Chapter 7

Singular elements of linear series

Keynote Questions

(a) If $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}_{t\in\mathbb{P}^1}$ is a general pencil of plane curves of degree d, how many of the curves C_t are singular? (Answer on page 253.)

(b) Let $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^2}$ be a general net of plane curves. What is the degree and geometric genus of the curve $\Gamma \subset \mathbb{P}^2$ traced out by the singular points of members of the net? What is the degree and geometric genus of the discriminant curve $\mathcal{D} = \{t \in \mathbb{P}^2 \mid C_t \text{ is singular}\}$? (Answer in Section 7.6.2.)

(c) Let $C \subset \mathbb{P}^r$ be a smooth nondegenerate curve of degree d and genus g. How many hyperplanes $H \subset \mathbb{P}^r$ have contact of order at least $r+1$ with C at some point? (Answer on page 268.)

(d) If $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^1}$ is a general pencil of plane curves of degree d, how many of the curves C_t have hyperflexes (that is, lines having contact of order 4 with C_t)? (Answer on page 405.)

(e) If $\{C_t \subset \mathbb{P}^2\}_{t\in\mathbb{P}^4}$ is a general four-dimensional linear system of plane curves of degree d, how many of the curves C_t have a triple point? (Answer on page 257.)

In this chapter we introduce the *bundle of principal parts* associated with a line bundle $\mathcal{L}$ on a smooth variety X. This is a vector bundle on X whose fiber at a point $p \in X$ is the space of Taylor series expansions around p of sections of the line bundle, up to a given order. We will use the techniques we have developed to compute the Chern classes of this bundle, and this computation will enable us to answer many questions about singular points and other special points of varieties in families. We will start out by discussing hypersurfaces in projective space, but the techniques we develop are much more broadly applicable to families of hypersurfaces in any smooth projective variety X, and in Section 7.4.2 we will see how to generalize our formulas to that case.

In the last section (Section 7.7) we introduce a different approach to such questions, the "topological Hurwitz formula."

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It is important to emphasize the standing hypothesis that the ground field $\Bbbk$ is of characteristic 0. In contrast to the preceding chapters, many of the theorems in this chapter are false as stated in characteristic $p > 0$. When it makes the geometric argument simpler, we will allow ourselves to work over the complex numbers, appealing to the "Lefschetz principle" to say that the results we obtain apply over any algebraically closed field of characteristic 0.

7.1 Singular hypersurfaces and the universal singularity

Before starting on this path, we will take a moment to talk about loci of singular plane curves, and more generally singular hypersurfaces in $\mathbb{P}^n$. Let $\mathbb{P}^N = \mathbb{P}^{\binom{d+n}{n}-1}$ be the projective space parametrizing all hypersurfaces of degree d in $\mathbb{P}^n$. Our primary object of interest is the *discriminant locus* $\mathcal{D} \subset \mathbb{P}^N$, defined as the set of singular hypersurfaces.

A central role in this chapter will be played by the *universal singular point* $\Sigma = \Sigma_{n,d}$ of hypersurfaces of degree d in $\mathbb{P}^n$, defined as follows:

$$\begin{array}{ccc} \Sigma = \{(Y, p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in Y_{\text{sing}}\} & \xrightarrow{\pi_2} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi_1} & & \\ \{\text{hypersurfaces } Y \text{ of degree } d \text{ in } \mathbb{P}^n\} = \mathbb{P}^N & & \end{array}$$

If we write the general form of degree d on $\mathbb{P}^n$ as $F = \sum a_I x^I$ and think of it as a bihomogeneous form of bidegree $(1, d)$ in the coordinates a_I of $\mathbb{P}^N$ and the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$, then Σ is defined by the bihomogeneous equations

$$F = 0 \quad \text{and} \quad \frac{\partial F}{\partial x_i} = 0 \ \text{ for } i = 0, \ldots, n,$$

so Σ is an algebraic set. Note that the first of these equations is implied by the others (in characteristic 0!); given the dimension statement of Proposition 7.1 below, this means that Σ is a complete intersection of $n+1$ hypersurfaces of bidegree $(1, d-1)$ in $\mathbb{P}^N \times \mathbb{P}^n$.

The image $\mathcal{D}$ of Σ in $\mathbb{P}^N$ is the set of singular hypersurfaces, called the *discriminant*. The next proposition shows that $\mathcal{D}$ is a hypersurface, and that $\Sigma \to \mathcal{D}$ is a resolution of singularities:

Proposition 7.1. *With notation as above, suppose that $d \geq 2$.*

(a) *The variety Σ is smooth and irreducible of dimension $N - 1$ (that is, codimension $n + 1$); in fact, the fibers of Σ over $\mathbb{P}^n$ are projective spaces $\mathbb{P}^{N-n-1}$.*

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(b) *The general singular hypersurface of degree d has a unique singularity, which is an ordinary double point. In particular, Σ is birational to its image $\mathcal{D} \subset \mathbb{P}^N$.*
(c) *$\mathcal{D}$ is an irreducible hypersurface in $\mathbb{P}^N$.*

Proof: Let $p \in \mathbb{P}^n$ be a point, and let $x_0, \dots, x_n$ be homogeneous coordinates on $\mathbb{P}^n$ such that $p = (1, 0, \dots, 0)$. Let $f(x_1/x_0, \dots, x_n/x_0) = x_0^{-d} F(x_0, \dots, x_n) = 0$ be the affine equation of the hypersurface $F = 0$. For $d \geq 1$, the $n+1$ coefficients of the constant and linear terms f_0 and f_1 in the Taylor expansion of f at p are equal to certain coefficients of F, so the fiber of Σ over p is a projective subspace of $\mathbb{P}^N$ of codimension $n+1$. The first part of the proposition follows from this, and implies that the discriminant $\mathcal{D} = \pi_1(\Sigma)$ is irreducible.

To prove the statements in the second part of the proposition, note that the fiber of Σ over a point $p \in \mathbb{P}^n$ contains the hypersurface that is the union of $d-2$ hyperplanes not containing p with a cone over a nonsingular quadric in $\mathbb{P}^{n-1}$ with vertex p. This hypersurface has an ordinary double point at p, and is generically reduced. By the previous argument, the hypersurfaces corresponding to points of the fiber of Σ over p form a linear system of hypersurfaces, with no base points other than p. Bertini's theorem shows that a general member of this system is smooth away from p. Thus the fiber of the map $\pi_1 : \Sigma \to \mathcal{D} \subset \mathbb{P}^N$ over a general point of $\mathcal{D}$ consists of just one point, showing that the map is birational onto its image. Since smoothness is an open condition on a quadratic form, the general member has only an ordinary double point at p.

The fact that Σ, which has dimension $N-1$, is birational to $\mathcal{D}$ shows that $\mathcal{D}$ also has dimension $N-1$, completing the proof. □

The defining equation of $\mathcal{D} \subset \mathbb{P}^N$ is difficult to write down explicitly, though of course it can be computed in principle by elimination theory. There are determinantal formulas in a few cases: see for example Gelfand et al. [2008] and Eisenbud et al. [2003]. Even in relatively simple cases such as $n = 1$ the discriminant locus has a lot of interesting features, as a picture of the real points of the discriminant of a quartic $f(a) = x^4 + ax^2 + bx + c$ in one variable suggests (see Figure 7.1). For a nice animation of the discriminant of a quartic polynomial, see `http://youtu.be/MV2uVYqGiNc`, created by Hans-Christian Graf v. Bothmer and Oliver Labs as part of their Geometrical Animations Advent Calendar.

In view of Proposition 7.1, we can rephrase the first keynote question of this chapter as asking for the number of points of intersection of a general line $L \subset \mathbb{P}^N$ with the hypersurface $\mathcal{D}$; that is, the degree of $\mathcal{D}$. How can we determine this if we cannot write down the form? As we will see below, Chern classes provide a mechanism for doing exactly this.

There is an interpretation of the discriminant hypersurface in $\mathbb{P}^N$ that relates $\mathcal{D}$ to an object previously encountered in Chapter 1. The d-th Veronese map ν_d embeds $\mathbb{P}^n$ in the dual $\mathbb{P}^{N*}$ of the projective space $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, in such a way that the intersection of $\nu_d(\mathbb{P}^n)$ with the hyperplane corresponding to a point $F \in \mathbb{P}^N$ is isomorphic, via ν_d,

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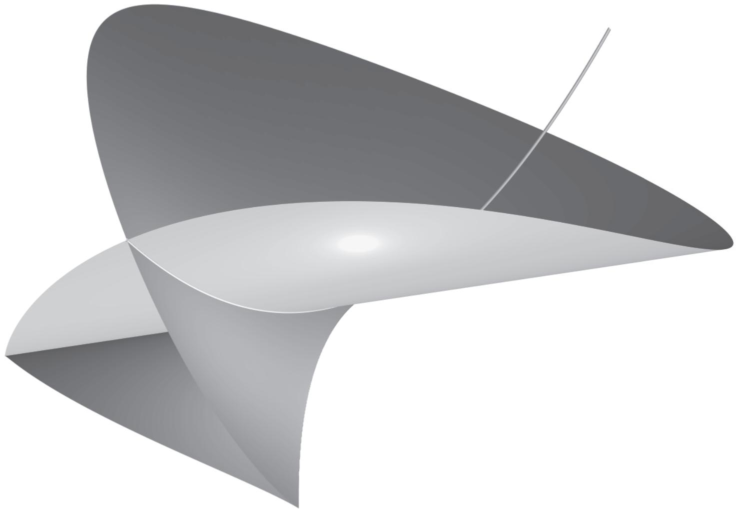

Figure 7.1 Real points of the discriminant of a quartic polynomial.

to the corresponding hypersurface $F = 0$ in $\mathbb{P}^n$. Thus the discriminant is the set of hyperplanes in $\mathbb{P}^{N*}$ that have singular intersection with $\nu_d(\mathbb{P}^n)$, or, equivalently, those that contain a tangent plane to $\nu_d(\mathbb{P}^n)$. This is the definition of the *dual variety* to $\nu_d(\mathbb{P}^n)$, which we first encountered in Section 2.1.3. Proposition 7.1 shows that the dual of $\nu_d(\mathbb{P}^n)$ is a hypersurface, and that the general tangent hyperplane is tangent at just one point, at which the intersection has an ordinary double point.

7.2 Bundles of principal parts

We can simplify the problem of describing the discriminant by linearizing it. We do not ask "is the hypersurface $X = V(F)$ singular?"; rather, we ask for each point $p \in \mathbb{P}^n$ in turn the simpler question "is X singular at p?". This is very much analogous to our approach to lines on hypersurfaces, where instead of asking "does X contains lines?" we asked for each line L "does X contain L?". As in that context, this approach converts a higher-degree equation in the coefficients of F into a family of systems of linear equations, whose solution set we can then express as the vanishing of a section of a vector bundle.

For each point $p \in \mathbb{P}^n$, we have an $(n+1)$-dimensional vector space

$$E_p = \frac{\{\text{germs of sections of } \mathcal{O}_{\mathbb{P}^n}(d) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 2 \text{ at } p\}}.$$

This space should be thought of as the vector space of first-order Taylor expansions of forms of degree d. We will see that the spaces E_p fit together to form a vector bundle, called the bundle of *first-order principal parts*, which we will write as $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$.

A form F of degree d will give rise to a section τ_F of this vector bundle whose value at the point p is the first-order Taylor expansion of F locally at p, and whose vanishing locus is thus the set of singular points of the hypersurface $F = 0$.

An important feature of the situation is that each vector space E_p has a naturally defined subspace: the space of germs vanishing at p. These subspaces will, as we will see, glue together into a subbundle of $\mathcal{P}^1$. Using the Whitney formula (Theorem 5.3), this will help evaluate the Chern classes of the bundle.

We can generalize this in two ways: We can replace "2" by "$m + 1$," with m any positive integer; and we can replace the forms of degree d by the sections of a coherent sheaf on an arbitrary variety X (though in practice we will be working almost exclusively with line bundles on smooth varieties). To make this precise, let $\mathcal{L}$ be a quasi-coherent sheaf on a $\Bbbk$-scheme X, and write $\pi_1, \pi_2 : X \times X \to X$ for the projections onto the two factors. Let $\mathcal{I}$ be the ideal of the diagonal in $X \times X$. We set

$$\mathcal{P}^m(\mathcal{L}) = \pi_{2*}(\pi_1^*\mathcal{L} \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{m+1}),$$

which is a quasi-coherent sheaf on X. When X is smooth we call this the *bundle of principal parts*. We will parse and explain this expression below, but first we list its very useful properties:

Theorem 7.2. *The sheaves $\mathcal{P}^m(\mathcal{L})$ have the following properties:*

(a) *If $p \in X$ is a closed point, then there is a canonical identification of the fiber $\mathcal{P}^m(\mathcal{L}) \otimes \kappa(p)$ of $\mathcal{P}^m(\mathcal{L})$ at p with the sections of the restriction of $\mathcal{L}$ to the m-th-order neighborhood of p; that is,*

$$\mathcal{P}^m(\mathcal{L}) \otimes \kappa(p) = H^0(\mathcal{L} \otimes \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1})$$

as vector spaces over $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} = \Bbbk$. In other words,

$$\mathcal{P}^m(\mathcal{L}) \otimes \kappa(p) = \frac{\{\text{germs of sections of } \mathcal{L} \text{ at } p\}}{\{\text{germs vanishing to order} \geq m+1 \text{ at } p\}}.$$

(b) *If $F \in H^0(\mathcal{L})$ is a global section, then there is a global section $\tau_F \in H^0(\mathcal{P}^m(\mathcal{L}))$ whose value at p is the class of F in $H^0(\mathcal{L} \otimes \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{m+1})$.*

(c) *$\mathcal{P}^0(\mathcal{L}) = \mathcal{L}$, and for each $m > 0$ there is a natural right exact sequence*

$$\mathcal{L} \otimes \mathrm{Sym}^m(\Omega_X) \longrightarrow \mathcal{P}^m(\mathcal{L}) \longrightarrow \mathcal{P}^{m-1}(\mathcal{L}) \longrightarrow 0,$$

where Ω_X denotes the sheaf of $\Bbbk$-linear differential forms on X.

(d) *If X is smooth and of finite type over $\Bbbk$ and $\mathcal{L}$ is a vector bundle on X, then $\mathcal{P}^m(\mathcal{L})$ is a vector bundle on X, and the right exact sequences of part (c) are left exact as well.*

Proof: Since the constructions all commute with restriction to open sets, we may harmlessly suppose that $X = \operatorname{Spec} R$ is affine. Thus also $X \times X = \operatorname{Spec} S$, where $S := R \otimes_{\Bbbk} R$. We may think of $\mathcal{L}$ as coming from an R-module L, and then $\pi_1^* L := L \otimes_K R$. Pushing a (quasi-coherent) sheaf $\mathcal{M}$ on $X \times X$ forward by π_{2*} simply means considering the corresponding S-module as an R-module via the ring map $R \to S$ sending r to $1 \otimes_{\Bbbk} r$.

In this setting, the sheaf of ideals $\mathcal{I}$ defining the diagonal embedding of X in $X \times X$ corresponds to the ideal $I \subset S$ that is the kernel of the multiplication map $S = R \otimes_{\Bbbk} R \to R$. If R is generated as a $\Bbbk$-algebra by elements x_i, then I is generated as an ideal of S by the elements $x_i \otimes 1 - 1 \otimes x_i$.

With this notation, we see that the R-module corresponding to the sheaf $\mathcal{P}^m(\mathcal{L})$ can be written as

$$P^m(L) = (L \otimes_{\Bbbk} R)/I^{m+1}(L \otimes_{\Bbbk} R),$$

regarded as an R-module by the action $f \mapsto 1 \otimes r$ as above.

Part (a) now follows: If the $\Bbbk$-rational point p corresponds to the maximal ideal

$$\mathfrak{m} = \operatorname{Ker}\bigl(R \xrightarrow{\varphi} \Bbbk\bigr), \quad \varphi : x_i \mapsto a_i,$$

then in $R/\mathfrak{m} \otimes_R S \cong R$ the class of $x_i \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} x_i$ is $x_i \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} a_i = x_i - a_i$. Thus

$$P^m(L) \otimes_R R/\mathfrak{m} = L/(\{x_i \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} x_i\})^{m+1} L = L/(\{x_i - a_i\})^{m+1} L,$$

as required.

Part (b) is similarly obvious from this point of view: The section τ_F can be taken to be the image of the element $F \otimes_{\Bbbk} 1$ in $(L \otimes_{\Bbbk} R)/I^{m+1}(L \otimes_{\Bbbk} R)$. As the construction is natural, these elements will glue to a global section when we are no longer in the affine case.

