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Affine Hopf Algebras, I

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INTRODUCTION

In this paper we develop some of the basic theory of ungraded coalgebras, bialgebras and Hopf algebras. For us a bialgebra is simply an algebra which is also a coalgebra such that the algebra and coalgebra structures are compatible, whereas a Hopf algebra is required to possess an antipode in the sense of ([*2*], page 222) or symmetry in the sense of ([*4*], page 27).

The antipode plays the role of the inverse map $x \mapsto x^{-1}$ for groups, and for the deeper results, its existence must be assumed. Unlike [*8*] we do not require an underlying grading. The Hopf algebras which are the principal models for our study—group algebra, universal enveloping algebra of an ungraded Lie algebra, the commutative coordinate ring and the cocommutative hyperalgebra of an affine algebraic group—are substantially different from the Hopf algebras in [*8*].

The first chapter introduces our comultiplication notation which plays the same indispensable role for computations involving coalgebras that the multiplication notation $a_1 \cdots a_n$ plays for algebras. The linear dual C^* of a coalgebra C is an algebra and C is a locally finite C^*-bimodule in a natural way. The subbimodules of C are precisely the sub-coalgebras. In §1.3 we show that there is a maximal subspace A^0 in the linear dual of an algebra which carries a natural coalgebra structure. It depends functorially on the

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algebra. In the particular case of a finitely generated commutative algebra A we show that A^0 is a dense subspace of the linear dual to A.

The language of Hopf algebras permits a parallel treatment of questions relating to groups and Lie algebras, cf. [1], [9], [10]. This shows itself first in §1.5 where for any Hopf algebra H we define a subgroup $G(H)$ of the group of invertible elements in H and a sub Lie algebra $P(H)$ of the Lie algebra underlying the associative algebra H. The elements of $G = G(H)$ are called grouplike and the elements of $\mathfrak{g} = P(H)$ are called primitive. The group algebra of G and the (restricted) universal enveloping algebra of $\mathfrak{g}$ are linearly disjoint sub Hopf algebras of H. We define the adjoint action of a Hopf algebra on itself. Under this action H becomes an H-module in such a way that G acts as inner automorphisms of H and $\mathfrak{g}$ acts as inner derivations of H.

The adjoint representation of H on itself is an example of an H-module algebra. This is an algebra A which is an H-module such that the algebra structure of A and coalgebra structure of H are compatible. The elements of G act as algebra automorphisms of an H-module algebra and the elements of $\mathfrak{g}$ act as derivations. If A is an H-module algebra we define the smash product $A \# H$, in §1.8. This is an algebra containing A and H and plays the role of the semi-direct product.

In Chapter II we define inductively nth order differential operators on a commutative algebra. The first order differential operators which vanish on the unit are exactly the derivations of the algebra. For an augmented algebra A we define differential functionals $\mathscr{D}(A)$, a subspace of the linear dual. If $\mathscr{A}$ is the coordinate ring of an affine algebraic group then $\mathscr{D}(\mathscr{A})$ is the hyperalgebra associated with the group and $\mathfrak{g}$, the primitive elements of $\mathscr{D}(\mathscr{A})$, is the Lie algebra of the algebraic group. In a natural way $\mathscr{A}$ is a $\mathscr{D}(\mathscr{A})$-module algebra and this action induces an isomorphism of $\mathscr{D}(\mathscr{A})$ with the right invariant differential operators on $\mathscr{A}$, and $\mathfrak{g}$ corresponds to the right invariant derivations on $\mathscr{A}$, (2.4.3). In (2.4.5) we prove that the algebra of differential operators on $\mathscr{A}$ is isomorphic to the smash product of $\mathscr{A}$ by $\mathscr{D}(\mathscr{A})$. This implies that each derivation is an $\mathscr{A}$-linear combination of invariant derivations, i.e., elements of the Lie algebra and generalizes Hochschild's result ([5], page 502, Lemma 4.1).

In the sequel we show that if $\mathscr{A}$ is a commutative bialgebra over a field of characteristic zero then $\mathscr{D}(\mathscr{A})$ is the universal enveloping algebra of its primitive elements. As a consequence we show that a commutative Hopf algebra over a field of characteristic zero is reduced. We show that if $\mathscr{A}$ is the coordinate ring of an affine algebraic group in any characteristic, then each primitive element of $\mathscr{D}(\mathscr{A})$ lies in an infinite sequence of divided powers.

In the third chapter we study irreducible and connected coalgebras. An irreducible coalgebra is one which contains a unique simple sub-coalgebra. A connected coalgebra is an irreducible coalgebra which contains a one-dimensional simple sub-coalgebra. This definition of connectedness for a coalgebra will be seen to generalize the definition given in [8] for the graded case. We show that every irreducible sub-coalgebra of a coalgebra lies in a unique irreducible component; moreover, a cocommutative coalgebra is the direct sum of its irreducible components. This permits us to prove Kostant's theorem that a cocommutative Hopf algebra H over an algebraically closed field is the smash product of a connected Hopf algebra by the group algebra of G. As a consequence if $\mathscr{A}$ is the coordinate ring of an affine algebraic group then $\mathscr{A}^0$ is the smash product of the hyperalgebra of the group by the group algebra of the group.

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1. Preliminaries

If V and W are vector spaces over k, we let $\mathrm{Hom}(V, W)$ denote the space of all k-linear maps from V into W, $V^* = \mathrm{Hom}(V, k)$ and $\mathrm{End}\, V = \mathrm{Hom}(V, V)$. We let $u^* \in \mathrm{Hom}(W^*, V^*)$ denote the transpose map of $u \in \mathrm{Hom}(V, W)$. If $f \in V^*$ and $v \in V$, it will sometimes be convenient to write $\langle f, v\rangle$ instead of $f(v)$. For example, if $S \subset V$ and $T \subset V^*$ we set

$$S^\perp = \{f \in V^* \mid \langle f, S\rangle = 0\}$$
$$T^\perp = \{v \in V \mid \langle T, v\rangle = 0\}$$

A subset T of V^* is *dense* if $T^\perp = 0$. Observe that $V^* \otimes W^*$ is a dense subset of $(V \otimes W)^*$. A subspace $W \subset V$ is *co-finite* if the space V/W is finite dimensional. If A and B are algebras over k, we let $\mathrm{Alg}(A, B)$ denote the set of all algebra morphisms $u : A \to B$. A vector space M over k is an A-module if $\pi \in \mathrm{Alg}(A, \mathrm{End}\, M)$ is given; M is *locally finite* if every finite dimensional subspace of M generates a finite dimensional A-submodule.

We let $\tau : V \otimes W \to W \otimes V$ denote the 'twist' map which carries $v \otimes w \to w \otimes v$.

1.1. We recall briefly some basic definitions.

Definition. An *algebra* (A, p, η) over a field k is a vector space A over k together with linear maps $p : A \otimes A \to A$ and $\eta : k \to A$ such that $p \circ (p \otimes I) = p \circ (I \otimes p) : A \otimes A \otimes A \to A$ and

$$p \circ (\eta \otimes I) = I = p \circ (I \otimes \eta) : k \otimes A = A = A \otimes k \to A.$$

A is commutative if $p \circ \tau = p : A \otimes A \to A$. If (A, p_A, η_A) and (B, p_B, η_B) are k-algebras then the vector space $A \otimes B$ is given the algebra structure $p_{A\otimes B}$, $\eta_{A\otimes B}$ where

$$p_{A\otimes B} = (p_A \otimes p_B) \circ (I \otimes \tau \otimes I) : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$$

and

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B : k \otimes k = k \to A \otimes B$$

A linear map $u : A \to B$ is an algebra morphism if

$$p_B \circ (u \otimes u) = u \circ p_A : A \otimes A \to B$$

and

$$u \circ \eta_A = \eta_B : k \to B.$$

A *coalgebra* (C, Δ, ϵ) is a vector space C over k together with linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ such that

$$(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta : C \to C \otimes C \otimes C$$

and

$$(\epsilon \otimes I) \circ \Delta = I = (I \otimes \epsilon) \circ \Delta : C \to k \otimes C = C = C \otimes k$$

Reversing the arrows from before, we say that C is co-commutative if $\tau \circ \Delta = \Delta : C \to C \otimes C$.

If $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ are k-coalgebras, we define a coalgebra structure $\Delta_{C\otimes D}$, $\epsilon_{C\otimes D}$ on the vector space $C \otimes D$ where

$$\Delta_{C\otimes D} = (I \otimes \tau \otimes I) \circ (\Delta_C \otimes \Delta_D) \qquad \text{and} \qquad \epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D .$$

A linear map $u : C \to D$ is a coalgebra morphism if

$$(u \otimes u) \circ \Delta_C = \Delta_D \circ u : C \to D \otimes D$$

and

$$\epsilon_D \circ u = \epsilon_C : C \to k$$

A subspace D of a coalgebra (C, Δ, ϵ) is a *subcoalgebra* if $\Delta D \subset D \otimes D$. D is then a coalgebra whose structure morphisms are the restrictions of those of C.

It should be noted that if $u : C \to D$ is a coalgebra morphism then the image of the linear map u is a subcoalgebra of D.

A necessary and sufficient condition for a subspace $D \subset C$ to be a subcoalgebra is that $D^{\perp} \subset C^*$ is a twosided ideal. If $I \subset C^*$ is any twosided ideal then $I^{\perp} \subset C$ is a subcoalgebra.

A subspace J of a coalgebra (C, Δ, ϵ) is a coideal if $J \subset \operatorname{Ker} \epsilon$ and $\Delta J \subset C \otimes J + J \otimes C$. If J is a coideal then C/J has a unique natural coalgebra structure where $\pi : C \to C/J$ is a coalgebra morphism. If $f : C \to D$ is any coalgebra morphism then $J = \operatorname{Ker} f$ is a coideal in C and there is a unique coalgebra morphism $\bar{f} : C/J \to D$ such that $\bar{f}\pi = f$.

A necessary and sufficient condition for a subspace $J \subset C$ to be a coideal is that $J^{\perp} \subset C^*$ is a subalgebra. If $A \subset C^*$ is any subalgebra then $A^{\perp} \subset C$ is a coideal.

If A and B are algebras and $f, g \in \operatorname{Alg}(A, B)$ then $\operatorname{Ker}(f - g)$ is a subalgebra of A. In fact

$$0 \to \operatorname{Ker}(f - g) \to A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is an equalizer diagram. Similarly if C and D are coalgebras and $f, g \in \operatorname{Coalg}(C, D)$ then $\operatorname{Im}(f - g)$ is a coideal in D so that $D/\operatorname{Im}(f - g) = \operatorname{Coker}(f - g)$ is a quotient coalgebra of D. Also,

$$C \underset{g}{\overset{f}{\rightrightarrows}} D \to D/\operatorname{Im}(f - g) = \operatorname{Coker}(f - g) \to 0$$

is a coequalizer diagram.

If (C, Δ, ϵ) is a coalgebra over k and F is any field extension of k then

$$(C_F = C \otimes_k F, \Delta_F = \Delta \otimes I, \epsilon_F = \epsilon \otimes I)$$

is a coalgebra over F. This is the scalar extension of C from k to F.

Remark. In the category of co-commutative coalgebras over k, the tensor product described above is simply the product in the category. Also the coalgebra k is a final object.

A *bialgebra* $(H, \Delta, p, \epsilon, \eta)$ is a vector space H over k which has both an algebra structure (H, p, η) and a coalgebra structure (H, Δ, ϵ) which are compatible in the sense that $\epsilon : H \to k$ and $\Delta : H \to H \otimes H$ are both algebra morphisms; this can easily be seen to be equivalent to requiring that $\eta : k \to H$ and $p : H \otimes H \to H$ should both be coalgebra morphisms. The bialgebra H is called *commutative* if the underlying algebra structure on H is commutative; it is called *co-commutative* if the underlying coalgebra structure on H is co-commutative. A linear map from one bialgebra into another is a morphism of bialgebras if it is both an algebra and a coalgebra morphism. Finally the tensor product of two bialgebras is again a bialgebra with the tensor product algebra and coalgebra structure.

1.1.1 *Remark.* It follows from our definitions that a co-commutative (respectively commutative) bialgebra is a monoid (respectively comonoid) in the category of co-commutative coalgebras (respectively commutative algebras). Similarly it will be seen that a group (respectively co-group) in the category of co-commutative coalgebras (respectively commutative algebras) is simply a co-commutative Hopf algebra (respectively commutative Hopf algebra).

1.2. If (C, Δ, ϵ) is a coalgebra, let $p : C^* \otimes C^* \to C^*$ be the map induced by restriction to $C^* \otimes C^*$ of the transpose map $\Delta^* : (C \otimes C)^* \to C^*$. The map p together with $\eta = \epsilon^* : k = k^* \to C^*$ give a natural algebra structure to C^*. Thus if $f, g \in C^*$, then $f \cdot g \in C^*$ is defined by

$$f \cdot g = (f \otimes g) \circ \Delta : C \to k \otimes k = k.$$

Notice that C^* is a commutative algebra if C is a co-commutative coalgebra.

If $c \in C$, we will write $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ to denote $\Delta c = \Delta_1 c$. We define

$$\Delta_n = (\Delta \otimes 1 \otimes \cdots \otimes 1) \circ \Delta_{n-1} : C \to \bigotimes^{n+1} C$$

inductively and write

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n)} \quad \text{to denote} \quad \Delta_{n-1}(c))$$

Finally, if $f : C \times C \times \cdots \times C \to V$ is an n-linear map into a k-vector space V which induces the linear map $\tilde{f} : \otimes^n C \to V$ we will write

$$\sum_{(c)} f(c_{(1)}, \ldots, c_{(n)}) \quad \text{to denote} \quad \tilde{f}(\Delta_{n-1}(c))$$

In this notation, for example, the product $f \cdot g$ of two functions $f, g \in C^*$ is given by: $(f \cdot g)(c) = \sum_{(c)} f(c_{(1)})\, g(c_{(2)})$.

1.3. If (A, p, η) is an algebra, the transpose map $p^* : A^* \to (A \otimes A)^*$ does not in general carry A^* into the *subspace* $A^* \otimes A^*$ (the natural imbedding of $A^* \otimes A^*$ into $(A \otimes A)^*$ is not surjective unless A is finite dimensional). But we shall see that there is a certain largest subspace A^0 of A^* which *is* carried into $A^* \otimes A^*$ by p^* and that in face $p^*(A^0) \subset A^0 \otimes A^0$.

If A is an algebra, define

$$A^0 = \{f \in A^* \mid f \in I^\perp \text{ for some cofinite twosided ideal } I \text{ of } A\}.$$

1.3.1. LEMMA. *Let A and B be k-algebras.*

(a) *If $u \in \mathrm{Alg}(A, B)$ and J is a cofinite ideal of B, then $u^{-1}(J)$ is a cofinite ideal of A and $u^*(B^0) \subset A^0$.*

(b) *The natural imbedding $A^* \otimes B^* \to (A \otimes B)^*$ induces an isomorphism of $A^0 \otimes B^0$ onto $(A \otimes B)^0$.*

Proof. (a) Let J be a cofinite ideal of B. Then the kernel $u^{-1}(J)$ of the composite algebra morphism $A \to B \to B/J$ is a cofinite ideal I of A and it is clear that $u^*(J^\perp) \subset I^\perp \subset A^0$.

(b) Let I be a cofinite ideal of A and J be a cofinite ideal of B. Then we have

$$\begin{aligned}(A \otimes J + I \otimes B)^\perp &= [A \otimes B/A \otimes J + I \otimes B]^* \\ &= [A/I \otimes B/J]^* \\ &= (A/I)^* \otimes (B/J)^* \\ &= I^\perp \otimes J^\perp\end{aligned}$$

This proves that $A \otimes J + I \otimes B$ is a cofinite ideal of $A \otimes B$ and that $I^\perp \otimes J^\perp = (A \otimes J + I \otimes B)^\perp \subset (A \otimes B)^0$ so that $A^0 \otimes B^0 \subset (A \otimes B)^0$. Conversely let K be a cofinite ideal of $A \otimes B$. Then by part (a) we have

$$I = \{a \in A \mid a \otimes 1 \in K\} \text{ is a cofinite ideal of } A$$

and

$$J = \{b \in B \mid 1 \otimes b \in K\} \text{ is a cofinite ideal of } B$$

Clearly K contains the cofinite ideal $A \otimes J + I \otimes B$ so that $K^\perp$ is contained in $(A \otimes J + I \otimes B)^\perp = I^\perp \otimes J^\perp \subset A^0 \otimes B^0$. Q.E.D.

1.3.2. Corollary. *If (A, p, η) is an algebra, then the map*

$$p^* : A^* \to (A \otimes A)^*$$

carries A^0 into $A^0 \otimes A^0$; if we define $\epsilon : A^0 \to k$ to be the restriction to A^0 of the map $\eta^* : A^* \to k^* = k$ then A^0 becomes a coalgebra. If I is a cofinite ideal of A then $p^*(I^\perp) \subset I^\perp \otimes I^\perp$ so that $I^\perp$ is a subcoalgebra of A^0. Finally if $u \in$ Alg(A, B) then $u^0 : B^0 \to A^0$ is a coalgebra morphism.

Proof. If I is a cofinite ideal of A then $A \otimes I + I \otimes A$ is a cofinite ideal of $A \otimes A$. Since $p(A \otimes I + I \otimes A) = I$ we conclude that

$$p^*(I^\perp) \subset (A \otimes I + I \otimes A)^\perp = I^\perp \otimes I^\perp \subset A^0 \otimes A^0$$

The last statement follows from part (a) of the lemma.

1.3.3. Corollary. *If (H, Δ, p) is a bialgebra then H^0 is a subalgebra of H^* and (H^0, p^*, Δ^*) is a bialgebra.*

Proof. Apply part (a) of the lemma to $\Delta \in$ Alg$(H, H \otimes H)$.

If A is an algebra, and $a \in A$, we define operators $L(a)$ and $R(a)$ in End(A^*) as follows:

$$\langle L(a)f, b\rangle = \langle f, ba\rangle \qquad \text{where} \qquad f \in A^*, \quad b \in A$$
$$\langle R(a)f, b\rangle = \langle f, ab\rangle$$

Under this action A^* becomes an $A - A$ bimodule; that is, if $a, b \in A$

$$L(ab) = L(a) \circ L(b) \qquad R(ab) = R(b) \circ R(a)$$
$$L(a) \circ R(b) = R(b) \circ L(a) \qquad L(1) = I = R(1)$$

1.3.4. Proposition. *Let A be an algebra and let $f \in A^*$. The following conditions are equivalent*:

1) $f \in A^0$

2) $p^*(f) \in A^0 \otimes A^0$

3) $p^*(f) \in A^* \otimes A^*$

4) $A \cdot f = \{L(a)f \mid a \in A\}$ is finite dimensional

5) $A \cdot f \cdot A =$ (*the A–A sub-bimodule of A^* generated by f*) *is finite dimensional.*

Proof. 1) implies 2) by Corollary 1.3.2 of the previous lemma.
It is clear that 2) implies 3).

3) implies 4). Assume $p^*(f) = \sum f_i \otimes g_i \in A^* \otimes A^*$ where $f_i, g_i \in A^*$. Then $\langle L(a)f, b\rangle = \langle f, ba\rangle = \langle p^*(f), b \otimes A\rangle = \sum \langle f_i, b\rangle\langle g_i, a\rangle$. Thus $L(a)f = \sum \lambda_i f_i$ where $\lambda_i = \langle g_i, a\rangle$ so that $L(a)f$ lies in the finite dimensional subspace of A^* spanned by the $\{f_i\}$.

