Faculty of Mathematics
Part III Essays: 2021-22

Titles 1 – 66

Department of Pure Mathematics
& Mathematical Statistics

Titles 67 – 100

Department of Applied Mathematics
& Theoretical Physics

Titles 101 – 127

Additional Essays

Introductory Notes

Overview. As explained in the Part III Handbook, in place of a three-hour end-of-year examination paper you may submit an essay written during the year. The Part III Essay Booklet contains details of the approved essay titles, together with general guidelines and instructions for writing an essay. A timetable of relevant events and deadlines is included on page (iii).

In the past the great majority of Part III students have chosen to write an essay: the work is an enjoyable change and is valuable training for research.

Credit. The essay is equivalent to one three-hour examination paper and marks are credited accordingly. As noted in Appendix III of the Part III Handbook, the Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students their essay mark has been amongst their highest marks across all examination papers, both because of the typical amount of effort devoted to the essay and the different skill set tested (compared to a time-limited written examination). The Faculty Board wishes that hard work and talent thus exhibited should be properly rewarded.

Essay Titles. The titles of essays in this booklet have been approved by the Part III Examiners.[1] If you wish to write an essay on a topic not covered in this booklet you should approach your Part III Subject Adviser/Departmental Contact or another member of the academic staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board (email: undergrad-office@maths.cam.ac.uk) **not later than 1 February** requesting that an essay on that topic be approved.[2] The new essay title will require the approval of the Part III Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional essays approved by the Part III Examiners will be announced and added to this booklet not later than 1 March. All essay titles are open to all candidates. If you request an essay title you are under no obligation to write the corresponding essay. Essay titles **cannot** be approved informally: the only allowed essay titles are those which appear in the final version of this document (available on the Faculty web site).[3]

Interaction with the Essay Setter. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions. There is a range of practices across the Faculty for the other two meetings depending on the nature of the essay and whether, say, there is a need for further references and/or advice about technical questions. The setter may comment on an outline of the essay (for example in the second meeting), and may offer general feedback (for example, on mathematical style in general terms, or on whether clearer references to other sources are required) on a draft of the essay in the final meeting. The setter is not allowed to give students an expected grade for their essay.

Content of Essay and Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography. Candidates are reminded that mathematical content is more important than style.

Presentation of Essay. Your essay should be legible and may be either handwritten or produced on a word processor. There is no prescribed length for the essay in the University Ordinances, but the

[1] The titles are also published in the University's journal of record, i.e. the Cambridge University Reporter.
[2] Regulation 17 of the Regulations for the Mathematical Tripos.
[3] All additional titles will also be published in the Cambridge University Reporter.

Faculty Board Advice to the Part III Examiners suggests that 5,000-8,000 words is a normal length, and exceptionally long essays (i.e. more than twice this maximum) are discouraged. If you are in any doubt as to the length of your essay, please consult either the essay setter or your Part III Subject Adviser/Departmental Contact.

Academic Misconduct and Plagiarism. Before starting your essay you must read

- both the University's statement on the *Definition of Academic Misconduct* available at the URL https://www.plagiarism.admin.cam.ac.uk/definition,
- and the *Faculty Guidelines on Plagiarism and Academic Misconduct* available at the URL https://www.maths.cam.ac.uk/facultyboard/plagiarism/; the latter is reproduced starting on page (v) of this document.

The University takes a very serious view of academic misconduct in University examinations. The powers of the University Disciplinary Panels extend to the amendment of academic results or the temporary or permanent removal of academic awards, and the temporary or permanent exclusion from membership of the University. Fortunately, incidents of this kind are very rare.

Signed Declaration. The essay submission process includes signing the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff who set the essay. If the setter is unsure about your situation you should consult the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) as soon as possible.

***Viva Voce* Examination.** The Part III Examiners have power, at their discretion, to examine a candidate *viva voce* (i.e. to give an oral examination) on the subject of her or his essay, although this procedure is not often used.

Time Management. It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

Final Decision on Whether to Submit an Essay. You are not asked to state which essay (if any) and which written papers you have chosen for examination until the beginning of the Easter term. At that point, you will be sent the appropriate form to complete. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 5 May 2022**. **This deadline will be strictly adhered to.**

Essay Submission. You should submit your essay to the Chair of Part III Examiners (c/o Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second

Thursday in Easter Full Term, which this year is **Thursday 5 May 2022**. **This deadline will be strictly adhered to.**

- Together with your essay you should include a completed and signed *Essay Submission Form* as found on page (iv) of this document.
- The title page of your essay should bear **only** the essay title. Please **do not** include your name or any other personal details on the title page or anywhere else on your essay.
- At the time of writing it has not been decided if essay submission will be electronic or in hard copy. If the latter, then it is important that you ensure that the pages of your essay are fastened together in an appropriate way, e.g. by stapling or binding them. However, please do not bind or staple the *Essay Submission Form* to your essay, but instead attach it loosely, e.g. with a paperclip.

Extension of Submission Deadline. If an extension is likely to be needed due to **exceptional and unexpected developments**, a letter of application and explanation demonstrating the nature of such developments is required from the candidate's Director of Studies. This application should be sent to the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) by the submission date as detailed above.[4] It is expected that such an extension would be (at most) to the following Monday at 12 noon. A student who is dissatisfied with the decision of the Director of Taught Postgraduate Education can request within seven days of the decision, or by the submission date (extended or otherwise), whichever is earlier, that the Chair of the Faculty review the decision. The provision of any such extension will be reported to the Part III Examiners.

Return of Essays. It is not possible to return essays to candidates. You are therefore advised to make your own copy before handing in your essay.

Further Guidance. Advice on writing an essay is provided in two Wednesday afternoon talks listed below. Slides from these talks will subsequently be made available on the Part III Academic Support Moodle (see https://www.vle.cam.ac.uk/course/view.php?id=203401).

Feedback. If you have suggestions as to how these notes might be improved, please write to the *Chair of Part III Examiners* (c/o Undergraduate Office, Centre for Mathematical Sciences).

Timetable of Relevant Events and Deadlines

Wednesday 10 November 4:15pm	Talk *Planning your essay: reading, understanding, structuring.*
Wednesday 26 January 4:15pm	Talk *Writing your essay: from outline to final product.*
Tuesday 1 February	Deadline for Candidates to request additional essays.
Thursday 5 May, Noon	Deadline for Candidates to return form stating choice of papers and essays.
Thursday 5 May, Noon	Deadline for Candidates to submit essays.
Thursday 2 June	Part III Examinations expected to begin.

[4] Alternatively, the University's procedure can be invoked via the Examination Access and Mitigation Committee; see the *Guidance notes for dissertation and coursework extensions* linked from https://www.student-registry.admin.cam.ac.uk/about-us/EAMC.

MATHEMATICAL TRIPOS, PART III 2022
Essay Submission Form

To the Chair of Examiners for Part III of the Mathematical Tripos

I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: .... Date: ....

Title of Essay: ....

....

Essay Number: ....

Your Name: .... College: ....

Assessor comments:
Any essay comments we receive will be sent to your College immediately following the publication of results. Comments are not mandatory, and your assessor may not provide them. If you would prefer to receive your comments by email, please provide your preferred email address below:

....

Appendix 1: Faculty of Mathematics: Guidelines on Plagiarism and Academic Misconduct

For the latest version of these guidelines please see
https://www.maths.cam.ac.uk/internal/faculty/plagiarism

University Resources

The University publishes information on *Plagiarism and Academic Misconduct*, including

- The *University definition of academic misconduct*;
- Information for students, covering
    - *Students' responsibilities*
    - *Why does academic integrity matter?*
    - *Collusion*
- information about *Referencing* and *Study skills*;
- information on *Resources and support*;
- the *University's statement on proofreading*;
- *Plagiarism FAQs*.

There are references to the University statement

- in the Part IB and Part II Computational Project Manuals, and
- in the Part III Essay Booklet (linked from the Part III Essays page),
- in the Computational Biology Handbook (linked from the Computational Biology Course page).

Please read the University statement carefully; it is your responsibility to read and abide by this statement.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to *Turnitin Plagiarism Detection Software)*.

The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person's language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition, you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline to avoid plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.
In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting

A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source;
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing

Paraphrasing means putting someone else's work into your own words. Para- phrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ..."). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness

When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources

You should use web sources as if you were using a book or journal article.
The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration

Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Links to University Information

- Information on *Plagiarism and Academic Misconduct*, including
    - *Students' responsibilities*;
    - *Information for staff*.

Appendix 2: Essay Descriptors for Part III of the Mathematical Tripos

The Part III Committee believes that the essay is a key component of Part III. It also believes that it is entirely reasonable and possible that candidates may obtain higher marks for essays than in their examination, both because of the typical amount of effort devoted to the essay, and also the different skill set which is tested compared to a time-limited written examination. In light of these beliefs, as well as the comments of both the internal examiners and the external examiners, the Part III Committee believes that it is appropriate to suggest the following descriptors for the various possible broad grade ranges for an essay. The committee trysts that these guidelines prove useful in guiding the judgement of the inevitably large numbers of assessors marking essays, and that these guidelines strengthen the mechanisms by which all essays are assessed uniformly. They are not meant to be either prescriptive or comprehensive, but rather general guidance consistent with long-standing practice within the faculty.

