Algebraic Topology Homework 9

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Last compiled November 3, 2022

§ *Problems from 2.1*

EXERCISE 22. Prove by induction on dimension the following facts about the homology of finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$

(c) If X has k n-cells, then $H_n(X)$ is generated by at most k-elements.

Proof: **(a)** First notice that we have the following series of isomorphisms:

$$H_i(X^k, X^{k-1}) \cong H_i(X^k/X^{k-1}) \cong H_i\left(\bigvee_\alpha S^k\right) \cong \bigoplus H_i(S^k) = \begin{cases} \bigoplus_\alpha \mathbb{Z} & i = k \\ 0 & \text{else} \end{cases}$$

where i and k are nonnegative integers and α indexes the k-cells of X. The first isomorphism is given by Proposition 2.22 (a special case of Theorem 2.13 when (X,A) is a good pair), the second isomorphism follows from the homeomorphism $X^k/X^{k-1} \cong \bigvee_\alpha S^k$ and the third is implied by Corollary 2.25. Now consider the following exact sequence

$$\ldots \longrightarrow H_i(X^{n-1}) \longrightarrow H_i(X^n) \longrightarrow H_i(X^n, X^{n-1}) \longrightarrow \ldots$$

induced by the inclusion $X^{n-1} \hookrightarrow X^n$. Since X is dimension n, $X = X^n$, and by the above isomorphisms, $H_i(X^n, X^{n-1}) \cong 0$ when $i > n$. We therefore appeal to induction: if $H^{n-1} = 0$, then by the exactness of the above sequence, we have $H_i(X^n) \cong H_i(X^n - 1) \cong 0$. For the base case, suppose X were a 0-dimensional CW-complex. Then X would be a disjoint union of points, and from previous results $H_i(X) = 0$ for all $i > 0$. This proves the first portion of part (a).

For the second part, we return to the exact sequence given by the inclusion $X^{n-1} \hookrightarrow X^n$. When $i = n$, we get that $H_n(X^{n-1}) = 0$ and so

$$\ldots \longrightarrow 0 \longrightarrow H_n(X) \longrightarrow \bigoplus_\alpha \mathbb{Z} \longrightarrow \ldots$$

is the relevant portion of the sequence. By exactness, the map $H_n(X) \longrightarrow \bigoplus_\alpha \mathbb{Z}$ must be injective, so $H_n(X)$ is isomorphic as an abelian group to its image in $\bigoplus_\alpha \mathbb{Z}$. Since all subgroups of a free abelian group are themselves free abelian, $H_n(X)$ is free abelian.

(b) Now let n be some fixed integer, not necessarily the dimension of X, and suppose X has no $(n-1)$-cells nor any $(n+1)$-cells. This means that the $(n-1)$-skeleton is equal to the $(n-2)$-skeleton, i.e. $X^{n-1} = X^{n-2}$ By part (a), it must then be the case that $H_n(X^{n-1}) \cong H_{n-1}(X^{n-1}) \cong 0$, and so the exact sequence

$$\ldots \longrightarrow H_n(X^{n-1}) \longrightarrow H_n(X) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow \ldots$$

can actually be written

$$... \longrightarrow 0 \longrightarrow H_n(X) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow 0 \longrightarrow ...$$

which implies that $H_n(X) \cong H_n(X, X^{n-1})$ by exactness.

As we saw in part (a), $H_n(X^n, X^{n-1})$ is the free group generated by n-cells, so if we can show $H_n(X, X^{n-1}) \cong H_n(X^n, X^{n-1})$ we'll be done. For this, we'll make use of the short exact sequence

$$0 \longrightarrow C_n(X^n, X^{n-1}) \longrightarrow C_n(X, X^{n-1}) \longrightarrow C_n(X, X^n) \longrightarrow 0$$

to obtain the following long exact sequence on homology:

$$... \longrightarrow H_{n+1}(X, X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_n(X, X^{n-1}) \longrightarrow H_n(X, X^n) \longrightarrow ...$$

By part (a), we know that $H_i(X^n, X^{n-1}) = 0$ whenever $i \neq n$. Furthermore, because there are no $(n+1)$-cells, $X^{n+1} = X^n$ and hence $H_{n+1}(X, X^n) = 0$. To see that $H_n(X, X^n) \cong 0$, consider the long exact sequence of the pair (X, X^n):

$$... \longrightarrow H_n(X^n) \xrightarrow{\alpha} H_n(X) \longrightarrow H_n(X) \longrightarrow H_n(X, X^n) \longrightarrow H_{n-1}(X_n) \xrightarrow{\beta} H_{n-1}(X) \longrightarrow ...$$

I claim that α is injective, β is surjective and hence by problem 2.1.15 $H_n(X, X^n) \cong 0$. Indeed, the exactness of

$$... \longrightarrow H_{i+1}(X^n, X^{n-1}) \longrightarrow H_i(X^{n-1}) \xrightarrow{\alpha} H_i(X^n) \xrightarrow{\beta} H_i(X^n, X^{n-1}) \longrightarrow ...$$

implies that α is surjective whenever $i > i - 1$ and β is injective whenever $n > i - 1$.

