5 Sheaves of Modules

Exercise 5.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathscr{E}$ denoted $\check{\mathscr{E}}$ to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.*

a *Show that $(\check{\mathscr{E}})^\vee \cong \mathscr{E}$.*

b *For any $\mathcal{O}_X$-module $\mathscr{F}$, we have $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes \mathscr{F}$.*

c *For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, we have $\hom_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \hom_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.*

d Projection Formula. *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.*

Solution. a Even without any conditions on $\mathscr{E}$ there is a canonical morphism $\mathscr{E} \to \mathscr{H}om(\mathscr{H}om(\mathscr{E}, \mathcal{O}_X), \mathcal{O}_X)$ defined by evaluation. Given an open subset U we want to define for every section, $s \in \mathscr{E}(U)$ a natural transformation $\hom(\mathscr{E}, \mathcal{O}_X)|_U \to \mathcal{O}_X|_U$. For every open subset $V \subseteq U$ we define an element of $\hom_{\mathcal{O}_X}(\mathscr{E}(V), \mathcal{O}_X(V)) \to \mathcal{O}_X(V)$ by evaluating at $s|_V$.

In the case where $\mathscr{E}$ is locally free, it can be seen that this canonical morphism is an isomorphism by looking at the stalks. On the stalks, this morphism is the canonical morphism of $\mathcal{O}_{X,x}$-modules,

$$\mathscr{E}_x \to \hom_{\mathcal{O}_{X,x}}(\hom_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathcal{O}_{X,x}))$$

Since $\mathscr{E}_x$ is free, this morphism is an isomorphism.

b Again, we have a canonical isomorphism

$$\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{F})$$

To define this consider the presheaf $U \mapsto \mathscr{H}om(\mathcal{O}_X, \mathscr{E})(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$. Denote this presheaf by $\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes^{pre}_{\mathcal{O}_X} \mathscr{F}$. Its sheafification is $\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes_{\mathcal{O}_X} \mathscr{F}$ so we need just define a morphism

$$\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes^{pre}_{\mathcal{O}_X} \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{F})$$

and then sheafification will give a morphism of the kind we require. Define the morphism of presheaves section wise by noticing that every section $s \in \mathscr{F}(U)$ gives a natural transformation $\mathcal{O}_X|_U \to \mathscr{F}|_U$ by multiplying by the restriction of s. Then we define

$$\begin{aligned} \hom(\mathcal{O}_X|_U, \mathscr{E}|_U) \otimes_{\mathcal{O}_X|_U} \mathscr{F}(U) &\to \hom(\mathscr{E}|_U, \mathscr{F}|_U) \\ \phi \otimes s &\mapsto \left(\mathscr{E}|_U \xrightarrow{\phi} \mathcal{O}_X|_U \xrightarrow{s} \mathscr{F}|_U \right) \end{aligned}$$

The stalk of the morphism we have just defined is the obvious canonical morphism of $\mathcal{O}_{X,x}$-modules. When $\mathscr{E}$ is locally free, $\mathscr{E}_x$ is free and so this is an isomorphism.

Exercise 5.2. *Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.*

- a *To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R KtoL$.*
- b *That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.*

Solution. a Since R is a discrete valuation ring, there are two nontrivial open subsets of $\operatorname{Spec} R$: the total space and $U = \{\eta\}$ the set containing only the generic point. So by definition, an $\mathcal{O}_X$-module consists of an $\mathcal{O}_X(X) = R$ module M and an $\mathcal{O}_X(U) = K$ module L, together with an R-module homomorphism $M \to L_R$ where L_R is L considered as an R-module. Since restriction and extension of scalars are adjoint the R-module homomorphism is that same as giving an K-module homomorpism $M \otimes_R K \to L$.

b Let $\mathscr{F}$ be the $\mathcal{O}_X$-module. If $\mathscr{F}$ is quasi-coherent then every point has a neighbourhood on which $\mathscr{F}$ has the form $\widetilde{M}$. The only neighbourhood of the unique closed point of $\operatorname{Spec} R$ is the whole space, so $\mathscr{F}$ is of the form $\widetilde{M}$ and therefore $L = \mathscr{F}(U) = M_{(0)} = M \otimes_R K$. Conversely, suppose $M \otimes_R K \to L$ is an isomorphism. $M \otimes_R K \to L$ is the adjunct morphism of $M \to L_R$ so we get a factorization $M \to M \otimes_R K \to L_R$ where the first morphism is the unit of the adjunction. This factorization means gives a morphism of sheaves $\widetilde{M} \to \mathscr{F}$ and since it is an isomorphism, the morphism of sheaves is. So $\mathscr{F}$ is quasi-coherent.

Exercise 5.3. *Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\sim$ and Γ are adjoint.*

Solution. We begin by defining a morphism of sheaves $\eta : \Gamma(X, \mathscr{F})^\sim \to \mathscr{F}$. On a distinguished affine open, $D(a)$ of $\operatorname{Spec} A$ we have $\Gamma(X, \mathscr{F})^\sim(D(a)) \cong \Gamma(X, \mathscr{F})_a$ and so restriction $\mathscr{F}(X) \to \mathscr{F}(D(a))$ induces a morphism $\Gamma(X, \mathscr{F})^\sim(D(a)) \to \mathscr{F}(D(a))$. If we have two distinguished open subsets $D(a), D(b)$ of X, it can be seen that the restriction of the morphisms agree on the intersection, and so we have defined a morphism of sheaves. Furthermore, since $X = D(1)$, on global sections we have the identity $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$.

Now for a morphism $\phi \in \hom_A(M, \Gamma(X, \mathscr{F}))$ we define a morphism in $\hom_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$ by $\eta \circ \widetilde{\phi}$. By the observation that $\widetilde{\phi}_X : \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$ is the identity, we see that $\Gamma : \hom_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F}) \to \hom_A(M, \Gamma(X, \mathscr{F}))$ is an inverse to this assignment. Hence we have a bijection.

Exercise 5.4. *Show that a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on a scheme X is quasi-coherent if ad only if every point of X has a neighbourhood U, such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank.*

Solution. First suppose that $\mathscr{F}$ is quasi-coherent. Then every point has a neighbourhood U on which $\mathscr{F}|_U \cong \widetilde{\mathscr{F}(U)}$. Since every module is a cokernel of a morphism between free modules, [1] $\mathscr{F}(U)$ is the cokernel of a morphism $F_1 \to F_0$ of free $\mathcal{O}_X(U)$-modules. Since the functor $\sim$ is a left adjoint it is right exact and therefore preserves cokernels. So $\mathscr{F}|_U$ is isomorphic to the cokernel of $\widetilde{F_1} \to \widetilde{F_0}$. The functor $\sim$ also preserves arbitrary products and so $\widetilde{F_1}, \widetilde{F_0}$ are free $\mathcal{O}_X$-modules.

Conversely, suppose that locally $\mathscr{F}$ is isomorphic to a cokernel of a morphism of free sheaves. Take an affine neighbourhood $U = \operatorname{Spec} A$ of a point x on which $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves $\mathscr{F}_1 \to \mathscr{F}_0$. Since the $\mathscr{F}_i$ are free, the adjunction morphisms $\widetilde{\mathscr{F}_i(U)} \to \mathscr{F}_i$ are isomorphisms. So we have a diagram

$$\begin{tikzcd}
\mathscr{F}_1 \arrow[r] \arrow[d, equal] & \mathscr{F}_0 \arrow[r] \arrow[d, equal] & \widetilde{\mathscr{F}(U)} \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
\mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \arrow[r] & \mathscr{F} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. So it follows from the five lemma that the adjunction morphism $\widetilde{\mathscr{F}(U)} \to \mathscr{F}$ is an isomorphism. Hence, $\mathscr{F}$ is quasi-coherent.

