Algebraic Topology Homework 9

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Last compiled October 28, 2022

§ *Problems from 2.1*

EXERCISE 2.1.20 Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof: The suspension SX of X is the disjoint union of two cones of X with bases identified. For instance, the suspension of S^n is S^{n+1}, so this proof will likely bear resemblance to the calculation of $H_i(S^n)$. Write $SX = C_1X \cup C_2X$ where C_1X and C_2X are the two cones in question, with the understanding that their bases are identified. The inclusion $C_2X \hookrightarrow SX$ induces a chain of short exact sequences whose rows are given by

$$0 \longrightarrow C_n(C_2X) \longrightarrow C_n(SX) \longrightarrow C_n(SX, C_2X) \longrightarrow 0,$$

and because $C_2(X)$ is contractible, $H_n(C_2X) \cong 0$ for all $n \geq 1$. This means the long exact sequence in homology induced by these short exact sequences of chains is

$$... \longrightarrow 0 \longrightarrow \tilde{H}_n(SX) \longrightarrow H_n(SX, C_2X) \longrightarrow 0 \longrightarrow ...$$

away from $n = 0$, so we have isomorphisms $\tilde{H}_n(SX) \cong \tilde{H}_n(SX, C_2X)$ whenever $n \geq 1$. Because SX is path connected through the tips of its cones, this isomorphism holds at $n = 0$ as well, where all the homology groups in question are simply $\mathbb{Z}$ and hence the reduced homology groups are all trivial.

Now notice that C_1X and C_2X share a base homeomorphic to X, so $C_1X \cap C_2X = X$. Applying the excision theorem (the version which looks at the inclusion $(B, B \cap A) \hookrightarrow (X, A))$ to the pairs $(C_1X, X) \hookrightarrow (SX, C_2X)$ induces an isomorphism $\tilde{H}_n(C_1X, X) \cong \tilde{H}_n(SX, C_2X) \cong \tilde{H}_n(SX)$ for all n. Technically, we actually need to take slightly enlarged copies of C_1X and C_2X, since we require the *interiors* of C_1X and C_2X cover SX for the excision theorem to apply. However, these enlarged copies can be chosen in such a way so that they deformation retract to C_1X and C_2X meaning that morally we are okay to ignore this detail. To get the desired isomorphism, we simply look at the long exact sequence of the pair (C_1X, X) which reads

$$... \longrightarrow \tilde{H}_{n+1}(C_1X) \longrightarrow \tilde{H}_{n+1}(C_1X, X) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(C_1X) \longrightarrow ...$$

Because C_1X is contractible, $\tilde{H}_n(C_1X) \cong 0$ for all n, and hence we have isomorphisms $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(C_1X, X) \cong \tilde{H}_{n+1}(SX)$.

The above argument relied on the fact that C_1X and C_2X were contractible and were glued along a base homeomorphic to X, so if we attach more copies of the cone of X to the same base, we ought to get a similar result. Let S_kX denote the "suspension of X" with k copies of CX attached to X. It's clear that $S_{k+1}X = S_kX \cup CX$, so we proceed inductively. Repeating the argument above with S_kX in the place of C_1X and CX in the place of C_2X, everything works up until we make use of the con tractability of C_1X. Indeed, S_kX need not be contractible. This does get us to the following point, however:

$$\tilde{H}_{n+1}(S_{k+1}X) \cong \tilde{H}_{n+1}(S_kX, X).$$

However, $CX/X \simeq SX$, where X is understood to be the base of CX. This is easy to see from the construction of CX: it is $X \times I/X \times \{1\}$. The suspension SX is obtained by identifying two copies of CX at the base, or equivalently, $SX \cong X \times I/(X \times \{1\} \cup X \times \{0\})$.

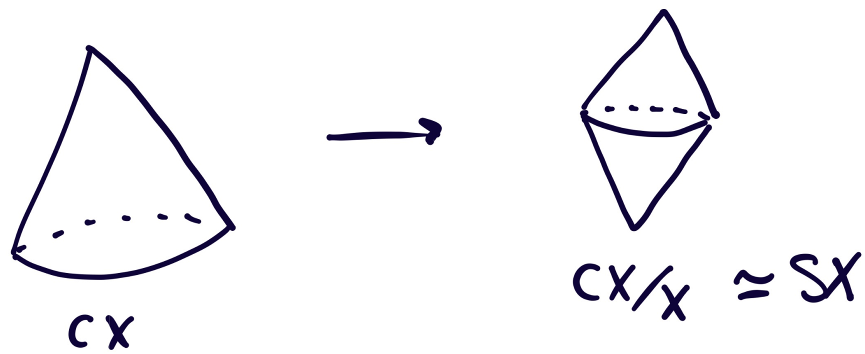

Figure 1: Collapsing the base of a cone to a point produces SX

Now consider what happens when we collapse the common base of the cones comprising S_kX. Each copy of CX becomes a copy of SX, and each SX meets every other SX at the newly formed point resulting from the quotient by X.