Finally, since both $H_n(X, X^{n-1})$ and $H_n(X^n, X^{n-1})$ are trivial, $H_n(X) \cong H_n(X, X^{n-1})$ which implies $H_n(X)$ is freely generated by n-cells.

(b) The injections β and surjections α imply that $H_n(X) \cong H_n(X^k)$ whenever $k > n$, and in particular $H_n(X) \cong H_n(X^{n+1})$. The long exact sequence

$$... \longrightarrow H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) \longrightarrow ...$$

arising from the pair (X^{n+1}, X^n) is then actually

$$... \longrightarrow H_{n+1}(X^{n+1}, X^n) \longrightarrow H_n(X^n) \longrightarrow H_n(X) \longrightarrow 0 \longrightarrow ...$$

from the isomorphism $H_n(X) \cong H_n(X^{n+1})$ above and $H_n(X^{n+1}, X^n) \cong 0$ from the beginning of part (a). Thus, $H_n(X^n)$ surjects onto $H_n(X)$. However, in part (a) we saw that $H_n(X^n)$ was generated by the n-cells of X, hence the number of n-cells of $H_n(X)$ puts an upper bound on the minimal number of generators needed to generate $H_n(X)$. $\square$

§ ***Problems from 2.2***

EXERCISE 5. Show that any two reflections of S^n across different n-dimensional hyperplanes are homotopic, in fact homotopic through reflections. [The linear algebra formula for a reflection in terms of inner products may be helpful.]

Proof: The linear algebra formula Hatcher alludes to is reflection in the direction of u given by $f_u(x) = x - 2u \cdot \frac{x \cdot u}{u^2}$. It is a map on $\mathbb{R}^{n+1}$ which reflects a point x across the hyper plane through the origin whose normal vector is $0 \neq u \in \mathbb{R}^{n+1}$. Notice that for any vector $0 \neq u$, the reflection f_u in the direction of u

(1) negates u

$$f_u(u) = u - 2u\frac{u \cdot u}{u^2} = u - 2u = -u,$$

(2) fixes the hyper plane $x \cdot u = 0$

$$x \cdot u = 0 \implies f_u(x) = x - 2u\frac{x \cdot u}{u^2} = x - 0 = x,$$

(3) is an involution

$$\begin{aligned}
f_u\left(f_u(x)\right) &= \left(x - 2u\frac{x \cdot u}{u^2}\right) - 2u\frac{\left(x - 2u\frac{x \cdot u}{u^2}\right) \cdot u}{u^2} \\
&= x - 2u\frac{x \cdot u}{u^2} - 2u\frac{x \cdot u}{u^2} + 2u\frac{2u^2\frac{x \cdot u}{u^2}}{u^2} \\
&= x - 4u\frac{x \cdot u}{u^2} + 4\frac{x \cdot u}{u^2} \\
&= x
\end{aligned}$$

(4) and is a norm-preserving isometry (is “norm preserving” redundant?)

$$\begin{aligned}
\|f_u(x)\|^2 &= \left(x - 2u\frac{x \cdot u}{u^2}\right) \cdot \left(x - 2u\frac{x \cdot u}{u^2}\right) \\
&= x^2 - 4x \cdot u\frac{x \cdot u}{u^2} + \frac{4(x \cdot u)^2}{u^2} \\
&= x^2 = \|x\|^2,
\end{aligned}$$

which should be enough to convince us that this is indeed a reflection. Because f_u is norm preserving, it is a continuous map which maps sends S^n to S^n, and for notational convenience we will redefine f_u to be the restriction $f_u|_{S^n} : S^n \longrightarrow S^n$.

Consider some other vector $0 \neq v \in \mathbb{R}^{n+1}$ and suppose that the line between v and u does not contain the origin. Let $\gamma : I \longrightarrow \mathbb{R}^{n+1}$ be the linear interpolation from u to v, i.e. the map $\gamma(t) = u \cdot t - (1-t) \cdot v$. Then the map $F : S^n \times I \longrightarrow S^n$ defined $F_t(x) = f_{\gamma(t)}(x)$ is continuous and satisfies $F_0(x) = f_v(x)$ and $F_1(x) = f_u(x)$; hence, it is a homotopy between f_u and f_v comprised itself entirely of reflection maps.