Part (c) requires another important idea: The module of $\Bbbk$-linear differentials $\Omega_{R/\Bbbk}$ is isomorphic, as an R-module, to I/I^2, which has a universal derivation $\delta : R \to I/I^2$ given by $\delta(f) = f \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} f$. This is plausible, since when X is smooth one can see geometrically that the normal bundle of the diagonal, which is $\operatorname{Hom}(I/I^2, R)$, is isomorphic to the tangent bundle of X, which is $\operatorname{Hom}(\Omega_{R/\Bbbk}, R)$. See Eisenbud [1995, Section 16.8] for further discussion and a general proof. Given this fact, the obvious surjection $\operatorname{Sym}^m(\Omega_{R/\Bbbk}) \cong \operatorname{Sym}^m(I/I^2) \to I^m/I^{m+1}$ yields the desired right exact sequence.

Finally, it is enough to prove part (d) locally at a point $q \in X \times X$. If q is not on the diagonal then, after localizing, I is the unit ideal, and the result is trivial, so we may assume that $q = (p, p)$. Locally at p, the module L is free, so it suffices to prove the result when $L = R$.

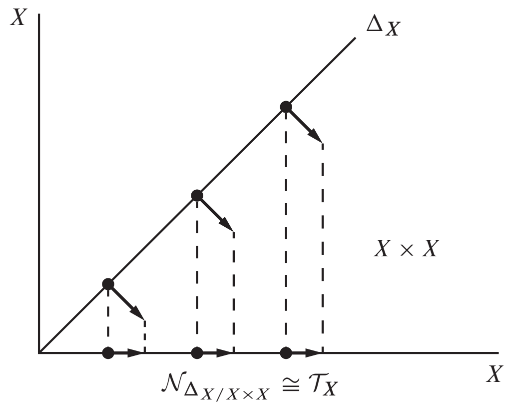

Figure 7.2 The normal bundle of the diagonal $\Delta_X \subset X \times X$ is isomorphic to the tangent bundle of X.

Write $d : R \to \Omega_{R/\Bbbk}$ for the universal $\Bbbk$-linear derivation of R. Since X is smooth, $\Omega_{R/\Bbbk}$ is locally free at p, and is generated there by elements $d(x_1), \ldots, d(x_n)$, where $x_1, \ldots, x_n$ is a system of parameters at p, and thus $\mathrm{Sym}^m(\Omega_{R/\Bbbk})$ is the free module generated by the monomials of degree m in the $d(x_i)$. Since R is a domain, I is a prime ideal.

Because $\mathrm{Sym}^m(\Omega_{R/\Bbbk})$ is free, it suffices to show that the map

$$\mathrm{Sym}^m(\Omega_{R/\Bbbk}) \to S/I^{m+1}$$

is a monomorphism (in fact, an isomorphism onto I^m/I^{m+1}) after localizing at the prime ideal I. Since $I/I^2 \cong \Omega_{R/\Bbbk}$ is free on the classes mod I^2 of the elements $x_i \otimes_\Bbbk 1 - 1 \otimes_\Bbbk x_i$ that correspond to the $d(x_i)$, Nakayama's lemma shows that, in the local ring S_I, I_I is generated by the images of the $x_i \otimes_\Bbbk 1 - 1 \otimes_\Bbbk x_i$ themselves, and it follows that these are a regular sequence. Thus the associated graded ring $S_I/I_I \oplus I_I/I_I^2 \oplus \cdots$ is a polynomial ring on the classes of the elements $x_i \otimes_\Bbbk 1 - 1 \otimes_\Bbbk x_i$, and in particular the monomials of degree m in these elements freely generate I_I^m/I_I^{m+1}. Consequently, the map $S_I \otimes_S \mathrm{Sym}^m(\Omega_{R/\Bbbk}) \to I_I^m/I_I^{m+1}$ is an isomorphism, as desired. □

Remark. The name "bundle of principal parts," first used by Grothendieck and Dieudonné, was presumably suggested by the (conflicting) usage that the "principal part" of a meromorphic function of one variable at a point is the sum of the terms of negative degree in the Laurent expansion of the function around the point — a finite power series, albeit in the inverse variables. It is not the only terminology in use: $\mathcal{P}^m(\mathcal{L})$ would be called the bundle of *m-jets of sections of $\mathcal{L}$* by those studying singularities of mappings (see for example Golubitsky and Guillemin [1973, II.2]) and some algebraic geometers (for example Perkinson [1996].) On the other hand, the m-jet terminology is in use in another conflicting sense in algebraic geometry: the "scheme of m-jets" of a scheme X is used to denote the scheme parametrizing mappings from $\mathrm{Spec}\,\Bbbk[x]/(x^{m+1})$ into X. So we have thought it best to stick to the Grothendieck–Dieudonné usage.

Example 7.3. We will not use this in any of the calculations below, but in the simplest and most interesting case, where $m = 1$, $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, it is possible to describe the bundle $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ very explicitly:

$$\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \cong \begin{cases} \Omega_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} & \text{if } d = 0, \\ \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} & \text{if } d \neq 0. \end{cases}$$

This curious dichotomy is explained by the answer to a more refined question: By part (d), we have a short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(d) \longrightarrow \mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0,$$

and we can ask for its class in

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(d), \Omega_{\mathbb{P}^n}(d)) \cong \operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n}) = H^1(\Omega_{\mathbb{P}^n}) = \Bbbk.$$

More generally, for any line bundle $\mathcal{L}$ on a smooth variety X, the short exact sequence in part (d) gives us a class in

$$\operatorname{Ext}^1_X(\mathcal{L}, \Omega_X \otimes \mathcal{L}) = H^1(\Omega_X),$$

called the *Atiyah class* of $\mathcal{L}$ and denoted $\operatorname{at}(\mathcal{L})$ (Atiyah [1957] and Illusie [1972]). The formula for $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ follows at once from the more refined and more uniform result that

$$\operatorname{at}(\mathcal{O}_{\mathbb{P}^n}(d)) = d \cdot \eta,$$

where $\eta \in \operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n})$ is the class of the tautological sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{(x_0,\dots,x_n)} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

See Perkinson [1996, 2.II] and Re [2012] for an analysis of all the $\mathcal{P}^m$.

7.3 Singular elements of a pencil

7.3.1 From pencils to degeneracy loci

Using the bundle of principal parts, we can tackle a slightly more general version of Keynote Question (a): How many linear combinations of general polynomials F and G of degree d on $\mathbb{P}^n$ have singular zero loci? By Proposition 7.1, none of the hypersurfaces $X_t = V(t_0 F + t_1 G)$ of the pencil will be singular at more than one point. Furthermore, no two elements of the pencil will be singular at the same point, since otherwise every member of the pencil would be singular there. Thus, the general form of the keynote question is equivalent to the question: For how many points $p \in \mathbb{P}^n$ is some element X_t of the pencil singular at p? This, in turn, amounts to asking at how many points $p \in \mathbb{P}^n$

are the values $\tau_F(p)$ and $\tau_G(p)$ in the fiber of $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ at p linearly dependent, given that they are dependent at finitely many points? We can do this with Chern classes, provided that the degeneracy locus is reduced; we will establish this first.

To start, consider the behavior of the sections τ_F and τ_G around a point $p \in \mathbb{P}^n$ where they are dependent. At such a point, some linear combination $t_0 F + t_1 G$ — which we might as well take to be F — vanishes to order 2. If G were also zero at p, then the scheme $V(F, G)$ would have (at least) a double point at p. But Bertini's theorem shows that a general complete intersection such as $V(F, G)$ is smooth, so this cannot happen; thus we can assume that $G(p) \neq 0$.

To show that $V(\tau_F \wedge \tau_G)$ is reduced at p, we restrict our attention to an affine neighborhood of p where all our bundles are trivial. By Proposition 7.1, the hypersurface $C = V(F)$ has a node at p, so if we work on an affine neighborhood where the bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is trivial, and take p to be the origin with respect to coordinates $x_1, \dots, x_n$, we may assume that the functions F and G have Taylor expansions at p of the form

$$\begin{aligned} f &= f_2 + (\text{terms of order} > 2), \\ g &= \ \ 1 + (\text{terms of order} \geq 1). \end{aligned}$$

The sections τ_F and τ_G are then represented locally by the rows of the matrix

$$\begin{pmatrix} f & \partial f/\partial x_1 & \cdots & \partial f/\partial x_n \\ g & \partial g/\partial x_1 & \cdots & \partial g/\partial x_n \end{pmatrix}.$$

The vanishing locus of $\tau_F \wedge \tau_G$ near p is, by definition, defined by the 2×2 minors of this matrix, and to prove that it is a reduced point we need to see that it contains n functions (vanishing at p) with independent linear terms. Suppressing all the terms of the functions in the matrix that could not contribute to the linear terms of the minors, we get the matrix

$$\begin{pmatrix} 0 & \partial f_2/\partial x_1 & \cdots & \partial f_2/\partial x_n \\ 1 & 0 & \cdots & 0. \end{pmatrix}.$$

Thus there are 2×2 minors whose linear terms are $\partial f_2/\partial x_1, \dots, \partial f_2/\partial x_n$, and these are linearly independent because $f_2 = 0$ is a smooth quadric and the characteristic is not 2.

As usual, if we assign multiplicities appropriately, we can extend the calculations to pencils whose degeneracy locus $V(\tau_F \wedge \tau_G)$ is nonreduced. In Section 7.7.2 we will see one way to calculate these multiplicities.

7.3.2 The Chern class of a bundle of principal parts

Once again, let F, G be general forms of degree d on $\mathbb{P}^n$. As we saw in the previous section, the linear combinations $t_0 F + t_1 G$ that are singular correspond exactly to points where the two sections τ_F and τ_G are dependent. The degeneracy locus of τ_F and τ_G is the n-th Chern class of the rank-$(n+1)$ bundle $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$, so we turn to the

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computation of this class. For brevity, we will shorten $\mathcal{P}^1(\mathcal{O}_{\mathbb{P}^n}(d))$ to $\mathcal{P}^1(d)$, but the reader should keep in mind that this is *not* "a bundle $\mathcal{P}^1$ tensored with $\mathcal{O}(d)$!"

Stated explicitly, if $\zeta \in A^1(\mathbb{P}^n)$ denotes the class of a hyperplane in $\mathbb{P}^n$, we want to compute the coefficient of ζ^n in

$$c(\mathcal{P}^1(d)) \in A(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1}).$$

Parts (c) and (d) of Theorem 7.2 give us a short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(d) \longrightarrow \mathcal{P}^1(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0,$$

so $c(\mathcal{P}^1(d)) = c(\mathcal{O}_{\mathbb{P}^n}(d)) \cdot c(\Omega_{\mathbb{P}^n}(d))$. (See Proposition 7.5 for the other $\mathcal{P}^m(d)$.) On the other hand, $\Omega_{\mathbb{P}^n}$ fits into a short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

Tensoring with $\mathcal{O}(d)$, we get an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0$$

similar to the one involving $\mathcal{P}^1(d)$. This does not mean that $\mathcal{P}^1(d)$ and $\mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}$ are isomorphic (they are not), but by the Whitney formula their Chern classes agree:

$$c(\mathcal{P}^1(d)) = c(\mathcal{O}_{\mathbb{P}^n}(d-1)^{n+1}) = (1 + (d-1)\zeta)^{n+1}.$$

Putting this formula together with the idea of the previous section, we deduce:

Proposition 7.4. *The degree of the discriminant hypersurface in the space of forms of degree d on $\mathbb{P}^n$ is*

$$\deg c_n(\mathcal{P}^1(d)) = (n+1)(d-1)^n,$$

and this is the number of singular hypersurfaces in a general pencil of hypersurfaces of degree d in $\mathbb{P}^n$.

In particular, this answers Keynote Question (a): A general pencil of plane curves of degree d will have $3(d-1)^2$ singular elements.

It is pleasant to observe that the conclusion agrees with what we get from elementary geometry in the cases where it is easy to check, such as those of plane curves ($n = 2$) with $d = 1$ or $d = 2$. For $d = 1$, the statement $c_2 = 0$ simply means that there are no singular elements in a pencil of lines. The case $d = 2$ corresponds to the number of singular conics in a general pencil $\{C_t\}$ of conics. To see that this is really $3(d-1)^2 = 3$, note that the pencil $\{C_t\}$ consists of all conics passing through the four (distinct) base points, and a singular element of the pencil will thus be the union of a line joining two of the points with the line joining the other two. There are indeed three such pairs of lines (see Figure 7.3).

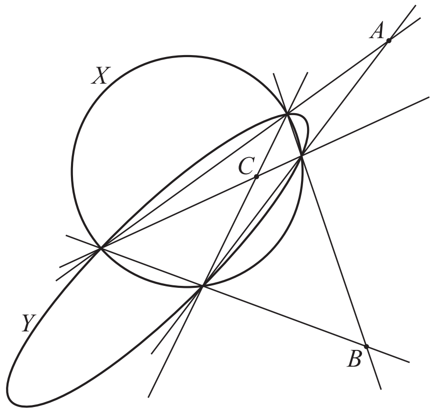

Figure 7.3 A, B, C are the singular elements of the pencil of plane conics containing X and Y.

We could also get the number 3 by viewing the pencil of conics as given by a 3×3 symmetric matrix M of linear forms on $\mathbb{P}^2$ whose entries vary linearly with a parameter t; the determinant of M will then be a cubic polynomial in t.

As the reader may have noticed, there is a simpler way to arrive at the formula of Proposition 7.4. We observed in Section 7.1 that the universal singularity

$$\Sigma = \{(Y, p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in Y_{\text{sing}}\}$$

is a complete intersection of $n + 1$ hypersurfaces of bidegree $(1, d - 1)$ in $\mathbb{P}^N \times \mathbb{P}^n$. Denoting by α and ζ the pullbacks to $\mathbb{P}^N \times \mathbb{P}^n$ of the hyperplane classes in $\mathbb{P}^N$ and $\mathbb{P}^n$, this means that Σ has class

$$\begin{aligned}[\Sigma] &= (\alpha + (d-1)\zeta)^{n+1} \\ &= \alpha^{n+1} + \binom{n+1}{1}\alpha^n (d-1)\zeta + \cdots + \binom{n+1}{n}\alpha^n (d-1)^n \zeta^n.\end{aligned}$$

When we push this class forward to $\mathbb{P}^N$, all the terms go to 0 except the last, from which we can conclude that the class of the discriminant hypersurface $\mathcal{D}$ is $[\mathcal{D}] = (n+1)(d-1)^n \alpha$; that is, $\mathcal{D} = \pi_1(\Sigma)$ has degree $(n+1)(d-1)^n$.

Why did we adopt the approach via principal parts, given this alternative? The answer is that, as we will see in Section 7.4.2, the principal parts approach can be applied to linear series on arbitrary smooth varieties; the alternative we have just given applies *only* to projective space.