An Essay of Distinction Standard

Typical characteristics expected of a distinction standard essay include:

- Demonstration of a clear mastery of all the underlying mathematical content of the essay.
- Demonstration of a deep understanding and synthesis of advanced mathematical concepts.
- A well-structured and well-written essay of appropriate length (5000-8000 words) with:
    - very few grammatical or presentational issues;
    - a clear introduction demonstrating an appreciation of the context of the central topic of the essay;
    - a coherent presentation of that central topic;
    - a final section which draws the entire essay to a clear and comprehensible end, summarizing well the key points while suggesting future work.

An essay of distinction standard would be consistent with the quality expected of an introductory chapter of a PhD thesis from a leading mathematics department. A more elegant presentation and synthesis than that presented in the underlying papers, perhaps in the form of a shorter or more efficient proof of some mathematical result would be one possible characteristic of an essay of distinction standard. Furthermore, it would be expected that an essay containing publishable results would be of a high distinction standard, but, for the avoidance of doubt, publishable results are **not necessary** for an essay to be of high distinction standard. An exceptionally high mark (α+) should be justified by a specific extra statement from the assessor highlighting precisely which section of the essay was of particularly distinguished quality.

An Essay of Merit Standard

Typical characteristics expected of a merit standard essay include:

- Demonstration of a good mastery of most of the underlying mathematical content of the essay.
- Demonstration of understanding and synthesis of mathematical concepts typical of the content of a Part III course.
- A largely well-structured essay of appropriate length (5000-8000 words) with:
    - some, but essentially minor, grammatical or presentational issues;
    - an introduction demonstrating an appreciation of a least some context of the central topic of the essay;
    - a reasonable presentation of that central topic;
    - a final section which draws the entire essay to a comprehensible end, summarizing the key points.

An essay of merit standard would be consistent with the quality expected of a first class standard final year project from a leading mathematics department. Such essays would not typically exhibit extensive reading beyond the suggested material in the essay description, or original content.

An Essay of Pass Standard

Typical characteristics expected of a pass standard essay include:

- Demonstration of understanding of some of the underlying mathematical content of the essay.
- An essay exhibiting some non-trivial flaws in presentation through, for example:
    - an inappropriate length;
    - repetition or lack of clarity;
    - lack of a coherent structure;
    - the absence of either an introduction or conclusion.

An essay of pass standard would be consistent with the quality expected of an upper second class standard final year project from a leading mathematics department. For the avoidance of doubt, an excessively long essay (i.e. of the order of twice the suggested maximum length or more) would be likely to be of (at best) pass standard. A key aspect of the essay is that the important mathematical content is presented clearly in (at least close to) the suggested length.

Contents

1. International Comparisons of Deaths .... Professor J.A.D. Aston

There has been considerable interest in comparing the performance of different countries during the COVID-19 pandemic (one among many examples is [1]). However, there are many issues with simply counting deaths or even deaths per capita. A proper statistical analysis needs to take into account what is actually being compared across countries and how different methods might yield different results. Some countries such as the UK's Office for National Statistics [2] publish considerable amounts of information around deaths while others publish less comprehensively.

There are a number of possibilities within this essay. Given the temporal nature of the data, it will be possible to develop models to look at how death numbers developed for different countries over the pandemic, or a critical review could be undertaken of methods used by National Statistical Institutes to determine relevant deaths. Alternative methods to examine the levels of death from COVID, including those going forward, could also be proposed and discussed.

Relevant Courses

Useful: Modern Statistical Methods

References

[1] Excess deaths associated with COVID-19 pandemic in 2020: age and sex disaggregated time series analysis in 29 high income countries British Medical Journal 2021; 373: n1137

[2] ONS COVID-19 Insights: Deaths (2021) `https://tinyurl.com/ONSCovidDeaths`

2. The Statistics of Manifold Data .... Professor J.A.D. Aston

Many observed data are constrained by their intrinsic features or geometry. This is especially true when the objects under analysis are shapes or images, as in many cases the angle of view and the magnification of the shape is unimportant [1]. This is also linked to the study of data which arises in particularly spaces, such as data observed on a (hyper-)sphere [2,3], or where the observations are types of matrices, such as those which are positive definite [4]. All these settings yield data that are inherently non-Euclidean, but most statistical analysis is predicated on the data coming from a Euclidean space.

The idea of this essay will be to review some of the recent advances in shape and related statistics, many of which are based on concepts from differential geometry and metric spaces [5]. There is then considerable scope in the essay. Theoretical investigation could be undertaken into some of the underlying metrics that are used in statistical shape analysis. Methodology for certain special cases of shape data could be compared, such as different methods for the statistical analysis of samples of positive definite matrices. Alternatively, data analysis could be undertaken for shape observations derived from images, for example.

Relevant Courses

Useful: Differential Geometry, Modern Statistical Methods, Topics in Statistical Theory, Functional Data Analysis

References

[1] IL Dryden and KV Mardia (1998). *Statistical Shape Analysis.* Wiley

[2] S Jung, IL Dryden and JS Marron. (2012) Analysis of principal nested spheres, *Biometrika* **99**:551–568.

[3] KV Mardia and P Jupp (2000). *Directional Statistics.* Wiley.

[4] D Pigoli, JAD Aston, IL Dryden and P Secchi. (2014) Distances and Inference for Covariance Functions, *Biometrika* **101**:409–422.

[5] P Dubey and H Müller (2020). Functional models for time-varying random objects (with discussion), *Journal of the Royal Statistical Society* **82**: 275–327.

3. Operads and long knots .... Dr R. Boyd

A knot is a smooth embedding of S^1 in S^3, and knot theory is often concerned with knots up to isotopy. In other words, it is the study of the path components of the space $\mathrm{Emb}(S^1, S^3)$ of all such embeddings. But each of these path components has interesting topology in its own right, and a complete answer for the homotopy type of each component follows from work of Hatcher [4], and a series of papers by Budney [1] [2] [3].

Central to this answer is the associated space of *long knots*—embeddings of $\mathbb{R}^1$ in $\mathbb{R}^3$ which are 'standard' outside the unit cube. Budney shows this space is a free algebra over the 'splicing operad', with generating space given by hyperbolic and torus knots.

In this essay you should read and understand the construction of the splicing operad in the setting of long knots, and the associated description of the homotopy type of each component in terms of torus and hyperbolic knots. This will involve learning background on operads, embedding spaces, and some knot theory. After this you could go on to explore low-dimensional results such as the proofs of the homotopy type for the torus and hyperbolic components, or study splicing operads for higher dimensional long knots i.e. the space $\mathrm{Emb}(\mathbb{R}^j, M)$ for some $j \geq 1$ and manifold M.

Relevant Courses

Essential: Part III Algebraic Topology

Useful: Part III Knots

References

[1] R. Budney *Little cubes and long knots*, Topology 46 (2007), no. 1, 1–27.

[2] R. Budney, *Topology of knot spaces in dimension 3*, Proc. Lond. Math. Soc. (3) 101 (2010), no. 2, 477–496.

[3] R. Budney, *An operad for splicing*, J. Topol. 5 (2012), no. 4, 945–976.

[4] A. Hatcher, *Topological Moduli Spaces of Knots*, available at https://pi.math.cornell.edu/~hatcher/Papers/knotspaces.pdf.

4. Deformation theory .... Dr C.J.B. Brookes

Algebraic deformation theory is primarily concerned with the interplay between homological algebra and the perturbations of algebraic structures. For example one might want to deform a commutative algebra to give a non-commutative one via 'quantisation'.

In a series of papers [2], [3], [4] and [5] Gerstenhaber developed deformation theory for associative algebras. The cohomology theory required in this context is Hochschild cohomology and one finds that 2-cocycles arise from deformations. A more modern approach is to consider additional algebraic structures, including that of a Lie algebra defined on the Hochschild cocomplex of the associative algebra. Deformations correspond to Maurer-Cartan elements. This approach has arisen in the work of Kontsevich on the deformation quantisation of Poisson manifolds.

The introductory article by Fox [1] is a good place to start. It first describes the classical theory of deformations of associative algebras and then moves on to more general algebraic structures. The papers of Gerstenhaber are also very readable.

Relevant Courses

Useful: Commutative Algebra

References

T. Fox, An introduction to algebraic deformation theory, J Pure and Applied Algebra, 84 (1993), 17-41.

M.Gerstenhaber, On the deformation theory of rings and algebras, Ann. Math. 78 (1963) 267-288.

M.Gerstenhaber, On the deformation theory of rings and algebras II, Ann. Math. 84 (1966) 1-19.

M.Gerstenhaber, On the deformation theory of rings and algebras III, Ann. Math. 88 (1968) 1-34.

M.Gerstenhaber, On the deformation theory of rings and algebras IV, Ann. Math. 99 (1974) 257-256.

