Algebraic Topology Homework 13

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§ ***Problems from 2.2***

EXERCISE 15. Show that if X is a CW complex then $H_n(X^n)$ is free by identifying it with the kernel of the cellular boundary map $H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$.

Proof: This one seems so brief that I imagine I'm missing something or lacking justification. Essentially, this problem entails looking at the diagram on page 139 of Hatcher immediately preceding Theorem 2.35. It gives us an exact sequence

$$0 \longrightarrow H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}).$$

Since j_n is injective, $H_n(X^n)$ is identified with $\operatorname{img} j_n = \ker \partial_n$. This alone should be enough to argue that $H_n(X^n)$ is injective, since $H_n(X^n, X^{n-1})$ is the free abelian group generated by the n-cells of X. However, as per the problem statement's instructions, we can also see this by identifying X with the kernel of $d_n : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$, since $\ker d_n = \ker \partial_n$ by the injectivity of j_n. Here's a screencap of the diagram for reference:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

Figure 1: Diagram on page 139 of Hatcher

□

EXERCISE 17. Show the isomorphism between cellular and singular homology is natural in the following sense: A map $f : X \longrightarrow Y$ that is *cellular* – satisfying $f(X^n) \subseteq Y^n$ for all n – induces a chain map f_* between the cellular chain complexes of X and Y, and the map $f_* : H_n^{CW}(W) \longrightarrow H_n^{CW}(Y)$ induced by this chain map corresponds to $f_* : H_n(X) \longrightarrow H_n(Y)$ under the isomorphism $H_n^{CW} \approx H_n$.

Proof: Let X and Y be cell complexes and let $f : X \longrightarrow Y$ be a cellular map so that $f(X^n) \subset Y^n$ for all n. The naturality of singular homology says that the diagram

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[d, "f_*"] \arrow[rr, bend left, "d_n"] & H_{n-1}(X^{n-1}) \arrow[d, "f_*"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "f_*"] \\
H_n(Y^n, Y^n - 1) \arrow[rr, bend right, "d_n"'] & H_{n-1}(Y^{n-1}) & H_{n-1}(Y^{n-1}, Y^{n-2})
\end{tikzcd}$$

commutes, where f_* is the map induced on singular homology. Thus, f induces a cellular chain map $f_\sharp$ between the cellular chain complexes of X and Y, that is, we have a commutative diagram

$$\begin{array}{ccccccccc}
... & \longrightarrow & H_n(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_n(X^{n-1}, X^{n-2}) & \longrightarrow & ... \\
 & & \downarrow f_\sharp & & \downarrow f_\sharp & & \downarrow f_\sharp & & \\
... & \longrightarrow & H_n(Y^{n+1}, Y^n) & \xrightarrow{d_{n+1}} & H_n(Y^n, Y^{n-1}) & \xrightarrow{d_n} & H_n(Y^{n-1}, Y^{n-2}) & \longrightarrow & ...
\end{array}$$

This means f induces a map on cellular homology, which we denote f_*^{CW} in adherence to the problem's stated notation. Using the long exact sequence of the pair (X^n, X^{n-1}) along with Lemma 2.34, we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \operatorname{img} \partial_{n+1} & \longrightarrow & H_n(X^n) & \xrightarrow{i_n} & H_n(X^n + 1) & \longrightarrow & 0 \\
 & & & & & & \downarrow i_{n+1} & & \\
 & & & & & & H_n(X) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

whose rows and columns are exact, where $i^n : X^n \longrightarrow X^{n+1}$ is the inclusion map. Since $i_{n+1} i_n$ is the inclusion $X_n \hookrightarrow X$ we also have a short exact sequence

$$0 \longrightarrow \operatorname{img} \partial_{n+1} \longrightarrow H_n(X^n) \xrightarrow{i_n} H_n(X) \longrightarrow 0.$$