4) implies 1). If $M = A \cdot f$ is a finite dimensional (left) A-module then $I = \{a \in A \mid L(a)M = 0\}$ is a cofinite ideal of A. Since $L(a)f = 0$ for $a \in I$ it follows from the equation $\langle L(a)f, 1\rangle = \langle f, a\rangle$ that $f \in I^\perp \subset A^0$.

1) implies 5). Let I be a two sided ideal of A. Then

$$\langle L(a)\, R(b) f, I\rangle = \langle f, bIa\rangle = 0$$

so that if $f \in I^\perp$ it follows that $L(a) R(b) f \in I^\perp$. Thus $I^\perp$ is an A–A sub-bimodule of A^*; it is finite dimensional if I is cofinite.

Finally it is clear that 5) implies 4). Q.E.D.

1.3.5. Corollary. *A^0 is a locally finite A–A sub-bimodule of A^*.*

1.3.6. Corollary. *Let (C, Δ, ϵ) be a coalgebra and let (A, p, η) be the dual* *algebra. If* $j : C \to C^{**} = A^*$ *is the natural injection then the image of* j *is contained in* A^0 *and* $j : C \to A^0$ *is a coalgebra morphism.*

Proof. If $A^* \otimes A^* \to (A \otimes A)^*$ is the canonical imbedding then the diagrams

$$\begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[d, "j"'] & & C \otimes C \arrow[d, "j\otimes j"] \\ A^* \arrow[r, "\mathfrak{p}^*"] & (A \otimes A)^* & A^* \otimes A^* \arrow[l] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "j"'] & k \\ A^* \arrow[ur, "\eta^*"'] & \end{tikzcd}$$

are commutative. The corollary follows at once from part 3) of the proposition. Q.E.D.

Remark. If $u \in \mathrm{Alg}(A, B)$ then $u^0 : B^0 \to A^0$ is a coalgebra morphism by Corollary 1.3.2. Clearly we obtain in this way a contravariant functor $A \to A^0$ from the category of k-algebras into the category of k-coalgebras. Similarly if $u : C \to D$ is a coalgebra map then $u^* : D^* \to C^*$ is an algebra map and we obtain a contravariant functor $C \to C^*$ from the category of k-coalgebras into the category of k-algebras. The two contravariant functors so defined are *adjoint* functors: if A is any k-algebra and C is any k-coalgebra then the mapping $u \mapsto u^0 \circ j$ defines a natural bijective correspondence from $\mathrm{Alg}(A, C^*)$ onto $\mathrm{Coalg}(C, A^0)$. (Note: here $j : C \to (C^*)^0$ is the coalgebra morphism of the corollary above.) The inverse mapping $\mathrm{Coalg}(C, A^0) \to \mathrm{Alg}(A, C^*)$ carries $v \mapsto v^* \circ k$ where $k : A \to A^0)^*$ is the algebra morphism $k = \lambda^* \circ \mu$ defined by the natural injections $\lambda : A^0 \to A^*$ and $\mu : A \to A^{**}$.

Definition. An algebra A is *proper* if A^0 is dense in A^*.

Clearly A is a proper algebra if and only if the cofinite ideals have intersection zero, or equivalently the finite dimensional representations of A 'separate' points of A.

1.3.7. Proposition. *A finitely generated commutative algebra is proper.*

Proof. Let a be a nonzero element of A and let $I = \{x \in A \mid xa = 0\}$. Then $I \neq A$ so there is a maximal ideal M containing I. Let $J = \bigcap_n M^n$. By the lemma of Artin-Rees $MJ = J$ and there is a $y \in M$ such that $(1 - y)J = 0$. If a were in J it would follow in particular that $(1 - y)a = 0$ so that $1 - y \in I \subset M$ and therefore $1 = y + (1 - y) \in M$. It follows that $a \notin J$ so that $a \notin M^n$ for some n. By the Hilbert nullstellensatz ([6], p. 255) M is a coffine ideal. By the Hilbert basis theorem it is a finitely generated ideal. It follows from the corollary to the following lemma that M^n is a cofinite ideal. Since $a \notin M^n$ we are done. Q.E.D.

1.3.8. LEMMA. *Let M be a left module for a not necessarily commutative algebra A. If M is a finitely generated A-module and V is a cofinite subspace of A then VM is a cofinite subspace of M.*

Proof. Since M is a finitely generated A-module there is a surjective A-module morphism

$$\pi : A \oplus \cdots \oplus A \to M.$$

Thus $VM = V\pi(A \oplus \cdots \oplus A) \supset \pi(V \oplus \cdots \oplus V)$ and we have the factorization

$$\begin{array}{ccc} A \oplus \cdots \oplus A & \xrightarrow{\pi} & M \\ \downarrow & & \downarrow \\ (A/V) \oplus \cdots \oplus (A/V) & \xrightarrow{\bar{\pi}} & M/VM. \end{array}$$

VM is cofinite since $\bar{\pi}$ is surjective and A/V is finite dimensional. Q.E.D.

1.3.9. COROLLARY. *If A is a (not necessarily commutative) algebra and I is a cofinite left ideal which is finitely generated as an ideal then I^n is a cofinite left ideal for all n.*

Proof. $I^n = I^{n-1} \cdot I$ so by induction and (1.3.8) I^n is cofinite in I. Since I is cofinite in A we are done. Q.E.D.

1.3.10. PROPOSITION. *Suppose B is a commutative algebra with subalgebra A where A is noetherian and B is a finitely generated A-module. The natural coalgebra morphism $\pi : B^0 \to A^0$ (induced by the inclusion $\iota : A \to B$) is surjective.*

Proof. Suppose I is a cofinite ideal of A. By the Artin-Rees theorem [Zariski and Samuel, Vol. II, p. 255, Th 4′] there is a number l where

$$I^{l+1}B \cap A = I(I^l B \cap A).$$

Set $J = I^{l+1}B$; by (1.3.8) J is a cofinite ideal of B where by the above formula $J \cap A \subset I$. Thus there is a *linear* map $t : B/J \to A/I$ making the following diagram commutative:

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ {\scriptstyle \iota}\uparrow & & \downarrow{\scriptstyle t} \\ A & \longrightarrow & A/I. \end{array}$$

If $f \in A^0$ where $\operatorname{Ker} f \supset I$ then f induces a unique linear map $\bar{f}: A/I \to k$ where the following diagram is commutative:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "f"'] & & A/I \arrow[dl, "\bar{f}."] \\ & k & \end{tikzcd}$$

Thus letting $F: B \to k$ be the composite $B \to B/J \xrightarrow{\iota} A/I \xrightarrow{\bar{f}} k$. We have that $F \in B^0$ (since $J \subset \operatorname{Ker} F$) and $\pi(F) = f$. Q.E.D.

1.4. Let C be a coalgebra and let $A = C^*$ with the induced algebra structure (1.2). For each $f \in A$ we have defined endomorphisms $R(f)$ and $L(f)$ of A^* and have seen that A^* is an A–A bimodule under this action. Since $A^* = C^{**}$, C is a subspace of A^*. (In fact C is contained in A^0 by Corollary 1.3.6 so that A is a proper algebra.)

1.4.1. Proposition. *If $c \in C$ let $\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. Then for any $f \in A$*

$$L(f)\, c = \sum \langle f, c_{(2)} \rangle\, c_{(1)} \in C$$

$$R(f)\, c = \sum \langle f, c_{(1)} \rangle\, c_{(2)} \in C$$

Thus C is a locally finite A–A sub-bimodule of A^*.

Proof. If $g \in A$ we have

$$\langle g, L(f)\, c \rangle = \langle gf, c \rangle = \langle g \otimes f, \Delta c \rangle = \sum_{(c)} \langle g, c_{(1)} \rangle \langle f, c_{(2)} \rangle.$$

This proves the first equality. The second is proved the same way.

1.4.2. Proposition. *Let D be a subspace of the coalgebra C. The following conditions are equivalent:*

1) *D is a subcoalgebra of C*
2) *D is an A–A sub-bimodule of C*
3) *$D^\perp$ is an ideal of the algebra A.*

Proof. The condition for D to be an A–A sub-bimodule of C is simply that $R(f)c \in D$ and $L(f)c \in D$ for all $f \in A^*$, $c \in D$. Thus (1) implies (2) by virtue of Proposition 1.4.1.

If $f \in A$, it is clear that $\langle D^\perp, R(f)D \rangle = 0$ if and only if $\langle fD^\perp, D \rangle = 0$. Since $D^{\perp\perp} = D$, it follows that $R(f)D \subset D$ for all $f \in A$ if and only if $D^\perp$ is a left ideal of A. Similarly $L(f)D \subset D$ for all $f \in A$ if and only if $D^\perp$ is a right ideal of A. This shows that (2) and (3) are equivalent.

Finally assume that $D^\perp$ is an ideal of A. Since

$$(D \otimes D)^\perp \cap (A \otimes A) = A \otimes D^\perp + D^\perp \otimes A$$

to prove that $\Delta D \subset D \otimes D$ it suffices to check that

$$\langle A \otimes D^\perp + D^\perp \otimes A, \Delta D\rangle = \langle A \cdot D^\perp + D^\perp \cdot A, D\rangle = \langle D^\perp, D\rangle = 0$$

Q.E.D.

1.4.3. COROLLARY. *If $\{C_i\}_{i\in I}$ is a family of subcoalgebras of C then $\sum_{i\in I} C_i$ and $\bigcap_{i\in I} C_i$ are subcoalgebras of C. Moreover every subspace V of C is contained in a unique minimal subcoalgebra called the coalgebra 'generated' by V. If V is finite dimensional, so is the coalgebra generated by V.*

Proof. The first assertions are clearly true for A–A sub-bimodules of C; the last assertion follows because C is a locally finite A–A bimodule.

1.5. If (C, Δ, ϵ) is a coalgebra and (A, p, η) is an algebra, we define an (associative) algebra structure called *convolution* on $\operatorname{Hom}(C, A)$. If $u, v \in \operatorname{Hom}(C, A)$ we define the product $u * v \in \operatorname{Hom}(C, A)$ by

$$u * v = p \circ (u \otimes v) \circ \Delta : C \to A$$

The unit of this algebra is the map $\eta \circ \epsilon : C \to A$ (which we will write ϵ, considering k being imbedded in A via η). The algebra $\operatorname{Hom}(C, A)$ is functorial in C and A in the sense that: $C \to \operatorname{Hom}(C, A)$ is a contravariant functor from the category of coalgebras to the category of algebra and $A \to \operatorname{Hom}(C, A)$ is a covariant functor from the category of algebras to the category of algebras. Thus if $u, v \in \operatorname{Hom}(C, A)$ and $f : D \to C$ is a coalgebra morphism and $g : A \to B$ is an algebra morphism then

$$g \circ (u * v) = (g \circ u) * (g \circ v) \in \operatorname{Hom}(C, B)$$

and

$$(u * v) \circ f = (u \circ f) * (v \circ f) \in \operatorname{Hom}(D, A)$$

Notice that if $A = k$ the algebra structure on $C^* = \operatorname{Hom}(C, k)$ agrees with the one already defined in 1.2.

In particular, if H is a bialgebra, then $\operatorname{End} H = \operatorname{Hom}(H, H)$ is an algebra (with unit ϵ) under *convolution* where if $u, v \in \operatorname{End} H$

$$(u * v)(h) = \sum_{(h)} u(h_{(1)})\, v(h_{(2)}) \qquad \text{for} \qquad h \in H.$$

DEFINITION. If the identity operator $I \in \operatorname{End} H$ has an inverse $s \in \operatorname{End} H$

in the convolution algebra, H is called a *Hopf algebra* and s is an *antipode*. Clearly s is then unique. Thus s is characterized by the equation

$$\epsilon(h) = \sum_{(h)} h_{(1)}s(h_{(2)}) = \sum_{(h)} s(h_{(1)})\, h_{(2)} \qquad \text{for all} \qquad h \in H$$

If H is a bialgebra, let

$$\begin{aligned} G(H) &= \{x \in H \mid x \neq 0 \text{ and } \Delta x = x \otimes x\}, \\ P(H) &= \{x \in H \mid \Delta x = x \otimes 1 + 1 \otimes x\}, \\ G_H &= G(H^0) = \mathrm{Alg}(H, k), \\ P_H &= P(H^0) = \{f \in H^* \mid \Delta f = f \otimes \epsilon + \epsilon \otimes f\}. \end{aligned}$$

An element of $G(H)$ is called *grouplike*; an element of $P(H)$ is called *primitive*.

1.5.1. Lemma. (a) *if $x, y \in G(H)$ then $xy \in G(H)$ and $\epsilon(x) = \epsilon(y) = 1$. If H is a Hopf algebra then $G(H)$ is a subgroup of the group of units of H and $s(x) = x^{-1}$ for $x \in G(H)$. Moreover $H \mapsto G(H)$ is a covariant functor from the category of bialgebras (resp. Hopf algebras) to the category of semi-groups (resp. groups).*

(b) *if $x, y \in P(H)$ then $[x, y] = xy - yx \in P(H)$ and $\epsilon(x) = \epsilon(y) = 0$. Moreover if $p > 0$ where p = characteristic k then $x \in P(H)$ implies $x^p \in P(H)$. Thus $P(H)$ is a (restricted) Lie algebra and $H \mapsto P(H)$ is a covariant functor from the category of bialgebras to the category of (restricted) Lie algebras. If $x \in P(H)$ where H is a Hopf algebra then $s(x) = -x$.*

The proof is immediate from the definitions.

1.5.2. Proposition. *If $(H, p, \Delta, \epsilon, \eta)$ is a Hopf algebra with antipode s then*

(1) *If $u : H \to A$ is an algebra morphism then u is invertible in the convolution algebra* $\mathrm{Hom}(H, A)$ *and $u^{-1} = u \circ s$. Moreover* $\mathrm{Alg}(H, A)$ *is a subgroup of the group of units of* $\mathrm{Hom}(H, A)$ *if A is commutative.*

(2) *If $v : C \to H$ is a coalgebra morphism then v is invertible in the convolution algebra* $\mathrm{Hom}(C, H)$ *and $v^{-1} = s \circ v$. Moreover the set* $\mathrm{Coalg}(C, H)$ *of coalgebra morphisms from C into H is a subgroup of the group of units of* $\mathrm{Hom}(C, H)$ *if C is co-commutative.*

(3) *If $u : H_1 \to H_2$ is a bi-algebra morphism where H_1 and H_2 are Hopf algebras then $u \circ s_1 = s_2 \circ u$.*

(4) $\epsilon \circ s = \epsilon = s \circ \epsilon : H \to k$

(5) $p \circ (s \otimes s) \circ \tau = s \circ p : H \otimes H \to H$

(6) $\tau \circ (s \otimes s) \circ \Delta = \Delta \circ s : H \to H \otimes H$

(7) $s \circ s = I$ *if H is a commutative or co-commutative Hopf algebra.*

Note that (4), (5), and (6) assert that $s : H \to H$ is an algebra (coalgebra) anti-morphism; i.e. $s : H \to H^{\text{op}}$ is a bi-algebra morphism where H^{op} has product and coproduct opposite to that of H. (7) accounts for the choice of the word 'antipode'.

Proof. (1) If $u \in \text{Alg}(H, A)$, then

$$u * (u \circ s) = (u \circ I) * (u \circ s) = u \circ (I * s) = u \circ \epsilon = \epsilon$$

and

$$(u \circ s) * u = (u \circ s) * (u \circ I) = u \circ (s * I) = u \circ \epsilon = \epsilon.$$

Thus u is invertible and $u^{-1} = u \circ s$. Since $p_A : A \otimes A \to A$ and $u \circ s : H \to A$ are algebra morphisms when A is commutative [by (5)], the last statement in (1) follows from the equations $u * v = p_A \circ (u \otimes v) \circ \Delta$ and $u^{-1} = u \circ s$.

(2) The proof is just like (1).

(3) If $u : H_1 \to H_2$ is a morphism of bialgebras we have $u^{-1} = u \circ s_1$ by (1) and $u^{-1} = s_2 \circ u$ by (2). Thus $s_2 \circ u = u \circ s_1$.

(4) $\epsilon : H \to k$ is a morphism of bialgebras so this follows from (3)

(5) Since $p_H : H \otimes H \to H$ is a coalgebra morphism, p is invertible in the convolution algebra $\text{Hom}(H \otimes H, H)$ and $p^{-1} = s \circ p$ by (2). It thus suffices to prove that $p \circ (s \otimes s) \circ \tau$ is also a (left) inverse of p. Now if $a \otimes b \in H \otimes H$,

$$[(p \circ (s \otimes s) \circ \tau) * p](a \otimes b) = \sum_{(a),(b)} s(b_{(1)})\, s(a_{(1)})\, a_{(2)}\, b_{(2)}$$

$$= \epsilon(a) \sum_{(b)} s(b_{(1)})\, b_{(2)} = \epsilon(a)\, \epsilon(b) = \epsilon(a \otimes b)$$

(6) Similarly the map $\Delta \in \text{Alg}(H, H \otimes H)$ is invertible in $\text{Hom}(H, H \otimes H)$ and $\Delta^{-1} = \Delta \circ s$ by (1). It thus suffices to check that $\tau \circ (s \otimes s) \circ \Delta$ is a (right) inverse of Δ. *If* $h \in H$,

$$[\Delta * (\tau \circ (s \otimes s) \circ \Delta)](h) = \sum_{(h)} h_{(1)}s(h_{(4)}) \otimes h_{(2)}s(h_{(3)})$$

$$= \sum h_{(1)}s(h_{(3)}) \otimes \epsilon(h_{(2)})$$

$$= \sum h_{(1)}s(h_{(2)}) \otimes 1 = \epsilon(h)(1 \otimes 1).$$

(7) If H is commutative then $s \in \text{Alg}(H, H)$ by (5) so that $s^{-1} = s \circ s$ by (1). If H is co-commutative we also have $s^{-1} = s \circ s$ by (6) and (2). But $s^{-1} = I$ by definition so that $s \circ s = I$ in either case.

Remarks. 1. We have seen (Corollary 1.3.3.) that a bi-algebra structure on H induces one on H^0. If in addition H is a Hopf algebra with antipode s, then $s(ab) = s(b)\,s(a)$ by (5) above. Just as in Lemma 1.3.1 (a) we conclude that $s^*(H^0) \subset H^0$ so that H^0 is also a Hopf algebra with the antipode induced on H^0 by restriction of $s^* : H^* \to H^*$.

2. If H is a Hopf algebra with augmentation ideal $\mathscr{M}$, then $s(\mathscr{M}) \subset \mathscr{M}$ by (4) so that $s(\mathscr{M}^{n-1}) \subset \mathscr{M}^{n-1}$ by (5). Thus s^* carries $(\mathscr{M}^{n+1})^\perp$ into itself.

3. If H_1 and H_2 are Hopf algebras with antipodes s_1, s_2, then the bialgebra $H_1 \otimes H_2$ is a Hopf algebra where $s(a \otimes b) = s_1(a) \otimes s_2(b)$.

