Algebraic Topology Homework 10

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§ *Problems from 2.1*

EXERCISE 16.

(a) Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.

(b) Show that $H_1(X, A) = 0$ if and only if $H_1(A) \longrightarrow H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof:

(a) Suppose first that A meets all path components of X. To show $H_0(X, A) = 0$, it suffices to show that all relative 0-cycles are equivalent to 0-cycles in A. First, note that a 0-simplex $\sigma : \Delta_0 \longrightarrow X$ is entirely determined by the image of the single point constituting Δ_0, that is, there is a one-to-one correspondence between points of X and 0-simplicies. Denote the 0-simplex sending the point in Δ_0 to $x \in X$ by σ_x. Given a point $x \in X$, there exists a path $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(1) = x$ and $\gamma(0) = a$ for some point $a \in A$, by the assumption that A meets every path component of X. Such a path can be realized as a 1-simplex via composition with an isomorphism $\Delta_1 \xrightarrow{\sim} [0, 1]$. Then $\delta(\gamma) = \sigma_x - \sigma_a$. As an element in $C_0(X, A)$, this is equivalent to σ_x since $-\sigma_a \in C_0(A)$. But then $\sigma_x \in \mathrm{img}(\delta_1)$, and hence represents the trivial class in $H_0(X, A)$. Since x was chosen arbitrarily, every 0-simplex represents the trivial class in $H_0(X, A)$, so $H_0(X, A) = 0$.

Now suppose that $H_0(X, A) = 0$. To show this implies A meets every path component of X, we consider the long exact sequence

$$... \longrightarrow H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} \xrightarrow{H}{}_0\ (X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0.$$

Recall that i_* acts by sending a class $[\alpha] \in H_0(A)$ represented by a cycle to $[i(\alpha)] \in H_0(X)$. Since $H_0(X, A)$ is 0, the map i_* induced by the inclusion $A \hookrightarrow X$ must be surjective. This means for each $[\beta] \in H_0(X)$ there is some $[\alpha] \in H_0(A)$ such that $i(\alpha) = \beta$. This implies that A has nontrivial intersection with the cycle β in X. Taking β to lie in a particular path component of X gives the result, for as we have seen, $H_0(X)$ can be generated by choosing one cycle for each path component of X.

(b) Note first that each path component of X contains at most one path component of A if and only if $H_0(A) \xrightarrow{i_*} H_0(X)$ is injective. Recall that the free abelian group $H_0(A)$ can be generated by choosing a single cycle in each path component of A. If two path components A_1 and A_2 of A are both intersect a path component X_1 of X nontrivially, then we can choose two cycles a_1 and a_2 corresponding in A_1 and A_2 respectively such that $i(a_1), i(a_2) \in X_1$. However, this means these are both cycles in a single path component of X, and hence represent the same element in $H_0(X)$.

If instead $i_* : H_0(A) \longrightarrow H_0(X)$ is not injective. Then we can find some element $[a] \neq 0 \in H_0(A)$ such that $i_*([a]) = 0$. Let I be an index set for the path components A_k of A and a_k be a cycle contained in A_k. Then we can find some $c_k \in \mathbb{Z}$ such that only finitely many are nonzero and

$$[a] = \sum_{k \in I} c_k [a_k].$$

Now applying i_* gives $\sum_{k \in I} c_k i_*([a_k]) = i_*([a]) = 0$. If each $i_*([a_k])$ belonged to a separate path component of X, then the c_k would necessarily be zero, but as this is not the case by the assumption that $[a] \neq 0$, at least two of the $i_*([a_k])$ lie in the same path component.

We now consider the same exact sequence as in part (a), but this time a little farther up:

$$... \longrightarrow H_2(X, A) \xrightarrow{\partial} H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \longrightarrow H_0(A) \longrightarrow ...$$

Suppose first that $H_1(X, A) = 0$. By the exactness of this sequence, $H_1(A) \xrightarrow{i_*} H_1(X)$ is necessarily surjective. Likewise, further down the sequence, $H_1(X, A) = 0$ implies that $H_0(A) \longrightarrow H_0(X)$ is injective, and hence by what we have already shown, each path component of X contains at most one path component of A.

Now suppose that $H_1(A) \longrightarrow H_1(X)$ is surjective and each path component of X contains at most one path component of A. The latter condition tells us that $H_0(A) \longrightarrow H_0(X)$ is injective, and hence the relevant exact sequence reads

$$...H_2(X, A) \longrightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow ...$$

and so by problem 2.1.15 we have that $H_1(X, A)$ is necessarily 0. To repeat a portion of the argument in this problem, the surjectivity of the first i_* implies that the kernel of j_* is all of $H_1(X)$, and so $\operatorname{img} j_* = 0$ in $H_1(X, A)$ and hence $\ker \partial = 0$. However, the injectivity of the latter i_* implies that $\operatorname{img} \partial = 0$, so in particular $\ker \partial = H_1(X, A)$. This then implies $H_1(X, A) = 0$.