The proof of 2.35 (used for inspiration in problem 15 as well) tells us that the isomorphism $\varphi_X : H_n(X) \longrightarrow H_n^{CW}(X)$ between singular and cellular homology is induced by j_n, i.e. it fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{img} \partial_{n+1} & \longrightarrow & H_n(X^n) & \xrightarrow{i_n} & H_n(X) & \longrightarrow & 0 \\
 & & \downarrow j_n & & \downarrow j_n & & \downarrow \varphi_X & & \\
0 & \longrightarrow & \operatorname{img} d_{n+1} & \longrightarrow & \ker d_n & \longrightarrow & H_n^{CW}(X) & \longrightarrow & 0
\end{array}$$

whose rows are exact. Imagine two copies of this diagram (I'm real tired of tikz), one for X and one for Y, with corresponding entries in the top row connected via f_* and entries in the bottom connected via $f_\sharp$. The top portion will commute by the naturality of singular homology, and by the isomorphism between singular and cellular homology the square

$$\begin{tikzcd}
H_n(X) \arrow[r, "f_*"] \arrow[d, "\varphi_X"] & H_n(Y) \arrow[d, "\varphi_Y"] \\
H_n^{CW}(X) \arrow[r, "f_*^{CW}"] & H_n^{CW}(Y)
\end{tikzcd}$$

commutes. This shows that the isomorphism between singular and cellular homology is indeed natural. □

EXERCISE 19. Compute $H_i(\mathbb{RP}^n/\mathbb{RP}^m)$ for $m < n$ by cellular homology, using the standard CW structure on $\mathbb{RP}^n$ with $\mathbb{RP}^m$ as its m-skeleton.

Proof: Recall that the standard cell structure on $\mathbb{RP}^n$ consists of a single k-cell for each $0 \leq k \leq n$. The quotient $\mathbb{RP}^n/\mathbb{RP}^m$ amounts to collapsing all $k \leq m$ cells to a point, and thus yields us a cellular chain complex of the form

$$\mathbb{Z} \xrightarrow{d_n} \dots \xrightarrow{d_{m+2}} \mathbb{Z} \xrightarrow{d_{m+1}} 0 \longrightarrow \dots \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0,$$

where $C_k^{CW} = 0$ for all $1 \leq k \leq m$ and is $\mathbb{Z}$ for all other indices, including $d = 0$. The chain maps d_k remain unchanged for $k > m$, i.e. we still have that

$$\ker(d_k) = \begin{cases} \mathbb{Z} & k \text{ is odd} \\ 0 & k \text{ is even} \end{cases} \qquad \text{and} \qquad \operatorname{img}(d_k) = \begin{cases} 0 & k \text{ is odd} \\ 2\mathbb{Z} & k \text{ is even} \end{cases},$$

whereas d_k is the trivial map for all $k \leq m$. Thus, working through the odd/even cases for both m and n gives us

$$H_k(\mathbb{RP}^n/\mathbb{RP}^m) = \begin{cases} \mathbb{Z} & k = 0,\ k = m+1 \text{ if } m \text{ is odd or } k = n \text{ if } n \text{ is odd} \\ \mathbb{Z}_2 & k \text{ is odd and } m+1 \leq k < n \\ 0 & \text{else} \end{cases}.$$

□

EXERCISE 20. For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Proof: For a finite cell complex X, let $c_n(X)$ denote the number of n-cells. Let X and Y both be cell complexes. The n-cells in $X \times Y$ are simply the products of i-cells in X and j-cells in Y where $i + j = n$, as given in Appendix A.6 in Hatcher. But with this realization, we immediately have that

$$\begin{aligned}
\chi(X \times Y) &= \sum_n (-1)^n c_n(X \times Y) \\
&= \sum_n \sum_{i+j=n} (-1)^{i+j} c_i(X) c_j(Y) \\
&= \left(\sum_i (-1)^i c_i(X) \right) \left(\sum_j (-1)^j c_j(X) \right) \\
&= \chi(X)\chi(Y),
\end{aligned}$$

so we are actually done. $\square$

EXERCISE 21. If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Proof: This one is quick, as is the next one. As in question 20, let $c_n(X)$ denote the number of n-cells in a CW complex X. If X is the union of two smaller subcomplexes, then $A \cap B$ is also a subcomplex of X consisting of cells that are contained in both A and B. We thus have that