It is easy to extend the Chern class computation in Proposition 7.4 to all the $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, and this will be useful in the rest of this chapter:

Proposition 7.5. $$c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\big) = (1 + (d-m)\zeta)^{\binom{n+m}{n}}.$$

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Proof: We will again use the exact sequences of Theorem 7.2. With the Whitney formula, they immediately give

$$c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\big) = \prod_{j=0}^{m} c(\mathrm{Sym}^j(\Omega_{\mathbb{P}^n})(d)).$$

To derive the formula we need, we apply Lemma 7.6 below to the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

and the line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(d)$. To simplify the notation, we set $\mathcal{U} = \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$. The lemma yields

$$c(\mathrm{Sym}^j(\Omega_{\mathbb{P}^n})(d)) = c(\mathrm{Sym}^j(\mathcal{U})(d)) \cdot c(\mathrm{Sym}^{j-1}(\mathcal{U})(d))^{-1}$$

for all $j \geq 1$. Combining this with the obvious equality $\mathrm{Sym}^0(\Omega_{\mathbb{P}^n})(d) = \mathcal{O}_{\mathbb{P}^n}(d)$, we see that the product in the formula for $c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\big)$ is

$$c(\mathcal{O}_{\mathbb{P}^n}(d)) \cdot \frac{c(\mathrm{Sym}^1(\mathcal{U})(d))}{c(\mathcal{O}_{\mathbb{P}^n}(d))} \cdot \frac{c(\mathrm{Sym}^2(\mathcal{U})(d))}{c(\mathrm{Sym}^1(\mathcal{U})(d))} \cdots,$$

which collapses to

$$c\big(\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))\big) = c(\mathrm{Sym}^m(\mathcal{U})(d)).$$

But

$$\begin{aligned} c(\mathrm{Sym}^m(\mathcal{U})(d)) = c(\mathrm{Sym}^m(\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1})(d)) &= c\big(\mathcal{O}_{\mathbb{P}^n}(-m)^{\binom{n+m}{n}}(d)\big) \\ &= c\big(\mathcal{O}_{\mathbb{P}^n}(d-m)^{\binom{n+m}{n}}\big) \\ &= (1 + (d-m)\zeta)^{\binom{n+m}{n}}, \end{aligned}$$

yielding the formula of the proposition. □

Lemma 7.6. *If*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is a short exact sequence of vector bundles on a projective variety X with rank $\mathcal{C} = 1$, *then, for any $j \geq 1$,*

$$c(\mathrm{Sym}^j(\mathcal{A}) \otimes \mathcal{L}) = c(\mathrm{Sym}^j(\mathcal{B}) \otimes \mathcal{L}) \cdot c(\mathrm{Sym}^{j-1}(\mathcal{B}) \otimes \mathcal{C} \otimes \mathcal{L})^{-1}.$$

Proof: For any right exact sequence of coherent sheaves

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

the universal property of the symmetric powers (see, for example, Eisenbud [1995, Proposition A2.2.d]) shows that for each $j \geq 1$ there is a right exact sequence

$$\mathcal{E} \otimes \mathrm{Sym}^{j-1}(\mathcal{F}) \longrightarrow \mathrm{Sym}^j(\mathcal{F}) \longrightarrow \mathrm{Sym}^j(\mathcal{G}) \longrightarrow 0.$$

Since $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are vector bundles, the dual of the exact sequence in the hypothesis is exact, and we may apply the result on symmetric powers with $\mathcal{G} = \mathcal{A}^*$, $\mathcal{F} = \mathcal{B}^*$ and $\mathcal{E} = \mathcal{C}^*$.

In this case, since $\operatorname{rank} \mathcal{E} = \operatorname{rank} \mathcal{C} = 1$, the sequence

$$0 \longrightarrow \mathcal{E} \otimes \operatorname{Sym}^{j-1}(\mathcal{F}) \longrightarrow \operatorname{Sym}^j(\mathcal{F}) \longrightarrow \operatorname{Sym}^j(\mathcal{G}) \longrightarrow 0$$

is left exact as well, as one sees by comparing the ranks of the three terms (this is a special case of a longer exact sequence, independent of the rank of $\mathcal{E}$, derived from the Koszul complex).

Since these are all bundles, dualizing preserves exactness, and we get an exact sequence

$$0 \longrightarrow \operatorname{Sym}^j(\mathcal{A}^*)^* \longrightarrow \operatorname{Sym}^j(\mathcal{B}^*)^* \longrightarrow \operatorname{Sym}^{j-1}(\mathcal{B}^*)^* \otimes \mathcal{C} \longrightarrow 0.$$

Of course the double dual of a bundle is the bundle itself, and the dual of the j-th symmetric power is naturally isomorphic to the j-th symmetric power of the dual, so all the *'s cancel, and we can deduce the lemma from the Whitney formula. □

7.3.3 Triple points of plane curves

We can adapt the preceding ideas to compute the number of points of higher order in linear families of hypersurfaces. By way of example we consider the case of triple points of plane curves.

Let $\mathbb{P}^N$ be the projective space of all plane curves of degree $d \geq 3$, and let

$$\Sigma' = \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid \operatorname{mult}_p(C) \geq 3\}.$$

The condition that a curve C have a triple point at a given point $p \in \mathbb{P}^2$ is six independent linear conditions on the coefficients of the defining equation of C, from which we see that the fibers of the projection map $\Sigma' \to \mathbb{P}^2$ on the second factor are linear spaces $\mathbb{P}^{N-6} \subset \mathbb{P}^N$, and hence that Σ' is irreducible of dimension $N-4$. It follows that the set of curves with a triple point is irreducible as well. An argument similar to that for double points also shows that a general curve $f = 0$ with a triple point has only one. In particular, the projection map $\Sigma' \to \mathbb{P}^N$ on the first factor is birational onto its image. It follows in turn that *the locus $\Phi \subset \mathbb{P}^N$ of curves possessing a point of multiplicity 3 or more is an irreducible variety of dimension $N-4$*. We also see that if C is a general curve with a triple point at p, then p is an *ordinary* triple point of C; that is, the projectivized tangent cone $\mathbb{T}C_pX$ is smooth or, equivalently, the cubic term f_3 of the Taylor expansion of f around p has three distinct linear factors.

We ask now for the degree of the variety of curves with a triple point, or, equivalently, the answer to Keynote Question (e): If $F_0, \dots, F_4$ are general polynomials of degree d on $\mathbb{P}^2$, for how many linear combinations $F_t = t_0F_0 + \cdots + t_4F_4$ (up to scalars) will the corresponding plane curve $C_t = V(F_t) \subset \mathbb{P}^2$ have a triple point?

If we write τ_F for the section defined by F in $\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))$, then C has a triple point at p if and only if τ_F vanishes at p. An argument analogous to the one given in Section 7.3.1, together with the smoothness of the tangent cone at a general triple point, shows that the 5×5 minors of the map $\mathcal{O}_{\mathbb{P}^2}^5 \to \mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))$ generate the maximal ideal locally at a general point where a linear combination of the F_i defines a curve with an ordinary triple point.

Thus the number of triple points in the family is the degree of the second Chern class $c_2\big(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))\big)$. By Proposition 7.5,

$$c\big(\mathcal{P}^2(\mathcal{O}_{\mathbb{P}^2}(d))\big) = (1 + (d-2)\zeta)^6 = 1 + 6(d-2)\zeta + 15(d^2 - 4d + 4)\zeta^2.$$

Proposition 7.7. *If $\Phi = \Psi_{d,n} \subset \mathbb{P}^N$ is the locus in the space of all curves of degree d in $\mathbb{P}^2$ of curves having a triple point, then for $d \geq 2$*

$$\deg(\Phi) = 15(d^2 - 4d + 4).$$

In case $d = 1$, the number 15 computed is of course meaningless, because the expected dimension $N - 4$ of Φ is negative — any five global sections τ_{F_i} of the bundle $\mathcal{P}^2(\mathcal{O}(1))$ are everywhere-dependent. On the other hand, the number 0 computed in the case $d = 2$, which is 0, really does reflect the fact that no conics have a triple point. For $d = 3$, the computation above gives 15, a number we already computed as the degree of the locus of “asterisks” in Section 2.2.3.

7.3.4 Cones

As we remarked, the calculation in the preceding section is a generalization of the calculation in Section 2.2.3 of the degree of the locus Φ parametrizing triples of concurrent lines (“asterisks”) in the space $\mathbb{P}^9$ parametrizing plane cubic curves. There is another generalization of this problem: We can ask for the degree, in the space $\mathbb{P}^N$ parametrizing hypersurfaces of degree d in $\mathbb{P}^n$, of the locus Ψ of *cones*. We are now in a position to answer that more general problem, which we will do here.

We will not go through the steps in detail, since they are exactly analogous to the last calculation; the upshot is that the degree of Ψ is the degree of the n-th Chern class of the bundle $\mathcal{P}^{d-1}(\mathcal{O}_{\mathbb{P}^n}(d))$. By Proposition 7.5,

$$c\big(\mathcal{P}^{d-1}(\mathcal{O}_{\mathbb{P}^n}(d))\big) = (1+\zeta)^{\binom{n+d-1}{n}},$$

and so we have:

Proposition 7.8. *If $\Psi = \Psi_{d,n} \subset \mathbb{P}^N$ is the locus of cones in the space of all hypersurfaces of degree d in $\mathbb{P}^n$, then*

$$\deg(\Psi) = \binom{\binom{n+d-1}{n}}{n}.$$

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Thus, for example, in case $d = 2$ we see again that the locus of singular quadrics in $\mathbb{P}^n$ is $n+1$, and in case $d = 3$ and $n = 2$ the locus of asterisks has degree 15. Likewise, in the space $\mathbb{P}^{14}$ of quartic plane curves, the locus of concurrent 4-tuples of lines has degree

$$\binom{\binom{5}{2}}{2} = \binom{10}{2} = 45.$$

Compare this to the calculation in Exercise 2.57!

7.4 Singular elements of linear series in general

Let X be a smooth projective variety of dimension n, and let $\mathcal{W} = (\mathcal{L}, W)$ be a linear system on X. We think of the elements of $\mathbb{P}W$ as divisors in X, and, as in Section 7.1, we introduce the incidence correspondence

$$\Sigma_{\mathcal{W}} = \{(Y, p) \in \mathbb{P}W \times X \mid p \in Y_{\text{sing}}\}$$

with projection maps $\pi_1 : \Sigma \to \mathbb{P}W$ and $\pi_2 : \Sigma \to X$. Also as in Section 7.1 we denote by $\mathcal{D} = \pi_1(\Sigma) \subset \mathbb{P}W$ the locus of singular elements of the linear series $\mathcal{W}$, which we again call the *discriminant*.

As mentioned in the introduction to this chapter, the techniques developed so far apply as well in this generality. What is missing is the analog of Proposition 7.1: We do not know in general that Σ is irreducible of codimension $n+1$, we do not know that it maps birationally onto $\mathcal{D}$ (as we will see more fully in Section 10.6, the discriminant $\mathcal{D}$ may have dimension strictly smaller than that of Σ) and we do not know that the general singular element of $\mathcal{W}$ has one ordinary double point as its singularity. Thus the formulas we derive in this generality are only *enumerative formulas*, in the sense of Section 3.1: They apply subject to the hypothesis that the loci in question do indeed have the expected dimension, and even then only if multiplicities are taken into account.

That said, we can still calculate the Chern classes of the bundle of principal parts $\mathcal{P}^1(\mathcal{L})$, and derive an enumerative formula for the number of singular elements of a pencil of divisors (that is, the degree of $\mathcal{D} \subset \mathbb{P}W$, in case $\mathcal{D}$ is indeed a hypersurface); we will do this in Section 7.4.1 below.

We note one interpretation of $\mathcal{D}$ in case the linear series $\mathbb{P}W$ is very ample. If $X \subset \mathbb{P}^N$ is a smooth variety and

$$\mathcal{W} = (\mathcal{O}_X(1), W) \quad \text{with } W = H^0(\mathcal{O}_{\mathbb{P}^N}(1))|_X$$

is the linear series of hyperplane sections of X, then a section in W is singular if and only if the corresponding hyperplane is tangent to X. Thus the set of points in $\mathbb{P}W$

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corresponding to such sections is the *dual variety* to X, and the number of singular elements in a general pencil of these sections is the degree of the dual variety. We will treat dual varieties more thoroughly in Section 10.6.

7.4.1 Number of singular elements of a pencil

Let X be a smooth projective variety of dimension n and $\mathcal{W} = (\mathcal{L}, W)$ a pencil of divisors on X (typically, a general pencil in a larger linear series). We can use the Chern class machinery to compute the expected number of singular elements of $\mathcal{W}$. To simplify the notation, we will denote the first Chern class of the line bundle $\mathcal{L}$ by $\lambda \in A^1(X)$, and the Chern classes of the cotangent bundle Ω_X of X simply by $c_1, c_2, \ldots, c_n$.

From the exact sequence

$$0 \longrightarrow \Omega_X \otimes \mathcal{L} \longrightarrow \mathcal{P}^1(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow 0$$

and Whitney's formula, we see that the Chern class of $\mathcal{P}^1(\mathcal{L})$ is the Chern class of $\mathcal{L} \otimes (\mathcal{O}_X \oplus \Omega_X)$. Since $c_i(\mathcal{O}_X \oplus \Omega_X) = c_i(\Omega_X) = c_i$, we may apply the formula for the Chern class of a tensor product of a line bundle (Proposition 5.17) to arrive at

$$c_k(\mathcal{P}^1(\mathcal{L})) = \sum_{i=0}^{k} \binom{n+1-i}{k-i} \lambda^{k-i} c_i.$$

In particular,

$$\begin{aligned} c_n(\mathcal{P}^1(\mathcal{L})) &= \sum_{i=0}^{n} (n+1-i)\lambda^{n-i} c_i \\ &= (n+1)\lambda^n + n\lambda^{n-1} c_1 + \cdots + 2\lambda c_{n-1} + c_n. \end{aligned} \tag{7.1}$$

As remarked above, this represents only an enumerative formula for the number of singular elements of a pencil. But the calculations of Section 7.3.1 hold here as well: A singular element Y of a pencil corresponds to a reduced point of the relevant degeneracy locus if Y has just one ordinary double point as its singularity. Thus we have the following:

Proposition 7.9. *Let X be a smooth projective variety of dimension n. If $\mathcal{W} = (\mathcal{L}, W)$ is a pencil of divisors on X having finitely many singular elements $D_1, \ldots, D_\delta$ such that*

(a) each D_i has just one singular point,
(b) that singular point is an ordinary double point, and
(c) that singular point is not contained in the base locus of the pencil,

then the number δ of singular elements is the degree of the class

$$\gamma(\mathcal{L}) := c_n(\mathcal{P}^1(\mathcal{L})) = (n+1)\lambda^n + n\lambda^{n-1} c_1 + \cdots + 2\lambda c_{n-1} + c_n \in A^n(X),$$

where $\lambda = c_1(\mathcal{L})$ and $c_i = c_i(\Omega_X)$.

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Naturally, there will be occasions when we want to apply this formula but may not be able to verify hypotheses (a)–(c) of Proposition 7.9 — for example, as we will see in Section 10.6, these hypotheses are not necessarily satisfied by a general pencil of hyperplane sections of a smooth projective variety $X \subset \mathbb{P}^r$. It is worth asking, accordingly, what can we conclude from the enumerative formula in the absence of these hypotheses.

First off, if the class $\gamma(\mathcal{L})$ is nonzero, then we can conclude that *the pencil $\mathcal{W}$ must have singular elements*; this applies to any pencil on any variety. Secondly, if $\mathcal{W}$ is a general pencil in a very ample linear series $\mathcal{V} = (\mathcal{L}, V)$, we can form the *universal singular point*, as in Section 7.1:

$$\Sigma = \{(v, p) \in \mathbb{P}V \times X \mid p \in (D_v)_{\mathrm{sing}}\},$$

where $D_v \subset X$ is the divisor corresponding to the element $v \in \mathbb{P}V$. As in the proof of Proposition 7.1, we see that the fibers of Σ over X are projective spaces of codimension $n+1$ in $\mathbb{P}V$, and hence that Σ has codimension $n+1$ in $\mathbb{P}V \times X$; it follows that the preimage of a general pencil $\mathbb{P}W \subset \mathbb{P}V$ is Σ will be finite. We can conclude, therefore, that in this situation *the degree of the class $\gamma(\mathcal{L})$ must be nonnegative, and if it is 0 then $\mathcal{W}$ will have no singular elements* — in other words, the locus of singular elements of the linear system $\mathcal{V}$ has codimension > 1 in $\mathbb{P}V$, and every singular element will have positive-dimensional singular locus. We will see an example of a situation where this is the case in Exercise 7.28 below, and investigate the question in more detail in Section 10.6.

Finally, we will see in Section 7.7.2 below a way of calculating multiplicities of the relevant degeneracy locus topologically, so that even in case the singular elements of $\mathcal{W}$ do not satisfy the hypothesis of having only one ordinary double point we can say something about the number of singular elements. (The conclusions of Section 7.7.2 are stated only for pencils of curves on a surface, but analogous statements hold in higher dimension as well.)

7.4.2 Pencils of curves on a surface

By way of an example, we will apply the results of Proposition 7.9 to pencils of curves on surfaces. For the case $d = 2$, see Exercises 7.22 and 7.23.

Suppose that $X \subset \mathbb{P}^3$ is a smooth surface of degree d and that V is the linear series of intersections of X with surfaces of degree e, so that $\mathcal{L} = \mathcal{O}_X(e)$.

We claim that the three hypotheses of Proposition 7.9 are satisfied for a general pencil $W \subset V$:

(a) The fact that a general singular element of $\mathcal{W}$ (equivalently, of $\mathcal{V}$) has only one singularity in the case $e = 1$ is somewhat subtle; it is equivalent to the statement that the Gauss map from the surface to its dual variety is birational. (This is sometimes false in characteristic p!) This statement is proven for all smooth hypersurfaces in Corollary 10.21.

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The case $e > 1$ can be deduced from the case $e = 1$ by using Bertini's theorem and Proposition 7.10.