The proof for the coherent case is the same. To get a cokernel of finite rank free modules we do the following. M is finitely generated so there is a surjective morphism $R^n \to M$ that sends each standard basis element to a generator. Its kernel is not finitely generated a priori but we have assumed that R is noetherian, hence R^n is a noetherian module, hence every submodule is finitely generated. So we can find a morphism $R^m \to R^n$ that is surjective onto the kernel of $R^n \to M$. Hence, M is a cokernel of finite rank free R-modules.

Exercise 5.5. *Let $f : X \to Y$ be a morphism of schemes.*

a *Show by example that if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.*

b *Show that a closed immersion is a finite morphism.*

c *If f is a finite morphism of noetherian schemes, and if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ is coherent on Y.*

Solution. a Consider the pushforward of the structure sheaf under the morphism $\operatorname{Spec} k(t) \to \operatorname{Spec} k$ for a field k. Certainly, $\mathcal{O}_{\operatorname{Spec} k(t)}$ is coherent on $k(t)$ but since $k(t)$ is not a finitely generated k-module, its pushforward is not coherent.

[1]Take F_0 to be the free R-module with basis the underlying set of M equipped with the obviousmorphism $F_0 \to M$. Now let F_1 be the free R-module with underlying set the elements of the kernel of $F_0 \to M$. This comes with a morphism $F_1 \to F_0$ and the cokernel of this morphism is, of course, M.

b Let $i : Z \to X$ be a closed immersion of schemes. Let $\{U_j = \operatorname{Spec} A_j\}$ be an open affine cover of X. The restrictions $i^{-1}U_j \to U_j$ are also closed immersions and so by Exercise II.3.11(b) the are of the form $\operatorname{Spec}(A_j/I_j) \to \operatorname{Spec} A_j$ for suitable ideals $I_j \subset A_j$. Since each A_j/I_j is a finitely generated A_j-module, this shows that i is finite.

c Let $\{\operatorname{Spec} B_i\}$ be an open affine cover of Y. Since f is finite, each $f^{-1} \operatorname{Spec} B_i$ is affine (say $\operatorname{Spec} A_i$) and since $\mathscr{F}$ is coherent and X noetherian, the sheaf $\mathscr{F}$ is of the form $\widetilde{M_i}$ on each $\operatorname{Spec} A_i$ (Proposition II.5.4) where the A_i are finitely generated B_i-modules and the M_i are finitely generated A_i-modules. On $\operatorname{Spec} B_i$ we have $f_*\mathscr{F}|_{\operatorname{Spec} B_i} \cong ({}_{B_i}M_i)^\sim$ by Proposition II.5.2(d). Since A_i is a finitely generated B_i-module and M_i is a finitely generated A_i-module it follows that ${}_{B_i}M_i$ is a finitely generated B_i-module. Hence, $f_*\mathscr{F}$ is coherent.

Exercise 5.6. Support.

a Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathscr{F} = \widetilde{M}$. For any $m \in M$ show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$.

b Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.

c The support of a coherent sheaf on a noetherian scheme is closed.

d For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^\sim \cong \mathscr{H}^0_Z(\mathscr{F})$, where $Z = V(\mathfrak{a})$ and $\mathscr{F} = \widetilde{M}$.

e Let X be a noetherian scheme, and let Z be a closed subset. If $\mathscr{F}$ is a quasi-coherent (respectively coherent) $\mathcal{O}_X$-module, then $\mathscr{H}^0_Z(\mathscr{F})$ is also quasi-coherent (respectively coherent).

Solution. a By definition $\operatorname{Supp} m$ is the set of points $\mathfrak{p} \in \operatorname{Spec} A$ such that $m_{\mathfrak{p}} \neq 0$. This condition is equivalent to asking that $sm \neq 0$ for all $s \notin \mathfrak{p}$. If $\mathfrak{p} \in V(\operatorname{Ann} m)$ then $\mathfrak{p} \supseteq \operatorname{Ann} m$ and so $sm \neq 0$ for all $s \notin \mathfrak{p}$ and therefore $\mathfrak{p} \in \operatorname{Supp} m$. Conversely, if $\mathfrak{p} \in \operatorname{Supp} m$ then there is some nonzero $s \notin \mathfrak{p}$ such that $sm = 0$ and therefore $\mathfrak{p} \not\supseteq \operatorname{Ann} m$ so $p \notin V(\operatorname{Ann} m)$.

b By definition $\operatorname{Supp} \mathscr{F}$ is the set of primes $\mathfrak{p} \in \operatorname{Spec} A$ such that $M_{\mathfrak{p}} \neq 0$. Suppose $\mathfrak{p} \in \operatorname{Supp} \mathscr{F}$. If $\mathfrak{p} \notin V(\operatorname{Ann} M)$ then there is some $s \notin \mathfrak{p}$ such that $sm = 0$ for all m. This means that $M_{\mathfrak{p}} = 0$ which contradicts the assumption that $\mathfrak{p} \in \operatorname{Supp} \mathscr{F}$. Hence, $\operatorname{Supp} \mathscr{F} \subseteq V(\operatorname{Ann} M)$. Conversely, suppose that $\mathfrak{p}$ is not in the support of M, so $M_{\mathfrak{p}} = 0$. Then for each element $m \in M$ there is some $s \in A \backslash \mathfrak{p}$ such that $sm = 0$. In particular, if $\{m_i\}$ is a finite set of generators for M then there are $s_i \in A \backslash \mathfrak{p}$ such that $s_i m_i = 0$. This means that $s = \prod s_i \in A \backslash \mathfrak{p}$ is in $\operatorname{Ann} M$. Hence, $\mathfrak{p} \not\supseteq \operatorname{Ann} M$.

c The support of a sheaf is the union of the supports of the sheaf resricted to each element of an open cover. Take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for some finitely generated A_i-modules. Then by the previous part, for each i, the support intersected with U_i is a closed subset of U_i. This implies that the support is closed in X. [2]

d Let $U = X - Z$ and $j : U \to X$ the inclusion. Exercise I.1.20(b) gives us an exact sequence:

$$0 \to \mathscr{H}_Z^0(\mathscr{F}) \to \mathscr{F} \to j_*\mathscr{F}$$

Since j is quasi-compact and separated, we can apply Proposition II.5.8(c) to find that $j_*\mathscr{F}$ is quasi-coherent. Using a similar diagram to that in the proof of Exercise II.6.4 we see that $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent. So we only need to show that the module of global sections are isomorphic. That is, we want an isomorphism between the following two modules

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$$

$$\Gamma_Z(\mathscr{F}) = \{m \in M | \operatorname{Supp} m \subseteq Z\}$$

First suppose that $m \in \Gamma_{\mathfrak{a}}(M)$. Then $\mathfrak{a}^n \subseteq \operatorname{Ann} m$ for some $n > Z$ so $V(\mathfrak{a}^n) \supseteq V(\operatorname{Ann} m)$. But $V(\mathfrak{a}^n) = V(\mathfrak{a})$ (Lemma II.2.1(a)) and by the first part of this exercise $V(\operatorname{Ann} m) = \operatorname{Supp} m$. Furthermore, by definition $Z = V(\mathfrak{a})$. So our inclusion $V(\mathfrak{a}^n) \supseteq V(\operatorname{Ann} m)$ becomes $Z \supseteq \operatorname{Supp} m$. Hence, $m \in \Gamma_Z(\mathscr{F})$.