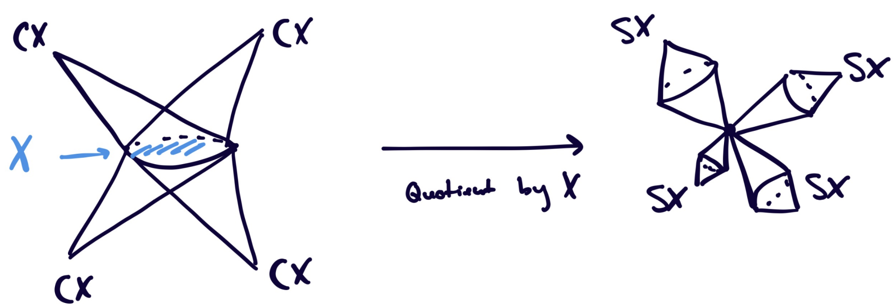

Figure 2: Collapsing the common base of $C_1X \cup C_2X \cup ... \cup C_kX$ to a point produces $SX \vee SX \vee ... \vee SX$.

Hence,

$$\tilde{H}_{n+1}(S_kX, X) \cong \tilde{H}_{n+1}(S_kX/X) \cong \tilde{H}_{n+1}(SX \vee ... \vee SX) \cong \bigoplus_{i=1}^{k} H_{n+1}(SX) \cong \bigoplus_{i=1}^{k} \tilde{H}_n(X)$$

for all n, where we have used the fact $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$. □

EXERCISE 2.1.21 Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \longrightarrow C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \longrightarrow \tilde{H}_{n+1}(SX)$.

Proof: Consider the following maps of chain complexes:

$$C_n(X) \xrightarrow{\alpha} C_{n+1}(CX, X) \xrightarrow{\beta} C_{n+1}(SX).$$

We previously showed that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(CX, X) \cong \tilde{H}_{n+1}(SX)$, so this seems like a good sequence to consider. How are these chain maps defined? The map α takes a singular simplex $\sigma : \Delta^n \longrightarrow X$ to $C\sigma : C\Delta^n \longrightarrow CX$. However, the cone of any n-simplex is an $n+1$-simplex, so $C\Delta^n \longrightarrow CX$ can instead be considered as a mapping $\Delta^{n+1} \longrightarrow CX$. The map $C\sigma$ can be written down explicitly: given any point $(t_0, ..., t_{n+1}) \in \Delta^{n+1} = C\Delta^n$,

$$C\sigma(t_0, ..., t_{n+1}) = \sum_{i=1}^{n} t_i \sigma(v_i) + t_{n+1} p$$

where $p \in CX$ is the cone point, i.e. the point resulting from collapsing $X \times \{1\}$. $C\sigma$ first takes the simplex σ into the copy of X comprising the base of CX and then takes the cone.

Anyways, removing v_i from the simplex $[v_0, ..., v_{n+1}]$ is equivalent to setting $t_i = 0$, so from the definition of $C\sigma$ we immediately get

$$(C\sigma)|_{[v_0,...,\hat{v}_i,...,v_{n+1}]} = C(\sigma|_{[v_0,...,\hat{v}_i,...,v_{n+1}]})$$

whenever $0 \leq i \leq n$ and

$$(C\sigma)|_{[v_0,...,v_n,\hat{v}_{n+1}]} = \sigma$$

for $i = n+1$, where σ here is interpreted as the original map $\sigma : \Delta^n \longrightarrow X$ composed with the homeomorphism $X \longrightarrow X \times \{1\} \subseteq CX$. We can now compute $\partial(C\sigma)$:

$$\begin{aligned} \partial(C\sigma) &= \sum_{i=0}^{n+1} (-1)^i (C\sigma)|_{[v_0,...,\hat{v}_i,...,v_{n+1}]} \\ &= \sum_{i=0}^{n} (-1)^i C(\sigma|_{[v_0,...,\hat{v}_i,...,v_{n+1}]}) + (-1)^{n+1}\sigma \\ &= C(\partial\sigma) + (-1)^{n+1}\sigma. \end{aligned}$$