If the line between v and u does contain the origin, then choose some other nonzero point $w \in \mathbb{R}^{n+1}$ which is not on the linear subspace spanned by u and v. By what we have already shown, $f_u \simeq f_w$ and $f_v \simeq f_w$, and since homotopy equivalence is an equivalence relation, $f_u \simeq f_v$. ◻

EXERCISE 7. For an invertible linear transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ is id or $-\mathrm{id}$ according to whether the determinant of f is positive or negative. [Use Gaussian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with $\pm_1$'s on the diagonal.]

Proof: Let us first fix a basis for $\mathbb{R}^n$ so that we can talk about the matrix $A \in \mathrm{GL}_n(\mathbb{R})$ which gives our linear map f. We'll write A instead of f, and we note here at the beginning that the restriction of A to $\mathbb{R}^n \setminus \{0\}$ maps surjectively into $\mathbb{R}^n \setminus \{0\}$. This means A induces a map $A_* : H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ on reduced homology groups, and because it is an automorphism of $\mathbb{R}^n$, it is an automorphism of homology (its inverse on $\mathbb{R}^n$ induces its inverse on homology).

Let $r : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$ be the retraction $r(x) = \frac{x}{\|x\|}$ of the punctured plane to S^{n-1} and let $\iota : S^{n-1} \longrightarrow \mathbb{R}^n \setminus \{0\}$ be the inclusion map. The punctured plane deformation retracts to S^{n-1} and r is a homotopy equivalence, that is $r \circ \iota = \mathrm{id}_{S^{n-1}}$ and $\iota \circ r \simeq \mathrm{id}_{R^n \setminus \{0\}}$. Functoriality implies that the induced maps $r_* \circ \iota_*$ and $\iota_* \circ r_*$ are the identity on homology, implying that both r_* and ι_* are isomorphisms.

The naturality of the long exact sequence diagram for reduced homology groups applied to the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ implies that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & & \tilde{H}_{n-1}(S^{n-1}) & & \\
 & & & & \downarrow \iota_* & & \\
0 = H_n(\mathbb{R}^n) & \longrightarrow & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\partial} & \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \longrightarrow & \tilde{H}_{n-1}(\mathbb{R}^n) = 0 \\
 & & \downarrow A_* & & \downarrow A_* & & \\
0 = H_n(\mathbb{R}^n) & \longrightarrow & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\partial} & \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \longrightarrow & \tilde{H}_{n-1}(\mathbb{R}^n) = 0 \\
 & & & & \downarrow r_* & & \\
 & & & & \tilde{H}_{n-1}(S^{n-1}) & &
\end{array}$$

All of these homology groups are isomorphic to $\mathbb{Z}$. Furthermore, ι_*, A_* and r_* are all isomorphisms as previously discussed, and ∂ is itself an isomorphism by exactness. Let us now examine $\mathrm{GL}_n(\mathbb{R})$ as a topological space.

Claim: $\mathrm{GL}_n(\mathbb{R})$ has exactly two path components: the set of matrices with positive determinant and the set of matrices with negative determinant.

Assuming this claim, if $\det(A) > 0$ then there is a path $\gamma : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ such that $\gamma(0) = A$ and $\gamma(1) = B^+$, where $B^+ = I$ is the identity matrix; and if $\det(A) < 0$ then there is a path γ such that $\gamma(0) = A$ and $\gamma(1) = B^-$ where B^- is the diagonal matrix with -1 in the first entry and 1 everywhere else. In either case, the map $F : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$ given by $F_t(x) = \gamma(t)x$ is a homotopy from the map A to the map $B^{\pm}$. This means A and $B^{\mathrm{sgn}(\det(A))}$ induce the same map on horology. The map $B^+ = I$ is the identity on $\mathbb{R}^n$ and is hence the identity on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \mathbb{Z}$, or in other words, it is multiplication by 1. Now consider B^-. This maps a vector $(x_1, ..., x_n)$ to $(-x_1, x_2, ..., x_n)$, is a reflection in the 1st coordinate and thus induces the map $n \mapsto -n$ on $H_n(\mathbb{R}^n, \mathbb{R} \setminus \{0\})$.

END OF MAIN PROOF

Proof of claim: We will thus first show that each elementary matrix is path connected to either B^+ or B^- and then generalize the result to an arbitrary $A \in \mathrm{GL}_n(\mathbb{R})$. There are three cases to consider.

Case 1. Suppose E is an elementary matrix corresponding to the row operation "multiply row i by constant c". Let M_t be the matrix identical to E except in the i^{th} diagonal entry, which we set to $[M_t]_{ii} = (1-t)c + t$. Then $\gamma : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ defined $\gamma(t) = M_t$ is a path from E to B^+. Likewise, if $c < 0$, then setting $[M_t]_{ii} = (1-t)c - t$ yields a path from E to B^-.

