \alpha_k z^p + \cdots$ and we obtain that $\alpha_{k+1} = \epsilon^p \alpha_k$, hence $\alpha_k = \epsilon^{kp} \alpha_1$. Therefore in the case when p and q are coprime the numbers α_k are distinct.

Let us consider pairs (P, Q) where P is a differential operator of order p, Q is a normalized differential operator of order q, the orders p and q are coprime and $[P, Q] = \hbar$. It follows from the above considerations that the order of $b(z)$ is equal to p, therefore the leading coefficient of the matrix $B_i^j(z)$ (coinciding with the leading coefficient of the matrix $z^{j-i}M_i^j(z^q)$) has distinct eigenvalues. This allows us to describe the moduli space of such pairs and to prove that this moduli space does not depend on $\hbar$ (see [18]). We can formulate this description in the following way.

Let us say that the polynomial $q \times q$ matrix $M_i^j(z^q)$ is regular if the leading coefficient of the matrix $z^{j-i}M_i^j(z^q)$) has q distinct eigenvalues. If this leading coefficient has degree p we say that a regular matrix M belongs to the space $\mathcal{M}_{p,q}$. We say that a solution of the equation $[P, Q] = \hbar$ where P is a differential operator of order p, Q is a normalized differential operator of order q is regular if the companion matrix M is regular. It follows from the above consideration that the moduli space of regular solutions does not depend on $\hbar$. It can be considered as a q-fold covering of the space $\mathcal{M}_{p,q}/\mathcal{T}$ where $\mathcal{T}$ denotes the group of triangular matrices. If p and q are coprime all solutions of the equation $[P, Q] = \hbar$ are regular.

In the above statements we have used the choice of z^q-basis specified by the condition $v_i = c_i z^i$+lower order terms. We say that a companion matrix in general z^q-basis is regular, if the corresponding matrix in the preferred basis is regular. The condition of regularity for other choices of z^q-basis is not so simple. However, one can give a necessary condition of regularity that is valid in any basis. Namely, we should consider the characteristic polynomial

$$\det(M - \lambda \cdot 1) = \sum A_i(z^q)\lambda^i.$$

Let us suppose that $\det M = A_0(z^q)$ is a polynomial of degree p with respect to z^q. If M is regular then the degree of A_i with respect to z^q is less or equal than $[\frac{p(q-i)}{q}]$. This follows from the remark that the asymptotic behavior of the characteristic polynomial of matrix M for large z does not depend on the choice of basis.

The results we have obtained for scalar differential operators can be generalized to the case of matrix differential operators. Instead of Grassmannian Gr we should consider the vector Sato Grassmannian Gr_s. It consists of subspaces V of $\mathcal{H}^s = \mathcal{H} \otimes \mathbb{C}^s$ (of direct sum of s copies of $\mathcal{H}$) such that the natural projection of V to $\mathcal{H}_+^s = \mathcal{H}_+ \otimes \mathbb{C}^s$ is an isomorphism.

It is easy to generalize Theorems 2.1 and 2.3 to this case; see [14], Th. 6.2, [11], Prop 2.1.

Theorem 2.5. *If $V \in Gr_s$ is invariant with respect to the operator of multiplication by z^q and with respect to the operator $\tilde{P} = \hbar\frac{d}{dz^q} + b(z)$ we can construct a matrix differential operator P and a normalized matrix differential operator Q of order q obeying $[P, Q] = \hbar$. Here $b(z)$ is a Laurent series having $s \times s$ matrices as coefficients; if its leading term has degree p then the operator P is of order p.*

Notice that as in the scalar case one can derive the Virasoro constraints on the fermionic state Ψ_V and on the corresponding tau-function.

Again we can define the companion matrix of the pair (P, Q) as a matrix of $\tilde{P}$ in a z^q-basis of V. It is convenient to use a z^q-basis obeying

$v_{i\alpha} = c_{i\alpha} z^i e_\alpha$+lower order terms.

Here $0 \le i < q$,$1 \le \alpha \le s$, e_α denotes the standard basis of $\mathbb{C}^s$ and $c_{i\alpha}$ are non-vanishing constants.

The companion matrix in this basis can be regarded as a $q \times q$ matrix with with entries that are $s \times s$ matrices depending polynomially on z^q. We denote this matrix as M_i^j. It is defined by the Eq. (4) where v_i denotes now a q-dimensional vector having s-dimensional vectors as components. Introducing the notation $u_{i\alpha} = z^{-i} v_{i\alpha}$ we obtain the Eq. (5) where the matrix B is defined by the same formula as in scalar case.

We will prove the following theorem:

Theorem 2.6. *Let us suppose that the entries of the leading coefficient of the matrix $z^{j-i} M_i^j(z^q)$ are scalar matrices. In other words we assume that ${}^0B$ (the leading coefficient of the matrix B) has the form ${}^0B = \sigma \otimes I_s$ where σ is a $q \times q$ matrix with complex entries and I_s stands for unit $s \times s$ matrix; we assume that σ has q distinct eigenvalues. Then there exist q pairs of matrix differential operators obeying $[P, Q] = \hbar$ and having M as the companion matrix.*

We derive this statement from Theorem 2.5 generalizing the second proof of Theorem 2.4. It is sufficient to check that the Eq. (5) has q solutions obeying

$u_{i\alpha} = c_{i\alpha}$+lower order terms.