Examples

1.6.1. If $k[G]$ is the semi-group algebra of the semi-group G, then the semi-group morphisms $G \to G \times G$ and $G \to \{e\}$ which map $g \mapsto (g, g)$ and $g \mapsto e$ respectively induce algebra morphisms

$$\Delta : k[G] \to k[G \times G] = k[G] \otimes k[G] \qquad \text{and} \qquad \epsilon : k[G] \to k[\{e\}] = k.$$

$k[G]$ is a bialgebra where for $g \in G$, $\Delta g = g \otimes g$ and $\epsilon(g) = 1$ and the group-like elements of $k[G]$ are precisely those of G. It is a Hopf algebra (with antipode $s(g) = g^{-1}$ for $g \in G$) if and only if G is a group.

1.6.2. If L is a (restricted) Lie algebra with (restricted) universal enveloping algebra $U(L)$ then the Lie algebra morphisms $L \to L \times L$, and $L \to \{0\}$ (which map $x \mapsto (x, x)$ and $x \mapsto 0$ respectively) induce algebra morphisms $\Delta : U(L) \to U(L \times L) = U(L) \otimes U(L)$ and $\epsilon : U(L) \to U(\{0\}) = k$ where for $x \in L$, $\Delta(x) = 1 \otimes x + x \otimes 1$, and $\epsilon(x) = 0$. $U(L)$ is then a Hopf algebra with antipode given by $s(x) = -x$ for $x \in L$. When k has characteristic zero or $U(L)$ is the restricted universal enveloping algebra of a restricted Lie algebra the primitive elements of $U(L)$ are precisely the elements of L.

1.6.3. Let $A(G)$ be the coordinate ring of an affine algebraic group G. The (regular) maps $G \times G \to G$ (multiplication) and $\{e\} \to G$ (unit) induce algebra morphisms

$$\Delta : A(G) \to A(G \times G) = A(G) \otimes A(G) \qquad \text{and} \qquad A(G) \to A(\{e\}) = k.$$

$A(G)$ is a Hopf algebra with antipode induced by the regular map $G \to G$ which carries $x \mapsto x^{-1}$.

1.6.4. Let H be a Hopf algebra (not necessarily commutative!) whose underlying algebra structure is finitely generated and let $G_H = \mathrm{Alg}(H, k)$. We have seen (§1.5) that G_H is a multiplicative group with unit ϵ. By Corollary 1.4.3 there is a finite dimensional subcoalgebra C of H which contains a set of generators of H and from §1.3 C is a (left) H^*-submodule of H. Since G_H is contained in H^* we may consider C as a G_H module. This representation of G_H on C is faithful (and finite dimensional) and the image of G_H in End C is an affine algebraic group: in fact a closed subgroup of the algebraic group Aut C.

1.7. Let H be a bialgebra and let V, W be (left) H-modules. Then $V \otimes W$ is an $H \otimes H$ module; by pull back along the coproduct $\Delta : H \to H \otimes H$, $V \otimes W$ then becomes an H module where $h \cdot (v \otimes w) = \sum_{(h)} (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w)$. Also k is a (trivial) H-module by pull-back along $\epsilon : H \to k$.

DEFINITION. Let (B, p, η) be an algebra which is also an H-module. B is called an *H-module algebra* if $p : B \otimes B \to B$ and $\eta : k \to B$ are both H-module morphisms: that is

$$h \cdot (bb') = \sum_{(h)} (h_{(1)} \cdot b)(h_{(2)} \cdot b') \qquad \text{and} \qquad h \cdot 1 = \epsilon(h)\, 1$$

for all $h \in H$, $b, b' \in B$.

Remark. If B is an H-module algebra and $h \in H$ then the map $b \mapsto h \cdot b$ from B into B is an algebra morphism (respectively an algebra *derivation*) if $h \in G(H)$ (respectively $h \in P(H)$).

1.7.1. *Example.* If H is a bialgebra and $B = H^*$ then we have seen in 1.3 that H^* is a left H-module where $h \cdot f = L(h)f$. Under this action H^* is an H-module algebra. For if $h \in H$ it is easy to check that

$$L(\Delta h)(f \otimes g) = \sum_{(h)} L(h_{(1)})f \otimes L(h_{(2)})\, g \qquad \text{where} \qquad f, g \in H^*$$

Then if $k \in H$ we have:

$$\begin{aligned} \langle L(h)(fg), k\rangle &= \langle fg, kh\rangle = \langle f \otimes g, \Delta k \cdot \Delta h\rangle = \langle L(\Delta h)(f \otimes g), \Delta k\rangle \\ &= \sum_{(h)} \langle L(h_{(1)})f \otimes L(h_{(2)})\, g, \Delta k\rangle \\ &= \Big\langle \sum_{(h)} (L(h_{(1)})f)(L(h_{(2)})\, g), k \Big\rangle \end{aligned}$$

and $\langle L(h)\epsilon, k\rangle = \langle \epsilon, kh\rangle = \epsilon(h)\langle \epsilon, k\rangle$ so that $L(h)\epsilon = \epsilon(h)\epsilon$.

If now A is a bialgebra and we let $H = A^0$, then A is a (left) H-submodule of H^* by Proposition 1.4.1 so that A is an A^0-module algebra in this way.

1.7.2. *Example.* Let H be a bialgebra, B an algebra and $\sigma : H \to B$ an algebra morphism which has an inverse $\tau : H \to B$ in the convolution algebra Hom(H, B). Then B is an $H \otimes H$ module where $(h \otimes h')(b) = \sigma(h)\, b\tau(h')$. By pull back along $\Delta : H \to H \otimes H$, B becomes an H-module denoted ${}^\sigma B$.

Explicitly the module action of H on ${}^{\sigma}B$ [called the *adjoint action* associated with $\sigma \in \mathrm{Alg}(H, B)$] is given by:

$$[\mathrm{ad}(h)](b) = \sum_{(h)} \sigma(h_{(1)})\, b\tau(h_{(2)}) \qquad \text{where} \qquad h \in H \qquad \text{and} \qquad b \in B$$

Then ${}^{\sigma}B$ is an H-module algebra. Thus if $h \in H$, $b, b' \in B$ we have:

$$\begin{aligned}\sum_{(h)} [\mathrm{ad}\, h_{(1)}(b)][\mathrm{ad}\, h_{(2)}(b')] &= \sum_{(h)} \sigma(h_{(1)})\, b\tau(h_{(2)})\, \sigma(h_{(3)})\, b'\tau(h_{(4)}) \\ &= \sum_{(h)} \epsilon(h_{(2)})\, \sigma(h_{(1)})\, bb'\tau(h_{(3)}) \\ &= \sum_{(h)} \sigma(h_{(1)})\, bb'\tau(h_{(2)}) = [ad\ h](bb')\end{aligned}$$

and $[\mathrm{ad}\, h](1) = \sum_h \sigma(h_{(1)})\, 1\tau(h_{(2)}) = \epsilon(h) \cdot 1$.

In particular if H is a Hopf algebra, then by part (1) of proposition 1.5.2 any $\sigma \in \mathrm{Alg}(H, B)$ is invertible in the convolution algebra $\mathrm{Hom}(H, B)$ and the adjoint action of H on ${}^{\sigma}B$ is given by

$$[\mathrm{ad}\, h](b) = \sum \overline{h_{(1)}}b\, \overline{s(h_{(2)})} \qquad \text{where} \qquad \bar{h} = \sigma(h) \in B$$

Notice that if $h \in G(H)$ then $[\mathrm{ad}\, h](b) = \bar{h}b\bar{h}^{-1}$ and if $h \in P(H)$ then $[\mathrm{ad}\, h](b) = \bar{h}b - b\bar{h}$.

1.8. Let B be an H-module algebra. We now define the *semi-direct product* or *smash product* of B by H as follows:

1) $B \# H$ is an algebra with given algebra isomorphisms $i : B \to B \# H$, $j : H \to B \# H$.

2) If ${}^{j}B \# H$ denotes the H-module algebra structure as in example 1.7.2 then $B \xrightarrow{i} {}^{j}B \# H$ is an H-module (algebra) morphism,

(i.e. $i \in \mathrm{Alg}_H\, (B, {}^{j}B \# H)$).

3) If A is any algebra with maps $g \in \mathrm{Alg}(H, A)$, $f \in \mathrm{Alg}_H\, (B, {}^{g}A)$ then there is a unique algebra morphism $M : B \# H \to A$ such that the following diagram is commutative.

$$\begin{tikzcd} & B \# H \arrow[dd, "M"] & \\ B \arrow[ur, "i"] \arrow[dr, "f"'] & & H \arrow[ul, "j"'] \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

We may realize $B \# H$ by putting an algebra structure on $B \otimes H$ as follows: (we write $b \# h$ for $b \otimes h$)

$$(b \# h)(b' \# h') = \sum_{(h)} h(h_{(1)} \cdot b') \otimes h_{(2)}h$$

where $b, b' \in B$ $h, h' \in H$.

$$i : B \to B \# H, \qquad b \mapsto b \# 1.$$
$$j : H \to B \# H, \qquad h \mapsto 1 \# h.$$

Routine verification shows that the algebra structure is associative with unit $1 \# 1$, i and j are algebra morphisms and i is an H-module algebra morphism. Given A, f and g as in (3) we define $M : B \# H \to A$, $b \# h \mapsto f(b)\, g(h) \in A$. Another routine check shows that the condition in (3) is satisfied with regard to M.

Of course $(B \# H, i, j)$ is unique up to isomorphism.

If $\pi : H \to \operatorname{End} B$ is the representation of H inducing the given H-module algebra structure we shall sometimes write $B \#_\pi H$ for $B \# H$.

In Example 1.7.1 we may form $B \#_L H$. It is then easy to check that the map $b \# h \mapsto \tilde{b}L(h)$ is an algebra morphism from $B \#_L H \to \operatorname{End} B$. ($\tilde{b} : B \to B$, $c \mapsto bc$.)

In Example 1.7.2 we denote the smash product by $B \#_\sigma H$. In this case the map $b \# h \mapsto b\sigma(h) = b\bar{h}$ is an algebra morphism from $B \#_\sigma H \to B$. If B_0 is a subalgebra of B such that $\operatorname{ad} h(B_0) \subset B_0$ for all $h \in H$, then B_0 is an H-module subalgebra of B, and we may form $B_0 \# H$. Since the imbedding $B_0 \# H \to B \#_\sigma H$ is an algebra morphism, the composition map $B_0 \# H \to B \#_\sigma H \to B$ is an algebra morphism.

Examples

1.8.1. If F is a Galois extension of a field k and G is the Galois group of F over k then F becomes a $k[G]$ module-algebra. The algebra $F \# k[G]$ is simply the usual cross-product $F \# G$ (with trivial cocycle); hence, $F \# k[G]$ is isomorphic to $\operatorname{End} F$.

1.8.2. If σ is a homomorphism from a group G into the group of automorphisms of another group G' then σ extends in a natural way to a homomorphism from G into the group of automorphisms $\operatorname{Aut} k[G']$ of the group algebra $k[G']$. This map in turn lifts uniquely to an algebra morphism from $k[G] \to \operatorname{End} k[G']$. The action λ of $k[G]$ on $k[G']$ so defined makes $k[G']$ into a $k[G]$ module algebra. One can show that

$$k[G'] \#_\lambda k[G] \cong k[G' \times_\sigma G]$$

where $G' \times_\sigma G$ is the 'semi-direct product' group.

1.8.3. If σ is a homomorphism from a Lie algebra L into the Lie algebra of derivations of another Lie algebra L', then σ extends in a natural way to a homomorphism from L into the Lie algebra of derivations of the universal enveloping algebra $\mathscr{U}(L')$. This map in turn lifts uniquely to an algebra morphism from $\mathscr{U}(L)$ into End $\mathscr{U}(L')$. The action λ of $\mathscr{U}(L)$ on $\mathscr{U}(L')$ so defined makes $\mathscr{U}(L')$ into a $\mathscr{U}(L)$-module algebra. One can show that

$$\mathscr{U}(L') \#_\lambda \mathscr{U}(L) \cong \mathscr{U}(L' \times_\sigma L)$$

where $L' \times_\sigma L$ is the 'semi-product' Lie algebra.

2. Differential Operators

2.1. Let A be a commutative algebra over k and let M and N be A-modules. Let $\mathrm{Hom}_A(M, N)$ consist of those $u \in \mathrm{Hom}(M, N)$ such that $u(am) = au(m)$ for all $a \in A$, $m \in M$. For $u \in \mathrm{Hom}(M, N)$ and $a \in A$ we define $[a, u] \in \mathrm{Hom}(M, N)$ by:

$$[a, u](m) = a \cdot u(m) - u(am) \qquad \text{where} \qquad m \in M.$$

Definition. We filter $\mathrm{Hom}(M, N)$ by increasing subspaces $\{\mathscr{D}_A{}^n(M, N)\}$ where $\mathscr{D}_A{}^n(M, N)$ is defined inductively by:

$$\mathscr{D}_A{}^0(M, N) = \{u \in \mathrm{Hom}(M, N) \mid [a, u] = 0 \ \text{ for all } \ a \in A\} = \mathrm{Hom}_A(M, N)$$

$$\mathscr{D}_A^{n+1}(M, N) = \{u \in \mathrm{Hom}(M, N) \mid [a, u] \in \mathscr{D}_A{}^n(M, N) \qquad \text{for all} \qquad a \in A\}$$

Observe in particular that if $a \in A$ then $\tilde{a} \in \mathscr{D}_A{}^0(M, M)$ where $\tilde{a}(m) = am$.

2.1.1. Lemma. (a) *Let M, N, P be A-modules. Then the composition map* $\mathrm{Hom}(M, N) \times \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P)$ *which sends* $(u, v) \mapsto v \circ u$ *carries* $\mathscr{D}_A{}^r(M, N) \times \mathscr{D}_A{}^s(N, P)$ *into* $\mathscr{D}_A^{r+s}(M, P)$. *If* $u \in \mathrm{Hom}_A(M, N)$ *and* $v \in \mathrm{Hom}_A(N, P)$ *then*

$$v \circ \mathscr{D}_A{}^n(M, N) \subset \mathscr{D}_A{}^n(M, P) \qquad \textit{and} \qquad \mathscr{D}_A{}^n(N, P) \circ u \subset \mathscr{D}_A{}^n(M, P).$$

(b) *Under composition of endomorphisms,* $\{\mathscr{D}_A{}^n(M, M)\}$ *is a filtered subalgebra of* End M. *If* $u \in \mathscr{D}_A{}^r(M, M)$ *and* $v \in \mathscr{D}_A{}^s(M, M)$ *then* $[u, v] = u \circ v - v \circ u \in \mathscr{D}_A^{r+s-1}(M, M)$ *so that* $u \circ v \equiv v \circ u \bmod \mathscr{D}_A^{r+s-1}(M, M)$; *thus the associated graded algebra is commutative.*

(c) *Let A and B be commutative k-algebras and let M, N be A-modules and M', N' be B modules (so that $M \otimes M'$ and $N \otimes N'$ are $A \otimes B$ modules). Then the natural injection*

$$\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(M', N') \to \mathrm{Hom}(M \otimes M', N \otimes N')$$

carries

$$\mathscr{D}_A{}^r(M, N) \otimes \mathscr{D}_B{}^s(M', N') \qquad \textit{into} \qquad \mathscr{D}^{r+s}_{A\otimes B}(M \otimes M', N \otimes N')$$

Proof. (a) If $a \in A$, $u \in \mathrm{Hom}(M, N)$, $v \in \mathrm{Hom}(N, P)$, the first part of the lemma follows by induction on $r + s$ using the identity:

$$[a, v \circ u] = [a, v] \circ u + v \circ [a, u] \in \mathrm{Hom}(M, P).$$

The second part of (a) then follows from the first because

$$\mathrm{Hom}_A(M, N) = \mathscr{D}_A{}^0(M, N).$$

(b) The first part of (b) follows from (a) by taking $M = N = P$. The second part follows by induction on $r + s$ using the identity

$$[a, [u, v]] = [[a, u], v] + [u, [a, v]].$$

(c) Let $u \in \mathrm{Hom}(M, N)$ and $v \in \mathrm{Hom}(M', N')$. The result stated in (c) follows by induction on $r + s$ using the identity:

$$[a \otimes b, u \otimes v] = [a, u] \otimes (\tilde{b} \circ v) + (u \circ \tilde{a}) \otimes [b, v]$$

where $a \in A$, $b \in B$.

Here $\tilde{a} \in \mathscr{D}_A{}^0(M, M)$ sends $m \mapsto am$
and $\tilde{b} \in \mathscr{D}_A{}^0(N', N')$ sends $n' \mapsto bn'$.

Observe that $u \circ \tilde{a} \in \mathscr{D}_A{}^r(M, N)$ if $u \in \mathscr{D}_A{}^r(M, N)$ using the result already established in (a). Q.E.D.

2.2. If M is a vector space over k then $A \otimes M$ has an A-module structure defined by $a \cdot (a' \otimes m) = aa' \otimes m$.

Let $r_M : M \to A \otimes M$ be the linear map $m \mapsto 1 \otimes m$. It is well known that the pair $(A \otimes M, r_M)$ has the following universal property:

$[U]$: For any A-module N and linear map $u \in \mathrm{Hom}(M, N)$ there is a unique $f \in \mathrm{Hom}_A(A \otimes M, N)$ such that $u = f \circ r_M$. Thus the map $f \mapsto f \circ r_M$ from $\mathrm{Hom}_A(A \otimes M, N)$ into $\mathrm{Hom}(M, N)$ is bijective. The inverse map

$$\Phi : \mathrm{Hom}(M, N) \to \mathrm{Hom}_A(A \otimes M, N)$$

carries $u \in \mathrm{Hom}(M, N)$ into $\Phi(u) \in \mathrm{Hom}_A(A \otimes M, N)$ where

$$\Phi(u)(a \otimes m) = au(m).$$

Define algebra morphisms

$$l : A \to A \otimes A \quad \text{by} \quad l(a) = a \otimes 1$$

and

$$r : A \to A \otimes A \quad \text{by} \quad r(a) = 1 \otimes a$$

When we speak of the A-module structure associated with any $A \otimes A$ module we shall always understand the A-module structure induced by pull-back along l (rather than r). In particular if M is any A-module then $A \otimes M$ is naturally an $A \otimes A$ module and the A-module structure induced on $A \otimes M$ in this way is the module structure associated with the vector space M above. Taking $M = A$, $A \otimes A$ becomes an A-module where $a \in A$ acts on the left hand member of $A \otimes A$. Then $l : A \to A \otimes A$ is a morphism of A-modules.

If M and N are A-modules, then $\mathrm{Hom}(M, N)$ and $\mathrm{Hom}(A \otimes M, N)$ become $A \otimes A$ modules (and therefore A-modules) if we define:

$$(a \otimes b)u : m \mapsto au(bm) \quad \text{where} \quad a, b \in A, \quad u \in \mathrm{Hom}(M, N) \quad \text{and} \quad m \in M.$$

$$(a \otimes b)f : a' \otimes m \mapsto f(aa' \otimes bm) = f((a \otimes b) \cdot (a' \otimes m)) \text{ where}$$

$$f \in \mathrm{Hom}(A \otimes M, N).$$

2.2.1. LEMMA. (a) *If* $u \in \mathrm{Hom}(M, N)$ *and* $a \in A$ *then*

$$[a, u] = (a \otimes 1 - 1 \otimes a) \cdot u.$$

(b) $\mathscr{D}_A{}^n(M, N)$ *is an* $A \otimes A$*-submodule* (*and so an* A*-submodule*) *of* $\mathrm{Hom}(M, N)$.