Case 2. Now suppose that E is an elementary matrix corresponding to swapping row i with row j, that is,

$$E_{ij} = 1,\ E_{ji} = 1,\ E_{ii} = 0,\ E_{jj} = 0$$

and E matches the identity matrix in all other entries. The determinant of this matrix is -1, so we seek to connect it to B^- with a path. Let M_t be the matrix equal to I in all entries except for the four above, which we define

$$[M_t]_{ij} = \cos\left(\frac{\pi}{2}t\right),\ [M_t]_{ji} = \cos\left(\frac{\pi}{2}t\right),\ [M_t]_{ii} = \sin\left(\frac{\pi}{2}t\right),\ [M_t]_{jj} = -\sin\left(\frac{\pi}{2}t\right).$$

Then $\det(M_t) = -\left(\cos^2\left(\frac{\pi}{2}t\right) + \sin^2\left(\frac{\pi}{2}t\right)\right) = -1$ for every $t \in [0,1]$. Furthermore, $\gamma : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ defined $\gamma(t) = M_t$ is continuous and satisfies $\gamma(0) = E$ and $\gamma(1) = E'$, where E' is the elementary matrix given by scaling row j by -1. By the previous case, E' is connected to B^- via some continuous path, so the composition of this path with γ yields a path from E to B^-.

Case 3. Finally, suppose E is the elementary matrix given by adding the multiple of row i by constant $c \in \mathbb{R}$ to row j. Then E is equal to the identity matrix $I = B^+$ in all entries except for the $(ji)^{\text{th}}$, and instead $E_{ji} = c$. Set M_t to be the identity matrix in all entries except the $(ji)^{\text{th}}$ as well, where $[M_t]_{ji} = (1-t)c$. Then $\gamma : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ defined M_t is a path from E to $I = B^+$.

Now consider a general invertible matrix $A \in \mathrm{GL}_n(\mathbb{R})$. If $\det(A) > 0$, then there is a sequence of elementary matrices $E_1, ..., E_n$ such that $B^+ = E_n \cdot ... \cdot E_1 A$. Let $B_i = B^{\mathrm{sgn}(\det(E_i))}$, that is, either B^+ or B^- depending on whether E_i has positive or negative determinant. Further let $\gamma_i : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ be the path connecting B_i to E_i so that $\gamma_i(0) = B_i$ and $\gamma_i(1) = E_i$. Then $\gamma : [0,1] \longrightarrow \mathrm{GL}_n(\mathbb{R})$ defined

$$\gamma(t) = \gamma_n(t) \cdot ... \cdot \gamma_1(t) A$$

is a path from $B_n ... B_1 A = (-1)^k A$ to $E_n ... E_1 A = B^+$, where k is equal to the number of occurrences of B^- in the list $B_1, ..., B_n$. Since $\det(B^+) = \det(E_n) \cdot ... \cdot \det(E_1)\det(A) > 0$ and $\det(A) > 0$, k must be even and hence γ is a path from A to B^+.

Following the same setup but with $\det(A) < 0$ gives us a path γ from $B_n ... B_1 A$ to B^+, as before. However, since $\det(B^+) = 1 > 0$ and $\det(A) < 0$, k must be odd and hence $B_n ... B_1 A = B^- A$, or in other words, A with the first diagonal entry replaced with its additive inverse. We can fix this by multiplying γ by B^-: the path $\lambda(t) = B^-\gamma(t)$ satisfies $\lambda(0) = B^-(B^-A) = A$ to $\lambda(1) = B^-B^+ = B^-$. $\square$

EXERCISE 8. A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \longrightarrow \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \longrightarrow S^2$. Show that the degree of $\hat{f}$ equals the degree of

f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Proof: Recall that any polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ can be written $f(z) = c(z - a_1)^{d_i} \cdot ... \cdot (z - a_k)^{d_k}$ where $a_1, ..., a_k \in \mathbb{C}$ are the roots of f and $d_1 + ... + d_k = \deg f$. Let $\hat{f}$ denote the map induced on S^2 by f, and notice that by Proposition 2.30, the degree of $\hat{f}_*$ is equal to the sum of $\deg \hat{f}_*|a_i$, that is, the total degree of $\hat{f}_*$ can be found by summing the degrees of $\hat{f}_*$ restricted to small neighborhoods of its zeros. Note also that the extension of f to the Riemann sphere is given by sending $\infty \mapsto \infty$, i.e. on any open set U of the Riemann sphere *not* containing ∞, $\hat{f}|_U = f|_U$.