5. Bieri-Strebel invariants and nonarchimedean amoebas .... Dr C.J.B. Brookes

In 1980 Bieri and Strebel [3] classified metabelian groups for which there is a presentation in terms of finitely many generators and finitely many relations. (A metabelian group G is one in which there is an abelian normal subgroup K where G/K is also abelian.) For a finitely generated group the space of homomorphisms of G to the additive group of the reals forms a finite dimensional real vector space V and the method was to associate with G a subset of V. This geometric set is related to the logarithmic limit set defined previously by Bergman [1] for ideals of group algebras of free abelian groups (or Laurent polynomial rings); via conjugation K

may be regarded as a module over the integral group algebra of G/K. In [2] Bieri and Groves showed that these geometric sets are all polyhedral. There were extensions of the theory to general groups where the points of the set represent certain tree actions of the group [3] and [6].

Such sets also arose as the nonarchimedean amoebae of Gelfand, Kapranov and Zevelinsky (see [8] and [10] for further developments), and are related to compactifications of subvarieties of tori (see [11]). An overview can be found in ine first chapter of the book by Maclagan and Sturmfels [9].

The essayist could concentrate on the group-theoretic aspects - see for example the essay by Strebel in [7], or consider primarily the amoebic approach.

Relevant Courses

Essential: Commutative Algebra

References

GM Bergman, The logarithmic limit set of an algebraic variety, Trans. Amer. Math. Soc 157 (1971),459 - 469.

R Bieri and JRJ Groves, The geometry of the set of characters induced by val uations, J. Reine Angew. Math. 347 (1984) , 168 - 195.

R Bieri, W Neumann and R Strebel, A geometric invariant of discrete groups, Invent math. 90 (1987), 451 - 477.

R Bieri and R Strebel, Valuations and finitely presented metabelian groups, Proc London Math Soc 41 (1980), 439 - 464.

R Bieri and R Strebel A geometric invariant of modules over an abelian group, J Reine Angew. Math. 322 (1981), 170 - 189.

KS Brown, Trees, valuations and the Bieri- Neumann -Strebel invariant, Invent. Math. 90 (1987), 479 - 504.

Group theory, essays for Philip Hall, ed KW Gruenberg and JE Roseblade, Acad emic Press 1984.

M Einsiedler, M Kapranov, D Lind, Non-archimedean amoebas and tropical varieties, J. Reine Angew. Math. 601 (2006) 139 - 157.

D.Maclagan, B. Sturmfels, Introduction to tropical geometry, Graduate studies in math. 161, American Math. Soc. 2015

S.Payne, Adelic amoebas disjoint from open halfspaces, J. Reine Angew. Math. 625 (2008) 115 - 123.

J. Tevelev, Compactifications of subvarieties of tori, American J. Math. 129 (2007), 1087 - 1104.

6. Kähler-Einstein metrics via quantisation .... R. Dervan

One of the most celebrated results in geometry over the last decade is the characterisation of the existence of Kähler-Einstein metrics on Fano varieties through K-stability; this is known as the Yau-Tian-Donaldson conjecture and was proven by Chen-Donaldson-Sun in 2012. Fano varieties are a special type of complex projective variety, on which the Kähler-Einstein condition can be viewed as giving a canonical choice of metric, while the K-stability condition is purely algebro-geometric. Thus the conjecture links solvability of a certain PDE on a complex manifold to an algebraic condition.

The aim of this essay is not to review the proof of Chen-Donaldson-Sun, but rather to understand a newer, strikingly short proof due to K. Zhang [3]. Zhang uses "quantisation", which is loosely the finite dimensional approximation of infinite dimensional objects. Understanding of Zhang's work will require learning quite a bit of background material; Székelyhidi's book [2] is one excellent reference. Context and helpful related work can also be found in [1].

Relevant Courses

Essential: Complex Manifolds, Algebraic Geometry.

References

[1] Y.A. Rubinstein, G. Tian, K. Zhang, Basis divisors and balanced metrics, arXiv:2008.08829, J. Reine Angew. Math. 778 (2021), 171–218.

[2] G. Székelyhidi *An introduction to extremal Kähler metrics.* Graduate Studies in Mathematics, 152. American Mathematical Society, Providence, RI, 2014. xvi+192, a shorter version also available at `https://www3.nd.edu/~gszekely/notes.pdf`

[3] K. Zhang, *A quantization proof of the uniform Yau-Tian-Donaldson conjecture.* Available at `arXiv:2102.02438`.

7. Counting cubic number fields .... Professor T. A. Fisher

Let $N_d(X)$ be the number of degree d number fields whose discriminant has absolute value at most X. It is conjectured that $N_d(X) \sim c_d X$ for some constant c_d as $X \to \infty$. This was proved in the case $d = 3$ by Davenport and Heilbronn [4], and extended to the cases $d = 4$ and $d = 5$ by Bhargava.

This essay should concentrate on explaining the proof for $d = 3$, for which the references [3] and [5] may also be helpful. If time and space permit, the essay might also discuss extensions to larger d, second order terms, or computational aspects (for the latter see [1] and [2]).

Relevant Courses

Useful: Local Fields

References

[1] K. Belabas, A fast algorithm to compute cubic fields, *Math. Comp.* 66 (1997), no. 219, 1213-1237.

[2] K. Belabas, On quadratic fields with large 3-rank, *Math. Comp.* 73 (2004), no. 248, 2061-2074.

[3] M. Bhargava, A. Shankar and J. Tsimerman, On the Davenport-Heilbronn theorems and second order terms, *Invent. Math.* 193 (2013), no. 2, 439-499.

[4] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields. II, *Proc. Roy. Soc. London Ser. A* 322 (1971), no. 1551, 405-420.

[5] M.M. Wood, Asymptotics for number fields and class groups, *Directions in number theory*, 291-339, Assoc. Women Math. Ser., 3, Springer, 2016.

8. Classical invariant theory and moduli of genus 2 curves .... Professor T. A. Fisher

This essay should begin by reviewing the classical (i.e. 19th century) invariant theory of binary forms of degree n (see for example [4] or [7]) with particular reference to the cases $n = 4$ and $n = 6$. The case $n = 4$ is related to elliptic curves (see [2] or [8]), which are classified up to isomorphism (over an algebraically closed field) by their j-invariant. The case $n = 6$ leads to the definition of the Igusa (or Igusa-Clebsch) invariants that likewise classify genus 2 curves. The main aim of the essay should be to describe the algorithm of Mestre [6] for recovering the equation for a genus 2 curve from its Igusa invariants. If time and space permit, the connection to Siegel modular forms, or extensions such as those in [1] or [3], could also be considered.

Relevant Courses

Useful: Elliptic Curves, Algebraic Geometry

References

[1] G. Cardona and J. Quer, Field of moduli and field of definition for curves of genus 2, in *Computational aspects of algebraic curves*, World Scientific Publishing Co., 2005.

[2] J.E. Cremona, Classical invariants and 2-descent on elliptic curves, *J. Symbolic Comput.* 31 (2001), no. 1-2, 71–87.

[3] J. Gutierrez and T. Shaska, Hyperelliptic curves with extra involutions, *LMS J. Comput. Math. 8* (2005), 102–115.

[4] D. Hilbert, *Theory of algebraic invariants*, CUP, 1993.

[5] J. Igusa, Arithmetic variety of moduli for genus two, *Ann. of Math.* (2) 72, (1960), 612–649.

[6] J.-F. Mestre, Construction de courbes de genre 2 à partir de leurs modules, in *Effective methods in algebraic geometry*, Birkhäuser, 1991.

[7] P.J. Olver, *Classical invariant theory*, CUP, 1999.

[8] A. Weil, Remarques sur un mémoire d'Hermite, *Arch. Math.* (Basel) 5, (1954), 197–202.

9. Countable Ordinals, Proof theory and Fast-growing Functions .... Thomas Forster

A good point of departure for this essay would be Exercise 10 on Sheet 4 of Prof Johnstone's Part II set Theory and Logic course in 2012/3, `https://www.dpmms.cam.ac.uk/study/II/Logic/2012-2013/LSqns4.pdf` in which the student is invited to show how, for every countable ordinal, a subset of $\mathbb{R}$ can be found that is wellordered to that length in the inherited order. The obvious way to do this involves induction on countable ordinals and leads swiftly to the discovery of fundamental sequences. These can be put to work immediately in the definition of hierarchies of fast-growing functions. This leads in turn to the Schmidt conditions, which the student should explain carefully. There is a wealth of material on how proofs of totality for the faster-growing functions in this hierarchy have significant—indeed *calibratable*—consistency strength. One thinks of Goodstein's function and Con(PA), or of Paris-Harrington. There is plenty here from which the student can choose what to cover.

The Doner-Tarski hierarchy of functions (addition, multiplication, exponentiation ...) invites a transfinite generalisation and supports a generalisation of Cantor Normal Form for ordinals. Nevertheless, the endeavour to notate ordinals beyond ϵ_0 does not use those ideas, but rather the enumeration of fixed points: such is the *Veblen hierarchy*. From this one is led to the impredicative Bachmann notation, with Ω and the ϑ function.