(c) $\mathrm{Hom}_A(A \otimes M, N)$ *is an* $A \otimes A$ *submodule of* $\mathrm{Hom}(A \otimes M, N)$ *and the map* $\Phi : \mathrm{Hom}(M, N) \to \mathrm{Hom}_A(A \otimes M, N)$ *described in* [U] *above is an isomorphism of* $A \otimes A$ *modules.*

Proof. (a) and (c) follow immediately from the definitions. (b) follows from 2.1.1 (b) and the observation that $(a \otimes b) \cdot u = \tilde{a} \circ u \circ \tilde{b}$ where $\tilde{b} \in \mathscr{D}_A{}^0(M, M)$ and $\tilde{a} \in \mathscr{D}_A{}^0(N, N)$.

Let $p : A \otimes A \to A$ be the algebra structure map and define $I = \text{kernel } p$. Then p is both an algebra morphism and an A-module morphism so that I is both an ideal and an A-submodule of the algebra $A \otimes A$. Moreover $l : A \to A \otimes A$ is also a morphism of algebras as well as A-modules and $p \circ l : A \to A$ is the identity map. Thus the sequence

$$0 \to I \to A \otimes A \underset{l}{\overset{p}{\rightleftarrows}} A \to 0$$

is split exact sequence of A-modules and in particular we obtain the A-module decomposition $A \otimes A = I \oplus l(A)$.

Define $q = 1 - l \circ p : A \otimes A \to I$ ($1 : A \otimes A \to A \otimes A$ is the identity) so that q is the projection of $A \otimes A$ onto I corresponding to the section l; explicitly $q(a \otimes a') = a \otimes a' - aa' \otimes 1$. Of course $q \in \mathrm{Hom}_A(A \otimes A, I)$.

Define $\delta = q \circ r$ as the composition map $A \to A \otimes A \to I$ so that $\delta a = 1 \otimes a - a \otimes 1 \in I$. Clearly $\delta = r - l : A \to I$.

If K is any subset of the set $S_n = \{0, 1, 2,..., n\}$, let K' be the complement of K in S_n and let $|K|$ denote the cardinality of K. Then if $a_0, ..., a_n$ is a sequence of $n + 1$ elements of A, set $a_K = \prod_{k \in K} a_k$ and let $a_\phi = 1$.

2.2.2. Lemma. (a) *I is the A-submodule (and hence a fortiori the ideal) of $A \otimes A$ generated by $\delta a = 1 \otimes a - a \otimes 1$ where $a \in A$.*

(b) *I^{n+1} is the A-submodule of $A \otimes A$ generated by $(\delta a_0) \cdots (\delta a_n) = \sum_K (-)^{|K|} a_K \otimes a_{K'}$ where $a_0, ..., a_n \in A$ and $K \subset S_n$.*

Proof. Let $\omega = \sum a_i \otimes a_i' \in I$ where $a_i, a_i' \in A$. Then $0 = p(\omega) = \sum a_i a_i'$ so that

$$\omega = \sum (a_i \otimes 1)[1 \otimes a_i' - a_i' \otimes 1] = \sum a_i \cdot (1 \otimes a_i' - a_i' \otimes 1) = \sum a_i \delta(a_i').$$

This proves (a). The second statement (b) follows immediately by induction.

2.2.3. Proposition. *Let M and N be A-modules and let $u \in \mathrm{Hom}(M, N)$.*

(a) *The following conditions are equivalent:*

(1) $u \in \mathscr{D}_A^n(M, N)$

(2) *$I \cdot u$ is contained in $\mathscr{D}_A^{n-1}(M, N)$*

(3) $I^{n+1} \cdot u = 0$

(4) $\Phi(u)[I^{n+1} \cdot (A \otimes M)] = 0$

(b) *$u \in \mathscr{D}_A^n(A, M)$ if and only if $\Phi(u)(I^{n+1}) = 0$.*

Proof. We know $u \in \mathscr{D}_A^n(M, N)$ if and only if

$$\delta(a) \cdot u = (1 \otimes a - a \otimes 1) \cdot u = -[a, u] \in \mathscr{D}_A^{n-1}(M, N).$$

Since $\mathscr{D}_A^{n-1}(M, N)$ is an $A \otimes A$ submodule of $\mathrm{Hom}(M, N)$ and I is generated by elements of the form $\delta(a)$, it follows that (1) and (2) are equivalent. Since $u \in \mathscr{D}_A^0(M, N)$ if and only if $I \cdot u = 0$, (2) is equivalent to (3). Finally the equivalence of (3) and (4) follows from the isomorphism

$$\Phi : \mathrm{Hom}(M, N) \to \mathrm{Hom}_A(A \otimes M, N)$$

of $A \otimes A$ modules. Part (b) then follows from the equivalence of (1) and (4).

2.2.4. *Remark.* From 2.2.2 (b) it follows that $u \in \mathscr{D}_A^n(A, M)$ if and only if

$\Phi(u)[\delta(a_0) \cdots \delta(a_n)] = \sum_{K \subset S_n} (-)^{|K|} a_K \cdot u(a_{K'}) = 0$ for every sequence $a_0, ..., a_n$ of $n + 1$ elements of A. For $n = 1$ this reduces to

$$u(ab) = au(b) + bu(a) - abu(1) \qquad \text{for all} \quad a, b \in A.$$

Define $J_n(M) = (A \otimes M)/(I^{n+1} \cdot (A \otimes M))$ so that $J_n(A) = (A \otimes A)/I^{n+1}$

$$\begin{aligned} \Omega_n(A) &= J_n(A) \otimes_{A \otimes A} I = I/I^{n+1} \\ \mathrm{Der}^n(A, M) &= \{u \in \mathscr{D}_A{}^n(A, M) \mid u(1) = 0\} \qquad \text{and} \\ \mathrm{Der}(A, M) &= \mathrm{Der}^1(A, M) \end{aligned}$$

Let $\prod_n : A \otimes M \to J_n(M)$ and $\prod'_n : I \to \Omega_n(A)$ be the canonical epimorphisms and let $j_n : M \to J_n(M)$ and $d_n : A \to \Omega_n(A)$ be the composition maps:

$$j_n = \textstyle\prod_n \circ r_M : M \to A \otimes M \to J_n(M)$$

$$d_n = \textstyle\prod'_n \circ \delta : A \to I \to \Omega_n(A)$$

Then $J_n(M)$ and $\Omega_n(A)$ are $A \otimes A$ modules and therefore A-modules. Moreover the A-module decomposition $A \otimes A = I \oplus l(A)$ induces by passage to quotients an A-module decomposition

$$J_n(A) = (A \otimes A)/I^{n+1} = \Omega_n(A) \oplus l(A)$$

and we obtain a commutative diagram:

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\Pi'_n"'] & A \otimes A \arrow[r, "q"] \arrow[d, "\Pi_n"] & I \arrow[d, "\Pi'_n"] \\ \Omega_n(A) \arrow[r] & J_n(A) \arrow[r, "q_n"] & \Omega_n(A) \end{tikzcd}$$

where the horizontal maps on the left are the natural inclusions and q_n is the map induced by passage to quotients from q. The composition of the upper (lower) horizontal maps is the identity.

2.2.5. *Remarks.* (a) $\prod_n \in \mathrm{Hom}_A(A \otimes M, J_n(M))$ and $j_n = \prod_n \circ r_M$ so that $\Phi(j_n) = \prod_n$ from the (universal) defining property of Φ. Also $\prod'_n \in \mathrm{Hom}_A(I, \Omega_n(A))$ and $q \in \mathrm{Hom}_A(A \otimes A, I)$. Since $d_n = \prod'_n \circ \delta = \prod'_n \circ q \circ r$ it follows similarly that $\Phi(d_n) = \prod'_n \circ q : A \otimes A \to \Omega_n(A)$.

(b) From 2.2.4 it follows that $\mathrm{Der}^0(A, M) = 0$ and $u \in \mathrm{Der}(A, M)$ if and only if $u(ab) = au(b) + bu(a)$ for all $a, b \in A$.

(c) If $u \in \mathrm{Hom}(A, M)$ then $\Phi(u) \in \mathrm{Hom}_A(A \otimes A, M)$; thus we have $\Phi(u)(l(A)) = \Phi(u)(A \otimes 1) = A \cdot u(1)$. By 2.2.3 (b) it follows that $u \in \mathrm{Der}^n(A, M)$ if and only if $\Phi(u)(I^{n+1}) = 0 = \Phi(u)(l(A))$.

2.2.6. THEOREM. (a) $j_n \in \mathscr{D}_A{}^n(M, J_n(M))$. *Moreover* j_n *is universal: if* $u \in \mathscr{D}_A{}^n(M, N)$ *then there is a unique* $\alpha \in \mathrm{Hom}_A(J_n(M), N)$ *such that* $\alpha \circ j_n = u$. *Thus the map* $\mathrm{Hom}_A(J_n(M), N) \to \mathscr{D}_A{}^n(M, N)$ *which carries* $\alpha \mapsto \alpha \circ j_n$ *is bijective.*

(b) $d_n \in \mathrm{Der}^n(A, \Omega_n(A))$. *Moreover* d_n *is universal: if* $D \in \mathrm{Der}^n(A, M)$ *there is a unique* $\beta \in \mathrm{Hom}_A(\Omega_n(A), M)$ *such that* $D = \beta \circ d_n$. *Thus the map* $\mathrm{Hom}_A(\Omega_n(A), M) \to \mathrm{Der}^n(A, M)$ *which carries* $\beta \mapsto \beta \circ d_n$ *is bijective.*

Proof. (a) $\Phi(j_n) = \prod_n$ by 2.2.5 (a). Since $\prod_n : A \otimes M \to J_n(M)$ vanishes on $I^{n+1} \cdot (A \otimes M)$ it follows by the criterion (4) in 2.2.3 (a) that $j_n \in \mathscr{D}_A{}^n(M, J_n(M))$. Now let $u \in \mathscr{D}_A{}^n(M, N)$. By the universal property [U] there is a unique map $\Phi(u) \in \mathrm{Hom}_A(A \otimes M, N)$ such that $\Phi(u) \circ r_M = u$. By 2.2.3 (a) it follows that $\Phi(u)$ vanishes on $I^{n+1} \cdot (A \otimes M)$ so that $\Phi(u) = \alpha \circ \prod_n$ for a unique $\alpha \in \mathrm{Hom}_A(J_n(M), N)$. Thus $u = \Phi(u) \circ r_M = \alpha \circ \prod_n \circ r_M = \alpha \circ j_n$. We may summarize this argument in the following commutative diagram:

$$\begin{tikzcd}
 & & J_n(M) \arrow[dd, "\alpha"] \\
M \arrow[urr, "j_n"] \arrow[r] \arrow[drr, "u"'] & A \otimes M \arrow[ur, "\Pi_n"] \arrow[dr, "\Phi(u)"] & \\
 & & N
\end{tikzcd}$$

(b) By 2.2.5 (a) we have $\Phi(d_n) = \prod_n' \circ q : A \otimes A \to \Omega_n(A)$. Since q is the identity on I and $\prod_n'$ vanishes on I^{n+1} it follows that $\Phi(d_n)(I^{n+1}) = 0$. Also $\delta(1) = 1 \otimes 1 - 1 \otimes 1 = 0$ so that $d_n(1) = 0$ and hence $d_n \in \mathrm{Der}^n(A, M)$ by 2.2.3 (b). The fact that any $D \in \mathrm{Der}^n(A, M)$ can be uniquely expressed in the form $D = \beta \circ d_n$ follows from an argument similar to that given in part (a) and the commutativity of the diagram.

$$\begin{tikzcd}
 & & \Omega_n(A) \arrow[dd, "\beta"] \\
A \arrow[urr, "d_n"] \arrow[r, "r"] \arrow[drr, "D"'] & A \otimes A \arrow[ur] \arrow[dr, "\Phi(D)"] & \\
 & & M
\end{tikzcd}$$

where the unlabeled map $A \otimes A \to \Omega_n(A)$ is the composition

$$\prod_n{}' \circ q : A \otimes A \to I \to \Omega_n(A). \qquad \text{Q.E.D.}$$

2.3. Let A be an *augmented* commutative algebra with augmentation map $\epsilon \in \mathrm{Alg}(A, k)$ and let $\mathscr{M}$ denote the kernel of ϵ. If V is a vector space over k then V is a (trivial) A-module by pull back along ϵ. We shall find it convenient to write

$$\begin{array}{llll} & \mathscr{D}_\epsilon{}^n(A, V) & \text{instead of} & \mathscr{D}_A{}^n(A, V) \\ \text{and} & \mathscr{D}^n(A) & \text{instead of} & \mathscr{D}_\epsilon{}^n(A, k). \end{array}$$

2.3.1. PROPOSITION. (a) *If* $u \in \mathrm{Hom}(A, V)$ *then* $u \in \mathscr{D}_\epsilon{}^n(A, V)$ *if and only if* $u(\mathscr{M}^{n+1}) = 0$. *If* $\prod_n : A \to A/\mathscr{M}^{n+1}$ *is the canonical epimorphism then* $\phi \mapsto \phi \circ \prod_n$ *is a bijection from* $\mathrm{Hom}(A/\mathscr{M}^{n+1}, V)$ *onto* $\mathscr{D}_\epsilon{}^n(A, V)$.

(b) $\mathscr{D}^n(A) = (\mathscr{M}^{n+1})^\perp$ *and the transpose map* $\prod_n^* : (A/\mathscr{M}^{n+1})^* \to A^*$ *induces an isomorphism from* $(A/\mathscr{M}^{n+1})^*$ *onto* $\mathscr{D}^n(A)$.

(c) *If* $\mathscr{M}$ *is a finitely generated ideal of* A *then* $\mathscr{D}^n(A)$ *is a finite dimensional subcoalgebra of* A^0 *and the natural map* $V \otimes \mathscr{D}^n(A) \to \mathscr{D}_\epsilon{}^n(A, V)$ *is bijective. Explicitly* $v \otimes f \in V \otimes \mathscr{D}^n(A)$ *then* $v \otimes f$ *corresponds to the map in* $\mathscr{D}^n(A, V)$ *which carries* $a \mapsto \langle f, a\rangle v$.

Proof. (a) If $u \in \mathrm{Hom}(A, V)$ and $a \in A$ then $[a, u] \in \mathrm{Hom}(A, V)$ is the map which sends $b \mapsto u(\epsilon(a)b - ab)$. Since $\mathscr{M}$ is generated by elements of the form $\epsilon(a)1 - a$ we conclude

$$[a, u](\mathscr{M}^n) = 0 \qquad \text{for all} \quad a \in A \quad \text{if and only if} \quad u(\mathscr{M}^{n+1}) = 0.$$

Thus by induction $u \in \mathscr{D}_\epsilon{}^n(A, V)$ if and only if $u(\mathscr{M}^{n+1}) = 0$. This proves (a). Part (b) follows from (a) by choosing $V = k$.

If $\mathscr{M}$ is finitely generated, then $\mathscr{M}^{n+1}$ is cofinite and $\mathscr{D}_n(A) = (\mathscr{M}^{n+1})^\perp$ is a finite dimensional subcoalgebra of A^0 by 1.3.9. Using the fact that the natural injection $V \otimes W^* \to \mathrm{Hom}(V, W)$ is bijective when W is finite dimensional we conclude (when $\mathscr{M}$ is finitely generated) that the map $V \otimes (A/\mathscr{M}^{n+1})^* \to \mathrm{Hom}(A/\mathscr{M}^{n+1}, V)$ is bijective. Thus each of the individual maps in the chain

$$V \otimes \mathscr{D}^n(A) \to V \otimes (A/\mathscr{M}^{n+1})^* \to \mathrm{Hom}(A/\mathscr{M}^{n+1}, V) \to \mathscr{D}_\epsilon{}^n(A, V)$$

is bijective and therefore the composite map from $V \otimes \mathscr{D}^n(A) \to \mathscr{D}_\epsilon{}^n(A, V)$ is bijective. A routine check shows that the map has the stated explicit form. Q.E.D.

If $\epsilon \in \mathrm{Alg}(A, k)$, we have agreed above to write $\mathscr{D}^n(A)$ or $\mathscr{D}_\epsilon{}^n(A, k)$ instead of $\mathscr{D}_A{}^n(A, k)$. More generally, if A and B are commutative k-algebras and $\sigma \in \mathrm{Alg}(A, B)$, then B is an A-module by pull back along σ. We shall then write:

$$\begin{array}{lrll} & \mathscr{D}_\sigma{}^n(A, B) & \text{instead of} & \mathscr{D}_A{}^n(A, B) \\ & \mathrm{Der}_\sigma{}^n(A, B) & \text{instead of} & \mathrm{Der}^n(A, B) \\ \text{and} & \mathrm{Diff}_n(A) & \text{instead of} & \mathscr{D}_I{}^n(A, A) \end{array}$$

where $I : A \to A$ is the identity map. Note that $\sigma \in \mathscr{D}_\sigma{}^0(A, B)$. Also by 2.3.1 (b) $\mathrm{Der}_\epsilon{}^1(A, k) = (\mathscr{M}^2)^\perp \cap (1)^\perp$.

From 2.1.1. we have:

2.3.2. Lemma. (a) $\mathrm{Diff}\, A = \{\mathrm{Diff}_n\, A\}$ *is a filtered algebra under composition*; $\mathrm{Diff}_1\, A$ *is a Lie algebra where* $[u, v] = u \circ v - v \circ u$.

(b) *If* $\sigma \in \mathrm{Alg}(A, B)$ *and* $\tau \in \mathrm{Alg}(B, C)$ *where* A, B *and* C *are commutative algebras then*

$$\mathscr{D}_\tau{}^n(B, C) \circ \sigma \subset \mathscr{D}^n_{\tau\circ\sigma}(A, C) \qquad \text{and} \qquad \tau \circ \mathscr{D}_\sigma{}^n(A, B) \subset \mathscr{D}^n_{\tau\circ\sigma}(A, C)$$

(c) *If* $\sigma, \tau \in \mathrm{Alg}(A, B)$ *then* $\sigma \otimes \tau \in \mathrm{Alg}(A \otimes A, B \otimes B)$ *and*

$$\mathscr{D}_\sigma{}^r(A, B) \otimes \mathscr{D}_\tau{}^s(A, B) \subset \mathscr{D}^{r+s}_{\sigma\otimes\tau}(A \otimes A, B \otimes B).$$

2.4. Throughout the remainder of this section $\mathscr{A}$ will denote a commutative bialgebra and B a commutative algebra. We have seen in (1.5) that $\mathrm{Hom}(\mathscr{A}, B)$ is an algebra under convolution.