Then we can define V as a subspace spanned by $z^{qm} v_{i\alpha}$ where $v_{i\alpha} = z^i u_{i\alpha}$.

The recursion formula (11) for the solution of (5) can be written in more detail in the form

$$({}^0b_\alpha^\beta \delta_i^j - {}^0B_{i\alpha}^{j\beta})({}^k u_{j\beta}) = ({}^k B_{i\alpha}^{j\beta} - {}^k b_\alpha^\beta \delta_i^j)({}^0 u_{j\beta}) + \text{known terms.}$$

In particular,

$$({}^0b_\alpha^\beta \delta_i^j - {}^0B_{i\alpha}^{j\beta})({}^0 u_{j\beta}) = 0.$$

Recall that ${}^0B_{i\alpha}^{j\beta} = \sigma_i^j \delta_\alpha^\beta$; if $\sigma_i^j s_j = \lambda s_i$ (i.e. s is an eigenvector of σ) we can take ${}^0u_{i\alpha} = s_i$. Let us denote by ρ the eigenvector of the transposed matrix σ having eigenvalue λ; it follows from our assumptions that $< \rho, s > \neq 0$. The inner product of LHS of the recursion relation with ρ vanishes; this allows us to calculate ${}^k b_\alpha^\beta$. Then the recursion formula gives us ${}^k u_{i\alpha}$.

We have assumed in Theorem 2.6 that σ has q distinct eigenvalues. This condition is satisfied if the degree p of the leading term of B is coprime with q; the proof is the same as in scalar case.

3. Difference Operators: Discrete Quantum Curves

We would like to solve the equation $KL = qLK$ where K and L are difference operators. Our consideration will be based on the notion of pseudodifference operator.

Let us consider linear operators acting on the space of Laurent series $\mathcal{H}$. We say that a (doubly infinite) sequence $a = (a_k)$ specifies a diagonal operator transforming a sequence c_k into a sequence $a_k c_k$ (or equivalently a series $\sum_{-\infty<k<<\infty} c_k z^k$ into the series $\sum_{-\infty<k<<\infty} a_k c_k z^k$); we denote this operator by the same letter a. The shift operator Λ transforms a sequence c_k into the sequence c_{k-1} (equivalently the series $c(z) = \sum_{-\infty<k<<\infty} c_k z^k$ goes to the series $zc(z) = \sum_{-\infty<k<<\infty} c_{k-1} z^k$)). In other words Λ can be interpreted as multiplication by z.

Notice, that one can consider k as a continuous parameter $k \in \mathbb{R}$. Then we can modify the definition of the shift operator considering the operator $\Lambda_\hbar$ that transforms $c(k)$ into

$c(k-\hbar)$. Of course, for fixed $\hbar$ this makes no difference. However, in applications we should consider $\hbar$ as a small parameter and work with power series with respect to $\hbar$.

One defines pseudodifference operators by the formula

$$L = \sum_{-\infty < s << \infty} a(s)\Lambda^s$$

where $a(s)$ are diagonal operators. If the sum is finite then L is a difference operator. Restricting the summation to negative s we obtain the operator L_-. Taking the sum over $s \geq 0$ we obtain the operator L_+; notice that L_+ is a difference operator.

An operator of the form

$$\Lambda^n + \sum_{-\infty < s < n} a(s)\Lambda^s$$

is a monic pseudodifference operator of order n. Monic pseudodifference operators of order zero form a group denoted by $\mathcal{S}$.

The space $\mathcal{H}$ has natural decreasing filtration $H_m = z^m\mathcal{H}_+ = span(z^m, z^{m+1}, \ldots)$. One can characterize difference operators as pseudodifference operators compatible with this filtration.[5] One can give another characterization of difference operators as pseudodifference operators transforming the space of Laurent polynomials $H = \bigcup H_m$ into itself.

Let us say that a flag in $\mathcal{H}$ is a decreasing filtration V_m such that $z^{-m}V_m \in Gr$. In other words we assume that the natural projection $\pi_m : V_m \to H_m$ is an isomorphism. Let us denote by $w_m(z)$ the point of V_m obeying $\pi_m(w_m) = z^m$. A flag is specified by an arbitrary sequence of series $w_m(z)$ obeying the condition

$$w_m(z) = z^m + \sum_{k<m} w_{m,k} z^k,$$

hence $w_{m,k}$ can be considered as coordinates in the space $\mathcal{F}$ of flags. (We should take $V_m = \mathrm{span}(w_m, w_{m+1}, \ldots)$.)

The notion of flag was essentially used in [1], the theorems below are closely related to the results of this paper. However, they admit simple independent proofs.

Theorem 3.1. *For every monic pseudodifference operator of order zero $S \in \mathcal{S}$ we can construct a flag V in $\mathcal{H}$ taking $V_m = SH_m$. This construction gives a one-to-one correspondence between $\mathcal{S}$ and the space of flags $\mathcal{F}$.*

To prove this fact we notice that the operator $S = 1 + \sum_{r>0} s(r)\Lambda^{-r}$ where $s(r)$ are diagonal operators transforms z^m into

$$S(z^m) = z^m + \sum_{r>0} s_{m-r}(r) z^{m-r}.$$

We see immediately that $SH_m = \mathrm{span}(S(z^m), S(z^{m+1}), \ldots)$ is a flag in $\mathcal{H}$. The map $\mathcal{S} \to \mathcal{F}$ is bijective (we can identify the coordinates in these spaces using the formula $s_{m-r}(r) = w_{m,m-r}$).

