Chapter 5

Elements of the Calculus of Variations

This is a very exciting subject lying at the frontier between mathematics and physics. The limited space we will devote to this subject will hardly do it justice, and we will barely touch its physical significance. We recommend to anyone looking for an intellectual feast the Chapter 16 in vol.2 of "The Feynmann Lectures on Physics" [35], which in our opinion is the most eloquent argument for the raison d'être of the calculus of variations.

5.1 The least action principle

5.1.1 *The 1-dimensional Euler-Lagrange equations*

From a very "dry" point of view, the fundamental problem of the calculus of variations can be easily formulated as follows.

Consider a smooth manifold M, and let $L : \mathbb{R} \times TM \to \mathbb{R}$ by a smooth function called the *lagrangian*. Fix two points p_0, $p_1 \in M$. The *action* of a piecewise smooth path $\gamma : [0,1] \to M$ connecting these points is the real number $S(\gamma) = S_L(\gamma)$ defined by

$$S(\gamma) = S_L(\gamma) := \int_0^1 L(t, \dot{\gamma}(t), \gamma(t))dt.$$

In the calculus of variations one is interested in those paths as above with minimal action.

Example 5.1.1. Given a smooth function $U : \mathbb{R}^3 \to \mathbb{R}$ called the *potential*, we can form the lagrangian

$$L(\dot{q}, q) : \mathbb{R}^3 \times \mathbb{R}^3 \cong T\mathbb{R}^3 \to \mathbb{R},$$

given by

$$L = Q - U = \text{kinetic energy} - \text{potential energy} = \frac{1}{2}m|\dot{q}|^2 - U(q).$$

The scalar m is called the *mass*. The action of a path (trajectory) $\gamma : [0,1] \to \mathbb{R}^3$ is a quantity called the *Newtonian action*. Note that, as a physical quantity, the Newtonian action is measured in the same units as the energy. □

Example 5.1.2. To any Riemann manifold (M, g) one can naturally associate two lagrangians $L_1,\ L_2 : TM \to \mathbb{R}$ defined by

$$L_1(v, q) = g_q(v, v)^{1/2} \quad (v \in T_q M),$$

and

$$L_2(v, q) = \frac{1}{2} g_q(v, v).$$

We see that the action defined by L_1 coincides with the length of a path. The action defined by L_2 is called the *energy* of a path. □

Before we present the main result of this subsection we need to introduce a bit of notation.

Tangent bundles are very peculiar manifolds. Any collection $(q^1, \dots, q^n)$ of local coordinates on a smooth manifold M automatically induces local coordinates on TM. Any point in TM can be described by a pair (v, q), where $q \in M,\ v \in T_q M$. Furthermore, v has a decomposition

$$v = v^i \partial_i, \ \text{ where } \ \partial_i := \frac{\partial}{\partial q^i}.$$

We set $\dot{q}^i := v^i$ so that

$$v = \dot{q}^i \partial_i.$$

The collection $(\dot{q}^1, \dots, \dot{q}^n; q^1, \dots, q^n)$ defines local coordinates on TM. These are said to be *holonomic* local coordinates on TM. This will be the only type of local coordinates we will ever use.

Theorem 5.1.3 (The least action principle). *Let $L : \mathbb{R} \times TM \to \mathbb{R}$ be a lagrangian, and $p_0,\ p_1 \in M$ two fixed points. Suppose $\gamma : [0, 1] \to M$ is a* smooth *path such that the following hold.*

(i) $\gamma(i) = p_i,\ i = 0, 1.$
(ii) $S_L(\gamma) \le S_L(\tilde{\gamma})$, *for any smooth path $\tilde{\gamma} : [0, 1] \to M$ joining p_0 to p_1.*

Then the path γ satisfies the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\gamma}} L(t, \dot{\gamma}, \gamma) = \frac{\partial}{\partial \gamma} L(t, \dot{\gamma}, \gamma).$$

More precisely, if $(\dot{q}^j, q^i)$ are holonomic local coordinates on TM such that $\gamma(t) = (q^i(t))$, and $\dot{\gamma} = (\dot{q}^j(t))$, then γ is a solution of the system of nonlinear ordinary differential equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^k}(t, \dot{q}^j, q^i) = \frac{\partial L}{\partial q^k}(t, \dot{q}^j, q^i), \ \ k = 1, \dots, n = \dim M.$$

Definition 5.1.4. A path $\gamma : [0,1] \to M$ satisfying the Euler-Lagrange equations with respect to some lagrangian L is said to be an *extremal* of L. □

To get a better feeling of these equations consider the special case discussed in Example 5.1.1

$$L = \frac{1}{2}m|\dot{q}|^2 - U(q).$$

Then

$$\frac{\partial}{\partial \dot{q}}L = m\dot{q}, \;\; \frac{\partial}{\partial q}L = -\nabla U(q),$$

and the Euler-Lagrange equations become

$$m\ddot{q} = -\nabla U(q). \tag{5.1.1}$$

These are precisely Newton's equation of the motion of a particle of mass m in the force field $-\nabla U$ generated by the potential U.

In the proof of the least action principle we will use the notion of *variation of a path.*

Definition 5.1.5. Let $\gamma : [0,1] \to M$ be a smooth path. A *variation* of γ is a smooth map

$$\alpha = \alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M,$$

such that $\alpha_0(t) = \gamma(t)$. If moreover, $\alpha_s(i) = p_i \; \forall s$, $i = 0,1$, then we say that α is a *variation rel endpoints.* □

Proof of Theorem 5.1.3. Let α_s be a variation of γ rel endpoints. Then

$$S_L(\alpha_0) \leq S_L(\alpha_s) \quad \forall s,$$

so that

$$\frac{d}{ds}\,|_{s=0}\; S_L(\alpha_s) = 0.$$

Assume for simplicity that the image of γ is entirely contained in some open coordinate neighborhood U with coordinates $(q^1, \dots, q^n)$. Then, for very small $|s|$, we can write

$$\alpha_s(t) = (q^i(s,t)) \;\text{ and }\; \frac{d\alpha_s}{dt} = (\dot{q}^i(s,t)).$$

Following the tradition, we set

$$\delta\alpha := \frac{\partial \alpha}{\partial s}\,|_{s=0} = \frac{\partial q^i}{\partial s}\partial_i \quad \delta\dot{\alpha} := \frac{\partial}{\partial s}\,|_{s=0}\; \frac{d\alpha_s}{dt} = \frac{\partial \dot{q}^j}{\partial s}\frac{\partial}{\partial \dot{q}^j}.$$

The quantity $\delta\alpha$ is a vector field along γ called *infinitesimal variation* (see Figure 5.1). In fact, the pair $(\delta\alpha; \delta\dot{\alpha}) \in T(TM)$ is a vector field along $t \mapsto (\gamma(t), \dot{\gamma}(t)) \in TM$. Note that $\delta\dot{\alpha} = \frac{d}{dt}\delta\alpha$, and at endpoints $\delta\alpha = 0$.