We used the linearity of C on chains to move the sum inside the argument of C. A small neighborhood $X \times [0, \epsilon)$ of $X \times \{0\}$ inside CX deformation retracts to $X \times \{0\}$, so the pair (CX, X) is a good pair (where X is identified with $X \times \{0\}$, as usual) and gives us the following long exact sequence on homology:

$$... \longrightarrow \tilde{H}_{n+1}(CX) \longrightarrow \tilde{H}_{n+1}(CX, X) \xrightarrow{\partial} \tilde{H}_n(X) \longrightarrow \tilde{H}_n(CX) \longrightarrow ...$$

which gives us isomorphisms $\partial : \tilde{H}_{n+1}(CX, X) \xrightarrow{\sim} \tilde{H}_n(X)$ since CX is contractible and hence $\tilde{H}_n(CX) = 0$.

Now consider the linear extension f of α, the map taking $\sigma \mapsto C\sigma$ above. We show that $f_*\partial$ is the identity on $\tilde{H}_{n+1}(CX, X)$. Given a relative cycle $\gamma \in C_{n+1}(CX, X)$ the boundary $\partial\gamma$ lies in $C_n(X)$. However, $f_*[\partial(\gamma)] = [f_*(\partial\gamma)]$, and f acts on $\partial\gamma$ by extending it to a cone. This means f precisely "undoes" the action of ∂, so

$$f_*\partial[\gamma] = f_*[\partial\gamma] = [\gamma].$$

Since $f_*\partial$ is the identity and ∂ is an isomorphism, f_* must also be an isomorphism.

Now, because (CX, X) is a good pair, the quotient $q : (CX, X) \longrightarrow (CX/X, X/X) \cong (SX, \text{pt})$ induces an isomorphism $q_* : \tilde{H}_*(CX, X) \cong \tilde{H}_*(SX, \text{pt})$. This gives us an isomorphism $q_* \circ f_* : \tilde{H}_n(X) \longrightarrow \tilde{H}_{n+1}(SX, \text{pt}) \cong \tilde{H}_{n+1}(SX)$ corresponding to the chain map $q \circ f$. $\square$

EXERCISE 2.1.23 Show that the second barycentric subdivision of a Δ-complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a Δ-complex with the property that each simplex has all its vertices distinct, then show that for a Δ-complex with this property, barycentric subdivision produces a simplicial complex.

Proof: I was stuck on this one for a while, I had to find a stack overflow post for the following hint (https://math.stackexchange.com/questions/1050085/a-question-about-hatcher-exercise-2-1-23). It suggested splitting this argument up into two parts:

(A) We show that if X is a Δ-complex with k-skeleton X^k, then $B(X^k)$ is comprised of simplicies whose vertices are distinct.

(B) We show that if X is a Δ-complex such that each k-simplex has distinct vertices, then $B(X)$ is a simplicial complex.

If both implications are true, then the second barycentric subdivision of an arbitrary Δ-complex will be a simplicial complex.

(A) We argue inductively on the dimension of X, the Δ-complex.

In the base case, X is a 1-simplex homeomorphic to an interval $[a, b]$. Barycentric subdivision gives us *two* intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. These two 1-simplicies have distinct vertices, a and b.

Now suppose that (A) holds for all $k < n$, i.e. that if X is a Δ-complex then for all $k < n$ the k-simplicies of $B(X)$ are comprised of distinct vertices. For each n-simplex $X^n = [x_0, ..., x_n]$ with barycenter b in X, we get n subsimplices $X_i^n = [b, x_0, ..., \hat{x}_i, ..., x_n]$. Passing to the boundary maps, we get that

$$\partial X_n^i = \sum_{j \neq i} (-1)^j [b, x_0, ..., \hat{x}_j, ..., x_n]$$

noting that x_i is omitted from $[b, x_0, ..., \hat{x}_j, ..., x_n]$. For $j \neq \ell$, $[b, x_0, ..., \hat{x}_j, ..., x_n] \neq [b, x_0, ..., \hat{x}_\ell, ..., x_n]$ by the inductive hypothesis. Since each X^i omits a different vertex x_i, this shows that $X_i^n = X_{i'}^n$ if and only if $i = i'$, meaning that all the subsimplices have distinct vertices. Hence $B(X^n)$ is made up of n-simplicies whose vertices are all distinct from one another.

(B) Now suppose that X^n is the n-skeleton of a Δ-complex X whose k-simplicies are comprised of distinct vertices. We prove that $B(X^n)$ is a simplicial complex by induction.

