Example Sheet 4

Isaac Martin

EXERCISE 3. Let $A \subseteq B$ be a subring of B with the property that $B \setminus A$ is a multiplicatively closed set. Show that A is integrally closed in B.

> *Proof:* Let $S = B \setminus A$ and suppose we have some element $x \in B$ integral over A, i.e. such that $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ for some elements $a_0, \ldots, a_{n-1} \in A$. Choose n to be as small as possible. This equation implies
>
> $$b\left(b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_1\right) = -a_0 \in A.$$
>
> The elements b and $b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_1$ cannot both avoid A, otherwise their product would also avoid A. If $b \in A$ then we are done, so suppose $b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_1 \in A$. In particular, by subtracting a_1, we see that $b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_2 b \in A$. Setting $-a_1' = b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_2 b$, we see that
>
> $$b^{n-1} + a_{n-1}b^{n-2} + \ldots + a_2 b^1 + a_1' = 0,$$
>
> hence b is the root of a monic polynomial of degree strictly smaller than n. Hence this polynomial is trivial and $n = 1$, implying that $b \in A$. We conclude that A is integrally closed in B. □

EXERCISE 8. Let k be a field, and Γ any totally ordered abelian group. Let $A := k[\Gamma]$ denote the group ring of Γ, i.e. $k[\Gamma]$ is the k-vector space with basis $z^\gamma \mid \gamma \in \Gamma$ and multiplication determined by

$$z^\gamma \cdot z^{\gamma'} = z^{\gamma+\gamma'}.$$

Show that A is an integral domain.

Define $v_0 : A \setminus 0 \to \Gamma$ by

$$v_0\left(\sum_{i \in I} \alpha_i z^{\gamma_i}\right) = \min\{\gamma_i \mid i \in I\}$$

where I is a finite index set and $\alpha_i \in k \setminus \{0\}$ for each i. Show v_0 satisfies conditions (a) and (b) of Questions 7. Now let K be the field of fractions of A. Show that v_0 can be extended to $v : K^* \to \Gamma$ so that v is a valuation with value group Γ.