Chapter 3

Introduction to Grassmannians and lines in $\mathbb{P}^3$

Keynote Questions

(a) Given four general lines $L_1, \dots, L_4 \subset \mathbb{P}^3$, how many lines $L \subset \mathbb{P}^3$ will meet all four? (Answer on page 110.)

(b) Given four curves $C_1, \dots, C_4 \subset \mathbb{P}^3$ of degrees $d_1, \dots, d_4$, how many lines will meet general translates of all four? (Answer on page 112.)

(c) If $C, C' \subset \mathbb{P}^3$ are two general twisted cubic curves, how many chords do they have in common? That is, how many lines will meet each twice? (Answer on page 115.)

(d) If $Q_1, \dots, Q_4 \subset \mathbb{P}^3$ are four general quadric surfaces, how many lines are tangent to all four? (Answer on page 125.)

3.1 Enumerative formulas

In this chapter we introduce Grassmannian varieties through enumerative problems, of which the keynote questions above are examples. To clarify this context we begin by discussing enumerative problems in general and their relation to the intersection theory described in the preceding chapters.

In Section 3.2 we lay out the basic facts about Grassmannians in general. (Sections 3.2.5 and 3.2.6 may be omitted on the first reading, but will be important in later chapters.)

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Starting in Section 3.3 we focus on the Grassmannian of lines in $\mathbb{P}^3$. We calculate the Chow ring and then, in Sections 3.4 and 3.6, use this to solve some enumerative problems involving lines, curves and surfaces in $\mathbb{P}^3$. In Section 3.5 we introduce the key technique of *specialization*, using it to re-derive some of these formulas.

3.1.1 What are enumerative problems, and how do we solve them?

Enumerative problems in algebraic geometry ask us to describe the set Φ of objects of a certain type satisfying a number of conditions — for example, the set of lines in $\mathbb{P}^3$ meeting each of four given lines, as in Keynote Question (a), or meeting each of four given curves $C_i \subset \mathbb{P}^3$, as in Keynote Question (b). In the most common situation, we expect Φ to be finite and we ask for its cardinality, whence the name enumerative geometry. Enumerative problems are interesting in their own right, but — as van der Waerden is quoted as saying in the introduction — they are also a wonderful way to learn some of the more advanced ideas and techniques of algebraic geometry, which is why they play such a central role in this text.

There are a number of steps common to most enumerative problems, all of which will be illustrated in the examples of this chapter. If we are asked to describe the set Φ of objects of a certain type that satisfy a number of conditions, we typically carry out the following five steps:

- *Find or construct a suitable parameter space $\mathcal{H}$ for the objects we seek.* Suitable, for us, will mean that $\mathcal{H}$ should be projective and smooth, so that we can carry out calculations in the Chow ring $A(\mathcal{H})$. Most importantly, though, for each condition imposed, the locus $Z_i \subset \mathcal{H}$ of objects satisfying that condition should be a closed subscheme (which means in turn that the set $\Phi = \bigcap Z_i$ of solutions to our geometric problem will likewise have the structure of a subscheme of $\mathcal{H}$).

  In our examples, the natural choice of parameter space $\mathcal{H}$ is the Grassmannian $G = \mathbb{G}(1,3)$ parametrizing lines in $\mathbb{P}^3$, which we will construct and describe in Sections 3.2.1 and 3.2.2 below; as we will see, it is indeed smooth and projective of dimension 4. As we will see in Sections 3.3.1 and 3.4.2, moreover, the locus $\Sigma_C \subset G$ of lines $\Lambda \subset \mathbb{P}^3$ meeting a given curve $C \subset \mathbb{P}^3$ will indeed be a closed subscheme of codimension 1.

- *Describe the Chow ring $A(\mathcal{H})$ of $\mathcal{H}$.* This is what we will undertake in Section 3.3 below; in the case of the Grassmannian $\mathbb{G}(1,3)$, we will be able to give a complete description of its Chow ring. (In some circumstances, we may have to work with the cohomology ring rather than the Chow ring, as in Appendix D, or with a subring of $A(\mathcal{H})$ including the classes of the subschemes Z_i, as in Chapter 8.)

- *Find the classes $[Z_i] \in A(\mathcal{H})$ of the loci of objects satisfying the conditions imposed.* Thus, in the case of Keynote Question (b), we have to determine the class in $A(G)$ of the locus $Z_i \subset G$ of lines meeting the curve C_i; the answer is given in Section 3.4.2.

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- *Calculate the product of the classes found in the preceding step*. If we have done everything correctly up to this point, this should be a straightforward combination of the two preceding steps.

At this point, we have what is known as an *enumerative formula*: It describes the class, in $A(\mathcal{H})$, of the scheme $\Phi \subset \mathcal{H}$ of solutions to our geometric problem, *under the assumption that this locus has the expected dimension and is generically reduced* — that is, the cycles $Z_i \subset \mathcal{H}$ intersect generically transversely. (If the cycles Z_i are all locally Cohen–Macaulay, then by Section 1.3.7 the enumerative formula describes the class of the subscheme $\Phi \subset \mathcal{H}$ under the weaker hypothesis that Φ has the expected dimension; that is, the cycles Z_i are dimensionally transverse.)

- *Verify that the set of solutions, viewed as a subscheme of $\mathcal{H}$, indeed has the expected dimension, and investigate its geometry*. We will discuss, in the following section, what exactly we have proven if we simply stop at the conclusion of the last step. But ideally we would like to complete the analysis and say when the cycles $Z_i \subset \mathcal{H}$ do in fact meet generically transversely or dimensionally transversely. In particular, if the geometric problem posed depends on choices — the number of lines meeting each of four curves C_i, for example, depends on the C_i — we would like to be able to say that for general choices the corresponding scheme Φ is indeed generically reduced.

Thus, for example, in the case of Keynote Question (b), the analysis described above and carried out in Section 3.4.2 will tell us that if the subscheme $\Phi \subset G$ of lines meeting each of four curves $C_i \subset \mathbb{P}^3$ is zero-dimensional then it has degree $2\prod \deg(C_i)$. But it does *not* tell us that the actual number of lines meeting each of the four curves is in fact $2\prod \deg(C_i)$ for general C_i, or for that matter for any. That is addressed in Section 3.4.2 in characteristic 0; we will also see another approach to this question in Exercises 3.30–3.33 that also works in positive characteristic.

One reason this last step is sometimes given short shrift is that it is often the hardest. For example, it typically involves knowledge of the local geometry of the subschemes $Z_i \subset \mathcal{H}$ — their smoothness or singularity, and their tangent spaces or tangent cones accordingly — and this is usually finer information than their dimensions and classes. But it is necessary, if the result of the first four steps is to give a description of the actual set of solutions, and it is also a great occasion to learn some of the relevant geometry.

3.1.2 The content of an enumerative formula

Because the last step in the process described above is sometimes beyond our reach, it is worth saying exactly what has been proved when we carry out just the first four steps in the process.

In general, the computation of the product $\alpha = \prod [Z_i] \in A(\mathcal{H})$ of the classes of some effective cycles Z_i in a space $\mathcal{H}$ tells us the following:

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(a) If $\alpha \neq 0$ (for example, if $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) \neq 0$), *we can conclude that the intersection* $\bigcap Z_i$ *is nonempty*. This is the source of many applications of enumerative geometry; for example, it is the basis of the Kempf/Kleiman–Laksov proof of the existence half of the Brill–Noether theorem, described in Appendix D.

(b) If the cycles Z_i intersect in the expected dimension, then the class α is a positive linear combination of the classes of the components of the intersection $\bigcap Z_i$. In particular, if $\alpha \in A_0(\mathcal{H})$ has dimension 0, then *the number of points of* $\bigcap Z_i$ *is at most* $\deg(\alpha)$. This in turn implies:

(i) If $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) < 0$, we may conclude that the intersection $\bigcap Z_i$ is infinite rather than finite. More generally, if α is not the class of an effective cycle, *we can conclude that* $\bigcap Z_i$ *has dimension greater than the expected dimension.*

(ii) If $\alpha \in A_0(\mathcal{H})$ and $\deg(\alpha) = 0$, then the intersection $\bigcap Z_i$ must either be empty or infinite. (In general, if $\alpha = 0$ we can conclude that either $\bigcap Z_i = \varnothing$ or $\bigcap Z_i$ has dimension greater than the expected dimension.)

So, suppose we have carried out the first four steps in the process of the preceding section in the case of Keynote Question (a): We have described the Grassmannian $G = \mathbb{G}(1,3)$ and its Chow ring, found the class $\sigma_1 = [Z]$ of the cycle Z of lines meeting a given line $L \subset \mathbb{P}^3$, and calculated that $\deg(\sigma_1^4) = 2$. What does this tell us?

Without a verification of transversality, the formula $\deg \sigma_1^4 = 2$ really only tells us that the number of intersections is either infinite or 1 or 2. Beyond this, it says that *if* the number of "solutions to the problem" — in this case, lines in $\mathbb{P}^3$ that meet the four given lines — is finite, then there are two *counted with multiplicity* — that is, either two solutions with multiplicity 1, or one solution with multiplicity 2. In order to say more, we need to be able to say when the intersection $\bigcap Z_i$ has the expected dimension; we need to be able to detect transversality and, ideally, to calculate the multiplicity of a given solution. (The third of these is often the hardest. For example, in the calculation of the number of lines meeting four given curves $C_i \subset \mathbb{P}^3$, we see in Exercises 3.30–3.33 how to check the condition of transversality, but there is no simple formula for the multiplicity when the intersection is not transverse.)

A common aspect of enumerative problems is that they themselves may vary with parameters: If we ask how many lines meet each of four curves C_i, the problem varies with the choice of curves C_i. In these situations, a good benchmark of our understanding is whether we can count the actual number of solutions for a *general* such problem: for example, whether we can prove that if $C_1, \dots, C_4$ are general conics, then there are exactly 32 lines meeting all four. Thus, in most of the examples of enumerative geometry we will encounter in this book, there are two aspects to the problem. The first is to find the "expected" number of solutions by carrying out the first four steps of the preceding section to arrive at an enumerative formula. The second is to verify transversality — in other words, that the actual cardinality of the set of solutions is indeed this expected number — when the problem is suitably general.

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3.2 Introduction to Grassmannians

A *Grassmann variety*, or *Grassmannian*, is a projective variety whose closed points correspond to the vector subspaces of a certain dimension in a given vector space. Projective spaces, which parametrize one-dimensional subspaces, are the most familiar examples. In this chapter we will begin the study of Grassmannians in general, and then focus on the geometry and Chow ring of the Grassmannian of lines in $\mathbb{P}^3$, the first and most intuitively accessible example beyond projective spaces.

Our goal in doing this is to introduce the reader to some ideas that will be developed in much greater generality (and complexity) in later chapters: the Grassmannian (as an example of parameter spaces), the methods of *undetermined coefficients* and *specialization* for computing intersection products more complicated than those mentioned in Chapter 2, and questions of transversality, treated via the tangent spaces to parameter spaces. For more information about Grassmannians, the reader may consult the books of Harris [1995] for basic geometry of the Grassmannian, Griffiths and Harris [1994] for the basics of the Schubert calculus and Fulton [1997] for combinatorial formulas, as well as the classic treatment in the second volume of Hodge and Pedoe [1952].

As a set, we take the Grassmannian $G = G(k, V)$ to be the set of k-dimensional vector subspaces of the vector space V. We give this set the structure of a projective variety by giving an inclusion in a projective space, called the *Plücker embedding*, and showing that the image is the zero locus of a certain collection of homogeneous polynomials.

A k-dimensional vector subspace of an n-dimensional vector space V is the same as a $(k-1)$-dimensional linear subspace of $\mathbb{P}V \cong \mathbb{P}^{n-1}$, so the Grassmannian $G(k, V)$ could also be thought of as parametrizing $(k-1)$-dimensional subspaces of $\mathbb{P}V$. We will write the Grassmannian $G(k, V)$ as $\mathbb{G}(k-1, \mathbb{P}V)$ when we wish to think of it this way. When there is no need to specify the vector space V but only its dimension, say n, we will write simply $G(k, n)$ or $\mathbb{G}(k-1, n-1)$. Note also that there is a natural identification

$$G(k, V) = G(n-k, V^*)$$

sending a k-dimensional subspace $\Lambda \subset V$ to its annihilator $\Lambda^\perp \subset V^*$.

There are two points of potential confusion in the notation. First, if $\Lambda \subset V$ is a k-dimensional vector subspace of an n-dimensional vector space V, we will often use the same symbol Λ to denote the corresponding point in $G = G(k, V)$. When we need to make the distinction explicit, we will write $[\Lambda] \in G$ for the point corresponding to the plane $\Lambda \subset V$. Second, when we consider the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ we will sometimes need to work with the corresponding vector subspaces of V. In these circumstances, if $\Lambda \subset \mathbb{P}V$ is a k-plane, we will write $\widetilde{\Lambda}$ for the corresponding $(k+1)$-dimensional vector subspace of V.

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3.2.1 The Plücker embedding

To embed the set of k-dimensional vector subspaces of a given vector space V in a projective space, we associate to a k-dimensional subspace $\Lambda \subset V$ the one-dimensional subspace

$$\textstyle\bigwedge^k \Lambda \subset \bigwedge^k V;$$

that is, if Λ has basis $v_1, \dots, v_k$, we associate to it the point of $\mathbb{P}(\bigwedge^k V)$ corresponding to the line spanned by $v_1 \wedge \cdots \wedge v_k$. This gives us a map of sets

$$G(k, V) \to \mathbb{P}(\textstyle\bigwedge^k V) \cong \mathbb{P}^{\binom{n}{k}-1},$$

called the *Plücker embedding*. To see that this map is one-to-one, observe that if $v_1, \dots, v_k$ are a basis of $\Lambda \subset V$, then a vector v annihilates $\eta = v_1 \wedge \cdots \wedge v_k$ in the exterior algebra if and only if v is in the span Λ of $v_1, \dots, v_k$; thus η determines Λ.

Concretely, if we choose a basis $\{e_1, \dots, e_n\}$ for V, and so identify V with $\mathbb{k}^n$, we may represent Λ as the row space of a $k \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{pmatrix}.$$

In these terms, a basis for $\bigwedge^k V$ is given by the set of products

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n},$$

and if $v_1, \dots, v_k$ is a basis for Λ then we may write a nonzero element of $\bigwedge^k \Lambda$ in the form

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\dots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Here the scalar $p_{i_1,\dots,i_k}$ is the determinant of the submatrix (that is, *minor*) of A made from the columns $i_1, \dots, i_k$. These $p_{i_1,\dots,i_k}$ are called the *Plücker coordinates* of Λ.

The matrix A is not unique, since we can multiply on the left by any invertible $k \times k$ matrix Ω without changing the row space, but the collection of $k \times k$ minors of A, viewed as a vector in $\mathbb{k}^{\binom{n}{k}}$, is well-defined up to scalars: Multiplying by Ω multiplies each such minor by $\det(\Omega)$. (Conversely, if another matrix A' has the same row space, then we can write $A' = \Omega A$ for some invertible $k \times k$ matrix Ω.)

A quick-and-dirty way to see that the image $G \hookrightarrow \mathbb{P}(\bigwedge^k V)$ of the Plücker embedding — the locus of vectors $\eta \in \bigwedge^k V$ that are expressible as a wedge product $v_1 \wedge \cdots \wedge v_k$ of k vectors $v_i \in V$ — is a closed algebraic set is to use the ring structure of the exterior algebra $\bigwedge V$. Writing out an element $\eta \in \bigwedge^k V$ in coordinates as above, we see that $e_i \wedge \eta = 0$ if and only if η can be written as $e_i \wedge \eta'$ for some $\eta' \in \bigwedge^{k-1} V$. Since there

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is nothing special about the vector $e_i \in V$ we could replace it by any nonzero element $v \in V$. Repeating this idea, we see that a nonzero element $\eta \in \bigwedge^k V$ can be written in the form $v_1 \wedge \cdots \wedge v_k$ for some (necessarily independent) $v_1, \ldots, v_k \in V$ if and only if the kernel of the multiplication map

$$V \xrightarrow{\wedge \eta} \bigwedge^{k+1} V$$

has dimension at least k. That is, the image of the Plücker embedding is

$$G = \{\eta \in \bigwedge^k V \mid \operatorname{rank}(V \xrightarrow{\wedge \eta} \bigwedge^{k+1} V) \leq n - k\},$$

and this is the zero locus of the homogeneous polynomials of degree $n-k+1$ on $\bigwedge^k V$ that are the $(n-k+1)$-st-order minors of the map $\wedge \eta : V \to \bigwedge^{k+1} V$ written out as a matrix.

Once we know that G is an algebraic set, it follows that G is a variety: Its ideal is the kernel of the map of polynomial rings

$$\Bbbk[p_{i_1,\ldots,i_k}]_{1 \leq i_1 < \cdots < i_k \leq n} \to \Bbbk[x_{i,j}]_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}}$$

sending $p_{i_1,\ldots,i_k}$ to the corresponding Plücker coordinate of the generic matrix $(x_{i,j})$, and is thus prime.