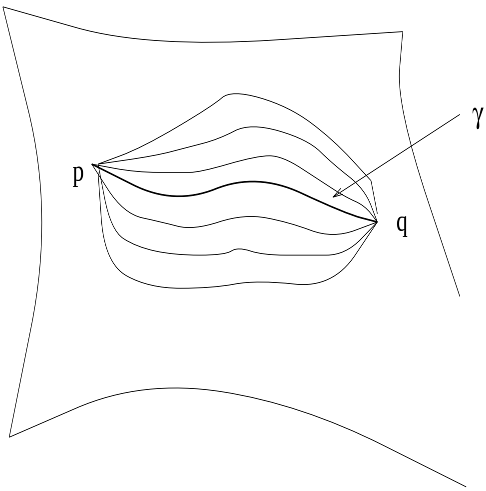

Fig. 5.1 *Deforming a path rel endpoints*

Exercise 5.1.6. Prove that if $t \mapsto X(t) \in T_{\gamma(t)}M$ is a smooth vector field along γ, such that $X(t) = 0$ for $t = 0, 1$ then there exists at least one variation rel endpoints α such that $\delta\alpha = X$.

Hint: Use the exponential map of some Riemann metric on M. □

We compute (at $s = 0$)

$$0 = \frac{d}{ds} S_L(\alpha_s) = \frac{d}{ds}\int_0^1 L(t, \dot{\alpha}_s, \alpha_s) = \int_0^1 \frac{\partial L}{\partial q^i}\delta\alpha^i dt + \int_0^1 \frac{\partial L}{\partial \dot{q}^j}\delta\dot{\alpha}_s^j dt.$$

Integrating by parts in the second term in the right-hand side we deduce

$$\int_0^1 \left\{ \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right\} \delta\alpha^i dt. \tag{5.1.2}$$

The last equality holds *for any variation* α. From Exercise 5.1.6 deduce that it holds for any vector field $\delta\alpha^i\partial_i$ along γ. At this point we use the following classical result of analysis.

If $f(t)$ is a continuous function on [0,1] such that

$$\int_0^1 f(t)g(t)dt = 0 \quad \forall g \in C_0^\infty(0,1),$$

then f is identically zero.

Using this result in (5.1.2) we deduce the desired conclusion. □

Remark 5.1.7. (a) In the proof of the least action principle we used a simplifying assumption, namely that the image of γ lies in a coordinate neighborhood. This is true locally, and for the above arguments to work it suffices to choose only a special type of variations, localized on small intervals of $[0,1]$. In terms of infinitesimal variations this means we need to look only at vector fields along γ supported in local coordinate neighborhoods. We leave the reader fill in the details.

(b) The Euler-Lagrange equations were described using holonomic local coordinates. The minimizers of the action, if any, are objects independent of any choice of local coordinates, so that the Euler-Lagrange equations have to be independent of such choices. We check this directly.

If (x^i) is another collection of local coordinates on M and $(\dot{x}^j, x^i)$ are the coordinates induced on TM, then we have the transition rules

$$x^i = x^i(q^1, \dots, q^n), \quad \dot{x}^j = \frac{\partial x^j}{\partial q^k}\dot{q}^k,$$

so that

$$\frac{\partial}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\frac{\partial}{\partial x^j} + \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k\frac{\partial}{\partial \dot{x}^j}$$

$$\frac{\partial}{\partial \dot{q}^j} = \frac{\partial \dot{x}^i}{\partial \dot{q}^j}\frac{\partial}{\partial \dot{x}^j} = \frac{\partial x^j}{\partial q^i}\frac{\partial}{\partial \dot{x}^j}$$

Then

$$\frac{\partial L}{\partial q^i} = \frac{\partial x^j}{\partial q^i}\frac{\partial L}{\partial x^j} + \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k\frac{\partial L}{\partial \dot{x}^j}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{d}{dt}\left(\frac{\partial x^j}{\partial q^i}\frac{\partial L}{\partial \dot{x}^j}\right) = \frac{\partial^2 x^j}{\partial q^k \partial q^i}\dot{q}^k\frac{\partial L}{\partial \dot{x}^j} + \frac{\partial x^j}{\partial q^i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^j}\right).$$

We now see that the Euler-Lagrange equations in the q-variables imply the Euler-Lagrange in the x-variable, i.e., these equations are *independent of coordinates.*

The æsthetically conscious reader may object to the way we chose to present the Euler-Lagrange equations. These are intrinsic equations we formulated in a coordinate dependent fashion. Is there any way of writing these equation so that the intrinsic nature is visible "on the nose"?

If the lagrangian L satisfies certain nondegeneracy conditions there are two ways of achieving this goal. One method is to consider a natural nonlinear connection ∇^L on TM as in [77]. The Euler-Lagrange equations for an extremal $\gamma(t)$ can then be rewritten as a "geodesics equation"

$$\nabla^L_{\dot{\gamma}}\dot{\gamma}.$$

The example below will illustrate this approach on a very special case when L is the lagrangian L_2 defined in Example 5.1.2, in which the extremals will turn out to be precisely the geodesics on a Riemann manifold.

Another far reaching method of globalizing the formulation of the Euler-Lagrange equation is through the Legendre transform, which again requires a nondegeneracy condition on the lagrangian. Via the Legendre transform the Euler-Lagrange equations become a system of *first order equations* on the cotangent bundle T^*M known as *Hamilton equations.*

These equations have the advantage that can be formulated on manifolds more general than the cotangent bundles, namely on *symplectic manifolds*. These are manifolds carrying a closed 2-form whose restriction to each tangent space defines a symplectic duality (see Subsection 2.2.4).

Much like the geodesics equations on a Riemann manifold, the Hamilton equations carry a lot of information about the structure of symplectic manifolds, and they are currently the focus of very intense research. For more details and examples we refer to the monographs [5, 25]. □

Example 5.1.8. Let (M, g) be a Riemann manifold. We will compute the Euler-Lagrange equations for the lagrangians L_1, L_2 in Example 5.1.2.

$$L_2(\dot{q}, q) = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j,$$

so that

$$\frac{\partial L_2}{\partial \dot{q}^k} = g_{jk}\dot{q}^j \quad \frac{\partial L_2}{\partial q^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j.$$

The Euler-Lagrange equations are

$$\ddot{q}^j g_{jk} + \frac{\partial g_{jk}}{\partial q^i}\dot{q}^i\dot{q}^j = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j. \tag{5.1.3}$$

Since $g^{km}g_{jm} = \delta_j^m$, we get

$$\ddot{q}^m + g^{km}\left(\frac{\partial g_{jk}}{\partial q^i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^i\dot{q}^j = 0. \tag{5.1.4}$$

When we derivate with respect to t the equality $g_{ik}\dot{q}^i = g_{jk}\dot{q}^j$ we deduce

$$g^{km}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^i\dot{q}^j = \frac{1}{2}g^{km}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^i\dot{q}^j.$$