Relevant Courses

Essential: Part II Set Theory and Logic

References

[1] Schwichtenberg and Wainer, *Proofs and Computations*, Cambridge University Press, (http://www.cambridge.org/us/academic/subjects/mathematics/logic-categories-and-sets/proofs-and-computations)

[2] Doner and Tarski, *An extended arithmetic of ordinal numbers*, Fundamenta Mathematicae 1969, (http://www.math.ucsb.edu/~doner/articles/Doner-Tarski.pdf)

[3] Forster, *A Tutorial on (mainly countable) Ordinals*, (http://www.dpmms.cam.ac.uk/~tf/ordinalsforwelly.pdf)

10. Wellquasiorders and Betterquasiorders .... Dr. T. E. Forster

A well-quasi-order is a reflexive transitive relation with no infinite descending chains and no infinite antichains. Although this may not sound natural there are many natural examples, at least one of which is famous: the theorem of Seymour and Robertson that finite graphs under the graph minor relation form a WQO. There is also Laver's theorem that the isomorphism types of scattered total orders (orders in which the rationals cannot be embedded) form a WQO. Finite trees with nodes labelled with elements of a WQO are also WQO-ed. The class of WQO's lacks certain nice closure properties and this leads to a concept of *Better*-quasi-ordering. The class of BQOs is algebraically nicer.

These combinatorial ideas have wide ramifications in graph theory, logic and computer science (lack of infinite descending chains is always liable to be connected with termination of processes) and the area has a good compact literature and some meaty theorems. Recommended for those of you who liked the Logic course and the Combinatorics course.

A Big plus for this topic is that there is no textbook! I have a project to write one, and the obvious place to start is with my draught (itemised below) which has coverage of most aspects and an extensive bibliography, much of which I have photocopies of. Interested students should discuss this with me.

Relevant Courses

Essential: None

Useful: Combinatorics, Logic and Set Theory

References

[1] Thomas Forster, Nathan Bowler and Monika Seisenberger, *An Introduction to WQO and BQO Theory*, (https:www.dpmms.cam.ac.uk/~tf/BQObok.pdf)

11. D-modules, Hodge theory, Representation theory .... I. Grojnowski

The aim of this essay is to understand the basics of the theory of D-modules, and some concrete applications to either representation theory or the topology of algebraic varieties.

Begin by learning the basic properties of holonomic D-modules on algebraic varieties—Bernstein's lemma on b-functions, and its consequences (the formalism of the six operations). This is the basic language of algebraic geometry and modern representation theory, and has applications throughout mathematics and physics.

Then, either

1) prove the Beilinson-Bernstein theorem, describing the category of representations of a semisimple Lie algebra in terms of D-modules on the flag variety. This is an extraordinary result, which generalizes the Borel-Weil-Bott theorem (describing finite dimensional representations in terms of line bundles on the flag variety).

or

2) Study the mixed Hodge structure on D-modules, beginning by computing the Kashiwara-Malgrange filtration on vanishing cycles in some interesting cases. (If you do this, you'll have to learn about Hodge theory, too!).

References

Many textbook expositions of D-modules now exist. The two best are by the originators of the subject—Kashiwara and Bernstein (the latter are printed notes, available on the web somewhere).

The original Beilinson-Bernstein paper is 3 pages long, it is

A. Beilinson, J Bernstein, Localisation de $\mathfrak{g}$-modules, C. R. Acad. Sci. Paris. 292 (1981), no. 1, 15–18.

but there are many expositions which are probably easier to read; for background on Hodge theory there are Deligne's extraordinary papers:

P. Deligne, Theorie des Hodges II, III. Inst. Hautes Etudes Sci. Publ. Math. No. 40 (1971), 5–57; Inst. Hautes Etudes Sci. Publ. Math. No. 44 (1974), 5–77.

P. Deligne, Travaux de Griffiths, Seminar Bourbaki 376, Lecture Notes in Math 180, Springer Verlag 1970, 213–237

12. The geometry of Bun_G and Verlinde's formula .... I. Grojnowski

This is an essay about the geometry and representation theory of the stack of G bundles on an algebraic curve, where G is a reductive group. When $G = GL_n$, this is the moduli of vector bundles on the curve.

This is one of the central objects in the geometric Langlands program; it is a geometric avatar of one of the central objects of number theory, and it is also fundamental for modern physics.

How you should study this depends on how much algebraic geometry you know.

If your only exposure to algebraic geometry is from the Part III course, this essay is a great excuse to learn about concrete and subtle examples of algebraic varieties, and some of the advanced technology (GIT, moduli, and stacks) that modern geometers use.

In that case, you should take Mukai's book as your guide; you want to understand chapters 10 and 11, and some of the references quoted there. (This probably means you should read the entire book. It is a lovely book!)

If you already know a lot of algebraic geometry, the topics that an ambitious essay would cover:

The Drinfeld-Simpson theorem. Define the stack Bun_G, and its coarse quotients. Uniformise Bun_G analytically. Define the Harder-Narismhan stratification.

Explicitly describe the stack of G-bundles on $\mathbb{P}^1$ and on an elliptic curve E, and using this describe the cohomology of line bundles on the stack of bundles over these curves.

Degenerate curves to nodal rational curves, and state and prove the Verlinde formula.

Compute $Pic(Bun_G)$, and Pic of the coarse moduli spaces.

In any case, if you are interested in the essay we can discuss the exact details of what you will learn.

References for Bun_G.

0. Mukai, S. An Introduction to Invariants and Moduli, Cambridge University Press, 2012.

1. Drinfeld, V. G., Simpson, C. B-structures on G-bundles and local triviality. Math. Res. Lett. 2, 823–829 (1995).

2. Heinloth, J. Uniformization of $\mathcal{G}$-bundles. arXiv:0711.4450

3. Heinloth, J. Hilbert-Mumford stability on algebraic stacks and applications to $\mathcal{G}$-bundles on curves. arXiv:1609.06058

4. Faltings, G. Vector Bundles on curves. Bonn lecture notes, 1995. (Available on the internet)

5. Zhu, X. An introduction to affine Grassmannians and the geometric Satake equivalence. arXiv:1603.05593

G-Bundles on curves, Pseudo traces, GIT : Geometric Langlands after Drinfeld and Lafforgue

13. Computational Algebraic Geometry ....
Prof M Gross

The theory of Gröbner bases, which may be viewed as a vast generalization of the standard long division algorithm, is a powerful tool which has made many concepts in commutative algebra and algebraic geometry amenable to computation. *Macaulay 2* and SINGULAR are now widely used packages which allow almost any concept taught in a standard algebraic geometry course to be calculated in specific examples.

At its most basic, Gröbner bases give a solution to the ideal membership question: given an ideal $I \subseteq k[x_1, \ldots, x_n] = S$ and an element $f \in S$, is $f \in I$? However the theory allows solutions to a wide range of questions, such as computing the kernels of module homomorphisms, computing intersections of ideals, computing the Hilbert polynomials of modules, calculating Ext and Tor groups, etc.

I envisage this essay should involve two parts. First, I would like you to absorb and exposit the theory of Gröbner bases. Chapter 15 of [1] provides an excellent introduction, while there are other books more exclusively focused on Gröbner bases, such as [2] and [3].

Second, I would like you to familiarise yourself with one of the computational packages; personally I am used to *Macaulay 2*. This is free for download. You should then set yourself an interesting computation, understand how to carry it out, and be sure to explain all the theory behind the particular calculations you have chosen. Possible computations include but are not limited to construction of some interesting varieties in projective space, calculating cohomology groups, etc. Reference [4] should give an excellent introduction as to what is possible.

Relevant Courses

Essential: Part III Commutative Algebra and Algebraic Geometry.

References

[1] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Spring-Verlag, 1995.

[2] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms: An introduciton to computational algebraic geometry and commutative algebra*, Springer-Verlag, 4th edition 2016.

[3] T. Becker and V. Weispfennig, *Gröbner bases: A computational approach to commutative algebra*, Springer-Verlag, GTM 141.

[4] D. Eisenbud, D. Grayson, M. Stillman, B. Sturmfels (Ed.), *Computations in Algebraic Geometry with Macaulay 2*, Springer-Verlag.

14. Mirror Symmetry ....
Prof M Gross

Mirror symmetry is a geometric phenomenon first noticed by string theorists circa 1990. In string theory, one expects a 10-dimensional universe, and hence one would posit a space-time geometry $\mathbf{R}^4 \times X$ where X is a a very small compact six-dimensional manifold. Various physical considerations lead to X often in fact being a non-singular three-dimensional variety over the complex numbers of a special sort, known as a *Calabi-Yau manifold.* Initial evidence suggested that such manifolds came in pairs $X, \check{X}$, with a relationship on Hodge numbers given by $h^{p,q}(X) = h^{3-p,q}(\check{X})$. Calculations by Candelas, de la Ossa, Green and Parkes [1] then suggested that certain invariants of X, namely counts of holomorphic curves in X, could be calculated via a very different procedure on $\check{X}$, namely so-called period integrals. [1] carried this calculation out for X the quintic threefold in projective four-space, and obtained predictions for the number of rational curves in X of every degree. For algebraic geometers, this was a surprising result, and has led to what is now a huge field.