Conversely, suppose that $m \in \Gamma_Z(\mathscr{F})$, so $\operatorname{Supp} m \subseteq Z$. By what we have just written we immediately see that this implies that $V(\operatorname{Ann} m) \subseteq V(\mathfrak{a})$. By Lemma II.2.1(c) this implies that $\sqrt{\operatorname{Ann} m} \supseteq \sqrt{\mathfrak{a}}$ and so $\sqrt{\operatorname{Ann} m} \supseteq \mathfrak{a}$. Now since A is noetherian, $\mathfrak{a}$ is finitely generated by, say n elements $\{a_i\}$. Since $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann} M}$ there is some j_i such that for eacj i we have $a_i^{j_i} \in \operatorname{Ann} M$. Let $N = \frac{\max\{j_i\}}{n}$. Now every element of $\mathfrak{a}$ can be written as a polynomial in the a_i with no constant term, and so every element of $\mathfrak{a}^N$ can be written as a polynomial in the a_i where the degree of the smallest homogeneous part is N. For a polynomial of this form, every mononial contains a factor of the form a_i^k where $k \geq \max\{j_i\} \geq j_i$ by definition of N. Hence, every element of $\mathfrak{a}^N$ is in $\operatorname{Ann} M$ and so $\mathfrak{a}^N m = 0$. Hence, $m \in \Gamma_{\mathfrak{a}}(M)$.

e Let $\{U_i\}$ be an affine cover on which $\mathscr{F}$ is locally of the form $\widetilde{M_i}$. Since X is noetherian we can apply the previous part of this question to find that $\mathscr{H}_Z^0(\mathscr{F})|_{U_i} \cong \Gamma_{\mathfrak{a}_i}(M_i)^\sim$ where $\mathfrak{a}_i$ is the ideal of $Z \cap U_i$ (see Exercise II.3.11(b)). Hence, $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent. The same proof works for the coherent case.

[2] Let Z denote the support of X and Z^c its complement. For each i since $Z \cap U_i$ is closed in U_i we see that $Z^c \cap U_i$ is open in U_i. Since U_i is open in X this implies that $Z^c \cap U_i$ is open in X as well. So $Z^c = \cup(Z^c \cap U_i)$ is a union of open sets, and therefore open itself. Hence, Z is closed.

Exercise 5.7. *Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.*

a *If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighbourhood U of x such that $\mathscr{F}|_U$ is free.*

b *$\mathscr{F}$ is ocally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.*

c *$\mathscr{F}$ is invertible (i.e. locally free of rank 1) if and only if there is a cogerent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$.*

Solution. a Consider a neighbourhood of x on which $\mathscr{F}$ has the form $\widetilde{M}$ (where M is a finitely generated A-module where A is the ring of global sections of said neighbourhood), so that $\mathscr{F}_x \cong M_\mathfrak{p}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. Hence, we have an isomorphism $M_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$. Let e_i be the images in $M_\mathfrak{p}$ of the standard basis elements (note that we can choose the isomorphism so that $e_i \in M$). Let $\{m_i\}$ be a finite set of elements that generate M and $(\frac{a_{i1}}{s_{i1}}, \ldots, \frac{a_{in}}{s_{in}})$ their images in $A_\mathfrak{p}^{\oplus n}$. Let $s = \prod_{ij} s_{ij}$ and consider the open affine subset $D(s)$ of $\operatorname{Spec} A$. As s is invertible in M_s so are all the s_{ij} and so we have the relation $m_i = \sum \frac{a_{ij}}{s_{ij}} e_j$. This shows that we have a surjective morphism $\phi : A_s^{\oplus n} \to M_s$. We wish it to be injective as well. Consider the kernel. Since A is noetherian, every submodule of $A_s^{\oplus n}$ is finitely generated and so there is a morphism $A_s^{\oplus m} \to A_s^{\oplus n}$ whose image is the kernel of ϕ. This can be represented by a matrix with entries in A_s and multiplying the basis of $A_s^{\oplus m}$ by a suitable invertible element of A_s (the inverse of the product of the denominators of the entries of the matrix of ϕ would do nicely) we can assume the entries of the matrix b_{ij} are in A. Now when we restrict (= tensor with) to $A_\mathfrak{p}$, the kernel vanishes as $id_{A_\mathfrak{p}} \times \phi$ is our original isomorphism so this means that every element b_{ij} is zero in $A_\mathfrak{p}$. This means that there is some t_{ij} for each b_{ij} such that $t_{ij} b_{ij} = 0$ in M. Let $t = s \prod t_{ij}$ and consider $D(t) \subseteq D(s)$. Tensoring our exact sequence with A_t now kills ϕ, by our choice of t and so we obtain an isomorphism $A_t^{\oplus n} \cong M_t$.

b If $\mathscr{F}$ is locally free then by definition the stalks are free $\mathcal{O}_x$-modules for all $x \in X$. Conversely, if the stalks are all free $\mathcal{O}_x$-modules then by part (a) each point has a neighbourhood U on which $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module, hence $\mathscr{F}$ is locally free.

c If $\mathscr{F}$ is invertible then consider $\mathscr{G} = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$. There is a canonical morphism $\mathscr{F} \otimes \hom(\mathscr{F}, \mathcal{O}_X) \to \mathcal{O}_X$ defined by evaluation and on the stalks, this is an isomorphism since $\mathscr{F}$ is locally free of rank 1.

Conversely, suppose that there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$. Let x be a point of X. The vector space $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x) \otimes_{\mathcal{O}_{X,x}} k(x)$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)) \otimes_{k(x)} (\mathscr{G}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ as well as $k(x)$. Hence, the vector space $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ is of dimension one, and similarly for $\mathscr{G}$. Consider an affine neighbourhood of x on which $\mathscr{F}$ has the form

$\widetilde{M}$ and $\mathscr{G}$ the form $\widetilde{N}$ and let $\mathfrak{p} \in A$ be the prime corresponding to x. Since $\mathscr{F}$ and $\mathscr{G}$ are coherent, M and N are finitely generated and so by Nakayama's lemma, a set of generators for $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)) = M_\mathfrak{p} \otimes k(\mathfrak{p})$ lifts to a set of generators for $M_\mathfrak{p}$. We have seen that $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ is a vector space of dimension one and so it follows that $M_\mathfrak{p}$ is generated by a single element, say $m \in M$, as an $A_\mathfrak{p}$-module, and similarly, $N_\mathfrak{p}$ is generated by a single element, say n, as an $A_\mathfrak{p}$-module. Hence, $M_\mathfrak{p} \otimes N_\mathfrak{p}$ is generated by $m \otimes n$. Recall that it is also isomorphic to $A_\mathfrak{p}$. We define three morphisms:

$$\begin{array}{ccc} A_\mathfrak{p} \to M_\mathfrak{p} & M_\mathfrak{p} \to M_\mathfrak{p} \otimes N_\mathfrak{p} & M_\mathfrak{p} \otimes N_\mathfrak{p} \to A_\mathfrak{p} \\ \frac{a}{s} \mapsto \frac{a}{s} m & \frac{m'}{s} \mapsto \frac{m'}{s} \otimes n & \frac{a}{s}(m \otimes n) \mapsto \frac{a}{s} \end{array}$$

By recalling that the first morphism is surjective we see that the composition of the second two is an inverse to the first. Hence $\mathscr{F}_x \cong \mathcal{O}_{X,x}$ and so $\mathscr{F}$ is locally free of rank one.