[5] We say that an operator A and a decreasing filtration F_m are compatible if for some integer a we have $AF_m \subset F_{m-a}$ for all n.

Corollary 3.1. *The representation $V_m = SH_m$ induces a correspondence between pseudodifference operators compatible with the flag V_m and difference operators (=pseudodifference operators compatible with the flag H_m).*

We define the flag space V as the union of spaces V_m ; in particular, H is a flag space of the flag H_m.

Corollary 3.2. *The representation $V_m = SH_m$ induces a correspondence between pseudodifference operators preserving the flag space $V = \bigcup V_m$ and difference operators.*

Theorem 3.2. *Every monic pseudodifference operator L of order n can be transformed into operator Λ^n by means of monic pseudodifference operator of order zero:*

$$\Lambda^n = SLS^{-1}, \tag{12}$$

where $S \in \mathcal{S}$. If L is a difference operator then the flag space $V = SH$ is Λ^n-invariant. If L is an invertible difference operator (i.e. L^{-1} exists and also is a difference operator) then V is Λ^{-n}-invariant.

This theorem can be used to give a general construction of invertible difference operators.

Corollary 3.3. *Let us suppose the flag space $V = \bigcup V_m$ is invariant with respect to multiplication by z^n and by z^{-n}. Then representing V in the form $V = SH$ where $S \in \mathcal{S}$ we can construct an invertible difference operator L using the formula $L = S^{-1}\Lambda^n S$.*

To check this fact we notice that H is invariant with respect to L and $L^{-1} = S^{-1}\Lambda^{-n}S$.

The commutative Lie algebra γ of polynomials $\sum_{-\infty<<i<<\infty} t_i z^i$ acts on $\mathcal{H}$, hence on Gr and on the space of flags $\mathcal{F}$. As in Sect. 2 this follows from the remark that for $g(t) = \exp(\sum t_i z^i)$ where t_i are nilpotent parameters and a flag $V = (V_m)$ we can consider a flag $g(t)V = (g(t)V_m)$.

We can use Theorem 3.1 to define the action of the Lie algebra γ on $\mathcal{G}$ and Theorem 3.2 to define the action on monic pseudodifference operators of order n. These actions can be written in the form

$$\frac{\partial S}{\partial t_m} = (S\Lambda^m S^{-1})_- S \tag{13}$$

$$\frac{\partial L}{\partial t_m} = [L_+^{\frac{m}{n}}, L] \tag{14}$$

These formulas are called discrete KP equations (or Toda equations) [1]. They are very similar to formulas of Sect. 2. However, here $m \in \mathbb{Z}$, all operators are pseudodifference operators, D is replaced by Λ. Notice, that formula (14) can be considered also as an action of γ on the space of monic difference operators.

To solve the equation $KL = qLK$ where L is a monic difference operator we consider the flag $V_m = SH_m$ where S is defined by the formula (12). If K is a difference operator then the flag $V_m = SH_m$ is compatible with operators $\tilde{K} = SKS^{-1}$ and $\Lambda^n = SLS^{-1}$. Using Theorem 3.1 we obtain

Theorem 3.3. *For every pair (K, L) of difference operators obeying $KL = qLK$ where L is monic of order n one can construct a flag V_m in $\mathcal{H}$ such that it is compatible with operator Λ^n and with pseudodifference operator $\tilde{K}$ obeying $\tilde{K}\Lambda^n = q\Lambda^n\tilde{K}$.*

Conversely, if we have a flag V_m in $\mathcal{H}$ such that it is compatible with operator Λ^n and with pseudodifference operator $\tilde{K}$ obeying $\tilde{K}\Lambda^n = q\Lambda^n\tilde{K}$ we can construct a pair (K, L) of difference operators obeying $KL = qLK$.

Recall that the operator Λ^n acts as multiplication by z^n. The operator $\tilde{K}$ can be represented in the form

$$\tilde{K} = ({}^0b)\Lambda^p + ({}^1b)\Lambda^{p-1} + \cdots$$

where ${}^m b$ are diagonal operators obeying ${}^m b_k = q({}^m b_{k-n})$.

It follows from Theorem 3.3 that the Lie algebra γ acts on the moduli space $\mathcal{P}$ of pairs (K, L) of difference operators obeying $KL = qLK$ (we assume that L is monic). In other words, the moduli space we consider is invariant with respect to the discrete KP-hierarchy. In the proof of Theorem 3.3 we use the characterization of difference operators as pseudodifferential operators compatible with the filtration H_m. We can use instead the characterization of difference operators as pseudodifference operators preserving H.

We obtain

Theorem 3.4. *There exists one-to-one correspondence between pairs (K, L) of difference operators obeying $KL = qLK$ and L is monic of order n and pairs $(\tilde{K}, V)$ where V is a flag space invariant with respect to Λ^n and pseudodifference operator $\tilde{K}$ obeying $\tilde{K}\Lambda^n = q\Lambda^n\tilde{K}$*

If the flag space V is invariant also with respect Λ^{-n} then L^{-1} is a difference operator. Conversely, if L^{-1} is a difference operator then V is Λ^{-n}-invariant.