$$c_n(X) = c_n(A) + c_n(B) - C_n(A \cap B),$$

and hence

$$\chi(X) = \sum_n (-1)^n c_n(X)$$

$$\sum_n (-1)^n (c_n(A) + c_n(B) - C_n(A \cap B))$$

$$= \sum_n (-1)^n c_n(A) \; + \; \sum_n (-1)^n c_n(B) \; - \; \sum_n (-1)^n c_n(A \cap B)$$

$$= \chi(A) + \chi(B) - \chi(A \cap B).$$

$\square$

EXERCISE 23. Show that if the closed orientable surface M_g of genus g is a covering space of M_h, then $g = n(h-1)+1$ for some n, namely, n is the number of sheets in the covering. [Conversely, if $g = n(h-1)+1$] then there is an n-sheeted covering $M_g \longrightarrow M_h$, as we saw in Example 1.41.]

Proof: Note that because M_g is compact, any covering space $M_g \longrightarrow M_h$ is finite sheeted. Consider such an n-sheeted covering space $M_g \longrightarrow M_h$. We then have that

$$2 - 2g = \chi(M_g) = n\chi(M_h) = n(2 - 2h)$$

by exercise 2.2.22, which says that $\chi(\tilde{X}) = n\chi(X)$ when $p : \tilde{X} \longrightarrow X$ is an n-sheeted covering of finite cell complexes. Solving for g, we obtain the desired result $g = n(h-1) + 1$. $\square$

EXERCISE 41. For X a finite CW complex and F a field, show that the Euler characteristic $\chi(X)$ can also be computed by the formula $\chi(X) = \sum_n (-1)^n \dim H_n(X; F)$, the alternating sum of the dimensions of the vector spaces $H_n(X; F)$.

Proof: Let c_n denote the number of n-cells in X. Working over a field F gives us a chain complex

$$\ldots \longrightarrow F^{\oplus c_n} \xrightarrow{d_n} F^{\oplus c_{n-1}} \xrightarrow{d_{n-1}} \ldots \longrightarrow F^{\oplus c_0} \longrightarrow 0.$$

Following the proof of Theorem 2.44, we have that $\dim F^{\oplus c_n} = c_n = \dim(\ker d_n) + \dim(\operatorname{img} d_n)$ and $\dim(\ker d_n) = \dim(\operatorname{img} d_{n+1}) + \dim(H_n(X; F))$ by the rank-nullity theorem and the fact that $H_n(X; F) = \ker d_n / \operatorname{img} d_{n+1}$. Substituting the second equation into the first and multiplying by $(-1)^n$ gives us

$$(-1)^n c_n = (-1)^n \big(\dim(\operatorname{img} d_{n+1}) + \dim(H_n(X; F)) + \dim(\operatorname{img} d_n) \big),$$

and summing over n leads to cancellation of the $\dim(\operatorname{img} d_n)$ terms, yielding

$$\chi(X) = \sum_n (-1)^n c_n = \sum_n (-1)^n \dim H_n(X; F).$$

$\square$

§ *Problems from 2.C*

EXERCISE 2. Use the Lefschetz fixed point theorem to show that a map $S^n \longrightarrow S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $x \mapsto -x$.

Proof: We first show that if X is a path-connected simplicial complex then the map $f_* H_0(X) \longrightarrow H_0(X)$ induced by a simplicial map $f : X \longrightarrow X$ is the identity. The 0^{th} homology group is $H_0(X) = C_0(X)/\operatorname{img}(\partial_1) = \mathbb{Z}$ by path-connectedness, so all vertices lie in the same homology class. Since f_* is simplicial, we thus have that $f_*([v]) = [f(v)] = [v]$ for any vertex v of X, hence f_* is the identity. By simplicial approximation, it follows that any continuous map $f : X \longrightarrow X$ is homotopic to a simplicial map as long as X is a finite simplicial complex, after barycentric subdividing if necessary, at least. Hence, any map $f : X \longrightarrow X$ in this setup satisfies $\operatorname{tr}(f_*) = 1$. In particular, this holds when $X = S^n$.

