# 8 Differentials

**Exercise 8.1.** *a Generalize (8.7) as follows. Let $B$ be a local ring containing a field $k$, and assume that the residue field $k(B) = M/\mathfrak{m}$ of $B$ is a separable generated extension of $k$. Then the exact sequence of (8.4A),*

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

*is exact on the left also.*

*b Generalize (8.8) as follows. With $B, k$ as above, assume furthermore that $k$ is perfect, and that $B$ is a localization of an algebra of finite type over $k$. Then show that $B$ is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \operatorname{tr.d.} k(B)/k$.*

*c Strengthen (8.15) as follows. Let $X$ be an irreducible scheme of finite type over a perfect field $k$, and let $\dim X = n$. For any point $x \in X$, not necassarily closed, show that the local ring $\mathcal{O}_x$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at $x$ is free of rank $n$.*

*d Strengthen (8.16) as follows. If $X$ is a variety over an algebraically closed field $k$, then $U = \{x \in X | \mathcal{O}_x$ is a regular local ring$\}$ is an open dense subset of $X$.*

*Solution.* a To show that $\delta$ is injective is equivalent to showing that the morphism of vector spaces

$$\delta^* : \hom_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \to \hom_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

is surjective. Note the isomorphisms:

$$\hom_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \cong \hom_B(\Omega_{B/k}, k(B)) \cong \operatorname{Der}_k(B, k(B))$$

So given a $k(B)$-linear homomorphism $h : \mathfrak{m}/\mathfrak{m}^2 \to k(B)$ we want to find a $k$-derivation $d' : B \to k(B)$ such that pushing it through the isomorphisms and then $\delta^*$ gives the original $h$. First we describe the image of a derivation $d'$ through the isomorphisms and then $\delta^*$. The deriviation $d' : B \to k(B)$ first becomes the $B$-homomophism described by $db \mapsto d'b$ (use the expression of $\Omega$ as a free module generated by the $db$ modulo the suitable relations). This then becomes the $k(B)$-homomorphism $db \otimes c \mapsto cd'b$ and then applying $\delta^*$ gives $b \mapsto d'b$. So a derivation $d'$ just gets mapped to its restriction to $\mathfrak{m}$ (note that if $b \in \mathfrak{m}^2$ then $b = \sum a_i c_i$ for some $a_i, c_i \in \mathfrak{m}$ and so $d'b = \sum a_i d'c_i + c_i d'a_i = 0$ in $k(B) = B/\mathfrak{m}$).

Now given a $k(B)$-linear homomorphism $h : \mathfrak{m}/\mathfrak{m}^2 \to k(B)$ we describe a $k$-derivation $d' : B \to k(B)$. For $b \in B$, write $b = c + \lambda$ with $\lambda \in k(B), c \in \mathfrak{m}$ in the unique way using the section
$\begin{tikzcd} k(B) \arrow[r] \arrow[rr, bend left, "id"] & B \arrow[r] & k(B) \end{tikzcd}$
from Theorem 8.25A. Then define $d'(b) = h(c)$.

b Suppose that $\Omega_{B/k}$ is free of the given rank. Then we have the exact sequence from part (a):

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

We know that the dimension of $\Omega_{B/k} \otimes k(B)$ is $\dim B + \operatorname{tr.d.} k(B)/k$ by assumption and that $\dim \Omega_{k(B)/k} = \operatorname{tr.d.} k(B)/k$ by Theorem II.8.6A ($k(B)$ is separably generated over $k$ since we have assumed $k$ perfect, see Theorem I.4.8A). Hence, the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is $\dim B$ and so by definition $B$ is regular.