(b) The fact that the singularity of a general singular element of $\mathcal{W}$ is an ordinary double point is also tricky. Again, it follows for $e > 1$ from the case $e = 1$, and when $e = 1$ it can be done for a general surface $X \subset \mathbb{P}^3$ by an incidence correspondence/dimension count argument (Exercise 7.42). For an arbitrary X, however, it requires the introduction of the *second fundamental form*; we will describe this in the following section and use it to prove the statement we want in Theorem 7.11.

(c) Finally, the third hypothesis of Proposition 7.9 follows much as in the case of plane curves: By Bertini's theorem, the base locus of a general pencil in a very ample linear series is smooth — in this case a set of reduced points — and a reduced point on a smooth surface cannot be the intersection of two divisors if one of them is singular.

We have used:

Proposition 7.10. *Let $\Gamma \subset \mathbb{P}^n$ be a finite subscheme with homogeneous coordinate ring S_Γ.*

(a) *Γ imposes independent conditions on forms of degree* $\deg \Gamma - 1$.

(b) *Γ fails to impose independent conditions on forms of degree $\Gamma - 2$ if and only if Γ is contained in a line.*

(c) *Let l be a general linear form, and set $R_\Gamma := S_\Gamma/(l)S_\Gamma$. In general, Γ imposes independent conditions on forms of degree e in $\mathbb{P}^n$ if and only if R_Γ is 0 in degree e.*

Since R_Γ is generated as an S_Γ-module in degree 0, the range of integers e such that $(R_\Gamma)_e \neq 0$ is an interval in $\mathbb{Z}$ of the form $[0, \dots, r]$, and the number r is called the *Castelnuovo–Mumford regularity* of R_Γ. See Eisenbud [2005, Chapter 4] for more on this important notion.

Proof: The condition that Γ imposes independent conditions on forms of degree s is equivalent to the statement that $(\dim S_\Gamma)_s = \deg \Gamma$.

Using the exact sequences

$$0 \longrightarrow (S_\Gamma)_{t-1} \xrightarrow{\cdot l} (S_\Gamma)_t \longrightarrow (R_\Gamma)_t \longrightarrow 0,$$

we see that $\dim_{\Bbbk}(S_\Gamma)_s = \sum_{t=0}^{s} \dim_{\Bbbk}(R_\Gamma)_t$.

By Eisenbud [1995, Section 1.9], the dimension of R_Γ is the degree of the scheme Γ, proving Part (c). Part (a) is an immediate consequence. If Γ is not contained in a line, then $(R_\Gamma)_1 \geq 2$, so $(R_\Gamma)_{\deg \Gamma - 1} = 0$, proving Part (b). See Eisenbud and Harris [1992]. □

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Letting $\zeta \in A^1(X)$ denote the restriction of the hyperplane class, we have $c_1(\mathcal{L}) = e\zeta$, and, as we have seen,

$$\begin{aligned} c(\mathcal{T}_X) &= \frac{c(\mathcal{T}_{\mathbb{P}^3})}{c(\mathcal{N}_{X/\mathbb{P}^3})} \\ &= \frac{(1+\zeta)^4}{1+d\zeta} \\ &= (1+4\zeta+6\zeta^2)(1-d\zeta+d^2\zeta^2) \\ &= 1+(4-d)\zeta+(d^2-4d+6)\zeta^2. \end{aligned}$$

Thus $c_1 = (d-4)\zeta$ and $c_2 = (d^2-4d+6)\zeta^2$. From (7.1), above we see that

$$c_2(\mathcal{P}(\mathcal{L})) = (3e^2+2(d-4)e+d^2-4d+6)\zeta^2.$$

Finally, since $\deg(\zeta^2) = d$, the number of singular elements in the pencil of curves on a smooth surface $X \subset \mathbb{P}^3$ of degree d cut by a general pencil of surfaces of degree e is

$$\deg c_2(\mathcal{P}(\mathcal{L})) = d(3e^2+2(d-4)e+d^2-4d+6). \tag{7.2}$$

As explained above, this will be the degree of the dual surface of the e-th Veronese image $\nu_e(X)$ of X. For example, when $e = 1$ this reduces to

$$\deg X^* = d(d-1)^2,$$

as calculated in Section 2.1.3.

When $e = 2$, we are computing the expected number of singular points in the intersection of X with a general pencil $\{Q_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ of quadric surfaces in $\mathbb{P}^3$, and we find that it is equal to

$$d^3 + 2d.$$

The reader should check the case $d = 1$ directly! We invite the reader to work out some more examples, and to derive analogous formulas in higher (and lower!) dimensions, in Exercises 7.22–7.27.

7.4.3 The second fundamental form

A useful tool in studying singularities of elements of linear series is the *second fundamental form* S_X of a smooth variety $X \subset \mathbb{P}^n$. The notion was first considered in differential geometry, and is usually described using a metric, but we give a purely algebro-geometric treatment. We will explain the definition and an application; more information can be found, for example, in Griffiths and Harris [1979].

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As we shall see at the end of this section, the second fundamental form is closely related to the *Gauss map* $\mathcal{G}_X : X \to \mathbb{G}(k, n)$, which sends each point $p \in X$ to its tangent plane $\mathbb{T}_p X \subset \mathbb{P}^n$: The information S_X carries is equivalent to that of the differential

$$d\mathcal{G}_X : \mathcal{T}_X \to \mathcal{G}_X^* \mathcal{T}_{\mathbb{G}(k,n)}.$$

(See Section 2.1.3 for the definition of the Gauss map for hypersurfaces, and below for the general case.)

Since we will be dealing with both duals and pullbacks of vector bundles in this section, we will write the dual of a bundle $\mathcal{E}$ as $\mathcal{E}^\vee$ instead of our more usual $\mathcal{E}^*$.

Throughout this section X will denote a smooth subvariety of dimension k in $\mathbb{P}^n$.

We will define $S = S_X$ to be a map of sheaves on X

$$S : \mathcal{I}_X/\mathcal{I}_X^2 \to \operatorname{Sym}^2(\mathcal{T}_X^\vee).$$

We regard $\operatorname{Sym}^2(\mathcal{T}_X^\vee)$ as the bundle of quadratic forms on the tangent spaces to X. Let f be a function on an open subset of $\mathbb{P}^n$ defined in a neighborhood of $p \in X$ and vanishing on X, so that f is a local section of $\mathcal{I}_X$. When restricted to the tangent space $\mathbb{T}_p X \subset \mathbb{P}^n$ of X at p, the function f is singular at p, so the restriction $\overline{f} = f|_{\mathbb{T}_p X}$ vanishes together with all its first derivatives at p. Because of this, the quadratic part of the Taylor expansion of f at p is independent of the choice of coordinates. Via the identification $T_p(\mathbb{T}_p X) = T_p X$, we define $S(f)_p$, the value of $S(f)$ at p, to be the quadratic term in the expansion of $\overline{f}$ at p.

We claim that $S(f)$ vanishes if $f \in \mathcal{I}_X^2$, and thus that S defines a map $\mathcal{I}_X/\mathcal{I}_X^2 \to \operatorname{Sym}^2(\mathcal{T}_X^\vee)$. We work in local coordinates z_i at p. In these terms, $S(f)$ is the quadratic form defined by the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(p)\right).$$

If $f = gh$ is the product of two functions vanishing on X, then the locally defined function $\partial f/\partial z_i = g(\partial h/\partial z_i) + h(\partial g/\partial z_i)$ vanishes on X, so the Hessian matrix is identically 0 on X. Since S is linear, this suffices to prove the claim.

Recall from Eisenbud [1995, Chapter 16] that there is an exact sequence

$$0 \longrightarrow \mathcal{I}_{X/\mathbb{P}^n}/\mathcal{I}_{X/\mathbb{P}^n}^2 \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \Omega_X \longrightarrow 0.$$

Since X is smooth this is a short exact sequence of vector bundles. Composing the inclusion $\mathcal{I}_{X/\mathbb{P}^n}/\mathcal{I}_{X/\mathbb{P}^n}^2 \to \Omega_{\mathbb{P}^n}|_X$ with the first map of the restriction of the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \mathcal{O}_X^{n+1}(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we get an inclusion of bundles $\iota : \mathcal{I}_{X/\mathbb{P}^n}/\mathcal{I}_{X/\mathbb{P}^n}^2 \to \mathcal{O}_X^{n+1}$. The Gauss map $\mathcal{G} : X \to G(n-k, n+1)$ may be defined as the unique map such that the pullback $\mathcal{G}^*(\mathcal{S})$ of the inclusion of the universal subbundle on $G(n-k, n+1)$ is ι.

(The more usual definition of the Gauss map is dual to this one: Starting from the derivative $T_X \to T_{\mathbb{P}^n}|_X$ of the inclusion map, one takes the pullback of the image under the surjection

$$\mathcal{O}_X^{n+1} \to T_{\mathbb{P}^n}|_X;$$

the two descriptions are related by the duality isomorphism $G(k+1, n+1) \cong G(n-k, n+1)$.)

As explained in Section 3.2.5, the derivative of this map takes a derivation ∂ to the result of applying ∂ to the entries of a matrix representing ι and then projecting to $\mathcal{O}^{n+1}/\mathcal{E}$. We can put all these actions into the composite map

$$\mathcal{T}_X \otimes (\mathcal{I}_{X/\mathbb{P}^n}/\mathcal{I}^2_{X/\mathbb{P}^n}) \to \Omega_{\mathbb{P}^n}|_X \to \Omega_X,$$

or equivalently the map

$$\mathcal{I}_{X/\mathbb{P}^n}/\mathcal{I}^2_{X/\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}|_X \otimes \Omega_X \to \Omega_X \otimes \Omega_X.$$

A local computation in coordinates x_i on X shows that the image of the class of a function $f \in \mathcal{I}_{X/\mathbb{P}^n}$ has the form

$$\sum_{i,j} \frac{\partial f}{\partial x_i \partial x_j} dx_i \otimes dx_j,$$

and is thus a symmetric tensor, an element of $\mathrm{Sym}^2(\Omega_X) \subset \Omega_X \otimes \Omega_X$; in fact, it is the quadratic term of the Taylor expansion of f.

We can now complete the proof of the result of Section 7.4.2, based on Corollary 10.21, which will be proven independently:

Theorem 7.11. *Let $X \subset \mathbb{P}^n$ be any smooth hypersurface of degree $d > 1$. The set of points $X_i \subset X$ where the rank of the quadratic form $S(f)_p$ is at most i is an algebraic subset of dimension at most i. In particular:*

(a) There are at most finitely many points where the tangent hyperplane section $Y = X \cap \mathbb{T}_p X$ has multiplicity 3 or more at p; that is, $S(f)_p = 0$.

(b) If $p \in X$ is a general point, then the tangent hyperplane section $Y = X \cap \mathbb{T}_p X$ has an ordinary double point at p; that is, for general p the rank of $S(f)_p$ is equal to the dimension of X.

Proof: Since X is a hypersurface, the ideal of X near p is generated by a single function f, so we may regard $S(f)$ as a map from X to the total space of a twist of the vector bundle $\mathrm{Sym}^2(\mathcal{T}_X^\vee)$. The locus X_i is thus the (reduced) preimage of the closed algebraic set of forms of rank $\leq i$.

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It suffices to prove the general statement. Suppose that X_i had dimension $> i$ for some $i \geq 0$, and let p be a general (and in particular smooth) point of X_i. Since p is general in X_i, the null-space $T'_{X,q} \subset T_{X,q}$ of $S(f)_q$ has constant dimension $\geq \dim X - i$ for q in a neighborhood $U \subset X_i$ of p, and these null-spaces form a subbundle $\mathcal{T}'_U \subset \mathcal{T}_X|_U$. The tangent spaces to U also form a subbundle, and by our hypothesis on the dimension of X_i these two subbundles intersect in a subbundle $\mathcal{T}_U \cap \mathcal{T}'_X$ of rank ≥ 1.

We may assume that the ground field is the complex numbers. Integrating a local analytic vector field inside $\mathcal{T}_U \cap \mathcal{T}'_X$, we obtain the germ of a curve in X along which the Gauss map has derivative 0. This contradicts the assertion of Corollary 10.21 that $\mathcal{G}_X$ is a finite mapping from X to its dual. $\square$

7.5 Inflection points of curves in $\mathbb{P}^r$

Bundles of principal parts are very useful for studying maps of curves to projective space. The connection with "singular elements of linear series" comes from the fact that a hyperplane in projective space is tangent to a nondegenerate curve if and only if its intersection with the curve — an element of the linear system corresponding to the embedding — is singular. If the plane meets the curve with a higher degree of tangency — think of the tangent line at a flex point of a plane curve — then that will be reflected in a higher-order singularity. Thus the technique we developed in Section 7.2 will allow us to solve the third of the keynote questions of this chapter: How to extend the notion of flexes to curves in $\mathbb{P}^n$, and how to count them.

Recall that if $C \subset X$ is a reduced curve on a scheme X and $D \subset X$ an effective Cartier divisor on C, then for any closed point $p \in D \cap C$ we defined the multiplicity of intersection of C with D at p to be the length (or the dimension over the ground field $\Bbbk$, which will be the same since we are supposing that $\Bbbk$ is algebraically closed) of $\mathcal{O}_{C,p}/\mathcal{I}(D) \cdot \mathcal{O}_{C,p}$. Thus, for example, when $p \notin C \cap D$ the multiplicity is 0, and the multiplicity is 1 if and only if C and D are both smooth at p and meet transversely there.

For the purpose of this chapter it is convenient to expand this notion. Suppose that C is a smooth curve, $f : C \to X$ is a morphism and D is any subscheme of X such that $f^{-1}(D)$ is a finite scheme. We define the *order of contact* of D with C at $p \in C$ to be

$$\operatorname{ord}_p f^{-1}(D) := \dim_{\kappa(p)} \mathcal{O}_{C,p}/f^*(\mathcal{I}(D)).$$

Since we have assumed that C is smooth, the local ring $\mathcal{O}_{C,p}$ is a discrete valuation ring, so $\operatorname{ord}_p f^{-1}(D)$ is the minimum of the lengths of the algebras $\mathcal{O}_{C,p}/f^*(g)$, where g ranges over the local sections of $\mathcal{I}(D)$ at p, or over the generators of this ideal.

If f is the inclusion map of a smooth curve $C \subset X = \mathbb{P}^r$, and $D = \Lambda \subset \mathbb{P}^r$ is a linear subspace, then the order of contact $\operatorname{ord}_p f^{-1}(\Lambda)$ is the minimum, over the set of hyperplanes H containing Λ, of the intersection multiplicity $m_p(C, H)$.

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For example, if $p \in C \subset \mathbb{P}^2$ is a smooth point of a plane curve and $L \supset p$ is any line through p, then the order of contact of L with C at p is at least 1; L is tangent to C at p if and only if it is at least 2. The line L is called a *flex tangent* if the order is at least 3, and in this case p is called a *flex* of C. Carrying this further, we say that p is a *hyperflex* if the tangent line L at p meets C with order ≥ 4. We adopt similar definitions in the situation where $f : C \to \mathbb{P}^2$ is a nonconstant morphism from a smooth curve. For a curve in 3-space we can consider both the orders of contact with lines and the orders of contact with hyperplanes.

7.5.1 Vanishing sequences and osculating planes

We will systematize these ideas by considering a linear system $\mathcal{W} = (\mathcal{L}, W)$ on a smooth curve C with $\dim W = r + 1$. Given a point $p \in C$ and a section $\sigma \in W$, the order of vanishing $\operatorname{ord}_p \sigma$ of σ at p is defined to be the length of the $\mathcal{O}_{C,p}$-module $\mathcal{L}_p/(\mathcal{O}_{C,p}\sigma)$. Again, because the ground field $\Bbbk$ is algebraically closed we have $\kappa(p) = \Bbbk$, so

$$\operatorname{ord}_p \sigma = \dim_{\Bbbk} \mathcal{L}_p/(\mathcal{O}_{C,p}\sigma).$$

Given $p \in C$, consider the collection of all orders of vanishing of sections $\sigma \in W$ at p.