Consider X^1 for the base case. This is comprised solely of 1-simplicies which we enumerate X_i^1 so that $\partial X_i^1 = \{a_i, b_i\}$. Suppose two 1-simplicies have the same endpoints, i.e. if $a_i = a_j$ and $b_i = b_j$ for some i and j. Once we barycentric subdivide we add two points m_i and m_j so that $B(X_i^1) \cong [a_i, m_i] \cup [m_i, b_i]$ and $B(X_j^1) = [a_j, m_j] \cup [m_j, b_j]$. By the inductive hypothesis, $a_i \neq m_i \neq b_i$, and because X_i^1 and X_j^1 are distinct, $m_i \neq m_j$. This means $B(X_i^1)$ has at least one vertex distinct from $B(X_j^1)$ for all $i \neq j$, and hence $B(X^1)$ is a simplicial complex.

We now proceed to the inductive step. Suppose that $B(X^k)$ is a simplicial complex $\forall k < n$ and that $X_i^n = [x_0, ..., x_n]$ and $X_j^n = [y_0, ..., y_n]$ are two n-simplicies in X^n which share the same vertices. The barycentric subdivision again introduces two barycenters b_i and b_j which are necessarily distinct whenever $i \neq j$ (since X_i^n and X_j^n are distinct).

Using the same argument as in part (A), $\partial B(X_i^n)$ and $\partial B(X_j^n)$ are given by alternating sums of $n-1$ simplicies which are coned over b_i and b_j respectively. Since the barycenters aren't equal, each term of $\partial B(X_i^n)$ and $\partial B(X_j^n)$ are distinct. This means $B(X_i^n)$ And $B(X_j^n)$ have distinct vertices from one another whenever $i \neq j$.

Therefore, $B(X)$ is a simplicial complex by induction, and by applying barycentric subdivision to an arbitrary Δ-complex twice we obtain a simplicial complex. □

EXERCISE 2.1.29 Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Proof: We already know the homology groups of the torus:

$$H_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & n \in \{0,2\} \\ \mathbb{Z}^2 & n = 1 \\ 0 & else \end{cases},$$

so we must now compute the homology of $S^1 \vee S^1 \vee S^2$. Corollary 2.25 seems handy for this situation, it says that the n^{th} reduced horology of a wedge sum $\bigvee_\alpha X_\alpha A$ is isomorphic to $\bigoplus_\alpha \tilde{H}_n(X_\alpha)$, provided that the basepoints $x_\alpha \in X_\alpha$ of the identifications are all good pairs. Luckily, for any point $x \in S^1$ or $y \in S^2$, the pairs (S^1, x) and (S^2, y) are good pairs. One way to see this is that we may take a small ϵ-neighborhood of x in S^1 homeomorphic to an interval or of y in S^2 homeomorphic to an open disk, in either case, these neighborhoods deformation retract to x and y respectively. Hence, we can use Corollary 2.25 to calculate the homology of $S^1 \vee S^1 \vee S^2$ and get

$$\tilde{H}_n(S^1 \vee S^1 \vee S^1) \cong \tilde{H}_n(S^1)^{\oplus 2} \oplus \tilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^{\oplus 2} & n = 1 \\ 0 & else \end{cases}$$

from our knowledge of the homology of a sphere. Converting from reduced to "typical" homology gives us $H_0(S^1 \vee S^1 \vee S^2) \cong \mathbb{Z}$, so the homology groups of $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ do indeed match.

We now turn our attention to covering spaces. The torus $S^1 \times S^1$ can be realized as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, hence we have a quotient map $\pi : \mathbb{R}^2 \longrightarrow S^1 \times S^1$. This is easily seen to be a covering map as each open square $(n, n+1) \times (m, m+1) \subseteq \mathbb{R}^2$ maps homeomorphically to $S^1 \times S^1 - (S^1 \vee S^1)$. Since $\mathbb{R}^2$ is simply connected, it is thus the universal cover of $S^1 \times S^1$. The plane $\mathbb{R}^2$ is contractible, and hence has only trivial reduced homology groups.

Recall that we constructed the universal cover of $S^1 \vee S^1 \vee S^2$ in a previous exercise. It is obtained by attaching a copy of S^2 at every vertex of the universal cover of $S^1 \vee S^1$, as seen in Figure (3).