Exercise 5.8. *Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function*

$$\phi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following:

a *The function ϕ is upper semi-continuous. That is, for any $n \in \mathbb{Z}$ the set*

$$\Phi(n) = \{x \in X | \phi(x) \geq n\}$$

is closed.

b *If $\mathscr{F}$ is locally free, and X is connected, then ϕ is a constant function.*

c *Conversely, if X is reduced, and ϕ is constant, then $\mathscr{F}$ is locally free.*

Solution. a Since a set is closed only if it is closed on each element of an open cover, we need only prove the result for the case when X is affine. We will show that the set $\Phi(n)^c$ is open by showing that every point in it has a neighbourhood contained in it. Let $x \in \Phi(n)^c$. So $\dim_{k(\mathfrak{p})}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}) < n$ where $\mathfrak{p} \in \operatorname{Spec} A = X$ is the prime corresponding to x and $M = \Gamma(X, \mathscr{F})$. In fact, let m be this dimension. By Nakayama's lemma, a basis of the vector space $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ lifts to a set of generators $\{m_i\}$ for the $A_\mathfrak{p}$-module $M_\mathfrak{p}$, so $M_\mathfrak{p}$ is generated by $m < n$ elements [3]. Note that we can assume $m_i \in M$. Now let $\{n_i\}$ be a generating set for the A-module M. In $M_\mathfrak{p}$ we can write each n_i as $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. So setting $s = \prod s_{ij}$, we can write sn_i as $\sum a'_{ij} m_j$ for suitable a'_{ij}. None of the s_{ij} are in $\mathfrak{p}$ and so $s \notin \mathfrak{p}$, hence

[3] Let $\{\overline{v_i}\}$ be a set of generators for $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ where $v_i \in M_\mathfrak{p}$ represents the class of $\overline{v_i}$. Let N be the submodule of $M_\mathfrak{p}$ generated by the v_i. Then $M_\mathfrak{p} = \mathfrak{p}M_\mathfrak{p} + N$ since $N \to M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is surjective. So Nakayama's lemma says that $M = N$.

$\mathfrak{p} \in D(s)$. Consider another prime $\mathfrak{q} \in D(s)$. Since $s \notin \mathfrak{q}$, the element s is invertible in $A_\mathfrak{q}$ and so recalling our expressions $sn_i = \sum a'_{ij} m_j$ we can write n_i as an $A_\mathfrak{q}$-linear combination of the m_j. Since n_i generate M, they also generate $M_\mathfrak{q}$ and since the m_j generate the n_i in $M_\mathfrak{q}$ we see that the m_j generate $M_\mathfrak{q}$. Hence, $M_\mathfrak{q}$ is generated by $m < n$ elements and therefore, so is $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$. Hence $\mathfrak{q} \in \Phi(n)^c$. Since $\mathfrak{q}$ was arbitrarily chosen, this shows that $D(s) \subseteq \Phi(n)^c$. So every point in $\Phi(n)^c$ has an open neighbourhood contained in $\Phi(n)^c$, hence, $\Phi(n)^c$ is a union of open sets and therefore open itself. It follows that $\Phi(n)$ is closed.

b If $\mathscr{F}$ is locally free then for each point x there is an open nighbourhood U on which ϕ is constant. So $\Phi(n)$ is a union of open sets, and therefore open itself. In part (a) we have shown that it is also a closed set and so if X is connected, the set is either empty or the whole space. Since $\mathscr{F}$ is locally free it is locally of finite rank and so there is some n for which $\Phi(n)$ empty and therefore $\min_i\{i|\Phi(i) \neq \varnothing\}$ is a finite integer, say m. The set of points such that $\phi(x) = m$ is $\Phi(m)\backslash\Phi(m+1)$ and by definition of m this is the whole space. Hence, ϕ is constant.

c Let x be a point. We will find an open neighbourhood U of x such that $\mathscr{F}|_U$ is free of finite rank. Let $\operatorname{Spec} A$ be an affine neighbourhood of x, let $\mathfrak{p}$ be the prime of A corresponding to X, and let M be the finitely generated A-module corresponding to $\mathscr{F}|_{\operatorname{Spec} A}$. Since X is reduced, A has no nilpotents and similarly for all $A_\mathfrak{q}$ and A_f for $f \in A$, $\mathfrak{q} \in \operatorname{Spec} A$. Let $n = \dim M_\mathfrak{p} \otimes k(\mathfrak{p})$, choose a basis for this vector space and lift it to a set of generators $m_1, \dots, m_n$ for $M_\mathfrak{p}$ (using Nakayama's lemma as in the first part). Let $\{n_i\}$ be a finite set of generators for M. In $M_\mathfrak{p}$ these can each be written as $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. Setting $s = \prod s_{ij}$, these expressions hold also in A_s so we get a short exact sequence

$$0 \to \ker\phi \to A_s^{\oplus n} \xrightarrow{\phi} M_s \to 0$$

This sequence holds in each $A_\mathfrak{q}$ for $\mathfrak{q} \in D(s)$ but since ϕ is constant, each $M_\mathfrak{q} \otimes k(\mathfrak{q})$ has dimension n and so ϕ tensored with $k(\mathfrak{q})$ is an isomorphism. That is, $k(\mathfrak{q}) \otimes \ker\phi = 0$ for all $\mathfrak{q} \in D(s)$. This implies that for every element of $\ker\phi$, the components of the tuples are in $\mathfrak{q}A_s$ for all $\mathfrak{q} \in D(s)$. This implies that they are in the nilradical of A_s. But A_s is reduced, since X is reduced, so the nilradical is zero. Hence, $\ker\phi = 0$, and M_s is free.

Exercise 5.9. *Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.*

a Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$.

b Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees.

c *With the same hypothesis, we define an approxalence relation $\approx$ on graded S-modules by saying that $M \approx M'$ if there is an integer d such that $M_{\geq d} \cong M'_{\geq d}$. We will sa that a graded S-module M is quasi-finitely generated if it is approxalent to a finitely generated module. Now show that the functors $\sim$ and Γ_* induce an approxalence of categories between the category of quasi-finitely generated graded S-modules modulo the approxalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.*

Solution. a Since S_1 generates S, there is a cover of distinguished open affines $D_+(f)$ with $f \in S_1$. Now to give a global section of $M(n)^\sim$ is the same as giving a section on each $D_+(f)$ such that the intersections agree. For $m \in M_d$, the element m defines a section on each $D_+(f)$, as it has degree zero in $M(d)_{(f)} = \Gamma(D_+(f), M(n)^\sim)$, and these sections agree on the intersections where they are, again, $m \in M(d)_{(fg)} = \Gamma(D_+(fg), M(n)^\sim)$. Hence, they define a global section and we obtain a morphism of abelian groups $\alpha : M \to \Gamma_*(\widetilde{M})$.