Now suppose that $B$ is regular. By the argument just described, we know that $\dim_{k(B)} \Omega_{B/k} \otimes k(B)$ is $\dim B + \operatorname{tr.d.} k(B)/k$. If we can show also that $\dim_K \Omega_{B/k} \otimes K$ is $\dim B + \operatorname{tr.d.} k(B)/k$ (where $K$ is the quotient field of $B$) then we will be done by Lemma II.8.9 for the same reasons that it works in the proof of Theorem II.8.8. As in that proof we have $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$ by Proposition II.8.2A and since $k$ is perfect, $K$ is a separably generated extension of $k$ (Theorem I.4.8A) so $\dim_K \Omega_{K/k} = \operatorname{tr.d.} K/k$ by Theorem 8.6A. Hence $\dim_K \Omega_{B/k} \otimes K = \operatorname{tr.d.} K/k$. Now we have assumed that $B$ is the localization of an algebra $A$ of finite type over $k$, so $B = A_\mathfrak{p}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. This means that we have $\operatorname{Frac} A = \operatorname{Frac} B$ and height $\mathfrak{p} = \dim B$. So by Theorem I.1.8A we have $\operatorname{tr.d.} K/k = \dim A = \text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim B + \dim A/\mathfrak{p} = \dim B + \operatorname{tr.d.} \operatorname{Frac}(A/\mathfrak{p})/k = \dim B + \operatorname{tr.d.} k(B)$. So we have shown that $\dim_K \Omega_{B/k} \otimes K$ is $\dim B + \operatorname{tr.d.} k(B)/k$ and now we can happily apply Lemma II.8.9 to get the desired result.

c Take an affine neighbourhood $\operatorname{Spec} A$ of $x$ in which $x$ corresponds to the prime ideal $\mathfrak{p}$. Define $B = A_\mathfrak{p}$ and we have the hypotheses of part (b) satisfied so we see that $\mathcal{O}_x = B$ is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $\dim B + \operatorname{tr.d.} k(B)/k = \dim A = \dim X$ (see the proof of the previous part for the former equality). The stalk $(\Omega_{X/k})_x$ is $\Omega_{A/k} \otimes_A B$ and we have an isomorphism $\Omega_{B/k} \cong \Omega_{S^{-1}A/k} \cong S^{-1}\Omega_{A/k} \cong \Omega_{A/k} \otimes_A B$ by Proposition II.8.2A where $S$ is the multiplicative set of elements not in $\mathfrak{p}$, so $\mathcal{O}_x = B$ is a regular local ring if and only if $\Omega_{B/k} \cong \Omega_{A/k} \cong (\Omega_{X/k})_x$ is free of rank $\dim B + \operatorname{tr.d.} k(B)/k = \dim A = \dim X$.

d By (8.16) we know that there exists some open dense subset $V$ of $X$ which is nonsingular, hence $U$ is dense since it contains any such $V$. At every point $x$ of $U$, the coherent sheaf $\Omega_{X/k}$ is locally free by part (c) and so by Exercise II.5.7(a) there is an open neighbourhood $W$ of $x$ on which $\Omega_{X/k}|_W$ is free of rank $n$. This implies that at every point $w$ of $W$, the stalks $(\Omega_{X/k})_w$ are free of rank $n$ and therefore, again by part (c), $w \in U$. So every point of $U$ has an open neighbourhood contained in $U$, and therefore $U$ is open.

**Exercise 8.2.** *Let $X$ be a variety of dimension $n$ over $k$. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on $X$, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections* *which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence*

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

*where $\mathscr{E}'$ is locally free.*

*Solution.* Consider the scheme $X \times V$ and the subset of points $Z = \{(x, s) | s_x \in \mathfrak{m}_x \mathscr{E}_x\}$. We label the projections by $\pi_1 : Z \subset X \times V \to X$ and $\pi_2 : Z \subset X \times V \to V$. Now for any point $x \in X$, the preimage $\pi^{-1}x$ in $Z$ is the set of global sections $s$ in $V$ that $s_x \in \mathfrak{m}_x \mathscr{E}_x$. Otherwise said, it is the kernel of the $k(x)$-vector space morphism $V \otimes_k k(x) \to \mathscr{E}_x \otimes_{\mathcal{O}_x} k(x)$. Since $\mathscr{E}$ is generated by global sections this morphism is always surjective and since $\mathscr{E}$ is locally free of rank $r$ this kernel will then have rank $m - r$ where $m = \dim V$. Hence, the dimension of $Z$ as a closed subset of $X \times V$ is $n + m - r$. By assumption $r > n$ and so $n + m - r < m$. Hence, the second projection $\pi_2 : Z \to V$ cannot be surjective. Any point not in the image will be a global section with the required property.