If the flag space $V = span(\cdots, w_m, \cdots)$ is invariant with respect to the operators Λ^n and Λ^{-n} then the vectors $w_0, \ldots, w_{n-1}$ form a $z^{\pm n}$-basis of $V = \bigcup V_m$ (i.e. the vectors $z^{mn}w_i$ where $m \in \mathbb{Z}, 0 \le i < n$ form a basis of this space). To prove this fact we notice that the operators Λ^n and Λ^{-n} specify a structure of a torsion-free finitely generated module over the algebra of Laurent polynomials on V. Such a module is always free. (To prove this one can use the fact that the algebra of Laurent polynomials is isomorphic to the group algebra of $\mathbb{Z}$.) Using the dimension count one can check that the rank of this module is equal to n and the vectors $w_0, \ldots, w_{n-1}$ are free generators of the module. This is equivalent to the statement we need.

In the conditions of Theorem 3.4 the operator $\tilde{K}$ acts on $V = \bigcup V_m$, hence we can consider the matrix M_i^j of $\tilde{K}$ in the $z^{\pm n}$-basis we constructed (or more generally in any $z^{\pm n}$-basis). By definition this matrix is the companion matrix of the pair (K, L). (The entries of this matrix are polynomials of z^n and z^{-n}.) Another definition of the companion matrix is based on the remark that in the case when L is a monic difference operator of order n and L^{-1} is also a difference operator the elements z^i with $0 \le i < n$ constitute a (L, L^{-1})-basis of H. We can define the companion matrix as matrix of the operator K in this basis (or more generally in any (L, L^{-1})-basis. In this section we always consider the companion matrix in the special basis described above.

It follows from this definition that

$$\sum ({}^m b)\Lambda^{p-m} w_i = M_i^j w_j. \tag{15}$$

Our goal is to find a pair of difference operators having companion matrix M. It will be more convenient to work with $u_i = z^{-i}w_i$ and with the matrix $B_i^j(z) = z^{j-i}M_i^j(z^q)$. Then we should consider the equation

$$\sum ({}^m b)\Lambda^{p-m}(z^i u_i) = z^i B_i^j u_j \tag{16}$$

that can be written as

$$\sum_{m\geq 0,k\geq 0} {}^m b_{p-m-k+i}\, {}^k u_i z^{p-m-k} = B_i^j u_j. \tag{17}$$

Here

$$u_i = \sum_{k\geq 0} {}^k u_i z^{-k}$$

and

$${}^m b_k = q({}^m b_{k-n}). \tag{18}$$

We assume that the matrix

$$B_i^j = \sum_{m\geq 0} {}^m B_i^j z^{s-m}$$

is known. We should find ${}^m b_k$ and ${}^m u_i$ by induction. First of all looking at the leading terms we see that $p = s$ and

$$({}^0 b_{p+i})({}^0 u_i) = ({}^0 B_i^j)({}^0 u_j). \tag{19}$$

The recursion formula formula is similar to the formula in Sect. 2:

$$({}^0 b_{p-m+i})({}^m u_i) - ({}^0 B_i^j)({}^m u_j) = ({}^m B_i^j)({}^0 u_j) - ({}^m b_{p-m+i})({}^0 u_i) + \text{known terms}. \tag{20}$$

To guarantee the existence of solutions we assume that ${}^0 b_k$ is an eigenvalue of ${}^0 B_i^j$ for all k and that ${}^0 B_i^j$ has n distinct eigenvalues. As in Sect. 2 knowing that α is an eigenvalue of ${}^0 B_i^j$ and assuming that p and n are coprime we can say that all eigenvalues have the form $\epsilon^r \alpha$ where $\epsilon^n = 1$. Combining this statement with equality (18) we obtain that the solution can exist only in the case when $q^n = 1$. From the other side if $q^n = 1$ we can take as ${}^0 b_i$ for $0 \leq i < n$ arbitrary eigenvalues of ${}^0 B_i^j$; then the formula ${}^0 b_k = q({}^0 b_{k-n})$ specifies all other ${}^0 b_i$ as eigenvalues of ${}^0 B_i^j$. We obtain the following

Theorem 3.5. *Let us suppose that the leading term ${}^0 B_i^j$ of the matrix $B_i^j = z^{j-i} M_i^j(z^n)$ has order p that is coprime with n. Then it has n distinct eigenvalues. Let us suppose that for every k the number ${}^0 b_k$ is equal to one of these eigenvalues. Assume that ${}^0 b_k$ obey ${}^0 b_k = q({}^0 b_{k-n})$ and ${}^0 u_i$ obey (19). Then we can construct a pair of difference operators K, L with companion matrix $M_i^j(z^n)$ solving the recursion formula (20). The operator L^{-1} also will be a difference operator.*

The proof repeats the second proof of Theorem 2.4. Notice that the arguments used in this proof give us the numbers ${}^m b_k$ only for $k = p - m + i$ where $0 \leq i < n$. However, knowing these numbers we can find all ${}^m b_k$ from (18).