There is currently a vast literature on mirror symmetry; I include below several references providing an entrance into this literature. Reference [2] gives a good overall exposition of the state of the art in the 1990s, and my chapter in [3] covers in great detail the original calculations of [1]. Reference [4] gives a much narrower but more modern point of view leading to current developments in the field.

A successful essay should not attempt to communicate all aspects of mirror symmetry, and should stay narrowly focused on a few well-chosen topics. In addition, this essay *must* involve a mathematical rather than a string-theoretic discussion of the field, and no physics background is required for this essay. An incomplete list of possible directions would include: (1) The Candelas et al mirror symmetry calculation for genus 0 invariants. (2) An investigation of the Batyrev construction of mirror pairs. (3) An investigation of Gromov-Witten invariants and the proof of genus 0 mirror symmetry for the quintic. (4) An investigation of Homological Mirror Symmetry. (5) An investigation of approaches using tropical geometry.

(1) is covered in references [1] and [3], and (2),(3) are covered in [2], although there are more modern approaches to (3), e.g., due to Gathmann [5]. For (4), there are many possible references, but this would require a very solid grounding in symplectic geometry. [4] covers much of the material of (5).

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Toric Varietes.

Optional: Part III Complex Manifolds, Part III Symplectic Geometry.

References

[1] P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds,* Phys. Lett. B 258 (1991), no. 1-2, 118–126.

[2] David Cox, Sheldon Katz, *Mirror symmetry and algebraic geometry.* Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999. xxii+469 pp.

[3] Mark Gross, Daniel Huybrechts, Dominic Joyce, *Calabi-Yau manifolds and related geometries.* Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003.

[4] Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, **114**. American Mathematical Society, Providence, RI, 2011. xvi+317 pp

[5] Andreas Gathmann, *Relative Gromov-Witten invariants and the mirror formula,* Math. Ann. 325 (2003), no. 2, 393–412.

15. The C^0 Arnol'd conjecture .... Dr. Ailsa Keating

An important theorem of Eliashberg and Gromov is that given a symplectic manifold M, the group of symplectomorphisms $\mathrm{Symp}(M)$ is C^0 closed in the group of diffeomorphisms $\mathrm{Diff}(M)$ [1,2]. This is the start of C^0 symplectic topology: loosely, the study of the C^0 closure of $\mathrm{Symp}(M)$ in $\mathrm{Homeo}(M)$. A natural question is to understand which of the phenonema in the smooth setting carry over to the topological one. In particular, Hamiltonian diffeomorphisms are expected to have at least as many fixed points as the sum of the Betti numbers of M; this is called the Arnol'd Conjecture, and known for large classes of M. In contrast, a striking result of Buhovsky, Humiliere and Seyfaddini is that every closed and connected symplectic manifold of dimension at least 4 admits a Hamiltonian homeomorphism with a single fixed point [4].

The essay should open with a few pages of introduction to C^0 symplectic topology; starting with the Gromov non-squeezing theorem, it should give a proof of the theorem of Eliashberg and Gromov, for instance by fleshing out the version in [3]. It should then give an account of the construction in [4]. This should include a well-organised overview of the proof, and of how its components fits together; and fleshed out details, in the candidate's own words, of proofs of component(s) of their choosing, possibly including one or two running examples. Candidates are encouraged to discuss this with the essay setter. (Note that while long, [4] largely does not require background knowledge beyond the part III courses listed.)

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Symplectic Geometry

Useful: Topic in Symplectic Topology

References

[1] Yakov Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology*, Funktsional. Anal. i Prilozhen. 21, no. 3, p. 65–72, (1987).

[2] Mihkail Gromov,*Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 9 Springer-Verlag (1986).

[3] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology.* Third edition. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017.

[4] Lev Buhovsky, Vincent Humilière, Sobhan Seyfaddini, *A C^0 counterexample to the Arnold conjecture.* Invent. Math. 213 (2018), no. 2, 759–809.

16. Cyclic quatrilaterals and symplectic topology .... Dr. Ailsa Keating

Let Q be a quadrilateral in the Euclidean plane. We say that Q inscribes in a smooth Jordan curve if there is an orientation-preserving similarity of the plane taking the vertices of Q to points of the curve. Greene and Lobb [1] recently proved that if Q inscribes in a circle, then it also inscribes in any other smooth Jordan curve. Their proof relies on a thirty year-old theorem in symplectic topology, due independently to Polterovich and Viterbo [2,3], which constrains a topological invariant of embedded Lagrangian tori in $\mathbb{C}^2$: their 'minimum Maslov number' must be equal to 2. This theorem was an early application of Gromov's theory of holomorphic curves.

The essay should give an exposition of Greene–Lobb's article, and of the statement and proof of the underlying theorem by Polterovich and Viterbo. Background results from holomorphic curve theory can be treated as 'black boxes' as required.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry

Useful: Symplectic Geometry, Topic in Symplectic Topology

References

[1] Joshua Greene and Andrew Lobb, *Cyclic quadrilaterals and smooth Jordan curves*, arXiv:2011.05216

[2] Claude Viterbo, *A new obstruction to embedding Lagrangian tori*, Invent. Math. 100 (1990), no. 2, 301–320.

[3] Leonid Polterovich, *The Maslov Class of the Lagrange Surfaces and Gromov's Pseudo-Holomorphic Curves*, Transactions of the American Mathematical Society, Vol. 325, No. 1 (May, 1991), pp. 241-248

Background reading:

[4] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology.* Third edition. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017.

[5] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology.* American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.

17. Entropy and Information in Ergodic Theory .... Professor I. Kontoyiannis

Roughly and somewhat incorrectly speaking, for the purposes of this essay an ergodic system is the most general form of a stochastic process that satisfies the strong law of large numbers. Ergodic theory has its origins in the study of abstract dynamical systems. It is mostly analytical in flavor and it also shares many tools with probability theory and information theory. The *entropy* has played a central role in the development of ergodic theory, and it has provided deep and strong connections with probability and information theory.

This essay will explore and describe the foundations of the ergodic theory of discrete sample paths as outlined in the first four sections of [1], and the connections with entropy and information theory in the rest of Chapter 1 and in Chapter 2 of [1]. The material is sections I.9 and I.10 is interesting but not essential. More ambitious essays can consider discussing material from Chapter 5 of [2], Section 14 of [3], or the more recent work [4].

Relevant Courses

Essential: Probability and measure, Linear analysis

Useful: Analysis of functions, Coding and cryptography

References

[1] P.C. Shields, *The ergodic theory of discrete sample paths.* Vol. 13. American Mathematical Soc., 1996.

[2] D.J. Rudolph, *Fundamentals of measurable dynamics: Ergodic theory on Lebesgue spaces.* Oxford, UK: Clarendon Press, 1990.

[3] P. Billingsley, *Ergodic theory and information.* John Wiley & Sons, New York, 1965.

[4] E. Lindenstrauss and M. Tsukamoto. "From rate distortion theory to metric mean dimension: variational principle." *IEEE Transactions on Information Theory*, **64**, no. 5, pp. 3590-3609, 2018.

18. Yau's solution of the Calabi conjecture ....

Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be equivalently expressed as a differential form which is necessarily closed. Furthermore, the cohomology class defined by this form depends only on the complex manifold, but not on the choice of Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential geometry, Complex manifolds

Useful: Algebraic topology, Elliptic Partial Differential Equations

References

[1] D. Joyce, *Riemannian holonomy groups and calibrated geometry*, OUP 2007. Chapters 6 and 7.

[2] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, **31** (1978), 339–411.

[3] a good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

19. Stable differential forms ....

Dr A.G. Kovalev

The term 'stable form' was suggested by Hitchin for an alternating multilinear form $\omega \in \Lambda^p(\mathbf{R}^n)^*$ which has an open orbit in the natural action of $GL(n, \mathbb{R})$. Stable forms generalize the notion of non-degenerate bilinear forms to alternating forms of an arbitrary degree p and non-trivial examples include stable forms of degree $p = 3$ in dimensions $n = 6$ and 7. The essay could begin by exploring the work [1] and showing how stable differential 3-forms lead to Ricci-flat metrics on 6- and 7-dimensional manifolds, as critical points of a volume function. In dimension 6 this gives an alternative view on Calabi–Yau manifolds (of complex dimension 3), and in dimension 7 one recovers a geometry associated with the exceptional Lie group G_2. The essay could investigate the significance of stable forms in one of these two geometries and its relation to Ricci-flat metrics with reduced holonomy [2]. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential Geometry, Complex manifolds

Useful: Algebraic Topology

References

[1] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, arXiv:math.DG/0010054 or J. Differential Geom. **55** (2000), 547–576.

[2] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press 2000.

[3] a good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

20. Infinite Games in Set Theory ....

Professor B. Löwe

One of the most surprising developments in Foundations of Mathematics was that the theory of infinite games was revealed to be intimately and inextricably linked to the deep foundational questions of axiomatic set theory.