Using this global section $s$ we define a morphism $\mathcal{O}_X \to \mathscr{E}$ by sending $1 \mapsto s$, and define $\mathscr{E}'$ as the cokernel of $\mathcal{O}_X \to \mathscr{E}$. To see that we have an exact sequence as desired, consider the stalk at $x \in X$. We want to show that $\mathcal{O}_x \to \mathcal{O}_x^{\oplus r}$ is injective where we are using the isomorphism $\mathcal{O}_x^{\oplus r} \cong \mathscr{E}_x$; let $s_x = (a_1, \dots, a_r)$. This morphism sends $a \mapsto a(a_1, \dots, a_r)$ and so if $aa_i = 0$ for all $i$ then $a = 0$ or $a_i = 0$ for all $i$ (since $X$ is integral the local rings have no zero divizors) but we have chosen $s$ so that $s_x \notin \mathfrak{m}_x \mathscr{E}_x$ and so $a_i \notin \mathfrak{m}_x$ for some $i$, and therefore $a = 0$.

Now we must show that $\mathscr{E}'_x = \mathcal{O}_x^{\oplus r} / \mathcal{O}_x$ is free, then the local free-ness of $\mathscr{E}'$ will follow from Exercise II.5.7(b). We do this by explicitely constructing an isomorphism $\mathcal{O}_x^{\oplus (r-1)}$. We have assumed that one of $a_i$ is not in $\mathfrak{m}_x$. Without loss of generality we can assume that it is $a_r$. Writing $\mathcal{O}_x$ as $A_\mathfrak{p}$ for some affine $\operatorname{Spec} A$ containing $\mathfrak{p}$ we see that $a_r$ is invertible since it is not in $\mathfrak{p}A_\mathfrak{p} = \mathfrak{m}_x$. Now consider the composition $A_\mathfrak{p}^{r-1} \to \mathcal{A}_\mathfrak{p}^r \to \mathcal{A}_\mathfrak{p}^r / sA_\mathfrak{p}$ where the first morphism sends $(b_1, \dots, b_{r-1})$ to $(b_1, \dots, b_{r-1}, 0)$. Clearly the composition is injective for $(b_1, \dots, b_{r-1}, 0) \in sA_\mathfrak{p}$ contradicts the assumption that $a_r \notin \mathfrak{m}_x$. For surjectivity, let $b = (b_1, \dots, b_r)$ represent an element of $\mathcal{A}_\mathfrak{p}^r / sA_\mathfrak{p}$. Then $b - a_r^{-1} b_r s \in A_\mathfrak{p}^{r-1}$ and $(b - a_r^{-1} b_r s) - b \in sA_\mathfrak{p}$. So we are done.

**Exercise 8.3.** Product Schemes.

a *Let $X$ and $Y$ be schemes over another scheme $S$. Use (8.10) and (8.11) to show that $\Omega_{X \times Y/S} \cong p_1^* \Omega_{X/S} \oplus p_2^* \Omega_{Y/S}$.*

b *If $X$ and $Y$ are nonsingular varieties over a field $k$, show that $\omega_{X \times Y} \cong p_1^* \omega_X \otimes p_2^* \omega_Y$.*

c *Let $Y$ be a nonsingular plane cubic curve, and let $X$ be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I. Ex. 7.2).*

*Solution.* a From (8.10) it follows that $\Omega_{X\times Y/X} \cong p_2^*(\Omega_{Y/S})$ and $\Omega_{X\times Y/Y} \cong p_1^*(\Omega_{X/S})$. Combining these with Proposition (8.11) gives exact sequences

$$\Omega_{X\times Y/X} \to \Omega_{X\times Y/S} \to \Omega_{X\times Y/Y} \to 0$$

$$\Omega_{X\times Y/Y} \to \Omega_{X\times Y/S} \to \Omega_{X\times Y/X} \to 0$$

To see that the relavent morphisms actually do decompose $\Omega_{X\times Y/S}$ into $p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$ we go to Matsumura to find the definitions of these morphisms. It is enough to consider the affine case, so let $A$ and $B$ be rings over $C$. We want to know if the composition

$$\Omega_{A\otimes_C B/A} \stackrel{\sim}{\leftarrow} \Omega_{B/C} \otimes_B (B \otimes_C A) \to \Omega_{A\otimes_C B/C} \to \Omega_{A\otimes_C B/A}$$

is the identity. The first module is generated by elements of the form $dx$ where $x \in A \otimes_C B$. Since $d$ is a morphism of abelian groups and $d(a \otimes b) = d(a \otimes 1) + d(1 \otimes b) = d(1 \otimes b)$ it is enough to consider elements of the form $d(1 \otimes b)$. The first map takes such an element to $(1 \otimes 1) \otimes db$. This then gets taken to $d(1 \otimes b)$ which gets taken back to $d(1 \otimes b)$ so the composition is the identity.