One can modify the above consideration to study the solutions to the equation $KL = qLK$ for $q = e^{\hbar}$ in the limit $\hbar \to 0$ as power series with respect to $\hbar$. In this situation the equation (16) can be written as

$$\sum_{n\geq 0,k\geq 0} {}^n b_{p-n-k+i}\, {}^k u_i z^{p-n-k} = B_i^j u_j. \tag{21}$$

where

$$u_i = \sum_{k\geq 0, r\geq 0} {}^{k,r}u_i z^{-k}\hbar^r,$$
$${}^n b_k = \sum_{r\geq 0} {}^{n,r}b_k$$

and

$${}^{n,r}b_k = {}^{n,r}b_{k-q} + \sum_{s>0} {}^{n,r-s}b_{k-q}\frac{1}{s!}. \tag{22}$$

(The last equation follows from (18).)

We can find the coefficients ${}^{n,r}u_i$ and ${}^{n,r}b_k$ using double recursion. We are writing (21) as a system of equations for these coefficients. The equations for $r = 0$ coincide with the equations coming from (17) for $q = 1$; we have solved them by means of recursion with respect to n. Now we assume that we have found all coefficients with $r < s$. Then we can find the coefficients ${}^{n,r}u_i$ and ${}^{n,r}b_k$ using the recursion formula with respect to n.

We obtain

Theorem 3.6. *Let us suppose that the leading term* ${}^0B_i^j$ *of the matrix* $B_i^j = z^{j-i}M_i^j(z^n)$ *has n distinct eigenvalues. Then for every pair* (K, L) *of commuting difference operators with companion matrix* $M_i^j(z^q)$ *we can find a formal deformation* $(K_\hbar, L_\hbar)$ *having the same companion matrix and obeying* $K_\hbar L_\hbar = e^\hbar L_\hbar K_\hbar$.

(Saying that $(K_\hbar, L_\hbar)$ is a formal deformation of (K, L) we have in mind that $K_\hbar$ and $L_\hbar$ are power series with respect to $\hbar$ giving K, L for $\hbar = 0$.)

4. Quantization

Let us consider a pair (P, Q) of commuting differential operators, or a pair (K, L) of commuting difference operators (we assume that Q and L are monic and that L^{-1} is also a difference operator). We say that a pair $(P_\hbar, Q_\hbar)$ of differential operators obeying $[P_\hbar, Q_\hbar] = \hbar$ (quantum curve) is obtained by quantization of the pair (P, Q) if it has the same companion matrix. Similarly, a pair $(K_\hbar, L_\hbar)$ of difference operators obeying $K_\hbar L_\hbar = e^\hbar L_\hbar K_\hbar$ (discrete quantum curve) is obtained by quantization of the pair (K, L) if it has the same companion matrix. Notice that in these definitions we can work with matrix differential or difference operators.[6]

A pair (P, Q) of commuting differential operators satisfies an algebraic equation $A(P, Q) = 0$. This means that P and Q can be considered as meromorphic functions f, g on an algebraic curve $A(x, y) = 0$. Let us describe a procedure [12,17] that permits us to construct commuting differential operators starting with two meromorphic functions on an algebraic curve C. We start for simplicity with the case when the functions f, g have only one pole at a smooth point a (the function f has a pole of order p, the function g has a pole of order q). Let us suppose that we have found a subspace $\mathcal{E}$ of the space of meromorphic functions on C having the following properties a) the space $\mathcal{E}$ is spanned

[6] Recall that the companion matrix is not unique. We can defined as a matrix of P in the Q-basis of $\mathcal{H}_+$ or as a matrix of K in (L, L^{-1})-basis of H. In the above definition we have in mind the companion matrix in the preferred basis z^i where $0 \leq i < q$, where q stands for the order of Q or L.

by functions s_n that have a pole of order n at the point a (here $n \in \mathbb{N}$ or $n = 0$; saying that a function has a pole of order 0 we have in mind that it it is holomorphic and does not vanish at a), b) this space is invariant with respect to multiplication by f and g. We introduce a coordinate z in the neighborhood of the point a in such a way $g = z^q$ in this neighborhood (hence $z = \infty$ at a). Considering the Laurent series (with respect to z) of functions belonging to $\mathcal{E}$ we obtain a subspace $V \subset \mathcal{H}$; it is easy to check that we can apply Theorem 2.3 for $\hbar = 0$ to operators of multiplication acting on $V \in Gr$ to get commuting differential operators.

Now we should describe the construction of the space $\mathcal{E}$. The first idea is to take as $\mathcal{E}$ the space of all functions having a pole only at a. However, this does not work -the projection of the corresponding subspace of $\mathcal{H}$ on $\mathcal{H}_+$ has the index $\mathrm{g}(C) - 1$ where $\mathrm{g}(C)$ denotes the genus of the curve C (this follows from well known fact that the sequence of orders of poles has $\mathrm{g}(C)$ gaps). We should extend this space allowing functions with poles at the points of some divisor (the extended space is still invariant with respect to multiplication by f and g). Generic divisor of degree $g - 1$ (non-special divisor) will give the subspace we need (any divisor of degree $\mathrm{g}(C) - 1$ gives a subspace of index 0, generically it belongs to the big cell).