Infinite games had been used as a mathematical tool in topology and set theory since the 1920s. In the 1960s, Mycielski and Steinhaus had started the axiomatic investigation of determinacy axioms [5]; this was originally considered a niche and esoteric subfield of set theory since their main axiom, the *Axiom of Determinacy* AD contradicted the Axiom of Choice. Finally, in the 1980s, the famous Martin-Steel theorem established a very close connection between the large cardinal hierarchy and the hierarchy of axioms of determinacy [4].

The theory of infinite games was the subject of a Part III course in the academic years 2019/20 and 2020/21 and the lecture material is available online [2]. The goal of this essay is to explore one particular topic of the theory of infinite games, present the basic theory in an expository chapter and then provide an overview of the results concerning the chosen particular topic.

Particular topics could be from the following list:

(i) *Applications of infinite games in topology and measure theory.* Games are closely related to *regularity properties* of sets of reals such as the perfect set property, the Baire property, Lebesgues measurability, or more technical properties such as K_σ-regularity. Via numerous game constructions, determinacy axioms provide the regularity of simple sets [1, § 27].

(ii) *Large cardinals from determinacy.* In the somewhat arcane world of AD where the Axiom of Choice is false, some small cardinals (such as $\aleph_1$, $\aleph_2$ and others) have large cardinal properties. In turn, this yields inner models of ZFC with large cardinals [1, § 28].

(iii) *Determinacy from large cardinals.* Somewhat surprisingly, the existence of large cardinals implies that some determinacy axioms hold. The first proof of Borel determinacy (now known to be a theorem of ZFC) used large cardinals [3] and improvements of these results eventually led to the mentioned Martin-Steel theorem [1, §§ 31 & 32].

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent), Part III *Large Cardinals.*

Useful: Part III *Logic & Computability*, Part III *Model Theory* (reading course).

References

[1] A. Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

[2] B. Löwe. Infinite Games, Part III course in Lent term 2021, University of Cambridge, website.

[3] D. A. Martin. Measurable cardinals and analytic games. Fund. Math. 66 (1969/70), 287–291.

[4] D. A. Martin, J. R. Steel. A proof of projective determinacy. J. Amer. Math. Soc. 2 (1989), no. 1, 71–125.

[5] J. Mycielski, H. Steinhaus. A mathematical axiom contradicting the axiom of choice. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 1–3.

21. Constructibility & Large Cardinals ....

Professor B. Löwe

Gödel's famous 1938 proof of the consistency of the Continuum Hypothesis is based on his *constructible universe*, a set theoretic model construction that provides us with the minimal inner model of set theory $\mathbf{L}$; we are also going to consider two generalisations, the minimal inner model of set theory containing a set A, $\mathbf{L}(A)$, and the minimal inner model of set theory defined with an additional predicate that can test membership of a set A, $\mathbf{L}[A]$. These models provide well understood models of set theory with nice properties: often, they are models of the Axiom of Choice and the Generalised Continuum Hypothesis (cf. [2, Chapter VI].

In general, the minimality of these models means that they cannot contain too many or too big large cardinals. The goal of this essay is to explore the contrasting relationship between large cardinals and these minimal models of set theory. The essay should explore one particular topic concerning this relationship, present the basic theory of constructibility in an expository chapter, and then provide an overview of the results concerning the chosen particular topic.

Particular topics could be from the following list:

(i) *Large cardinals that cannot exist in minimal models.* The constructible universe cannot contain a measurable cardinal and this result generalises to other types of large cardinals with respect to variants of the constructible universe [1, §§ 5].

(ii) *Large cardinals that can exist in minimal models.* The smallest large cardinal notions (inaccessible cardinals, weakly compact cardinals, and some of the large cardinal notions defined in terms of partitions) can exist in the constructible universe; even more, these large cardinal properties are absolute between the universe and the constructible universe [2, Chapter IV], so if such a large cardinal exists, it will also exist in the constructible universe.

(iii) *Large cardinals and the generalised continuum hypothesis.* Since constructibility is our main method to show consistence of GCH, the incompatibility results in (i) are obstacles for showing that GCH is consistent with large cardinals. Kunen developed a theory of inner models of measurability using the $\mathbf{L}[A]$-construction [1, § 20].

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent), Part III *Large Cardinals.*

Useful: Part III *Logic & Computability*, Part III *Model Theory* (reading course).

References

[1] A. Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

[2] K. Kunen. *Set theory. An introduction to independence proofs.* Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1980.

22. Robust estimation via robust optimization .... Prof P. Loh

Many classical robust estimation procedures are based on M-estimators [6, 7]. Various theoretical results have been derived under the contamination setting where data are drawn i.i.d. from a mixture distribution $(1-\epsilon)P_\theta + \epsilon Q$, where P_θ is the uncontaminated distribution, Q is arbitrary, and the goal is to estimate θ with small statistical error [8].

More recently, several authors have proposed a new strategy for robust parameter estimation by introducing a robust estimation step directly into an iterative optimization procedure [5, 11, 10]. Rather than the classical approach of minimizing a robust loss function applied to individual data points, the idea is to minimize a non-robust loss function via an iterative optimization procedure which is itself robust. The proposed optimization procedure operates by replacing the gradient in successive gradient steps by a robust mean estimator for the population-level gradient vector [4, 9, 1]. Theoretically, statistical error bounds are proved which are valid even when the form of contamination is *adversarial* rather than in the form of a contaminated mixture.

The goal of this essay is to compare and contrast the aforementioned robust estimation techniques both empirically and theoretically. What are the tradeoffs between different procedures? How do they compare with classical M-estimation under different contamination models? Going beyond these proposals for robust gradient descent, how might other first-order optimization methods [3, 2] be made robust in an analogous manner, and what conclusions may be drawn for such procedures?

Relevant Courses

Essential: Robust Statistics

Useful: Optimisation

References

[1] S. Balakrishnan, S. S. Du, J. Li, and A. Singh. Computationally efficient robust sparse estimation in high dimensions. In *Conference on Learning Theory*, pages 169–212, 2017.

[2] A. Beck. *First-Order Methods in Optimization.* SIAM, 2017.

[3] D. P. Bertsekas. *Convex Optimization Algorithms.* Athena Scientific Belmont, 2015.

[4] I. Diakonikolas, G. Kamath, D. Kane, J. Li, A. Moitra, and A. Stewart. Robust estimators in high-dimensions without the computational intractability. *SIAM Journal on Computing*, 48(2):742–864, 2019.

[5] I. Diakonikolas, G. Kamath, D. Kane, J. Li, J. Steinhardt, and A. Stewart. SEVER: A robust meta-algorithm for stochastic optimization. In *International Conference on Machine Learning*, pages 1596–1606, 2019.

[6] P. J. Huber. Robust estimation of a location parameter. *The Annals of Mathematical Statistics*, pages 73–101, 1964.

[7] P. J. Huber. Robust regression: Asymptotics, conjectures and Monte Carlo. *The Annals of Statistics*, 1(5):799–821, 1973.

[8] P. J. Huber and E. M. Ronchetti. *Robust Statistics.* Wiley Series in Probability and Statistics. Wiley, 2011.

[9] K. A. Lai, A. B. Rao, and S. Vempala. Agnostic estimation of mean and covariance. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 665–674. IEEE, 2016.

[10] L. Liu, Y. Shen, T. Li, and C. Caramanis. High dimensional robust sparse regression. In *International Conference on Artificial Intelligence and Statistics*, pages 411–421. PMLR, 2020.

[11] A. Prasad, A. S. Suggala, S. Balakrishnan, and P. Ravikumar. Robust estimation via robust gradient estimation. *arXiv preprint arXiv:1802.06485*, 2018.

23. Moduli of Curves ....

Dr N. Nabijou

Moduli spaces are ubiquitous in algebraic geometry. They arise from attempts to parametrise some given class of geometric objects — for example, subschemes of projective space, or line bundles on a fixed algebraic curve. Miraculously, the set of such geometric objects is itself endowed with a natural scheme structure, reflecting how the parametrised objects relate to one another. This output is referred to as a moduli space.

Perhaps the most important example is the moduli space of algebraic curves. Though the objects being parametrised are relatively simple, the resulting moduli space is extremely intricate, and remains an active topic of research, with many open questions.

This essay will focus on the moduli space of smooth pointed curves $\mathcal{M}_{g,n}$, and the compactification given by the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. The emphasis will be on explicit examples and computations, and not on abstract generalities. As such, you will take the existence of the moduli space on faith, and focus instead on its behaviour.

There are countless interesting aspects to explore. You will focus on one of the following topics:

- Stable and semistable reduction.
- Intersection theory: psi, kappa and lambda classes.
- Hurwitz schemes and admissible covers.
- Topology: Euler characteristic, top-weight cohomology.
- Variants: Hassett spaces, wonderful models.

Which one you choose will depend on your specific strengths and inclinations. I am also open to considering additional suggestions.

Even a very good essay is not expected to contain original research. However, you will have the opportunity to demonstrate deep understanding, by providing detailed treatments of examples not covered in the existing literature.