2.4.1. Lemma. *If* $\sigma, \tau \in \mathrm{Alg}(\mathscr{A}, B)$ *then* $\sigma * \tau \in \mathrm{Alg}(\mathscr{A}, B)$ *and the convolution map* $\mathrm{Hom}(\mathscr{A}, B) \times \mathrm{Hom}(\mathscr{A}, B) \to \mathrm{Hom}(\mathscr{A}, B)$ *carries*

$$\mathscr{D}_\sigma{}^n(\mathscr{A}, B) \otimes \mathscr{D}_\tau{}^m(\mathscr{A}, B) \quad \textit{into} \quad \mathscr{D}^{n+m}_{\sigma*\tau}(\mathscr{A}, B).$$

Proof. As B is commutative, the algebra structure map $p_B : B \otimes B \to B$ is an algebra morphism; thus each of the individual maps in the chain

$$\sigma * \tau = p_B \circ (\sigma \otimes \tau) \circ \Delta_{\mathscr{A}} : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A} \to B \otimes B \to B$$

is an algebra morphism so that $\sigma * \tau \in \mathrm{Alg}(\mathscr{A}, B)$. If $u \in \mathscr{D}_\tau{}^n(\mathscr{A}, B)$ and $v \in \mathscr{D}_\tau{}^m(\mathscr{A}, B)$ then by part (c) of 2.3.2 we have

$$u \otimes v \in \mathscr{D}^{n+m}_{\sigma\otimes\tau}(\mathscr{A} \otimes \mathscr{A}, B \otimes B).$$

By 2.3.2 (b) we then have:

$$u * v = p \circ (u \otimes v) \circ \Delta \in \mathscr{D}^{n+m}_{p\circ(\sigma\otimes\tau)\circ\Delta}(\mathscr{A}, B) = \mathscr{D}^{n+m}_{\sigma*\tau}(\mathscr{A}, B)$$

2.4.2. Corollary. *$\mathscr{D}(\mathscr{A}) = \{\mathscr{D}^n(\mathscr{A})\}$ is a filtered subalgebra of $\mathscr{A}^*$. If the augmentation ideal $\mathscr{M}$ of $\mathscr{A}$ is finitely generated then $\mathscr{D}(\mathscr{A})$ is a sub-bialgebra of $\mathscr{A}^0$. If in addition $\mathscr{A}$ is a Hopf algebra then $\mathscr{D}(\mathscr{A})$ is a sub-Hopf algebra of $\mathscr{A}^0$.*

Proof. Since $\epsilon * \epsilon = \epsilon$ this follows from the lemma above and 1.3.9 together with Remark 2 of Section 1.5. Q.E.D.

If $\partial \in \mathscr{A}^*$, we have seen 1.4.1 that $R(\partial)$ and $L(\partial)$ define operators in End $\mathscr{A}$. We define

$$\mathrm{End}^R \mathscr{A} = \{u \in \mathrm{End}\, \mathscr{A} \mid u \circ R(\partial) = R(\partial) \circ u \text{ for all } \partial \in \mathscr{A}^*\}$$

and

$$\mathrm{Diff}_n{}^R \mathscr{A} = (\mathrm{Diff}_n \mathscr{A}) \cap \mathrm{End}^R (\mathscr{A})$$

Clearly $\mathrm{End}^R \mathscr{A}$ is a subalgebra of End $\mathscr{A}$ under composition; by 2.3.2 (a) $\mathrm{Diff}\, \mathscr{A} = \{\mathrm{Diff}_n(\mathscr{A})\}$ is a filtered subalgebra of End $\mathscr{A}$. Thus $\mathrm{Diff}^R \mathscr{A} = \{\mathrm{Diff}_n{}^R(\mathscr{A})\}$ is a filtered subalgebra of End $\mathscr{A}$ under composition.

2.4.3. Proposition. *Let $\mathscr{A}$ be a commutative bialgebra. Then the map $L : \partial \mapsto L(\partial)$ is an algebra isomorphism from $\mathscr{A}^*$ onto $\mathrm{End}^R \mathscr{A}$. Its restriction to $\mathscr{D}(\mathscr{A})$ gives an isomorphism of (filtered) algebras from $\mathscr{D}(\mathscr{A})$ onto $\mathrm{Diff}^R \mathscr{A}$.*

Proof. We have seen that $L(\partial_1\partial_2) = L(\partial_1) \circ L(\partial_2)$ for $\partial_1, \partial_2 \in \mathscr{A}^*$, so that $L : \mathscr{A}^* \to \mathrm{End}\, \mathscr{A}$ is an algebra morphism. The image of L is contained in $\mathrm{End}^R \mathscr{A}$ because $L(\partial_1) \circ R(\partial_2) = R(\partial_2) \circ L(\partial_1)$. Define

$$\phi : \mathrm{End}^R \mathscr{A} \to \mathscr{A}^* \quad \text{by} \quad \phi(u) = \epsilon \circ u \quad \text{where} \quad u \in \mathrm{End}^R \mathscr{A}$$

If $f \in \mathscr{A}$ and $\partial \in \mathscr{A}^*$ then

$$\langle \phi(L(\partial)), f\rangle = \langle \epsilon, L(\partial) f\rangle = \langle \partial, f\rangle$$

so that $\phi \circ L : \mathscr{A}^* \to \mathscr{A}^*$ is the identity map. Thus L is injective and ϕ is surjective.

To prove that ϕ is injective, let $u \in \mathrm{End}^R \mathscr{A}$ and assume $\phi(u) = 0$. If $f \in \mathscr{A}$ and $\partial \in \mathscr{A}^*$ then

$$0 = \langle \phi(u), R(\partial) f\rangle = \langle \epsilon, u[R(\partial) f]\rangle = \langle \epsilon, R(\partial)\, u(f)\rangle = \langle \partial, u(f)\rangle$$

Thus $u = 0$ so that ϕ is bijective. Since $\phi \circ L$ is the identity on $\mathscr{A}^*$ it follows that L is bijective. Hence L (and therefore also its inverse ϕ) is an algebra isomorphism.

Notice next that $\mathscr{D}^n(\mathscr{A}) = \mathscr{D}_\epsilon{}^n(\mathscr{A}, k)$ is naturally contained in $\mathscr{D}_\epsilon{}^n(\mathscr{A}, \mathscr{A})$. Also if $I : \mathscr{A} \to \mathscr{A}$ denotes the identity map then $I \in \mathscr{D}_I{}^0(\mathscr{A}, \mathscr{A})$. Thus if $\partial \in \mathscr{D}^n(\mathscr{A})$ we have

$$I * \partial \in \mathscr{D}_I{}^0(\mathscr{A}, \mathscr{A}) * \mathscr{D}_\epsilon{}^n(\mathscr{A}, \mathscr{A}) \subset \mathscr{D}_{I*\epsilon}^n(\mathscr{A}, \mathscr{A}) = \mathrm{Diff}_n(\mathscr{A}) \quad \text{by} \quad 2.4.1.$$

so that $I * \partial \in \mathrm{Diff}_n(\mathscr{A})$. But also if $f \in \mathscr{A}$ we have

$$L(\partial)f = \sum_{(f)} \langle \partial, f_{(2)} \rangle f_{(1)} = (I * \partial)(f)$$

so that $L(\partial) = I * \partial \in \mathrm{Diff}_n(\mathscr{A})$. As we have seen that $L(\partial) \in \mathrm{End}^R \mathscr{A}$ we conclude that $L(\partial) \in \mathrm{Diff}_n{}^R(\mathscr{A})$ when $\partial \in \mathscr{D}^n(\mathscr{A})$.

Conversely if $u \in \mathrm{Diff}_n{}^R(\mathscr{A})$ then

$$\phi(u) = \epsilon \circ u \in \epsilon \circ \mathscr{D}_I^n(\mathscr{A}, \mathscr{A}) \subset \mathscr{D}^n_{\epsilon\circ I}(\mathscr{A}, k) = \mathscr{D}^n(\mathscr{A}) \qquad \text{by 2.3.2 (b)}$$

Q.E.D.

2.4.4. COROLLARY. *The map $\partial \mapsto L(\partial)$ is a (Lie algebra) isomorphism from the Lie algebra $\mathrm{Der}_\epsilon{}^1(\mathscr{A}, k)$ onto the Lie algebra $\mathrm{Der}_I{}^R(\mathscr{A}, \mathscr{A})$ of right invariant derivations of $\mathscr{A}$. If $\mathscr{A}$ is a Hopf algebra then L induces a (group) isomorphism from the group $\mathrm{Alg}(\mathscr{A}, k)$ onto the group $\mathrm{Aut}^R \mathscr{A}$ of right invariant automorphisms of $\mathscr{A}$.*

Proof. If $\partial \in \mathscr{A}^*$ then $L(\partial)(1) = \langle \partial, 1 \rangle 1$. The first statement then follows from 2.4.3 and the definitions of $\mathrm{Der}_\epsilon{}^1(\mathscr{A}, k)$ and $\mathrm{Der}_I{}^R(\mathscr{A}, \mathscr{A})$. If $\mathscr{A}$ is a Hopf algebra then we have seen that $\mathrm{Alg}(\mathscr{A}, k)$ is a subgroup of the group of units of $\mathscr{A}^*$. As $\mathscr{A}$ is a left $\mathscr{A}^*$ module under the L-action, it is clear that $\partial \in \mathrm{Alg}(\mathscr{A}, k)$ implies $L(\partial) \in \mathrm{Aut}^R \mathscr{A}$. Conversely if $u \in \mathrm{Aut}^R \mathscr{A}$ then obviously $\phi(u) = \epsilon \circ u \in \mathrm{Alg}(\mathscr{A}, k)$. Since ϕ is the inverse of L we are done.

Q.E.D.

Recall that by 1.8 (and 2.4.2) the algebra $\mathscr{A} \#_L \mathscr{D}(\mathscr{A})$ is a subalgebra of $\mathscr{A} \#_L \mathscr{A}^0$ which acts in a natural way as endomorphisms of $\mathscr{A}$.

2.4.5. THEOREM. *Let $\mathscr{A}$ be a commutative Hopf algebra whose augmentation ideal $\mathscr{M}$ is finitely generated. Let B be a commutative algebra and $\sigma \in \mathrm{Alg}(\mathscr{A}, B)$. Then:* (a) *The map $B \otimes \mathscr{D}^n(\mathscr{A}) \to \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$ defined by $b \otimes \partial \mapsto \tilde{b} \circ \sigma \circ L(\partial)$ is an isomorphism of B-modules. [for $b \in B$ we let $\tilde{b} \in \mathrm{End}\, B$ be the map $b' \mapsto bb'$].*

(b) *The algebra morphism $H : \mathscr{A} \#_L \mathscr{D}(\mathscr{A}) \to \mathrm{Diff}(\mathscr{A})$ which carries $f \otimes \partial \to f \circ L(\partial)$ is an isomorphism of filtered algebras so that*

$$H(\mathscr{A} \#_L \mathscr{D}^n(\mathscr{A})) = \mathrm{Diff}_n \mathscr{A}.$$

Proof. (a) $\sigma \in \mathrm{Alg}(\mathscr{A}, B)$ is invertible in the convolution algebra $\mathrm{Hom}(\mathscr{A}, B)$ by 1.5.2. Let $\tau \in \mathrm{Alg}(\mathscr{A}, B)$ be the (left) inverse of σ so that $\tau * \sigma = \epsilon$. If $u \in \mathscr{D}_\epsilon{}^n(\mathscr{A}, B)$ then $\sigma * u \in \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$ by 2.4.1. Similarly if $v \in \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$ then $\tau * v \in \mathscr{D}_\epsilon{}^n(\mathscr{A}, B)$. It is then clear from the associativity of the convolution algebra that the map $u \mapsto \sigma * u$ from $\mathscr{D}_\epsilon{}^n(\mathscr{A}, B) \to \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$ is bijective. Also by 2.3.1 (c) the map $B \otimes \mathscr{D}^n(\mathscr{A}) \to \mathscr{D}_\epsilon{}^n(\mathscr{A}, B)$ which carries

$b \otimes \partial \mapsto \tilde{b} \circ \eta_B \circ \partial$ ($\eta_B : k \to B$ is the unit map of B) is bijective. Thus the composition map

$$B \otimes \mathscr{D}^n(\mathscr{A}) \to \mathscr{D}_c{}^n(\mathscr{A}, B) \to \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$$

is bijective. This map carries $b \otimes \partial \mapsto \sigma * (\tilde{b} \circ \eta \circ \partial) \in \mathscr{D}_\sigma{}^n(\mathscr{A}, B)$. If we let $v = \sigma * (\tilde{b} \circ \eta \circ \partial)$ then $v : \mathscr{A} \to B$ carries $f \in \mathscr{A}$ into

$$\sum_{(f)} \sigma(f_{(1)})\, b\langle \partial, f_{(2)} \rangle = b\sigma\Big(\sum_{(f)} \langle \partial, f_{(2)} \rangle f_{(1)}\Big) = b\sigma(L(\partial) f).$$

Thus $v = \tilde{b} \circ \sigma \circ L(\partial)$.

(b) We have seen in 1.7.1 that $\mathscr{A}$ is an $\mathscr{A}^0$-module algebra under the action $L : \mathscr{A}^0 \to \operatorname{End} \mathscr{A}$; in 1.8.1 we saw that the natural map $f \otimes \partial \mapsto \tilde{f} \circ L(\partial)$ is an algebra morphism from $\mathscr{A} \#_L \mathscr{A}^0 \to \operatorname{End} \mathscr{A}$. As $\mathscr{D}(\mathscr{A})$ is a Hopf subalgebra of $\mathscr{A}^0$, $\mathscr{A}$ is a $\mathscr{D}(\mathscr{A})$ module algebra under the same (restricted) action and we obtain by composition an algebra morphism

$$H : \mathscr{A} \#_L \mathscr{D}(\mathscr{A}) \to \mathscr{A} \#_L \mathscr{A}^0 \to \operatorname{End} \mathscr{A}$$

which carries $f \otimes \partial \in \mathscr{A} \#_L \mathscr{D}(\mathscr{A})$ into $\tilde{f} \circ L(\partial) \in \operatorname{End} \mathscr{A}$.

Taking $B = \mathscr{A}$ and $\sigma =$ the identity in part (a) we see that the map $H : f \otimes \partial \mapsto \tilde{f} \circ L(\partial)$ is a bijection from $\mathscr{A} \otimes \mathscr{D}^n(\mathscr{A})$ onto

$$\mathscr{D}_I{}^n(\mathscr{A}, \mathscr{A}) = \operatorname{Diff}_n(\mathscr{A}).$$

Thus $H : \mathscr{A} \#_L \mathscr{D}(\mathscr{A}) \to \operatorname{Diff} \mathscr{A}$ is an algebra isomorphism which carries $\mathscr{A} \otimes \mathscr{D}^n(\mathscr{A})$ onto $\operatorname{Diff}_n(\mathscr{A})$.

2.4.6. *Remark.* Notice in part (a), our proof requires only that $\mathscr{A}$ be a *bialgebra* such that $\sigma \in \operatorname{Alg}(\mathscr{A}, B)$ has a (left) inverse in the convolution algebra $\operatorname{Hom}(\mathscr{A}, B)$. We also require, of course, that the augmentation ideal of $\mathscr{A}$ should be finitely generated: an assumption that is satisfied when the algebra structure underlying $\mathscr{A}$ is finitely generated (or more generally is noetherian).

The map H in part (b) carries $A \# \mathscr{D}^1(\mathscr{A})$ isomorphically to $\operatorname{Diff}_1(\mathscr{A})$ and induces an isomorphism between $\mathscr{A} \otimes \operatorname{Der}_\epsilon{}^1(\mathscr{A}, k)$ and $\operatorname{Der}_I(\mathscr{A}, \mathscr{A})$. We thus have:

2.4.7. Corollary. *The map* $\mathscr{A} \otimes \operatorname{Der}_\epsilon{}^1(\mathscr{A}, k) \to \operatorname{Der}_I(\mathscr{A}, \mathscr{A})$ *which carries* $a \otimes \partial \mapsto \tilde{a}L \circ (\partial)$ *is a linear isomorphism. Hence,* $\operatorname{Der}_I(\mathscr{A}, \mathscr{A})$ *is a free (left)* $\mathscr{A}$*-module with rank equal to the dimension of* $\operatorname{Der}_\epsilon{}^1(\mathscr{A}, k)$ *which equals the dimension of* $\mathscr{M}/\mathscr{M}^2$.

2.4.8. Corollary. *If* $\mathscr{D}^1(\mathscr{A})$ *generates the algebra* $\mathscr{D}(\mathscr{A})$ *then* $\operatorname{Diff}_1 \mathscr{A}$ *generates the algebra* $\operatorname{Diff} \mathscr{A}$.

3. Split Coalgebras and Hopf Algebras

3.1. Definition. Let C be a coalgebra over a field k. C is *irreducible* if any two subcoalgebras of C have non-zero intersection.

C is *simple* if C has no proper subcoalgebras.
C is *pointed* if every simple subcoalgebra of C is one dimensional.
C is *connected* if C is pointed and irreducible.
C is *composite* if C is spanned by connected subcoalgebras.

A subcoalgebra D of C is irreducible (resp. simple, etc.) if D is irreducible, (resp. simple, etc.) with its induced coalgebra structure. Clearly a subcoalgebra of an irreducible (resp. connected or pointed) coalgebra is again irreducible (resp. connected or pointed). A maximal irreducible (resp. maximal connected) subcoalgebra of C is called an *irreducible component* (resp. *connected component*) of C.

Let $G(C) = \{x \in C \mid x \neq 0 \text{ and } \Delta x = x \otimes x\}$.

Finally, a finite dimensional algebra is *local* if it contains a unique maximal twosided ideal.

3.1.1. Lemma. (a) *A simple subcoalgebra of C is finite dimensional and any subcoalgebra of C contains at least one simple subcoalgebra.*

(b) *If k is algebraically closed, then any co-commutative coalgebra C is pointed.*

(c) *A coalgebra C is irreducible if and only if C contains exactly one simple subcoalgebra T; any subcoalgebra of C must then contain T.*

(d) *If $x \in G(C)$ then $k \cdot x$ is a one dimensional subcoalgebra of C and the map $x \mapsto k \cdot x$ is a bijection from $G(C)$ onto the set of one dimensional subcoalgebras of C. In particular a pointed coalgebra is connected if and only if $G(C)$ contains only one member.*

(e) *If T is a simple subcoalgebra of C, then $T^{\perp}$ is a cofinite maximal twosided ideal of algebra $A = C^*$.*

(f) *If C is finite dimensional then C is irreducible (respectively connected) if and only if the algebra $A = C^*$ is a local (respectively augmented local) algebra.*

(g) *A finite dimensional algebra A is local if and only if A contains a nilpotent maximal twosided ideal.*

Proof. (a) We have seen (Corollary 1.4.3) that every element of C generates a finite dimensional subcoalgebra. Hence every subcoalgebra contains finite dimensional subcoalgebras. (a) follows immediately.

(b) By Proposition 1.4.2 and cocommutivity a simple subcoalgebra of C is an irreducible A-submodule of C where $A = C^*$. The assertion then follows from the fact that any finite dimensional irreducible representation of a commutative algebra A over an algebraically closed field must be one dimensional.

(c) That irreducible coalgebras have the stated property is clear from the result established in 1.4.3 that the intersection of (simple) subcoalgebras is again a subcoalgebra. Conversely if T is the only simple subcoalgebra of C, then every subcoalgebra of C must contain T so that C is irreducible.

(d) Let T be a one dimensional subcoalgebra and let x be the unique element of T such that $\epsilon(x) = 1$. Then $\Delta x = x \otimes x$ so that $x \in G(C)$. For if $\Delta x = \lambda x \otimes x$ then $x = \epsilon(\lambda x)x = \lambda x$ whence $\lambda = 1$. Conversely if $x \in G(C)$, it is immediate that $\epsilon(x) = 1$ and x generates a one-dimensional subcoalgebra of C.