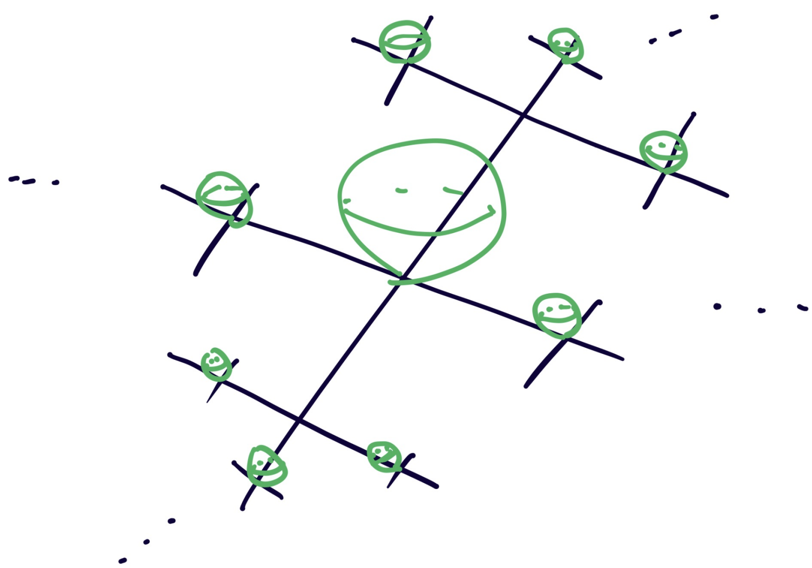

Figure 3: Portion of the universal cover of $S^1 \vee S^1 \vee S^2$

Contracting along the line segments between the spheres doesn't affect the homology groups and produces a countable union of spheres. This certainly doesn't have trivial homology, as once can see by Corollary 2.25 again for instance. $\square$

EXERCISE Exercise 2.2.28

(a) Use the Meyer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

(b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism from the boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

Proof:

(a) Let X be the space in question, a torus $T^2 = S^1 \times S^1$ with a Möbius band M glued to $S^1 \times \{x_0\}$ via its boundary. Let $U = S^1 \times (x_0 - \epsilon, x_0 + \epsilon)$ be a tubular neighborhood of $S^1 \times x_0$ and let V be a similar tubular neighborhood of ∂M. Set $A = M \cup U$ and $B = T^2 \cup V$, where T^2 and M have been identified with their images in X. By construction, U deformation retracts onto $S^1 \times \{x_0\} = \partial M$ and V deformation retracts onto $\partial M = S^1 \times \{x_0\}$, so A deformation retracts to M and B deformation retracts to T^2 via the same maps composed with the identity on M and T^2 respectively.

By definition, A and B are both open, $A \cup B = X$ and $A \cap B$ is an open neighborhood of $S^1 \times \{x_0\} = \partial M$ which deformation retracts onto $S^1 \times \{x_0\} = \partial M$. Applying the Meyer-Vietoris sequence therefore gives us

$$\dots \longrightarrow \tilde{H}_2(A \cap B) \longrightarrow \tilde{H}_2(A) \oplus \tilde{H}_2(B) \longrightarrow \tilde{H}_2(X) \longrightarrow \tilde{H}_1(A \cap B) \longrightarrow \dots$$
$$\dots \longrightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \longrightarrow \tilde{H}_1(X) \longrightarrow \tilde{H}_0(A \cap B) \longrightarrow \dots$$

so after applying our knowledge of these groups, we get

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_2(X) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^{\oplus 2} \longrightarrow \tilde{H}_1(X) \longrightarrow 0 \longrightarrow \ldots$$

The main part of this sequence demanding close inspection is the $n = 1$ portion. The map $\Phi : \tilde{H}_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B)$ is induced by the inclusion of $A \cap B$ into A and B. As discussed above, we have $A \cap B \simeq \partial M = S^1 \times \{x_0\}$, $A \simeq M$ and $B \simeq T^2$, so instead we may think of Φ as a map $\tilde{H}_1(\partial M) \longrightarrow \tilde{H}_1(M) \oplus \tilde{H}_1(T^2)$ induced by the inclusions $S^1 \times \{x_0\} \hookrightarrow T^2$ and $\partial M \hookrightarrow M$.