We define the *vanishing sequence* $a(\mathcal{W}, p)$ of the linear system $\mathcal{W}$ at p to be the sequence of integers that occur as orders of vanishing at p of sections in W, arranged in strictly increasing order:

$$a(\mathcal{W}, p) := (a_0(\mathcal{W}, p) < a_1(\mathcal{W}, p) < \cdots).$$

Since sections vanishing to distinct orders are linearly independent, $a(\mathcal{W}, p)$ has at most $\dim W$ elements. On the other hand, we can find a basis for W consisting of sections vanishing to distinct orders at p (start with any basis; if two sections vanish to the same order replace one with a linear combination of the two vanishing to higher order, and repeat). It follows that the number of elements in $a(\mathcal{W}, p)$ is exactly $\dim_{\Bbbk} W = r + 1$:

$$a(\mathcal{W}, p) = (a_0(\mathcal{W}, p) < \cdots < a_r(\mathcal{W}, p)).$$

We set $\alpha_i = a_i - i$, and call the associated weakly increasing sequence

$$\alpha(\mathcal{W}, p) = (\alpha_0(\mathcal{W}, p) \leq \cdots \leq \alpha_r(\mathcal{W}, p))$$

the *ramification sequence* of $\mathcal{W}$ at p. When the linear system $\mathcal{W}$ or the point p we are referring to is clear from context, we will drop it from the notation and write $a_i(p)$ or just a_i in place of $a_i(\mathcal{W}, p)$, and similarly for α_i.

For example, p is a base point of $\mathcal{W}$ if and only if $a_0(p) = \alpha_0(p) > 0$, and more generally $a_0(p)$ is the multiplicity with which p appears in the base locus of $\mathcal{W}$. If p is a base point of $\mathcal{W}$ then, since C is a smooth curve, we may remove it; that is, W is in the image of the monomorphism $H^0(\mathcal{L}(-a_0 p)) \to H^0(\mathcal{L})$, and we may thus consider W as defining a linear series $\mathcal{W}' := (\mathcal{L}(-a_0 p), W)$. In this way most questions about linear systems on smooth curves can be reduced to the base point free case.

When p is not a base point of $\mathcal{W}$, so that $\mathcal{W}$ defines a morphism $f : C \to \mathbb{P}^r$ in a neighborhood of p, we have $a_1(p) = 1$ if and only if f is an embedding near p. If $r = 2$ and $\mathcal{W}$ is very ample, so that f is an embedding, we thus have $\alpha_0(p) = \alpha_1(p) = 0$ for all p and $\alpha_2(p) > 0$ for some particular p if and only if there is a line meeting the embedded curve with multiplicity > 2 at p; that is, p is an inflection point of the embedded curve. The geometric meaning of the vanishing sequence is given in general by the next result:

Proposition 7.12. *Let $\mathcal{W} = (\mathcal{L}, W)$ be a linear series on a smooth curve C, and let $p \in C$. If p is not a base point of $\mathcal{W}$, we let $\mathcal{W}' = \mathcal{W}$; in general, let $\mathcal{W}' = \big(\mathcal{L}(-a_0(\mathcal{W}, p)p), W\big)$.*

(a) $a_i(\mathcal{W}', p) = a_i(\mathcal{W}, p) - a_0(\mathcal{W}, p)$.

(b) Choose $\sigma_0, \dots, \sigma_r \in W$ such that σ_j vanishes at p to order $a_j(\mathcal{W}', p)$, and let H_j be the hyperplane in $\mathbb{P}(W^*)$ corresponding to σ_j. The plane

$$L_i = H_{i+1} \cap \cdots \cap H_r$$

is the unique linear subspace of dimension i with highest order of contact with C at p, and that order is $a_{i+1}(\mathcal{W}, p)$.

The planes L_i are called the *osculating planes* to $f(C)$ at p. We always have $L_0 = p$. If $f(C)$ is smooth at $f(p)$ then L_1 is the tangent line, and in general it is the reduced tangent cone to the branch of $f(C)$ that is the image of an analytic neighborhood of $p \in C$.

Proof: (a) A section of $\mathcal{L}(-dp)$ that vanishes to order m as a section of $\mathcal{L}(-dp)$ will vanish to order $m + d$ at p as a section of $\mathcal{L}$.

(b) Writing f for the germ at p of the morphism defined by $\mathcal{W}'$, it follows from the definitions that $\operatorname{ord}_p L_i = a_{i+1}$. If there were an i-plane L' with higher order of contact, and we wrote

$$L' = H'_{i+1} \cap \cdots \cap H'_r$$

for some hyperplanes H'_r, then each H'_r would have order of contact with C at p strictly greater than a_{i+1}. But these would correspond to independent sections in W, and taking linear combinations of these sections we would get $r - i$ sections with vanishing orders at p strictly greater than a_{i+1}. This contradicts the assumption that the highest $r - i$ elements of the vanishing sequence are $a_{i+1}, \dots, a_r$. $\square$

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7.5.2 Total inflection: the Plücker formula

We say that p is an *inflection point* for a linear system $(\mathcal{L}, W)$ of dimension r if the ramification sequence $(\alpha_0, \dots, \alpha_r)$ is not $(0, \dots, 0)$, or, equivalently, if $\alpha_r > 0$, which is the same as $a_r > r$. When $\mathcal{W}$ arises from a morphism $f : C \to \mathbb{P}^r$ that is an embedding near p, p is an inflection point of $\mathcal{W}$ if and only if some hyperplane has contact $\geq r+1$ at p.

We define the *weight* of $p \in C$ with respect to $\mathcal{W}$ to be

$$w(\mathcal{W}, p) := \sum_{i=0}^{r} \alpha_i.$$

This number is a measure of what might be called the "total inflection" of $\mathcal{W}$ at p. We can compute the sum $\sum_{p \in C} w(\mathcal{W}, p)$ as a Chern class of the bundle of principal parts of $\mathcal{L}$.

Theorem 7.13 (Plücker formula)**.** *If $\mathcal{W}$ is a linear system of degree d and dimension r on a smooth projective curve C of genus g, then*

$$\sum_{p \in C} w(W, p) = (r+1)d + (r+1)r(g-1).$$

This is our answer to Keynote Question (c). Note that it is only an enumerative formula, in the sense that each hyperplane having contact of order $r+1$ or more with C at a point p has to be counted with multiplicity $w(\mathcal{W}, p)$. We might expect that if C is a suitably general curve — say, one corresponding to a general point on a component of the open subset of the Hilbert scheme parametrizing smooth, irreducible, nondegenerate curves — then all inflection points of C would have weight 1, but this is actually false (see Exercises 7.40–7.41). It can be verified in some cases, such as plane curves (see Exercise 7.32), and it is true also for complete intersections with sufficiently high multidegree (see Exercise 7.39 for a step forward in that direction); it remains an open problem to say when it holds in general.

Proof: The key observation is that both sides of the desired formula are equal to the degree of the first Chern class of the bundle $\mathcal{P}^r(\mathcal{L})$. We can compute the class of this bundle from Theorem 7.2 as

$$c(\mathcal{P}^r(\mathcal{L})) = \prod_{j=0}^{r} c(\operatorname{Sym}^j(\Omega_C) \otimes \mathcal{L}).$$

Since Ω_C is a line bundle we have $\operatorname{Sym}^j(\Omega_C) = \Omega_C^j$, and thus $c((\operatorname{Sym}^j(\Omega_C) \otimes \mathcal{L}) = 1 + jc_1(\Omega_C) + c_1(\mathcal{L})$. It follows that

$$c_1(\mathcal{P}^r(\mathcal{L})) = (r+1)c_1(\mathcal{L}) + \binom{r+1}{2} c_1(\Omega_C).$$

Since the degree of Ω_C is $2g-2$, the degree of this class is

$$\deg c_1(\mathcal{P}^r(\mathcal{L})) = (r+1)d + (r+1)r(g-1),$$

the right-hand side of the Plücker formula.

We may define a map $\varphi : \mathcal{O}_C^{r+1} \to \mathcal{P}^r(\mathcal{L})$ by choosing any basis $\sigma_0, \dots, \sigma_r$ of W and sending the i-th basis element of $\mathcal{O}_C^{r+1}$ to the section τ_{σ_i} of $\mathcal{P}^r(\mathcal{L})$ corresponding to σ_i. We will complete the proof by showing that for any point $p \in C$ the determinant of the map φ vanishes at p to order exactly $w(W, p)$, and that there are only finitely many points $w(W, p)$ where the determinant is 0.

To this end, fix a point $p \in C$. Since the determinant of φ depends on the choice of basis $\sigma_0, \dots, \sigma_r$ only up to scalars, we may choose the basis σ_i so that the order of vanishing $\operatorname{ord}_p(\sigma_i) = a_i$ at p is $a_i(W, p)$. Trivializing $\mathcal{L}$ in a neighborhood of p, we may think of the section σ_i locally as a function, and φ is represented by the matrix

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_r \\ \sigma_0' & \sigma_1' & \cdots & \sigma_r' \\ \vdots & \vdots & & \vdots \\ \sigma_0^{(r)} & \sigma_1^{(r)} & \cdots & \sigma_r^{(r)} \end{pmatrix},$$

where σ_i' denotes the derivative and $\sigma_i^{(r)}$ the r-th derivative. Because σ_i vanishes to order $\geq i$ at p, the matrix evaluated at p is lower-triangular, and the entries on the diagonal are all nonzero if and only if $a_i = i$ for each i; that is, if and only if p is not an inflection point for W.

We can compute the exact order of vanishing of $\det\varphi$ at an inflection point as follows: Denote by $v(z)$ the $(r+1)$-vector $(\sigma_0(z), \dots, \sigma_r(z))$, so that the determinant of φ is the wedge product

$$\det(\varphi) = v \wedge v' \wedge \cdots \wedge v^{(r)}.$$

Applying the product rule, the n-th derivative of $\det(\varphi)$ is then a linear combination of terms of the form

$$v^{(\beta_0)} \wedge v^{(\beta_1+1)} \wedge \cdots \wedge v^{(\beta_r+r)},$$

with $\sum \beta_i = n$. Now, $v^{(\beta_0)}(p) = 0$ unless $\beta_0 \geq \alpha_0$; similarly, $v^{(\beta_0)}(p) \wedge v^{(\beta_1)}(p) = 0$ unless $\beta_0 + \beta_1 \geq \alpha_0 + \alpha_1$, and so on. We conclude that *any derivative of* $\det(\varphi)$ *of order less than* $w = \sum \alpha_i$ *vanishes at* p, and the expression for the w-th derivative of $\det(\varphi)$ has exactly one term nonzero at p, namely

$$v^{(\alpha_0)} \wedge v^{(\alpha_1+1)} \wedge \cdots \wedge v^{(\alpha_r+r)}.$$

Since this term appears with nonzero coefficient, we conclude that $\det(\varphi)$ vanishes to order exactly w at p.

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It remains to show that not every point of C can be an inflection point for W — that is, that $\det\varphi$ is not identically zero. To prove this, suppose that $\det(\varphi)$ does vanish identically, that is, that

$$v \wedge v' \wedge \cdots \wedge v^{(k)} \equiv 0 \tag{7.3}$$

for some $k \leq r$. Suppose in addition that k is the smallest such integer, so that at a general point $p \in C$ we have

$$v(p) \wedge v'(p) \wedge \cdots \wedge v^{(k-1)}(p) \neq 0;$$

in other words, $v(p), \ldots, v^{(k-1)}(p)$ are linearly independent, but $v^{(k)}(p)$ lies in their span Λ. Again using the product rule to differentiate the expression (7.3), we see that

$$\frac{d}{dz}(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+1)} \equiv 0,$$

so that $v^{(k+1)}(p)$ also lies in the span of $v(p), \ldots, v^{(k-1)}(p)$. Similarly, taking the second derivative of (7.3), we see that

$$\frac{d^2}{dz^2}(v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k)}) = v \wedge v' \wedge \cdots \wedge v^{(k-1)} \wedge v^{(k+2)} \equiv 0,$$

where are all the other terms in the derivative are zero because they are $(k+1)$-fold wedge products of vectors lying in a k-dimensional space. Continuing in this way, we see that $v^{(m)}(p) \in \Lambda$ for all m; it follows by integration that $v(z) \in \Lambda$ for all z. This implies that the linear system W has dimension $k < r + 1$, contradicting our assumptions. □

Flexes of plane curves

Theorem 7.13 gives the answer to Keynote Question (c). We do not even need to assume C is smooth; if C is singular, as long as it is reduced and irreducible we view it as the image of the map $\nu : \widetilde{C} \to \mathbb{P}^r$ from its normalization. For example, when $r = 2$, if we apply the Plücker formula to the linear system corresponding to this map, we see that C has

$$(r+1)d + r(r+1)(g-1) = 3d + 6g - 6$$

flexes, where g is the genus of $\widetilde{C}$, that is, geometric genus of C. If the curve C is indeed smooth, then $2g - 2 = d(d-3)$, and so this yields

$$3d + 6g - 6 = 3d + 3d(d-3) = 3d(d-2).$$

To be explicit, this formula counts points $p \in \widetilde{C}$ such that, for some line $L \subset \mathbb{P}^2$, the multiplicity of the pullback divisor ν^*L at p is at least 3. In particular:

(a) It does not necessarily count nodes of C, even though at a node p of C there will be lines having intersection multiplicity 3 or more with C at p.
(b) It does count singularities where the differential $d\nu$ vanishes, for example cusps.

Some applications of the general Plücker formula appear in Exercises 7.35–7.37.

We mention that there is an alternative notion of a flex point of a (possibly singular) curve $C \subset \mathbb{P}^2$: a point $p \in C$ such that, for some line $L \subset \mathbb{P}^2$ through p, we have

$$m_p(C \cdot L) \geq 3.$$

In this sense, a node p of a plane curve C is a flex point, since the tangent lines to the branches of the curve at the node will have intersection multiplicity at least 3 with C at p. When we want to talk about flexes in this sense, we will refer to them as *Cartesian* flexes, since they are defined in terms of the defining equation of $C \subset \mathbb{P}^2$ rather than its parametrization by a smooth curve.

There is a classical way to calculate the number of flexes of a plane curve that does count Cartesian flexes. Briefly, if C is the zero locus of a homogeneous polynomial $F(X, Y, Z)$, we define the *Hessian* of C be the zero locus of the polynomial

$$H = \begin{vmatrix} \dfrac{\partial^2 F}{\partial X^2} & \dfrac{\partial^2 F}{\partial X \partial Y} & \dfrac{\partial^2 F}{\partial X \partial Z} \\ \dfrac{\partial^2 F}{\partial X \partial Y} & \dfrac{\partial^2 F}{\partial Y^2} & \dfrac{\partial^2 F}{\partial Y \partial Z} \\ \dfrac{\partial^2 F}{\partial X \partial Z} & \dfrac{\partial^2 F}{\partial Y \partial Z} & \dfrac{\partial^2 F}{\partial Z^2} \end{vmatrix}.$$

For a smooth plane curve C, the Cartesian flexes are exactly the points of intersection of C with its Hessian (it is even true that on a smooth curve C the weight of a flex p is equal to the intersection multiplicity of C with its Hessian at p). In Exercise 7.33, we will explore what happens to the flexes on a smooth plane curve when it acquires a node.

Hyperflexes

First, the bad news: We are not going to answer Keynote Question (d) here. The question itself is well-posed: We know that a general plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 4$ has only ordinary flexes, and it is not hard to see that the locus of those curves that do have a hyperflex is a hypersurface in the space $\mathbb{P}^N$ of all such curves (see Exercise 7.38). Surely the techniques we have employed in this chapter will enable us to calculate the degree of that hypersurface? Unfortunately, they do not, and indeed the reason we included Keynote Question (d) is so that we could point out the problem.

Very much by analogy with the analysis of lines on surfaces and singular points on curves, we would like to determine the class of the "universal hyperflex:" that is, in the universal curve

$$\Phi = \{(C, p) \in \mathbb{P}^N \times \mathbb{P}^2 \mid p \in C\},$$

the locus

$$\Gamma = \{(C, p) \in \Phi \mid p \text{ is a hyperflex of } C\}.$$

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Moreover, it seems as if this would be amenable to a Chern class approach: We would define a vector bundle $\mathcal{E}$ on Φ whose fiber at a point $(C, p) \in \Phi$ would be the vector space

$$\mathcal{E}_{(C,p)} = \frac{\{\text{germs of sections of } \mathcal{O}_C(1) \text{ at } p\}}{\{\text{germs vanishing to order} \geq 4 \text{ at } p\}}.$$

We would then have a map of vector bundles on Φ from the trivial bundle with fiber $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ to $\mathcal{E}$, and the degeneracy locus of this map would be the universal hyperflex Γ. Since this is the locus where three sections of a bundle of rank 4 are linearly dependent, we could conclude that

$$[\Gamma] = c_2(\mathcal{E}).$$

As we indicated, though, there is a problem with this approach. The description above of the fibers of $\mathcal{E}$ makes sense *as long as p is a smooth point of C, but not otherwise*. Reflecting this fact, if we were to try to define $\mathcal{E}$ by taking $\Delta \subset \Phi \times_{\mathbb{P}^N} \Phi$ the diagonal and setting

$$\mathcal{E} = \pi_{1*}(\pi_2^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\Phi \times_{\mathbb{P}^N} \Phi} / \mathcal{I}_\Delta^4),$$

the sheaf $\mathcal{E}$ would have fiber as desired over the open set $U \subset \Phi$ of (C, p) with C smooth at p, but would not even be locally free on the complement. The fact that bundles of principal parts do not behave well in families (except, of course, smooth families) is a real obstruction to carrying out this sort of calculation.