For each root a_i we may find an open disk U_i containing a_i which doesn't contain ∞ or any other roots of f. Because f is a polynomial, it is holomorphic and hence has a convergent Taylor series in U_i, provided we choose U_i to be small enough. This means that for $z \in U_i$,

$$\hat{f}(z) = (z - a_i)^{d_i}(c_0 + c_1(z - a_i) + c_2(z - a_i)^2 + ...).$$

Let V be a neighborhood of 0 such that $\hat{f}(U_i) = f(U_i) \subseteq V$ for all $1 \leq i \leq k$. Near a root a_i, $c_0(z - a_i)^{d_i}$ is a good approximation for $\hat{f}(z)$, i.e. for a sufficiently small circle $S_r(a_i) \subseteq \mathbb{C}$ of radius r centered at a_i parameterized by $\gamma : [0, 1] \longrightarrow S_r(a_i)$, $\hat{f} \circ \gamma$ will wrap around the circle $S_{c_0 r}(0)$ d_i times, potentially with some perturbation/wobble. This means the local degree of $\hat{f}_*$ at a_i is d_i, since $\hat{f}$ and f are equal when restricted to U_i. Applying Proposition 2.30, we see that

$$\deg \hat{f}_* = \sum_{i=1}^{k} \deg \hat{f}_*|_{U_i} = d_1 + ... + d_k = \deg f,$$

where $\deg f$ is the degree of f as a polynomial. This gives us good reason to call the degree of a map $S^n \longrightarrow S^n$ "degree" in the first place.

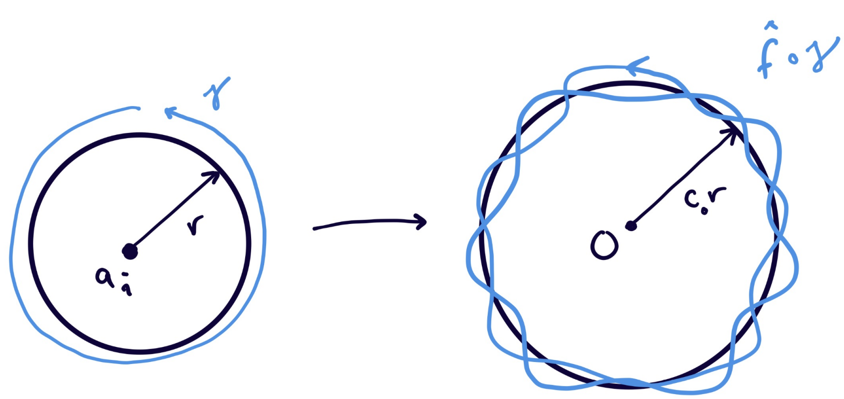

Figure 1: The map $\hat{f} \circ \gamma$ is $c_0(z - a_i)^{d_i}$ plus some wobble.

$\square$

Exercise 12. Show that the quotient map $S^1 \times S^1 \longrightarrow S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \longrightarrow S^1 \vee S^1$ is null homo topic.

Proof: We first note that $(S_1 \times S^1, S^1 \vee S^1)$ is a good pair. Indeed, we can take a tubular neighborhood of $S^1 \vee S^1$ inside $S^1 \times S^1$ which deformation retracts to $S^1 \vee S^1$, as illustrated in Figure 2.

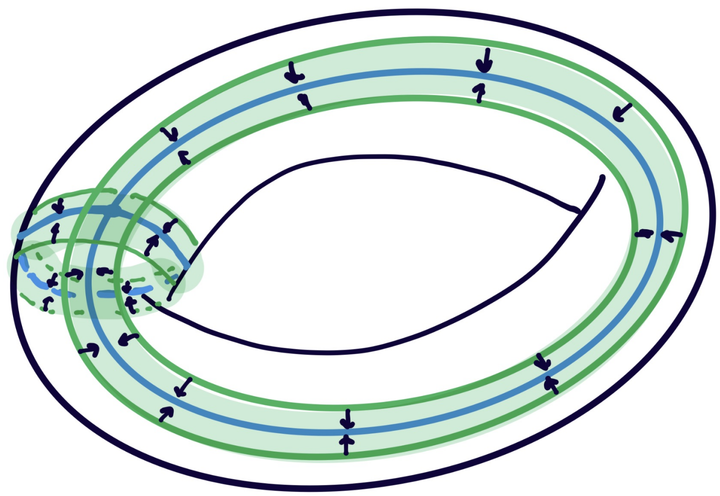

Figure 2: Tubular neighborhood deformation retracts onto $S^1 \vee S^1$.