$\square$

Exercise 17.

(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown. [What are X/A and X/B?]

Proof:

(a) Recall that if A is a finite set of points, say $|A| = m$ for instance, then

$$H_n(A) = \begin{cases} \mathbb{Z}^{\oplus m} & n = 0 \\ 0 & n > 0 \end{cases}.$$

We also know that

$$H_n(S^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{\oplus 2} & n = 1 \\ 0 & \text{else} \end{cases}.$$

Set $X = S^2$ and let $A \subseteq X$ be a finite subset consisting of m points. Then the long exact sequence of the pair (X, A) reads

$$...0 \longrightarrow H_2(A) \xrightarrow{i_*} H_2(X) \xrightarrow{j_*} H_2(X, A) \xrightarrow{\partial} H_1(A) \longrightarrow ...$$

in index 2. In all higher indices we have that $H_k(A) = H_k(X) = 0$, so this same exact sequence tells us $H_k(X, A) = 0$ whenever $k \geq 3$. Since $H_2(A) = H_1(A) = 0$, we get an isomorphism $H_2(X) \cong H_2(X, A)$. In the index 1 position we have

$$...0 \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow ...$$

consider the rightmost map above. The group $H_0(A)$ is the free abelian group on m generators $g_1, ..., g_m$ and are all mapped by i_* to the single generator g of $H_0(X) = \mathbb{Z}$, hence

$$\ker i_* = \left\{ \sum_{j=1}^{m-1} a_j g_j \; - \; g \sum_{j=1}^{m-1} a_j \;\middle|\; a_1, ..., a_j \in \mathbb{Z} \right\} \cong \mathbb{Z}^{m-1}.$$

This means $\operatorname{img} \partial = \ker i_* \cong \mathbb{Z}^{m-1}$. The long exact sequence gives us a short exact sequence

$$0 \longrightarrow H_1(X) \longrightarrow H_1(X, A) \longrightarrow \operatorname{img} \partial = \ker i_* \longrightarrow 0,$$

and because $\ker i_*$ is free, this short exact sequence splits giving us

$$H_1(X, A) \cong H_1(X) \oplus \ker i_* = 0 \oplus \mathbb{Z}^{m-1}.$$

For the final homology group, we have

$$... \longrightarrow H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0.$$

We have already seen that $H_0(A) \xrightarrow{i_*} H_0(X)$ is surjective, so $\ker(H_0(X) \longrightarrow H_0(X, A)) = \operatorname{img} i_* = H_0(X)$ and implies that j_* is the trivial map. However, exactness at $H_0(X, A)$ means that j_* is surjective, and hence $H_0(X, A) = 0$. To summarize,

$$H_0(S^2, A) = \begin{cases} H_2(S^2) \cong \mathbb{Z} & n = 2 \\ H_1(S^2) \oplus \mathbb{Z}^{m-1} \cong \mathbb{Z}^{m-1} & n = 1 \\ 0 & \text{else} \end{cases}.$$

Notice that nothing in our above argument relied on the fact that $X = S^2$. The only space-specific properties we needed were triviality of $H_n(X)$ for all $n \geq 3$ to get $H_n(X, A) = 0$ and the path connectedness of X in order to see $i_* : H_0(A) \longrightarrow H_0(X)$ was path connected. Both of these properties still hold for $X = S^1 \times S^1$, so by the same arguments as above,

$$H_0(S^1 \times S^1, A) = \begin{cases} H_2(S^1 \times S^1) \cong \mathbb{Z} & n = 2 \\ H_1(S^1 \times S^1) \oplus \mathbb{Z}^{m-1} \cong \mathbb{Z}^{m+1} & n = 1 \\ 0 & \text{else} \end{cases}.$$

(b)

□

§ *Problems from 3.3*

EXERCISE 3. Show that every covering space of an orientable manifolds is an orientable manifold.

EXERCISE 4. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

EXERCISE 7. For a map $f : M \longrightarrow N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \longrightarrow S^n$.

EXERCISE 8. For a map $f : M \longrightarrow N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subseteq N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \epsilon_i$ where ϵ_i is $+1$ or -1 according to whether $f : B_i \longrightarrow B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

§ *Problems from 3.3*