(e) If T is a simple subcoalgebra of C then T^* is a simple (finite dimensional) algebra. From the exact sequence $0 \to T^{\perp} \to C^* \to T^* \to 0$ we conclude that $T^{\perp}$ is a maximal ideal of C^*.

(f) We have seen that subcoalgebras D of C correspond to twosided ideals $D^{\perp}$ of $A = C^*$. By part (c) we conclude that C is irreducible if and only if A contains only one maximal ideal; i.e. A is a local algebra. Since an irreducible coalgebra is connected if and only if its simple subcoalgebra is one dimensional, the assertions of (f) are clear.

(g) Since A is finite dimensional, the Jacobson radical of A is the intersection of the maximal twosided ideals. The assertion then follows from the fact that the Jacobson radical is the largest nilpotent ideal of A.

3.1.2. Proposition. (a) *Let* $C = \sum C_i$ *where each* C_i *is a subcoalgebra of* C. *Then*

1. *Any simple subcoalgebra of* C *is contained in one of the* C_i.
2. C *is irreducible if and only if each* C_i *is irreducible and* $\cap C_i \neq 0$.
3. C *is pointed if and only if each* C_i *is pointed. In particular any composite coalgebra is pointed.*

(b) *Every irreducible (resp. connected) subcoalgebra of* C *is contained in a unique irreducible (resp. connected) component of* C.

(c) *The irreducible components are linearly independent. They span* C *if* C *is co-commutative. In particular a pointed co-commutative coalgebra is composite.*

(d) *If* $C = \coprod \{C_i \mid i \in I\}$ *where each* C_i *is irreducible then* $\{C_i \mid i \in I\}$ *are the irreducible components of* C. *Thus if* D *is an irreducible subcoalgebra of* C *then* $D \subset C_i$ *for some* i.

(e) *If every element of* C *is contained in an irreducible subcoalgebra then* C

is irreducible. If every element of C is contained in a connected subcoalgebra then C is connected.

(f) *Let $f : C \to D$ be a coalgebra morpkism. Then $f(C)$ is connected if C is connected; $f(C)$ is irreducible and co-commutative if C is irreducible and co-commutative. More generally if C is irreducible with co-commutative simple subcoalgebra T, then $f(C)$ is irreducible with simple subcoalgebra $f(T)$.*

Proof. (a) 1. Let T be a simple subcoalgebra of C. As T is finite dimensional, it is contained in the sum of a finite number of the C_i. Thus we may assume the sum $C = \sum C_i$ is itself finite. Then

$$T \subset C = \sum_{k=1}^{n} C_k$$

Taking annihilators gives

$$T^{\perp} \supset 0 = \bigcap_{k=1}^{n} I_k \qquad \text{where} \quad I_k = C_k^{\perp}$$

Since $T^{\perp}$ is maximal (3.1.1. (e)), $C^*/T^{\perp}$ is a simple ring so from the containment $I_1 \cdots I_n \subset T^{\perp}$ we obtain by passage to quotients that $I_k \subset T^{\perp}$ for some k. Thus

$$C_k = I_k^{\perp} \supset T.$$

(a) 2. If $C = \sum C_i$ is irreducible and T is the unique simple subcoalgebra of C, then each C_i is irreducible and contains T so that $\bigcap C_i \neq 0$.

Conversely if C_i is irreducible and $\bigcap C_i \neq 0$ choose a simple subcoalgebra

$$T \subset \bigcap C_i$$

Then $T \subset C_i$ so that T is the unique simple subcoalgebra of C_i. By (a)1, it follows that T is the only simple subcoalgebra of C so that C is irreducible.

(a) 3. follows immediately from (a) 1.

(b) The unique irreducible component containing an irreducible subcoalgebra D is simply the *sum* of all the irreducible subcoalgebras containing D. This is irreducible by (a) 2.

(c) Let $\{C_i\}_{i \in I}$ be the *distinct* irreducible components and let T_i be the simple subcoalgebra contained in C_i. We first prove independence.

If $C_i \cap \sum_{j \neq i} C_j = 0$ then $T_i \subset \sum_{j \neq i} C_j$. Hence by (a) 1, $T_i \subset C_j$ for some $j \neq i$. Then $C_i \cap C_j \neq 0$ so that $C_i + C_j$ is irreducible by (a) 2. This contradicts the fact that each C_k is maximal irreducible.

To prove that the C_i span C when C is co-commutative is equivalent to showing that if $x \in C$ then $x = \sum x_i$ where each x_i lies in an irreducible subcoalgebra. Since the subcoalgebra X generated by x is finite dimensional, it suffices to prove (c) for a *finite dimensional* coalgebra for we could then conclude that $x = \sum x_i$ where each x_i lies in an irreducible subcoalgebra of X. These are also irreducible subcoalgebras of C.

If then C is a finite dimensional coalgebra, $A = C^*$ is a direct product of local algebras.

$A = \prod_1^n A_i$ where each A_i is a local algebra ([*11*], pg. 205, Thm. 3). Thus $C = \prod_1^n C_i$ where $C_i = A_i{}^*$ is an irreducible coalgebra by 3.1.1. (f).

(d) If D_i is the irreducible component containing C_i it is clear $D_i \neq D_j$ if $i \neq j$. Since $\sum D_i = C$, it follows from part (c) that $D_i = C_i$ and $\{C_i\}$ exhausts the set of irreducible components.

(e) The first part of (e) follows immediately from (c) and (d). Since a split irreducible coalgebra is connected, the second part then follows from the first.

(f) It suffices to prove the last assertion. Let $y \in D$. Choose $x \in C$ such that $f(x) = y$ and let X be the subcoalgebra generated by x. Clearly $T \subset X$. It suffices by part (e) to show that $f(X)$ is irreducible with simple subcoalgebra $f(T)$. We may thus assume that C is finite dimensional and f is surjective. First assume that C is connected.

By 3.1.1 (f) and duality it suffices to show the following: if A is a finite dimensional local algebra with maximal twosided ideal $\mathcal{M}$ where $A/\mathcal{M}$ is commutative, then any subalgebra B is local with maximal ideal $B \cap \mathcal{M}$. Let $\mathcal{N} = B \cap \mathcal{M}$ then $\mathcal{N}$ is nilpotent since $\mathcal{M}$ is nilpotent. Moreover $B/\mathcal{N}$ is a subalgebra of the (finite-dimensional) field $A/\mathcal{M}$ so that $B/\mathcal{N}$ is a field and therefore $\mathcal{N}$ is maximal. It follows by 3.1.1 (g) that B is local with maximal ideal $\mathcal{N}$. Q.E.D.

DEFINITION. If $g \in G(C)$ let C^g denote the connected component of C which contains g (see 3.1.3 (b)).

3.1.3. COROLLARY. (a) *Let $C = \sum C_i$. Then C is connected if and only if each C_i is connected and $\bigcap C_i \neq 0$.*

(b) *Every connected subcoalgebra of C is contained in a unique connected component of C. The connected components of C are precisely $\{C^g \mid g \in G(C)\}$, or again those irreducible components of C whose simple subcoalgebras are one dimensional.*

(c) *If C is composite, then C is the direct sum of its connected components.*

Conversely let $S \subset G(C)$ and for each $g \in S$ let $C(g)$ be a connected subcoalgebra of C such that

$$C = \coprod_{g \in S} C(g)$$

Then C is composite, $S = G(C)$ and $C(g) = C^g$ is the connected component of C containing g.

(d) *Let $f : C \to D$ be a coalgebra morphism. If $g \in G(C)$ then $f(g) \in G(D)$ and $f(C^g) \subset D^{f(g)}$. Moreover $f(G(C)) = G(f(C))$ if C is composite.*

(e) *Let $f : C \to D$ be a coalgebra morphism. If C is composite then $f(C)$ is a composite subcoalgebra of D.*

Proof. (a) If each C_i is connected and $\bigcap C_i \neq 0$ then $\sum C_i$ is irreducible and therefore connected by part (a) 2 of the proposition.

(b) The sum of all the connected subcoalgebras containing a given connected subcoalgebra is connected by (a). The rest of (b) is then clear.

(c) This follows from parts (c), (d) and (a) 3 of the proposition and the fact that the connected components and the irreducible components coincide for a composite coalgebra.

(d) If $g \in G(C)$ then $\epsilon(f(g)) = \epsilon(g) = 1$ so that $f(g) \neq 0$. Since $\Delta f(g) = (f \otimes f)(g \otimes g)$ it follows that $f(g) \in G(D)$. Since $f(C^g)$ is connected by part (f) of the proposition and contains the one dimensional subcoalgebra $k \cdot f(g)$ we conclude that $f(C^g)$ is connected and $f(C^g) \subset D^{f(g)}$. Moreover if C is composite and $g' \in G(f(C))$ then $k \cdot g' \subset f(C) = \sum_g f(C^g)$ so that $g' \in f(C^g)$ for some $g \in G(C)$ by 3.1.2 (a). Then $g' = f(g)$ because $f(C^g)$ is connected.

(e) If C is composite then C is spanned by $\{C^g \mid g \in G\}$ so that $f(C)$ is spanned by $\{f(C^g) \mid g \in G\}$. As $f(C^g)$ is connected we are done. Q.E.D.

3.2. Let C be a coalgebra and let $C^+ = \ker \epsilon_C$. If $f : C \to D$ is a coalgebra morphism it follows from the identity $\epsilon_D \circ f = \epsilon_C$ that $f(C^+) \subset D^+$. Now assume that C is connected. Let 1 denote the unique element of $G(C)$ and let $\mathscr{M} = 1^\perp$ be the augmentation ideal of $A = C^*$; we obtain direct sum decompositions:

$$C = C^+ \oplus k \cdot 1 \qquad \text{and} \qquad A = \mathscr{M} \oplus k \cdot \epsilon.$$

These decompositions determine projections:

$$E^C : C \to C \quad \text{mapping} \quad c \mapsto c - \epsilon(c)1 \in C^+ \qquad \text{for all} \quad c \in C$$

$$(E^C)^* : A \to A \quad \text{mapping} \quad f \mapsto f - f(1)\epsilon \in \mathscr{M} \qquad \text{for all} \quad f \in A$$

Using the notation of 1.3, we define $\delta_n{}^C : C \to \otimes^{n+1} C$ as the composition

$$\delta_n{}^C = \overset{n+1}{\otimes} E^C \circ \Delta_n{}^C : C \to \overset{n+1}{\otimes} C$$

When the coalgebra is understood we write simply E, Δ_n, δ_n. We shall continue to write Δ for Δ_1 and δ for δ_1. Since $E(C) = C^+$ it follows that $\delta_n(C)$ is contained in $\otimes^{n+1} C^+$. Also $\delta_n 1 = 0$ so that $\delta_n c = \delta_n \bar{c}$ where $\bar{c} = E(c) \in C^+$. Define

$$C_n = \text{Kernel } \delta_n{}^C$$
$$C_n{}^+ = C_n \cap C^+$$
$$P(C) = C_1{}^+.$$

The elements of $P(C)$ are called *primitive*. Note that $\Delta_0 : C \to C$ is the identity map so that $\delta_0 = E$ and $C_0 = \text{Ker } \delta_0 = k \cdot 1$. Thus $C = C^+ \oplus C_0$. It will also be convenient to define $C_{-1} = 0$.

3.2.1. Lemma. *Let C and C' be connected coalgebras.*

(a) *If $c \in C$ then $\delta c = \Delta c - (c \otimes 1 + 1 \otimes c) + \epsilon(c) 1 \otimes 1$; thus in particular if $c \in C^+$ then $\delta c = \Delta c - (c \otimes 1 + 1 \otimes c) \in C^+ \otimes C^+$ and*

$$P(C) = \{c \in C \mid \Delta c = c \otimes 1 + 1 \otimes c\}$$

Also if $c \in C$ then $\delta c = \delta \bar{c} = \Delta \bar{c} - (\bar{c} \otimes 1 + 1 \otimes \bar{c})$ where $\bar{c} = Ec \in C^+$.

(b) *If $f : C \to C'$ is a coalgebra morphism then $E' \circ f = f \circ E : C \to C'$ and*

$$\Delta_n{}' \circ f = \overset{n+1}{\otimes} f \circ \Delta_n : C \to \overset{n+1}{\otimes} C'$$

so that

$$\delta_n{}' \circ f = \overset{n+1}{\otimes} f \circ \delta_n : C \to \overset{n+1}{\otimes} C'.$$

Finally $f(C_n) \subset C_n{}'$; moreover if f is injective then $f(C_n) = f(C) \cap C_n{}'$.

(c) *Let C, D be arbitrary coalgebras and C^g the connected component of C which contains $g \in G(C)$. If $f : C \to D$ is a coalgebra morphism then $f(C_n{}^g) \subset D_n^{f(g)}$.*

Proof. (a) If $c \in C$ then

$$\delta c = (E \otimes E) \circ \Delta c = \sum_{(c)} (c_{(1)} - \epsilon(c_{(1)})\, 1) \otimes (c_{(2)} - \epsilon(c_{(2)})\, 1)$$
$$= \Delta c - (c \otimes 1 + 1 \otimes c) + \epsilon(c)(1 \otimes 1)$$

Since $\delta 1 = 0$ it is clear that $\delta \bar{c} = \delta c$.

(b) The stated identities follow immediately from the definitions. If $x \in C_n$ then $\delta_n x = 0$ so that $\delta_n' \circ f(x) = \otimes^{n+1} f \circ \delta_n(x) = 0$ and therefore $f(x) \in C_n'$. Finally if f is injective then $\otimes^{n+1} f : \otimes^{n+1} C \to \otimes^{n+1} C'$ is injective. Thus if $x \in C$ then $f(x) \in C_n'$ if and only if $\delta_n' \circ f(x) = \otimes^{n+1} f \circ \delta_n(x) = 0$; i.e. if and only if $\delta_n x = 0$; that is, $x \in C_n$.

(c) follows at once from part (b) and 3.1.3 (d). Q.E.D.

3.2.2. Proposition. *Let C be a connected coalgebra and let $\mathscr{M} = 1^\perp$ be the augmentation ideal of the dual algebra $A = C^*$. Furthermore let B be a dense subalgebra of A and let $\mathscr{N} = B \cap \mathscr{M}$.*

(a) *If D is a subcoalgebra of C then D is connected and $D_n = D \cap C_n$*

(b) *$C_n = (\mathscr{N}^{n+1})^\perp$; in particular $C_n = (\mathscr{M}^{n+1})^\perp$*

(c) *If $c \in C$ then $c \in C_n$ if and only if $L(f)c \in C_{n-1}$ for all $f \in \mathscr{N}$ (recall for the case $n = 0$ that $C_{-1} = 0$)*

(d) *If D is an n-dimensional subcoalgebra of C then $D \subset C_{n-1}$*

(e) *$C = \bigcup C_n$*

(f) *If C^{op} is the coalgebra obtained from C by 'twisting' the coproduct $[\Delta^{\mathrm{op}} = \tau \circ \Delta]$ then C^{op} is connected and $(C^{\mathrm{op}})_n = (C_n)^{\mathrm{op}}$*

Proof. (a) It is clear from the definition that D is connected and $j : D \to C$ is a coalgebra morphism. The result stated then follows at once from 3.2.1 (b).

(b) We first prove that $C_n = (\mathscr{M}^{n+1})^\perp$. If $f_0, \dots, f_n \in A$ and $c \in C$ we have

$$\langle f_0 \otimes \cdots \otimes f_n, \delta_n(c)\rangle = \langle \bar{f}_0 \otimes \cdots \otimes \bar{f}_n, \Delta_n(c)\rangle = \langle \bar{f}_0 \cdots \bar{f}_n, c\rangle$$

where $\bar{f} = f \circ E = f - f(1)\epsilon \in \mathscr{M}$. This shows that $C_n = (\mathscr{M}^{n+1})^\perp$.

Since $\mathscr{N} \subset \mathscr{M}$ we have $\mathscr{N}^{n+1} \subset \mathscr{M}^{n+1}$ so that $C_n = (\mathscr{M}^{n+1})^\perp \subset (\mathscr{N}^{n+1})^\perp$. Now let $x \in (\mathscr{N}^{n+1})^\perp$ and choose a finite dimensional subcoalgebra D of C which contains x. Because B is dense in A it follows that the natural algebra morphism $\pi : B \to D^*$ is surjective and $\bar{\mathscr{N}} = \pi(\mathscr{N})$ is the augmentation ideal of D^*. As we have already established $C_n = (\mathscr{M}^{n+1})^\perp$ it follows that $D_n = (\bar{\mathscr{N}}^{n+1})^\perp$ so that $x \in D_n$. By part (a) we conclude $x \in C_n$. This proves (b).

(c) This follows from (b) and the fact that $\langle \mathscr{N}^n, L(f)c\rangle = 0$ for all $f \in \mathscr{N}$ if and only if $\langle \mathscr{N}^{n+1}, c\rangle = 0$.

(d) By 3.1.1 (f), D^* is an n-dimensional local algebra with augmentation ideal J. Then $J^n = 0$ so that $D = (J^n)^\perp = D_{n-1}$ by part (b). The result then from part (a).

(e) Since every element of C is contained in a finite dimensional subcoalgebra this is clear in view of part (d).

(f) is clear from the definitions. Q.E.D.

3.2.3. PROPOSITION. *Let F be a field containing k and let $C_F = F \otimes_k C$ be the coalgebra over F obtained from the coalgebra C by "extending scalars."*

(a) *If C is connected then C_F is connected and $(C_F)_n = F \otimes C_n$.*

(b) *If C is composite then C_F is composite and the connected components of C_F are $\{F \otimes C^g \mid g \in G(C)\}$. Moreover $G(C_F) = \{1 \otimes g \mid g \in G(C)\}$.*

Now let C be co-commutative and F purely inseparable over k.

(c) *If C is irreducible, then C_F is irreducible. More generally:*

(d) *The irreducible components of C_F are given by $\{F \otimes C_i \mid C_i$ irreducible component of $C\}$.*

Proof. (a) To prove that C_F is connected it suffices by 3.1.2 (e) to proof that every $x \in C_F$ is contained in a connected subcoalgebra of C_F. Since there is a finite dimensional subcoalgebra D of C with the property that $x \in D_F \subset C_F$, it suffices to show that D_F is connected. Thus we are reduced to proving that C_F is connected when C is *finite dimensional.* In this case we have

$$C_F{}^* = \operatorname{Hom}_F(C_F, F) = \operatorname{Hom}_k(C, F) = F \otimes C^*$$

It thus suffices by 3.1.1 (f) to show that if A is a finite dimensional augmented local algebra over k, then $A_F = F \otimes A$ is an augmented local algebra over F. If $\epsilon : A \to k$ is the augmentation map of A, then the kernel $\mathcal{M}$ of ϵ is a nilpotent ideal of A. The augmentation map of the algebra $A_F = F \otimes A$ is $1 \otimes \epsilon : F \otimes A \to F$. Since the kernel of this map is the nilpotent ideal $F \otimes \mathcal{M}$, we are done.

To prove that $(C_F)_n = F \otimes C_n$, observe that the algebra A_F is dense in $(C_F)^* = \operatorname{Hom}_F(C_F, F)$. If M is the augmentation ideal of $(C_F)^*$ and $\mathcal{M}$ is the augmentation ideal of A then $A_F \cap M = F \otimes \mathcal{M}$.

