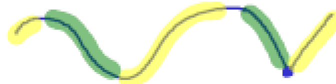
Corollary of MVT

f continuous on $[a,b]$, diff. on (a,b), then

if $f'(x)>0$ on $(a,b) \Rightarrow f$ is increasing on $[a,b]$

slope of tangent line



if $f'(x)<0$ on $(a,b) \Rightarrow f$ is decreasing on $[a,b]$

Determine in which intervals $\underbrace{g(x)}_{\text{diff on } (-\infty,\infty)} = x^2 e^{-3x^2}$ increases/decreases

$g'>0$ $\quad$ $g'<0$

$$g'(x) = 2xe^{-3x^2} - 6x^3e^{-3x^2} \qquad >0? \qquad <0?$$

$$= 2x\, e^{-3x^2}\,(1-3x^2)$$

$$= 2x\, \boxed{e^{-3x^2}}\,(1-\sqrt{3}x)(1+\sqrt{3}x)$$

$e^{-3x^2} > 0$

$g'(x)=0$

	$-\infty$		$\boxed{x=-\frac{1}{\sqrt{3}}}$		$\boxed{x=0}$		$\boxed{x=\frac{1}{\sqrt{3}}}$		$+\infty$
$2x>0$		−		−		+		+	
$1-\sqrt{3}x>0$		+		+		+		−	
$1+\sqrt{3}x>0$		−		+		+		+	
$g'(x)$		+	$-\frac{1}{\sqrt{3}}$	−	0	+	$\frac{1}{\sqrt{3}}$	−	
		↗		↘		↗		↘	

increasing in $\left(-\infty, -\frac{1}{\sqrt{3}}\right]$ & $\left[0, \frac{1}{\sqrt{3}}\right]$

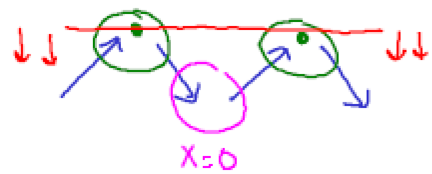
decreasing in $\left[-\frac{1}{\sqrt{3}}, 0\right]$ & $\left[\frac{1}{\sqrt{3}}, +\infty\right)$

Can we find local extrema of $g(x)$?

If so why & what are they?

If loc extrema exist

$\rightarrow$ must be at C.P.



$x = -\frac{1}{\sqrt{3}}$
local max
ABS

$\boxed{\begin{array}{l} x=0 \\ \text{local min} \end{array}}$

$x = \frac{1}{\sqrt{3}}$
local max
ABS

($g(x)$ is even function helps us checking our work) $\rightarrow$ $g\left(-\frac{1}{\sqrt{3}}\right) = g\left(\frac{1}{\sqrt{3}}\right)$

* to know whether at $x=0$ we have an absolute min, we must do a little bit of work

cannot be <0

$$\left\{\begin{array}{l} g(x) = \boxed{x^2} e^{-3x^2} \geq 0 \\ g(0) = 0 \end{array}\right\} \text{ at } x=0 \text{ we have ABS. MIN.}$$

Given $f(x)$ how to find local extrema?

$x = c$

① $x = c$ is critical point $\nearrow$ $f'(c) = 0$ or f' DNE

or ② $x = c$ is an endpoint

First Derivative Test for Local Extrema $\boxed{\text{SIGN of } f'}$

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

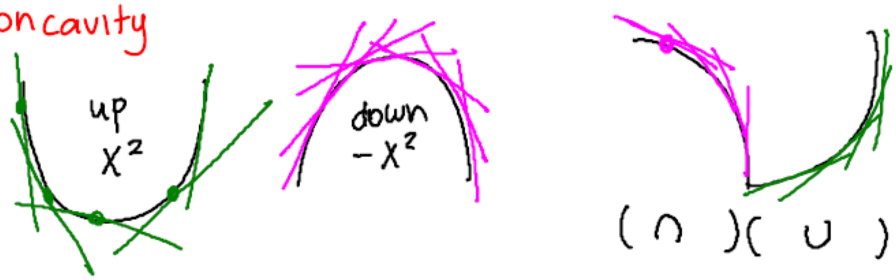
1. if f' changes from negative (dec $\searrow$) to positive (inc $\nearrow$) at c, then f has a local minimum at c; $-$ $+$
2. if f' changes from positive $\nearrow$ to negative $\searrow$ at c, then f has a local maximum at c; $+$ $-$
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c. $+$ $+$ $-$ $-$

If c is an endpoint for $f(x)$, then:
↳ always LOCAL extrema

		at $x=c$
• if $f' > 0$ in $(\dots, c]$, then	$\nearrow c$	MAX
• if $f' > 0$ in $[c, \dots)$, then	$c \nearrow$	min
• if $f' < 0$ in $(\dots, c]$, then	$\searrow c$	min
• if $f' < 0$ in $[c, \dots)$, then	$c \searrow$	MAX

Determine all extrema of $f(x) = \sqrt{4 - x^2} - x$

Concavity



The graph of a differentiable function $f(x)$ is

CONCAVE UP on an open interval I if $f'(x)$ is increasing in I
slopes are increasing

geometrically: tangent line is below graph

CONCAVE DOWN on an open interval I if $f'(x)$ is decr. in I
slopes are decr.

geometrically: tangent line is Above graph

EXAMPLE: $f(x) = x^3 + 2x$ where is concave up?

where $f'(x)$ is increasing

$f'(x) = \boxed{3x^2 + 2 = g(x)}$ how to find where $g(x)$ ↗ or ↘

$f''(x) = g'(x) > 0$ equiv. to $g(x)$ incr.