We substitute this equality in (5.1.4), and we get

$$\ddot{q}^m + \frac{1}{2}g^{km}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^i\dot{q}^j = 0. \tag{5.1.5}$$

The coefficient of $\dot{q}^i\dot{q}^j$ in (5.1.5) is none other than the Christoffel symbol Γ_{ij}^m so this equation is precisely the geodesic equation. □

Example 5.1.9. Consider now the lagrangian $L_1(\dot{q}, q) = (g_{ij}\dot{q}^i\dot{q}^j)^{1/2}$. Note that the action

$$S_L\big(q(t)\big) = \int_{p_0}^{p_1} L(\dot{q}, q)dt,$$

is independent of the parametrization $t \mapsto q(t)$ since it computes the length of the path $t \mapsto q(t)$. Thus, when we express the Euler-Lagrange equations for a minimizer γ_0 of this action, we may as well assume it is parametrized by arclength, i.e., $|\dot{\gamma}_0| = 1$. The Euler-Lagrange equations for L_1 are

$$\frac{d}{dt}\frac{g_{kj}\dot{q}^j}{\sqrt{g_{ij}\dot{q}^i\dot{q}^j}} = \frac{\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j}{2\sqrt{g_{ij}\dot{q}^i\dot{q}^j}}.$$

Along the extremal we have $g_{ij}\dot{q}^i\dot{q}^j = 1$ (arclength parametrization) so that the previous equations can be rewritten as

$$\frac{d}{dt}\left(g_{kj}\dot{q}^j\right) = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j.$$

We recognize here the equation (5.1.3) which, as we have seen, is the geodesic equation in disguise. This fact almost explains why the geodesics are the shortest paths between two nearby points. □

5.1.2 *Noether's conservation principle*

This subsection is intended to offer the reader a glimpse at a fascinating subject touching both physics and geometry. We first need to introduce a bit of traditional terminology commonly used by physicists.

Consider a smooth manifold M. The tangent bundle TM is usually referred to as the *space of states* or the *lagrangian phase space*. A point in TM is said to be a *state*. A lagrangian $L : \mathbb{R} \times TM \to \mathbb{R}$ associates to each state several meaningful quantities.

- The *generalized momenta*: $p_i = \frac{\partial L}{\partial \dot{q}^i}$.
- The *energy*: $H = p_i\dot{q}^i - L$.
- The *generalized force*: $F = \frac{\partial L}{\partial q^i}$.

This terminology can be justified by looking at the lagrangian of a classical particle in a potential force field, $F = -\nabla U$,

$$L = \frac{1}{2}m|\dot{q}|^2 - U(q).$$

The momenta associated to this lagrangian are the usual kinetic momenta of the Newtonian mechanics

$$p_i = m\dot{q}^i,$$

while H is simply the total energy

$$H = \frac{1}{2}m|\dot{q}|^2 + U(q).$$

It will be convenient to think of an extremal for an arbitrary lagrangian $L(t, \dot{q}, q)$ as describing the motion of a particle under the influence of the generalized force.

Proposition 5.1.10 (Conservation of energy). *Let $\gamma(t)$ be an extremal of a* time independent *lagrangian $L = L(\dot{q}, q)$. Then the energy is conserved along γ, i.e.,*

$$\frac{d}{dt}H(\gamma, \dot{\gamma}) = 0.$$

Proof. By direct computation we get

$$\frac{d}{dt}H(\gamma,\dot{\gamma}) = \frac{d}{dt}(p_i\dot{q}^i - L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i - \frac{\partial L}{\partial q^i}\dot{q}^i - \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i$$

$$= \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right\}\dot{q}^i = 0 \quad \text{(by Euler} - \text{Lagrange)}. \qquad \square$$

At the beginning of the 20th century (1918), Emmy Noether discovered that many of the conservation laws of the classical mechanics had a geometric origin: they were, most of them, reflecting a symmetry of the lagrangian!!!

This became a driving principle in the search for conservation laws, and in fact, conservation became almost synonymous with symmetry. It eased the leap from classical to quantum mechanics, and one can say it is a very important building block of quantum physics in general. In the few instances of conservation laws where the symmetry was not apparent the conservation was always "blamed" on a "hidden symmetry". What is then this Noether principle?

To answer this question we need to make some simple observations.

Let X be a vector field on a smooth manifold M defining a global flow Φ^s. This flow induces a flow Ψ^s on the tangent bundle TM defined by

$$\Psi^s(v, x) = \big(\,\Phi^s_*(v), \Phi^s(x)\,\big).$$

One can think of Ψ^s as defining an action of the additive group $\mathbb{R}$ on TM. Alternatively, physicists say that X is an *infinitesimal symmetry* of the given mechanical system described by the lagrangian L.

Example 5.1.11. Let M be the unit round sphere $S^2 \subset \mathbb{R}^3$. The rotations about the z-axis define a 1-parameter group of isometries of S^2 generated by $\frac{\partial}{\partial\theta}$, where θ is the longitude on S^2. $\square$

Definition 5.1.12. Let L be a lagrangian on TM, and X a vector field on M. The lagrangian L is said to be X-*invariant* if

$$L \circ \Psi^s = L, \;\; \forall s. \qquad \square$$

Denote by $\mathcal{X} \in \mathrm{Vect}\,(TM)$ the infinitesimal generator of Ψ^s, and by $\mathcal{L}_{\mathcal{X}}$ the Lie derivative on TM along $\mathcal{X}$. We see that L is X-invariant if and only if

$$\mathcal{L}_{\mathcal{X}}L = 0.$$

We describe this derivative using the local coordinates $(\dot{q}^j, q^i)$. Set

$$\big(\,\dot{q}^j(s), q^i(s)\,\big) := \Psi^s(\dot{q}^j, q^i).$$

Then

$$\frac{d}{ds}|_{s=0}\, q^i(s) = X^k \delta^i_k.$$

To compute $\frac{d}{ds}|_{s=0}\, \dot{q}^j(s) \frac{\partial}{\partial q^j}$ we use the definition of the Lie derivative on M

$$-\frac{d}{ds}\dot{q}^j \frac{\partial}{\partial q^j} = L_X(\dot{q}^i \frac{\partial}{\partial q^i}) = \left(X^k \frac{\partial \dot{q}^j}{\partial q^k} - \dot{q}^k \frac{\partial X^j}{\partial q^k} \right) \frac{\partial}{\partial q^j} = -\dot{q}^i \frac{\partial X^j}{\partial q^i} \frac{\partial}{\partial q^j},$$

since $\partial \dot{q}^j / \partial q^i = 0$ on TM. Hence

$$\mathfrak{X} = X^i \frac{\partial}{\partial q^i} + \dot{q}^k \frac{\partial X^j}{\partial q^k} \frac{\partial}{\partial \dot{q}^j}.$$

Corollary 5.1.13. *The lagrangian L is X-invariant if and only if*

$$X^i \frac{\partial L}{\partial q^i} + \dot{q}^k \frac{\partial X^j}{\partial q^k} \frac{\partial L}{\partial \dot{q}^j} = 0. \tag{5.1.6}$$

□

Theorem 5.1.14 (E. Noether). *If the lagrangian L is X-invariant, then the quantity*

$$P_X = X^i \frac{\partial L}{\partial \dot{q}^i} = X^i p_i$$

is conserved along the extremals of L.