A pair f, g of meromorphic functions on the curve C embeds this curve into $\mathbb{C} \times \mathbb{C}$. If the points of embedded curve satisfy the equation $A(x, y) = 0$ the pair (P, Q) of commuting differential operators we constructed obeys the same equation. We can quantize this pair constructing differential operators obeying the equation $[P_\hbar, Q_\hbar] = \hbar$ and having the same companion matrix in the preferred basis. It is important to notice that the explicit construction of the pair (P, Q) is not necessary in the quantization procedure. We can define the companion matrix directly in terms of functions f, g. Namely, we should find a g-basis of the space $\mathcal{E}$ and calculate the matrix of multiplication by f in this basis.

The construction we have described can be generalized to the case when functions f and g have multiple poles. Let us suppose, for example, that the function g has s poles, all of order q, at the points $a_1, \ldots, a_s$. Let us introduce the coordinate z_i in the neighborhood of a_i in such a way that $g = z_i^q$ in this neighborhood (hence $z_i = \infty$ at a_i). Then we should find a subspace $\mathcal{E}$ of the space of meromorphic functions on C having the following properties a) for arbitrary integers $n_1, \ldots n_s$ obeying $n_i \geq 0$ the space $\mathcal{E}$ contains a function that behaves as $z_i^{n_i}$ +lower order terms in the neighborhood of a_i and these functions span $\mathcal{E}$, b) this space is invariant with respect to multiplication by f and g. To every meromorphic function on C we should assign s Laurent series representing this function in the neighborhoods of points a_i. This construction sends $\mathcal{E}$ to a subspace $V \subset \mathcal{H}^s$; it follows from the condition a) that $V \in Gr_s$. Applying Theorem 2.5 for $\hbar = 0$ we obtain a pair of commuting differential matrix operators.

To construct the space $\mathcal{E}$ obeying the above conditions we start with the space of meromorphic functions having poles only at the points $a_1, \ldots, a_s$. If f has poles only at $a_1, \ldots, a_s$ this space is invariant with respect to multiplication by f and g (property b)). This property is preserved if we allow additional poles of order 1 at the points of some divisor. Choosing the divisor appropriately we can satisfy the property a).

Very similar construction works for difference operators [12,17]. Let us consider an algebraic curve C and two meromorphic functions f, g that are holomorphic everywhere except two smooth points a and b. We introduce a coordinate z in the neighborhood of a in such a way that $g = z^q$ in this neighborhood (we assume that $z = \infty$ at a hence g has a pole of order q). Let us suppose that a subspace $\mathcal{E}$ of the space of meromorphic functions on C has the following properties a) it is spanned by functions $s_n, n \in \mathbb{Z}$ such that s_n has a pole of order n at a for non-negative n and zero of order $-n$ for negative n, b) it is invariant with respect to multiplication by f and g.The functions s_n specify a flag in $\mathcal{H}$, the space $\mathcal{E}$ can be identified with the corresponding flag space. (To define this flag we consider Laurent series of s_n at the point a with respect to the coordinate z). This flag is compatible with multiplication by f and g. Applying Theorem 3.3 in the case $q = 1$ we obtain two commuting difference operators K, L. If the space $\mathcal{E}$ is invariant also with respect to the multiplication by g^{-1} the operator L^{-1} is also a difference operator.

Now we should explain how to find the subspace $\mathcal{E}$ satisfying the conditions a), b). Let us consider the space $\mathcal{K}$ consisting of meromorphic sections of some line bundle having poles only at points a, b (equivalently we can talk about meromorphic functions on C having poles of any order at a and b , that can also have simple poles at the points of some divisor). Let us denote by $\mathcal{K}(na)$ the subspace $\mathcal{K}$ consisting of functions of the form $z^n f(z)$ where f is finite at a; the space $\mathcal{K}(nb)$ is defined in similar way. One can prove that for appropriate choice of $\mathcal{K}$ the space $\mathcal{K}(na) \cap \mathcal{K}((-n+1)b)$ is one-dimensional; the function s_n can be defined as non-zero element of this space.

5. D-modules

In [4] quantum curves were studied from the viewpoint of D-modules. The approach of this paper is closely related to the approach of [18] and present paper. The construction of the point of Grassmannian used in [18] plays fundamental role also in [4]. The companion matrix specifies a meromorphic connection

$$\nabla = \hbar \frac{d}{dz^q} - B_i^j(z), \tag{23}$$

the flat sections of this connection can be identified with solutions to the equation (6). The flat connection ∇ can be identified with D-module studied in [4] (better to speak about the family of connections and a family of D-modules parametrized by $\hbar$ or about $D_\hbar$-module). An equivalent D-module can be defined by means of the matrix M_j^i. As we have noticed the matrices M defined by means of different z^q-bases are related by a gauge transformation; this means that M can be regarded as a connection and in the definition of D-module we can use any z^q-basis of V (any Q-basis of $\mathcal{H}_+$).