$f''(x) = g'(x) = 6x > 0$ when $x > 0$

$f(x)$ is concave up in $(0, +\infty)$

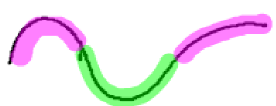
Second derivative test for concavity:

$y = f(x)$ graph of a twice-diff. function on I, then
↳ f'' must exist!

- if $\underbrace{f''(x) > 0}_{f' \nearrow}$ on I $\Rightarrow$ f concave up on I
- if $\underbrace{f''(x) < 0}_{f' \searrow}$ on I $\Rightarrow$ f concave down on I

" 1^{st} derivative test for monotonicity applied to 1^{st} der. of f "
increasing / decreasing

Q: function where we cannot use this test ⤴

2-diff
x^2 $\quad 2x \quad 2 > 0 \quad$ concave up
$e^x \quad e^x \quad e^x \ldots \quad$ concave up
poly. not lin. 
[$x \quad \boxed{1}$ const. $\quad 0 \quad \leftarrow$ never concave]

Ex: $f(x) = x^{\frac{5}{3}}$ doesn't have f'' at zero

$f'(x) = \frac{5}{3} x^{\frac{2}{3}}$

$f''(x) = \frac{10}{9} x^{-\frac{1}{3}}$

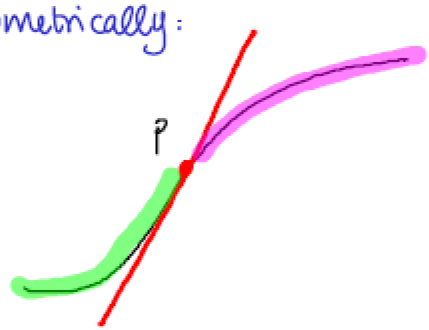
→ we cannot apply that test on intervals containing zero

$(-\infty, 0)$ ✓ or $(0, +\infty)$ ✓

or $[1, \pi]$

POINTS of inflection of f

Geometrically:

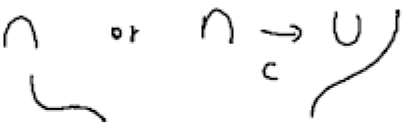


$P=(c, f(c))$ is inflection point if tangent line at P "CROSSES" the graph of f

$\Rightarrow$ at P the concavity changes

Let f be a function defined on $I=[a,b]$ and $c \in I$.
We say that $(c, f(c))$ is an inflection point of f if

a) concavity changes $\cup \underset{c}{\to} \cap$ or $\cap \underset{c}{\to} \cup$



[b) tangent line must exist (P cannot be a CORNER)]

Q: is b) equivalent to f being differentiable at P?

$\Downarrow$

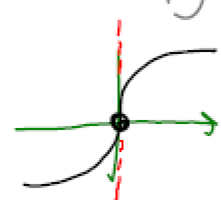
tangent line exists & NOT vertical!!!

$\sqrt{x}$ has vert tangent line but not diff.

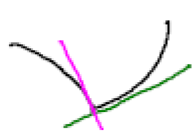
OBSERVATION:

Tangent line must exist at $(c, f(c))$ does NOT imply that the point is differentiable there!

if tangent line is vertical the function is not diff. but b) ✓ $f(x)=\sqrt[3]{x}$



the point (0,0) is an inflection point for $f(x)=\sqrt[3]{x}$ $\cup$ $\cap$



change conc. BUT NOT INFL. POINT!!!

THEOREM 5—Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval that contains $x = c$.

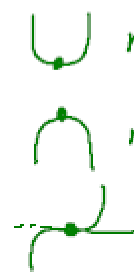
1. If $\boxed{f'(c) = 0}$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $\boxed{f'(c) = 0}$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $\boxed{f'(c) = 0}$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither. More work

⊖ . it can fail

$f = x^4$ $\quad f''(0) = 0$ min

$g = -x^4$ $\quad g''(0) = 0$ MAX

$h = x^5$ $\quad h''(0) = 0$ point with horiz. tang. not MAX, nor min.



(. only used for C.P where $f'(c) = 0$
→ cannot use this test at C.P. when f' DNE)

⊕ NO inequalities to solve, just PLUG in values

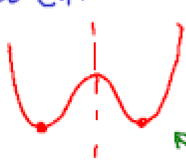
Find local min/max of $f(x) = x^4 - 2x^2 + 11$ using 2nd der test

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$ C.P. $x = 0$ $\quad x = \pm 1$

$f''(x) = 12x^2 - 4$

evaluate f'' at C.P. $\quad f''(0) = -4 < 0$ $\quad f''(1) = 8 > 0$ $\quad f''(-1) = 8 > 0$

MAX local $\quad$ min local $\quad$ min local



← to show that at $x = \pm 1$ abs. min need to do more work
at $x = 0$ NO ABS. MAX