Theorem 2.13 from Hitcher then implies that we have a long exact sequence

$$\ldots \longrightarrow H_2(S^1 \vee S^1) \longrightarrow H_2(S^1 \times S^1) \xrightarrow{p_*} H_2(S^2) \xrightarrow{\partial} H_1(S^1 \vee S^1) \longrightarrow H_2(S^1 \times S^1) \xrightarrow{p_*} \ldots$$

where $p : S^1 \times S^1 \longrightarrow S^1 \times S^1 / S^1 \vee S^1 \cong S^2$ is the quotient map in question. As is standard practice by now, let's fill in this sequence with friendlier looking symbols for our groups and consider the relevant maps more closely:

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{p_*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^2 \xrightarrow{i} \mathbb{Z}^2 \xrightarrow{p_*},$$

where i is the map induced by the inclusion $S^1 \vee S^1 \hookrightarrow S^1 \times S^1$. This is a cellular map which takes the two 1-cells of $S^1 \vee S^1$ to 1-cells in $S^1 \times S^1$; hence it is injective. This implies that $\operatorname{img} \partial = \ker i = 0$, and thus that $\operatorname{img} p_* = \ker \partial = \mathbb{Z}$. Since exactness on the left side of p_* implies that p_* is injective, we conclude that it is an isomorphism and hence that p is not nullhomotopic.

Now let $p : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ be the universal cover of $S^1 \times S^1$. Any map $f : S^2 \longrightarrow S^1 \times S^1$ satisfies $f_*(\pi_1(S^2, s_0)) = p_*(\pi_1(\mathbb{R}^2, x_0))$ since both $\pi_1(S^2, s_0)$ and $\pi_1(\mathbb{R}^2, x_0)$ are trivial (here we implicitly assume $f(s_0) = p(x_0)$, as basepoints need to be compatible). By the covering space lifting property (I remember the theorem but am not sure about the name) we have a lift $g : S^2 \longrightarrow \mathbb{R}^2$ of f, but g is nullhomotopic since $\mathbb{R}^2$ is contractible, so f must be as well. $\square$

§ ***Problems from 2.B***

EXERCISE 1. Compute $H_i(S^n - X)$ when X is a subspace of S^n homeomorphic to $S^k \vee S^\ell$ or to $S^k \sqcup S^\ell$.

Proof: Suppose first that $X \cong S^k \sqcup S^\ell$. Denote the two connected components of X inside S^n by $X_1 \cong S^k$

and $X_2 \cong S^\ell$. I hope that I can use the results in section 2B of Hatcher, for proposition 2B.1.b conveniently tells us that

$$\tilde{H}_i(S^n - X_1) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{else} \end{cases}, \quad \tilde{H}_i(S^n - X_2) \cong \begin{cases} \mathbb{Z} & i = n-\ell-1 \\ 0 & \text{else} \end{cases}.$$

Let A be an open set which deformation retracts to $S^n - X_1$ and similarly B an open set which deformation retracts to X_2; this can be done by the standard process of "thickening" X_1 and X_2, i.e. by choosing an appropriate tubular neighborhood. We then get that $A \cup B = S^n$ while $A \cap B$ deformation retracts to $X_1 \cup X^2$. This allows us to apply Mayer-Vietoris:

$$... \longrightarrow \tilde{H}_{i+1}(S^n) \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - X_1) \oplus \tilde{H}_i(S^n - X_2) \longrightarrow \tilde{H}_i(S^n) \longrightarrow ...$$

We have three obvious indices to look at, $i = n, \ell$ and k. Because $k, \ell < n$, we get that $\tilde{H}_i(S^n - X_1) = \tilde{H}_i(S^n - X_2) = 0$ by Proposition 2B.1.b, so this section of Mayer-Vietoris reads

$$... \longrightarrow 0 \longrightarrow \tilde{H}_n(S^n - X) \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow ...$$

and hence $\tilde{H}_n(S^n - X) \cong 0$. This same argument holds for all $i > n$ too.

Suppose now that both k and ℓ are nonzero. Then

$$... \longrightarrow 0 \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - X_1) \oplus \tilde{H}_i(S^n - X_2) \longrightarrow 0 \longrightarrow ...$$

is the relevant section of the Mayer-Vietoris sequence $i < n-1$. The portions to consider are the cases when $i = n-k-1$ and $i = n-\ell-1$ as these yield $\mathbb{Z}$'s in the sequence (in the case that either k or ℓ are 0, then we need to investigate $i = n-1$, which becomes complicated). In any case, when $i = n-\ell-1$ or $i = n-k-1$, we have

$$0 \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - X_1) \oplus \tilde{H}_i(S^n - X_2) \longrightarrow 0$$

which means $\tilde{H}_i(S^n - X)$ is $\mathbb{Z}$ if $k \neq \ell$ and $\mathbb{Z}^2$ if $k = \ell$ At all other positions except at $i = 0$ and $i = n-1$ we have only trivial terms in the sequence. When $i = n-1$ we have

$$... \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_{n-1}(S^n - X) \longrightarrow 0 \longrightarrow ...$$

which yields an isomorphism $\tilde{H}_{n-1}(S^n - X) \cong \mathbb{Z}$. At $i = 0$ we get

$$0 \longrightarrow \tilde{H}_0(S^n - X) \longrightarrow \tilde{H}_0(S^n - X_1) \oplus \tilde{H}_0(S^n - X_2) \longrightarrow 0$$

which implies $\tilde{H}_0(S^n - X) = 0 \implies H_0(S^n - X) \cong \mathbb{Z}$. To summarize, when $k, \ell \neq 0$,

$$H_i(S^n - X) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i \in \{n-k-1, n-\ell-1, n-1\} \text{ and } k \neq \ell \\ \mathbb{Z}^2 & i = n-k-1 \text{ and } k = \ell \\ 0 & \text{otherwise} \end{cases}.$$

