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K-THEORY SEMINAR

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Week 1: Vector bundles, clutching functions, Grassmanians

1.1 Vector Bundles

Let's discuss the idea of a vector bundle. Let X be a topological space and suppose to every $x \in X$ we have a vector space E_x. A *quasi-vector bundle* is the space $\bigsqcup E_x$ equipped with any topology such that the projection $\bigsqcup E_x \longrightarrow X$ is continuous.

Definition 1.1.1. A *vector bundle* is $\pi : E \longrightarrow X$ where each $E_x = \pi^{-1}(X)$ $x \in X$ is a vector space and for each $x \in X$ there exists an open neighborhood U of x and a homeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times E_x$ which restricts to a linear isomorphic $E_{x_0} \longrightarrow \{x_0\} \times E_x$. This map φ is called a local trivialization above U.

Example 1.1.2.

(a) if V is a vector space then the projection $p : X \times V \longrightarrow X$ is a vector bundle. We call this a *trivial vector bundle* since X is a local trivialization for any element $x \in X$.

(b) Consider the $\mathbb{Z}$ action on $\mathbb{R} \times \mathbb{R}$ given by $a \cdot (x, y) = (x + a, (-1)^a y)$. Then

$$p_1 : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$

where p_1 is the projection to the first coordinate is a vector bundle. The total space is the Möbius band and the base space is the central circle.

Definition 1.1.3. Given a continuous map $f : X \longrightarrow Y$, if we have a vector bundle $E \longrightarrow Y$, then we get a vector bundle f^*E over X via pull back:

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $(f^*E)_x = E_{f(x)}$.

If the map φ is of constant rank, i.e. if φ_x has the same rank for all $x \in X$, then $\ker \varphi$ and $\operatorname{coker} \varphi$ are both vector bundles over X as well.

Definition 1.1.4. Suppose we have two vector bundles $\pi : E \to X$ and $\pi' : E' \to X$ over the same base space. A *morphism* of vector bundles $\varphi : (\pi, E, X) \to (\pi', E', X)$ is a map $E \to E'$ which both commutes with the projection maps π and π' and restricts to a linear maps on each fiber: $\varphi_x : E_x \to E'_x$. Notice that E and E' are *not* required to be of the same rank.

Definition 1.1.5. The category of vector bundles with base field K over base X is denoted $\mathsf{Vect}_K(X)$. We denote by $\pi_0(\mathsf{Vect}_K(X))$ the category of vector bundles over X up to isomorphism.

Clutching Construction

Vector bundles can be defined locally and then glued together to give a global vector bundle. Here is the construction:

Theorem 1.1.6. *Let $\{U_\alpha\}_{\alpha \in A}$ be a cover for X. Denote by $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} = U_{\alpha\beta} \cap U_\gamma$ etc.*

Suppose we have a family of vector bundles $E_\alpha \to U_\alpha$ and isomorphisms $g_{\alpha\beta} : E_\alpha|_{U_{\alpha\beta}} \to E_\beta|_{U_{\alpha\beta}}$. If the gluing maps $g_{\alpha\beta}$ additionally satisfy the cocycle condition $g_{\alpha\gamma}|_{U_{\alpha\beta\gamma}} = g_{\alpha\beta} \circ g_{\beta\gamma}$, then we can glue the local vector bundles $E_\alpha \to U_\alpha$ on the intersections $U_{\alpha\beta}$ to get a total bundle $E \to X$ where

$$E = \bigsqcup_{\alpha \in A} E_\alpha \Big/ \left(e_\alpha \sim e_\beta \ \textit{ if } \ g_{\alpha\beta}(e_\alpha) = e_\beta\right).$$

This is identical to the gluing construction for sheaves, see Hartshorne Exercise 2.1.13.

The cocycle condition here exists purely to ensure $e_\alpha \sim e_\beta$ is transitive and is hence an equivalence relation.

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Week 2: More on Vector Bundles and K^0

Two more things about vector bundles. Given vector bundles E and E' over a space Y, we can define

- The **Whitney Sum**: this is defined fiberwise: $(E \oplus E')_x \cong E_x \oplus E'_x$
- The **tensor product**: $(E \otimes E')_x = E_x \otimes E'_x$
- The **pullback:** if $f : X \longrightarrow Y$ is continuous, then $f^*E = \{(x, e) \in X \times E \mid f(x) = p(e)\}$ where $p : E \longrightarrow X$ is the bundle map. Sections are given $(F^*E)_y = E_{f(x)}$ where $f(x) = y$.

I don't know what "varies continuously in morphisms" means, I'll include it later.

More generally any endofunctor $F : \mathsf{Vect} \longrightarrow \mathsf{Vect}$ which varies continuously in morphisms extends to an endofunctor in the category of vector bundles via its action on sections.

Examples

- Trivial bundles; these are vector bundles which are trivialized over the whole space. When M is a parallelizable manifold, TM is a trivial bundle.

This is often taken to be the definition of parallelizable. Alternatively one can define a manifold to be parallelizable if it has a global frame.

- Tautological vector bundles; $\mathbb{RP}^n, \mathbb{CP}^n$, Grassmanians.
- Subbundles; $S^1 \times \mathbb{R}^2$
- Normal bundles; $M \hookrightarrow N$ is a smooth embedding, a normal bundle to M is a subbundle of TN which picks out at each point the subspace which is normal to TM in TN.

Lemma 2.0.1. The restriction of a vector bundle $E \longrightarrow X \times I$ to $\times\{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.

Corollary 2.0.2. If $f : A \longrightarrow B$ is a homotopy equivalence of vector bundles then the pullback f^* induces a bijection $f^* : \mathsf{Vect}^n_K(B) \longrightarrow \mathsf{Vect}^n_K(A)$ (remember that $\mathsf{Vect}^n_K(X)$ is the set of isomorphism classes of rank n vector bundles with base field K over X).

Example 2.0.3. Consider an embedding $S^{n-2} \hookrightarrow S^n$. Then the open tubular neighborhoods of X^{n-2} are in bijection with points of the total space of rank 2 vector bundles over S^{n-2}.

2.1 K^0

$\mathsf{Vect}^n_{\mathbb{C}}(X)$ has the natural structure of an abelian monoid under the Whitney sum operation. This means we can complete it to a group, and this process is known as Groethendieck completion. For an arbitrary monoid it's this:

$$\mathrm{Gr}(M) = \mathbb{Z}\langle M\rangle / ([m] + [n] - [m+n], m, n \in M)$$

and we define $K^0(X) = \mathrm{Gr}(\mathsf{Vect}^n_{\mathbb{C}}(X))$.