If $s \in S_e$ and $m \in M_d$ then $s\alpha(m) \in \Gamma_*(\widetilde{M})$ is defined as the image of $m \otimes s$ in $\Gamma(X, M(d)^\sim \otimes \mathcal{O}_X(e))$ under the isomorphism $M(d)^\sim \otimes \mathcal{O}_X(e) \cong M(d+e)^\sim$. In our case (that is, where $\mathscr{F}$ from the definition on page 118 is of the form $\widetilde{M}$) the isomorphism is the one induced by $M(d) \otimes_S S(e) \cong M(d+e)$ and so $s\alpha(m) = \alpha(sm)$ and therefor α is a morphism of graded modules.

b

c Part (b) of this exercise shows that M is equivalent to $\Gamma_*(\widetilde{M})$ if M is finitely generated, and Proposition II.5.15 says that $\Gamma_*(\mathscr{F})^\sim$ is isomorphic to $\mathscr{F}$ for any quasi-coherent sheaf $\mathscr{F}$. So if Γ and $\sim$ have images in the appropriate subcategories we are done. That is, we want to show that for a quasi-finitely generated graded S-module M, the sheaf $\widetilde{M}$ is coherent, and for a coherent sheaf $\mathscr{F}$ that $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Suppose that M is a quasi-finitely generated graded S-module. Then there is a finitely generated graded S-module M' such that $M_{\geq d} \cong M'_{\geq d}$ for some d. This implies that for every element $f \in S_1$ we have $M_(f) \cong M'_{(f)}$ since $\frac{m}{f^n} = \frac{mf^d}{f^{n+d}}$. Since M' is finitely generated, $M'_{(f)}$ is finitely generated. S is generated by S_1 as an S_0 algebra so open subsets of the form $M_{(f)}$ cover $X = \operatorname{Proj} S$ and so there is a cover of X on which $\widetilde{M}$ is locally equivlanent to a coherent sheaf. Hence $\widetilde{M}$ is coherent.

Now consider a coherent $\mathcal{O}_X$-module $\mathscr{F}$. Then by Theorem II.5.17 $\mathscr{F}(n)$ is generated by a finite number of global sections for sufficiently large n. Let M' be the submodule of $\Gamma_*(\mathscr{F})$ generated by these sections. We have an inclusion $M' \hookrightarrow \Gamma_*(\mathscr{F})$ which induces an inclusion of sheaves $\widetilde{M'} \hookrightarrow \widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$ where the latter isomorphism comes from Proposition II.5.15. Tensoring with $\mathcal{O}(n)$ we have an inclusion $\widetilde{M(n)}' \hookrightarrow \mathscr{F}(n)$ that is actually an isomorphism since $\mathscr{F}(n)$ is generated by global sections in

M'. Tensoring again with $\mathcal{O}(-n)$ we then find that $\widetilde{M'}$ is isomorphic to $\mathscr{F}$. Now M' is finitely generated and so by part (b) there is a d_0 such that for all $d > d_0$ we have $M_d \cong \Gamma(X, \widetilde{M'}(d)) \cong \Gamma(X, \mathscr{F}(d)) = \Gamma_*(\mathscr{F})_d$. Hence, $M_{\geq d_0} \cong \Gamma_*(\mathscr{F})_{\geq d_0}$ and so $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Exercise 5.10. *Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \operatorname{Proj} S$.*

a *For any homogeneous ideal $I \subseteq S$, we define the saturation $\overline{I}$ of I to be $\{s \in S |$ for each $i = 0, \ldots, r$ there is an n such that $x_i^n s \in I\}$. Show that $\overline{I}$ is a homogeneous ideal.*

b *Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.*

c *If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathscr{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.*

d *There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.*

Solution. a Suppose that $s, t \in \overline{I}$ then for each i there is an n and an m such that $x_i^n s, x_i^m t \in I$. So $x_i^{n+m} st \in I$, $x_i^{n+m}(s+t) \in I$, and for any $a \in S$ we have $a x_i^n s \in I$. So $\overline{I}$ is certainly an ideal. Now write $s = s_0 + \cdots + s_k$ with each s_i homogeneous. Since x_i is homogeneous of degree 1, each $x_i^n s_k$ is homogeneous of degree $n + k$. Since I is a homogeneous ideal and $x_i^n(s_0 + \cdots + s_k) \in I$ it follows that $x_i^n s_k \in I$. Hence, $s_k \in \overline{I}$, so $\overline{I}$ is a homogeneous ideal.

b Suppose that two homogeneous ideals I_1 and I_2 define the same closed subscheme of X. Then by Proposition II.5.9 they define the same quasi-coherent sheaf of ideals $\mathscr{I}$ on X. Suppose s is a homogeneous element of I_1 of degree d. Then for each i, the element $\frac{s}{x_i^d}$ is a section of $\mathscr{I}(D_+(x_i))$. Since the sheaf of ideals of I_1 is the same as that of I_2, for each i there is some $t_i \in I_2$, homogeneous of degree d such that $\frac{s}{x_i^d} = \frac{t_i}{x_i^d}$, which implies that $x_i^{n_i}(s - t_i) = 0$ for some n_i. Since $t_i \in I_2$ so is $x_i^{n_i} t_i = x_i^{n_i} s$ and so s is in the saturation of I_2, hence $I_1 \subseteq \overline{I}_2$. By symmetry $I_2 \subseteq \overline{I}_1$ and since the operation of saturation is an idempotent we see that $\overline{I}_2 = \overline{I}_1$.

c Suppose $s \in S$ is a homogeneous element of degree d, in the saturation of $\Gamma_*(\mathscr{I}_Y)$. That is, for each i there is some n such that $x_i^n s \in \Gamma_*(\mathscr{I}_Y)$. There are only finitely many i and so we can assume it is the same n for all of them. Since $\mathscr{I}_Y$ is a subsheaf of $\mathcal{O}_X$, to show that $s \in \Gamma_*(\mathscr{I}_Y)_d = \Gamma(X, \mathscr{I}_Y(d))$ it will be enough to show that its restriction to each open $U_i = D_+(x_i)$ is in $\Gamma(U_i, \mathscr{I}_Y(d))$.

We know that $x_i^n s \in \Gamma(X, \mathscr{I}_Y(d+n))$ and so, $x_i^{-n} \otimes x_i^n s$ is a section in $\Gamma(U_i, \mathscr{I}_Y(d+n) \otimes \mathcal{O}(-n))$. But $\mathscr{I}_Y(d+n) \otimes \mathcal{O}(-n) \cong \mathscr{I}_Y(d)$ and under this isomorphism, $x_i^{-n} \otimes x_i^n s$ corresponds to $x_i^{-n} x_i^n s = s$. So $s \in \Gamma(U_i, \mathscr{I}_Y(d))$ for all i, hence $s \in \Gamma(X, \mathscr{I}_Y(d)) \subset \Gamma_*(\mathscr{I}_Y)$. So $\Gamma_*(\mathscr{I}_Y)$ is saturated.

d We have the following three sets and maps between them:

$$\left\{ \begin{array}{c} \text{homogeneous} \\ \text{ideals of } S \end{array} \right\} \underset{\sim}{\overset{\Gamma_*(-)}{\leftrightarrows}} \left\{ \begin{array}{c} \text{quasi-coherent} \\ \text{sheaves of ideals} \end{array} \right\} \underset{\text{Prop II.5.9}}{\overset{\mathscr{I}_-}{\leftrightarrows}} \left\{ \begin{array}{c} \text{closed} \\ \text{subschemes of} X \end{array} \right\}$$

Proposition II.5.9 says that the maps between the two rightmost sets are bijective, and Proposition II.5.15 says that the composition left, then right from the middle is an isomorphism. Keeping in mind the bijection between the two rightmost sets, part (b) of this exercise says that two homogeneous ideals determine the same quasi-coherent sheaf of ideals if and only if they have the same saturation. Since we already know that $\sim$ is surjective, and we now know that each preimage has a unique saturated homogeneous ideal in it, we see that $\sim$ defines a bijection between the saturated homogeneous ideals of S and quasi-coherent sheaves of ideals. Part (c) of this question says that $\Gamma_*(-)$ is its inverse.

