Toric Geometry: Example Sheet 1

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Last compiled February 3, 2022

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§ *Theory Problems*

EXERCISE 1. Given a cone $\sigma \subseteq N_{\mathbb{R}}$ prove that the double dual recovers the original cone:

$$(\sigma^{\vee})^{\vee} = \sigma.$$

This justifies the use of the word "dual".

> *Proof:* We provide two solutions to this problem.
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> **(1)** This is a rather inelegant solution which makes use of the identifications $V \cong V^{\vee} \cong (V^{\vee})^{\vee}$ in the case that V is a finite dimensional vector space. It nonetheless reflects how one typically thinks of the dual cone $\sigma^{\vee}$ geometrically.
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> Recall that for any field K and any K-vector space V of dimension $n < \infty$, we can find a non-canonical isomorphism $V \cong V^{\vee}$. One typically constructs such an isomorphism as follows.
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> First, fix a basis $\{e_1, ..., e_n\}$ for V and define $e_i^{\vee}$ to be the K-linear functional $e_i^{\vee}\left(\sum_{i=1}^n a_i e_i\right) = a_i$. It is straightforward to check that $\{e_1^{\vee}, ..., e_n^{\vee}\}$ forms a basis for the dual space $V^{\vee}$. We may similarly define the basis $\{e_1^{\vee\vee}, ..., e_n^{\vee\vee}\}$ of the double dual $V^{\vee\vee}$.
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> The pairing $\langle -, - \rangle : V^{\vee} \times V \longrightarrow K$ appearing in the definition of $\sigma^{\vee}$ is the bilinear map defined $\langle \lambda, v \rangle = \lambda(v)$. Adopting the above notation in the case that $V = N_{\mathbb{R}}$, we see that this pairing is simply the standard Euclidean inner product. Indeed, letting $\{e_i\}$ denote the standard basis on $\mathbb{R}^n \cong N_{\mathbb{R}}$, given any $v \in N_{\mathbb{R}}$ and $m \in M_{\mathbb{R}}$ and choosing $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ such that $v = \sum a_i e_i$ and $m = \sum b_i e_i^{\vee}$, we see that
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> $$\begin{aligned} \langle m, v \rangle &= m(v) \\ &= (b_1 e_1^{\vee} + ... + b_n e_n^{\vee})(v) \\ &= b_1 e_1^{\vee}(v) + ... + b_n e_n^{\vee}(v) \\ &= b_1 \cdot a_1 + ... + b_1 \cdot a_1. \end{aligned}$$
>
> By identifying $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ via $e_i \leftrightarrow e_i^{\vee}$, we may in fact *define* $\langle m, v \rangle$ to be the Euclidean inner product. This is useful because the Euclidean inner product is symmetric, i.e. $\langle m, v \rangle = \langle v, m \rangle$. By further identifying $\mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R}) = M_{\mathbb{R}}^{\vee}$ with $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ by $e_i \leftrightarrow e_i^{\vee} \leftrightarrow e_i^{\vee\vee}$, we see that for $v \in M_{\mathbb{R}}^{\vee}$ and $m \in M_{\mathbb{R}}$,
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> $$\langle v, m \rangle \geq 0 \iff \langle m, v \rangle \geq \iff \langle m, v' \rangle \geq 0$$
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> where v' is the unique element in $N_{\mathbb{R}}$ corresponding to $v \in M_{\mathbb{R}}^{\vee}$. Thus, under these identifications, we quite literally have that $\left(\sigma^{\vee}\right)^{\vee} = \sigma$.
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> **(2)** After reading Fulton more closely, I realized that it is perhaps more natural to define $\left(\sigma^{\vee}\right)^{\vee}$ to be a subset of σ rather than a subset of $\mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$. Given a subset $A \subseteq M_{\mathbb{R}}$, we first define the *predual* cone $A^{\vee} \subseteq N_{\mathbb{R}}$ of A to be
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> $$A^{\vee} = \{v \in N_{\mathbb{R}} \mid \lambda(v) \geq 0, \text{ for all } \lambda \in A\},$$

and then define the double dual $\left(\sigma^\vee\right)^\vee$ to be the predual cone of $\sigma^\vee$. Showing that $\left(\sigma^\vee\right)^\vee = \sigma$ is therefore equivalent to showing that for any $v_0 \in N_\mathbb{R} \setminus \sigma$, there is some $\lambda \in \sigma^\vee$ such that $\lambda(v_0) < 0$.

