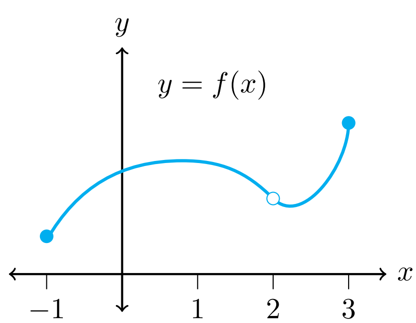
Calculus M408C

Fall 2023

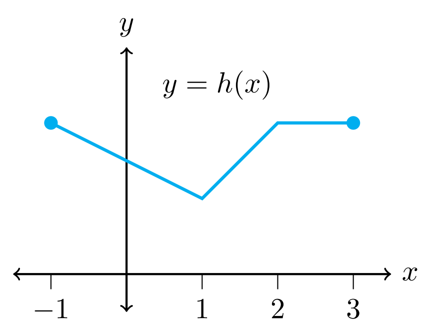
Continuity

1. A function $f(x)$ is **continuous at the point** $x = a$ if $\lim_{x \to a} f(x) = f(a)$. A function is **continuous** if it is continuous at every point in its domain. What we have been referring to as "nice" functions are precisely continuous functions. A common way of thinking about continuous functions is that their graphs can be drawn without taking your pen off the page. For the graphs below, determine if the functions they represent are continuous on the displayed interval. If they do not, locate the points where the functions are **discontinuous**.

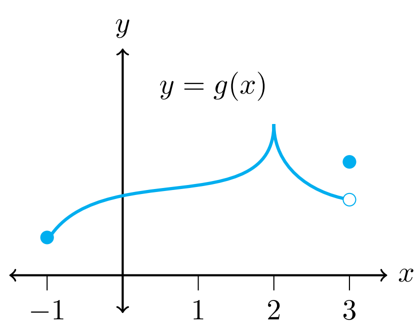
a)



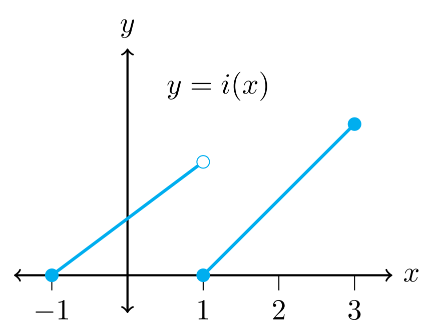
c)



b)



d)



2. Determine all points for which the following functions are continuous. If you find points where the functions is discontinuous, state the type of discontinuity that occurs.

a) $f(x) = \dfrac{1}{(x+2)^2} + 4$ b) $g(t) = \dfrac{t+3}{t^2 - 3t - 10}$ c) $h(s) = \begin{cases} s^2 - 2s - 1 & s \neq 1 \\ 0 & s = 1 \end{cases}$

d) $y = \dfrac{1}{|x|+1} - \dfrac{x^2}{2}$ e) $v(t) = \begin{cases} t^2 - 1 & t < 0 \\ 2t & t \geq 0 \end{cases}$ f) $a(t) = \dfrac{t+2}{\cos(t)}$

g) $f(z) = \dfrac{\sqrt{z^4+1}}{1+\sin^2(z)}$ h) $f(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$ i) $k(t) = \tan(t)$

j) $g(s) = \dfrac{\cos(s)}{s}$ k) $h(x) = \dfrac{x\tan(x)}{x^2+1}$ l) $y = |x-1| + \sin(x)$

3. For each of the following functions, define a piece-wise function that is continuous everywhere and equal to the given function on the given function's domain.

a) $u(t) = \dfrac{t^2-9}{t-3}$ b) $f(x) = \dfrac{x^2+3x-10}{x-2}$ c) $a(z) = \dfrac{z^3-1}{z^2-1}$

4. The **intermediate value theorem** states that if f is a continuous function on a closed interval $[a, b]$, and y_0 is any value between $f(a)$ and $f(b)$, then there is a point c in $[a, b]$ such that $f(c) = y_0$. One important application of the intermediate value theorem is when $f(a)$ and $f(b)$ have opposite signs (one is positive and the other, negative). In this special case the intermediate value theorem implies that there is a a value c for which $f(c) = 0$. For each of the following functions, find values of a and b, so that $|a - b| \leq 1$, for which $f(a)$ and $f(b)$ have opposite signs and use the intermediate value theorem to conclude there is a root of f in the interval (a, b). You may use a calculator for these questions. If you are to receive a problem like this on an exam, you will be given functions that you can evaluate to find exact answers without a calculator.

a) $f(x) = x^3 - 15x + 1$ b) $f(t) = \cos(x) - x$ c) $f(s) = s^3 + 2s - 5$

d) $f(z) = z^3 - \sqrt{z} - 20$

Answers

1. a) Discontinuous at $x = 2$.
   b) Discontinuous at $x = 3$.
   c) Continuous.
   d) Discontinuous at $x = 1$.