By 3.2.2. (b) we conclude

$$(C_F)_n = [(F \otimes \mathcal{M})^{n+1}]^{\perp} = (F \otimes \mathcal{M}^{n+1})^{\perp} = F \otimes C_n$$

(b) If C is composite then $C = \coprod \{C^g \mid g \in G(C)\}$. Then

$$C_F = F \otimes C = \coprod \{F \otimes C^g \mid g \in G(C)\}.$$

Since each $F \otimes C^g$ is a connected coalgebra over F by part (a), the result stated follows from 3.1.3 (c).

(c) Just as in part (a) we may assume C is finite dimensional. Then $(C_F)^* = F \otimes C^*$. By 3.1.1 (f) it thus suffices to show that if A is a finite dimensional commutative local k-algebra with maximal ideal $\mathcal{M}$ then $F \otimes A$ is again local. Since F is purely inseparable of characteristic p, if $x \in F \otimes A$

there is an integer n such that $x^{p^n} \in k \otimes A = A$. If $x^{p^n} \notin \mathscr{M}$, then x^{p^n} is invertible in A (A is local) so that x is invertible in $F \otimes A$. Thus

$$\{x \in F \otimes A \mid \text{some power of } x \text{ lies in } \mathscr{M}\}$$

is the unique maximal ideal of $F \otimes A$ so that $F \otimes A$ is local.

(d) If $C = \coprod \{C_i \mid i \in I\}$ where the C_i are the irreducible components of C then $C_F = \coprod \{F \otimes C_i \mid i \in I\}$. Since each $F \otimes C_i$ is irreducible by part (c), the result follows from 3.1.2 (d).

3.2.4. PROPOSITION. *Let C and D be coalgebras*

(a) *If C and D are each connected then $C \otimes D$ is connected and $(C \otimes D)_n = \sum_{k=0}^{n} C_k \otimes D_{n-k}$; moreover $(C \otimes D)_n \cap (C^+ \otimes D^+) = \sum_{k=1}^{n-1} C_k^+ \otimes D_{n-k}^+$.*

(b) *If C and D are each composite then $C \otimes D$ is composite and the connected components of $C \otimes D$ are $\{C^g \otimes D^h \mid (g, h) \in G(C) \otimes G(D)\}$. Moreover $G(C \otimes D) = \{g \otimes h \mid g \in G(C), h \in G(D)\}$.*

Proof. (a) To prove $C \otimes D$ is connected, it suffices by 3.1.2 (e) to prove that every finite dimensional subcoalgebra of $C \otimes D$ is connected. Since such a subcoalgebra is contained in one of the form $C' \otimes D'$ where C' and D' are finite dimensional subcoalgebras of C and D we may assume that C and D are themselves finite dimensional.

If $A = C^*$ and $B = D^*$ then A and B are local algebras with maximal twosided ideals $\mathscr{M}$ and $\mathscr{N}$, each nilpotent and of codimension one. Since the ideal $A \otimes \mathscr{N} + \mathscr{M} \otimes B$ is clearly nilpotent and of codimension one in the algebra $A \otimes B = (C \otimes D)^*$, we conclude by 3.1.1 that $C \otimes D$ is connected.

To prove that $(C \otimes D)_n = \sum_{r+s=n} C_r \otimes D_s$, let $A = C^*$ and $B = D^*$ (note that C and D are no longer assumed to be finite dimensional). If M is the augmentation ideal of the algebra $(C \otimes D)^*$ and $\mathscr{M}$, $\mathscr{N}$ are the augmentation ideals of A and B then we have

$$(A \otimes B) \cap M = A \otimes \mathscr{N} + \mathscr{M} \otimes B.$$

Since $A \otimes B$ is a dense subalgebra of $(C \otimes D)^*$ we conclude by 3.2.2 (b) that

$$\begin{aligned}
(C \otimes D)_n &= [(A \otimes \mathscr{N} + \mathscr{M} \otimes B)^{n+1}]^\perp = \Big[\sum_{k=-1}^{n} \mathscr{M}^{k+1} \otimes \mathscr{N}^{n-k}\Big]^\perp \\
&= \bigcap_{k=-1}^{n} (\mathscr{M}^{k+1} \otimes \mathscr{N}^{n-k})^\perp \\
&= \bigcap_{k=-1}^{n} (C_k \otimes D + C \otimes D_{n-k-1}) \qquad \text{where} \qquad C_{-1} = 0 = D_{-1} \\
&= \sum_{r+s=n} C_r \otimes D_s
\end{aligned}$$

Finally $C_r = C_0 \oplus C_r^+$ and $D_s = D_0 \oplus D_s^+$ so that

$$(C \otimes D)_n = \sum_{k=0}^{n} C_k \otimes D_{n-k} = \sum_{k=1}^{n-1} C_k^+ \otimes D_{n-k}^+ + T$$

where

$$T = C_n^+ \otimes D_0 + C_0 \otimes D_n^+ + C_0 \otimes D_0$$

Thus

$$(C \otimes D)_n \cap (C^+ \otimes D^+) = \sum_{k=1}^{n-1} C_k^+ \otimes D_{n-k}^+ + T \cap (C^+ \otimes D^+).$$

Since $C \otimes D = [C^+ \otimes D^+] \oplus [C^+ \otimes D_0 + C_0 \otimes D^+ + C_0 \otimes D_0]$ we have $T \cap (C^+ \otimes D^+) = 0$. This completes the proof of (a).

(b) If C and D are composite we have $C = \coprod \{C^g \mid g \in G(C)\}$ and $D = \coprod \{D^h \mid h \in G(D)\}$ so that

$$C \otimes D = \coprod \{C^g \otimes D^h \mid (g, h) \in G(C) \otimes G(D)\}.$$

Since each $C^g \otimes D^h$ is connected by part (a), it follows from 3.1.3 (c) that $C \otimes D$ is composite, $G(C \otimes D) = \{g \otimes h \mid (g, h) \in G(C) \otimes G(D)\}$ and that $C^g \otimes D^h$ is the connected component of $C \otimes D$ which contains $g \otimes h$.

3.2.5. COROLLARY. (a) *Let C be a connected coalgebra. Then $C \otimes C$ is connected and $\Delta C_n \subset (C \otimes C)_n$. Moreover the following conditions are equivalent*:

1. $x \in C_n$
2. $\delta x \in (C \otimes C)_n$
3. $\delta x \in \sum_{k=1}^{n-1} C_k^+ \otimes C_{n-k}^+$
4. $\delta x \in C_{n-1}^+ \otimes C_{n-1}^+$
5. $\delta x \in C_{n-1} \otimes C_{n-1}$

(b) *If C and D are connected (resp. composite subcoalgebras of a bialgebra (H, p, Δ) then $C \cdot D = p(C \otimes D)$ is a connected (resp. composite) subcoalgebra of H; moreover (when C and D are connected) $C_n \cdot D_m \subset (C \cdot D)_{n+m}$.*

(c) *Let $C^1, C^2, \ldots, C^r$ be connected coalgebras and $P_i = P(C^i)$. Then $C = C^1 \otimes \cdots \otimes C^r$ is a connected coalgebra and the map $P_1 \times \cdots \times P_r \to P(C)$*

$$(x_1, x_2, \ldots, x_r) \mapsto x_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1$$
$$+ 1 \otimes x_2 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes 1 \otimes x_r$$

is a linear isomorphism.

Proof. (a) To prove that $\Delta C_n \subset (C \otimes C)_n$ it suffices by 3.2.2 (b) to show (using the notation of 3.2.4 (a)) that if $\langle \mathscr{M}^{n+1}, x\rangle = 0$ then

$$\langle (A \otimes \mathscr{M} + \mathscr{M} \otimes A)^{n+1}, \Delta x\rangle = 0.$$

Since

$$(A \otimes \mathscr{M} + \mathscr{M} \otimes A)^{n+1} = \sum_{r+s=n+1} \mathscr{M}^r \otimes \mathscr{M}^s$$

and

$$\langle \mathscr{M}^r \otimes \mathscr{M}^s, \Delta x\rangle = \langle \mathscr{M}^{r+s}, x\rangle$$

this is clear. The implication 1 implies 2 then follows from the identity

$$\delta x = \Delta x - (x \otimes 1 + 1 \otimes x) + \epsilon(x) 1 \otimes 1$$

Since δC is contained in $C^+ \otimes C^+$ the implication 2 implies 3 follows from 3.2.4 (a). 3 implies 4 and 4 implies 5 are clear. Finally if $\delta x \in C_{n-1} \otimes C_{n-1}$ and $f \in \mathscr{M} = 1^\perp$ then $L(f)x = (I \otimes f)\,\Delta x \in C_{n-1}$ so that $x \in C_n$ by 3.2.2. (c).

(b) If (H, p, Δ) is a bialgebra then p: $H \otimes H \to H$ is a coalgebra morphism. If C and D are connected (resp. composite) then $C \otimes D$ is a connected (resp. composite) subcoalgebra of $H \otimes H$ by the last proposition and $C \cdot D = p(C \otimes D)$ is connected (resp. composite) by 3.1.2 (f) (resp. 3.1.3 (f)). Finally when C and D are connected then $C_n \otimes D_m$ is contained in $(C \otimes D)_{n+m}$ so that

$$C_n \cdot D_m = p(C_n \otimes D_m) \subset p[(C \otimes D)_{n+m}] \subset p(C \otimes D)_{n+m} = (C \cdot D)_{n+m}$$

by 3.2.1 (b).

For (c) it is clear by induction on r that

$$P(C) = P_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes P_2 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes P_r .$$

Since the map from $P_1 \times \cdots \times P_r \to P(C)$ is injective (it has for left inverse the map

$$(1 \otimes \epsilon \otimes \cdots \otimes \epsilon) \times \cdots \times (\epsilon \otimes \epsilon \otimes \cdots \epsilon \otimes 1) : P(C) \to P_1 \times \cdots \times P_r)$$

the corollary is proved.

The following is a very useful lemma:

3.2.6. Lemma. *Let $f: C \to D$ be a coalgebra morphism where C is connected. If f is injective when restricted to $P(C)$ (i.e.* $\operatorname{Ker} f \cap P(C) = 0$) *then f is injective.*

Proof. Since f is a coalgebra morphism we have $\epsilon(f(1)) = 1$ and

$\epsilon(f(P(C)) = 0$ so that f is injective on $C_1 = k \cdot 1 \oplus P(C)$. Suppose by induction that f is injective on C_n where $n \geqslant 1$ and let $x \in C_{n+1}$. By 3.2.5. (a)

$$\Delta x = x \otimes 1 + 1 \otimes x + T \qquad \text{where} \qquad T \in C_n \otimes C_n .$$

Thus if $f(x) = 0$, we have $0 = \Delta f(x) = (f \otimes f)(\Delta x) = (f \otimes f)(T)$. If $g = f \mid C_n$ then $g : C_n \to D$ is injective so that $g \otimes g : C_n \otimes C_n \to D \otimes D$ is injective. Since $0 = (f \otimes f)(T) = (g \otimes g)(T)$, we conclude $T = 0$ so that $x \in P(C)$. Thus $x = 0$ by hypothesis.

3.2.7. Corollary. *Let C be a connected coalgebra.*

(a) *Let I be a coideal of C. If $I \cap P(C) = 0$ then $I = 0$.*

(b) *Let $A = C^*$ be the dual algebra. If B is a subalgebra of A such that $B^\perp \cap P(C) = 0$, then B is dense in A.*

(c) *Let D be a coalgebra and let $f, g : D \to C$ be two coalgebra morphisms such that* $\operatorname{Im}(f - g) \cap P(C) = 0$. *Then $f = g$.*

Proof. (a) Apply the lemma to the canonical coalgebra morphism $\pi : C \to C/I$.

(b) If $I = B^\perp$ then I is a coideal of C by 1.1. The result then follows from part (a).

(c) If $I = \operatorname{Im}(f - g)$ then I is a coideal of C by 1.1. The result then follows from part (a).

3.3. Let $N = \{0, 1, 2, ...\}$ be the set of nonnegative integers. If I is any set let $N^{(I)}$ be the additive semi-group of functions $\alpha : I \to N$ which are zero for all but a finite number of $i \in I$. For each $i \in I$, define $\epsilon_i \in N^{(I)}$ by the equation

$$\begin{aligned} \epsilon_i(j) &= 1 \qquad j = i \\ &= 0 \qquad j \neq i. \end{aligned}$$

If $\alpha \in N^{(I)}$ then $\alpha = \sum_{i \in I} \alpha(i) \epsilon_i$ and $N^{(I)}$ is a *free* abelian semi-group with basis $\{\epsilon_i \mid i \in I\}$. If $\alpha, \beta \in N^{(I)}$ define $\operatorname{Supp} \alpha = \{i \in I \mid \alpha(i) \neq 0\}$ and

$$|\alpha| = \sum_{i \in I} \alpha(i) \in N$$

$$\alpha! = \prod_{i \in I} \alpha(i)!$$

$$(\alpha, \beta) = \frac{(\alpha + \beta)!}{\alpha! \, \beta!}$$

Say $\alpha \leqslant \beta$ if $\alpha(i) \leqslant \beta(i)$ for all $i \in I$ and $\alpha < \beta$ if $\alpha \leqslant \beta$ and $\alpha \neq \beta$. Then $i \in \operatorname{Supp} \alpha$ if and only if $\epsilon_i \leqslant \alpha$.

Let C be a co-commutative connected coalgebra, $A = C^*$ and $\mathscr{M}$ the augmentation ideal of A. Let $\{c_i\}_{i \in I}$ be a basis of $P(C)$. For each $i \in I$, choose $f_i \in \mathscr{M}$ such that $\langle f_i, c_j \rangle = \delta_{ij}$. Then define

$$f^{\epsilon_i} = f^i$$

and $f^\alpha = \prod_{i \in I} (f_i)^{n_i}$ where $\alpha = \sum n_i \epsilon_i \in N^{(I)}$. Clearly $f^{\alpha+\beta} = f^\alpha \cdot f^\beta$ where $\alpha, \beta \in N^{(I)}$ and $f^\alpha \in \mathscr{M}^n$ where $n = |\alpha|$.

Let B be the subalgebra of A generated by the family $\{f_i \mid i \in I\}$. Clearly B is the subspace of A spanned by the functionals $\{f^\alpha \mid \alpha \in N^{(I)}\}$.

3.3.1. Proposition. (a) *B is a dense subalgebra of A.*

(b) *If $c \in C$ then $c \in C_n$ if and only if $\langle f^\alpha, c \rangle = 0$ whenever $|\alpha| > n$.*

(c) *If D is a finite dimensional subcoalgebra of C_n then $\langle f^\alpha, D \rangle = 0$ for all but a finite number of $\{\alpha \in N^{(I)} : |\alpha| = n\}$.*

Proof. (a) It is clear from the definition that $B^\perp \cap P = 0$, so that (a) follows from 3.2.7. (b).

(b) If $\mathscr{N} = B \cap \mathscr{M}$, it is clear that $\mathscr{N}$ is spanned by $\{f^\alpha : |\alpha| > 0\}$ so that $\mathscr{N}^{r+1}$ is spanned by $\{f^\alpha : |\alpha| > r\}$. The result then follows from 3.2.2. (b).

(c) Since D is finite dimensional, $D_1 = k \cdot 1 + D \cap P(C)$ is finite dimensional. It is then clear from the definition of f_i that the set $\{i \in I \mid \langle f_i, D_1 \rangle \neq 0\}$ is a *finite* subset J of I : If $|\alpha| = n$ and $j \in \operatorname{Supp} \alpha$, let $\beta = \alpha - \epsilon_j$. Then $|\beta| = n - 1$ and $L(f^\beta) D \subset D_1$. Since

$$\langle f^\alpha, D \rangle = 0 \qquad \text{if and only if} \qquad \langle f_j, L(f^\beta)D \rangle = 0$$

it follows that $\langle f^\alpha, D \rangle = 0$ unless $\operatorname{Supp} \alpha \subset J$.

3.3.2. Corollary. *If $c \in C_n$ and $\langle f^\alpha, c \rangle = 0$ for all $\alpha \in N^{(I)}$ such that $|\alpha| = n$, then $c \in C_{n-1}$.*

Proof. If $|\beta| > n$ then $f^\beta \in \mathscr{M}^{n+1}$ so that $\langle f^\beta, c \rangle = 0$. The assertion then follows from (b).

3.4. Let A be a commutative algebra over a field k. A is *split* if every cofinite maximal ideal of A has codimension one. Clearly this is equivalent to the only simple finite dimensional A-modules being one dimensional.

Let $G_A = \operatorname{Alg}(A, k)$.

3.4.1. Lemma. *Let A be a commutative algebra over a field k.*

(a) *A is split if and only if A^0 is a pointed coalgebra.* (*Recall that A^0 is pointed* $\Leftrightarrow$ *A^0 is composite by* 3.1.2.)

(b) *If k is algebraically closed, then A is split.*

(c) $G_A = G(A^0)$ *and the map $g \mapsto k \cdot g$ is a bijection from G_A onto the set of one dimensional subcoalgebras of A^0.*

Proof. (a) Let $\mathscr{I}$ be the set of cofinite ideals of A and $\mathscr{C}$ the set of finite dimensional subcoalgebras of A^0. If $I \in \mathscr{I}$ then $I^\perp \in \mathscr{C}$ by 1.3.2 and the map $I \mapsto I^\perp$ is a bijection from $\mathscr{I}$ onto $\mathscr{C}$ (if $D \in \mathscr{C}$, then $D^\perp \in \mathscr{I}$ and $D^{\perp\perp} = D$ because D is finite dimensional: the map $I \mapsto I^\perp$ is therefore surjective). The assertion (a) then follows from the fact that $\dim I^\perp = \operatorname{codim} I$.

(b) follows directly or from (a) and 3.1.1 (b).

(c) If $\Delta : A^0 \to A^0 \otimes A^0$ is the coproduct map of A^0, we must show that $G_A = \mathrm{Alg}(A, k) = \{f \in A^* \mid f \neq 0 \text{ and } \Delta f = f \otimes f\}$. This is the set denoted $G(A^0)$ in §3.1. This equality follows from the formula

$$\langle \Delta f - f \otimes f, a \otimes b \rangle = f(ab) - f(a)f(b) \qquad \text{where} \qquad a, b \in A, \quad f \in A^*.$$

The result asserted is then a consequence of 3.1.1. d. Q.E.D.

Let $x \in G_A$ and let $\mathscr{M}_x = \operatorname{kernel} x$; thus $\mathscr{M}_x$ is an ideal of A having codimension one. In Section 2 we defined

$$\mathscr{D}_x{}^n(A) = \mathscr{D}_x{}^n(A, k)$$

and saw that if the ideal $\mathscr{M}_x$ is finitely generated then $\mathscr{D}_x{}^n(A)$ is a subcoalgebra of A^0.

Let $\mathscr{D}_x(A) = \bigcup_n \mathscr{D}_x{}^n(A)$.