Proof. Consider an extremal $\gamma = \gamma(q^i(t))$ of L. We compute

$$\frac{d}{dt} P_X(\gamma, \dot{\gamma}) = \frac{d}{dt}\left\{ X^i(\gamma(t)) \frac{\partial L}{\partial \dot{q}^i} \right\} = \frac{\partial X^i}{\partial q^k} \dot{q}^k \frac{\partial L}{\partial \dot{q}^i} + X^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right)$$

$$\stackrel{\text{Euler-Lagrange}}{=} \frac{\partial X^i}{\partial q^k} \dot{q}^k \frac{\partial L}{\partial \dot{q}^i} + X^i \frac{\partial L}{\partial q^i} \stackrel{(5.1.6)}{=} 0.$$ □

The classical conservation-of-momentum law is a special consequence of Noether's theorem.

Corollary 5.1.15. *Consider a lagrangian $L = L(t, \dot{q}, q)$ on $\mathbb{R}^n$. If $\frac{\partial L}{\partial q^i} = 0$, i.e., the i-th component of the force is zero, then $\frac{dp_i}{dt} = 0$ along any extremal, i.e., the i-th component of the momentum is conserved.*

Proof. Take $X = \frac{\partial}{\partial q^i}$ in Noether's conservation law. □

The conservation of momentum has an interesting application in the study of geodesics.

Example 5.1.16. (Geodesics on surfaces of revolution). Consider a surface of revolution S in $\mathbb{R}^3$ obtained by rotating about the z-axis the curve $y = f(z)$

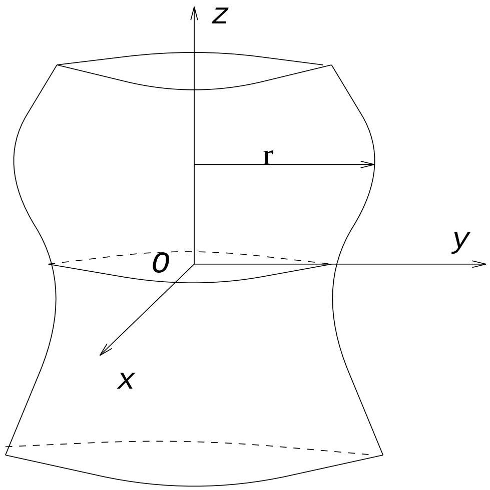

Fig. 5.2 *A surface of revolution*

situated in the yz plane. If we use cylindrical coordinates (r, θ, z) we can describe S as $r = f(z)$.

In these coordinates, the Euclidean metric in $\mathbb{R}^3$ has the form

$$ds^2 = dr^2 + dz^2 + r^2 d\theta^2.$$

We can choose (z, θ) as local coordinates on S, then the induced metric has the form

$$g_S = \{1 + (f'(z))^2\}dz^2 + f^2(z)d\theta^2 = A(z)dz^2 + r^2 d\theta^2, \;\; r = f(z).$$

The lagrangian defining the geodesics on S is

$$L = \frac{1}{2}\left(A\dot{z}^2 + r^2\dot{\theta}^2\right).$$

We see that L is independent of θ: $\frac{\partial L}{\partial \theta} = 0$, so that the generalized momentum

$$\frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$$

is conserved along the geodesics. □

This fact can be given a nice geometric interpretation. Consider a geodesic

$$\gamma(t) = (z(t), \theta(t)),$$

and compute the angle ϕ between $\dot{\gamma}$ and $\frac{\partial}{\partial \theta}$. We get

$$\cos\phi = \frac{\langle \dot{\gamma}, \partial/\partial\theta \rangle}{|\dot{\gamma}| \cdot |\partial/\partial\theta|} = \frac{r^2\dot{\theta}}{r|\dot{\gamma}|},$$

i.e., $r\cos\phi = r^2\dot{\theta}|\dot{\gamma}|^{-1}$. The conservation of energy implies that $|\dot{\gamma}|^2 = 2L = H$ is constant along the geodesics. We deduce the following classical result.

Theorem 5.1.17 (Clairaut). *On a surface of revolution $r = f(z)$ the quantity $r\cos\phi$ is constant along any geodesic, where $\phi \in (-\pi, \pi)$ is the oriented angle the geodesic makes with the parallels $z = const.$* □

Exercise 5.1.18. Describe the geodesics on the round sphere S^2, and on the cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$. □

5.2 The variational theory of geodesics

We have seen that the paths of minimal length between two points on a Riemann manifold are necessarily geodesics.

However, given a geodesic joining two points q_0, q_1 it may happen that it is not a minimal path. This should be compared with the situation in calculus, when a critical point of a function f may not be a minimum or a maximum.

To decide this issue one has to look at the second derivative. This is precisely what we intend to do in the case of geodesics. This situation is a bit more complicated since the action functional

$$S = \frac{1}{2}\int |\dot{\gamma}|^2 dt$$

is not defined on a finite dimensional manifold. It is a function defined on the "space of all paths" joining the two given points. With some extra effort this space can be organized as an infinite dimensional manifold. We will not attempt to formalize these prescriptions, but rather follow the ad-hoc, intuitive approach of [71].

5.2.1 *Variational formulae*

Let M be a connected Riemann manifold, and consider $p, q \in M$. Denote by $\Omega_{p,q} = \Omega_{p,q}(M)$ the space of all *continuous*, *piecewise smooth* paths $\gamma : [0,1] \to M$ connecting p to q.

An *infinitesimal variation* of a path $\gamma \in \Omega_{p,q}$ is a *continuous*, *piecewise smooth* vector field V along γ such that $V(0) = 0$ and $V(1) = 0$ and

$$\lim_{h \searrow 0} V(t \pm h)$$

exists for every $t \in [0,1]$, they are vectors $V(t)^{\pm} \in T_{\gamma}(t)M$, and $V(t)^+ = V^(t)^-$ for all but finitely many t-s. The space of infinitesimal variations of γ is an infinite dimensional linear space denoted by $T_\gamma = T_\gamma\Omega_{p,q}$.

Definition 5.2.1. Let $\gamma \in \Omega_{p,q}$. A *variation* of γ is a continuous map

$$\alpha = \alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$$

such that

(i) $\forall s \in (-\varepsilon, \varepsilon)$, $\alpha_s \in \Omega_{p,q}$.