Exercise 5.11. *Let S and T be two graded rings with $S_0 = T_0 = A$. We define the Cartesian product $S \times_A T$ to be the graded ring $\oplus_{d \geq 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$ and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.*

Solution. Let $f \in S_d$ and $g \in T_d$. We have a ring isomorphism $S_{(f)} \otimes_A T_{(g)} \to (S \times T)_{(f \otimes g)}$ defined by $\frac{s}{f^n} \otimes \frac{t}{g^m} \mapsto \frac{f^m s \otimes g^n t}{(f \otimes g)^{nm}}$ with inverse given by $\frac{s \otimes t}{(f \otimes g)^n} \mapsto \frac{s}{f^n} \otimes \frac{t}{g^n}$. Hence $D_+(f) \times D_+(g) \cong D_+(f \otimes g)$ and so composing with the inclusion $D_+(f) \times D_+(g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ we get morphisms $D_+(f \otimes g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ that are isomorphic onto their images.we get morphisms. Now consider $f' \otimes g' \in (S \times_A T)_{d'}$ and the restriction of the two morphisms $D_+(f \otimes g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ and $D_+(f' \otimes g') \to \operatorname{Proj} S \times \operatorname{Proj} T$ to their intersection $D_+(f'f \otimes g'g) = D_+(f \otimes g) \cap D_+(f' \otimes g')$. We have a diagram

$$\begin{tikzcd}
D_+(f \otimes g) \arrow[r] & D_+(f) \times D_+(g) \arrow[dr] & \\
D_+(f'f \otimes g'g) \arrow[u] \arrow[d] \arrow[r] & D_+(f'f) \times D_+(g'g) \arrow[u] \arrow[d] \arrow[r] & \operatorname{Proj} S \times \operatorname{Proj} T \\
D_+(f' \otimes g') \arrow[r] & D_+(f') \times D_+(g') \arrow[ur] &
\end{tikzcd}$$

with corresponding ring homomorphisms

$$\begin{tikzcd}
(S \times T)_{(f\otimes g)} \arrow[d] & S_{(f)} \otimes_A T_{(g)} \arrow[l] \arrow[d] \\
(S \times T)_{(f'f\otimes g'g)} & S_{(f'f)} \otimes_A T_{(g'g)} \arrow[l] \\
(S \times T)_{(f'\otimes g')} \arrow[u] & S_{(f')} \otimes_A T_{(g')} \arrow[l] \arrow[u]
\end{tikzcd}$$

Following an element through the upper square gives

$$\begin{tikzcd}
\frac{f^m s\otimes g^n t}{(f\otimes g)^{nm}} \arrow[d, squiggly] & \frac{s}{f^n} \otimes \frac{t}{g^m} \arrow[l, squiggly] \arrow[d, squiggly] \\
\frac{(f')^{nm} f^m s\otimes (g')^{nm} g^n t}{(f'f\otimes g'g)^{nm}} & \frac{(f')^n s}{(f'f)^n} \otimes \frac{(g')^m t}{(g'g)^m} \arrow[l, squiggly]
\end{tikzcd}$$

and so we see that the two squares commute. Hence, the restriction of the morphisms to intersections agree. Therefore, the morphisms patch together to give a global morphism $\operatorname{Proj} S \times T \to \operatorname{Proj} S \times \operatorname{Proj} T$ which we can see by the way we have defined it is an isomorphism.

To show that $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$ we use a similar method. That is, we provide an isomorphism on each of the distinguished opens of the form we have ben using and show that they agree on the intersections.

Exercise 5.12. *a Let X be a scheme over a scheme Y, and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L} \otimes \mathscr{M}$ is also very ample.*

b Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L} \otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

Exercise 5.13. *Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\oplus_{n\geq 0} S_n^{(d)}$ where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$ and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.*

Exercise 5.14. *Assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}_k^r$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.*

a Let $S = k[x_0, \ldots, x_r]/\Gamma_*(\mathscr{I}_X)$ be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure.

b *Use Exercise II.5.9 to show that $S_d = S'_d$ for all sufficiently large d.*

c *Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.*

d *As a corollary of (a), show that a closed subscheme $X \subseteq \mathbb{P}^r_A$ is projectively normal if and only if it is normal, and for every $n \geq)$ the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$.*

Exercise 5.15. Extension of coherent sheaves.

a *On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves.*

b *Let X be an affine noetherian scheme, U an open subset, and $\mathscr{F}$ coherent on U. Then there exists a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$.*

c *With $X, U, \mathscr{F}$ as in (b) suppose furthermore we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that we can find $\mathscr{F}'$ a coherent subsheaf of $\mathscr{G}$, with $\mathscr{F}'|_U \cong \mathscr{F}$.*

d

e

Solution. a Since the scheme is affine, a quasi-coherent sheaf corresponds to a module and a coherent sheaf a finitely generated module. So if $X = \operatorname{Spec} A$ showing that every A-module is a union of its finitely generated submodules is sufficient. But this is clear since for an A-module M, every element $m \in M$ is contained in a finitely generated submodule (take the submodule generated by m).

b Consider the pushforward $i_*\mathscr{F}$ where $i : U \to X$ is the inclusion. By Proposition 5.8(c) we know that it is at least quasi-coherent. Then by the previous part of this question it is the union of its coherent subsheaves, that is, $i_*\mathscr{F} = \cup_{\mathscr{G} \text{ coh.}} \mathscr{G}$. Restricting this union gives a system of subsheaves of $\mathscr{F}$ whose union is $\mathscr{F}$. But $\mathscr{F}$ is coherent on a Noetherian affine scheme, so the corresponding module is Noetherian. This means that the system of submodules corresponding to sheaves of the form $i^*\mathscr{G}$ (for $\mathscr{G}$ a coherent subsheaf of $i_*\mathscr{F}$) has a maximal element. But this system is directed and so the maximal element is the union. If $i^*\mathscr{F}'$ is the sheaf corresponding to this maximal element, then we have found a coherent subsheaf $\mathscr{F}'$ of $i_*\mathscr{F}$ such that $\mathscr{F}'|_U = \mathscr{F}$.

c We have a natural morphism $\mathscr{G} \to i_*i^*\mathscr{G}$ and so we can consider the subsheaf $\mathscr{G}'$ of $\mathscr{G}$ which is the preimage of $i_*\mathscr{F} \subseteq i_*(i^*\mathscr{G})$. On open sets V contained in U the morphism $\mathscr{G}(V) \to i_*i^*\mathscr{G}(V)$ is an isomorphism and so $\mathscr{G}'|_U = \mathscr{F}$. Consider the directed system of coherent subsheaves of $\mathscr{G}$ that are contained in $\mathscr{G}'$. Notice that by the following pullback diagram and the fact that the horizontal morphisms are injective, these are in one-to-one correspondence with coherent subsheaves of $i_*\mathscr{F}$, so their union is $\mathscr{G}'$.

$$\begin{tikzcd} \mathscr{G}' \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ i_*\mathscr{F} \arrow[r] & i_*i^*\mathscr{G} \end{tikzcd}$$