Though the equations for G just given have degree $n - k + 1$, the generators of the ideal of homogeneous forms vanishing on $G \subset \mathbb{P}(\bigwedge^k V)$ are actually quadratic polynomials, known as the *Plücker relations*. We will be able to describe these quadratic polynomials explicitly following Proposition 3.2; for fuller accounts (including a proof that they do indeed generate the homogeneous ideal of $G \subset \mathbb{P}(\bigwedge^k V)$) and some of their beautiful combinatorial structure, we refer the reader to De Concini et al. [1980, Section 2] or Fulton [1997, Section 9.1].

From here on, we will view $G(k, V)$ as being endowed with the structure of a projective variety via the Plücker embedding. As will follow from the description of its covering by affine spaces in the following subsection, it is a smooth variety of dimension $k(n-k)$. The smoothness statement follows in any case from the fact that $\mathrm{GL}(V)$ acts transitively on it by linear transformations of the projective space $\mathbb{P}(\bigwedge^k V)$.

Example 3.1. The first example of a Grassmannian other than projective space is the Grassmannian $G(2, 4) = \mathbb{G}(1, 3)$. Let V be a four-dimensional vector space, and consider the Plücker embedding of $G(2, V) = \mathbb{G}(1, \mathbb{P}V)$ in $\mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5$. Since (as we will see shortly) $\dim G(2, 4) = 4$, this will be a hypersurface. From the discussion above, we know that the equation of $G(2, 4)$ in this embedding is a polynomial relation among the minors $p_{i,j}$ of a generic 2×4 matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}.$$

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One way to obtain this relation is to note that the determinant of the 4×4 matrix with repeated rows

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{pmatrix}$$

must be 0. Expanding this determinant as a sum of products of minors of the first two rows and of the last two rows, all of which are Plücker coordinates, we obtain

$$p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0. \tag{3.1}$$

As this is an irreducible polynomial (and $\dim G(2,4) = 4$), it generates the homogeneous ideal of $G(2,4) \subset \mathbb{P}^5$, which is thus a smooth quadric.

In fact, for any n, the ideal of the Grassmannian of 2-planes $G(2,n)$ is cut out by quadratic polynomials in the Plücker coordinates similar to the polynomial (3.1) above. More precisely, if $e_1, \dots, e_n$ is a basis of V and $\eta = \sum p_{a,b} e_a \wedge e_b \in \bigwedge^2 V$, then the polynomials

$$\left\{ g_{a,b,c,d} := p_{a,b}p_{c,d} - p_{a,c}p_{b,d} + p_{a,d}p_{b,c} = 0 \mid 1 \le a < b < c < d \le n \right\}$$

minimally generate the ideal of the Grassmannian. These are the *Plücker relations* in the special case of the Grassmannian $G(2,n)$. We will describe the Plücker relations in general following Proposition 3.2.

Another way to characterize the collection of polynomials $\{g_{a,b,c,d}\}$ defining $\mathbb{G}(1,n)$, in characteristic not equal to 2, is that they are the coefficients of the element $\eta^2 = 0 \in \bigwedge^4 V$ — in other words, an element $\eta \in \bigwedge^2 V$ is decomposable if and only if $\eta \wedge \eta = 0$. These coefficients may be characterized (up to a factor of 2) as the *Pfaffians* of a skew-symmetric matrix.

Exercises 3.17–3.22 describe a number of aspects of the projective geometry of the Grassmannian in the Plücker embedding.

3.2.2 Covering by affine spaces; local coordinates

Like a projective space, a Grassmannian $G = G(k,V)$ can be covered by Zariski open subsets isomorphic to affine space. To see this, fix an $(n-k)$-dimensional subspace $\Gamma \subset V$, and let U_Γ be the subset of k-planes that do not meet Γ:

$$U_\Gamma = \{\Lambda \in G \mid \Lambda \cap \Gamma = 0\}.$$

This is a Zariski open subset of G: In fact, if we take $w_1, \dots, w_{n-k}$ to be any basis for Γ and set $\eta = w_1 \wedge \cdots \wedge w_{n-k}$, then we have

$$U_\Gamma = \{[\omega] \in G \subset \mathbb{P}(\textstyle\bigwedge^k V) \mid \omega \wedge \eta \neq 0\},$$

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from which we see that U_Γ is the complement of the hyperplane section of G corresponding to the vanishing of a Plücker coordinate (though not all hyperplane sections of G in the Plücker embedding have this form).

We claim now that the open set $U_\Gamma \subset G(k,n)$ is isomorphic to affine space $\mathbb{A}^{k(n-k)}$. To see this, we first choose an arbitrary point $[\Omega] \in U_\Gamma$ that will play the role of the origin; that is, fix a k-plane $\Omega \subset V$ complementary to Γ, so that we have a direct-sum decomposition $V = \Omega \oplus \Gamma$. Any k-dimensional subspace $\Lambda \subset V$ complementary to Γ projects to Γ modulo Ω — call this map π_Γ — and projects isomorphically to Ω modulo Γ — call this map π_Ω. Thus Λ is the graph of the linear map

$$\varphi : \Omega \xrightarrow{\pi_\Omega^{-1}} \Lambda \subset V = \Omega \oplus \Gamma \xrightarrow{\pi_\Gamma} \Gamma.$$

Conversely, the graph of any map $\varphi : \Omega \to \Gamma$ is a subspace $\Lambda \subset \Omega \oplus \Gamma = V$ complementary to Γ. These two correspondences establish a bijection

$$U_\Gamma \cong \operatorname{Hom}(\Omega, \Gamma) \cong \mathbb{A}^{k(n-k)}.$$

To make this explicit, suppose we choose a basis for V consisting of a basis $e_1, \dots, e_k$ for Ω followed by a basis $e_{k+1}, \dots, e_n$ for Γ. If $\Lambda \in U_\Gamma$ is a k-plane then the preimages $\pi_\Omega^{-1} e_1, \dots, \pi_\Omega^{-1} e_k \in \Lambda$ form a basis for Λ. Thus Λ is the row space of the matrix

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,1} & a_{1,2} & \dots & a_{1,n-k} \\ 0 & 1 & \dots & 0 & a_{2,1} & a_{2,2} & \dots & a_{2,n-k} \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{k,1} & a_{k,2} & \dots & a_{k,n-k} \end{pmatrix},$$

where $A = (a_{i,j})$ is the matrix representing the linear transformation $\varphi : \Omega \to \Gamma$ in the given bases. Since there is a unique vector in Λ projecting (mod Γ) to each $e_i \in \Omega$, this matrix representation is unique. The bijection defined above sends $\Lambda \in U_\Gamma$ to the linear transformation $\Omega \to \Gamma$ given by the transpose of the matrix $A = (a_{i,j})$.

If we start with any representation of Λ as the span of the rows of a $k \times n$ matrix B' with respect to the given basis of V, then the Plücker coordinate $p_{1,2,\dots,k}$, which is the determinant of the submatrix consisting of the first k columns of B', is nonzero. Multiplying B' on the left by the inverse of this submatrix gives us back the matrix B above, and thus the $k \times k$ minors of B are the $k \times k$ minors of B' multiplied by the inverse of the determinant of the first $k \times k$ minor of B'.

On the other hand, we can realize the entry $a_{i,j}$ of A, up to sign, as a $k \times k$ minor of B: It is (up to sign) the determinant of the $k \times k$ submatrix in which we take all the first k columns except for the i-th, and put in instead the $(k+j)$-th column. Thus we may write

$$\pm a_{i,j} = \frac{p_{1,\dots,i-1,\hat{i},i+1,\dots,k,k+j}(\Lambda)}{p_{1,\dots,k}(\Lambda)},$$

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and this expression shows that $a_{i,j}$ is a regular function on U_Γ. Thus the bijection $U_\Gamma \cong \mathbb{A}^{k(n-k)}$ is a biregular isomorphism.

More generally, it turns out that the ratios $p_{a_1,\dots,a_k}(\Lambda)/p_{1,2,\dots,k}(\Lambda)$ of Plücker coordinates are, up to sign, precisely the determinants of submatrices (of all sizes) of A. To express the result, suppose that I and J are sets of indices. Write A_I^J for the minor of the matrix A involving rows with indices in I and columns with indices in J. Write I' for the complement (in the set of row indices $\{1,\dots,k\}$) of I, and, if $J = \{j_1,\dots,j_t\}$, write $J+k$ for the "translated" set of indices $\{j_1+k,\dots,j_t+k\}$. With this notation, the $t\times t$ minor A_I^J of A is equal, up to sign, to the $k\times k$ minor of B involving the columns $I'\cup J$. To see this as a regular function on U_Γ we need only divide by the minor involving columns $1,\dots,k$:

Proposition 3.2. *With notation as above, suppose that $I = \{i_1,\dots,i_{k-t}\}$ are row indices and $J = \{j_1,\dots,j_t\}$ are column indices with each $j_i > k$. We have*

$$\pm \det A_I^J = \frac{p_{I'\cup(J+k)}(\Lambda)}{p_{1,\dots,k}(\Lambda)}.$$

For example, the 3×3 minor of the matrix

$$B = \begin{array}{c} \begin{array}{ccccccc} \mathbf{1} & 2 & 3 & 4 & \mathbf{5} & \mathbf{6} & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 1 & 0 & a_{2,1} & \boldsymbol{a_{2,2}} & \boldsymbol{a_{2,3}} & a_{2,4} \\ 0 & 0 & 1 & a_{3,1} & \boldsymbol{a_{3,2}} & \boldsymbol{a_{3,3}} & a_{3,4}, \end{pmatrix} \end{array}$$

involving columns 1, 5 and 6, is, up to sign, the 2×2 minor

$$\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}$$

of the matrix $(a_{i,j})$.

Proof: The expression in Plücker coordinates on the right is independent of the matrix representation chosen for Λ, so we may compute the two Plücker coordinates in terms of the matrix B in the form given above, so $p_{1,\dots,k}(\Lambda) = 1$ and $p_{I'\cup(J+k)}(\Lambda)$ is the minor of B involving the columns $I'\cup(J+k)$. Expanding this minor in terms of the $(k-t)\times(k-t)$ minors involving the rows of I', we see that all but the term $\pm 1\cdot A_I^J$ are zero. $\square$

Having established Proposition 3.2, it is easy to describe the *Plücker relations*, the quadratic polynomials in the Plücker coordinates that generate the homogeneous ideal of $G(k,V)\subset\mathbb{P}(\bigwedge^k V)$. With notation as above, consider the expansion of any $t\times t$ minor of A along one of its rows or columns. Replacing each factor of each term that appears by the ratio of two Plücker coordinates, with denominator $p_{1,\dots,k}(\Lambda)$, and multiplying through by $p_{1,\dots,k}(\Lambda)^2$, we get a homogeneous quadratic polynomial in the p_I satisfied identically in U_Γ and hence in all of $G(k,V)$. For more information we refer the reader to De Concini et al. [1980, Section 2] or Fulton [1997, Section 9.1].

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3.2.3 Universal sub and quotient bundles

In this section and the following, we will introduce the *universal bundles* on the Grassmannian $G(k,n)$ and show how to describe the tangent bundle to $G(k,n)$ in terms of them. These constructions are of fundamental importance in understanding the geometry of Grassmannians.

Let V be an n-dimensional vector space, $G = G(k,V)$ the Grassmannian of k-planes in V, and let $\mathcal{V} := G \times V$ be the trivial vector bundle of rank n on G whose fiber at every point is the vector space V (here we are thinking of a vector bundle as a variety, rather than as a locally free sheaf). We write $\mathcal{S}$ for the rank-k subbundle of $\mathcal{V}$ whose fiber at a point $[\Lambda] \in G$ is the subspace Λ itself; that is,

$$\mathcal{S}_{[\Lambda]} = \Lambda \subset V = \mathcal{V}_{[\Lambda]}.$$

$\mathcal{S}$ is called the *universal subbundle* on G; the quotient $\mathcal{Q} = \mathcal{V}/\mathcal{S}$ is called the *universal quotient bundle*. In the case $k = 1$—that is, $G = \mathbb{P}V \cong \mathbb{P}^{n-1}$—the universal subbundle $\mathcal{S}$ is the line bundle $\mathcal{O}_{\mathbb{P}V}(-1)$; similarly, in the case $k = n-1$ (so $G = \mathbb{P}V^*$) the universal quotient bundle $\mathcal{Q}$ is the line bundle $\mathcal{O}_{\mathbb{P}V^*}(1)$.

We have said "the rank-k subbundle of $\mathcal{V}$ whose fiber at a point $[\Lambda] \in G$ is the subspace Λ itself," and this certainly describes *at most* one bundle, since we have unambiguously defined a subset of $\mathcal{V} = G \times V$. Who would doubt that it is an algebraic subbundle of $\mathcal{V}$? To prove this, however, something more is necessary. Most primitively, we must check that it is trivial on an affine open cover, and that the transition functions are regular on the overlap of any two open sets of the cover. Alternatively, and equivalently, we may show that the subset $\mathcal{S}$ is an algebraic subset, and that over an open cover it is isomorphic, as an algebraic variety, to a trivial bundle. Here is a proof:

Proposition 3.3. *The subset $\mathcal{S}$ of $\mathcal{V}$ whose fiber over a point $[\Lambda] \in G = G(k,V)$ is the subspace $\Lambda \subset V$ is a vector bundle over G.*

Of course, it follows that $\mathcal{Q} = \mathcal{V}/\mathcal{S}$ is also a vector bundle.

Proof: Let S be the incidence correspondence

$$S = \{(\Lambda, v) \in G \times V \mid v \in \Lambda\}.$$

The set S is an algebraic subset of $G \times V$, since if we represent Λ by a vector $\eta \in \bigwedge^k \Lambda \subset \bigwedge^k V$, it is given by the equation $\eta \wedge v = 0 \in \bigwedge^{k+1} V$. Explicitly, if Λ is the row space of the matrix A, as in Section 3.2.2, then the condition $v \in \Lambda$ is equivalent to the vanishing of the $(k+1)$-st-order minors of the matrix obtained from A' by adjoining v as the $(k+1)$-st row. These minors can be expressed (by expanding along the new row of A') as bilinear functions in the coordinates of v and the Plücker coordinates, proving that S is an algebraic subset.

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Now pick a subspace $\Gamma \subset V$ of dimension $n - k$ and consider the preimage of $U_\Gamma \subset G$. Choosing a complement Ω to Γ as before, we can identify U_Γ with $\mathrm{Hom}(\Omega, \Gamma)$. Moreover, if $\Lambda \in U_\Gamma$ then the projection $\beta_{\Omega,\Gamma} : V \to \Omega$ with kernel Γ takes $S_{[\Lambda]} = \Lambda \subset V$ isomorphically to Ω. In other words, this projection gives an isomorphism S_{U_Γ} to the trivial bundle $\Omega \times U_\Gamma$. This proves that S is actually a vector bundle, which we identify as $\mathcal{S}$. □

The following result is the reason that we refer to $\mathcal{S}$ as the universal subbundle. A proof may be found in Eisenbud and Harris [2000].

Theorem 3.4. *If X is any scheme then the morphisms $\varphi : X \to G$ are in a one-to-one correspondence with rank-k subbundles $\mathcal{F} \subset V \otimes \mathcal{O}_X$ such that φ corresponds to the bundle $\mathcal{F} = \varphi^* \mathcal{S}$.*

There is also a projective analog of the vector bundle $\mathcal{S}$. Viewing G as $\mathbb{G}(k-1, \mathbb{P}V)$ (that is, as parametrizing $(k-1)$-planes in $\mathbb{P}V$), we set

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}.$$

The space Φ can also be realized as the projectivization of the universal subbundle $\mathcal{S}$, where by the *projectivization* of a vector bundle $\mathcal{E}$ on a scheme X we mean $\mathbb{P}\mathcal{E} := \mathrm{Proj}(\mathrm{Sym}\, \mathcal{E}^*)$ — a locally trivial fiber bundle over X whose fiber over a point $p \in X$ is $\mathbb{P}(\mathcal{E}_p)$. (We will see more of the space Φ in Section 4.8.1, where we will discuss flag manifolds in general and Φ in particular; we will deal with projective bundles in general in Chapter 9.) $\Phi = \mathbb{P}\mathcal{S}$ is called the *universal k-plane* over G.

Theorem 3.4 may be interpreted as saying that *the Grassmannian represents the functor of families of k-dimensional subspaces of V*, in the sense that the contravariant functor from schemes to sets given on objects by $X \mapsto \mathrm{Mor}(X, G(k, V))$ is naturally isomorphic to the functor given by $X \mapsto \{\text{rank-}k \text{ subbundles of } V \otimes \mathcal{O}_X\}$. Again, in the language that we will develop in Section 6.3, this says that *the Grassmannian $\mathbb{G}(k-1, \mathbb{P}V)$ is the Hilbert scheme of $(k-1)$-planes in $\mathbb{P}V$*. See Eisenbud and Harris [2000, Chapter 6] for an introduction to these ideas and a proof of this statement.

