Toric Geometry:
Theorems and Definitions

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1 Dictionary

Toric geometry is concerned with the construction of varieties and schemes given by specifying semigroups and fans and other combinatorial objects. It is therefore useful to fix certain symbols.

- N: We define $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ and note that $N \cong \mathbb{Z}^n$.
- M : We define M to be the dual lattice of N, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$.
- $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$: We define $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}^n$.

2 What makes a toric variety?

2.1 Tori

2.2 Toric Varieties

2.3 Cones and Fans

Throughout this section, let $T \cong \left(\mathbb{C}^*\right)^n$ and $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$. Note that N is the collection of 1-parameter subgroups of T, or the set of cocharacters if you prefer that terminology. In addition, every variety is an integral separated scheme of finite type over $\mathrm{Spec}\,\mathbb{C}$ unless otherwise specified.

Definition 2.1. A *rational polyhedral cone* σ in N is a set $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ given by the positive span of some finite subset of $N_{\mathbb{R}}$, i.e. a set

$$\sigma = \mathrm{cone}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{k} c_i v_i \;\middle|\; c_i \in \mathbb{R}_{\geq 0} \right\}.$$

By rescaling the cone basis set, we may assume $v_i \in N$ for each $1 \leq i \leq k$, and from now on will do so.

Definition 2.2. Let $\sigma = \mathrm{cone}\{v_1, \ldots, v_k\}$ be a rational polyhedral cone. The *span* of σ is the smallest vector subspace V containing σ. We have that

$$V = \sigma + (-\sigma) = \{v_1, \ldots, v_k\} = \{\sigma\}.$$

The *dimension* of σ is the dimension of the span of σ. We say that σ is *full-dimensional* if $\dim\sigma = \dim N_{\mathbb{R}} = n$.

Definition 2.3. A rational polyhedral cone is said to be *strictly convex* if it doesn't contain a line, i.e. if it doesn't contain a one dimensional affine subspace of $N_{\mathbb{R}}$.

Unless otherwise specified, by "cone" we mean "strictly convex rational polyhedral cone".

Definition 2.4. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, the *dual cone* $\sigma\vee \subseteq M_{\mathbb{R}}$ is defined

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, v\rangle \geq 0,\ \forall v \in \sigma\}.$$

The pairing $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is simply the evaluation map $\langle m, u\rangle = m(u)$.

We further define the double dual $(\sigma^{\vee})^{\vee}$ by

$$(\sigma^{\vee})^{\vee} = \left\{ v \in N_{\mathbb{R}} \;\middle|\; \langle m, v\rangle \geq 0,\ \forall m \in \sigma^{\vee} \right\}$$

The following are fundamental facts regarding σ and $\sigma^{\vee}$.

Proposition 2.5. Let σ be a cone in N and $\sigma^{\vee}$ be its dual.

(a) $\sigma^{\vee}$ is a rational polyhedral cone in M (not necessarily strictly convex)

(b) $(\sigma^{\vee})^{\vee} = \sigma$

(c) σ is full-dimensional if and only if $\sigma^{\vee}$ is strictly convex

Definition 2.6. A *fan* Σ in N is a collection of cones in N such that

(i) if $\sigma \in \Sigma$ then every face of σ belongs to Σ

(ii) if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

We wish to construct varieties from cones and fans. Starting with a cone σ in N, we will associate to it an affine variety $X_\sigma = \operatorname{Spec} R_\sigma$. Given a fan Σ, we will construct a variety X_Σ by gluing together X_{σ_1} and X_{σ_2} along $X_{\sigma_1 \cap \sigma_2}$.

We focus first on building a variety X_σ from a cone σ in N.

- Start with a