(ii) There exists a partition $0 = t_0 < t_1 \cdots < t_{k-1} < t_k = 1$ of [0,1] such that the restriction of α to each $(-\varepsilon, \varepsilon) \times (t_{i-1}, t_i)$ is a smooth map. □

Every variation α of γ defines an *infinitesimal variation*

$$\delta\alpha := \frac{\partial \alpha_s}{\partial s}\,|_{s=0}\ .$$

Exercise 5.2.2. Given $V \in T_\gamma$ construct a variation α such that $\delta\alpha = V$. □

Consider now the *energy functional*

$$E : \Omega_{p,q} \to \mathbb{R},\ \ E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt.$$

Fix $\gamma \in \Omega_{p,q}$, and let α be a variation of γ. The velocity $\dot\gamma(t)$ has a finite number of discontinuities, so that the quantity

$$\Delta_t\dot\gamma = \lim_{h\to 0^+}(\dot\gamma(t+h) - \dot\gamma(t-h))$$

is nonzero only for finitely many t's.

Theorem 5.2.3 (The first variation formula).

$$E_*(\delta\alpha) := \frac{d}{ds}\,|_{s=0}\ E(\alpha_s) = -\sum_t \langle(\delta\alpha)(t), \Delta_t\dot\gamma\rangle - \int_0^1 \langle \delta\alpha, \nabla_{\frac{d}{dt}}\dot\gamma\rangle dt, \tag{5.2.1}$$

where ∇ denotes the Levi-Civita connection. (Note that the right-hand side depends on α only through $\delta\alpha$ so it is really a linear function on T_γ.)

Proof. Set $\dot\alpha_s = \frac{\partial\alpha_s}{\partial t}$. We differentiate under the integral sign using the equality

$$\frac{\partial}{\partial s}|\dot\alpha_s|^2 = 2\langle \nabla_{\frac{\partial}{\partial s}}\dot\alpha_s, \dot\alpha_s\rangle,$$

and we get

$$\frac{d}{ds}\,|_{s=0}\ E(\alpha_s) = \int_0^1 \langle \nabla_{\frac{\partial}{\partial s}}\dot\alpha_s, \dot\alpha_s\rangle\,|_{s=0}\ dt.$$

Since the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ commute we have $\nabla_{\frac{\partial}{\partial s}}\frac{\partial\alpha}{\partial t} = \nabla_{\frac{\partial}{\partial t}}\frac{\partial\alpha}{\partial s}$.

Let $0 = t_0 < t_2 < \cdots < t_k = 1$ be a partition of [0,1] as in Definition 5.2.1. Since $\alpha_s = \gamma$ for $s = 0$ we conclude

$$E_*(\delta\alpha) = \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \langle \nabla_{\frac{\partial}{\partial t}}\delta\alpha, \dot\gamma\rangle.$$

We use the equality

$$\frac{\partial}{\partial t}\langle \delta\alpha, \dot\gamma\rangle = \langle \nabla_{\frac{\partial}{\partial t}}\delta\alpha, \dot\gamma\rangle + \langle \delta\alpha, \nabla_{\frac{\partial}{\partial t}}\dot\gamma\rangle$$

to integrate by parts, and we obtain

$$E_*(\delta\alpha) = \sum_{i=1}^{k} \langle \delta\alpha, \dot\gamma\rangle \Big|_{t_{i-1}}^{t_i} - \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \delta\alpha, \nabla_{\frac{\partial}{\partial t}} \dot\gamma\rangle dt.$$

This is precisely equality (5.2.1). □

Definition 5.2.4. A path $\gamma \in \Omega_{p,q}$ is called *critical* if

$$E_*(V) = 0, \ \ \forall V \in T_\gamma.$$ □

Corollary 5.2.5. *A path $\gamma \in \Omega_{p,q}$ is critical if and only if it is a geodesic.* □

Exercise 5.2.6. Prove the above corollary. □

Remark 5.2.7. Note that, a priori, a critical path may have a discontinuous first derivative. The above corollary shows that this is not the case: the criticality also implies smoothness. This is a manifestation of a more general analytical phenomenon called *elliptic regularity*. We will have more to say about it in Chapter 11. □

The map $E_* : T_\gamma \to \mathbb{R}$, $\delta\alpha \mapsto E_*(\delta\alpha)$ is called the first derivative of E at $\gamma \in \Omega_{p,q}$. We want to define a second derivative of E in order to address the issue raised at the beginning of this section. We will imitate the finite dimensional case which we now briefly analyze.

Let $f : X \to \mathbb{R}$ be a smooth function on the finite dimensional smooth manifold X. If x_0 is a critical point of f, i.e., $df(x_0) = 0$, then we can define the *Hessian* at x_0

$$f_{**} : T_{x_0}X \times T_{x_0}X \to \mathbb{R}$$

as follows. Given $V_1, V_2 \in T_{x_0}X$, consider a smooth map $(s_1, s_2) \mapsto \alpha(s_1, s_2) \in X$ such that

$$\alpha(0,0) = x_0 \ \text{ and } \ \frac{\partial\alpha}{\partial s_i}(0,0) = V_i, \ \ i = 1, 2. \tag{5.2.2}$$

Now set

$$f_{**}(V_1, V_2) = \frac{\partial^2 f(\,\alpha(s_1, s_2)\,)}{\partial s_1 \partial s_2}\,|_{(0,0)}\,.$$

Note that since x_0 is a critical point of f, the Hessian $f_{**}(V_1, V_2)$ is independent of the function α satisfying (5.2.2).

We now return to our energy functional $E : \Omega_{p,q} \to \mathbb{R}$. Let $\gamma \in \Omega_{p,q}$ be a critical path. Consider a 2-parameter variation of γ

$$\alpha := \alpha_{s_1,s_2} : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times [0, 1] \to M, \ \ (s_1, s_2, t) \mapsto \alpha_{s_1,s_2}(t).$$

Set $U := (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^2$ and $\gamma := \alpha_{0,0}$. The map α is continuous, and has second order derivatives everywhere except maybe on finitely many "coordinate" planes $s_i =$ const, or $t =$ const. Set $\delta_i\alpha := \frac{\partial\alpha}{\partial s_i}\,|_{(0,0)}$, $i = 1, 2$. Note that $\delta_i\alpha \in T_\gamma$.

Exercise 5.2.8. Given $V_1, V_2 \in T_\gamma$ construct a 2-parameter variation α such that $V_i = \delta_i\alpha$. □

We can now define the *Hessian* of E at γ by

$$E_{**}(\delta_1\alpha, \delta_2\alpha) := \frac{\partial^2 E(\alpha_{s_1,s_2})}{\partial s_1 \partial s_2}\,|_{(0,0)}\ .$$

Theorem 5.2.9 (The second variation formula).

$$E_{**}(\delta_1\alpha, \delta_2\alpha) = -\sum_t \langle \delta_2\alpha, \Delta_t \delta_1\alpha\rangle - \int_0^1 \langle \delta_2\alpha, \nabla^2 \frac{\partial}{\partial t}\delta_1\alpha - R(\dot\gamma, \delta_1\alpha)\dot\gamma\rangle dt, \quad (5.2.3)$$

where R denotes the Riemann curvature. In particular, E_{**} is a bilinear functional on T_γ.