3.2.4 The tangent bundle of the Grassmannian

Knowledge of the tangent bundle of the Grassmannian is the key to its geometry. It turns out that the tangent bundle can be expressed in terms of the universal bundles $\mathcal{S}$ and $\mathcal{Q}$:

Theorem 3.5. *The tangent bundle $\mathcal{T}_G$ to the Grassmannian $G = G(k, V)$ is isomorphic to $\mathcal{H}om_G(\mathcal{S}, \mathcal{Q})$, where $\mathcal{S}$ and $\mathcal{Q}$ are the universal sub and quotient bundles.*

Proof: Consider the open affine set

$$U_\Gamma = \{\Lambda \in G \mid \Lambda \cap \Gamma = 0\}$$

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described in Section 3.2.2, where Γ is a subspace of V of dimension $n-k$. Fixing a point $[\Omega] \in U_\Gamma$ and decomposing V as $\Omega \oplus \Gamma$, we get an identification of U_Γ with the vector space $\mathrm{Hom}(\Omega, \Gamma)$ under which the point $[\Omega]$ goes to the linear transformation 0. In particular, the tangent bundle $\mathcal{T}_G$ restricted to U_Γ is the trivial bundle and the fiber over $[\Omega]$ is $\mathrm{Hom}(\Omega, \Gamma)$.

The bundle $\mathcal{S}|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Omega \times U_\Gamma$ by the composite map

$$\mathcal{S}|_{U_\Gamma} \to V \times U_\Gamma \to V/\Gamma \times U_\Gamma = \Omega \times U_\Gamma,$$

and the bundle $\mathcal{Q}|_{U_\Gamma}$ is isomorphic to the trivial bundle $\Gamma \times U_\Gamma$ via the tautological projection $V \otimes \mathcal{O}_G \to \mathcal{Q}$. This gives an identification of fibers, depending on Γ:

$$(\mathcal{T}_G)_\Omega = \mathrm{Hom}(\Omega, \Gamma) = \mathrm{Hom}(\mathcal{S}_\Omega, \mathcal{Q}_\Omega).$$

To prove that these identifications extend to an isomorphism $\mathcal{T}_G \cong \mathcal{H}om_G(\mathcal{S}, \mathcal{Q})$, we must check that the gluing map for $\mathcal{T}_G$ and that for $\mathcal{H}om_G(\mathcal{S}, \mathcal{Q})$ on an intersection $U = U_\Gamma \cap U_{\Gamma'}$ containing the point $[\Omega]$ agree on the fiber over $[\Omega]$ (and thus agree as maps of bundles). We may regard $U \subset U_\Gamma = \mathrm{Hom}(\Omega, \Gamma)$ as the set of linear transformations whose graphs do not meet Γ', and this representation is related to the representation of $U \subset U_{\Gamma'}$ by the isomorphisms $\Gamma \underset{\alpha}{\longrightarrow} V/\Omega \underset{\beta}{\longleftarrow} \Gamma'$. The gluing

$$d\varphi : (\mathcal{T}_G|_{U_\Gamma})|_{U_{\Gamma'}} \xrightarrow{\sim} (\mathcal{T}_G|_{U_{\Gamma'}})|_{U_\Gamma}$$

along this set is by the differential of the composite linear transformation

$$\varphi : \mathrm{Hom}(\Omega, \Gamma) \xrightarrow{\alpha} \mathrm{Hom}(\Omega, V/\Omega) \xrightarrow{\beta^{-1}} \mathrm{Hom}(\Omega, \Gamma')$$

induced by these isomorphisms. Of course, the differential of a linear transformation is the same linear transformation. The same isomorphisms give the gluing of the bundle $\mathcal{H}om_G(\mathcal{S}, \mathcal{Q})$. □

From the identification of tangent vectors to $G = G(k, V)$ at Λ with the space $\mathrm{Hom}(\Lambda, V/\Lambda)$, we can see that not all tangent vectors at a given point are alike: We can associate to any tangent vector its *rank*, and this will be preserved under automorphisms of G (see Exercise 3.24 and, for a nice application, Exercise 3.23). In particular, this means that when $1 < k < \dim V - 1$ the automorphism group of $G(k, V)$ does *not* act transitively on nonzero tangent vectors, and hence Kleiman's theorem (Theorem 1.7) does not apply in positive characteristic. Nevertheless, the conclusions it gives for intersections of Schubert cycles are correct in all characteristics (and may be proven by a different method).

The Euler sequence on $\mathbb{P}^n$

The isomorphism of Theorem 3.5 is already useful in the case of projective space $\mathbb{P}^n = \mathbb{G}(0, n)$. In this setting Theorem 3.5 gives rise to the *Euler sequence*.

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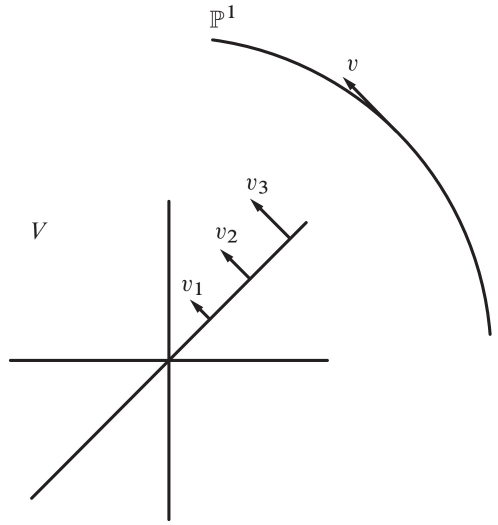

Figure 3.1 The tangent vectors v_1, v_2, v_3 to $\mathbb{A}^2$ all map to the tangent vector v to $\mathbb{P}^1$.

Let V be an $(n+1)$-dimensional vector space and $\mathbb{P}^n = \mathbb{P}V$ its projectivization. We consider the quotient map

$$q : U = V \setminus \{0\} \to \mathbb{P}^n$$

sending a nonzero vector $v \in V$ to the corresponding point $p = [v] \in \mathbb{P}^n$. The tangent space to U at v is the same as the tangent space to V at v, which is to say the vector space V itself, and the kernel of the differential

$$dq_v : T_v U \to T_p \mathbb{P}^n$$

is the one-dimensional subspace $\tilde{p} = \langle v \rangle \subset V$ spanned by v. Thus dq_v induces an isomorphism

$$V/\tilde{p} \xrightarrow{\sim} T_p \mathbb{P}^n,$$

as illustrated in Figure 3.1.

This isomorphism does not, however, give a natural identification of the vector spaces $V/\tilde{p}$ and $T_p \mathbb{P}^n$. Even though both these vector spaces depend only on the point $p \in \mathbb{P}^n$, *the isomorphism dq_v between them depends on the choice of the vector v.* Indeed, if λ is any nonzero scalar, the differential $dq_{\lambda v}$ is equal to dq_v divided by λ. But, by the same token, if $l : \langle v \rangle \to \Bbbk$ is any linear functional, then the map $l(v) \cdot dq_v$ *is* independent of the choice of v, and so we have a natural identification

$$\langle v \rangle^* \otimes V/\langle v \rangle \xrightarrow{\sim} T_{[v]} \mathbb{P}^n.$$

This is the identification

$$\mathcal{T}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{Q} = \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{Q}) = \mathcal{H}om(\mathcal{S}, \mathcal{Q})$$

asserted (more generally) in Theorem 3.5.

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To put it another way, in terms of coordinates $x_0, \dots, x_n$ on V, a constant vector field $\partial/\partial x_i$ on V does not give rise to a vector field on $\mathbb{P}V$, but the vector field

$$w(x) = x_j \frac{\partial}{\partial x_i}$$

on V does. This gives us a map

$$\mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to \mathcal{T}_{\mathbb{P}V},$$

whose kernel is the Euler vector field

$$e(x) = \sum x_i \frac{\partial}{\partial x_i}.$$

The resulting exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V} \xrightarrow{1 \mapsto e} \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \longrightarrow \mathcal{T}_{\mathbb{P}V} \longrightarrow 0$$

is called the *Euler sequence*. To relate this to the identification of the tangent bundle above, start with the universal sequence on $\mathbb{P}V$:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{P}V} \otimes V \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Now tensor with the line bundle $\mathcal{S}^* = \mathcal{O}_{\mathbb{P}V}(1)$; since $\mathcal{S} \otimes \mathcal{S}^* \cong \mathcal{O}_{\mathbb{P}V}$, we arrive at the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \longrightarrow \mathcal{S}^* \otimes \mathcal{Q} \longrightarrow 0.$$

By Theorem 3.5 the term on the right is $\mathcal{T}_{\mathbb{P}^n}$, and we obtain the Euler sequence again.

3.2.5 The differential of a morphism to the Grassmannian

Suppose that the morphism $\varphi : X \to G(k,n)$ corresponds via the universal property to a subbundle $\mathcal{E} \subset \mathcal{O}_X^n$, so that $\mathcal{E}$ is the pullback of the universal subbundle $\mathcal{S}$ on $G(k,n)$. Set $\mathcal{F} = \mathcal{O}^n/\mathcal{E}$, so that $\mathcal{F}$ is the pullback of the universal quotient bundle $\mathcal{Q}$ on $G(k,n)$.

The differential of φ is by definition a homomorphism of vector bundles

$$d\varphi : \mathcal{T}_X \to \varphi^*\mathcal{T}_{G(k,n)} = \varphi^* \mathcal{H}om_{G(k,n)}(\mathcal{S}, \mathcal{Q}) = \mathcal{H}om_X(\mathcal{E}, \mathcal{F}).$$

The local description of the Grassmannian above makes it easy to identify this homomorphism locally.

A global section of the $\mathcal{T}_X$ is called a *vector field*. Recall that a vector field may be identified with a derivation $\partial : \mathcal{O}_X \to \mathcal{O}_X$ (Eisenbud [1995, Chapter 16]). This works even if X is singular: In that case we *define* $\mathcal{T}_X$ to be the dual of the sheaf of differential forms; of course then $\mathcal{T}_X$ is a coherent sheaf, not necessarily a vector bundle. (A famous question posed by Zariski (see Lipman [1965]) asks whether, with this definition — and always assuming that the characteristic is 0 — $\mathcal{T}_X$ is a vector bundle if and only if X is smooth. See Hochster [1977] for some partial results.)

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Proposition 3.6. *Let X be a variety and $\varphi : X \to G(k,n)$ the morphism corresponding to a subbundle $i : \mathcal{E} \to \mathcal{O}_X^n$; set $\mathcal{F} = \mathcal{O}_X^n/\mathcal{E}$ and let $\pi : \mathcal{O}_X^n \to \mathcal{F}$ be the projection. Let $U \subset X$ be an open subset over which $\mathcal{E}$ is trivial and $\psi : \mathcal{O}_U^k \cong \mathcal{E}_U$ a trivialization of $\mathcal{E}$ over U. If ∂ is a vector field on U, then in U the homomorphism $(d\varphi)(\partial) \in \mathcal{H}om_U(\mathcal{E},\mathcal{F})$ is the composition $\pi \circ \partial(\alpha)$, where $\partial(\alpha)$ is the derivative with respect to ∂ of the composite map*

$$\alpha = i \circ \varphi : \mathcal{O}_U^k \xrightarrow{\psi} \mathcal{E}_U \xrightarrow{i} \mathcal{O}_U^n;$$

that is, $(d\varphi)(\partial)$ is the map obtained by applying ∂ to each entry of a matrix representing α and composing the result with the projection $\pi : \mathcal{O}_U^n \to \mathcal{F}_U$.

Note that the map $(d\varphi)(\partial)$ described above depends only on the subbundle $\mathcal{E}$, and not on the trivialization $\mathcal{O}_X^k \cong \mathcal{E}$ chosen: If $\beta : \mathcal{O}_X^k \to \mathcal{O}_X^k$ is an invertible matrix over $\mathcal{O}_X$, then

$$\partial(\alpha\beta) = (\partial\alpha)\beta + \alpha(\partial\beta),$$

and the second term vanishes when we project to $\mathcal{F}$.

Proof: The desired result follows at once from the description of the affine spaces covering the Grassmannian: We can change bases in $\mathcal{O}_U^n$ so that α is given by the $k \times n$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n-k} \\ 0 & 1 & \cdots & 0 & \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \alpha_{k,1} & \alpha_{k,2} & \cdots & \alpha_{k,n-k} \end{pmatrix},$$

where the $\alpha_{i,j}$ are functions on U, and give the morphism φ in local coordinates. The derivative of φ, applied to ∂, is then by definition obtained by applying ∂ to each of the coordinate functions $\alpha_{i,j}$. □

3.2.6 Tangent spaces via the universal property

There is another way to approach the tangent space, which depends on a pretty and well-known bit of algebra. Let $\mathcal{O}$ be a local ring with maximal ideal $\mathfrak{m}$, and suppose for simplicity that $\mathcal{O}$ contains a copy of its residue field $\Bbbk$.

Proposition–Definition 3.7. *There are natural one-to-one correspondences between the following sets:*

(a) $\mathrm{Hom}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2, \Bbbk)$ *(homomorphisms of $\Bbbk$-vector spaces).*
(b) $\mathrm{Der}_{\Bbbk}(\mathcal{O}, \Bbbk)$ *($\Bbbk$-linear derivations; that is, $\Bbbk$-vector space homomorphisms $d : \mathcal{O} \to \Bbbk$ that satisfy Leibniz' rule $d(fg) = fd(g) + gd(f)$).*
(c) $\mathrm{Hom}_{\Bbbk\text{-algebras}}(\mathcal{O}, \Bbbk[\epsilon]/(\epsilon^2))$ *(homomorphisms of $\Bbbk$-algebras).*
(d) $\mathrm{Mor}(D, \mathrm{Spec}\,\mathcal{O})$, *where $D = \mathrm{Spec}\,\Bbbk[\epsilon]/(\epsilon^2)$ (morphisms of schemes).*

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The first two of these are naturally $\mathbb{k}$*-vector spaces, and the correspondence preserves this structure; we regard the other two as equipped with this structure as well. Any of these spaces, with its vector space structure, is called the* Zariski tangent space *of* Spec $\mathcal{O}$ *at its closed point.*

When $\mathcal{O}$ is the local ring $\mathcal{O}_{X,x}$ of a variety (or scheme) at a closed point x, we think of the Zariski tangent space of $\mathcal{O}$ as the Zariski tangent space of X at x, and of course the set in (d) is the same as the set of morphisms of $\mathbb{k}$-schemes carrying the (unique) point of D — which we will call 0 — to x.

Proof: The sets in (c) and (d) are the same by definition. If φ is as in (c), then $\varphi|_{\mathfrak{m}}$ annihilates $\mathfrak{m}^2$ and induces a vector space homomorphism $\overline{\varphi} : \mathfrak{m}/\mathfrak{m}^2 \to (\epsilon)/(\epsilon^2) \cong \mathbb{k}$ as in (a). Similarly, a derivation d as in (b) induces a $\mathbb{k}$-linear map $\overline{d|_{\mathfrak{m}}} : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{k}$. We leave to the reader the construction of the inverse correspondences. □

Consider the case $X = G(k, V)$. The tangent space at $x = [\Lambda]$ is, by the argument above, the collection of maps $D \to G(k, V)$ sending 0 to $[\Lambda]$. By Theorem 3.4, giving a map $D \to G(k, V)$ is the same as the giving a rank-k subbundle $\mathcal{W}$ of $V \times D$; the map takes $0 \in D$ to $[\Lambda] \in \mathbb{G}(k, V)$ if and only if the fiber $\mathcal{W}_0$ is equal to Λ.

We can understand the identification of the tangent space $T_\Lambda G(k, V)$ to the Grassmannian with the space $\operatorname{Hom}(\Lambda, V/\Lambda)$ using this description together with the universal property of the Grassmannian described in Theorem 3.4.

Since D is affine, a vector bundle over D is the same as a locally free module over $\mathbb{k}[\epsilon]/(\epsilon^2)$. Since this ring is local, Nakayama's lemma shows that such a module is free (see for example Eisenbud [1995, Exercise 4.11]). Thus only the inclusion $\Lambda \times D \to V \times D$ varies.