b Suppose that the dimensions of $X$ and $Y$ are $n$ and $m$ respectively. Then we have

$$\begin{aligned}
\omega_{X\times Y} &= \wedge^{nm}\Omega_{X\times Y} && \text{(by definition)}\\
&\cong \wedge^{nm}(p_1^*(\Omega_X) \oplus p_2^*(\Omega_Y)) && \text{(part (a))}\\
&\cong (\wedge^n p_1^*(\Omega_X)) \otimes (\wedge^m p_2^*(\Omega_Y)) && \text{(Exercise I.5.16(d))}\\
&\cong (p_1^*(\wedge^n\Omega_X)) \otimes (p_2^*(\wedge^m\Omega_Y)) && \text{(Exercise I.5.16(e))}\\
&\cong p_1^*(\omega_X) \otimes p_2^*(\omega_Y) && \text{(by definition)}
\end{aligned}$$

c In Example 8.20.3 we see that $\omega_Y \cong \mathcal{O}_Y$ and so by part (b) we have $\omega_{Y\times Y} \cong p_1^*\omega_Y \otimes p_2^*\omega_Y \cong p_1^*\mathcal{O}_Y \otimes p_2^*\mathcal{O}_Y \cong \mathcal{O}_{Y\times Y}$. By Exercise II.4.5(d) the vector space of global sections of the structure sheaf of $Y \times Y$ has dimension one.

In Exercise I.7.2 we calculate the arithmetic genus of a plane cubic curve to be 1 in part (b) and then the arithmetic genus of $Y \times Y$ is calculated in part (e) as $1 - 1 - 1 = -1$.

**Exercise 8.4.** Complete Intersections in $\mathbb{P}^n$.

**Exercise 8.5.** Blowing up a Nonsingular Subvariety. *As in (8.24), let $X$ be a nonsingular variety, let $Y$ be a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \widetilde{X} \to X$ be the blowing up of $X$ along $Y$, and let $Y' = \pi^{-1}(Y)$.*

a *Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$, and $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$ defined by $n \mapsto$ class of $nY'$, give rise to an isomorphism $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.*

b *Show that $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.*

*Solution.* a Since $X$ is nonsingular we can associate each each invertible sheaf to a class of divisors (Remark II.6.11.1A). Then from Proposition II.6.5 we have the exact sequence and isomorphism:

$$\mathbb{Z} \to \operatorname{Cl} \widetilde{X} \to \operatorname{Cl} U \to 0 \qquad \operatorname{Cl} U \cong \operatorname{Cl} X$$

where $U = X - Y$. The composition $\operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} U$ is the same as the composition $\operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} U$ and so $\operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X$ is the identity. Furthermore, the composition $\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X$ is zero as a direct consequence of the exact sequence. So it remains only to find a splitting for $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$. Consider the embedding $j : Y' \to \widetilde{X}$. This provides a morphism $\operatorname{Pic} \widetilde{X} \to \operatorname{Pic} Y'$. We know by Theorem II.8.24(b) that $Y'$ is a projective bundle over $Y$ and then from Exercise II.7.9 that $\operatorname{Pic} Y' \cong \operatorname{Pic} Y \oplus \mathbb{Z}$. We follow 1 through the composition $\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} Y' \to \operatorname{Pic} Y \oplus \mathbb{Z} \to \mathbb{Z}$. We have 1 gets sent to $\mathscr{L}(Y') \in \operatorname{Pic} \widetilde{X}$ which by Proposition II.6.18 is isomorphic to $\mathscr{I}_{Y'}^{-1}$ which we know is $\mathcal{O}_{\widetilde{X}}(-1)$ (from the proof of (7.13) for example). This then becomes $\mathcal{O}_{Y'}(-1)$ which is then sent to $-1$. So our composition is not the identity, but is an isomorphism, and we only wanted to find a splitting for $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$ so compose with $1 \mapsto -1$ and we obtain our desired splitting.