Let b_1 and b_2 be the generators for $\tilde{H}_1(B) = \tilde{H}_1(T^2)$ representing the circle $S^1 \times \{x_0\}$ and $\{\text{pt}\} \times S^1$ respectively, a the sole generator for $\tilde{H}_1(A) = \tilde{H}_1(M)$ represented by the central circle of M, and c the sole generator for $\tilde{H}_1(\partial M)$. The circle given by ∂M wraps around the central circle of M twice, so the inclusion of ∂M into M induces a map sending c to $2a$ on the level of homology. However, $\partial M = S^1 \times \{x_0\}$ *is* a generator b_1 in $B \simeq T^2$, so the inclusion of $S^1 \times \{x_0\}$ into T^2 induces a map $c \mapsto b_1$ on the level of homology. Together, this means $\Phi(c) = 2a - b_1$, where we pick up a negative sign in the second summand due to the quirks of the Meyer-Vietoris sequence. Since $\tilde{H}_1(A \cap B) \cong \mathbb{Z}$, this fully determines the map Φ, meaning $\operatorname{img} \Phi \cong \mathbb{Z}(2a - b_1)$. Combining this with the exactness of the above sequence, we get that

$$\tilde{H}_1(X) \cong \tilde{H}_1(A) \oplus \tilde{H}_1(B) / \operatorname{img} \Phi \cong \mathbb{Z}^{\oplus 3} / \mathbb{Z}(2a - b_1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Our analysis of Φ is also invaluable for computing $\tilde{H}_2(X)$: since Φ took the sole generator of $\mathbb{Z}$ to a nonzero element of another free group, it is injective and hence has trivial kernel. This means the map $\beta : \tilde{H}_2(X) \longrightarrow \mathbb{Z}$ is trivial. The exactness of $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \tilde{H}_2(X)$ tells us that α is injective, and an additional application of exactness yields $\mathbb{Z} \cong \operatorname{img} \alpha = \ker \beta = \tilde{H}_2(X)$.

Because X is path-connected, we know $\tilde{H}_0(X) = 0$. For $n \geq 3$, Meyer-Vietoris tells us

$$\ldots \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(A \cap B) \longrightarrow \ldots$$

which reduces to $0 \longrightarrow \tilde{H}_n(X) \longrightarrow 0 \implies \tilde{H}_n(X) \cong 0$, since $\tilde{H}_{n-1}(A \cap B)$, $\tilde{H}_n(A)$ and $\tilde{H}_n(B)$ are all trivial for $n \geq 3$. In summary,

$$\tilde{H}_n(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}.$$

(b) First note that $\mathbb{RP}^1 \simeq S^1$. One can see this by recalling $\mathbb{RP}^1$ is obtained from S^1 by identifying antipodal points. Traversing S^1, we do not encounter a previously traversed point until we have moved π-radians, at which point we have returned to the chosen basepoint of S^1.

Let X be the space in question. We define tubular neighborhoods of $\partial M = \mathbb{RP}^1$ in a similar way as above: take U to be a neighborhood of $\mathbb{RP}^1$ in $\mathbb{RP}^2$ which deformation retracts onto $\mathbb{RP}^1$ and let $V \subseteq M$ be as in part (a). Defining $A = M \cup U \subseteq X$ and $B = \mathbb{RP}^2 \cup V$ means that

- A and B are open,
- $A \cup B = X$ and

- $A \cap B$ deformation retracts onto $\partial M = \mathbb{RP}^1$ in X.

Since $\mathbb{RP}^1$ and M are both homotopic to the circle, we have

$$\tilde{H}_n(A \cap B) \cong \tilde{H}_n(A) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

and

$$\tilde{H}_n(B) \cong \tilde{H}_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}.$$

For the same reasons as before, $\tilde{H}_n(X) = 0$ for $n \geq 3$ and $n = 0$, so the only relevant portion of the Meyer-Vietoris sequence is

$$... \longrightarrow 0 \longrightarrow \tilde{H}_2(X) \longrightarrow \overbrace{\tilde{H}_1(A \cap B)}^{\mathbb{Z}} \xrightarrow{\Phi} \overbrace{\tilde{H}_1(A)}^{\mathbb{Z}} \oplus \overbrace{\tilde{H}_1(B)}^{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \tilde{H}_1(X) \longrightarrow 0 \longrightarrow ...$$

Let a be the generator of $\tilde{H}_1(A)$ (representing the central circle of M), b the generator of $\tilde{H}_1(B)$, and c the generator of $\tilde{H}_1(A \cap B)$. Applying a similar argument as before, we see that c wraps around a twice and hence

$$\Phi(c) = 2a - b.$$

The exactness of the sequence tells us that Φ is injective so $\ker \Phi = 0$, and therefore $\operatorname{img}(\tilde{H}_2(X) \longrightarrow \tilde{H}_1(A \cap B)) = \ker \Phi = 0$. However, exactness tells us the kernel of $\tilde{H}_2(X) \longrightarrow \tilde{H}_1(A \cap B)$ must also be 0, meaning $\tilde{H}_2(X) = 0$. We therefore have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Phi}\!\!\longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{H}_1(X) \longrightarrow 0,$$

so

$$\tilde{H}_1(X) \cong \mathbb{Z}\langle a, b\rangle \;/\; \mathbb{Z}\langle 2a - b, 2b\rangle \cong \langle a, b\rangle \;/\; \langle 2a - b, 4a\rangle \cong \mathbb{Z}/4\mathbb{Z}.$$