There is a way around this problem: Ziv Ran [2005a; 2005b] showed that — at least over the preimage $\mathcal{C} \subset \Phi$ of a general line $\mathbb{P}^1 \subset \mathbb{P}^N$ — the vector bundle $\mathcal{E}|_{U \cap \mathcal{C}}$ extends to a locally free sheaf on a blow-up of $\mathcal{C}$, realized as a subscheme of the relative Hilbert scheme of $\mathcal{C}$ over $\mathbb{P}^1$. This approach does yield an answer to Keynote Question (d), and indeed applies far more broadly, albeit at the expense of a level of difficulty that places it outside the range of this text.

And now, the good news: there is another way to approach Keynote Question (d), and we will explain it in Section 11.3.1.

7.5.3 The situation in higher dimension

Is there an analog of the Plücker formula for linear series on varieties of dimension greater than 1? Assuming that the linear series yields an embedding $X \subset \mathbb{P}^r$, we might ask, for a start, what sort of singularities we should expect the intersection $X \cap \Lambda$ of X with linear spaces $\Lambda \subset \mathbb{P}^r$ of a given dimension to have at a point p, and ask for the locus of points that are "exceptional" in this sense.

We do not know satisfying answers to these questions in general. One issue is that, while the singularities of subschemes of a smooth curve are simply classified by their multiplicity, there is already a tremendous variety of singularities of subschemes of surfaces. (If we have a particular class of singularities in mind, such as the A_n-singularities described in Section 11.4.1, then these questions do become well-posed; see for example the beautiful analysis of elements of a linear system having an A_n-singularity in Russell [2003].) Another problem is that the analog of the final step in the proof of Theorem 7.13 — showing that not every point on a smooth curve $C \subset \mathbb{P}^r$ can be an inflection point — may not hold. For example, a dimension count might lead us to expect that for a general point p on a smooth, nondegenerate surface $S \subset \mathbb{P}^5$ no hyperplane $H \subset \mathbb{P}^5$ intersects S in a curve $C = H \cap S$ with a triple point at p, but there are such surfaces for which this is false, and we do not know a classification of such surfaces.

We will revisit this question in Chapter 11, where we will describe the behavior of plane sections of a general surface $S \subset \mathbb{P}^3$.

7.6 Nets of plane curves

We now want to consider larger-dimensional families of plane curves, and in particular to answer the second keynote question of this chapter. A key step will be to compute the class of the universal singular point $\Sigma = \{(C, p) \mid p \in C_{\text{sing}}\}$ as a subvariety of $\mathbb{P}^N \times \mathbb{P}^2$, where $\mathbb{P}^N = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(d))$.

7.6.1 Class of the universal singular point

Let $W = H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, so that $\mathbb{P}W$ is the projective space of hypersurfaces of degree d in $\mathbb{P}^n$, and consider the universal m-fold point

$$\Sigma = \Sigma_{n,d,m} = \{(X, p) \in \mathbb{P}W \times \mathbb{P}^n \mid \operatorname{mult}_p(X) \geq m\},$$

and let

$$\begin{tikzcd} \mathbb{P}W \times \mathbb{P}^n \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{P}^n \\ \mathbb{P}W & \end{tikzcd}$$

be the projection maps. We can express the class $[\Sigma] \in A(\mathbb{P}W \times \mathbb{P}^n)$ in terms of Chern classes:

Proposition 7.14. *$\Sigma_{n,d,m}$ is the zero locus of a section of the vector bundle*

$$\mathcal{P}^m := \pi_1^*\mathcal{O}_{\mathbb{P}W}(1) \otimes \pi_2^*\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)),$$

which has Chern class

$$c(\mathcal{P}^m) = (1 + (d-m)\zeta_n + \zeta_W)^{\binom{n+m}{n}},$$

where ζ_n and ζ_W are the pullbacks of the hyperplane classes on $\mathbb{P}^n$ and $\mathbb{P}W$ respectively.

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Thus the class of $\Sigma_{n,d,m}$ in $A(\mathbb{P}W \times \mathbb{P}^n)$ is the sum of the terms of total degree $\binom{n+m}{n}$ in this expression. For example, in the case $n = 2, m = 1$ this is

$$[\Sigma] = \zeta_W^3 + 3(d-1)\zeta_2\zeta_W^2 + 3(d-1)^2\zeta_2^2\zeta_W \in A^3(\mathbb{P}W \times \mathbb{P}^2).$$

Proof: The computation is similar to the one used in the calculation of the class of the universal line in Section 6.6. Since every polynomial $F \in W$ defines a section τ_F of $\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$, we have a map

$$W \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d))$$

of vector bundles on $\mathbb{P}^n$. Likewise, we have the tautological inclusion

$$\mathcal{O}_{\mathbb{P}W}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}W}$$

on $\mathbb{P}W$. We pull these maps back to the product $\mathbb{P}W \times \mathbb{P}^2$ and compose them to obtain a map

$$\pi_1^*\mathcal{O}_{\mathbb{P}W}(-1) \to \pi_2^*\mathcal{P}^m(\mathcal{O}_{\mathbb{P}^n}(d)),$$

or, equivalently, a section of the bundle $\mathcal{P}^m$. The zero locus of this map is $\Sigma \subset \mathbb{P}W \times \mathbb{P}^n$, so the class of $\Sigma_{n,d,m}$ in $A(\mathbb{P}W \times \mathbb{P}^n)$ is the class of a section of $\mathcal{P}^m$, as claimed.

To compute the Chern class of $\mathcal{P}^m$, we follow the argument of Proposition 7.5, pulling back the sequences

$$0 \longrightarrow \mathrm{Sym}^i(\Omega_{\mathbb{P}^n})(d) \longrightarrow \mathcal{P}^i(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow \mathcal{P}^{i-1}(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow 0$$

and tensoring with the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}W}(1)$ to get

$$c(\mathcal{P}^m) = \prod_{j=0}^{m} c(\mathrm{Sym}^j(\pi_2^*\Omega_{\mathbb{P}^n}) \otimes \mathcal{O}(d\zeta_n + \zeta_W)),$$

where we write $\mathcal{O}(d\zeta_n + \zeta_W)$ as shorthand for the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}W}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^n}(d)$. Using the exact sequences

$$0 \longrightarrow \mathrm{Sym}^i(\Omega_{\mathbb{P}^n}) \longrightarrow \mathrm{Sym}^i(\mathcal{O}_{\mathbb{P}^n}(-1)) \longrightarrow \mathrm{Sym}^{i-1}(\mathcal{O}_{\mathbb{P}^n}(-1)) \longrightarrow 0$$

we get a collapsing product as before, yielding the desired formula for the Chern class of $\mathcal{P}^m$. To deduce the special case at the end of the proposition, it suffices to remember that since ζ_2 is the pullback from a two-dimensional variety we have $\zeta_2^3 = 0$. $\square$

7.6.2 The discriminant of a net of plane curves

We return to the case of a net of plane curves of degree d. Throughout this section we fix a general net of plane curves of degree d, that is, the family of curves associated to a general linear subspace $W \subset H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ of dimension 3, parametrized by $\mathcal{B} = \mathbb{P}W \cong \mathbb{P}^2$.

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Let $\mathcal{D} \subset \mathcal{B}$ be the set of singular curves, called the *discriminant curve* of the net $\mathcal{B}$. Since $\mathcal{D}$ is the intersection of $\mathcal{B}$ with the discriminant hypersurface in $\mathbb{P}W$, its degree is $\deg \mathcal{D} = 3(d-1)^2$ by Proposition 7.4. Next, let $\Gamma \subset \mathbb{P}^2$ be the plane curve traced out by the singular points of members of the net, so that if we set

$$\Sigma_{\mathcal{B}} := \Sigma \cap (\mathcal{B} \times \mathbb{P}^2),$$

then the projection maps π_i on Σ restrict to surjections

$$\begin{tikzcd} \Sigma_{\mathcal{B}} \arrow[r, "\pi_2|_{\mathcal{B}}"] \arrow[d, "\pi_1|_{\mathcal{B}}"'] & \Gamma \\ \mathcal{D} & \end{tikzcd}$$

Since Σ is smooth of codimension 3, Bertini's theorem shows that $\Sigma_{\mathcal{B}}$ is a smooth curve in $\mathcal{B} \times \mathbb{P}^2$. Since the generic singular plane curve is singular at only one point, the map $\Sigma_{\mathcal{B}} \to \Gamma$ is birational. Since the fiber of Σ over a given point $p \in \mathbb{P}^2$ is a linear space of dimension $N-3$, the general 2-plane $\mathcal{B}$ containing a curve singular at p will contain a unique such curve. Thus the map $\Sigma_{\mathcal{B}} \to \mathcal{D}$ is also birational, and $\Sigma_{\mathcal{B}}$ is the normalization of each of Γ and $\mathcal{D}$. In particular the geometric genus of $\mathcal{D}$ and that of Γ are the same as the genus of $\Sigma_{\mathcal{B}}$.

From the previous section, we know that $\Sigma_{\mathcal{B}}$ is the zero locus of a section of the rank-3 bundle $\mathcal{P}^1|_{\mathcal{B}\times\mathbb{P}^2}$ on $\mathcal{B} \times \mathbb{P}^2$. This makes it easy to compute the degree and genus of $\Sigma_{\mathcal{B}}$, and we will derive the degree and genus of Γ, answering Keynote Question (b):

Proposition 7.15. *With notation as above, the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism, so both curves are smooth. The curve Γ has degree $3d-3$, and thus has genus $\binom{3d-4}{2}$. When $d \geq 2$, the curve $\mathcal{D}$ is singular.*

We will see how the singularities of $\mathcal{D}$ arise, what they look like and how many there are in Chapter 11.

Proof: We begin with the degree of Γ, the number of points of intersection of Γ with a line $L \subset \mathbb{P}^2$. Since $\Sigma_{\mathcal{B}} \to \Gamma$ is birational, this is the same as the degree of the product $[\Sigma_{\mathcal{B}}]\zeta_2 \in A^4(\mathcal{B} \times \mathbb{P}^2)$. (More formally, $\pi_{2*}[\Sigma_{\mathcal{B}}] = \Gamma$ and $\pi_{2*}[\Sigma_{\mathcal{B}}][L] = [\Sigma_{\mathcal{B}}]\zeta_2$.) Write $\zeta_{\mathcal{B}}$ for the restriction of ζ_W, the pullback of the hyperplane section from $\mathbb{P}^N$, to $\mathcal{B} \times \mathbb{P}^2$. The degree of a class in $\mathcal{B} \times \mathbb{P}^2$ is the coefficient of $\zeta_2^2\zeta_{\mathcal{B}}^2$ in its expression in

$$A(\mathcal{B} \times \mathbb{P}^2) = \mathbb{Z}[\zeta_2, \zeta_{\mathcal{B}}]/(\zeta_2^3, \zeta_{\mathcal{B}}^3).$$

Since $\zeta_{\mathcal{B}}^3 = 0$, the last formula in Proposition 7.14 gives

$$\begin{aligned} \deg(\Gamma) &= \deg \zeta_2\big(3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}\big) \\ &= \deg 3(d-1)\zeta_2^2\zeta_{\mathcal{B}}^2 \\ &= 3d-3. \end{aligned}$$

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Since Γ is a plane curve, the arithmetic genus of the curve Γ is $\binom{3d-4}{2}$.

Next we compute the genus $g_{\Sigma_{\mathcal{B}}}$ of the smooth curve $\Sigma_{\mathcal{B}}$. The normal bundle of $\Sigma_{\mathcal{B}}$ in $\mathbb{P}^2 \times \mathcal{B}$ is the restriction of the rank-3 bundle $\mathcal{P}^1$, and the canonical divisor on $\mathbb{P}^2 \times \mathcal{B}$ has class $-3\zeta_2 - 3\zeta_{\mathcal{B}}$, so by the general adjunction formula (Part (c) of Proposition 6.15) the degree of the canonical class of $\Sigma_{\mathcal{B}}$ is the degree of the line bundle obtained by tensoring the canonical bundle of $\mathcal{B} \times \mathbb{P}^2$ with $\bigwedge^3 \mathcal{P}^1$ and restricting the result to $\Sigma_{\mathcal{B}}$. This is the degree of the class

$$(-3\zeta_2 - 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1))[\Sigma_B] = (-3\zeta_2 - 3\zeta_{\mathcal{B}} + c_1(\mathcal{P}^1)) \cdot \left(3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}\right).$$

Substituting the value $c_1(\mathcal{P}^1) = 3((d-1)\zeta_2 + \zeta_W)$ from Proposition 7.14 and taking account of the fact that $\zeta_W\zeta_{\mathcal{B}} = \zeta_{\mathcal{B}}^2$, this becomes

$$(3d-6)\zeta_2 \cdot \left(3(d-1)\zeta_2\zeta_{\mathcal{B}}^2 + 3(d-1)^2\zeta_2^2\zeta_{\mathcal{B}}\right) = (3d-6)(3d-3)\zeta_2^2\zeta_{\mathcal{B}}^2$$

with degree $2g_{\Sigma_B} - 2 = (3d-3)(3d-6)$, and we see that

$$g(\Sigma_{\mathcal{B}}) = \frac{(3d-4)(3d-5)}{2} = \binom{3d-4}{2}.$$

Since this coincides with the arithmetic genus of Γ computed above, we see that Γ is smooth and the map $\Sigma_{\mathcal{B}} \to \Gamma$ is an isomorphism. On the other hand the degree $3(d-1)^2$ of $\mathcal{D}$ is different from that of Γ for all $d \geq 2$, so in these cases the arithmetic and geometric genera of $\mathcal{D}$ differ, and $\mathcal{D}$ must be singular, completing the proof. □

Here is a different method for computing the degree of Γ: The net $\mathcal{B}$ of curves, having no base points, defines a regular map

$$\varphi_{\mathcal{B}} : \mathbb{P}^2 \to \Lambda,$$

where $\Lambda \cong \mathbb{P}^2$ is the projective plane dual to the plane parametrizing the curves in the net $\mathcal{B}$. This map expresses $\mathbb{P}^2$ as a d^2-sheeted branched cover of Λ, and the curve $\Gamma \subset \mathbb{P}^2$ is the ramification divisor of this map.

By definition,

$$\varphi^*\mathcal{O}_\Lambda(1) = \mathcal{O}_{\mathbb{P}^2}(d);$$

so that, if we denote by ζ_Λ the hyperplane class on Λ, we have $\varphi^*\zeta_\Lambda = d\zeta$.

Pulling back a 2-form via the map $\varphi : \mathbb{P}^2 \to \Lambda$ we see that

$$K_{\mathbb{P}^2} = \varphi^* K_\Lambda + \Gamma,$$

and since $K_\Lambda = -3\zeta_\Lambda$, this yields

$$-3\zeta = -3d\zeta + [\Gamma]$$

or $[\Gamma] = (3d-3)\zeta$.

These ideas work for a net $\mathcal{W} = (\mathcal{L}, W)$ on an arbitrary smooth projective surface S, as long as we know the classes $c_1(\Omega_S), c_2(\Omega_S)$ and $\lambda = c_1(\mathcal{L})$ and can evaluate the degrees of the relevant products in $A(S)$. See Exercise 7.31 for an example.

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7.7 The topological Hurwitz formula

In this section we will work explicitly over the complex numbers, so that we can use the topological Euler characteristic. Using this tool, we will give a different approach to questions of singular elements of linear series. It sheds additional light on the formula of Proposition 7.4, and is applicable in many circumstances in which Proposition 7.4 cannot be used. In addition, it will allow us to describe the local structure of the discriminant hypersurface, such as its tangent planes and tangent cones. By the Lefschetz principle (see for example Harris [1995, Chapter 15]), moreover, the purely algebro-geometric consequences of this analysis, such as Propositions 7.19 and 7.20, hold more generally over an arbitrary algebraically closed field of characteristic 0. (There are also alternative ways of defining an Euler characteristic with the desired properties algebraically.)