Relevant Courses

Essential: Part III Algebraic Geometry, Part II Algebraic Geometry.

Useful: Part II Riemann Surfaces, Part III Commutative Algebra, Part III Algebraic Topology.

References

[1] Harris, J. and Morrison, I. *Moduli of curves.*
Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.

[2] Arbarello, E., Cornalba, M. and Griffiths, P. A. *Geometry of algebraic curves, volume II.*
With a contribution by Joseph Daniel Harris. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 268. Springer, Heidelberg, 2011.

[3] Pandharipande, R. *A calculus for the moduli space of curves.*
Algebraic geometry: Salt Lake City 2015, 459–487, Proc. Sympos. Pure Math., 97.1, Amer. Math. Soc., Providence, RI, 2018

[4] Chan, M. *Moduli spaces of curves: classical and tropical.*
https://www.math.brown.edu/mchan2/Mg.pdf

[5] Hassett, B. *Moduli spaces of weighted pointed stable curves.*
Adv. Math. 173 (2003), no. 2, 316–352.

24. Random Walks on Height Functions ....
Professor J. R. Norris

According to Donsker's invariance principle, any zero-mean, finite-variance random walk on the integers converges weakly under diffusive scaling to a Brownian motion. The diffusivity of the limit Brownian motion is simply the one-step variance of the random walk. The essay will examine the phenomenon of convergence to Brownian motion in a more general setting.

Suppose we are given a finite bipartite graph G with edge set E. Let us say that a function $f : G \to \mathbb{Z}$ is a height function if $|f(x) - f(y)| = 1$ whenever $(x, y) \in E$, and say that two height functions f and g are neighbours if $|f(x) - g(x)| = 1$ for all $x \in G$. Consider the random walk $(F_n)_{n\geq 0}$ on the set of height functions, that is, the random process which moves in each time step from its present state to a randomly chosen neighbour.

The aim of the essay is to show that the average height process $\bar{F}_n = \frac{1}{|G|}\sum_{x\in G} F_n(x)$ converges under diffusive scaling to a Brownian motion and to determine, at least in some special cases, the diffusivity of the limit. See Chapter 7 in [2] for an introduction to diffusion approximation. Ideas from [3] on correctors may also be useful. Some aspects of this essay may be open problems. Original work will receive special credit but is not necessary for an essay of Distinction standard.

Relevant Courses

Essential: None

Useful: Advanced Probability

References

[1] E. Boissard, S. Cohen, Th. Espinasse & J. Norris. Diffusivity of a random walk on random walks. Random Structures Algorithms 47 (2015), no. 2, 267–283.

[2] S.N. Ethier & T.G. Kurtz. *Markov processes: characterization and convergence.* Wiley (2005).

[3] M.J. Luczak & J.R. Norris. Averaging over fast variables in the fluid limit for Markov chains: application to the supermarket model with memory. Ann. Appl. Probab. 23 (2013), no. 3, 957–986.

25. Brownian Motion on a Riemannian Manifold ....

Professor J. R. Norris

Brownian motion on a Riemannian manifold is the unique random process which satifies the two conditions that it is a martingale and that its quadratic variation is given by the metric tensor. Properties of this process are then closely related to both local and global properties of the manifold.

The essay will present an account of one or more constructions of Brownian motion on a Riemannian manifold and will discuss ways to characterize Brownian motion in terms of discrete approximations, as a Markov process, using stochastic differential equations, and via the heat equation. Then some further topics can be chosen in which the behaviour of Brownian motion is analysed. Examples of such topics are: recurrence and transience, behaviour under projections, Brownian bridge and geodesics, long-time behaviour, the case of Lie groups.

The nature of this essay is a synthesis of material in a well developed field. Given the availability of many relevant sources, special credit will be given for an attractive and coherent account.

Relevant Courses

Essential: None

Useful: Advanced Probability, Stochastic Calculus and Applications, Differential Geometry

References

[1] Elworthy, K. D. Stochastic differential equations on manifolds. London Mathematical Society Lecture Note Series, 70. Cambridge University Press, Cambridge-New York, 1982

[2] Emery, Michel. Stochastic calculus in manifolds. Universitext. Springer-Verlag, Berlin, 1989.

[3] Grigor'yan, Alexander. Heat kernel and analysis on manifolds. AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009

[4] Hsu, Elton P. Stochastic analysis on manifolds. Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

26. Wavefront sets .... Gabriel P. Paternain

In this essay you will study the wavefront set of a distribution and its main properties.

For a distribution u on a manifold X, the wavefront set $\mathrm{WF}(u)$ is a subset of $T^*X \setminus \{0\}$ which is conic with respect to multiplication by positive scalars in the fibres of the cotangent bundle T^*X. It describes not only the position of the singularities of u but also their directions. The notion is fundamental for defining operations with distributions (calculus of wavefront sets), particularly products.

The main objective of this essay is to describe in detail the calculus of wavefront sets, including in particular Hörmander's condition for the product of distributions. Once this is established, some options will arise, like a discussion of Hörmander's theorem for the propagation of singularities of solutions of partial differential equations with real principal symbol (with a general outline of its proof) or a presentation of the use of wavefront sets on Feynman propagators (with motivation and background).

Chapter 11 of [3] and [2] are very friendly introductions to the topic.

Relevant Courses

Differential Geometry, Analysis of Partial Differential Equations, Distribution Theory and Applications

References

[1] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis* . Second edition. Springer Study Edition. Springer-Verlag, Berlin, 1990.

[2] C. Brouder, N.V. Dang, F. Hélin, *A smooth introduction to the wavefront set.* J. Phys. A **47** (2014), no. 44, 443001, 30 pp.

[3] F. G. Friedlander, M. Joshi, *Introduction to the theory of distributions.* Second Edition CUP 1998.

[4] A. Strohmaier, *Microlocal analysis. Quantum field theory on curved spacetimes*, 85–127, Lecture Notes in Phys., **786**, Springer, Berlin, 2009.

[5] P. Hintz, *Introduction to microlocal analysis.* Notes available at https://math.mit.edu/ phintz/18.157-S19/18.157.pdf

[6] G. Eskin, *Lectures on linear partial differential equations.* Graduate Studies in Mathematics, **123**. American Mathematical Society, Providence, RI, 2011.

27. Isomorphism theorems for random walks ....
Dr A. Prévost

Consider a random walk X on a finite graph $G = (V, E)$. Let $\ell_x(t)$ be the total time spent by X at $x \in V$ before time t, and let $\tau_u = \inf\{t \geq 0 : \ell_{x_0}(t) \geq u\}$ be the first time the walk has spent total time u at x_0, for some fixed vertex $x_0 \in V$ and $u > 0$. Let ϕ and γ be Gaussian free fields on G, that is Gaussian fields whose covariance function is the Green function, conditioned on taking the value 0 in x_0. The second Ray-Knight theorem [1] is an isomorphism relating the occupation times $\ell_\cdot(\tau_u)$ to the Gaussian free fields. More precisely, it states that the sum $2\ell_\cdot(\tau_u) + \phi^2$ is equal to $(\gamma + \sqrt{2u})^2$, the square of the field γ shifted by $\sqrt{2u}$, for an appropriate coupling of these objects under which ϕ and X are independent.

This isomorphism is however not very explicit about the precise construction of the coupling between the objects (X, ϕ) and γ, and a deeper understanding of this coupling have been obtained in [2] and [3], where the following questions are answered. Can one construct explicitely the Gaussian free field γ conditionally on the random walk X and the Gaussian free field ϕ? Conditionally on $(\gamma + \sqrt{2u})^2$, can one reconstruct X and ϕ? Can one also reconstruct these objects when conditioning on γ instead?

A successfull essay will give an overview of the different isomorphisms, present a proof of the initial second Ray-Knight theorem from [1], and of some of the newer isomorphisms.

Relevant Courses

Essential: Advanced probability

References

[1] Eisenbaum, N., Kaspi, H., Marcus, M.B., Rosen, J. and Shi, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann. Probab.* 28(4): 1781–1796.

[2] Sabot, C. and Tarrès, P. (2015). Inverting Ray-Knight identity. *Probab. Theory Relat. Fields* 165: 559–580.

[3] Lupu, T., Sabot, C. and Tarrès, P. (2019). Inverting the coupling of the signed Gaussian free field with a loop-soup. *Electron. J. Probab.* 24:Paper No. 70: 28.

28. Equivariant K-theory ....
Professor O. Randal-Williams

K-theory $K(-)$ is an invariant of topological spaces analogous to cohomology, but whose elements are described in terms of complex vector bundles rather than cochains. Due to this highly geometric description, for a compact Lie group G it is quite easy to invent a corresponding invariant $K_G(-)$ of topological spaces equipped with G-action, in terms of G-equivariant vector bundles. The G-equivariant K-theory of the one-point space turns out to be the commutative ring $R(G)$ of complex representations of G, making $K_G(-)$ into an $R(G)$-module. This leads to a theory which is algebraically richer than ordinary cohomology.