There are a few extraneous cases that fit into the $k, \ell \neq 0$ situation that we have not considered. For instance, if either $k = n-1$ or $\ell = n-1$ a similar thing occurs as in the $k = \ell$ case occurs; we get $H_i(S^n - X) = \mathbb{Z}^2$

in the $i = 0$. Additionally, if $n = 1$, then $k = \ell = n - 1 = 0$. This means $S^n - X$ is really the removal of two copies of S^0 from S^1, which results in four path components and hence $H_0(S^1 - X) \cong \mathbb{Z}^4$.

Now suppose that $k \neq 0$ and $\ell = 0$. Our homology remains unchanged except in the case that $i = n - \ell - 1 = n - 1$, where the Mayer-Vietoris sequence becomes

$$... \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_{n-1}(S^n - X) \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow ...$$

where we note that $\tilde{H}_{n-1}(S^n - X^k) \cong 0$ since $k \neq 0$. Because $\mathbb{Z}$ is free, this sequence splits and we get that $\tilde{H}_i(S^n - X) \cong \mathbb{Z} \oplus \mathbb{Z}$. All other homology groups remain unchanged from before.

Finally, if both $\ell = k = 0$ then we get

$$... \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_{n-1}(S^n - X) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow ...$$

when $i = n - 1$. This also splits because $\mathbb{Z} \oplus \mathbb{Z}$ is still free, so $\tilde{H}_{n-1}(S^n - X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. To summarize all these cases in a succinct way:

$$H_i(S^n - X) \cong \begin{cases} \mathbb{Z} & i = 0, n-k-1, n-\ell-1, n-1 \\ 0 & \text{otherwise} \end{cases}$$

and we merge the $\mathbb{Z}$'s together with direct sums if any of the $0, n - \ell - 1, n - k - 1, n - 1$ terms coincide.

Now recall that this is only half of the problem – we still need to treat the $X \cong S^k \vee S^\ell$ case. Luckily, there are not as many cases to think about here. Once again, denote by X_1 the copy of S^k in X and by X_2 the copy of S^ℓ in X. We still have that

$$\tilde{H}_i(S^n - X_1) \cong \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{else} \end{cases}, \quad \tilde{H}_i(S^n - X_2) \cong \begin{cases} \mathbb{Z} & i = n-\ell-1 \\ 0 & \text{else} \end{cases}.$$

by Proposition 2B.1.b. Using the same A and B as before, we get $A \cup B = S^n - \{p\}$ where p is the point at which X_1 and X_2 meet. It's still true that $A \cap B$ deformation retracts to X. The removal of one point from S^n results in $\mathbb{R}^n$, which is contractible, so $A \cup B \simeq \{pt\}$. Mayer-Vietoris therefore gives

$$... \longrightarrow 0 \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - X_1) \oplus \tilde{H}_i(X^n - X_2) \longrightarrow 0 \longrightarrow ...$$

for *all* i (yay!), and we get $\tilde{H}_i(S^n - X) \cong \tilde{H}_i(S^n - X_1) \oplus \tilde{H}_i(X^n - X_2) \cong \mathbb{Z} \oplus \mathbb{Z}$. In summary,

$$H_i(S^n - X) \cong \begin{cases} \mathbb{Z} & i = 0, n-k-1, n-\ell-1 \\ 0 & \text{else} \end{cases}.$$

$\square$

EXERCISE 2. Show that $\tilde{H}_i(S^n - X) \cong \tilde{H}_{n-i-1}(X)$ when X is homeomorphic to a finite connected graph. [First do the case that X is a tree.]

Proof: Starting with a tree is nice because trees are always contractible. This means in particular $\tilde{H}_{n-i-1}(X) \cong 0$. Intuitively, deleting a tree from the surface of S^n gives us a space homeomorphic to $\mathbb{R}^n$, which is also contractible. This nonetheless demands more justification, so we proceed inductively on the number of vertices contained in X, a number which we denote by V_X.