Summarizing,

$$\tilde{H}_n(X) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

□

EXERCISE 2.2.31 Use the Mayer-Vietoris sequence to show that there are isomorphisms $\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$ if the basepoints of X and Y that are identified in $X \vee Y$ are deformation retracts of neighborhoods $U \subseteq X$ and $V \subseteq Y$.

Proof: This is a straightforward application of Mayer-Vietoris. Let $x_0 \in X$ and $y_0 \in Y$ be the basepoints identified in the wedge sum $X \vee Y$, and let $U \subseteq X$ and $V \subseteq Y$ be open neighborhoods of x_0 and y_0 respectively which deformation retract to x_0 and y_0. Define $A = X \cup V \subseteq X \vee Y$ and $B = U \cup Y \subseteq X \vee Y$.

Then $A \cup B = X \vee Y$ and $A \cap B = U \cup V$. This latter set deformation retracts onto the basepoint of $X \vee Y$; simply deformation retract U onto x_0 in X and V onto y_0 in Y. This map will be continuous on $U \cup V$ since $U \cap V = \{x_0 = y_0\}$ and each individual deformation retract leaves the basepoint fixed.

The Mayer-Vietoris sequence then gives us

$$... \longrightarrow \tilde{H}_n(A \cap B) \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(A \cup B) \longrightarrow H_{n-1}(A \cap B) \longrightarrow ...$$

but because $A \cap B$ deformation retracts to a point and $A \cup B = X \vee Y$, we actually have

$$... \longrightarrow 0 \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(A \cup B) \longrightarrow 0 \longrightarrow ...$$

for each n. The exactness of the Mayer-Vietoris sequence then implies that this map is an isomorphism. □

EXERCISE 2.2.36 Show that $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ by definition. Namely, show that $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ and $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$.

Proof: Given the suggestion in the problem statement, we split up this proof into lemmas.

Lemma 1.1. $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$.

Proof. First note that $X \times \{x_0\}$ is a retraction of $X \times S^n$ via the map $r = \mathrm{id}_X \times \pi_{x_0}$ which sends X to itself and all of S^n to x_0. By problem 2.1.11 (from the last homework) this means the inclusion $\iota : X \times \{x_0\} \longrightarrow X \times S^n$ induces an inclusion $\iota_* : H_i(X \times \{x_0\}) \longrightarrow H_i(X \times S^n)$ on homology. With this in mind, consider the long exact sequence on relative homology:

$$... \longrightarrow H_{i+1}(B, A) \xrightarrow{\partial} H_i(A) \xrightarrow{\iota_*} H_i(B) \xrightarrow{j_*} H_i(B, A) \xrightarrow{\partial} H_{i-1}(A) \xrightarrow{\iota_*} ...$$

where we have written $B = X \times S^n$ and $A = X \times \{x_0\}$ to save space. By exactness and the injectivity of ι_*, $\operatorname{img} \partial = 0$. This in turn implies $\ker \partial = H_i(B, A)$ and so j_* is surjective, and hence we can insert zeros between each pair of $H_i(B, A)$ and $H_{i-1}(A)$ terms, giving us a short exact sequence

$$0 \longrightarrow H_i(X \times \{x_0\}) \xrightarrow{\iota_*} H_i(X \times S^n) \longrightarrow H_i(X \times S^n, X \times \{x_0\}) \longrightarrow 0$$

for each i. Furthermore, because $r \circ \iota = \mathrm{id}$, $r_* \circ \iota_* = \mathrm{id}$ which means ι_* is a split map on homology. This gives us an isomorphism

$$H_i(X \times S^n) \cong H_i(X \times \{x_0\}) \oplus H_i(X \times S^n, X \times \{x_0\}),$$

and because $X \times \{x_0\} \cong X$, $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$. □

Now we turn our attention to the $H_i(X \times S^n, X \times \{x_0\})$ term in the above direct sum.

Lemma 1.2. $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$.