b By (a) we can write $\omega_{\widetilde{X}}$ as $f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M} \in \operatorname{Pic} X$ and some integer $q$. We have an isomorphisms $X - Y \cong \widetilde{X} - Y'$ (Proposition II.7.13) and so $\omega_{\widetilde{X}}|_{\widetilde{X}-Y'} \cong \omega_U \cong \omega_X|_{X-Y}$. We also have an isomorphism $\operatorname{Pic} X \cong \operatorname{Pic} U$ (Proposition II.6.5) and so if $\mathscr{M}|_{X-Y} \cong \omega_X|_{X-Y}$, which it is, then $\mathscr{M} \cong \omega_X$. Now by Proposition II.8.20 we have $\omega_{Y'} \cong \omega_{\widetilde{X}} \otimes \mathscr{L}(Y') \otimes \mathcal{O}_{Y'} \cong f^*\omega_X \otimes \mathscr{L}((q+1)Y') \otimes \mathcal{O}_{Y'}$. Then by Proposition II.6.18 $\mathscr{L}((q+1)Y') \cong \mathscr{I}_{Y'}^{-q-1}$ and we know that $\mathscr{I}_{Y'} = \mathcal{O}_{\widetilde{X}}(1)$ (from the proof of (7.13) for example). Putting all this together we get $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Now we take a closed point $y \in Y$ and let $Z$ be the fibre of $Y'$ over $y$; that is, $Z = y \times_Y Y'$. We can use Exercise II.8.3(b) to find that $\omega_Z \cong \pi_1^*\omega_y \otimes \pi_2^*\omega_{Y'} \cong \pi_2^*(f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) \cong \mathcal{O}_Z(-q-1)$ since $\omega_y = \mathcal{O}_y$ and pulling $\omega_X$ back to $Z$ can be done via $y$ on which it becomes the structure sheaf. Now $Z$ is just projective space of dimension $r-1$ (Theorem II.8.24) and so $\omega_Z \cong \mathcal{O}_Z(-r)$ (Example II.8.20.1) so $q = r - 1$. Hence $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.

**Exercise 8.6.** Infinitesimal Lifting Property. *Let $k$ be an algebraically losed field, let $A$ be a finitely generated $k$-algebra such that* $\operatorname{Spec} A$ *is a nonsingular variety over $k$. Let $0 \to I \to B' \to B \to 0$ be an exact seqeuence, where $B'$ is a $k$-algebra, and $I$ is an ideal with $I^2 = 0$. Finally suppose given a $k$-algebra homomorphism $f : A \to B$. Then there exists a $k$-algebra homomorphism $g : A \to B'$ lifting $f$.*

*a First suppose that $g : A \to B'$ is a given homomorphism lifting $f$. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a $k$-derivation of $A$ into $I$, which we can consider as an element of*

$\hom_A(\Omega_{A/k}, I)$. *Conversely, for any* $\theta \in \hom_A(\Omega_{A/k}, I)$, *show that* $g' = g + \theta$ *is another homomorphism lifting* $f$.

b *Now let* $P = k[x_1, \ldots, x_n]$ *be a polynomial ring over* $k$ *of which* $A$ *is a quotient, and let* $J$ *be the kernel. Show that there does exist a homomorphism* $j : P \to B'$ *making a commutative diagram*

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ J & & I \\ \downarrow & & \downarrow \\ P & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

*and show that* $h$ *induces an* $A$*-linear map* $\overline{h} : J/J^2 \to I$.

c *Conclude by finding the desired morphism* $g : A \to B'$ *(Hartshorne essentially walks the reader through the proof of this part in his statement of the exercise).*

*Solution.* a Since $g$ and $g'$ both lift $f$, the difference $g - g'$ is a lift of zero, and therefore, the image lands in the submodule $I$ of $B'$. The homomorphisms $g$ and $g'$ are algebra homomorphisms and so they both send 1 to 1, hence the difference sends 1 to 0 and so for any $c \in k$ we have $\theta(k) = k\theta(1) = 0$. For the Leibniz rule we have

$$\begin{aligned} \theta(ab) &= g(ab) - g'(ab) \\ &= g(a)g(b) - g'(a)g'(b) \\ &= g(a)g(b) - g'(a)g'(b) + (g'(a)g(b) - g'(a)g(b)) \\ &= g(b)\theta(a) + g'(a)\theta(b) \end{aligned}$$

We can consider it as an element of $\hom_A(\Omega_{A/k}, I)$ by the universal property of the module of relative differentials.