Putting this together, we get a new way to look at the identification of the tangent spaces to the Grassmannian:

Proposition 3.8. *Let* $\Lambda \subset V$ *be a* k*-dimensional subspace, and let* $\varphi : \Lambda \to V/\Lambda$ *be a homomorphism. As an element of the tangent space to the Grassmannian* $G(k, V)$ *at the point* $[\Lambda]$*,* φ *corresponds to the free submodule*

$$\Lambda \otimes \mathbb{k}[\epsilon]/(\epsilon^2) \to V \otimes \mathbb{k}[\epsilon]/(\epsilon^2), \quad v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon,$$

where $\varphi' : \Lambda \to V$ *is any map that when composed with the projection* $V \to V/\Lambda$ *gives* φ*.*

Proof: Any map $\Lambda \times D \to V \times D$ that reduces to the inclusion modulo ϵ has the form $v \otimes 1 \mapsto v \otimes 1 + \varphi'(v) \otimes \epsilon$ for some φ'. If we work in the affine coordinates corresponding to a subspace Γ complementary to Λ and use the splitting $V = \Lambda \oplus \Omega$,

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then the point $\Lambda \subset V$ corresponds to the matrix

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

In this matrix representation φ is represented by the last $n-k$ columns of φ', and taking a different lifting of φ corresponds to making a different choice of the first $k \times k$ block of φ'.

We can do row operations to clear all the ϵ terms from the first $k \times k$ block, adding a multiple of ϵ times certain rows to other rows. This corresponds to composing with an automorphism of $\Lambda \times D$, and thus does not change the image of $\Lambda \times D \to V \times D$. Since we add after multiplying by ϵ, this does not change the block representing φ. Thus we may assume that the first $k \times k$ block of φ' is 0; equivalently, the first $k \times k$ block corresponding to the map $\Lambda \times D \to V \times D$ is the identity. □

3.3 The Chow ring of $\mathbb{G}(1, 3)$

Before launching into the geometry of general Grassmannians in the next chapter, we will spend the remainder of this chapter studying the geometry of $\mathbb{G}(1, 3)$, the Grassmannian of lines in $\mathbb{P}^3$. This is the simplest case beyond the projective spaces. The general results are in many ways similar, but more combinatorics is involved, and in the case of lines in $\mathbb{P}^3$ it is possible to visualize more of what is going on. Once the reader has absorbed the case of $\mathbb{G}(1, 3)$ the general results will seem more natural.

3.3.1 Schubert cycles in $\mathbb{G}(1, 3)$

To start, we fix a *complete flag* $\mathcal{V}$ on $\mathbb{P}^3$; that is, a choice of a point $p \in \mathbb{P}^3$, a line $L \subset \mathbb{P}^3$ containing p, and a plane $H \subset \mathbb{P}^3$ containing L (Figure 3.2).

We can give a stratification of $\mathbb{G}(1, 3)$ by considering the loci of lines $\Lambda \in \mathbb{G}(1, 3)$ having specified dimension of intersection with each of the subspaces p, L and H. These are called *Schubert cells* and their closures, which are irreducible subvarieties, are called *Schubert cycles* (or sometimes Schubert varieties); the classes of these cycles are the *Schubert classes*. As we shall see, the Schubert cells form an affine stratification of $\mathbb{G}(1, 3)$, and it will follow from Proposition 1.17 that the Schubert classes generate the Chow group $A(\mathbb{G}(1, 3))$. Using intersection theory, we will be able to show that in fact $A(\mathbb{G}(1, 3))$ is a free $\mathbb{Z}$-module having the Schubert classes as free generators. In the next chapter, we will see that the same situation is repeated for all Grassmannians.

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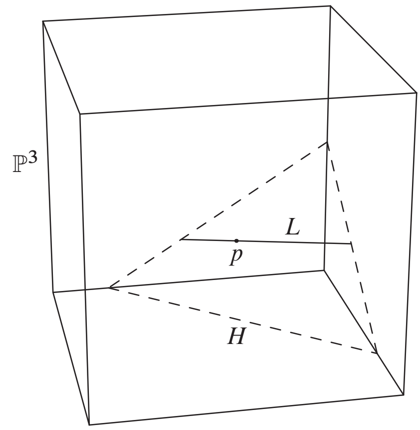

Figure 3.2 A complete flag $p \subset L \subset H$ in $\mathbb{P}^3$.

More formally, we begin not with the Schubert cells but with the Schubert cycles:

$$\begin{aligned}
\Sigma_{0,0} &= \mathbb{G}(1,3), \\
\Sigma_{1,0} &= \{\Lambda \mid \Lambda \cap L \neq \varnothing\}, \\
\Sigma_{2,0} &= \{\Lambda \mid p \in \Lambda\}, \\
\Sigma_{1,1} &= \{\Lambda \mid \Lambda \subset H\}, \\
\Sigma_{2,1} &= \{\Lambda \mid p \in \Lambda \subset H\}, \\
\Sigma_{2,2} &= \{\Lambda \mid \Lambda = L\}.
\end{aligned}$$

The four nontrivial ones are illustrated in Figure 3.3. In each case we take the reduced scheme structure. To show, for example, that $\Sigma_{1,0}$ is an irreducible variety, we note first that $\Sigma_{1,0}$ is the image of the incidence correspondence

$$\Gamma = \{(L', p) \in \mathbb{G}(1,3) \times L \mid p \in L'\}.$$

The fiber of Γ under projection to L, the set of lines through a given point $p \in L$, is isomorphic to $\mathbb{P}^2$; since all fibers are irreducible and of the same dimension and the projection is proper, it follows that Γ, and with it $\Sigma_{1,0}$, are irreducible. The proof that the other Schubert cycles are irreducible follows in exactly the same way.

Thus $\Sigma_{a,b}$ denotes the set of lines meeting the $(2-a)$-dimensional plane of $\mathcal{V}$ in a point and the $(3-b)$-dimensional plane of $\mathcal{V}$ in a line. This system of indexing may seem peculiar at first; the reasons for it will be clearer when we discuss Schubert cycles in general in the following chapter. For now, we will mention that the codimension of $\Sigma_{a,b}$ is $a+b$, as will be clear in examples and as we will prove in general in Theorem 4.1.

We often drop the second index when it is 0, writing for example Σ_1 instead of $\Sigma_{1,0}$. When the choice of flag is relevant, we will sometimes indicate the dependence by writing $\Sigma(\mathcal{V})$, or simply note the dependence on the relevant flag elements by writing, for example, $\Sigma_1(L)$ for the cycle of lines Λ meeting L.

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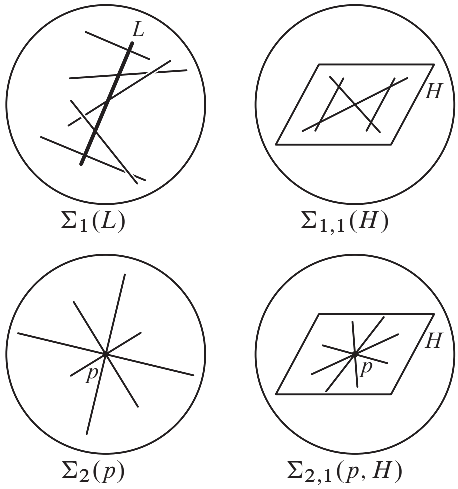

Figure 3.3 Schubert cycles in $\mathbb{G}(1,3)$.

It is easy to see that there are inclusions

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr, hook] & & \\
\{L\} = \Sigma_{2,2} \arrow[r, hook] & \Sigma_{2,1} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[r, hook] & \mathbb{G}(1,3). \\
 & & \Sigma_{1,1} \arrow[ur, hook] & &
\end{tikzcd}$$

For each index (a, b) we define the Schubert cell $\Sigma^\circ_{a,b}$ to be the complement in $\Sigma_{a,b}$ of the union of all the other Schubert cycles properly contained in $\Sigma_{a,b}$. To show that the $\Sigma_{a,b}$ form an affine stratification, it suffices to show that each $\Sigma^\circ_{a,b}$ is isomorphic to an affine space. We will do the most complicated case, leaving the others for the reader (the general case of a Schubert cell in $G(k, n)$ is done in Theorem 4.1).

Example 3.9. We will show that the set

$$\Sigma^\circ_1 = \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1}) = \{\Lambda \mid \Lambda \cap L \neq \varnothing \text{ but } p \notin \Lambda \text{ and } \Lambda \not\subset H\}$$

is isomorphic to $\mathbb{A}^3$. Let H' be a general plane containing the point p but not containing the line L. Any line meeting L but not passing through p and not contained in H meets H' in a unique point contained in $H' \setminus (H' \cap H)$ (Figure 3.4). Thus we have maps

$$\Sigma^\circ_1 \to (L \setminus \{p\}) \cong \mathbb{A}^1 \quad \text{and} \quad \Sigma^\circ_1 \to (H' \setminus (H \cap H')) \cong \mathbb{A}^2$$

sending Λ to $\Lambda \cap L$ and $\Lambda \cap H'$ respectively. The product of these maps gives us an isomorphism

$$\Sigma^\circ_1 \cong \mathbb{A}^1 \times \mathbb{A}^2 = \mathbb{A}^3.$$

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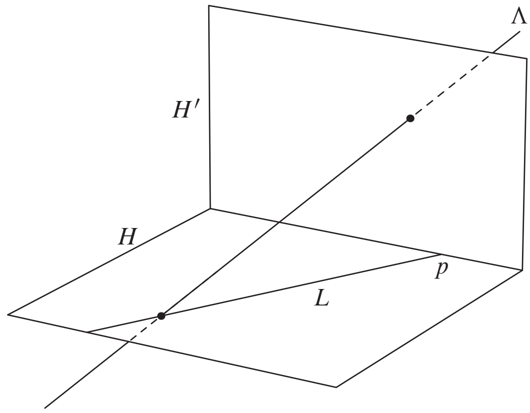

Figure 3.4 The map $\Lambda \mapsto (\Lambda \cap L, \Lambda \cap H')$ defines an isomorphism $\Sigma_1^\circ \to (L \setminus \{p\}) \times (H \setminus H \cap H') \cong \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3$.

By Theorem 1.7, the class $[\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,3))$ does not depend on the choice of flag, since any two flags differ by a transformation in GL_4; we will denote the class of $\Sigma_{a,b}$ by

$$\sigma_{a,b} = [\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,3)).$$

By Proposition 1.8, the group $A^0(\mathbb{G}(1,3))$ is isomorphic to $\mathbb{Z}$ and is generated by the fundamental class $\sigma_{0,0} = [\mathbb{G}(1,3)]$; by Theorem 1.7, the group $A^4(\mathbb{G}(1,3))$ is also isomorphic to $\mathbb{Z}$ and is generated by the class $\sigma_{2,2}$ of a point in $\mathbb{G}(1,3)$. (In particular any two points in $\mathbb{G}(1,3)$ are rationally equivalent.)

3.3.2 Ring structure

We can now determine the structure of the Chow ring of $\mathbb{G}(1,3)$ completely:

Theorem 3.10. *The six Schubert classes $\sigma_{a,b} \in A^{a+b}(\mathbb{G}(1,3))$, $0 \le b \le a \le 2$, freely generate $A(\mathbb{G}(1,3))$ as a graded abelian group, and satisfy the multiplicative relations*

$$\begin{aligned} \sigma_1^2 &= \sigma_{1,1} + \sigma_2 && (A^1 \times A^1 \to A^2); \\ \sigma_1\sigma_{1,1} &= \sigma_1\sigma_2 = \sigma_{2,1} && (A^1 \times A^2 \to A^3); \\ \sigma_1\sigma_{2,1} &= \sigma_{2,2} && (A^1 \times A^3 \to A^4); \\ \sigma_{1,1}^2 &= \sigma_2^2 = \sigma_{2,2}, \quad \sigma_{1,1}\sigma_2 = 0 && (A^2 \times A^2 \to A^4). \end{aligned}$$

From these formulas we deduce that $\sigma_1^3 = 2\sigma_{2,1}, \sigma_1^4 = 2\sigma_{2,2}$, and $\sigma_1^2\sigma_{1,1} = \sigma_1^2\sigma_2 = \sigma_{2,2}$. Since $\dim(\mathbb{G}(1,3)) = 4$, any product that would have degree > 4, such as $\sigma_2\sigma_{2,1}$, is 0.

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Proof of Theorem 3.10: As we said, we know by Proposition 1.17 that the Schubert classes $\sigma_{a,b}$ generate $A(\mathbb{G}(1,3))$. That they are free generators follows for $A^4(\mathbb{G}(1,3))$ from Proposition 1.21, and will follow for the remaining Chow groups from the intersections products above: For example, the formulas show that the matrix of the intersection pairing on $\sigma_{1,1}$ and σ_2 is nonsingular, so $A^2(\mathbb{G}(1,3))$ is freely generated by these two classes.

It remains to prove the formulas. We will consider the intersections of pairs of cycles, taking these with respect to generically situated flags $\mathcal{V}, \mathcal{V}'$. To simplify notation we will henceforth write $\Sigma_{a,b}$ and $\Sigma'_{a,b}$ for $\Sigma_{a,b}(\mathcal{V})$ and $\Sigma_{a,b}(\mathcal{V}')$, respectively.

We begin with the case of cycles of complementary dimension, starting with the intersection number of σ_2 with itself. By Kleiman transversality we have

$$\sigma_2^2 = \#(\Sigma_2 \cap \Sigma'_2) \cdot \sigma_{2,2},$$

and since the intersection

$$\Sigma_2 \cap \Sigma'_2 = \{\Lambda \mid p \in \Lambda \text{ and } p' \in \Lambda\}$$

consists of one point (corresponding to the unique line $\Lambda = \overline{p, p'}$ through p and p'), we conclude that

$$\sigma_2^2 = \sigma_{2,2}.$$

Similarly,

$$\sigma_{1,1}^2 = \#(\Sigma_{1,1} \cap \Sigma'_{1,1}) \cdot \sigma_{2,2};$$

since

$$\Sigma_{1,1} \cap \Sigma'_{1,1} = \{\Lambda \mid \Lambda \subset H \text{ and } \Lambda \subset H'\}$$

consists of the unique line $\Lambda = H \cap H'$, we conclude that

$$\sigma_{1,1}^2 = \sigma_{2,2}$$

as well. On the other hand, $\Sigma_2 = \{\Lambda \mid p \in \Lambda\}$ and $\Sigma'_{1,1} = \{\Lambda \mid \Lambda \subset H'\}$ are disjoint, since $p \notin H'$, so that

$$\sigma_2\sigma_{1,1} = 0.$$

Finally,

$$\Sigma_1 \cap \Sigma'_{2,1} = \{\Lambda \mid \Lambda \cap L \neq \varnothing \text{ and } p' \in \Lambda \subset H'\}.$$

Since L will intersect H' in one point q, and any line Λ satisfying all the above conditions can only be the line $\overline{p', q}$ (Figure 3.5), this intersection is again a single point. Thus

$$\sigma_1\sigma_{2,1} = \sigma_{2,2}$$

as well.

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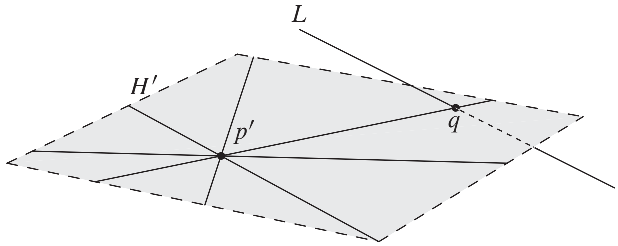

Figure 3.5 $\Sigma_1(L) \cap \Sigma'_{2,1}(p', H') = \{\overline{p', q}\}$.

We now turn to the intersections of cycles whose codimensions sum to less than 4. First, the intersection $\Sigma_1 \cap \Sigma'_2$ is the locus of lines Λ meeting L and containing the point p', which is to say the Schubert cycle $\Sigma_{2,1}$ with respect to a flag containing the point p' and the plane $\overline{p', L}$, so we have

$$\sigma_1\sigma_2 = \sigma_{2,1}.$$

In a similar fashion, the intersection of Σ_1 and $\Sigma'_{1,1}$ is a cycle of the form $\Sigma_{2,1}$ with respect to a certain flag; specifically, it is the locus of lines containing the point $L \cap H'$ and lying in H', so that

$$\sigma_1\sigma_{1,1} = \sigma_{2,1}.$$

The last and most interesting computation to be made is the product σ_1^2. (This is such a crucial case that we will prove it twice: here and in Section 3.5.1!) The difference between this case and the preceding ones is that the locus $\Sigma_1 \cap \Sigma'_1$ of lines meeting each of the two general lines L and L' is not a Schubert cycle.

We will use the method of undetermined coefficients, first introduced in Section 2.1.6. We have by now established that $A^2(\mathbb{G}(1,3))$ is freely generated by the classes $\sigma_{1,1}$ and σ_2, so that we may write

$$\sigma_1^2 = \alpha\sigma_2 + \beta\sigma_{1,1} \tag{3.2}$$

for some (unique) α and $\beta \in \mathbb{Z}$. We can then determine the coefficients α and β by taking the product of both sides of (3.2) with classes of complementary dimension.