This approach is based on the following simple observation:

Proposition 7.16. *Let X be a smooth projective variety over $\mathbb{C}$, and $Y \subset X$ a divisor. If we denote by χ_{top} the topological Euler characteristic (in the classical, or analytic, topology), then*

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(Y) + \chi_{\mathrm{top}}(X \setminus Y).$$

Proof: This will follow from the Mayer–Vietoris sequence applied to the covering of X by $U = X \setminus Y$ and a small open neighborhood V of Y.

Let $\mathcal{L} = \mathcal{O}_X(Y)$, and let σ be the section of $\mathcal{L}$ vanishing on Y. Introducing Hermitian metrics on X and the line bundle $\mathcal{L}$, we can use the gradient of the absolute value of σ to define a $\mathcal{C}^\infty$ map $V \to Y$ expressing V as a fiber bundle over Y with fiber a disc D^2, and simultaneously expressing $V \cap U$ as a bundle over Y with fiber a punctured disc. It follows that

$$\chi_{\mathrm{top}}(V) = \chi_{\mathrm{top}}(Y) \quad \text{and} \quad \chi_{\mathrm{top}}(V \cap U) = 0,$$

and we deduce the desired relation. □

It is a surprising fact that the formula $\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(Y) + \chi_{\mathrm{top}}(X \setminus Y)$ applies much more generally to an arbitrary subvariety Y of an arbitrary X; see for example Fulton [1993, pp. 93–95, 142].

Now let X be a smooth projective variety, and let $f : X \to B$ be a map to a smooth curve B of genus g. This being characteristic 0, there are only a finite number of points $p_1, \dots, p_\delta \in B$ over which the fiber X_{p_i} is singular. We can apply the relation on Euler characteristics to the divisor

$$Y = \bigcup_{i=1}^{\delta} X_{p_i} \subset X.$$

Naturally, $\chi_{\mathrm{top}}(Y) = \sum \chi_{\mathrm{top}}(X_{p_i})$, and on the other hand the open set $X \setminus Y$ is a fiber bundle over the complement $B \setminus \{p_1, \dots, p_\delta\}$, so that

$$\chi_{\mathrm{top}}(X \setminus Y) = \chi_{\mathrm{top}}(X_\eta)\chi_{\mathrm{top}}(B \setminus \{p_1, \dots, p_\delta\}) = (2 - 2g - \delta)\chi_{\mathrm{top}}(X_\eta),$$

where again η is a general point of B. Combining these, we have

$$\begin{aligned}\chi_{\mathrm{top}}(X) &= (2 - 2g - \delta)\chi_{\mathrm{top}}(X_\eta) + \sum_{i=1}^{\delta} \chi_{\mathrm{top}}(X_{p_i}) \\ &= \chi_{\mathrm{top}}(B)\chi_{\mathrm{top}}(X_\eta) + \sum_{i=1}^{\delta}(\chi_{\mathrm{top}}(X_{p_i}) - \chi_{\mathrm{top}}(X_\eta)).\end{aligned}$$

In this form, we can extend the last summation over all points $q \in B$. We have proven:

Theorem 7.17 (Topological Hurwitz formula)**.** *Let $f : X \to B$ be a morphism from a smooth projective variety to a smooth projective curve; let $\eta \in B$ be a general point. Then*

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(B)\chi_{\mathrm{top}}(X_\eta) + \sum_{q \in B}(\chi_{\mathrm{top}}(X_q) - \chi_{\mathrm{top}}(X_\eta)).$$

In English: The Euler characteristic of X is what it would be if X were a fiber bundle over B — that is, the product of the Euler characteristics of B and the general fiber X_η — with a "correction term" coming from each singular fiber, equal to the difference between its Euler characteristic and the Euler characteristic of the general fiber.

To see why Theorem 7.17 is a generalization of the classical Riemann–Hurwitz formula (see for example Hartshorne [1977, Section IV.2]), consider the case where X is a smooth curve of genus h and $f : X \to C$ a branched cover of degree d. For each point $p \in C$, we write the fiber X_p as a divisor:

$$f^*(p) = \sum_{q \in f^{-1}(p)} m_q \cdot q.$$

We call the integer $m_q - 1$ the *ramification index* of f at q; we define the *ramification divisor* R of f to be the sum

$$R = \sum_{q \in X}(m_q - 1) \cdot q,$$

and we define the *branch divisor* B of f to be the image of R (as a divisor, not as a scheme!) — that is,

$$B = \sum_{p \in C} b_p \cdot p, \quad \text{where } b_p = \sum_{q \in f^{-1}(p)} m_q - 1.$$

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Now, since the degree of any fiber $X_p = f^{-1}(p)$ of f is equal to d, for each $p \in C$ the cardinality of $f^{-1}(p)$ will be $d - b_p$, so its contribution to the topological Hurwitz formula is $-b_p$. The formula then yields

$$2 - 2h = d(2 - 2g) - \deg(B),$$

the classical Riemann–Hurwitz formula.

7.7.1 Pencils of curves on a surface, revisited

To apply the topological Hurwitz formula to Keynote Question (a), suppose that $\{C_t = V(t_0 F + t_1 G) \subset \mathbb{P}^2\}$ is a general pencil of plane curves of degree d. Since the polynomials F and G are general, the base locus $\Gamma = V(F, G)$ of the pencil will consist of d^2 reduced points, and the total space of the pencil — that is, the graph

$$X = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid p \in C_t\}$$

of the rational map $[F, G] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ — is the blow-up of $\mathbb{P}^2$ along Γ. In particular, X is smooth, so Theorem 7.17 can be applied to the map $f : X \to \mathbb{P}^1$ that is the projection on the first factor.

Since X is the blow-up of $\mathbb{P}^2$ at d^2 points, we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(\mathbb{P}^2) + d^2 = d^2 + 3.$$

Next, we know that a general fiber C_η of the map f is a smooth plane curve of degree d; as we saw in Example 2.17, its genus is $\binom{d-1}{2}$ and hence

$$\chi_{\text{top}}(C_\eta) = -d^2 + 3d.$$

We know from Proposition 7.1 that each singular fiber C appearing in a general pencil of plane curves has a single node as singularity. By the calculation in Section 2.4.6, then, its normalization $\widetilde{C}$ will be a curve of genus $\binom{d-1}{2} - 1$ and hence Euler characteristic $-d^2 + 3d + 2$. Since C is obtained from $\widetilde{C}$ by identifying two points, we have

$$\chi_{\text{top}}(C) = -d^2 + 3d + 1,$$

so the contribution of each singular fiber of f to the topological Hurwitz formula is exactly 1. It follows that the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\text{top}}(X) - \chi_{\text{top}}(\mathbb{P}^1)\chi_{\text{top}}(C_\eta) \\ &= d^2 + 3 - 2(-d^2 + 3d) \\ &= 3d^2 - 6d + 3,\end{aligned}$$

as we saw before.

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This same analysis can be applied to a pencil of curves on any smooth surface S. Let $\mathcal{L}$ be a line bundle on S with first Chern class $c_1(\mathcal{L}) = \lambda \in A^1(S)$, and let $W = \langle \sigma_0, \sigma_1 \rangle \subset H^0(\mathcal{L})$ be a two-dimensional vector space of sections with

$$\{C_t = V(t_0\sigma_0 + t_1\sigma_1) \subset S\}_{t \in \mathbb{P}^1}$$

the corresponding pencil of curves. We make — for the time being — two assumptions:

(a) The base locus $\Gamma = V(\{\sigma\}_{\sigma \in W})$ of the pencil is reduced; that is, it consists of $\deg(\lambda^2)$ points.
(b) Each of the finitely many singular elements of the pencil has just one node as singularity.

We also denote by $c_i = c_i(\Omega_S)$ the Chern classes of the cotangent bundle to S.

Given this, the calculation proceeds as before: we let X be the blow-up of S along Γ, and apply the topological Hurwitz formula to the natural map $f : X \to \mathbb{P}W^* \cong \mathbb{P}^1$. To start, we have

$$\chi_{\text{top}}(X) = \chi_{\text{top}}(S) + \#(\Gamma) = c_2 + \lambda^2$$

(we omit the "deg" here for simplicity). Next, by the adjunction formula, the Euler characteristic of a smooth member C_η of the pencil is given by

$$\chi_{\text{top}}(C_\eta) = -\deg(\omega_{C_\eta}) = -(c_1 + \lambda) \cdot \lambda = -\lambda^2 - c_1\lambda,$$

and, by Section 2.4.6, as in the plane curve case the Euler characteristic of each singular element of the pencil is 1 greater than the Euler characteristic of the general element. In sum, then, the number of singular fibers is

$$\begin{aligned}\delta &= \chi_{\text{top}}(X) - \chi_{\text{top}}(\mathbb{P}^1)\chi_{\text{top}}(C_\eta)\\ &= \lambda^2 + c_2 - 2(-\lambda^2 - c_1\lambda)\\ &= 3\lambda^2 + 2\lambda c_1 + c_2,\end{aligned}$$

agreeing with our previous calculation.

We will see how this may be applied in higher dimensions in Exercises 7.43–7.44.

7.7.2 Multiplicities of the discriminant hypersurface

One striking thing about this derivation of the formula for the number of singular elements in a pencil is that it gives a description of the multiplicities with which a given singular element counts that allows us to determine these multiplicities at a glance.

In the derivation of the formula, we assumed that the singular elements of the pencil had only nodes as singularities. But what if an element C of the pencil has a cusp? In that case the calculation of Section 2.4.6 says that the geometric genus of the curve — the genus of its normalization $\widetilde{C}$ — is again 1 less than the genus of the smooth fiber, but this time instead of identifying two points of $\widetilde{C}$ we are just "crimping" the curve at one point. (In the analytic topology, C and $\widetilde{C}$ are homeomorphic.) Thus,

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(\widetilde{C}) = \chi_{\text{top}}(C_\eta) + 2,$$

and the fiber C "counts with multiplicity 2," in the sense that its contribution to the sum in the right-hand side of Theorem 7.17 is 2. Similarly, if C has a tacnode, we have $g(\widetilde{C}) = g(C_\eta) - 2$, so that $\chi_{\text{top}}(\widetilde{C}) = \chi_{\text{top}}(C_\eta) + 4$, but we identify two points of $\widetilde{C}$ to form C, so in all

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 3,$$

and the contribution of the fiber C to the rightmost term in Theorem 7.17 is thus 3. If C has an ordinary triple point — consisting of three smooth branches meeting at a point — then $g(\widetilde{C}) = g(C_\eta) - 3$, but we identify three points of $\widetilde{C}$ to form C, so

$$\chi_{\text{top}}(C) = \chi_{\text{top}}(C_\eta) + 4,$$

and the contribution of the fiber C is 4. Moreover, if a fiber has more than one isolated singularity, the same analysis shows that the multiplicity with which it appears in the formula above is just the sum of the contributions coming from the individual singularities.

In addition to giving us a way of determining the contribution of a given singular fiber to the expected number, this approach tells us something about the geometry of the discriminant locus $\mathcal{D} \subset \mathbb{P}^N$ in the space $\mathbb{P}^N$ of plane curves of degree d. To see this, suppose that $C \subset \mathbb{P}^2$ is any plane curve of degree d with isolated singularities. Let D be a general plane curve of the same degree, and consider the pencil $\mathcal{B}$ of plane curves they span — in other words, take $\mathcal{B} \subset \mathbb{P}^N$ a general line through the point $C \in \mathbb{P}^N$. By what we have said, the number of singular elements of the pencil $\mathcal{B}$ other than C will be $3(d-1)^2 - (\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta))$, where C_η is a smooth plane curve of degree d; it follows that the intersection multiplicity $m_p(\mathcal{B}, \mathcal{D})$ of $\mathcal{B}$ and $\mathcal{D}$ at C is $\chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta)$.

Proposition 7.18. *Let $C \subset \mathbb{P}^2$ be any plane curve of degree d with isolated singularities. Then*

$$\operatorname{mult}_C(\mathcal{D}) = \chi_{\text{top}}(C) - \chi_{\text{top}}(C_\eta),$$

where C_η is a smooth plane curve of degree d

Thus a plane curve with a cusp (and no other singularities) corresponds to a double point of $\mathcal{D}$, a plane curve with a tacnode is a triple point, and so on. A curve C with one node and no other singularities is necessarily a smooth point of $\mathcal{D}$.

7.7.3 Tangent cones of the discriminant hypersurface

We can use the ideas above to describe the tangent spaces and tangent cones to the discriminant hypersurface $\mathcal{D} \subset \mathbb{P}^N$. To do this, we have to remove the first assumption in our application of the topological Hurwitz formula to pencils of curves, and deal with pencils whose base loci are not reduced.

To consider the simplest such situation, suppose that $p \in L \subset \mathbb{P}^2$ are a point and a line in the plane, and that F, G are general forms of degree d such that $V(F)$ and $V(G)$ pass through p and are tangent to L at p. Let $\Gamma = V(F, G)$ be the base locus of the pencil $C_t = V(t_0 F + t_1 G)$, so that Γ will be a scheme of degree d^2 consisting of $d^2 - 2$ reduced points and one scheme of degree 2 supported at p. Since being singular at p is one linear condition on the elements of the pencil, exactly one member of the pencil (which we may take to be C_0 after re-parametrizing the pencil) will be singular at p.

We could arrive at such a pencil by taking F to be the equation of a general curve with a node p and G a general polynomial vanishing at p; thus for the general pencil above, the singular element C_0 of the pencil will have a node at p, with neither branch tangent to L, while all the others elements are smooth at p and have a common tangent line $\mathbb{T}_p(C_t) = L$ at p.

Let X be the minimal smooth blow-up resolving the indeterminacy of the rational map φ from $\mathbb{P}^2$ to $\mathbb{P}^1$ associated to the pencil — that is, X is obtained by blowing up $\mathbb{P}^2$ at Γ_{red} and then blowing up the resulting surface at the point p' on the exceptional divisor corresponding to the common tangent line L to the smooth members of the pencil at p. (This is *not* the blow-up of S along the scheme Γ, which is singular! See for example Eisenbud and Harris [2000, IV.2.3].) Note that we are blowing up $\mathbb{P}^2$ a total of d^2 times, so that the Euler characteristic $\chi_{\mathrm{top}}(X)$ is equal to $3 + d^2$, just as in the general case.

What is different is the fiber of the map $X \to \mathbb{P}^1$ over $t = 0$: Rather than being a copy of the curve $C_0 = V(F)$, it is the union of the proper transform of C_0 and the proper transform E of the first exceptional divisor, that is, the union of the normalization $\widetilde{C}_0$ of C_0 and a copy E of $\mathbb{P}^1$, meeting at the two points of $\widetilde{C}_0$ lying over the node p (See Figure 7.4).

In sum, the Euler characteristic of the fiber over $t = 0$ is

$$\chi_{\mathrm{top}}(\widetilde{C}_0) + \chi_{\mathrm{top}}(E) - 2 = \chi_{\mathrm{top}}(\widetilde{C}_0) = \chi_{\mathrm{top}}(C_\eta) + 2,$$

and the fiber counts with multiplicity 2. We can use this to analyze the tangent planes to $\mathcal{D}$ at its simplest points:

Proposition 7.19. *Let C be a plane curve with a node at p and no other singularities. The tangent plane $\mathbb{T}_C \mathcal{D} \subset \mathbb{P}^N$ is the hyperplane $H_p \subset \mathcal{D}$ of curves containing the point p.*

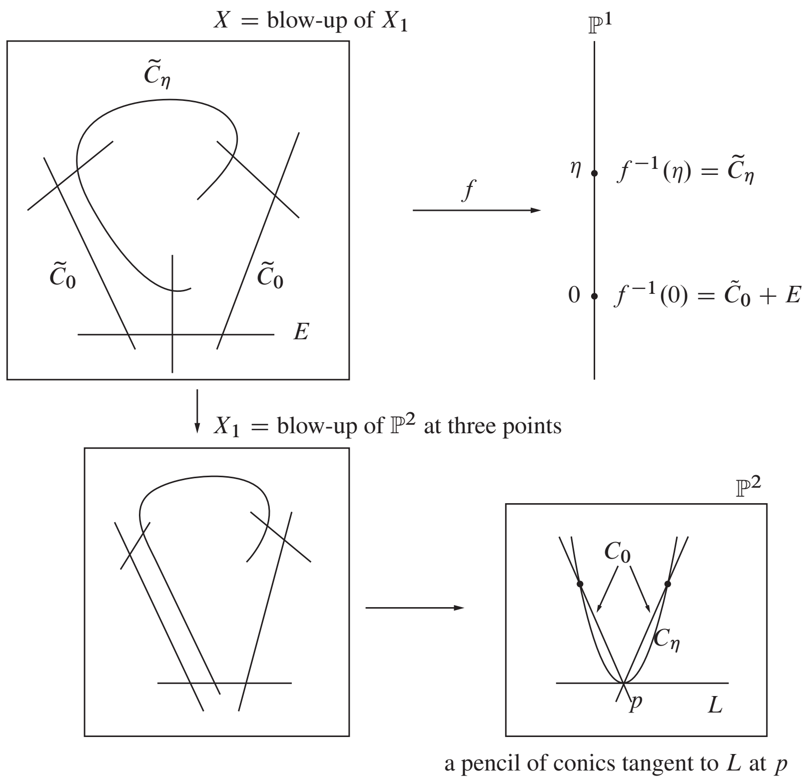

Figure 7.4 The morphism $f : X \to \mathbb{P}^1$ coming from the pencil of conics tangent to L at p.