Now we calculate the Lefschetz number for a map $f : S^n \longrightarrow S^n$:

$$\begin{aligned}\tau(f) &= \sum_i (-1)^i \operatorname{tr}(f_* : H_i(S^n) \longrightarrow H_i(S^n)) \\ &= \operatorname{tr}(f_* : H_0(S^n) \longrightarrow H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \longrightarrow H_n(S^n)) \\ &= 1 + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \longrightarrow H_n(S^n))\end{aligned}$$

because $H_i(S^n) = 0$ unless $i = 0, n$. By the Lefschetz fixed point theorem, there exists some fixed point whenever $\tau(f) \neq 0$. We see that $\tau(f) = 0$ if and only if $\deg(f) = (-1)^{n+1}$; that is, if f has the same degree as the antipodal map. □

EXERCISE 4. This one was just too long, I couldn't finish it in time.

EXERCISE 5. Let M be a closed orientable surface embedding in $\mathbb{R}^3$ in such a way that reflection across a plane P defines a homeomorphism $r : M \longrightarrow M$ fixing $M \cap P$, a collection of circles. Is it possible to homotope r to have no fixed points?

Proof: First consider a homeomorphism $h : S^1 \times I \longrightarrow S^1 \times I$, where

$$(e^{i\theta}, \alpha) \mapsto \begin{cases} (e^i\theta + 2\pi\alpha, \alpha) & \alpha \in [0, 1/2] \\ (e^{i\theta + 2\pi(1-\alpha)}, \alpha) & \alpha \in (1/2, 1] \end{cases},$$

i.e. as α moves from 0 to $1/2$ the circle attached to α is rotated an increasing amount until we reach $\alpha = 1/2$, at which point the circles rotate back to the original position. This map fixes the boundary of $S^1 \times I$, since $S^1 \times \{0, 1\}$ experiences no rotation.

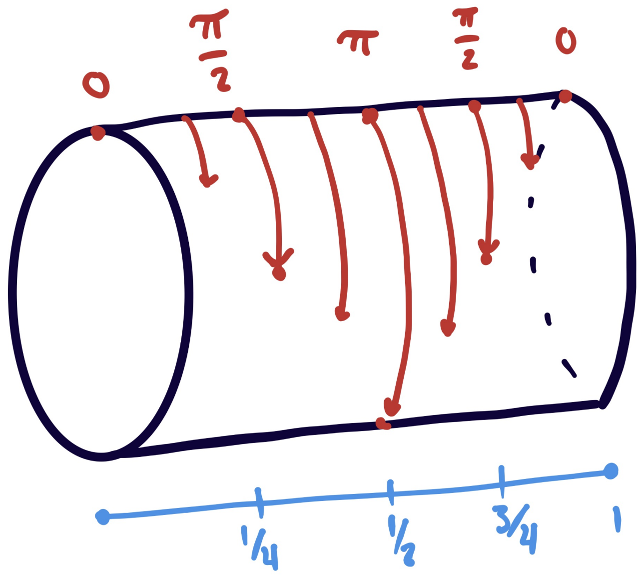

Figure 2: Illustration of the homeomorphism h on the cylinder

Note that h is homotopic to the identity on $S^1 \times I$, since we can simply interpolate each angle of rotation at a constant rate to 0.

For each circle in $M \cap P$, consider a band $N \cong S^1 \times I$ contained in M such that $M \cap P$ is identified with $S^1 \times \{1/2\}$ and apply the homeomorphism h. Crucially, no points of $M \cap P$ are fixed by h. Doing this for each circle and extending by the identity on $M \setminus P$ yields a homeomorphism $\overline{h} : M \longrightarrow M$, which is homotopic to the identity via the same homotopy yielding $h \simeq \mathrm{id}_{S^1 \times I}$ by extending by the identity outside of $M \cap P$. Thus, $r \circ \overline{h} \simeq r \circ \mathrm{id}_M$ and has no fixed points. $\square$