Now the argument of the previous part goes through. Since $\mathscr{G}'$ is the union of our directed system, and the restriction of this union to U is $\mathscr{F}$, there is a maximal element $\mathscr{F}'$ whose restriction to U is $\mathscr{F}$. So we have found a coherent subsheaf of $\mathscr{G}$ whose restriction to U is $\mathscr{F}$

d Let $\{U_i\}$ be an affine cover of X. Since X is noetherian, we can assume the the cover is finite. Restricting to U_1 and $U \cap U_1$, the hyotheses of the previous part are satisfied and so we can find a coherent subsheaf $\mathscr{F}_1$ of $\mathscr{G}|_{U_1}$ such that the restriction to $U_1 \cap U$ is isomorphic to $\mathscr{F}|_{U_1}$. Now consider $\mathscr{G}|_{U_1 \cup U_2}$ (note the union, not intersection!). Setting $X' = U_2$ and $U' = U_2 \cap (U \cup U_1)$ we have a quasi-coherent sheaf $\mathscr{G}|_{U_2}$ on $X' = U_2$ and a coherent subsheaf $\mathscr{F}_1|_{U'}$ on U'. The conditions of the previous part are satisfied and so we can find a coherent subsheaf $\mathscr{F}_2$ of $\mathscr{G}|_{U_2}$ whose restriction to U' is isomorphic to $\mathscr{F}|_{U'}$. In particular, the restriction to $U_1 \cap U_2$ is the same as that of F_1 so their "union" is a coherent subsheaf of $\mathscr{G}|_{U_1 \cap U_2}$ whose restriction to $U \cap (U_1 \cup U_2)$ is isomorphic to $\mathscr{F}|_{U \cap (U_1 \cup U_2)}$. Continuing in this was we eventually run out of U_i and end up with a coherent subsheaf $\mathscr{F}'$ of $\mathscr{G}$ such that the restriction to U is isomorphic to $\mathscr{F}$. In general, for the iterative step we will have $X' = U_i$ and $U' = U_i \cap (U \cup U_1 \cup \cdots \cup U_{i-1})$.

e If s is a section of $\mathscr{F}$ over an open set U, we apply (d) to the subsheaf of $\mathscr{F}|_U$ generated by s. In this way, for every open subset U and every section $s \in \mathscr{F}(U)$ there is a coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ such that $s \in \mathscr{F}'(U)$. Hence, $\mathscr{F}$ is the union of all of these.

Exercise 5.16. Tensor Operations on Sheaves.

a *Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.*

b *Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\wedge^r\mathscr{F} \otimes \wedge^{n-r}\mathscr{F} \to \wedge^n\mathscr{F}$ is a perfect pairing for ane r, i.e., it induces an isomorphism of $\wedge^r\mathscr{F}$ with $(\wedge^{n-r}\mathscr{F})^\vee \otimes \wedge^n\mathscr{F}$.*

c *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$,*

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p-1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

d *Same statement as (c), with exterior powers instead of symmetric powers.*

e *Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathscr{F}$.*

Solution. a Suppose that $\mathscr{F}$ is a free sheaf with basis global sections $e_1, \dots, e_n$. That is, the e_i are global sections and for each open set U, we have $\mathscr{F}(U) \cong \mathcal{O}_X(U)e_1|_U \oplus \cdots \oplus \mathcal{O}_X(U)e_n|_U$, and these isomorphisms respect the restriction homorphisms. Then the presheaf $U \mapsto \Phi(\mathscr{F}(U))$ (where Φ is one of $T^r, S^r, \bigwedge^r$) is free with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_r} | 1 \le i_1, \dots, i_r \le n\}, \{e_{i_1}e_{i_2} \dots e_{i_r} | 1 \le i_1 \le i_2 \le \cdots \le i_r \le n\}, \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 0 < i_1 < i_2 < \cdots < i_r < n+1\}$ respectively. As this presheaf of $\mathcal{O}_X$-modules is free, it is a sheaf. Now if $\mathscr{F}$ is an arbitrary locally free sheaf, we take a cover $\{U_i\}$ of X on which each $\mathscr{F}|_{U_i}$ is free. Then $\Phi(\mathscr{F})|_{U_i} = \Phi(\mathscr{F}|_{U_i})$, and so $\Phi(\mathscr{F})$ is locally free.

The ranks of $T^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$ are straightforward from the description of the basis: for T^r we have n choices for each of the i_j, of which there are r, and so there are n^r; for $\bigwedge^r$ the basis global sections are in one-to-one correspondence with subsets of $\{1, \dots, n\}$ of size r. For the rank of $S^r(\mathscr{F})$ we want to count how many tuples $(i_1, \dots, i_r) \in \{1, \dots, n\}^r$ there are such that $i_j \le i_{j+1}$. Tuples of this form are in one-to-one correspondence with subsets of $\{1, \dots, n+r-1\}$ of size $n-1$. To see this, choose a subset $\{k_1, \dots, k_{n-1}\}$ and suppose that the indexing is chosen so that $k_i < k_{i+1}$ for all i. Now define $k_i - k_{i-1} - 1$ to be the number of times that i appears in the tuple $(i_1, \dots, i_r)$. That is, our basis global section is $e_1^{k_1-1}e_n^{n+r-1-k_{n-1}} \prod_{i=2}^{n-1} e_i^{k_i-k_{i-1}-1}$. Conversely, given such a tuple $(i_1, \dots, i_r)$ we define $k_i = \sum_{j=1}^{i}(1 + \#\{i_\ell | i_\ell = j\})$. It can be seen that these are inverse operations.

b Suppose that $\mathscr{F}$ is free of rank n with basis of global sections $e_1, \dots, e_n$. Then the pairing is defined by $\omega \otimes \lambda \mapsto \omega \wedge \lambda$. Since $\mathscr{F}$ is free of rank n we have an isomorphism $\mathcal{O}_X \to \bigwedge^n \mathscr{F}$ given by $f \mapsto f(e_1 \wedge \cdots \wedge e_n)$. Every global section λ of $\bigwedge^{n-r} \mathscr{F}$ defines a morphism $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$ via $\omega \mapsto \omega \wedge \lambda$. Alternatively, given a morphism of $\mathcal{O}_X$-modules $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$ we have a morphism of global sections $\phi : \bigwedge^r \mathscr{F}(X) \to \bigwedge^n \mathscr{F}(X) \cong \mathcal{O}_X(X)$ and so we can define a global section of $\bigwedge^{n-r} \mathscr{F}$ by $\sum (-1)^{\kappa_I} \phi(e_{i_1} \wedge \cdots \wedge e_{i_r}) e_{j_1} \wedge \cdots \wedge e_{j_{n-r}}$ where the j_k are the elements of $\{1, \dots, n\}$ that don't appear as i_ℓ for some ℓ and κ_I is an appropriately chosen integer depending on the i_ℓ. It can be shown that these operations are inverses using the fact that if λ, μ are two basis global sections of $\bigwedge^r \mathscr{F}$ and $\bigwedge^{n-r} \mathscr{F}$ then $\lambda \wedge \mu$ is zero unless μ has all the complement elements to λ, in which case it is $\pm e_1 \wedge \cdots \wedge e_n$ (this is how we choose κ_I).

If $\mathscr{F}$ is not free, but still locally free, we can define such isomorphisms $\wedge^r \mathscr{F}|_{U_i} \cong (\wedge^{n-r} \mathscr{F})^\vee \otimes \wedge^n \mathscr{F}|_{U_i}$ locally on an open cover $\{U_i\}$ on which $\mathscr{F}$ is free. Then we need to check that the isomorphisms agree on their restrictions to $U_i \cap U_j$ for each i, j. But notice that when we defined the morphism $\bigwedge^r \mathscr{F} \to (\bigwedge^{n-r} \mathscr{F})^\vee$ we didn't explicitely use the basis. So since the inverse exists locally, it exists globally, by virtue of the fact that it is the inverse to an isomorphism of sheaves.