Proof: If $C \subset \mathbb{P}^2$ is a plane curve with one node p and no other singularities, then, by Proposition 7.18, C is a smooth point of $\mathcal{D}$. It thus suffices to show that H_p is contained in the tangent space to $\mathcal{D}$ at C_0. But, as we have seen, if $\mathcal{B} \subset \mathbb{P}^N$ is a general pencil including C and having p as a base point, $\mathcal{B}$ will meet $\mathcal{D}$ in exactly $3(d-1)^2 - 1$ points — in other words, a general line $\mathcal{B} \subset \mathbb{P}^N$ through C and lying in H_p will be tangent to $\mathcal{D}$ somewhere.

As above, we may suppose that $\mathcal{B}$ is spanned by a curve $F = 0$ with a node at p and no other singularities and a smooth curve $G = 0$ that passes through p.

To complete the argument — to show that such a line is indeed tangent to $\mathcal{D}$ specifically at C, and not somewhere else — we have to do two things: We have to relate the pencil $\mathcal{B}$ to nearby general pencils, and we have to localize the Euler characteristic. For the first, choose a general polynomial G', and consider the family of pencils $\{\mathcal{B}_s\}$ with $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_s$ the pencil spanned by F and a linear combination $G_s = G + sG'$; that is,

$$\mathcal{B}_s = \{V(F + t(G + sG')) \mid t \in \mathbb{P}^1\}.$$

For each s, we let X_s be the total space of the pencil $\mathcal{B}_s$ and $f_s : X_s \to \mathbb{P}^1$ the map $[F, G + sG']$.

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For general s, the pencil $\mathcal{B}_s$ will be a general pencil of curves of degree d; in particular, if $\mu > 0$ is sufficiently small, then for any $0 < |s| < \mu$ the pencil $\mathcal{B}_s$ will intersect $\mathcal{D}$ transversely in exactly $3(d-1)^2$ points. Moreover, by our description of $\mathcal{B}_0 \cap \mathcal{D}$, we know that as s approaches 0, two of these $3(d-1)^2$ points will approach a particular element $C_{t_0} \in \mathcal{B}$ — the point of tangency of $\mathcal{B}$ with $\mathcal{D}$ — and the remaining $3(d-1)^2 - 2$ will remain distinct from each other and from C_{t_0}.

For the second component of the argument (localizing the Euler characteristic), we cover the t-line $\mathbb{P}^1_t$ by a pair of open sets: $U = (|t| < \epsilon)$ a disc around the point $t = 0$, and $V = \mathbb{P}^1 \setminus (|t| \geq \epsilon/2)$ the complement of a smaller closed disc. We can choose ϵ small enough so that no singular fiber of $\mathcal{B}$ other than C_0 lies in $\overline{U}$; in particular, no singular fiber lies in $U \cap V$. It follows that, for some $\mu > 0$, the same is true for all $\mathcal{B}_s$ with $|s| < \mu$: none of the singular fibers of $\mathcal{B}_s$ lie in the overlap $U \cap V$. For any $0 < |s| < \mu$, accordingly, the number of singular fibers of $\mathcal{B}_s$ in U is the intersection multiplicity $m_{C_0}(\mathcal{B}, \mathcal{D})$ of $\mathcal{B}$ and $\mathcal{D}$ at C_0, which we claim is 2.

Now consider the total space of our family of pencils:

$$\Phi = \{(s, t, p) \in \Delta \times \mathbb{P}^1 \times \mathbb{P}^2 \mid F(p) + t(G(p) + sG'(p)) = 0\}.$$

Let Φ^V be the preimage of V in Φ. Since the fiber Φ^V_s of Φ^V over each $s \in \Delta$ is smooth, Φ^V is a fiber bundle over Δ and, in particular, all the Φ^V_s have the same Euler characteristic. We know that Φ^V_0 has exactly $3(d-1)^2 - 2$ singular fibers, each a curve with a single node, so that by Theorem 7.17

$$\chi_{\text{top}}(\Phi^V_0) = -d(d-3) + 3(d-1)^2 - 2;$$

since for $s \neq 0$ the Φ^V_s have the same Euler characteristic, the same logic tells us that they also have exactly $3(d-1)^2 - 2$ singular fibers over V. It follows that Φ^V_s has two singular fibers for $0 < |s| < \mu$, completing the argument. □

This argument shows that, more generally, if C is a plane curve with a unique singular point p, the tangent cone to $\mathcal{D}$ at C will be a multiple of the hyperplane H_p, and, more generally still, if C has isolated singularities $p_1, \ldots, p_\delta$, the tangent cone $\mathbb{T}_C\mathcal{D}$ is supported on the union of the planes H_{p_i}.

There is also a sort of converse to Proposition 7.18:

Proposition 7.20. *The smooth locus of $\mathcal{D}$ consists exactly of those curves with a single node and no other singularity.*

Proof: Proposition 7.18 gives one inclusion: if C has a node and no other singularity, it is a smooth point of $\mathcal{D}$. Moreover, if C has more than one (isolated) singular point, then the projection map $\Sigma \to \mathcal{D}$ is finite but not one-to-one over C; it is intuitively clear (and follows from Zariski's main theorem) that $\mathcal{D}$ is analytically reducible and hence singular at C. Moreover, we observe that if $d \geq 3$ any curve with multiple components is a limit of curves with isolated singularities and at least three nodes — just deform each multiple

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component $mC_0 \subset C$ to a union of m general translates of C_0 — so these must also lie in the singular locus of $\mathcal{D}$.

It remains to see that if C is a singular curve having a singularity p other than a node, then $\mathcal{D}$ is singular at C. This follows from an analysis of plane curve singularities: If C has isolated singularities including a point p of multiplicity $k \geq 3$, then, as we saw in Section 2.4.6, the genus of the normalization $\widetilde{C}$ is at most

$$g(\tilde{D}) \leq \binom{d-1}{2} - \frac{k(k-1)}{2}$$

and, since at most k points of the normalization lying over p are identified in C,

$$\chi_{\text{top}}(C) \geq 2 - 2g(\widetilde{C}) + k - 1 \geq -d(d-3) + (k-1)^2.$$

As for double points p other than a node, we have already done the case of a cusp; other double points will drop the genus of the normalization by 2 or more, and since we have at most two points of the normalization lying over p, we must have $\chi_{\text{top}}(C) \geq -d(d-3) + 3$. □

Finally, note that the techniques of this section can be applied in exactly the same way in one dimension lower!

Proposition 7.21. *Let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree d on $\mathbb{P}^1$, and $\mathcal{D} \subset \mathbb{P}^d$ the discriminant hypersurface, that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d-2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p.*

We leave the proof via the topological Hurwitz formula as an exercise; for an algebraic proof, see Proposition 8.6.

We add that there are many, many problems having to do with the local geometry of $\mathcal{D}$ and its stratification by singularity type, only a small fraction of which we know how to answer. The statements above barely scratch the surface; for more, see for example Brieskorn and Knörrer [1986] or Teissier [1977].

7.8 Exercises

Exercise 7.22. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S, where $a, b > 0$. What is the expected number of curves C_t that are singular? (Make sure your answer agrees with (7.2) in the case $(a, b) = (1, 1)$!)

Exercise 7.23. Prove that the number found in the previous exercise is the actual number of singular elements; that is, prove the three hypotheses of Proposition 7.9 in the case of $S = \mathbb{P}^1 \times \mathbb{P}^1$ and the line bundle $\mathcal{O}(a, b)$.

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Exercise 7.24. Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $L \subset S$ a line. Let $\{C_t\}_{t \in \mathbb{P}^1}$ be the pencil of conics on S cut out by the pencil of planes $\{H_t \subset \mathbb{P}^3\}$ containing L. How many of the conics C_t are singular? Use this to answer the question of how many other lines on S meet L.

Exercise 7.25. Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves singular at p — in other words, let F and G be two general polynomials vanishing to order 2 at p, and take $C_t = V(t_0 F + t_1 G)$. How many of the curves C_t will be singular somewhere else as well?

Exercise 7.26. Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. If $\{H_t \subset \mathbb{P}^4\}_{t \in \mathbb{P}^1}$ is a general pencil of hyperplanes in $\mathbb{P}^4$, find the expected number of singular hyperplane sections $S \cap H_t$. (Equivalently: if $\Lambda \cong \mathbb{P}^2 \subset \mathbb{P}^4$ is a general 2-plane, how many tangent planes to S intersect Λ in a line?)

Exercise 7.27. Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Using formula (7.1), find the expected number of singular hyperplane sections of X in a pencil. Again, compare your answer to the result of Section 2.1.3.

Exercise 7.28. Let $X \cong \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ be the Segre threefold. Using formula (7.1), find the number of singular hyperplane sections of X in a pencil.

Exercise 7.29. Let $S = X_1 \cap X_2 \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees e and f. What is the expected number of hyperplane sections of S having a triple point? (Check this in the case $e = f = 2$!)

Exercise 7.30. Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g; let e and f be the degrees of the classes $c_1(\mathcal{T}_S)^2, c_2(\mathcal{T}_S) \in A^2(S)$. Find the class of the cycle $T_1(S) \subset \mathbb{G}(1,n)$ of lines tangent to S in terms of d, e, f and g; from Exercise 4.21, we need only the intersection number $[T_1(S)] \cdot \sigma_3$. *Hint:* Consider instead the variety of tangent planes $T_2(S) \subset \mathbb{G}(2,n)$, and find the intersection with σ_2 as the intersection with $(\sigma_1)^2$ minus the intersection with $\sigma_{1,1}$.

Exercise 7.31. Let $S \subset \mathbb{P}^3$ be a general surface of degree d and $\mathcal{B}$ a general net of plane sections of S (that is, intersections of X with planes containing a general point $p \in \mathbb{P}^3$). What are the degree and genus of the curve $\Gamma \subset S$ traced out by singular points of this net? What are the degree and genus of the discriminant curve? Use this to describe the geometry of the finite map $\pi_p : S \to \mathbb{P}^2$ given by projection from p.

Exercise 7.32. Verify that for a general curve $C \subset \mathbb{P}^2$ of degree d the number $3d(d-2)$ is the actual number of flexes of C, that is, that all inflection points of C have weight 1.

Exercise 7.33. Let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves of degree $d \geq 3$; suppose C_0 is a singular element of C (so that in particular by Proposition 7.1 C_0 will have just one node as singularity). By our formula, C_0 will have six fewer flexes than the general member C_t of the pencil. Where do the other six flexes go? If we consider the incidence correspondence

$$\Phi = \{(t, p) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid C_t \text{ is smooth and } p \text{ is a flex of } C_t\},$$

what is the geometry of the closure of Φ near $t = 0$? Bonus question: Describe the geometry of

$$\widetilde{\Phi} = \{(t, p, L) \in \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^{2*} \mid C_t \text{ smooth, } p \text{ a flex of } C_t \text{ and } L = \mathbb{T}_p C_t\}$$

near $t = 0$.

Exercise 7.34. Find the points on $\mathbb{P}^1$, if any, that are ramification points for the maps $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(s, t) \mapsto (s^3, s^2t, st^2, t^3) \in \mathbb{P}^3 \quad \text{and} \quad (s, t) \mapsto (s^4, s^3t, st^3, t^4) \in \mathbb{P}^3.$$

Exercise 7.35. Show that the only smooth, irreducible and nondegenerate curve $C \subset \mathbb{P}^r$ with no inflection points is the rational normal curve.

Exercise 7.36. We define an *elliptic normal curve* to be a smooth irreducible nondegenerate curve of genus 1 and degree $r + 1$ in $\mathbb{P}^r$. Observe that for an elliptic normal curve E the Plücker formula yields the number $(r + 1)^2$ of inflection points. Show that these are exactly the images of any one under the group of translations of order $r + 1$ on E, each having weight 1.

Exercise 7.37. Let C be a smooth curve of genus $g \geq 2$. A point $p \in C$ is called a *Weierstrass point* if there exists a nonconstant rational function on C with a pole of order g or less at p and regular on $C \setminus \{p\}$.

(a) Show that the Weierstrass points of C are exactly the inflection points of the canonical map $\varphi : C \to \mathbb{P}^{g-1}$.
(b) Use this to count the number of Weierstrass points on C.

Exercise 7.38. Let $\mathbb{P}^N$ be the space of all plane curves of degree $d \geq 4$, and let $H \subset \mathbb{P}^N$ be the closure of the locus of smooth curves with a hyperflex. Show that H is a hypersurface. (We will be able to calculate the degree of this hypersurface once we have developed the techniques of Chapter 11.)

Exercise 7.39. To prove that a general complete intersection $C \subset \mathbb{P}^3$ does not have weight-2 inflection points, we need to prove that it does not have flex lines (lines with multiplicity-3 intersection with the curve) or planes with a point of contact of order 5. Prove the first statement: that a general complete intersection of two surfaces S_1 and S_2 of degrees $d_1 \geq d_2 > 1$ does not have a flex line.

The following two exercises show how to construct an example of a component of the Hilbert scheme whose general member is a smooth, irreducible, nondegenerate curve having inflection points of weight > 1.

Exercise 7.40. Let $S = \overline{p, E} \subset \mathbb{P}^n$ be a cone over an elliptic normal curve $E \subset \mathbb{P}^{n-1}$ (that is, a smooth curve of genus 1 embedded by a complete linear system of degree n), and let $L_1, \dots, L_{n-1} \subset S$ be lines of the ruling. Show that, for $n > 9$ and $m \gg 0$:

(a) The residual intersection C of S with a general hypersurface $X \subset \mathbb{P}^n$ of degree m containing $L_1, \dots, L_{n-1}$ is a smooth, irreducible and nondegenerate curve.
(b) Every deformation of C also lies on a cone over an elliptic normal curve. (The condition $n > 9$ is necessary to ensure that the surface S has itself no deformations other than cones. This follows from the classification of *del Pezzo surfaces*; see for example Beauville [1996].)

Thus the smooth, irreducible and nondegenerate curves C constructed in this fashion form an open subset of the Hilbert scheme of curves in $\mathbb{P}^n$.

Exercise 7.41. Let $C \subset S \subset \mathbb{P}^n$ be a curve as constructed in the preceding problem. Show that C has inflection points of weight > 1 (look at points where C is tangent to a line of the ruling of S).

Exercise 7.42. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 2$, $p \in S$ a general point and $H = \mathbb{T}_p S \subset \mathbb{P}^3$ the tangent plane to S at p. Show by an elementary dimension count (not using the second fundamental form or quoting Theorem 7.11) that the intersection $H \cap S$ has an ordinary double point at p.

Exercise 7.43. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\{C_t \subset S\}_{t \in \mathbb{P}^1}$ be a general pencil of curves of type (a, b) on S. Use the topological Hurwitz formula to say how many of the curves C_t are singular. (Compare this with your answer to Exercise 7.22.)

Exercise 7.44. Let $p \in \mathbb{P}^2$ be a point, and let $\{C_t \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be a general pencil of plane curves of degree d singular at p, as in Exercise 7.25. Use the topological Hurwitz formula to count the number of curves in the pencil singular somewhere else.

Exercise 7.45. Let $\mathbb{P}^5$ be the space of conic plane curves and $\mathcal{D} \subset \mathbb{P}^5$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double line. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?

Exercise 7.46. Now, let $\mathbb{P}^{14}$ be the space of quartic plane curves and $\mathcal{D} \subset \mathbb{P}^{14}$ the discriminant hypersurface. Let $C \in \mathcal{D}$ be a point corresponding to a double conic. What is the multiplicity of $\mathcal{D}$ at C, and what is the tangent cone?