3.4.2. Proposition. (a) *If $\mathscr{M}_x$ is a finitely generated ideal of A, then $\mathscr{D}_x(A)$ is the connected component of A^0 containing $x \in G_A$. Moreover*

$$[\mathscr{D}_x(A)]_n = \mathscr{D}_x{}^n(A)$$

that is, the natural filtration defined in 3.2.1 (*for any connected coalgebra*) *agrees on $\mathscr{D}_x(A)$ with the filtration defined in Section* 2 *for differential operators.*

(b) *If A is a finitely generated split commutative algebra then the set $\{\mathscr{D}_x(A) \mid x \in G_A\}$ are the connected components of A^0 and*

$$A^0 = \coprod_x \mathscr{D}_x(A)$$

Proof. (a) If $\mathscr{M}_x$ is finitely generated, then we have seen that

$$\mathscr{D}_x{}^n(A) = (\mathscr{M}_x^{n+1})^\perp \subset A^0.$$

Moreover $x \in \mathscr{D}_x{}^0(A) = k \cdot x$. To prove that $\mathscr{D}_x(A)$ is connected, it suffices by 3.1.1. (c) to show that any simple subcoalgebra D of $\mathscr{D}_x{}^n(A)$ must contain x. But if

$$D \subset \mathscr{D}_x^n(A) = (\mathscr{M}_x^{n+1})^\perp$$

then

$$M \supset (\mathscr{M}_x^{n+1})^{\perp\perp} = \mathscr{M}_x^{n+1} \qquad \text{where} \qquad M = D^\perp.$$

Since M is a maximal ideal of A, it is prime and so $M \supset \mathscr{M}_x$. Since $\mathscr{M}_x$ is maximal, $M = \mathscr{M}_x$ and so $D = M^\perp = \mathscr{M}_x{}^\perp = k \cdot x$. Thus $\mathscr{D}_x(A)$ is a connected subcoalgebra of A^0.

Let $A_x{}^0$ denote the connected component of A^0 which contains x. Then we have shown that $\mathscr{D}_x(A) \subset A_x{}^0$. To prove equality let D be a finite dimensional subcoalgebra of $A_x{}^0$. Then $k \cdot x$ is the only simple subcoalgebra contained in D. If $I = D^\perp$ then I is a cofinite ideal of A and $\mathscr{M}_x = x^\perp$ is the only maximal ideal containing I. Thus A/I is a finite dimensional local algebra having maximal ideal $\mathscr{M}_x/I$. Hence $\mathscr{M}_x/I$ is nilpotent so that $\mathscr{M}_x^{r+1} \subset I$ for some r. Since D is finite dimensional we then have $D = I^\perp \subset (\mathscr{M}_x^{r+1})^\perp = \mathscr{D}_x{}^r(A)$. This proves that $\mathscr{D}_x(A) = A_x{}^0$.

For notational convenience let C denote $\mathscr{D}_x(A)$. Then the natural algebra morphism from A into C^* obviously carries A onto a *dense* subalgebra B of C^*. If $\mathscr{N}_x$ denotes the augmentation ideal of C^* then this algebra morphism obviously carries $\mathscr{M}_x$ onto $\mathscr{N}_x \cap B$. We know $\mathscr{D}_x(A)_n = C_n = (\mathscr{N}_x^{n+1})^\perp$ and $\mathscr{D}_x{}^n(A) = (\mathscr{M}_x^{n+1})^\perp = (\mathscr{N}_x^{n+1} \cap B)^\perp$. By 3.2.2. (b) these two expressions are equal. This completes the proof of (a).

(b) If A is a finitely generated algebra, then of course each $\mathscr{M}_x$ is a finitely generated ideal so that $\mathscr{D}_x(A)$ is the connected component of A^0 containing $x \in G_A$. Since A is a split, A^0 is a composite coalgebra by 3.4.1 and A^0 is the direct sum of its connected components. Putting these results together gives (b). Q.E.D.

3.5. DEFINITION. A co-commutative bialgebra H is *composite* if the underlying coalgebra of H is composite and is *connected* if the underlying coalgebra structure of H is connected. A commutative bialgebra is *split* if the underlying algebra structure is split.

3.5.1. LEMMA. *Let C be a connected coalgebra and let B be an arbitrary algebra. If $v \in \mathrm{Hom}(C, B)$ and $v(1) = 0$ then $\epsilon - v$ is invertible in the convolution algebra $\mathrm{Hom}(C, B)$ and $(\epsilon - v)^{-1} = \sum_0^\infty v^n$. More generally, a linear map $u \in \mathrm{Hom}(C, B)$ is invertible if and only if $u(1)$ is an invertible element of B.*

Proof. Clearly

$$(*) \qquad \begin{aligned} \epsilon - v^{n+1} &= (\epsilon - v) * (\epsilon + v + \cdots + v^n) \\ &= (\epsilon + v + \cdots + v^n) * (\epsilon - v) \end{aligned}$$

It is easy to check that $v(C_0) = 0$ implies $v^{n+1}(C_n) = 0$. Since any $c \in C$ is contained in some C_n by 3.2.2. (e), $\sum v^n$ determines a map w in $\mathrm{Hom}(C, B)$. Moreover $\epsilon = (\epsilon - v) * w = w * (\epsilon - v)$ since this identity reduces to (*) on each C_n.

In particular it follows that if $w \in \mathrm{Hom}(C, B)$ is a linear map such that $w(1) = 1$ then w is invertible [if $v = \epsilon - w$ then $v(1) = 0$ so that $\epsilon - v = w$ is invertible.]

Now assume that $u \in \mathrm{Hom}(C, B)$ carries $1 \in C$ into an invertible element of B. Choose $u' \in \mathrm{Hom}(C, B)$ such that $u'(1) = u(1)^{-1}$. Then

$$(u' * u)(1) = 1 = (u * u')(1)$$

so that $u' * u$ and $u * u'$ are both invertible. It follows that u is invertible. Conversely if $u * v = v * u = \epsilon$ then $1 = \epsilon(1) = (u * v)(1) = u(1)\, v(1)$ and $v(1)\, u(1) = 1$ so that $u(1)$ is an invertible element of B. Q.E.D.

Let B_* denote the group of invertible elements of B.

3.5.2. PROPOSITION. *Let C be a composite coalgebra and let B be an arbitrary algebra. Then $u \in$ $\mathrm{Hom}(C, B)$ is invertible in the convolution algebra* $\mathrm{Hom}(C, B)$ *if and only if $u(G(C)) \subset B_*$.*

Proof. Since $C = \coprod_g C^g$ where C^g is the connected component of C containing $g \in G(C)$, it follows that $\mathrm{Hom}(C, B) = \prod_g \mathrm{Hom}(C^g, B)$. 3.5.2 then follows from 3.5.1 and the fact that an element in a product ring is invertible if and only if each component is invertible.

3.5.3. COROLLARY. *Let H be a composite bialgebra.*
The following conditions are equivalent:

(a) *H is a Hopf algebra.*

(b) *$G(H)$ is a subgroup of H_*.*

(c) *$G(H) \subset H_*$.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. Finally if $G(H) \subset H_*$ then $I \in \mathrm{Hom}(H, H)$ is invertible by the proposition so that (c) $\Rightarrow$ (a).

3.5.4. COROLLARY. (a) *A connected bialgebra is a Hopf algebra.*

(b) *Let $f : H \to H'$ be a* surjective *morphism of bialgebras. If H is a composite* *(resp. connected) Hopf algebra then H' is also a composite (resp. connected) Hopf algebra.*

(c) *A bi-ideal I in a composite Hopf algebra is a Hopf ideal: $s(I) \subset I$.*

(d) *If H and H' both composite (resp. connected) Hopf algebras then $H \otimes H'$ is also a composite (resp. connected) Hopf algebra.*

Proof. Parts (a), (b) and (d) follow immediately from 3.5.3, 3.1.3 and 3.2.4. Part (c) then follows from (b) and part (3) of Proposition 1.5.2.

3.5.5. Proposition. *Let C be a subcoalgebra of a bialgebra J and let H be the subalgebra of J generated by C.*

(a) *H is a subbialgebra of J which is cocommutative if C is cocommutative.*

(b) *If C is connected and $1 \in C$ then H is a connected Hopf algebra.*

(c) *If C is composite then H is a composite bialgebra and $G(H)$ is the sub-semigroup (with unit) of $G(J)$ generated by $G(C)$.*

(d) *If C is composite and $G(C) \subset H_*$, then H is a composite Hopf algebra.*

Proof. (a) Let $A = \{x \in H \mid \Delta x \in H \otimes H\}$ and $B = \{x \in H \mid \tau \circ \Delta x = \Delta x\}$. Then A and B are each subalgebras of H which contain C. Thus $A = H = B$. This proves (a).

(b) Let $C^1 = C$ and $C^i = p(C \otimes C^{i-1})$. Then C^i is connected by 3.2.5. (b) and induction. Clearly $C^1 \subset C^2 \subset \cdots$. Since $H = \sum C^i$ and $1 \in \bigcap C^i$ we conclude by 3.1.3. (a) that H is connected and so H is a Hopf algebra by 3.5.4. (a).

(c) By replacing C by $k \cdot 1 + C$ if necessary we may assume $1 \in C$. Let $G(C)^n = \{g_1 g_2 \cdots g_n \mid g_i \in G(C)\}$. Then by 3.2.5. (b) and induction C^i is split. By 3.2.4. (b) and 3.1.3. (d) $G(C^i) = G(C)^i$. Since $H = \sum C^i = \bigcup C^i$, H is co-split and $G(H) = \bigcup G(C)^i$ by 3.1.2. (a).

(d) If $G(C) \subset H_*$ then by the results of part (c), $G(H) \subset H_*$ so that the result stated then follows from 3.5.3.

3.5.6. Corollary. *If H is the sum of all the*

composite

cocommutative

composite and cocommutative

connected and containing 1

cocommutative, connected and containing 1 *subcoalgebras of J then H is the unique maximal*

composite
cocommutative
composite and cocommutative
connected
cocommutative and connected

subcoalgebra of J. By the proposition H is a sub-bialgebra of J.

3.5.7. PROPOSITION. *Let J be a bialgebra. Then J has a unique maximal (co-commutative) composite sub Hopf algebra H. H is the sum of all the (co-commutative) composite subcoalgebras C such that $G(C) \subset J_*$.*

Proof. Let $G_0 = G(J) \cap J_*$. Then $x \in G_0$ implies $x^{-1} \in G_0$ so that G_0 is a subgroup of J_*. For if $xy = yx = 1$, then

$$(x \otimes x)(y \otimes y) = 1 \otimes 1 = \Delta(xy) = (x \otimes x)\, \Delta y$$

so that $\Delta y = y \otimes y$ is the inverse of $x \otimes x \in J \otimes J$ and $y \in G_0$.

Let $\mathscr{F}$ be the family of all (co-commutative) composite subcoalgebras C of J and such that $G(C) \subset G_0$ and let $H = \sum C$ where $C \in \mathscr{F}$. Then H is a (co-commutative) composite subcoalgebra of J and $G(H) \subset G_0$ by 3.1.2. (a) so that $H \in \mathscr{F}$. Since it is clear that $G_0 \subset G(H)$ we have $G_0 = G(H)$. If $\bar{H}$ is the algebra generated by H, then $\bar{H}$ is a (co-commutative) composite bialgebra and $G(\bar{H}) = G_0$ by 3.5.5. By the maximality of H, we conclude $H = \bar{H}$ so that H is a bialgebra. In fact H is a Hopf algebra by 3.5.3. since $G(H) = G_0$ is a group. H obviously has the stated maximality property. Q.E.D.

Let H be a bialgebra. If $x \in G(H)$, let H^x denote the connected component of H containing x and let $H_n{}^x$ denote the nth term in the filtration of the connected coalgebra H^x. The following theorem for composite co-commutative Hopf algebras is due to B. Kostant (unpublished).

3.5.8. THEOREM. *Let H be a bialgebra.*

(a) *If $x, y \in G(H)$ then $H_n{}^x \cdot H_m{}^y \subset H^{xy}_{n+m}$. In particular H^1 is a co-connected sub-Hopf algebra of H.*

(b) *If H is a Hopf algebra then $s(H_n{}^x) \subset H_n^{s(x)}$ where $s(x) = x^{-1} \in G(H)$. Moreover $x \cdot H_n{}^1 = H_n{}^1 \cdot x = H_n{}^x$ and H^1 is a sub-Hopf algebra of H which is a $k[G]$ module algebra under the adjoint representation. The natural algebra morphism*

$$H^1 \# k[G] \to H$$

which carries $\partial \otimes x \mapsto \partial \cdot x$ is injective.

(c) *If H is a composite Hopf algebra, the algebra morphism described in* (b) *is an isomorphism.*

Proof. (a) Since H^x and H^y are connected subcoalgebras of H it follows from 3.2.5. (b) that $H^x \cdot H^y$ is a connected subcoalgebra of H (which contains $x \cdot y$) and so $H^x \cdot H^y$ is contained in H^{xy}. By 3.2.5. (b) we have

$$H_n{}^x \cdot H_m{}^y \subset (H^x \cdot H^y)_{n+m} .$$

Thus $H_n{}^x \cdot H_m{}^y \subset H_{n+m}^{xy}$. This proves (a). Note in particular that (since $x \in H_0{}^x$) we have $x \cdot H_m{}^y \subset H_m^{xy}$.

(b) Let $K = H^{\mathrm{op}}$ be the coalgebra (Hopf algebra) obtained from H by twisting the co-product $[\Delta_K = \tau \circ \Delta_H]$. Then $s : H \to K$ is a coalgebra map by part (6) of 1.5.2. We then have $s(H_n{}^x) \subset K_n^{s(x)}$ by 3.2.1. (c). The first containment then follows from 3.2.2. (f).
If $x \in G(H)$, we have by part (a):

$$H_n{}^1 = x^{-1} \cdot xH_n{}^1 \subset x^{-1}H_n{}^x \subset H_n{}^1$$

so that $x \cdot H_n{}^1 = H_n{}^x$. For each $x \in G$, the algebra morphism $H^1 \# k[G] \to H$ carries $H^1 \otimes x$ onto $H^1x = H^x$. The family of subspaces $\{H^x \mid x \in G(H)\}$ are among the family of irreducible components of H and these form a direct sum (coalgebra) decomposition of H [3.1.2.]. Thus the map $H^1 \# k[G] \to H$ is injective.

(c) If H is composite then $H = \coprod_x H^x$. Since the map $H^1 \# k[G] \to H$ sends $H^1 \otimes x$ onto $H^1 \cdot x = H^x$, the map is surjective. Q.E.D.

Remark. If H is a Hopf algebra and if $H^1 \# k[G]$ is given the tensor product coalgebra structure, then $H^1 \# k[G]$ is a Hopf algebra and the map described in (b) is a morphism of Hopf-algebras.

Let $\mathscr{A}$ be a commutative bialgebra and let $G_{\mathscr{A}} = \mathrm{Alg}(\mathscr{A}, k) = G(\mathscr{A}^0)$; if $x \in G_{\mathscr{A}}$, let $\mathscr{D}_x{}^n(\mathscr{A}) = \mathscr{D}_x{}^n(\mathscr{A}, k)$ be the differential operators defined in §2.4. Applying 2.4.1 in the case where $B = k$ we have the containment

$$\mathscr{D}_x{}^n(\mathscr{A}) \cdot \mathscr{D}_y{}^m(\mathscr{A}) \subset \mathscr{D}_{xy}^{n+m}(\mathscr{A}) \qquad \text{where} \qquad x, y \in G_{\mathscr{A}} = \mathrm{Alg}(\mathscr{A}, k)$$

3.5.9. THEOREM. *Assume the augmentation ideal of the commutative bialgebra $\mathscr{A}$ is finitely generated.*

(a) *If $x \in G_{\mathscr{A}}$ is invertible then $\mathscr{D}_x{}^n(\mathscr{A}) = x \cdot \mathscr{D}_\epsilon{}^n(\mathscr{A}) = \mathscr{D}_\epsilon{}^n(\mathscr{A}) \cdot x$ and $\mathscr{D}_x(\mathscr{A})$ is a connected subcoalgebra of $\mathscr{A}^0$; in fact it is the connected component of $\mathscr{A}^0$ which contains x. Moreover $\mathscr{D}_x(\mathscr{A})_n = \mathscr{D}_x{}^n(\mathscr{A})$ so that the filtration on* *the connected coalgebra $\mathscr{D}_x(\mathscr{A})$ defined in §3.2 coincides with the one defined in §2 for differential operators. In particular $\mathscr{D}_\epsilon(\mathscr{A})$ is a connected sub-bialgebra of $\mathscr{A}^0$.*

(b) *If $\mathscr{A}$ is a Hopf algebra then $G_{\mathscr{A}}$ is a group so the results described in* (a) *hold for each $x \in G_{\mathscr{A}}$. Moreover $s(\mathscr{D}_x{}^n(\mathscr{A})) = \mathscr{D}^n_{s(x)}(\mathscr{A})$ for each $x \in G$ (where $s(x) = x^{-1}$). $\mathscr{D}_\epsilon(\mathscr{A})$ is a $k[G_{\mathscr{A}}]$ module-algebra under the adjoint action and the natural algebra morphism $\mathscr{D}_\epsilon(\mathscr{A}) \#_{\mathrm{ad}} k[G_{\mathscr{A}}] \to \mathscr{A}^0$ which sends $\partial \otimes x \to \partial \cdot x$ is injective.*

(c) *If $\mathscr{A}$ is a split Hopf algebra then $\mathscr{A}^0 = \coprod_x \mathscr{D}_x(\mathscr{A})$ is the decomposition of the coalgebra $\mathscr{A}^0$ into its connected components; and the algebra morphism from $\mathscr{D}_\epsilon(\mathscr{A}) \# k[G] \to \mathscr{A}^0$ is an isomorphism.*

Proof. (a) By 3.4.2. (a) we know that $\mathscr{D}_\epsilon(\mathscr{A})$ is the connected component of $\mathscr{A}^0$ containing ϵ and also $\mathscr{D}_\epsilon{}^n(\mathscr{A}) = [\mathscr{D}_\epsilon(\mathscr{A})]_n$. If $x \in G_{\mathscr{A}}$ is invertible we have

$$\mathscr{D}_\epsilon{}^n(\mathscr{A}) = x^{-1} \cdot x \cdot \mathscr{D}_\epsilon{}^n(\mathscr{A}) \subset x^{-1} \cdot \mathscr{D}_x{}^n(\mathscr{A}) \subset \mathscr{D}_\epsilon{}^n(\mathscr{A})$$

so that

$$\mathscr{D}_x{}^n(\mathscr{A}) = x \cdot \mathscr{D}_\epsilon{}^n(\mathscr{A})$$

Since $x \in G_{\mathscr{A}}$ is invertible the map $\partial \mapsto x \cdot \partial$ from $\mathscr{A}^0 \to \mathscr{A}^0$ is a coalgebra *automorphism* which we have just observed carries $\mathscr{D}^n(\mathscr{A})$ onto $\mathscr{D}_\epsilon{}^n(\mathscr{A})$. This shows that $\mathscr{D}_x(\mathscr{A})$ is the connected component of $\mathscr{A}^0$ containing x and also that $[\mathscr{D}_x(\mathscr{A})]_n = \mathscr{D}_x{}^n(\mathscr{A})$. This completes the proof of (a).

(b) and (c). Since $\mathscr{A}$ is a commutative Hopf algebra, $\mathscr{A}^0$ is a co-commutative Hopf algebra. By 3.4.1. (c), $G(\mathscr{A}^0) = G_{\mathscr{A}}$ and $\mathscr{A}^0$ is a composite Hopf algebra by 3.4.1. (a) if $\mathscr{A}$ is a split Hopf algebra. In view of the identification established in (a) between the connected components of $\mathscr{A}^0$ and the family $\{\mathscr{D}_x(\mathscr{A}) \mid x \in G_{\mathscr{A}}\}$, the results asserted in (b) and (c) then follow at once from 3.5.6. Q.E.D.

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