2. a) Discontinuous at $x = -2$.
   b) Discontinuous at $t = -2, 5$.
   c) Discontinuous at $s = 1$.
   d) Continuous.
   e) Discontinuous at $t = 0$.
   f) Discontinuous at $t = \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.
   g) Continuous.
   h) Discontinuous at $x = 0$.
   i) Discontinuous at $t = \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.
   j) Discontinuous at $s = 0$.
   k) Discontinuous at $t = \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.
   l) Continuous.

3. a) $U(t) = t + 3$
   b) $F(x) = x + 5$
   c) Does not exist.

4. a) $(0, 1)$
   b) $(0, 1)$
   c) $(1, 2)$
   d) $(2, 3)$

Calculus 408C

Fall 2023

Limits at Infinity

1. Calculate the following limits.

a) $\lim_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

b) $\lim_{t\to-\infty} \left(\dfrac{t^2+t-1}{8t^2-3}\right)^{1/3}$

c) $\lim_{y\to-5} \dfrac{2y}{2y+10}$

d) $\lim_{x\to\infty} \dfrac{\sin^2(x)}{x^2}$

e) $\lim_{u\to\infty} \dfrac{\sqrt{u^2+1}}{u+1}$

f) $\lim_{v\to-\infty} \dfrac{\sqrt[3]{v}-5v+3}{2v+v^{2/3}-4}$

g) $\lim_{x\to\infty} \dfrac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

h) $\lim_{s\to7} \dfrac{4}{(s-7)^2}$

i) $\lim_{t\to0} \dfrac{2}{t^{1/5}}$

j) $\lim_{s\to-\infty} \dfrac{\sqrt[3]{s}-\sqrt[5]{s}}{\sqrt[3]{s}+\sqrt[5]{s}}$

k) $\lim_{a\to0} \dfrac{-1}{a^2(a+1)}$

l) $\lim_{t\to\infty} \dfrac{\cos^6\left(t^5+7t+1\right)}{t^2+3}$

m) $\lim_{x\to0} \dfrac{x^2-3x+2}{x^3-2x^2}$

n) $\lim_{t\to0} \dfrac{\sqrt{t^6+3t+1}}{t^3+t^2}$

o) $\lim_{y\to0} \dfrac{y^{17}+y^{11}-5y^2}{y^{16}-y^5+2y^2}$

p) $\lim_{z\to0} \dfrac{\sqrt[4]{z^{24}+z^{16}-z^{15}+1}}{\sqrt[3]{z^{18}-32z^2-1}}$

2. Recall we have seen two types of asymptotes so far: vertical and horizontal. The function $y=f(x)$ has a *vertical asymptote* at $x=a$ if at least one of the limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ is ∞ or $-\infty$. The function $y=f(x)$ has a *horizontal* asymptote at $y=c$ if at least one of $\lim_{x\to-\infty} f(x)$ or $\lim_{x\to\infty} f(x)$ is equal to c.

For the following functions, find all vertical and horizontal asymptotes.

a) $f(x)=\dfrac{4x^3+8x^2-11x+3}{x^3+2x^2-x-2}$

b) $v(t)=\dfrac{3t^3-21t-18}{2t^3+2t^2-42t-90}$

c) $g(z)=\dfrac{-z^2+3z-2}{z^4-5z^2+4}$

d) $u(t)=\dfrac{\pi\left(t^2-1\right)\left(t^2-9\right)\left(t^2-16\right)}{et^2(t+1)(t+2)(t+3)(t+4)}$

3. Carefully calculate (algebraically) the following limits.

a) $\lim_{x\to\infty} \sqrt{x+9}-\sqrt{x+4}$

b) $\lim_{t\to-\infty} \sqrt{x^2+3}+x$

c) $\lim_{y\to\infty} \sqrt{y^2+3y}-\sqrt{y^2-2y}$

d) $\lim_{a\to-\infty} 2a+\sqrt{4a^2+3a-2}$

Answers

1. a) 2 b) $\frac{1}{2}$ c) DNE

   d) 0 e) 1 f) $-\frac{5}{2}$

   g) ∞ h) ∞ i) DNE

   j) -1 k) $-\infty$ l) 0

   m) $-\infty$ n) ∞ o) $-\frac{5}{2}$

   p) -1

2. a) Vertical: $x = -2, -1, 1$, Horizontal: $y = 4$ b) Vertical: $t = -3, 5$, Horizontal: $y = \frac{3}{2}$

   c) Vertical: $z = -2, -1$, Horizontal: $y = 0$ d) Vertical: $t = -2, 0$, Horizontal: $y = \frac{\pi}{e}$

3. a) 0 b) 0

   c) $\frac{5}{2}$ d) $-\frac{3}{4}$