Proof. According to the first variation formula we have

$$\frac{\partial E}{\partial s_2} = -\sum_t \langle \delta_2\alpha, \Delta_t \frac{\partial \alpha}{\partial t}\rangle - \int_0^1 \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\rangle dt.$$

Hence

$$\frac{\partial^2 E}{\partial s_1 \partial s_2} = -\sum_t \langle \nabla_{\frac{\partial}{\partial s_1}} \delta_2\alpha, \Delta_1 \dot\gamma\rangle - \sum_t \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}} \left(\Delta_t \frac{\partial\alpha}{\partial t}\right)\rangle$$

$$-\int_0^1 \langle \nabla_{\frac{\partial}{\partial s_1}} \delta_2\alpha, \nabla_{\frac{\partial}{\partial t}} \dot\gamma\rangle dt - \int_0^1 \langle \delta_2\alpha, \nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial\alpha}{\partial t}\rangle dt. \quad (5.2.4)$$

Since γ is a geodesic, we have

$$\Delta_t \dot\gamma = 0 \ \text{ and } \ \nabla_{\frac{\partial}{\partial t}} \dot\gamma = 0.$$

Using the commutativity of $\frac{\partial}{\partial t}$ with $\frac{\partial}{\partial s_1}$ we deduce

$$\nabla_{\frac{\partial}{\partial s_1}} \left(\Delta_t \frac{\partial\alpha}{\partial t}\right) = \Delta_t \left(\nabla_{\frac{\partial}{\partial s_1}} \frac{\partial\alpha}{\partial t}\right) = \Delta_t \left(\nabla_{\frac{\partial}{\partial t}} \delta_1\alpha\right).$$

Finally, the definition of the curvature implies

$$\nabla_{\frac{\partial}{\partial s_1}} \nabla_{\frac{\partial}{\partial t}} = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s_1}} + R(\delta_1\alpha, \dot\gamma).$$

Putting all the above together we deduce immediately the equality (5.2.3). □

Corollary 5.2.10.

$$E_{**}(V_1, V_2) = E_{**}(V_2, V_1), \ \ \forall V_1, V_2 \in T_\gamma. \qquad \square$$

5.2.2 *Jacobi fields*

In this subsection we will put to work the elements of calculus of variations presented so far. Let (M, g) be a Riemann manifold and $p, q \in M$.

Definition 5.2.11. Let $\gamma \in \Omega_{p,q}$ be a geodesic. A *geodesic variation* of γ is a smooth map $\alpha_s(t) : (-\varepsilon, \varepsilon) \times [0,1] \to M$ such that, $\alpha_0 = \gamma$, and $t \mapsto \alpha_s(t)$ is a geodesic for all s. We set as usual $\delta\alpha = \frac{\partial \alpha}{\partial s}|_{s=0}$. □

Proposition 5.2.12. *Let $\gamma \in \Omega_{p,q}$ be a geodesic and (α_s) a geodesic variation of γ. Then the infinitesimal variation $\delta\alpha$ satisfies the* Jacobi equation

$$\nabla_t^2 \delta\alpha = R(\dot\gamma, \delta\alpha)\dot\gamma \quad (\nabla_t = \nabla_{\frac{\partial}{\partial t}}).$$

Proof.

$$\nabla_t^2 \delta\alpha = \nabla_t\left(\nabla_t \frac{\partial \alpha}{\partial s}\right) = \nabla_t\left(\nabla_s \frac{\partial \alpha}{\partial t}\right)$$

$$= \nabla_s\left(\nabla_t \frac{\partial \alpha}{\partial t}\right) + R(\dot\gamma, \delta\alpha)\frac{\partial \alpha}{\partial t} = R(\dot\gamma, \delta\alpha)\frac{\partial \alpha}{\partial t}. \quad \square$$

Definition 5.2.13. A smooth vector field J along a geodesic γ is called a *Jacobi field* if it satisfies the *Jacobi equation*

$$\nabla_t^2 J = R(\dot\gamma, J)\dot\gamma. \quad \square$$

Exercise 5.2.14. Show that if J is a Jacobi field along a geodesic γ, then there exists a geodesic variation α_s of γ such that $J = \delta\alpha$. □

Exercise 5.2.15. Let $\gamma \in \Omega_{p,q}$, and J a vector field along γ.
(a) Prove that J is a Jacobi field if and only if

$$E_{**}(J, V) = 0, \;\; \forall V \in T_\gamma.$$

(b) Show that a vector field J along γ which *vanishes at endpoints*, is a Jacobi field if any only if $E_{**}(J, W) = 0$, for *all* vector fields W along γ. □

Exercise 5.2.16. Let $\gamma \in \Omega_{p,q}$ be a geodesic. Define $\mathcal{J}_p$ to be the space of Jacobi fields V along γ such that $V(p) = 0$. Show that $\dim \mathcal{J}_p = \dim M$, and moreover, the evaluation map

$$\mathbf{ev}_q : \mathcal{J}_p \to T_pM \quad V \mapsto \nabla_t V(p)$$

is a linear isomorphism. □

Definition 5.2.17. Let $\gamma(t)$ be a geodesic. Two points $\gamma(t_1)$ and $\gamma(t_2)$ on γ are said to be *conjugate* along γ if there exists a nontrivial Jacobi field J along γ such that $J(t_i) = 0$, $i = 1, 2$. □

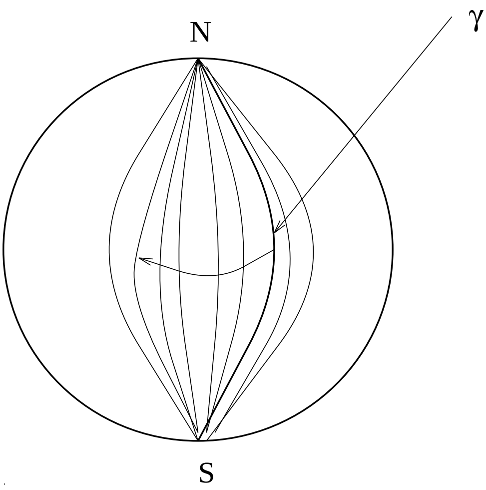

Fig. 5.3 *The poles are conjugate along meridians.*

Example 5.2.18. Consider $\gamma : [0, 2\pi] \to S^2$ a meridian on the round sphere connection the poles. One can verify easily (using Clairaut's theorem) that γ is a geodesic. The counterclockwise rotation by an angle θ about the z-axis will produce a new meridian, hence a new geodesic γ_θ. Thus (γ_θ) is a geodesic variation of γ with fixed endpoints. $\delta\gamma$ is a Jacobi field vanishing at the poles. We conclude that the poles are conjugate along any meridian (see Figure 5.3). □

Definition 5.2.19. A geodesic $\gamma \in \Omega_{p,q}$is said to be *nondegenerate* if q is not conjugated to p along γ. □

The following result (partially) explains the geometric significance of conjugate points.

Theorem 5.2.20. *Let $\gamma \in \Omega_{p,q}$ be a* nondegenerate, minimal geodesic. *Then p is conjugate with no point on γ other than itself. In particular, a geodesic segment containing conjugate points* cannot be minimal !