Proof. As hinted by Hatcher, this is a straightforward application of the Meyer-Vietoris sequence. The only potentially tricky part is choosing a good open cover for $X \times S^n$. Given that we've been prompted to find an isomorphism from something involving S^n to something involving S^{n-1}, it makes sense to choose open sets

U and V which cover S^n and whose intersection deformation retracts to S^{n-1}. This can be accomplished by letting U and V be the lower and upper disks D^n extending slightly past the equator of S^n. More explicitly, let $\epsilon > 0$ be small and set

$$U = \{(s_0, ..., s_n) \in S^n \mid s_0 < \epsilon\} \quad \text{and} \quad V = \{(s_0, ..., s_n) \in S^n \mid s_0 > -\epsilon\}.$$

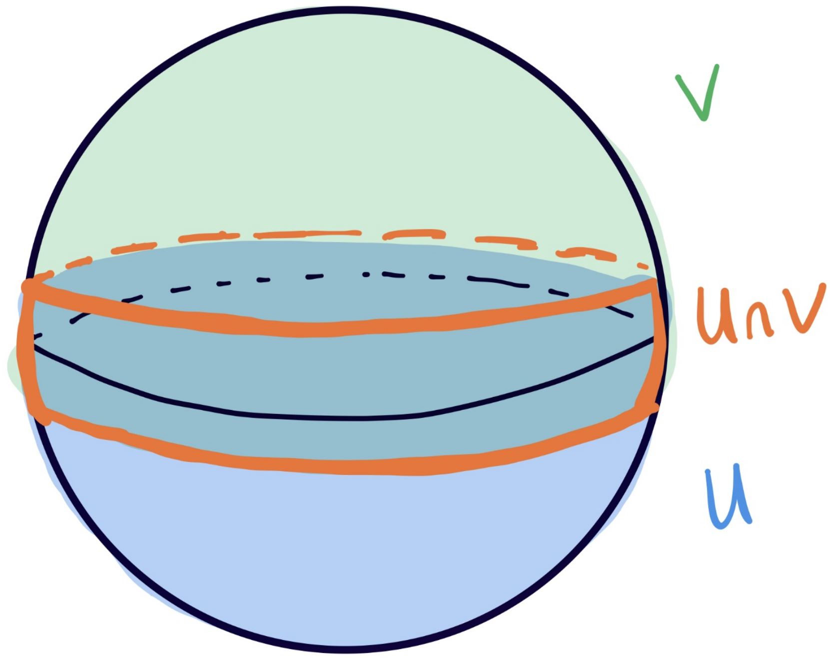

Figure 4: Good choice of open cover for S^n

Then $U \cup V = S^n$ and $U \cap V$ deformation retracts onto $S^{n-1} = \{(0, s_1, ..., s_n) \in S^n\}$. Now define $A = X \times U$ and $B = X \times V$, so that $A \cup B = X \times S^n$ and $A \cap B \simeq X \times S^{n-1}$. Applying the relative Meyer-Vietoris to this yields the long exact sequence

$$... \longrightarrow H_n(A \cap B, C) \longrightarrow H_n(A, C) \oplus H_n(B, C) \longrightarrow H_n(A \cup B, C) \longrightarrow ...$$

where $C = X \times \{x_0\}$ with x_0 chosen to lie on the equator of S^n. Because both U is a copy of D^n and is hence contractible, we have that

$$H_n(A, C) = H_n(X \times U, X \times \{x_0\}) \cong H_n(X \times \{x_0\}) \cong 0.$$

We get something similar for $B = X \times V$. This implies that the above relative Meyer-Vietoris sequence is actually

$$... \longrightarrow 0 \longrightarrow H_n(X \times S^n, X \times \{x_0\}) \longrightarrow H_{n-1}(X \times S^{n-1}, X \times \{x_0\}) \longrightarrow 0 \longrightarrow ...$$

for each n, which gives us the desired isomorphic by exactness. $\square$

With these two lemmas out the way, we can quickly complete the problem. Using Lemma 1.2 inductively, we get that

$$H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-n}(X \times S^0, X \times \{x_0\}).$$

However, S^0 is a set containing two points. We may assume one of these is x_0, and hence

$$H_{i-n}(X \times S^0, X \times \{x_0\}) \cong H_{i-n}(X \times \{x_1\}) \cong H_{i-n}(X)$$

since X is homeomorphic to $X \times \{x_1\}$. Applying Lemma 1.1, we conclude that

$$H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\}) \cong H_i(X) \oplus H_{i-n}(X).$$

$\square$