To do this, we use a version of the Hahn-Banach theorem I came across on Wikipedia. I'm not entirely sure this works, as I'm taking for granted that $N_\mathbb{R} \cong \mathbb{R}^n$ as a *topological* vector space. Here is the theorem:

Theorem 0.1. *Let A and B be non-empty convex subsets of a real locally convex topological vector space X. If $\operatorname{Int}(A) \neq \emptyset$ and $B \cap \operatorname{Int}(A) = \emptyset$, then there exists a continuous linear functional $f : X \longrightarrow \mathbb{R}$ such that $\sup f(A) \leq \inf f(B)$ and $|f(a)| < \inf f(B)$ for all $a \in \operatorname{Int}(A)$.*

Let v_0 be any element of $N_\mathbb{R}$ not in σ. Let A be an open ball centered at v_0 such that $A \cap \sigma = \emptyset$. This exists because σ is a closed subset of $N_\mathbb{R}$ which does not contain v_0, meaning the distance from v_0 to σ is positive. By Hahn-Banach, there exists a linear functional $\lambda \in M_\mathbb{R}$ such that $\lambda(v_0) < M = \inf \lambda(B)$. We show that $M = v_0$, hence $\lambda \in \sigma^\vee$.

We must have that $M \leq 0$ since $\lambda(0) = 0$ and $0 \in \sigma$. If $M < 0$, then there would necessarily be some $x \in \sigma$ such that $\lambda(x) < 0$. Assuming this to be the case, set $a = \frac{2\lambda(v_0)}{\lambda(x)}$, noting that $a > 0$ since $\lambda(x), \lambda(v_0) < 0$. This means that $ax \in \sigma$. However, recalling that $\lambda(v_0) < 0$, we have that

$$\lambda(ax) = a\lambda(x) = 2\lambda(v_0) < \lambda(v_0),$$

which is impossible since $\lambda(v_0) < \lambda(u)$ for all $u \in \sigma$. Hence, by contradiction, $M = 0$ and λ is nonnegative on all of σ. This means $\lambda \in \sigma^\vee$, so we are done.

I sincerely hope there is another proof besides the two provided here. The first feels highly unnatural and the second seems non-trivial. Given that both Cox-Little-Schneck and Fulton omit a proof of this fact in their book and that neither includes this problem as an exercise, I expect there exists a more natural, obvious proof of this fact that I am missing. $\square$

EXERCISE 2. Given a cone $\sigma \subseteq N_\mathbb{R}$ prove that σ is full-dimensional if and only if $\sigma^\vee$ is strictly convex.

EXERCISE 3. Let Σ and Σ' be fans in vector spaces $N_\mathbb{R}$ and $N'_\mathbb{R}$. Work out for yourself the correct definition of the product fan $\Sigma \times \Sigma'$ in $N_\mathbb{R} \oplus N'_\mathbb{R}$. Show that there is a natural isomorphism:

$$X_{\Sigma \times \Sigma'} \cong X_\Sigma \times_{\operatorname{Spec}\mathbb{C}} X_{\Sigma'}.$$

Slogan: "The construction of a toric variety from a fan commutes with products."

EXERCISE 4. In lectures, we claimed that the toric variety X_σ is smooth if and only if σ is generated by a subset of a $\mathbb{Z}$-basis for N. Complete the proof of this statement. Give an example to show that if σ is generated by a subset of a $\mathbb{Q}-$*basis* then X_σ need not be smooth.

EXERCISE 5. Let X be a *not-necessarily-normal* toric variety with dense torus T. Recall that we partitioned the lattice $N = \operatorname{Hom}_{\mathrm{AlgGrp}}(\mathbb{C}^*, T)$ of T based on the limits of one-parameter subgroups of T inside X. If X were normal, this would give the fan of X and therefore determine X uniquely. Give examples to show that, without the normality assumption, this data does not uniquely determine X.

EXERCISE 6. Let $S \subseteq M$ be an affine semigroup. The *saturation* of S is defined to be:

$$S^{\text{sat}} = \{m \in M \ : \ cm \in S \text{ for some } c \in \mathbb{Z}_{\geq 1}\}.$$

Clearly S^{sat} is saturated, and S is saturated if and only if $S = S^{\text{sat}}$. Consider the inclusion

$$\mathbb{C}[S] \subseteq \mathbb{C}[S^{\text{sat}}].$$

Look up "integral closure" of an integral domain, and prove that $\mathbb{C}[S^{\text{sat}}]$ is the integral closure of $\mathbb{C}[S]$. The dual morphism is known as the normalization of $\operatorname{Spec}\mathbb{C}[S]$. In each of the following examples, write down equations in affine space for both $\operatorname{Spec}\mathbb{C}[S]$ and its normalization $\operatorname{Spec}\mathbb{C}[S^{\text{sat}}]$, and study the morphism between them:

(a) $S = 2\mathbb{N} + 3\mathbb{N} \subseteq \mathbb{Z}$,

(b) $S = (1,1)\mathbb{N} + (1,0)\mathbb{N} + (0,2)\mathbb{N} \subseteq \mathbb{Z}^2$.

Proof: Suppose $x \in \operatorname{Frac}(\mathbb{C}[S])$ satisfies the monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ where $f \in \mathbb{C}[S][t]$. $\square$

§ ***Practice Problems***

EXERCISE 1. First problem

EXERCISE 10