We consider connections in the neighborhood of $z = \infty$ (connections on the punctured formal disk). A classification of connections on the punctured formal disk is well known (see [13]; we follow [7] in the description of the classification). Such a connection can be written in the form $\nabla = \frac{d}{dz} + A(z)$ where $A(z)$ stands for the matrix with entries from the field of Laurent series $K = \mathbb{C}((z))$. (The elements of K are formal series $\sum a_n z^n$ where $a_n = 0$ for $n >> 0$.) Notice that ∇ can be characterized as obeying Leibniz identity $\mathbb{C}$-linear operator in finite -dimensional vector space over K. Let us denote this vector space by V; the dimension of this space over K is called the dimension of connection. We say that two connections ∇ and ∇' are gauge equivalent if there exists an invertible matrix g with entries from K obeying $g\nabla = \nabla' g$.

Together with the field K we will consider the field $K_q = \mathbb{C}((v))$ where $v^q = z$. Notice K_q can be considered as a q -dimensional vector space over K. An important example of connection is specified by an operator $\nabla_{f,q} = \frac{d}{dz} + \frac{f(v)}{z}$ acting in K_q. Here $\frac{d}{dz} = \frac{1}{qv^{q-1}} \frac{d}{dv}$ and f stands for an operator of multiplication by Laurent series $f \in K_q$. The operator $\nabla_{f,q}$ determines a q-dimensional connection over K. Without loss of generality we can assume that

$$f = \sum_{m \in A} c_m v^m + c_0 = \sum_{m \in A} c_m z^{\frac{m}{q}} + c_0 \tag{24}$$

where the set A consists of natural numbers and $0 \le c_0 < 1$. (Connections with other f are gauge equivalent to connections with f in (24).) All irreducible connections are gauge equivalent to connections of the form (24). However, not all connections of this kind are irreducible; the condition of reducibility is the existence of non-trivial common divisor of the numbers $m \in A$ and q (if such a divisor does exist then we can represent the connection in similar form with q replaced by $q' < q$).

A more general example of connection is given by an operator

$$\nabla_{f,q,J} = \frac{d}{dz} + \frac{f(v)}{z} + \frac{J}{z}$$

acting on the space K_q^n. Here J stands for $n \times n$ matrix with complex entries, $f \in K_q$.

One can prove that every connection is gauge equivalent to a direct sum of connections of the form $\nabla_{f,q,J}$ where J is a Jordan block.

Let us consider a pair P, Q of differential operators obeying $[P, Q] = \hbar$ assuming that Q is normalized and the orders p, q of P, Q are coprime. If P, Q are scalar operators then it follows from our considerations that the corresponding connection is irreducible over the field $K(z^q)$ (over the field $K(z)$ it can be represented as a direct sum of q one-dimensional connections; this statement is equivalent to the diagonalization of the equation (7) used in Sect. 2). If P, Q are matrix operators of the size $r \times r$ then the connection is a direct sum of r irreducible connections.

It is well known that every D-module defined by meromorphic connection in dimension 1 is cyclic (see, for example, [9]); it can be represented in the form $\mathcal{D}/\mathcal{D} \cdot P$ where $\mathcal{D}$ denotes the algebra of differential operators with meromorphic coefficients (i.e. polynomials with respect to ∂_z with coefficients that are meromorphic with respect to z) and $P \in \mathcal{D}$. This is the form mostly used in [4].

The above statement means that the system of equations for flat sections $\nabla u = 0$ is equivalent to a single differential equation. One can use this statement to rewrite the system (6) or (4) as a single equation

$$\hat{A} w_0 = 0,$$

where $\hat{A}$ is a differential operator with meromorphic coefficients. Namely, we should consider $w = (w_0, \dots, w_{q-1})$ as an element of $\mathcal{F}^q$ where $\mathcal{F}$ denotes the field of meromorphic functions. Then $w_0 = \langle e_0, w \rangle$ where $e_0 = (1, 0, \dots, 0)$ and $\langle a, b \rangle = \sum a_i b_i \in \mathcal{F}$. Defining ∇_* by the formula

$$\nabla_* = \hbar \frac{d}{dz^q} + B_i^j(z),$$

and using $\nabla w = 0$ we obtain that

$$(\hbar \frac{d}{dz^q})^s w_0 = \langle \nabla_*^s e_0, w \rangle.$$

To find $\hat{A}$ we notice that the vectors $\nabla_*^s e_0$ with $s = 0, \ldots, q$ are linearly dependent in q-dimensional vector space over $\mathcal{F}$. If

$$\sum_{0 \le s \le q} a_s(z, \hbar) \nabla_*^s e_0 = 0$$

we can take

$$\hat{A} = \sum_{0 \le s \le q} a_s(z, \hbar)(\hbar \frac{d}{dz^q})^s. \tag{25}$$

Notice that knowing an operator annihilating w_0 it is easy to find an operator annihilating $u_0 = v_0$:

$$(\hat{A} - b(z))u_0 = 0.$$

It was suggested in the paper [8] that one should understand a quantum curve as an equation

$$\hat{A}\Psi = 0 \tag{26}$$

where $\hat{A}$ has the form (25) with polynomial coefficients a_s. It follows from the above consideration that starting with a quantum curve in our sense one can construct a quantum curve in the sense of [8] if one allows meromorphic coefficients a_s. If $\hat{A}$ has an irregular singular point at infinity one can construct formal solutions of (26) in the form