One way to do this is by invoking the associativity of $A(\mathbb{G}(1,3))$ and the previous calculations: We have

$$(\alpha\sigma_2 + \beta\sigma_{1,1})\sigma_2 = \sigma_1^2 \cdot \sigma_2 = \sigma_1(\sigma_1\sigma_2) = \sigma_1\sigma_{2,1} = \sigma_{2,2},$$

and since $\sigma_2^2 = \sigma_{2,2}$ and $\sigma_{1,1}\sigma_2 = 0$ we get $\alpha = 1$. Similarly, from

$$(\alpha\sigma_2 + \beta\sigma_{1,1})\sigma_{1,1} = \sigma_1^2 \cdot \sigma_{1,1} = \sigma_1(\sigma_1\sigma_{1,1}) = \sigma_1\sigma_{2,1} = \sigma_{2,2}$$

and $\sigma_{1,1}^2 = \sigma_{2,2}$ we see that $\beta = 1$. In sum, we have

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1},$$

and this completes our description of the Chow ring $A(\mathbb{G}(1,3))$. □

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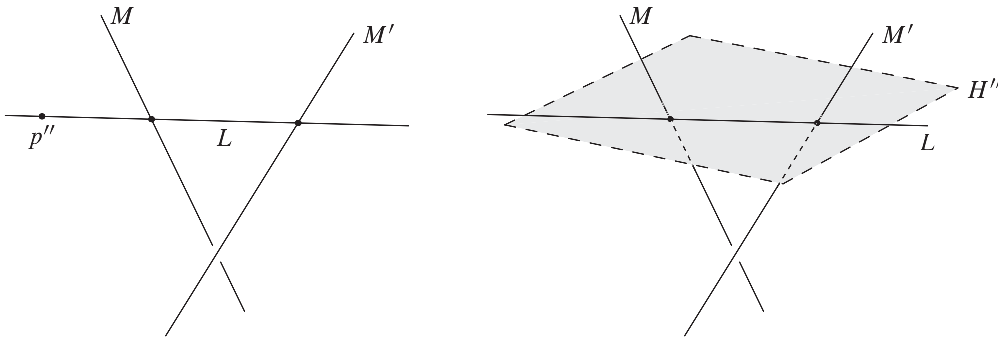

Figure 3.6 $\Sigma_2(p'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$; $\Sigma_{1,1}(H'')\cap\Sigma_1(M)\cap\Sigma_1(M') = \{L\}$.

It is instructive to compute $\sigma_1^2\sigma_2$ and $\sigma_1^2\sigma_{1,1}$ geometrically, without invoking associativity as in the proof above. To determine α, we used

$$\sigma_1^2 \cdot \sigma_2 = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_2 = \alpha\sigma_{2,2}.$$

By Kleiman transversality, we have

$$\alpha = \#\left\{\Lambda \;\middle|\; \begin{array}{l} \Lambda \cap L \neq \varnothing, \\ \Lambda \cap L' \neq \varnothing \text{ and} \\ p'' \in \Lambda \end{array}\right\}$$

for L and L' general lines and p'' a general point in $\mathbb{P}^3$. Any line Λ satisfying the three conditions must lie in each of the planes $\overline{p'', L}$ and $\overline{p'', L'}$, and so must be their intersection; thus $\alpha = 1$.

Similarly, to determine β we used

$$\sigma_1^2 \cdot \sigma_{1,1} = (\alpha\sigma_2 + \beta\sigma_{1,1}) \cdot \sigma_{1,1} = \beta\sigma_{2,2}.$$

Again, by generic transversality, we get:

$$\beta = \#\left\{\Lambda \;\middle|\; \begin{array}{l} \Lambda \cap L \neq \varnothing, \\ \Lambda \cap L' \neq \varnothing \text{ and} \\ \Lambda \subset H'' \end{array}\right\}$$

for L and L' general lines and H'' a general plane in $\mathbb{P}^3$. The only line Λ satisfying these conditions is the line joining the points $L \cap H''$ and $L' \cap H''$, so again $\beta = 1$ (Figure 3.6).

Tangent spaces to Schubert cycles

The generic transversality of the cycles $\Sigma_{a,b}$ and $\Sigma_{a',b'}$, guaranteed by Kleiman's theorem in characteristic 0, played an essential role in the computation above. By

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describing the tangent spaces to the Schubert cycles, we can prove this transversality directly and hence extend the results to characteristic p.

We will carry this out here for the intersection $\Sigma_2 \cap \Sigma_2'$. Tangent spaces to other Schubert cycles in $\mathbb{G}(1,3)$ are described in Exercises 3.26 and 3.27; they will be treated in general in Theorem 4.1. The key identification is given in the following result:

Proposition 3.11. *Let $\Sigma = \Sigma_2(p)$ be the Schubert cycle of lines in $\mathbb{P}^3 = \mathbb{P}V$ that contain p, and suppose that $L \in \Sigma_2(p)$. Writing $\widetilde{L} \subset V$ for the two-dimensional subspace corresponding to L, and identifying $T_L\,\mathbb{G}(1,3)$ with* $\mathrm{Hom}(\widetilde{L}, V/\widetilde{L})$*, we have*

$$T_L\Sigma = \{\varphi \mid \varphi(\tilde{p}) = 0\}.$$

Given Proposition 3.11, it follows immediately that for general $p, p' \in \mathbb{P}^3$ the cycles $\Sigma_2(p)$ and $\Sigma_2(p')$ meet transversely: If $p \neq p'$, then at the unique point $L = \overline{p, p'}$ of intersection of the Schubert cycles $\Sigma = \Sigma_2(p)$ and $\Sigma' = \Sigma_2(p')$, we have

$$T_{[L]}\Sigma \cap T_{[L]}\Sigma' = \{\varphi \mid \varphi(\tilde{p}) = \varphi(\tilde{p}') = 0\} = \{0\},$$

since $\tilde{p}$ and $\tilde{p}'$ span $\widetilde{L}$.

Proof of Proposition 3.11: We choose a subspace $\Gamma \subset V$ complementary to $\widetilde{L}$ and identify the open subset

$$U_\Gamma = \{\Lambda \in \mathbb{G}(1,3) \mid \Lambda \cap \Gamma = 0\}$$

with the vector space $\mathrm{Hom}(\widetilde{L}, \Gamma)$ by thinking of a 2-plane $\Lambda \in U_\Gamma$ as the graph of a linear map from $\widetilde{L}$ to Γ, just as in the beginning of the proof of Theorem 3.5. It is immediate from the identification that $U_\Gamma \cap \Sigma_2$ is the linear space in $\mathrm{Hom}(\widetilde{L}, \Gamma)$ consisting of maps φ such that $\tilde{p} \subset \mathrm{Ker}(\varphi)$. Thus its tangent space at a point $[L]$ has the same description. □

As a consequence of Theorem 3.10, we have the following description of the Chow ring of $\mathbb{G}(1,3)$:

Corollary 3.12.
$$A(\mathbb{G}(1,3)) = \frac{\mathbb{Z}[\sigma_1, \sigma_2]}{(\sigma_1^3 - 2\sigma_1\sigma_2,\ \sigma_1^2\sigma_2 - \sigma_2^2)}.$$

We will generalize this to the Chow ring of any Grassmannian, and prove it by applying the theory of Chern classes, in Chapter 5. A point to note is that the given presentation of the Chow ring has the same number of generators as relations — that is, given that the Chow ring $A(\mathbb{G}(1,3))$ has Krull dimension 0, it is a *complete intersection*. The analogous statement is true for all Grassmannians.

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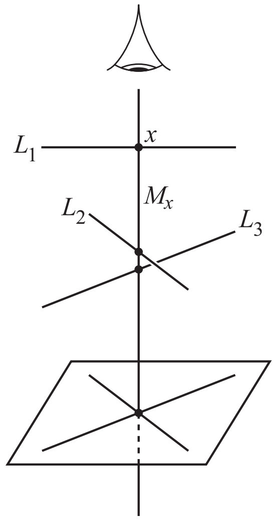

Figure 3.7 An apparent double point: when viewed from x, the lines L_2 and L_3 appear to cross at a point in the direction M_x, and therefore there is a unique line through x meeting L_2 and L_3.

3.4 Lines and curves in $\mathbb{P}^3$

In this section and the next we present several applications of the computations above.

3.4.1 How many lines meet four general lines?

This is Keynote Question (a). Since σ_1 is the class of the locus $\Sigma_1(L)$ of lines meeting a given line L, and generic translates of $\Sigma_1(L)$ are generically transverse, the number is

$$\deg \sigma_1^4 = 2.$$

We can see the geometry behind this computation — and answer more refined questions about the situation — as follows. Suppose that the lines are $L_1, \ldots, L_4$, and consider first the lines that meet just the first three. To begin with, we claim that *if $x \in L_1$ is any point, there is a unique line $M_x \subset \mathbb{P}^3$ passing through x and meeting L_2 and L_3*, as in Figure 3.7. To see this, note that if we project L_2 and L_3 from x to a plane H, we get two general lines in H, and these lines meet in a unique point y. The line $M_x := \overline{x, y}$ is then the unique line in $\mathbb{P}^3$ containing x and meeting L_2 and L_3. (Informally: If we look at L_2 and L_3, sighting from the point x, we see an "apparent crossing" in the direction of the line M_x — see Figure 3.7.) Moreover, if $x \neq x'$ then the lines $M_x, M_{x'}$ are disjoint: If they had a common point, they would lie in a plane, and all three of L_1, L_2, L_3 would be coplanar, contradicting our hypothesis of generality.

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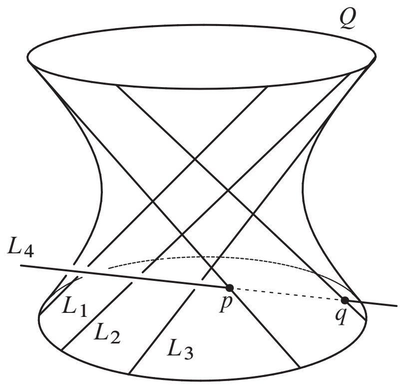

Figure 3.8 Two lines that meet each of $L_1, \dots, L_4$.

The union of the lines M_x is a surface that we can easily identify. There is a three-dimensional family of quadratic polynomials on each $L_i \cong \mathbb{P}^1$. Each restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_{L_i}(2))$ is linear, so its kernel has codimension ≤ 3. Since there is a 10-dimensional vector space of quadratic polynomials on $\mathbb{P}^3$, there is thus at least one quadric surface Q containing L_1, L_2 and L_3. By Bézout's theorem, any line meeting each of L_1, L_2, L_3, and thus meeting Q at least three times, must be contained in Q. Since the union of the lines M_x is a nondegenerate surface (the lines are pairwise disjoint, and so cannot lie in a plane), it follows that Q is unique, and is equal to the disjoint union

$$\coprod_{x \in L_1} M_x = Q.$$

Since the degree of Q is 2, and L_4 is general, L_4 meets Q in two distinct points p and q; the two lines M_p and M_q passing through p and q are the unique lines meeting all of $L_1, \dots, L_4$ (Figure 3.8). Thus we see again that the answer to our question is 2.

For which sets of four lines are there more or fewer than two distinct lines meeting all four? The geometric construction above will enable the reader to answer this question; see Exercise 3.29.

3.4.2 Lines meeting a curve of degree d

We do not know a geometric argument such as the above one for four lines that would enable us to answer the corresponding question for four curves, Keynote Question (b). In this case, intersection theory is essential. The basic computation is the following:

Proposition 3.13. *Let $C \subset \mathbb{P}^3$ be a curve of degree d. If*

$$\Gamma_C := \{L \in \mathbb{G}(1,3) \mid L \cap C \neq \varnothing\}$$

is the locus of lines meeting C, then the class of Γ_C is

$$[\Gamma_C] = d \cdot \sigma_1 \in A^1(\mathbb{G}(1,3)).$$

Proof: To see that Γ_C is a divisor, consider the incidence correspondence

$$\Sigma = \{(p, L) \in C \times \mathbb{G}(1,3) \mid p \in L\},$$

whose image in $\mathbb{G}(1,3)$ is Γ_C. The fibers of Σ under the projection to C are all projective planes, so Σ has pure dimension 3. On the other hand, the projection to Γ_C is generically one-to-one, so Γ_C also has pure dimension 3. (See Exercise 3.20 for a generalization.)

Now let γ_C be the class of Γ_C in $A^1(\mathbb{G}(1,3))$, and write

$$\gamma_C = \alpha \cdot \sigma_1$$

for some $\alpha \in \mathbb{Z}$. To determine α, we intersect both sides with the class $\sigma_{2,1}$ and get

$$\deg \gamma_C \cdot \sigma_{2,1} = \alpha \deg(\sigma_1 \cdot \sigma_{2,1}) = \alpha.$$

If (p, H) is a general pair consisting of a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ containing p, the Schubert cycle

$$\Sigma_{2,1}(p, H) = \{L \mid p \in L \subset H\}$$

will intersect the cycle Γ_C transversely. (This follows from Kleiman's theorem in characteristic 0, and can be proven in all characteristics by using the description of the tangent spaces to $\Sigma_1(p, H)$ in Exercise 3.27 and of the tangent spaces to Γ_C in Exercise 3.30.) Therefore

$$\alpha = \#(\Gamma_C \cap \Sigma_{2,1}(p, H)) = \#\{L \mid p \in L \subset H \text{ and } L \cap C \neq \varnothing\}.$$

To evaluate this number, note that H (being general) will intersect C transversely in d points $\{q_1, \ldots, q_d\}$; since $p \in H$ is general, no two of the points q_i will be collinear with p. Thus the intersection $\Gamma_C \cap \Sigma_{2,1}(p, H)$ will consist of the d lines $\overline{p, q_i}$, as in Figure 3.9. It follows that $\alpha = d$, so

$$\gamma_C = d \cdot \sigma_1. \qquad \square$$

We will revisit Proposition 3.13 in Section 3.5.3, where we will see how to calculate γ_C by the method of *specialization*.

Proposition 3.13 makes it easy to answer Keynote Question (b): If $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are general translates of curves of degrees $d_1, \ldots, d_4$, then the cycles Γ_{C_i} are generically transverse by Kleiman transversality, so the number of lines meeting all four is

$$\deg \prod_{i=1}^{4} [\Gamma_{C_i}] = \deg \prod_{i=1}^{4} (d_i \sigma_1) = 2 \prod_{i=1}^{4} d_i.$$

One can verify the necessary transversality by using our description of the tangent spaces, too; as a bonus, we can see exactly when transversality fails. This is the content of Exercises 3.30–3.33.

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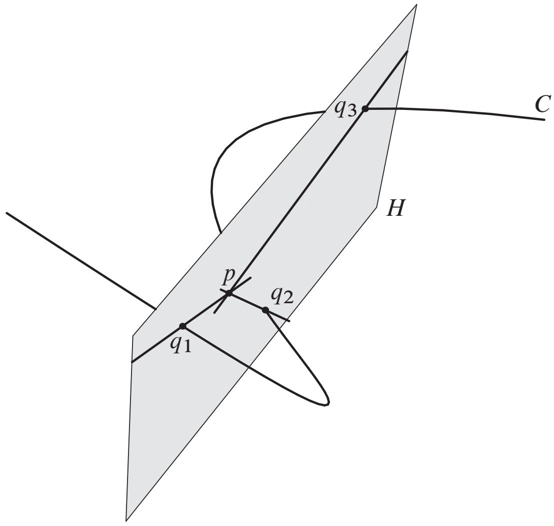

Figure 3.9 The intersection of Γ_C with $\Sigma_{2,1}(p, H)$.

3.4.3 Chords to a space curve

Consider now a smooth, nondegenerate space curve $C \subset \mathbb{P}^n$ of degree d and genus g. We define the locus $\Psi_2(C) \subset \mathbb{G}(1, n)$ of *chords*, or *secant lines* to C, to be the closure in $\mathbb{G}(1, n)$ of the locus of lines of the form $\overline{p, q}$ with p and q distinct points of C. Inasmuch as $\Psi_2(C)$ is the image of the rational map $\tau : C \times C \dashrightarrow \mathbb{G}(1, n)$ sending (p, q) to $\overline{p, q}$, we see that $\Psi_2(C)$ will have dimension 2.

Note that we could also characterize $\Psi_2(C)$ as the locus of lines $L \subset \mathbb{P}^n$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2. As we will see in Exercise 3.38, this characterization differs from the definition given when we consider singular curves, or (as we will see in Exercise 3.39) higher-dimensional secant planes to curves; but for smooth curves in $\mathbb{P}^n$ we will show in Exercise 3.37 they agree, and we can adopt either one. (For much more about secant planes to curves in general, see the discussion in Section 10.3.)