Proof. We argue by contradiction. Let $p_1 = \gamma(t_1)$ be a point on γ conjugate with p. Denote by J_t a Jacobi field along $\gamma\,|_{[0,t_1]}$ such that $J_0 = 0$ and $J_{t_1} = 0$. Define $V \in T_\gamma$ by

$$V_t = \begin{cases} J_t, & t \in [0, t_1] \\ 0, & t \geq t_1. \end{cases}$$

We will prove that V_t is a Jacobi field along γ which contradicts the nondegeneracy of γ.

Step 1.

$$E_{**}(U, U) \geq 0 \quad \forall U \in T_\gamma. \tag{5.2.5}$$

Indeed, let α_s denote a variation of γ such that $\delta\alpha = U$. One computes easily that

$$\frac{d^2}{ds^2} E(\alpha_{s^2}) = 2E_{**}(U, U).$$

Since γ is *minimal* for any small s we have length $(\alpha_{s^2}) \geq$ length (α_0) so that

$$E(\alpha_{s^2}) \geq \frac{1}{2}\left(\int_0^1 |\dot{\alpha}_{s^2}| dt\right)^2 = \frac{1}{2}\text{length}\,(\alpha_{s^2})^2 \geq \frac{1}{2}\text{length}\,(\alpha_0)^2$$

$$= \frac{1}{2}\text{length}\,(\gamma)^2 = E(\alpha_0).$$

Hence

$$\frac{d^2}{ds^2}\,|_{s=0}\; E(\alpha_{s^2}) \geq 0.$$

This proves (5.2.5).

Step 2. $E_{**}(V, V) = 0$. This follows immediately from the second variation formula and the fact that the nontrivial portion of V is a Jacobi field.

Step 3.

$$E_{**}(U, V) = 0, \;\; \forall U \in T_\gamma.$$

From (5.2.5) and Step 2 we deduce

$$0 = E_{**}(V, V) \leq E_{**}(V + \tau U, V + \tau U) = f_U(\tau) \quad \forall \tau.$$

Thus, $\tau = 0$ is a global minimum of $f_U(\tau)$ so that

$$f_U'(0) = 0.$$

Step 3 follows from the above equality using the bilinearity and the symmetry of E_{**}. The final conclusion (that V is a Jacobi field) follows from Exercise 5.2.15. □

Exercise 5.2.21. Let $\gamma : \mathbb{R} \to M$ be a geodesic. Prove that the set

$$\left\{\, t \in \mathbb{R}\;;\; \gamma(t) \text{ is conjugate to } \gamma(0) \,\right\}$$

is discrete. □

Definition 5.2.22. Let $\gamma \in \Omega_{p,q}$ be a geodesic. We define its *index*, denoted by ind (γ), as the cardinality of the set

$$C_\gamma = \left\{\, t \in (0, 1)\;;\; \text{is conjugate to } \gamma(0) \,\right\}$$

which by Exercise 5.2.21 is finite. □

Theorem 5.2.20 can be reformulated as follows: the index of a nondegenerate minimal geodesic is zero.

The index of a geodesic obviously depends on the curvature of the manifold. Often, this dependence is very powerful.

Theorem 5.2.23. *Let M be a Riemann manifold with non-positive* sectional curvature, *i.e.,*

$$\langle R(X, Y)Y, X\rangle \leq 0 \;\; \forall X, Y \in T_xM \;\; \forall x \in M. \tag{5.2.6}$$

Then for any $p, q \in M$ and any geodesic $\gamma \in \Omega_{p,q}$, ind $(\gamma) = 0$.

Proof. It suffices to show that for any geodesic $\gamma : [0,1] \to M$ the point $\gamma(1)$ is not conjugated to $\gamma(0)$.

Let J_t be a Jacobi field along γ vanishing at the endpoints. Thus

$$\nabla_t^2 J = R(\dot\gamma, J)\dot\gamma,$$

so that

$$\int_0^1 \langle \nabla_t^2 J, J\rangle dt = \int_0^1 \langle R(\dot\gamma, J)\dot\gamma, J\rangle dt = -\int_0^1 \langle R(J,\dot\gamma)\dot\gamma, J\rangle dt.$$

We integrate by parts the left-hand side of the above equality, and we deduce

$$\langle \nabla_t J, J\rangle|_0^1 - \int_0^1 |\nabla_t J|^2 dt = -\int_0^1 \langle R(J,\dot\gamma)\dot\gamma, J\rangle dt.$$

Since $J(\tau) = 0$ for $\tau = 0, 1$, we deduce using (5.2.6)

$$\int_0^1 |\nabla_t J|^2 dt \leq 0.$$

This implies $\nabla_t J = 0$ which coupled with the condition $J(0) = 0$ implies $J \equiv 0$. The proof is complete. $\square$

The notion of conjugacy is intimately related to the behavior of the exponential map.

Theorem 5.2.24. *Let (M, g) be a connected, complete, Riemann manifold and $q_0 \in M$. A point $q \in M$ is conjugated to q_0 along some geodesic if and only if it is a critical value of the exponential map*

$$\exp_{q_0} : T_{q_0}M \to M.$$

Proof. Let $q = \exp_{q_0} v$, $v \in T_{q_0}M$. Assume first that q is a critical value for $\exp_{q_0}$, and v is a critical point. Then $D_v \exp_{q_0}(X) = 0$, for some $X \in T_v(T_{q_0}M)$. Let $v(s)$ be a path in $T_{q_0}M$ such that $v(0) = v$ and $\dot v(0) = X$. The map $(s,t) \mapsto \exp_{q_0}(tv(s))$ is a geodesic variation of the radial geodesic $\gamma_v :\ t \mapsto \exp_{q_0}(tv)$. Hence, the vector field

$$W = \frac{\partial}{\partial s}|_{s=0} \exp_{q_0}(tv(s))$$

is a Jacobi field along γ_v. Obviously $W(0) = 0$, and moreover

$$W(1) = \frac{\partial}{\partial s}|_{s=0} \exp_{q_0}(v(s)) = D_v \exp_{q_0}(X) = 0.$$

On the other hand this is a nontrivial field since

$$\nabla_t W = \nabla_s|_{s=0} \frac{\partial}{\partial t} \exp_{q_0}(tv(s)) = \nabla_s v(s)|_{s=0} \neq 0.$$

This proves q_0 and q are conjugated along γ_v.

Conversely, assume v is not a critical point for $\exp_{q_0}$. For any $X \in T_v(T_{q_0}M)$ denote by J_X the Jacobi field along γ_v such that

$$J_X(q_0) = 0. \tag{5.2.7}$$

The existence of such a Jacobi field follows from Exercise 5.2.16. As in that exercise, denote by $\mathfrak{J}_{q_0}$ the space of Jacobi fields J along γ_v such that $J(q_0) = 0$. The map

$$T_v(T_{q_0}M) \to \mathfrak{J}_{q_0} \quad X \mapsto J_X$$

is a linear isomorphism. Thus, a Jacobi field along γ_v vanishing at both q_0 and q must have the form J_X, where $X \in T_v(T_{q_0}M)$ satisfies $D_v \exp_{q_0}(X) = 0$. Since v is not a critical point, this means $X = 0$ so that $J_X \equiv 0$. □

Corollary 5.2.25. *On a complete Riemann manifold M with non-positive sectional curvature the exponential map $\exp_q$ has no critical values for any $q \in M$.* □

We will see in the next chapter that this corollary has a lot to say about the topology of M.