$$\Psi = e^{Q(t)} t^{\rho} \psi(t)$$

where $z = t^r$, $Q(t)$ is a polynomial and ψ is a series with respect to t and a polynomial with respect to $\log t$. These solutions can be put together in equivalence classes; roughly speaking an equivalent solution can be obtained by means of substitution $t \to \epsilon t$ where $\epsilon^r = 1$. Instead of $\hat{A}$ and equation (26) one can consider a connection and corresponding flat sections. The description of the formal solutions of (26) follows then from the representation of connection as a direct sum of connections of the form $\nabla_{f,q,J}$.(If the Jordan block has size $j \times j$ then the corresponding solutions constitute j classes; each class consists of q elements). If $\hat{A}$ comes from scalar differential operators obeying $[P, Q] = \hbar$ and Q is a monic differential operator of order q then the corresponding connection has the form $\nabla_{f,q}$, hence all solutions are equivalent. Conversely, if all solutions to the equation (26) are equivalent then the corresponding connection is gauge equivalent to $\nabla_{f,q}$ and $\hat{A}$ comes from scalar differential operators obeying $[P, Q] = \hbar$.

The paper [2] is devoted to description of an algorithm that allows us not only to find formal solutions of (26) , but also to find their equivalence classes (rational Newton algorithm). The algorithm is based on the consideration of Newton polygon, equivalent solutions correspond to the same edge of this polygon. If all solutions are equivalent the Newton polygon has only one edge. This gives another proof of the condition satisfied by companion matrix in scalar case. (See the end of Sect. 2). From the other side, if the order of the symmetry group of $\hat{A}$ is equal to the order of $\hat{A}$ considered as differential operator and symmetry transformations acting on solutions do not have fixed points , then there exists only one equivalence class.

In particular, the polynomial $\hat{A} = \hbar^2 \frac{d^2}{dz^2} + s(z)$ is obviously symmetric with respect to the transformation changing the sign of the derivative $\frac{d}{dz}$, hence we can use the above remark to construct a quantum curve in the sense of present paper. Polynomials of this kind were considered in the paper [5] that is devoted to the generalization and analysis of Eynard–Orantin construction for Hitchin fibration. The authors of this paper show that the quantum curves in the sense of [8] corresponding to these polynomials appear in the neighborhood of branch points. The above remark shows that this is true also for quantum curves in our sense.

Acknowledgements. I am indebted to S. Gukov, M. Luu, M. Mulase, N. Orantin and V. Vologodsky for useful discussions. I was partially supported by the NSF grant DMS-0805989.

References

1. Adler, M., van Moerbeke, P.: Vertex operator solutions to the discrete KP hierarchy. Commun. Math. Phys. **203**, 185–210 (1999)
2. Barkatou, M.A.: Rational Newton algorithm for computing formal solutions of linear differential equations. Symbo. Algebraic Comput. **358**, 183–195 (1989)
3. Bouchard, V., Klemm, A., Marino, M., Pasquetti, S.: Remodeling the B-model. Commun. Math. Phys. **287**, 117–178 (2009)
4. Dijkgraaf, R., Hollands, L., Sulkowski, P.: Quantum curves and $\mathcal{D}$-modules. JHEP **0911**, 047 (2009)
5. Dumitrescu, O., Mulase, M.: Quantum curves for Hitchin fibrations and the Eynard–Orantin theory. Lett. Math. Phys. **104**, 635–671 (2014)
6. Eynard, B., Orantin, N., Invariants of algebraic curves and topological expansion. arXiv:math-ph/0702045
7. Graham-Squire, A.: Calculation of local formal Fourier transforms. Arkiv för Matematik **51**, 71–84 (2013)
8. Gukov, S., Sulkowski, P.: A-polynomial, B-model, and quantization. JHEP **1202**, 070 (2012)
9. Haefliger, A.: Local theory of meromorphic connections in dimension one (Fuchs theory). In: Borel, A., et al., (eds.) Algebraic D-modules, pp. 129–149. Academic Press (1987)
10. Kac, V., Schwarz, A.: Geometric interpretation of the partition function of 2D gravity. Phys. Lett. B **257**, 329–334 (1991)
11. Kimura, M., Vanhaecke, P.: Commuting matrix differential operators and loop algebras. Bulletin des sciences mathematiques, 125, 407 (2001)
12. Krichever, I.M.: Algebro-geometric construction of the Zaharov? Shabat equations and their periodic solutions, Doklady Akad, Nauk SSSR (1976)
13. Levelt A. H.M.: Jordan decomposition for a class of singular differential operators. Arkiv for Math. **13.1**, 1–27 (1975)
14. Li, Y., Mulase, M.: Prym varieties and integrable systems. Commun. Anal. Geom. **5**, 279 (1997)
15. Moore, G.: Geometry of the string equations. Commun. Math. Phys. **133**, 26 (1990)
16. Mulase, M.: Algebraic theory of KP equations. In: Penner, R., Yau, S.T. (eds.) Perspectives in Mathematical Physics, pp. 157–223 (1994)
17. Mumford, D.: An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations. In: Proceedings of the International Symposium on Algebraic Geometry, 1977, pp. 115–153. Kinokuniya Publ., Kyoto (1978)
18. Schwarz A.: On solutions to the string equation. Mod. Phys. Lett. A **6**(29), 2713–2726 (1991). arXiv:hep-th/9109015
19. Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations, Dover Phoenix edn. (2002)

Communicated by N. A. Nekrasov