When $V_X = 1$, $S^n - X$ truly is just the one-point-uncompactification of S^n and is thus contractible, so our base case is clear. Now suppose that $\tilde{H}_i(S^n - X) = 0$ when $V_X \leq k$. Consider the case then that $V_X = k + 1$. Since X is a tree, there is some vertex v with only one edge attached. Call this one edge e and the other vertex to which it connects w. Consider the deformation retract of X which collapses e; this produces a new tree Y with only k vertices. Set $A = S^n - e$ and $B = S^n - Y$, noting that these are both open sets. Then $A \cup B = S^n - \{v\}$ and $A \cap B = S^n - X$, so we are in a good position to apply Mayer-Vietoris:

$$... \longrightarrow \tilde{H}_{i+1}(S^n - \{v\}) \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - e) \oplus \tilde{H}_i(S^n - Y) \longrightarrow \tilde{H}_i(S^n - \{v\}) \longrightarrow ...$$

Applying the inductive hypothesis gives us

$$... \longrightarrow 0 \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

and so $\tilde{H}_i(S^n \setminus X) \cong 0$, as desired. Notice that we haven't proven that $S^n - X$ is contractible, only that it has trivial homology.

Now let X be homeomorphic to some finite graph. Denote by T the maximal subtree of X. If $T = X$ then we are done by what we have already shown; assume then that $T \subsetneq X$. We'll proceed by induction on the number of edges of $X \setminus T$. Think of the size of this set as indicating "how far X is from being a tree." Our base case is out of the way already by the $X \setminus T = \emptyset$ case, so assume that $\tilde{H}_i(S^n - X) \cong \tilde{H}_{n-i-1}(X)$ whenever $|E(X \setminus T)| \leq k$ and that we have $|E(X \setminus T)| = k + 1$. Write $X = X' \cup e$, where e is one of the edges not in T and X' is a graph with k edges not in T. It follows from the inductive hypothesis that $\tilde{H}_i(S^n - X') \cong \tilde{H}_{n-i-1}(X')$. This second group is actually computable; since X' has k edges not in the maximal subtree, after collapsing T to a point we obtain a wedge of k circles. Hence

$$\tilde{H}_i(X') \cong \begin{cases} \mathbb{Z}^m & i = 1 \\ 0 & else \end{cases}.$$

Because X is a simple graph, it has no loops formed by an edge connecting twice to a single vertex. This implies that $X' \cap e$ consists of precisely two points v and u. Both of these points are in T since maximal subtrees contain every vertex and in X' since $T \subseteq X'$. We now apply Mayer-Vietoris using open sets $A = S^n - e$ and $B = S^n - Y$. We have $A \cup B = S^n - \{v, u\} \cong S^{n-1}$ and $A \cap B = S^n \setminus X$. MV yields

$$... \longrightarrow \tilde{H}_{i+1}(S^{n-1}) \longrightarrow \tilde{H}_i(S^n - X) \longrightarrow \tilde{H}_i(S^n - e) \oplus \tilde{H}_i(S^n - Y) \longrightarrow \tilde{H}_i(S^{n-1}) \longrightarrow ...$$

If $i \neq n - 1$ or $n - 2$, then we get

$$... \longrightarrow 0 \longrightarrow \tilde{H}_i(S^n \setminus X) \longrightarrow \tilde{H}_i(S^n - X') \longrightarrow 0 \longrightarrow ...$$

which gives us an isomorphism $\tilde{H}_i(S^n - X) \cong \tilde{H}_i(S^n - X')$. Because $\tilde{H}_i(S^n - X') = 0$ unless $i = n - 2$, a case that we are not in, $\tilde{H}_i(S^n - X) = 0$ for all $i \neq n - 1, n - 2$.

Now consider $i = n - 1$. The relevant portion of MV is

$$... \longrightarrow 0 \longrightarrow \tilde{H}_i(S^n \setminus X) \longrightarrow \tilde{H}_i(S^n - X') \longrightarrow \mathbb{Z} \longrightarrow ...$$

but since $\tilde{H}_n(S^n - X') \cong 0$, we get $\tilde{H}_{n-1}(S^n - X) = 0$ too. All that remains is the $i = n - 2$ case:

$$... \longrightarrow 0 \longrightarrow \tilde{H}_{n-1}(S^{n-1}) \longrightarrow \tilde{H}_{n-2}(S^n - X) \longrightarrow \tilde{H}_{n-2}(S^n - Y) \longrightarrow 0...$$

where we have used the facts that $\tilde{H}_{n-1}(S^{n-1}) = \tilde{H}_{n-2}(S^{n-1}) = 0$. The two outer groups are both free, so we have a short exact sequence which splits:

$$... \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{H}_{n-2}(S^n - X) \longrightarrow \mathbb{Z}^m \longrightarrow 0 \longrightarrow ...$$

and so $\tilde{H}_{n-2}(S^n - X) = \mathbb{Z}^{m+1}$ when $i = n - 2$ and is 0 everywhere else. This is identical to $\tilde{H}_{n-i-1}(X)$, which we can compute in the same way as X' by collapsing T to a point and realizing we've obtained a wedge of $m + 1$ circles. $\square$