Let us now restrict ourselves to the case $n = 3$ of smooth, nondegenerate space curves $C \subset \mathbb{P}^3$, and ask: What is the class, in $A^2(\mathbb{G}(1, 3))$, of the locus $\Psi_2(C)$ of secant lines to C? We can answer this question by intersecting with Schubert cycles of complementary codimension (in this case, codimension 2). We know that

$$[\Psi_2(C)] = \alpha\sigma_2 + \beta\sigma_{1,1}$$

for some integers α and β. To find the coefficient β we take a general plane $H \subset \mathbb{P}^3$, and consider the Schubert cycle

$$\Sigma_{1,1}(H) = \{L \in \mathbb{G}(1, 3) \mid L \subset H\}.$$

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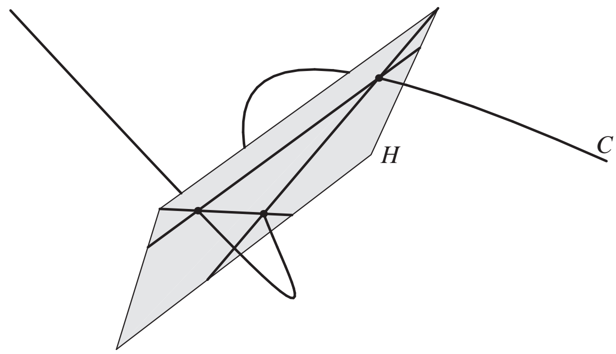

Figure 3.10 $\Sigma_{1,1}(H) \cap \Psi_2(C)$ consists of $\binom{\deg C}{2}$ lines.

By our calculation of $A(\mathbb{G}(1,3))$ and Kleiman transversality, we have

$$\beta = \deg\big(\sigma_{1,1} \cdot [\Psi_2(C)]\big) = \#(\Sigma_{1,1}(H) \cap \Psi_2(C)).$$

The cardinality of this intersection is easy to determine: The plane H will intersect C in d points $p_1, \dots, p_d$, no three of which will be collinear (Arbarello et al. [1985, Section 3.1]), so that there will be exactly $\binom{d}{2}$ lines $\overline{p_i, p_j}$ joining these points pairwise; thus

$$\beta = \binom{d}{2}$$

(see Figure 3.10).

Similarly, to find α we let $p \in \mathbb{P}^3$ be a general point and

$$\Sigma_2(p) = \{L \in \mathbb{G}(1,3) \mid p \in L\};$$

we have as before

$$\alpha = \deg\big(\sigma_2 \cdot [\Psi_2(C)]\big) = \#(\Sigma_2(p) \cap \Psi_2(C)).$$

To count this intersection — that is, the number of chords to C through the point p — consider the projection $\pi_p : C \to \mathbb{P}^2$. This map is birational onto its image $\overline{C} \subset \mathbb{P}^2$, which will be a curve having only nodes as singularities (see Exercise 3.34), and these nodes correspond exactly to the chords to C through p. (These chords were classically called the *apparent nodes* of C (Figure 3.11): If you were looking at C with your eye at the point p, and had no depth perception, they are the nodes you would see.) By the genus formula for singular curves (Section 2.4.6), this number is

$$\alpha = \binom{d-1}{2} - g.$$

Thus we have proven:

Proposition 3.14. *If $C \subset \mathbb{P}^3_{\mathbb{C}}$ is a smooth nondegenerate curve of degree d and genus g, then the class of the locus of chords to C is*

$$[\Psi_2(C)] = \left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma_{1,1} \in A^2(\mathbb{G}(1,3)).$$

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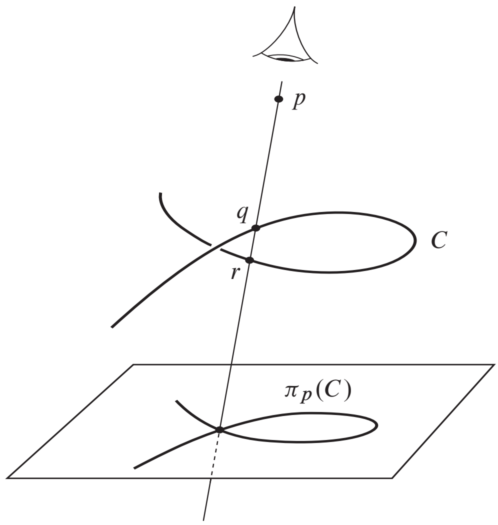

Figure 3.11 Another apparent node.

We can use this to answer the third of the keynote questions of this chapter: If C and C' are general twisted cubic curves, by Kleiman's theorem the cycles $S = \Psi_2(C)$ and $S' = \Psi_2(C')$ intersect transversely; since the class of each is $\sigma_2 + 3\sigma_{1,1}$, we have

$$\#(S \cap S') = \deg(\sigma_2 + 3\sigma_{1,1})^2 = 10.$$

Exercises 3.40 and 3.41 explain how to use the tangent space to the Grassmannian to prove generic transversality, and thus verify this result, in all characteristics.

3.5 Specialization

There is a another powerful approach to evaluating the intersection products of interesting subvarieties: *specialization*. In this section we will discuss some of its variations.

3.5.1 Schubert calculus by static specialization

As a first illustration we show how to compute the class $\sigma_1^2 \in A(\mathbb{G}(1,3))$ by specialization. The reader will find a far-reaching generalization to the Chow rings of Grassmannians and even to more general flag varieties in the algorithms of Vakil [2006a] and Coşkun [2009].

The idea is that instead of intersecting two general cycles $\Sigma_1(L)$ and $\Sigma_1(L')$ representing σ_1, we choose a *special* pair of lines L, L'. The goal is to choose L and L' special enough that the class of the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is readily identifiable, but at the same time not so special that the intersection fails to be generically transverse.

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We do this by choosing L and L' to be distinct but incident. The intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is easy to describe: If $p = L \cap L'$ is the point of intersection of the lines and $H = \overline{L, L'}$ the plane they span, then a line Λ meeting L and L' either passes through p or lies in H (since it then meets L and L' in distinct points). Thus, as sets, we have

$$\begin{aligned}\Sigma_1(L) \cap \Sigma_1(L') &= \{\Lambda \mid \Lambda \cap L \neq \varnothing \text{ and } \Lambda \cap L' \neq \varnothing\} \\ &= \{\Lambda \mid p \in \Lambda \text{ or } \Lambda \subset H\} \\ &= \Sigma_2(p) \cup \Sigma_{1,1}(H).\end{aligned}$$

If we now show that the intersection is generically transverse, we get the desired formula $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$. To check this transversality, we can use the description of the tangent spaces to $\Sigma_1(L)$ and $\Sigma_1(L')$ given in Exercise 3.26. First, suppose Λ is a general point of the component $\Sigma_2(p)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, that is, a general line through p; we will let $K = \overline{\Lambda, L}$ and $K' = \overline{\Lambda, L'}$ be the planes spanned by Λ together with L and L'. Viewing the tangent space $T_\Lambda(\mathbb{G}(1,3))$ as the vector space of linear maps $\varphi : \widetilde{\Lambda} \to V/\widetilde{\Lambda}$, we have

$$T_\Lambda(\Sigma_1(L)) = \left\{\varphi \mid \varphi(\tilde{p}) \subset \widetilde{K}/\widetilde{\Lambda}\right\} \quad \text{and} \quad T_\Lambda(\Sigma_1(L')) = \left\{\varphi \mid \varphi(\tilde{p}) \subset \widetilde{K}'/\widetilde{\Lambda}\right\}.$$

Since K and K' are distinct, they intersect in Λ, so that the intersection is

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \{\varphi \mid \varphi(\tilde{p}) = 0\}$$

Since this is two-dimensional, the intersection $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

Similarly, if Λ is a general point of the component $\Sigma_{1,1}(H)$ of $\Sigma_1(L) \cap \Sigma_1(L')$, so that Λ meets L and L' in distinct points q and q', we have

$$T_\Lambda(\Sigma_1(L)) = \left\{\varphi \mid \varphi(\tilde{q}) \subset \widetilde{H}/\widetilde{\Lambda}\right\} \quad \text{and} \quad T_\Lambda(\Sigma_1(L')) = \left\{\varphi \mid \varphi(\tilde{q}') \subset \widetilde{H}/\widetilde{\Lambda}\right\},$$

so

$$T_\Lambda(\Sigma_1(L)) \cap T_\Lambda(\Sigma_1(L')) = \left\{\varphi \mid \varphi(\widetilde{\Lambda}) \subset \widetilde{H}\right\}.$$

Again this is two-dimensional and we conclude that $\Sigma_1(L) \cap \Sigma_1(L')$ is transverse at $[\Lambda]$.

Before going on, we mention that the computation of σ_1^2 given here is an example of the simplest kind of specialization argument, what we may call *static specialization*: We are able to find cycles representing the two given classes that are special enough that the class of the intersection is readily identifiable, but general enough that they still intersect properly.

In general, we may not be able to find such cycles. Such situations call for a more powerful and broadly applicable technique, called *dynamic specialization*. There, we consider a one-parameter family of pairs of cycles (A_t, B_t) specializing from a "general" pair to a special pair (A_0, B_0), which may not intersect dimensionally transversely at all! The key idea is to ask not for the intersection $A_0 \cap B_0$ of the limiting cycles, but rather for the limit $\lim_{t\to 0}(A_t \cap B_t)$ of their intersections. For an example of

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dynamic specialization, see Section 4.4 of the following chapter, where we consider in the Grassmannian $\mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$ the self-intersection of the cycle of lines meeting a given line in $\mathbb{P}^4$.

3.5.2 Dynamic projection

Problems situated in projective space tend to be especially amenable to specialization techniques: We can use the large automorphism group of $\mathbb{P}^n$ to morph the objects we are dealing with into potentially simpler, more tractable ones. One fundamental example of this is the technique of *dynamic projection*, which we will describe here and use in the following section to re-derive the formulas for the class of the locus of lines incident to a curve.

Fix two disjoint planes A (the "attractor") and R (the "repellor") that span $\mathbb{P}^n$. Choose coordinates $x_0, \dots, x_r, y_0, \dots, y_a$ on $\mathbb{P}^n$ so that the equations of A are $\{x_i = 0\}$ and the equations of R are $\{y_i = 0\}$, and consider the action Ψ of the multiplicative group G_m on $\mathbb{P}^n$ given by

$$\begin{aligned}\psi_t : (x_0, \dots, x_r, y_0, \dots, y_a) \mapsto\ & (tx_0, \dots, tx_r, y_0, \dots, y_a)\\ =\ & (x_0, \dots, x_r, t^{-1}y_0, \dots, t^{-1}y_a).\end{aligned}$$

(In what follows, we will abbreviate $(x_0, \dots, x_r, y_0, \dots, y_a)$ to (x, y); for example, we will write $\psi_t(x, y) = (tx, y)$.) It is clear that the points of A and R remain fixed under the action of G_m. On the other hand, we can say intuitively that a point not in A or R will "flow toward A" as t approaches zero, and will "flow toward R" as t approaches ∞. More precisely, note that any point $p \notin A \cup R$ lies on a unique line that meets both A and R. (The span $\overline{p, A}$, being an $(a+1)$-plane, must meet R. Since A and R are disjoint, $\overline{p, A}$ can meet R only in a point $q \in R$; the line $\overline{p, q}$ is then the unique line containing p and meeting A and R.) This line is the closure of the orbit of p under the given action of G_m. In particular, any point in $\mathbb{P}^n \setminus R$ has a well-defined limit in A as t approaches zero.

Now suppose $X \subset \mathbb{P}^n$ is any variety. We consider the images of X under the automorphisms ψ_t, and in particular their flat limit as $t \to 0$. In other words, we set

$$Z^\circ = \{(t, p) \in \mathbb{A}^1 \times \mathbb{P}^n \mid t \neq 0 \text{ and } p \in \psi_t(X)\};$$

we let $Z \subset \mathbb{A}^1 \times \mathbb{P}^n$ be the closure of Z°, and look at the fiber X_0 of Z over $t = 0$. (Note that even if X is a variety, X_0 may well be nonreduced.) We think of X_0 as the limit of the varieties $\psi_t(X)$ as t approaches 0 (see Figure 3.12 for an illustration; see also Eisenbud and Harris [2000, Chapter 2] for a discussion of flat limits in general).

The following properties of the limit X_0 make it easy to analyze some interesting cases:

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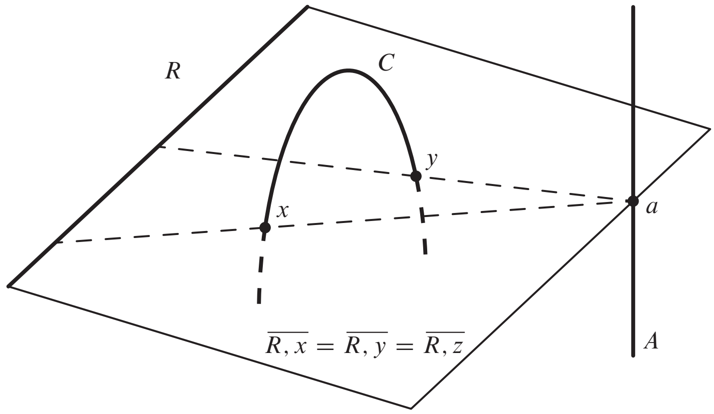

Figure 3.12 Dynamic projection of a conic in $\mathbb{P}^3$ from a line R to a line A.

Proposition 3.15. *With notation as above:*

(a) $X_0 \subset \mathbb{P}^n$ *is stable under the action of* G_m.
(b) $X_0 \cap R = X \cap R$.
(c) $(X_0)_{\mathrm{red}}$ *is contained in the cone over* $X_0 \cap R$ *with base* A *(in case* $X_0 \cap R = \varnothing$*, we take this to mean* $(X_0)_{\mathrm{red}} \subset A$*).*

In addition, we know (as for any rational equivalence) that X_0 is equidimensional and $\dim X_0 = \dim X$, and, by Eisenbud [1995, Exercise 6.11], that the Hilbert polynomial of X_0 is the same as the Hilbert polynomial of X.

Proof: For the first part, consider the action of G_m on the product $\mathbb{A}^1 \times \mathbb{P}^n$ given as the product of the standard action of G_m on $\mathbb{A}^1$ and the action Ψ above of G_m on $\mathbb{P}^n$; that is,

$$\varphi_t : (s, p) \to (ts, \psi_t(p)).$$

This carries Z to itself and the fiber $\{0\} \times \mathbb{P}^n$ to itself, so it carries X_0 to itself. But it acts on the fiber $\{0\} \times \mathbb{P}^n$ via the action Ψ above; thus X_0 is invariant under Ψ.

The second point is more subtle. (In particular, it is asymmetric: The same statement, with R replaced by A, would be false.) It is not, however, intuitively unreasonable: Since points in $\mathbb{P}^n \setminus R$ flow away from R as $t \to 0$, the only way a point $p \in R$ can be a limit of points $\psi_t(p_t)$ is if it is there all along, that is, if $p \in X \cap R$.

In any case, note first that one inclusion is immediate: Since R is fixed pointwise by the automorphisms ψ_t, we have

$$\mathbb{P}^1 \times (X \cap R) \subset Z$$

and hence $X \cap R \subset X_0 \cap R$. To see the other inclusion, we want to show that the ideal $I(X \cap R)$ is contained in $I(X_0 \cap R)$. Let $f(x) \in I(X \cap R)$. We can then write

$$f(x) = g(x, 0) \quad \text{for some } g(x, y) \in I(X).$$

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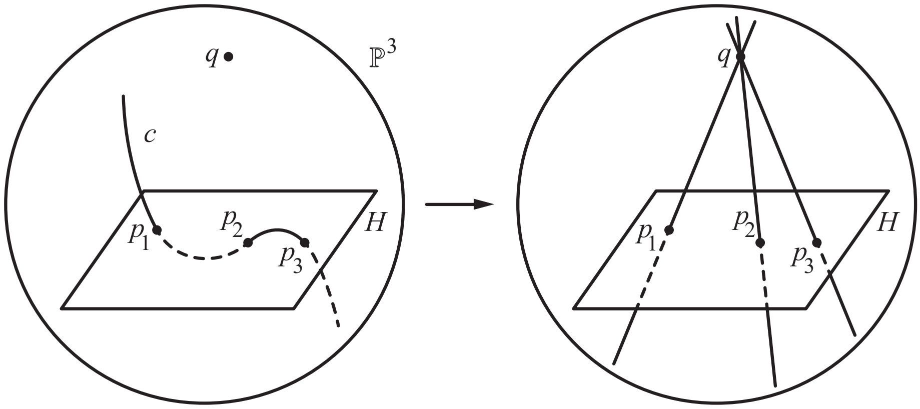

Figure 3.13 A space curve C specializes to a union of lines.

Now observe that

$$I(Z) \supset \{f(x,ty) \mid f \in I(X)\},$$

so $h(t,x,y) = g(x,ty) \in I(Z)$. Setting $t = 0$, we see that $g(x,0) \in I(X_0)$, and hence $f \in I(X_0 \cap R)$.