Exercise 5.17. Affine Morphisms.

a *Show that $f : X \to Y$ is an affine morphism if and only if for every open affine $V \subseteq Y$, the open subscheme $f^{-1}V$ of X is affine.*

b *An affine morphism is quasi-compact and separated. Any finite morphism is affine.*

c *Let Y be a scheme, and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Show that there is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \subset V$ of open affines of Y, the morphism $f^{-1}(U) \to f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$.*

d *If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X \to \mathbf{Spec}\,\mathcal{A} \to Y$ is an affine morphism, and $\mathcal{A} \cong f_*\mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \mathbf{Spec}\,\mathcal{A}$.*

e *Let $f : X \to Y$ be an affine morphism, and let $\mathcal{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivlance of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{A}$-modules.*

Solution. a Let $\{V_i\}$ be an open affine cover of Y such that $f^{-1}V_i$ is affine for all i. Given another open affine subset $V \subseteq Y$ we can consider the intersections $V \cap V_i$. These are open subsets of the affines V_i and so are covered by distinguished open affines $D(f_{ij})$ of the V_i. Let $A_i = \Gamma(V_i, \mathcal{O}_Y)$ and $B_i = \Gamma(f^{-1}V_i, \mathcal{O}_X)$. Then since both V_i and $f^{-1}V_i$ are affine, the morphism $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is induced by a ring homomorphism $\phi_i : A_i \to B_i$ and the preimage of $D(f_{ij})$ is $D(\phi f_{ij})$, also affine. So we have found an open cover (the $D(f_{ij})$) of V for which the preimage of every element in the cover is affine. Hence, the restricted morphism $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is affine. Now if the result holds for Y affine, then it will hold for this restricted morphism and in particular, the preimage of the whole space V_i will be affine. Hence, we just need to show that the result holds for Y affine.

So suppose that $Y = \operatorname{Spec} B$ is affine, and that the morphism f is affine. So there is an open cover $\{\operatorname{Spec} B_i\}$ of Y such that each of the preimages $f^{-1} \operatorname{Spec} B_i$ is an affine subscheme of X. We will show first that X is affine using the criterion of Exercise II.2.17. First refine the open affine cover

$\{\operatorname{Spec} B_i\}$ to one which consists of distinguished open subsets $D(f_i)$ of Y. The preimages will still be affine as the preimage of a distinguished open subset under a morphism of affine schemes is still a distinguished open affine (as we just saw above). Now since Y is affine, it is quasi-compact, so we can find a subcover of the $\{D(f_i)\}$ which is finite. So now we have a finite set of elements $\{f_i\}$ of B, which generate the unit ideal, and the preimage of each $D(f_i)$ under $f : X \to Y$ is an open affine subscheme of X. The sheaf morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a morphism of global sections, and since unity in B is a finite linear combination of the f_i, unity in $\Gamma(X, \mathcal{O}_X)$ is a finite linear combination of their images g_i under this morphism of global sections. It remains only to see that $X_{g_i} = f^{-1}D(f_i)$, and restricting to an open affine cover of X shows this. [4]

So now we know that X is affine. Open immersions are preserved by base change and so are morphisms between affine schemes. So the preimage $f^{-1}U = U \times_X Y$ of under of an open affine subset $U \subseteq X$ of X is affine.

b Let $f : X \to Y$ be an affine morphism. Take an open affine cover $\{V_i\}$ of Y. Since f is affine, each $U_i = f^{-1}V_i$ is affine, and since every affine scheme is quasi-compact, we have found a cover of Y for which each of the preimges is quasi-compact. Hence, f is quasi-compact.

Now consider the diagonal morphism $\Delta : X \to X \times_Y X$. This factors through the open subscheme $\bigcup U_i \times_{V_i} U_i \hookrightarrow X \times_Y X$ and so if $X \to \bigcup U_i \times_{V_i} U_i$ is a closed immersion, then so is Δ and f will be separated. The preimage of each $U_i \times_{V_i} U_i$ is U_i and so we just want to see that $U_i \to U_i \times_{V_i} U_i$ is a closed immersion. But this is a morphism of affine schemes whose corresponding ring homomorphism of global sections is surjective, hence, it is a closed immersion. So f is separated.

If f is finite then it follows from the definition that f is affine.

c

d That f is affine follows from the definition of $\mathbf{Spec}\,\mathcal{A}$. If $U \subseteq Y$ is an open subset, then by definition $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}U) = \mathcal{O}_X(\operatorname{Spec} \mathcal{A}(U)) = \mathcal{A}(U)$. So $f_*\mathcal{O}_X = \mathcal{A}$.

Conversely, suppose $f : X \to Y$ is an affine morphism. Let $\{V_i\}$ be an open affine cover of Y. Since f is affine $f^{-1}V_i$ is affine for each i, say $f^{-1}V_i = \operatorname{Spec} A_i$. We have $(f_*\mathcal{O}_X)|_{V_i} = f_*\mathcal{O}_{U_i}$ which is $f_*(\widetilde{A_i})$. By Proposition 5.2(d) this is $({}_{B_i}A_i)^\sim$ where $B_i = \mathcal{O}_Y(V_i)$, hence $f_*\mathcal{O}_X$ is a

[4] Explicitely, let $\operatorname{Spec} A$ be an open affine subset of X and $\rho : \Gamma(X, \mathcal{O}_X) \to A$ be the restriction morphism. Then $X_{g_i} \cap \operatorname{Spec} A = D(\rho g_i)$. But $f^{-1}D(f_i) \cap \operatorname{Spec} A$ is the preimage of the composition $\operatorname{Spec} A \to X \to \operatorname{Spec} B$. This composition gives an induced morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ and the morphism $B \to A$ of global sections of this restricted morphism factors into the morphism of global sections follows by restriction $B \to \Gamma(X, \mathcal{O}_X) \to B$. The preimage of $D(f_i)$ under the morphism $specA \to \operatorname{Spec} B$ is the distinguised open corresponding to the image of f_i in A, that is, $D(\rho g_i)$. So $f^{-1}D(f_i) \cap \operatorname{Spec} A = X_{g_i} \cap \operatorname{Spec} A$. This works for any open affine and so taking a cover of them, we see that the intersection of the two subsets $f^{-1}D(f_i), X_{g_i}$ with every element in an open cover is the same, therefore they are the same.

quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. To see that $\operatorname{Spec} \mathcal{A} \cong X$ we need to check that (i) for every open affine $V = \operatorname{Spec} B$ of Y we have $f^{-1}(V) = \operatorname{Spec} \mathcal{A}(V)$, and (ii) that for every inclusion of open affines $V' \subseteq V$ of Y the morphism $f^{-1}V' \subseteq f^{-1}V$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(V')$.

For (i) since f is affine know that $f^{-1}(V)$ as affine and therefore $f^{-1}(V) = \operatorname{Spec} \mathcal{O}_X(f^{-1}(V)) = \operatorname{Spec}(f_*\mathcal{O}_X)(V)$. For (ii), again since f is affine we know that $f^{-1}(V)$ and $f^{-1}(V')$ are affine and so $f^{-1}V' \hookrightarrow f^{-1}V$ corresponds to the ring homomorphism $\mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(f^{-1}V')$ which is none other than $f_*\mathcal{O}_X(V) \to f_*\mathcal{O}_X(V')$. That is, $\mathcal{A}(V) \to \mathcal{A}(V')$.

e

Exercise 5.18. Vector Bundles.