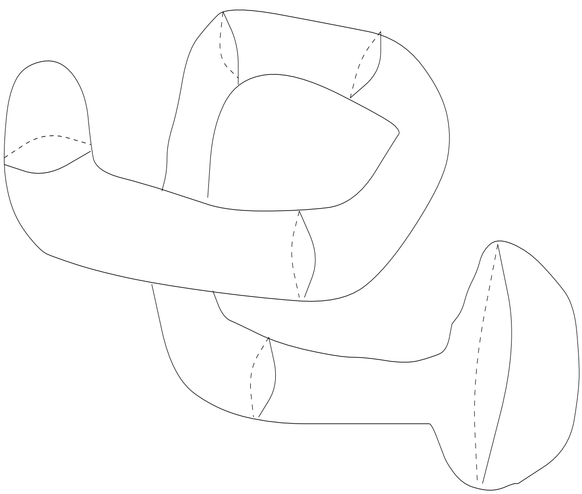

Fig. 5.4 *Lengthening a sphere.*

Consider now the following experiment. Stretch the round two-dimensional sphere of radius 1 until it becomes "very long". A possible shape one can obtain may look like in Figure 5.4. The long tube is very similar to a piece of cylinder so that the total (= scalar) curvature is very close to zero, in other words is very small. The lesson to learn from this intuitive experiment is that the price we have to pay for lengthening the sphere is decreasing the curvature. Equivalently, a highly curved surface cannot have a large diameter. Our next result offers a more quantitative description of this phenomenon.

Theorem 5.2.26 (Myers). *Let M be an n-dimensional complete Riemann manifold. If for all $X \in \mathrm{Vect}\,(M)$*

$$\mathrm{Ric}\,(X,X) \geq \frac{(n-1)}{r^2}|X|^2,$$

then every geodesic of length $\geq \pi r$ has conjugate points and thus is not minimal. Hence

$$\mathrm{diam}\,(M) = \sup\{\mathrm{dist}\,(p,q)\ ;\ p,q \in M\} \leq \pi r,$$

and in particular, Hopf-Rinow theorem implies that M must be compact. □

Proof. Fix a minimal geodesic $\gamma : [0,\ell] \to M$ of length ℓ, and let $e_i(t)$ be an orthonormal basis of vector fields along γ such that $e_n(t) = \dot\gamma(t)$. Set $q_0 = \gamma(0)$, and $q_1 = \gamma(\ell)$. Since γ is minimal we deduce

$$E_{**}(V,V) \geq 0 \quad \forall V \in T_\gamma.$$

Set $W_i = \sin(\pi t/\ell)e_i$. Then

$$E_{**}(W_i,W_i) = -\int_0^\ell \langle W_i, \nabla_t W_i + R(W_i,\dot\gamma)\dot\gamma\rangle dt$$

$$= \int_0^\ell \sin^2(\pi t/\ell)\left(\pi^2/\ell^2 - \langle R(e_i,\dot\gamma)\dot\gamma, e_i\rangle\right)dt.$$

We sum over $i = 1,\dots,n-1$, and we obtain

$$\sum_{i=1}^{n-1} E_{**}(W_i,W_i) = \int_0^\ell \sin^2 \pi t/\ell\left((n-1)\pi^2/\ell^2 - \mathrm{Ric}\,(\dot\gamma,\dot\gamma)\right)dt \geq 0.$$

If $\ell > \pi r$, then

$$(n-1)\pi^2/\ell^2 - \mathrm{Ric}\,(\dot\gamma,\dot\gamma) < 0,$$

so that,

$$\sum_{i=1}^{n-1} E_{**}(W_i,W_i) < 0.$$

Hence, at least for some W_i, we have $E_{**}(W_i,W_i) < 0$. This contradicts the minimality of γ. The proof is complete. □

We already know that the Killing form of a *compact, semisimple* Lie group is positive definite; see Exercise 4.1.19. The next result shows that the converse is also true.

Corollary 5.2.27. *A semisimple Lie group G with* positive definite *Killing pairing is compact.*

Proof. The Killing form defines in this case a bi-invariant Riemann metric on G. Its geodesics through the origin $1 \in G$ are the 1-parameter subgroups $\exp(tX)$ which are defined for all $t \in \mathbb{R}$. Hence, by Hopf-Rinow theorem G has to be complete.

On the other hand, we have computed the Ricci curvature of the Killing metric, and we found

$$\mathrm{Ric}\,(X,Y) = \frac{1}{4}\kappa(X,Y) \quad \forall X,Y \in \mathcal{L}_G.$$

The corollary now follows from Myers' theorem. □

Exercise 5.2.28. Let M be a Riemann manifold and $q \in M$. For the unitary vectors $X, Y \in T_qM$ consider the family of geodesics

$$\gamma_s(t) = \exp_q t(X + sY).$$

Denote by $W_t = \delta\gamma_s$ the associated Jacobi field along $\gamma_0(t)$. Form $f(t) = |W_t|^2$. Prove the following.
(a) $W_t = D_{tX}\exp_q(Y)$ = Frechet derivative of $v \mapsto \exp_q(v)$.
(b) $f(t) = t^2 - \frac{1}{3}\langle R(Y,X)X, Y\rangle_q t^4 + O(t^5)$.
(c) Denote by $\boldsymbol{x}^i$ a collection of normal coordinates at q. Prove that

$$g_{k\ell}(\boldsymbol{x}) = \delta_{k\ell} - \frac{1}{3}R_{kij\ell}\boldsymbol{x}^i\boldsymbol{x}^j + O(3).$$

$$\det g_{ij}(\boldsymbol{x}) = 1 - \frac{1}{3}R_{ij}\boldsymbol{x}^i\boldsymbol{x}^j + O(3).$$

(d) Let

$$\mathbf{D}_r(q) = \{x \in T_qM\ ;\ |x| \le r\}.$$

Prove that if the Ricci curvature is negative definite at q then

$$\mathrm{vol}_0\left(\mathbf{D}_r(q)\right) \le \mathrm{vol}_g\left(\exp_q(\mathbf{D}_r(q)\right)$$

for all r sufficiently small. vol_0 denotes the Euclidean volume in T_qM while vol_g denotes the volume on the Riemann manifold M. □

Remark 5.2.29. The interdependence "curvature-topology" on a Riemann manifold has deep reaching ramifications which stimulate the curiosity of many researchers. We refer to [27] or [71] and the extensive references therein for a presentation of some of the most attractive results in this direction. □

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