To prove the third assertion, note that the G_m-orbit of any point not contained in $R \cup A$ is a straight line joining a point of R to a point of A. Since X_0 is stable under G_m, it is the union of such lines, together with any points of $A \cup R$ it contains. □

We sometimes call this construction *dynamic projection*: We are realizing the projection map

$$\pi_R : \mathbb{P}^n \setminus R \to A,$$
$$(x,y) \mapsto (0,y),$$

as the limit of a family of automorphisms ψ_t of $\mathbb{P}^n$. As we will see, though, considering the limit of the images $\psi_t(X)$ yields more information than simply taking the projection $\pi_R(X)$. See Figures 3.12 and 3.13 for examples.

Example 3.16. Let $X \subset \mathbb{P}^n$ be a subvariety of dimension m and degree d. We will exhibit a dynamic projection of X whose limit is a d-fold m-plane (that is, a scheme whose support is an m-plane and that has multiplicity d at the general point of that plane), and another whose limit is the generically reduced union of d distinct m-planes containing a fixed $(m-1)$-plane.

To make the first construction, let $A \subset \mathbb{P}^n$ be any m-dimensional subspace, and choose R to be an $(n-m-1)$-plane $R \subset \mathbb{P}^n$ disjoint from X and from A. Since $X \cap R = \varnothing$, we see that $X_0 \cap R = \varnothing$ as well, and it follows that $(X_0)_{\text{red}} \subset A$. Since $\dim X_0 = \dim X = \dim A$, we see that the support of X_0 is exactly A, and computing the degree we have $\langle X_0 \rangle = d\langle A \rangle$, as claimed.

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To make the second construction, choose the repellor subspace R to be a general plane of dimension $n - m$, so that $R \cap X$ consists of $\deg X$ distinct points, and take A to be an $(m-1)$-plane disjoint from R. We see that $(X_0)_{\text{red}}$ is contained in the union of the m-planes that are the cones over the $\deg X$ points of $X \cap R = X_0 \cap R$. Also, X_0 must contain all these points. Since X_0 is equidimensional and has degree equal to $\deg X$, it follows that X_0 is the generically reduced union of these distinct planes, as required.

Note that while the multiplicities of X_0 are determined in both cases, the actual scheme structure of X_0 will depend very much on the geometry of X.

3.5.3 Lines meeting a curve by specialization

As an example of how dynamic projection can be used in specialization arguments, we revisit the computation of the class γ_C of the locus $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a curve $C \subset \mathbb{P}^3$ from Proposition 3.13.

Let $C \subset \mathbb{P}^3$ be a curve of degree d. Choose a plane $H \subset \mathbb{P}^3$ intersecting C transversely in points $p_1, \ldots, p_d$, and $q \in \mathbb{P}^3$ any point not lying on H. Consider the one-parameter group $\{A_t\} \subset \mathrm{PGL}_4$ with repellor plane H and attractor q; that is, choose coordinates $[Z_0, \ldots, Z_3]$ on $\mathbb{P}^3$ such that $q = [1,0,0,0]$ and H is given by $Z_0 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the closure of the locus

$$\Phi^\circ = \{(t,p) \in \mathbb{A}^1 \times \mathbb{P}^3 \mid t \neq 0 \text{ and } p \in C_t\}.$$

As we saw in the preceding section, the limit of the curves C_t as $t \to 0$ (that is, the fiber of Φ over $t = 0$) is supported on the union of the d lines $\overline{p_i, q}$, and has multiplicity 1 at a general point of each, as shown in Figure 3.13.

We can use this construction to give a rational equivalence between the cycle Γ_C and the sum of the Schubert cycles $\Sigma_1(\overline{p_i, q})$ in $\mathbb{G}(1,3)$. Explicitly, take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\{(t,\Lambda) \in \mathbb{A}^1 \times \mathbb{G}(1,3) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

As we will verify in Exercises 3.35 and 3.36, the fiber Ψ_0 of Ψ over $t = 0$ is supported on the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$ and has multiplicity 1 along each, establishing the rational equivalence $\gamma_C = d \cdot \sigma_1$.

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The fiber of Φ over $t = 0$ (that is, the flat limit $\lim_{t\to 0} C_t$ of the curves C_t) is *not* necessarily equal to the union of the d lines $\overline{p_i, q}$: it may have an embedded point at the point q. Nonetheless, the fiber Ψ_0, being a divisor in $\mathbb{G}(1,3)$, will not have embedded components.

3.5.4 Chords via specialization: multiplicity problems

One of the main difficulties in using specialization is the appearance of multiplicities. We will now illustrate this problem by trying to compute, via specialization, the class of the chords to a smooth curve in $\mathbb{P}^3$.

Consider again the family of curves $C_t := A_t(C)$ described in the previous section. What is the limit as $t \to 0$ of the cycles $\Psi_2(C_t) \subset \mathbb{G}(1,3)$ of chords to C_t? To interpret this question, let $\Pi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ be the closure of the locus

$$\Pi^\circ = \{(t,\Lambda) \in \mathbb{A}^1 \times \mathbb{G}(1,3) \mid t \neq 0 \text{ and } \Lambda \in \Psi_2(C_t)\}.$$

What is the fiber Π_0 of this family?

The support of Π_0 is easy to identify. It is contained in the locus of lines whose intersection with the flat limit $C_0 = \lim_{t\to 0} C_t$ contains a scheme of degree at least 2, which is to say the union of the Schubert cycles $\Sigma_{1,1}(\overline{p_i, p_j, q})$ of lines lying in a plane spanned by a pair of the lines $\overline{p_i, q}$, and the Schubert cycle $\Sigma_2(q)$ of lines containing the point q. Moreover one can show that the Schubert cycles $\Sigma_{1,1}(\overline{p_i, p_j, q})$ all appear with multiplicity 1 in the limiting cycle Π_0, from which we can deduce that the coefficient of $\sigma_{1,1}$ in the class of $\Psi_2(C)$ is $\binom{d}{2}$.

The hard part is determining the multiplicity with which the cycle $\Sigma_2(q)$ appears in Π_0: This will depend in part on the multiplicity of the embedded point of C_0 at q, which will in turn depend on the genus g of C (see for example Exercises 3.43 and 3.44). Note the contrast with the calculation in Section 3.5.3 of the class of the locus Γ_C of incident lines via specialization: There, the embedded component of the limit scheme $\lim_{t\to 0} \Gamma_{C_t}$ also depended on the genus of C, but did not affect the limiting cycle.

An alternative approach to this problem would be to use a different specialization to capture the coefficient of σ_2: Specifically, we could take the one-parameter subgroup with repellor a general point q and attractor a general plane $H \subset \mathbb{P}^3$. The limiting scheme $C_0 = \lim_{t\to 0} C_t$ will be a plane curve of degree d with $\delta = \binom{d-1}{2} - g$ nodes $r_1, \dots, r_\delta$, with a spatial embedded point of multiplicity 1 at each node. The limit of the corresponding cycles $\Psi_2(C_t) \subset \mathbb{G}(1,3)$ will correspondingly be supported on the union of the Schubert cycle $\Sigma_{1,1}(H)$ and the δ Schubert cycles $\Sigma_2(r_i)$. In this case the coefficient of the Schubert cycle $\Sigma_{1,1}(H)$ is the mysterious one (though calculable: given that a general line $\Lambda \subset H$ meets C_0 in d points, we can show that it is the limit of $\binom{d}{2}$ chords to C_t as $t \to 0$). On the other hand, one can show that the Schubert cycles $\Sigma_2(r_i)$ all appear with multiplicity 1 in the limit of the cycles $\Psi_2(C_t)$, from which we can read off the coefficient δ of σ_2 in the class of $\Psi_2(C)$.

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We will fill in some of the details involved in this calculation in Exercise 3.45.

3.5.5 Common chords to twisted cubics via specialization

To illustrate the artfulness possible in specialization arguments, we give a different specialization approach to counting the common chords of two twisted cubics: We will not degenerate the twisted cubics; we will just specialize them to a general pair of twisted cubic curves C, C' lying on the same smooth quadric surface Q, of types $(1, 2)$ and $(2, 1)$ respectively.

The point is, no line of either ruling of Q will be a chord of both C and C' (the lines of one ruling are chords of C but not of C', and vice versa for lines of the other ruling). But since $C \cup C' \subset Q$, any line meeting $C \cup C'$ in three or more distinct points must lie in Q. It follows that *the only common chords to C and C' will be the lines joining the points of intersection $C \cap C'$ pairwise*; since the number of such points is $\#(C \cap C') = \deg([C][C']) = 5$, the number of common chords will be $\binom{5}{2} = 10$. Of course, to deduce the general formula from this analysis, we have to check that the intersection $\Psi_2(C) \cap \Psi_2(C')$ is transverse; we will leave this as Exercise 3.46.

What would happen if we specialized C and C' to twisted cubics lying on Q, both having type $(1, 2)$? Now there would only be four points of $C \cap C'$, giving rise to $\binom{4}{2} = 6$ common chords. But now the lines of one ruling of Q would *all* be common chords to both. Thus $\Psi_2(C) \cap \Psi_2(C')$ would have a positive-dimensional component: Explicitly, $\Psi_2(C) \cap \Psi_2(C')$ would consist of six isolated points and one copy of $\mathbb{P}^1$. It might seem that in these circumstances we could not deduce anything about the intersection number $\deg([\Psi_2(C)] \cdot [\Psi_2(C')])$ from the actual intersection, but in fact the excess intersection formula of Chapter 13 can be used in this case to determine $\deg([\Psi_2(C)] \cdot [\Psi_2(C')])$; see Exercise 13.35.

3.6 Lines and surfaces in $\mathbb{P}^3$

3.6.1 Lines lying on a quadric

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and $F = F_1(Q) \subset \mathbb{G}(1, 3)$ the locus of lines contained in Q. In this section we will determine the class $[F] \in A(\mathbb{G}(1, 3))$. ($F$ is an example of a *Fano scheme*, a construction that will be treated extensively in Chapter 6.)

Via the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, the lines on Q are fibers of the two projections maps $Q \to \mathbb{P}^1$; in particular, we see that $\dim F = 1$. Since $A^3(\mathbb{G}(1, 3))$ is generated by $\sigma_{2,1}$, we must have

$$[F] = \alpha \cdot \sigma_{2,1}$$

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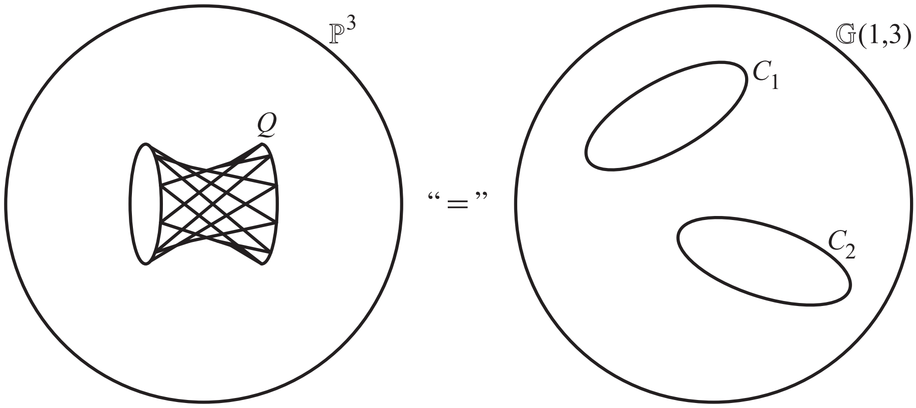

Figure 3.14 The rulings of a quadric surface $Q \subset \mathbb{P}^3$ correspond to conic curves $C_i \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$; thus $[F_1(Q)] = 4\sigma_{2,1}$.

for some integer α. If $L \subset \mathbb{P}^3$ is a general line and $\Sigma_1(L) \subset \mathbb{G}(1,3)$ the Schubert cycle of lines meeting L, then by Kleiman transversality we have

$$\begin{aligned}\alpha &= \deg([F] \cdot \sigma_1)\\ &= \#(\Sigma_1(L) \cap F)\\ &= \#\{M \in \mathbb{G}(1,3) \mid M \subset Q \text{ and } M \cap L \neq \varnothing\}.\end{aligned}$$

Now L, being general, will intersect Q in two points, and through each of these points there will be two lines contained in Q; thus we have $\alpha = 4$ and

$$[F] = 4\sigma_{2,1}.$$

We will see how to calculate the class of the locus of linear spaces on a quadric hypersurface more generally in Section 4.6.

The variety F is actually the union $C_1 \cup C_2$ of two disjoint curves in the Grassmannian, corresponding to the two rulings of Q; each of these curves has class $2\sigma_{2,1}$, and thus has degree 2 as a curve in the Plücker embedding in $\mathbb{P}^5$ (see Figure 3.14). For details see Eisenbud and Harris [2000].

3.6.2 Tangent lines to a surface

Next, let $S \subset \mathbb{P}^3$ be any smooth surface of degree d, and consider the locus $T_1(S) \subset \mathbb{G}(1,3)$ of lines tangent to S. Let Φ be the incidence correspondence

$$\Phi = \{(q, L) \in S \times \mathbb{G}(1,3) \mid q \in L \subset \mathbb{T}_q S\},$$

where $\mathbb{T}_q S$ denotes the projective plane tangent to S at q. The projection $\Phi \to S$ on the first factor expresses Φ as a $\mathbb{P}^1$-bundle over S, from which we deduce that Φ, and hence its image $T_1(S)$ in $\mathbb{G}(1,3)$, is irreducible of dimension 3.

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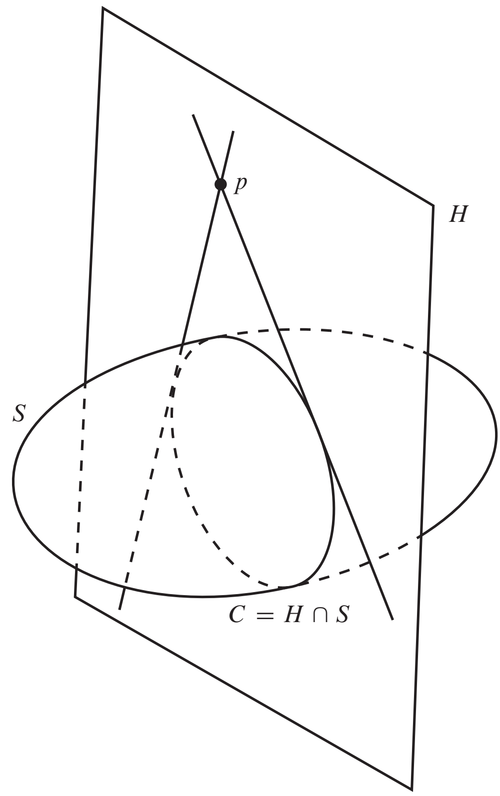

Figure 3.15 $\deg(\Sigma_{2,1}(p, H) \cap T_1(S)) = 2$.

To find the class of $T_1(S)$, we write

$$[T_1(S)] = \alpha \cdot \sigma_1,$$

and choose a general plane $H \subset \mathbb{P}^3$ and a general point $p \in H$. By Kleiman transversality,

$$\begin{aligned}\alpha &= [T_1(S)] \cdot \sigma_{2,1} \\ &= \#(\Sigma_{2,1}(p, H) \cap T_1(S)) \\ &= \#\{M \in \mathbb{G}(1,3) \mid q \in M \subset \mathbb{T}_q S \text{ for some } q \in S \text{ and } p \in M \subset H\}.\end{aligned}$$

Now H, being general, will intersect S in a smooth plane curve $C \subset H \cong \mathbb{P}^2$ of degree d, and, p being general in H, the line $p^* \subset \mathbb{P}^{2*}$ dual to p will intersect the dual curve $C^* \subset \mathbb{P}^{2*}$ transversely in $\deg(C^*)$ points. By Proposition 2.9, we have

$$\deg(C^*) = d(d-1)$$

and hence

$$[T_1(S)] = d(d-1)\sigma_1$$

(see Figure 3.15).

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This gives the answer to the last keynote question of this chapter: How many lines are tangent to each of four general quadric surfaces Q_i? Once more, Kleiman's theorem assures us that the cycles $T_1(Q_i)$ intersect transversely, a fact we can verify in all characteristics by explicit calculation. The answer is thus

$$\deg \prod [T_1(Q_i)] = \deg(2\sigma_1)^4 = 32.$$

3.7 Exercises

Exercise 3.17. Let $\Lambda, \Gamma \in G$ be two points in the Grassmannian $G = G(k, V)$. Show that the line $\overline{\Lambda, \Gamma} \subset \mathbb{P}(\bigwedge^k V)$ is contained in G if and only if the intersection $\Lambda \cap \Gamma \subset V$ of the corresponding subspaces of V has dimension $k-1$.

Exercise 3.18. Using the fact that the Grassmannian

$$G = G(k, V) \subset \mathbb{P}(\textstyle\bigwedge^k V)$$

is cut out by quadratic equations, show that if $[\Lambda] \in G$ is the point corresponding to a k-plane Λ then the tangent plane $\mathbb{T}_{[\Lambda]}G \subset \mathbb{P}(\bigwedge^k V)$ intersects G in the locus

$$G \cap \mathbb{T}_{[\Lambda]}G = \{\Gamma \mid \dim(\Gamma \cap \Lambda) \geq k-1\};$$

that is, the locus of k-planes meeting Λ in codimension 1.