Conversely, for any $\theta \in \hom_A(\Omega_{A/k}, I)$ we obtain a derivation $\theta \circ d : A \to I$ which we can compose with the inclusion $I \to B'$ to get a $k$-linear morphism from $A$ into $B'$ Since the sequence is exact, this $\theta$ vanishes on composition with $B' \to B$ and so $g + \theta$ is another $k$-linear homomorphism lifting $f$ and we just need to show that it is actually a morphism of $k$-algebras; that is, that it preserves multiplication.

$$\begin{aligned} g(ab) + \theta(ab) &= g(ab) + \theta(a)g(b) + g(a)\theta(b) \\ &= g(ab) + \theta(a)g(b) + g(a)\theta(b) + \theta(a)\theta(b) \qquad \text{since } I^2 = 0 \text{ and } \theta(a), \theta(b) \in I \\ &= (g(a) + \theta(a))(g(b) + \theta(b)) \end{aligned}$$

b A $k$-homomorphism out of $P$ is uniquely determined by the images of the $x_i$, which can be anything. So for each $i$ choose a lift $b_i$ of $f(x_i)$ in $B'$ and we obtain a morphism $h$ by sending $x_i$ to $b_i$ and extending to a $k$-algebra homomorphism. If $a \in P$ is in $J$ then by commutivity, the image of $h(a)$ in $B$ will be zero, implying that $h(a) \in I$ so we have at least a $k$-linear map $J \to I$. If $a \in J^2$ then $h(a) \in I^2 = 0$ so this map descends to $\overline{h} : J/J^2 \to I$. The last thing to check is that the map $\overline{h}$ is $A$-linear, and this follows from $h$ preserving multiplication.

c Applying the global sections functor to the exact sequence of (8.17) with $X = \operatorname{Spec} P$, $Y = \operatorname{Spec} A$ gives an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

which is exact on the right as well by (8.3A). Now since $A$ is nonsingular, $\Omega_{A/k}$ is locally free and and therefore projective so $\operatorname{Ext}^i(\Omega_{A/k}, I) = 0$ for all $i > 0$. So the exact sequence

$$0 \to \hom_A(\Omega_{A/k}, I) \to \hom_A(\Omega_{P/k} \otimes A, I) \to \hom_A(J/J^2, I) \to \operatorname{Ext}^1_A(\Omega_{A,k}, I) \to \dots$$

shows that $\hom(\Omega_{P/k} \otimes A, I) \to \hom(J/J^2, I)$ is surjective. So we can find a $P$-morphism $\theta : \Omega_{P/k} \to I$ whose image is $\overline{h}$ from part (b). We then define $\theta'$ as the composition $P \xrightarrow{d} \Omega_{P/k} \to I \to B'$ to obtain a $k$-derivation $P \to B'$. Let $h' = h - \theta$. For any element $b \in J$ we have $h'(b) = h(b) - \theta(b) = \overline{h}(b) - \overline{h}(b) = 0$ so $h'$ descends to a morphism $g : A \to B'$ which lifts $f$.

**Exercise 8.7.** *If $X$ is affine and nonsingular, then show that any extension of $X$ by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one.*

*Solution.* Since everything is affine, the problem restated is this: given a ring $A'$, an ideal $I \subset A'$ such that $I^2$, and an isomorphism $A'/I \cong A$, such that $I \cong M$ as an $A$-module (where $M$ is the finitely generated $A$-module corresponding to $\mathscr{F}$), show that $A' \cong A \oplus M$ as an abelian group, with multiplication defined by $(a, m)(a', m') = (aa', am' + a'm)$.

Using the infinitesimal lifting property we obtain a morphism $A \to A'$ that lifts the given isomorphism $A'/I \cong A$. This together with the given data provides the isomorphism $A \oplus M \cong A'$ of abelian groups where we use the isomorphism $M \cong I$ to associate $M$ with $I$ as an $A$-module. If $a \in A$ then $(a, 0)(a', m') = (aa', am')$ using the $A$-module structure on $A$ and $M \cong I$. If $m \in M \cong I$ then $(0, m)(a', m') = (0, a'm)$ since $mm' \in I^2$. So we have the required isomorphism.

