Problems from Hartshorne Chapter 2.2

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EXERCISE 2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \longrightarrow K$.

Proof: Suppose first that we have a map $f : \operatorname{Spec} K \longrightarrow X$. Topologically, this is determined solely by choosing an image $x \in f(P)$ for the sole point $P \in \operatorname{Spec} K$. Sheaf theoretically, this consists of a map $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_K$ (by $\mathcal{O}_K$ we mean $\mathcal{O}_{\operatorname{Spec} K}$). This induces a local ring map on the stalk at P: $f^\sharp_P : \mathcal{O}_{X,x} \longrightarrow (f_*\mathcal{O}_K)_P = K$, meaning that the maximal ideal $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$ is sent to the maximal ideal $(0) \subseteq K$, meaning that $\mathfrak{m}_x = \ker f^\sharp_P$. This in turn implies that $f^\sharp_P$ factors through the quotient $\pi : \mathcal{O}_{X,x} \mapsto k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and hence induces a map $k(x) \longrightarrow K$. This map is necessarily an inclusion since every ring homomorphism of fields is injective.

Now suppose we have an injection $p : k(x) \hookrightarrow K$. We can then define a map $f^\sharp_x : \mathcal{O}_{X,x} \longrightarrow K$ by $f^\sharp = p \circ \pi$, where $\pi : \mathcal{O}_{X,x} \longrightarrow k(x)$ is the quotient map. This is precisely a map on between the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{K,P}$. If we define $f : \operatorname{Spec} K \longrightarrow X$ by $P \mapsto x$ and $f^\sharp(U) : \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_K(U) = K$ by $f^\sharp(U) = f^\sharp_x \circ \iota$ where $\iota : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$ is the natural localization map, then $(f, f^\sharp)$ is a map of schemes. Note that for any open set $U \subseteq X$ not containing x the map $f^\sharp : \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_K(U)$ is necessarily the zero map, since $f_*\mathcal{O}_K(U) = \mathcal{O}_K(f^{-1}(U)) = \mathcal{O}_K(\varnothing) = 0$. $\square$

EXERCISE 2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof: The ring $k[x]$ is a PID since k is a field, so the prime ideals are all principally generated by irreducible polynomials $f \in k[x]$. $\square$