Exercise 3.19. Let V be an $(n+1)$-dimensional vector space, and consider the universal k-plane over $G = \mathbb{G}(k, \mathbb{P}V)$ introduced in Section 3.2.3:

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}.$$

Show that this is a closed subvariety of $G \times \mathbb{P}V$ of dimension $k + (k+1)(n-k)$, and that it is cut out on $G \times \mathbb{P}V$ by bilinear forms on $\mathbb{P}(\bigwedge^{k+1} V) \times \mathbb{P}V$.

Exercise 3.20. Use the preceding exercise to show that, if $X \subset \mathbb{P}^n$ is any subvariety of dimension $l < n-k$, then the locus

$$\Gamma_X = \{\Lambda \in \mathbb{G}(k, n) \mid X \cap \Lambda \neq \varnothing\}$$

of k-planes meeting X is a closed subvariety of $\mathbb{G}(k, n)$ of codimension $n-k-l$.

Exercise 3.21. Let $l < k < n$, and consider the locus of nested pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k:

$$\mathbb{F}(l, k; n) = \{(\Gamma, \Lambda) \in \mathbb{G}(l, n) \times \mathbb{G}(k, n) \mid \Gamma \subset \Lambda\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l, n) \times \mathbb{G}(k, n)$, and calculate its dimension. (These are examples of a further generalization of Grassmannians called *flag manifolds*, which we will explore further in Section 4.8.1.)

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Exercise 3.22. Again let $l < k < n$, and for any $m \le l$ consider the locus of pairs of linear subspaces of $\mathbb{P}^n$ of dimensions l and k intersecting in dimension at least m:

$$\mathbb{F}(l,k;m;n) = \{(\Gamma,\Lambda) \in \mathbb{G}(l,n) \times \mathbb{G}(k,n) \mid \dim(\Gamma \cap \Lambda) \ge m\}.$$

Show that this is a closed subvariety of $\mathbb{G}(l,n) \times \mathbb{G}(k,n)$ and calculate its dimension.

Exercise 3.23. Let $B \subset \mathbb{G}(1,n)$ be a curve in the Grassmannian of lines in $\mathbb{P}^n$, with the property that all nonzero tangent vectors to B have rank 1. Show that the lines in $\mathbb{P}^n$ parametrized by B either

(a) all lie in a fixed 2-plane;
(b) all pass through a fixed point; or
(c) are all tangent to a fixed curve $C \subset \mathbb{P}^n$.

(Note that the last possibility actually subsumes the first.)

Exercise 3.24. Show that an automorphism of $G(k,n)$ carries tangent vectors to tangent vectors of the same rank (in the sense of Section 3.2.4), and hence for $1 < k < n$ the group of automorphisms of $G(k,n)$ cannot act transitively on nonzero tangent vectors. Show, on the other hand, that the group of automorphisms of $G(k,n)$ *does* act transitively on tangent vectors of a given rank.

Exercise 3.25. In Example 3.9, we demonstrated that the open Schubert cell $\Sigma_1^\circ = \Sigma_1 \setminus (\Sigma_2 \cup \Sigma_{1,1})$ is isomorphic to the affine space $\mathbb{A}^3$. For each of the remaining Schubert indices a,b, show that the Schubert cell $\Sigma_{a,b}^\circ \subset \mathbb{G}(1,3)$ is isomorphic to the affine space of dimension $4-a-b$.

Exercise 3.26. Consider the Schubert cycle

$$\Sigma_1 = \{\Lambda \in \mathbb{G}(1,3) \mid \Lambda \cap L \ne \varnothing\}.$$

Suppose that $\Lambda \in \Sigma_1$ and $\Lambda \ne L$, so that $\Lambda \cap L$ is a point q and the span $\overline{\Lambda, L}$ a plane K. Show that Λ is a smooth point of Σ_1, and that its tangent space is

$$T_\Lambda(\Sigma_1) = \left\{\varphi \in \operatorname{Hom}(\widetilde{\Lambda}, V/\widetilde{\Lambda}) \mid \varphi(\tilde{q}) \subset \widetilde{K}/\widetilde{\Lambda}\right\}.$$

Exercise 3.27. Consider the Schubert cycle

$$\Sigma_{2,1} = \Sigma_{2,1}(p,H) = \{\Lambda \in \mathbb{G}(1,3) \mid p \in \Lambda \subset H\}.$$

Show that $\Sigma_{2,1}$ is smooth, and that its tangent space at a point Λ is

$$T_\Lambda(\Sigma_{2.1}) = \left\{\varphi \in \operatorname{Hom}(\widetilde{\Lambda}, V/\widetilde{\Lambda}) \mid \varphi(\tilde{p}) = 0 \text{ and } \operatorname{Im}(\varphi) \subset \widetilde{H}/\widetilde{\Lambda}\right\}.$$

Exercise 3.28. Use the preceding two exercises to show in arbitrary characteristic that general Schubert cycles $\Sigma_1, \Sigma_{2,1} \subset \mathbb{G}(1,3)$ intersect transversely, and deduce the equality $\deg(\sigma_1 \cdot \sigma_{2,1}) = 1$.

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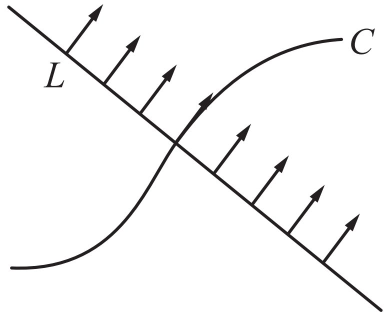

Figure 3.16 Deformation of a line L preserving incidence with a curve C.

Exercise 3.29. Let $L_1, \ldots, L_4 \subset \mathbb{P}^3$ be four pairwise skew lines and $\Lambda \subset \mathbb{P}^3$ a line meeting all four; set

$$p_i = \Lambda \cap L_i \quad \text{and} \quad H_i = \overline{\Lambda, L_i}.$$

Show that $[\Lambda] \in G$ fails to be a transverse point of intersection of the Schubert cycles $\Sigma_1(L_i)$ exactly when the cross-ratio of the four points $p_1, \ldots, p_4 \in \Lambda$ equals the cross-ratio of the four planes $H_1, \ldots, H_4$ in the pencil of planes containing Λ.

Exercises 3.30–3.33 deal with a question raised in Section 3.4.2: If $C_1, \ldots, C_4 \subset \mathbb{P}^3$ are general translates of four curves in $\mathbb{P}^3$, do the corresponding cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ of lines meeting the C_i intersect transversely?

To start with, we have to identify the smooth locus of the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting a given curve C, and its tangent spaces at these points; this is the content of the next exercise, which is a direct generalization of Exercise 3.26 above.

Exercise 3.30. Let $C \subset \mathbb{P}^3$ be any curve, and $L \subset \mathbb{P}^3$ a line meeting C at one smooth point p of C and not tangent to C. Show that the cycle $\Gamma_C \subset \mathbb{G}(1,3)$ of lines meeting C is smooth at the point $[L]$, and that its tangent space at $[L]$ is the space of linear maps $\widetilde{L} \to \Bbbk^4/\widetilde{L}$ carrying the one-dimensional subspace $\tilde{p} \subset \widetilde{L}$ to the one-dimensional subspace $(\widetilde{\mathbb{T}}_p C + \widetilde{L})/\widetilde{L}$ of $\Bbbk^4/\widetilde{L}$ (see Figure 3.16).

Next, we have to verify that, for general translates C_i of any four curves, the corresponding cycles Γ_{C_i} are smooth at each of the points of their intersection. A key fact will be the irreducibility of the relevant incidence correspondence:

Exercise 3.31. Let $B_1, \ldots, B_4 \subset \mathbb{P}^3$ be four irreducible curves and let $\varphi_1, \ldots, \varphi_4 \in \mathrm{PGL}_4$ be four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the incidence correspondence

$$\Phi = \{(\varphi_1, \ldots, \varphi_4, L) \in (\mathrm{PGL}_4)^4 \times \mathbb{G}(1,3) \mid L \cap \varphi_i(B_i) \neq \varnothing \text{ for all } i\}$$

is irreducible.

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Using this, we can prove the following exercise — asserting that for general translates C_i of four given curves and any line L meeting all four, the cycles Γ_{C_i} are smooth at $[L]$ — simply by exhibiting a single collection $(\varphi_1, \dots, \varphi_4, L)$ satisfying the conditions in question:

Exercise 3.32. Let $B_1, \dots, B_4 \subset \mathbb{P}^3$ be four curves and $\varphi_1, \dots, \varphi_4 \subset \mathrm{PGL}_4$ four general automorphisms of $\mathbb{P}^3$; let $C_i = \varphi_i(B_i)$. Show that the set of lines $L \subset \mathbb{P}^3$ meeting C_1, C_2, C_3 and C_4 is finite, and that, for any such L,

(a) L meets each C_i at only one point p_i;
(b) p_i is a smooth point of C_i; and
(c) L is not tangent to C_i for any i.

Exercise 3.33. Let $C_1, \dots, C_4 \subset \mathbb{P}^3$ be any four curves, and $L \subset \mathbb{P}^3$ a line meeting all four and satisfying the conclusions of Exercise 3.32. Use the result of Exercise 3.30 to give a necessary and sufficient condition for the four cycles $\Gamma_{C_i} \subset \mathbb{G}(1,3)$ to intersect transversely at $[L]$, and show directly that this condition is satisfied for all lines meeting $C_1, \dots, C_4$ when the C_i are general translates of given curves.

Exercise 3.34. Let $C \subset \mathbb{P}^3$ be a smooth curve and $p \in \mathbb{P}^3$ a general point. Show that

(a) p does not lie on any tangent line to C;
(b) p does not lie on any trisecant line to C; and
(c) p does not lie on any *stationary secant* to C (that is, a secant line $\overline{q,r}$ to C such that the tangent lines $\mathbb{T}_q C$ and $\mathbb{T}_r C$ meet).

Deduce from these facts that the projection $\pi_p : C \to \mathbb{P}^2$ is birational onto a plane curve $C_0 \subset \mathbb{P}^2$ having only nodes as singularities. (Note that as a consequence the same is true for the projection of a smooth curve $C \subset \mathbb{P}^n$ from a general $(n-3)$-plane to $\mathbb{P}^2$.)

Exercises 3.35 and 3.36 deal with the approach, described in Section 3.5.3, to calculating the class of the variety $\Sigma_C \subset \mathbb{G}(1,3)$ of lines incident to a space curve $C \subset \mathbb{P}^3$ by specialization. Recall from that section that we choose a general plane $H \subset \mathbb{P}^3$ meeting C at d points p_i and a general point $q \in \mathbb{P}^3$, and let $\{A_t\}$ be the one-parameter subgroup of PGL_4 with attractor q and repellor H; we let $C_t = A_t(C)$ and take $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ to be the closure of the locus

$$\Psi^\circ = \{(t, \Lambda) \in \mathbb{A}^1 \times \mathbb{G}(1,3) \mid t \neq 0 \text{ and } \Lambda \cap C_t \neq \varnothing\}.$$

Exercise 3.35. Show that the support of the fiber Ψ_0 is exactly the union of the Schubert cycles $\Sigma_1(\overline{p_i, q})$.

Exercise 3.36. Show that Ψ_0 has multiplicity 1 at a general point of each Schubert cycle $\Sigma_1(\overline{p_i, q})$.

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Exercise 3.37. Let $C \subset \mathbb{P}^r$ be a smooth curve. Show that the rational map $\varphi : C^2 \dashrightarrow \mathbb{G}(1,r)$ sending (p,q) to the line $\overline{p,q}$ when $p \neq q$ actually extends to a regular map on all of C^2 sending (p,p) to the projective tangent line $\mathbb{T}_p C$. Use this to show that the image of φ coincides with the locus of lines $L \subset \mathbb{P}^r$ such that the scheme-theoretic intersection $L \cap C$ has degree at least 2.

Exercise 3.38. Show by example that the conclusion of the preceding exercise is false in general if we do not assume $C \subset \mathbb{P}^r$ to be smooth. Is it still true if we allow C to have mild singularities, such as nodes?

Exercise 3.39. Similarly, show by example that the conclusion of Exercise 3.37 is false if we consider higher-dimensional secant planes: For example, the image of the rational map

$$\varphi : C^3 \dashrightarrow \mathbb{G}(2,r),$$
$$(p,q,r) \mapsto \overline{p,q,r},$$

need not coincide with the locus of 2-planes $\Lambda \subset \mathbb{P}^r$ whose scheme-theoretic intersection with C has degree at least 3.

Exercise 3.40. Show that the smooth locus of $S = \Psi_2(C)$ contains the locus of lines $L \subset \mathbb{P}^3$ such that the scheme-theoretic intersection $L \cap C$ consists of two reduced points, and for such a line L identify the tangent plane $T_L S$ as a subspace of $T_L \mathbb{G}$. (When is a tangent line to C a smooth point of $\Psi_2(C)$?)

Exercise 3.41. Use the result of the preceding exercise to show that if $C, C' \subset \mathbb{P}^3$ are two general twisted cubic curves, then the varieties $\Psi_2(C), \Psi_2(C') \subset \mathbb{G}(1,3)$ of chords to C and C' intersect transversely.

Exercise 3.42. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve of degree d and genus g, and let $L, M \subset \mathbb{P}^3$ be general lines.

(a) Find the number of chords to C meeting both L and M by applying Proposition 3.14.
(b) Verify this count by considering the product morphism

$$\pi_L \times \pi_M : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

(where $\pi_L, \pi_M : C \to \mathbb{P}^1$ are the projections from L and M) and comparing the arithmetic and geometric genera of the image curve.

Exercise 3.43. Let $C \subset \mathbb{P}^3$ be a smooth, irreducible nondegenerate curve of degree d, and let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^3$ be the family of curves specializing C to a scheme supported on the union of lines joining a point $p \in \mathbb{P}^3$ to the points of a plane section of C, as constructed in Section 3.5.3. Show that C_0 may have an embedded point at p, and that the multiplicity of this embedded point may depend on the genus of the curve C, by considering the examples of curves of degrees 4 and 5.

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Exercise 3.44. In the situation of the preceding problem, let $\Psi_2(C_t) \subset \mathbb{G}(1,3)$ be the locus of chords to C_t for $t \neq 0$. Suppose that the degree of C is 4. Show that the component $\Sigma_2(p)$ will be in the flat limit with multiplicity depending on the genus of C.

Exercise 3.45. Again, suppose $C \subset \mathbb{P}^3$ is any curve of degree d; choose a general plane $H \subset \mathbb{P}^3$ and point $p \in \mathbb{P}^3$, and consider the one-parameter group $\{A_t\} \subset \mathrm{PGL}_4$ with repellor point p and attractor plane H — that is, choose coordinates $[Z_0, \dots, Z_3]$ on $\mathbb{P}^3$ such that $p = [0,0,0,1]$ and H is given by $Z_3 = 0$, and consider for $t \neq 0$ the automorphisms of $\mathbb{P}^3$ given by

$$A_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}.$$

Let $C_t = A_t(C)$, and for $t \neq 0$ let $\Psi_2(C_t) \subset \mathbb{G}(1,3)$ be the locus of chords to C_t. Show that the Schubert cycle $\Sigma_{1,1}(H)$ appears as a component of multiplicity $\binom{d}{2}$ in the limiting scheme $\lim_{t\to 0} \Psi_2(C_t)$.

Hint: Let $\Psi \subset \mathbb{A}^1 \times \mathbb{G}(1,3)$ be the closure of the family

$$\Psi^\circ = \{(t, L) \mid t \neq 0 \text{ and } L \in \Psi_2(C_t)\},$$

and show that if $L \subset H$ is a general line then in a neighborhood of the point $(0, L) \in \mathbb{A}^1 \times G$ the family Ψ consists of the union of $\binom{d}{2}$ smooth sheets, each intersecting the fiber $\{0\} \times \mathbb{G}(1,3)$ transversely in the Schubert cycle $\Sigma_{1,1}(H)$.

Exercise 3.46. Let $C, C' \subset Q \subset \mathbb{P}^3$ be general twisted cubic curves lying on a smooth quadric surface Q, of types $(1,2)$ and $(2,1)$ respectively. Show that the intersection $\Psi_2(C) \cap \Psi_2(C')$ of the corresponding cycles of chords is transverse.

Exercise 3.47. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, and let $T(C) \subset \mathbb{G}(1,3)$ be the locus of its tangent lines. Find the class $[T(C)] \in A^3(\mathbb{G}(1,3))$ of $T(C)$ in the Grassmannian $\mathbb{G}(1,3)$.

Exercise 3.48. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, and let $S \subset \mathbb{P}^3$ be a general surface of degree e. How many tangent lines to C are tangent to S?

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