**Exercise 8.8.** *Using the method of (8.19), show that $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$ and $h^{q,0} = \dim_k \Gamma(X, \wedge^q \Omega_{X/k})$ are birational invariants of $X$, a projective nonsingular variety over $k$.*

*Solution.* The proof of (8.19) translates almost verbatim.

Suppose that we have another nonsingular, projective variety $X'$, birationally equivalent to $X$. Consider a birationally invertible map $X \to X'$ and let $V \subset X$ be the largest open subset of $X$ on which it is representable, and $f : V \to X'$ a representative morphism. We obtain a morphism of sheaves $f^*\Omega_{X'} \to \Omega_V$ via Proposition II.8.11. These are locally free sheaves of rank $n = \dim X$ and so we obtain morphisms $f^*\omega_{X'}^{\otimes n} \to \omega_V^{\otimes n}$ and $f^*\Omega_{X'}^q \to \Omega_V^q$ both of which induce morphisms of global sections. By (I, 4.5) there is an open subset $U$ of $V$ that is mapped isomorphically onto its image in $X'$ by $f$. This $\Omega_V|_U \cong \Omega'_X|_{f(U)}$ via $f$. We have a commutative square

$$\begin{array}{ccc} \Gamma(\omega_{X'}^{\otimes n}, X') & \longrightarrow & \Gamma(\omega_V^{\otimes n}, V) \\ \downarrow & & \downarrow \\ \Gamma(\omega_{f(U)}^{\otimes n}, f(U)) & \longrightarrow & \Gamma(\omega_U^{\otimes n}, U) \end{array}$$

and a similar one for $f^*\Omega_{X'}^q \to \Omega_V^q$. Since $f(U)$ is dense and open in $X'$, and a nonzero global section cannot vanish on a dense open subset, we see that the morphisms

$$\Gamma(\omega_{X'}^{\otimes n}, X') \to \Gamma(\omega_V^{\otimes n}, V) \qquad \Gamma(\Omega_{X'}^q, X') \to \Gamma(\Omega_V^q, V)$$

are both injective.

Now we compare $\Gamma(V, -)$ to $\Gamma(X, -)$. First we claim that $X - V$ has codimension $> 1$ in $X$. This follows from the valuative criterion of properness (4.7). If $P \in X$ is a point of codimension 1 then $\mathcal{O}_{X,P}$ is a discrete valuation ring because $X$ is nonsingular. The map from the generic point $\eta_X$ of $X$ to that of $X'$ fits into a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K(X) \arrow[r] \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{X,P} \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd}$$

and so we can extend $V$ to include $P$ and so by the definition of $V$, it already includes $P$.

To show that $\Gamma(V, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$ for the sheaves $\mathscr{F}$ that we are interested in, it suffices to show that $\Gamma(V \cap U, \mathscr{F}|_{U \cap V}) \cong \Gamma(U, \mathscr{F}|_U)$ for each open $U$ in a cover of $X$ (use the sequences $0 \to \Gamma(X, -) \to \oplus\Gamma(U_i, -) \to \oplus\Gamma(U_{ij}, -)$). Choose the open cover $\{U_i\}$ such that on each $U_i$ the sheaf $\mathscr{F}$ ($= \Omega_X^q$ or $\omega_X^{\otimes n}$) is free, and each $U_i$ is affine. Then what we need to show is that for each of these $U_i$, the morphism $\Gamma(U_i, \mathcal{O}_{U_i}) \to \Gamma(U \cap V, \mathcal{O}_{U_i \cap V})$ is bijective. Since $X$ is nonsingular, and therefore normal, and since $U_i - U_i \cap V$ has codimension $> 1$ in $U_i$, this is a consequence of (6.3A).

So the culmination is that we have an injective morphism $\Gamma(X', \mathscr{F}_{X'}) \to \Gamma(V, \mathscr{F}_X|_V)$ and a bijective morphism $\Gamma(X, \mathscr{F}_X) \to \Gamma(V, \mathscr{F}_X|_V)$ (where $\mathscr{F}_- =$

$\Omega^q_-$ or $\omega_-^{\otimes n}$). Hence, $P_n(X') \leq P_n(X)$ and $h^{q,0}(X') \leq h^{q,0}(X)$. By symmetry we get inequalities in the other direction and so these inequalities are actually equalities.