In this essay you will introduce the basics of equivariant K-theory following [2], and then study the proofs of two important results about it: the localisation theorem, and the Atiyah–Segal completion theorem; both concern the interplay of algebraic constructions applied to $K_G(X)$ and topological constructions applied to X. The first explains the meaning of localising $K_G(X)$ at a prime ideal $\mathfrak{p} \subset R(G)$ in terms of fixed points of the G-action on X, and you should follow the proof in [2]. The second, in its most general form, explains how the failure of $K_G(-)$ to be homotopy invariant can be improved by completing it at various ideals of $R(G)$. For this you should follow the short but dense proof in [1], for which you will need to unpack many details. For this and the rest of the essay, [3, Chapter XIV] could be used as a guide.

Finally, you should calculate some examples to exhibit these phenomena.

Relevant Courses

Essential: Part III Algebraic Topology. You will need to be comfortable with e.g. Part II Representation Theory.

Useful: Some commutative algebra (but less than Part III Commutative Algebra). This can be learnt along the way if necessary.

References

[1] J. F. Adams, J.-P. Haeberly, S. Jackowski, J. P. May, *A generalization of the Atiyah-Segal completion theorem.* Topology 27 (1988), no. 1, 1–6.

[2] G. Segal, *Equivariant K-theory.* Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 129–151.

[3] J. P. May, *Equivariant homotopy and cohomology theory.* CBMS Regional Conference Series in Mathematics, 91. `www.math.uchicago.edu/~may/BOOKS/alaska.pdf`

29. Resolution of singularities for varieties and families .... Dr D Ranganathan

The theorem on embedded resolution of singularities for varieties in characteristic 0 states that given a singular subvariety X embedded in a smooth variety Y, one can perform a sequence of birational surgeries to the geometry of Y, known as blowups, in such a way that the incarnation of X inside this modification $\widetilde{Y}$, namely its strict transform, is now non-singular. It is one of the most frequently used results in algebraic geometry. The theorem was proved by Hironaka in 1964 by an extremely complicated induction. In particular, the most naive resolution algorithm, which proceeds at each step by performing a blowup at the most singular locus, fails dramatically.

In 2019, a breakthrough paper of Abramovich, Temkin, and Włodarczyk gave a new approach to the problem by using a simple idea: to slightly alter the type of birational surgeries to a class of “stack theoretic blowups”. The resulting algorithm is extremely efficient and essentially elementary – repeatedly perform surgery on the worst singularities, and ask the singularity to tell you what type of surgery it wants you to perform. The process is also functorial and leads to several new directions, for example related to notions of resolutions of singularities for morphisms of varieties. The paper takes the theorem on resolution from a black box that many practitioners use but never learn, to an extremely concrete and transparent result.

This essay will explore one of many potential topics related to resolution of singularities for varieties and morphisms. The simplest direction would be to give an account of the ATW method and implement it in some basic examples of interest. Other potential directions could include an account of Abramovich and Karu's proof of the weak semistable reduction theorem, which is the appropriate form of resolution for morphisms of varieties, or an exploration of de Jong's method of alterations, which proves to be a useful substitute for resolution in positive characteristic.

Any essay on this topic must contain concrete examples and explicit calculations, for example for surfaces, hypersurfaces, or for toric varieties. An introduction to the topic can be found in Dan Abramovich's lecture at the 2018 International Congress of Mathematics, which is available on the internet.

Relevant Courses

Essential: Part III Algebraic Geometry.

Useful: Toric Geometry.

References

[1.] Abramovich, Dan and Karu, Kalle. Weak semistable reduction in characteristic 0. Inventiones mathematicae, 139 (2000) 241–273.

[1] Abramovich, Dan, Temkin, Michael, and Wlodarczyk, Jaroslaw. Functorial resolution via weighted blowings up. arXiv: 1906:07106.

[3.] de Jong, Johan. Smoothness, semi-stability and alterations. Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 51–93.

[4.] Kollár, János. Lectures on resolution of singularities. Annals of Mathematical Studies, 166. Princeton University Press, Princeton, NJ, 2007. vi+208 pp.

30. Equivariant geometry and moduli theory .... Dr D Ranganathan

A theme in modern algebraic geometry is study a projective manifold X by first associating an auxiliary space $\mathcal{M}(X)$ whose points parameterise geometric objects naturally associated to X, and then probing X by using the geometry and topology of $\mathcal{M}(X)$. For instance, $\mathcal{M}(X)$ might be the set of algebraic curves in X of a fixed topological type, or the set of vector bundles on X with given Chern data, and one might try to calculate the Euler characteristics of these auxiliary spaces $\mathcal{M}(X)$. Numbers that are extracted from such a process are called *virtual enumerative invariants*. The moduli spaces $\mathcal{M}(X)$ often have a beautiful geometric structure, but their global geometry is typically beyond the present technology of the subject, and that makes the actual calculation of the enumerative invariants difficult in practice.

An influential idea going back to Atiyah and Bott is that if X has a large group G of continuous symmetries, for example those coming from a sufficiently interesting torus action, the cohomology of the moduli spaces $\mathcal{M}(X)$ can be understood by information in a neighbourhood of a much simpler subset, namely the fixed point set of the induced G-action on $\mathcal{M}(X)$. The key insights that lifted Atiyah and Bott's original idea to the context of moduli spaces were due to Graber and Pandharipande, and is known as "virtual localisation".

This essay will focus on an application of the virtual localisation theorem to the study moduli spaces of curves in X (Gromov–Witten theory) or sheaves on X (Donaldson–Thomas theory). The goal of the essay will not be to give an account of all the subtleties of equivariant cohomology and virtual fundamental classes. Instead, an essay should give a rapid working overview of these background concepts, and then proceed directly to the use of these equivariant methods to study moduli spaces. There are several possibilities for the precise direction, but some are below:

- A classical route would be to use the localisation theorem to give a deformation of the multiplication rule in the class algebra of the symmetric group, giving a connection to representation theory.
- A more "virtual" direction would be to study Hurwitz theory on the moduli space of curves by localisation on the moduli space of maps from curves to $\mathbb{P}^1$. This could include a proof of the ELSV formula, which was shown by Okounkov and Pandharipande to imply Witten's conjecture.
- The study of the degree 0 Donaldson–Thomas theory of toric threefolds, which shows that a certain natural generating function for enumerative invariants is essentially given by the generating function for three dimensional partitions, known as the MacMahon function.

There are many other possibilities, which vary in their balance of theory and computation, but any essay in this direction should have a substantial amount of both.

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Algebraic Topology.

Useful: Toric Geometry, Symplectic Geometry.

References

[1] Atiyah, Michael and Bott. "The moment map and equivariant cohomology", Topology 23(1) (1984) 1–28.

[3] Cavalieri, Renzo. "A topological quantum field theory of intersection numbers on moduli spaces of admissible covers", Algebra Number Theory 1 (1) (2007) 35–66.

[2] Graber, Tom and Pandharipande, Rahul. "Localization of virtual classes", Inventiones Mathematicae, 135 (2) (1999) 487–518.

[4] Maulik, Davesh, Okounkov, Andrei, Nekrasov, Nikita, and Pandharipande, Rahul. "Gromov–Witten theory and Donaldson–Thomas theory I and II", Compositio Mathematica 142 (5) (2006) 1263–1304.

[5] Graber, Tom and Vakil, Ravi. "Hodge integrals and Hurwitz numbers via virtual localization." Compositio Mathematica, 135 (1) (2006) 25–36.

31. On compressible fluid dynamics with swirl .... Prof. P RAPHAEL

Singularity formation in fluid mechanics problem is the frontier of current research and a source of amazing challenges at the frontier of mechanics, physics and mathematics. The recent series of works [2], [3] has opened a breach for the qualitative description of singularity formation for compressible fluids.

This essay is an introduction to the subject. We will focus on the two dimensional problem. In spherical symmetry, regular self similar solutions can be constructed through the study of a classical two dimensional phase portrait, [1]. Regular solutions through the sonic line are constructed in [2] in a suitable regime of parameters. The aim of this essay is to start the analysis of self similar solutions in the presence of a non trivial swirl. The starting point is to analyze the corresponding three dimensional phase portrait.

Relevant Courses

Introduction to non linear Analysis. Introduction to PDE's.

References

[1] Guderley, G., Starke kugelige und zylindrische verdichtungsstösse in der nähe des kugelmittelpunktes bzw. der zylinderachse, Luftahrtforschung **19**, 302 (1942).

[2] Merle, F.; Raphaël, P.; Rodnianski, I.; Szeftel, J., *On smooth self similar solutions to the compressible Euler equations*, arXiv:1912.10998.

[3] Merle, F.; Raphaël, P.; Rodnianski, I.; Szeftel, J., *On the implosion of a three dimensional compressible fluid*, arXiv:1912.11009.

32. Self similar blow up for energy super critical wave equations .... Prof. P RAPHAEL

The aim of this essay is to explore the mechanism of self similar blow up for the energy super critical focusing wave equation

$$\partial_t^2 u - \Delta u - u^p = 0.$$

The existence of self similar blow up profiles reduces to an ordinary differential equation which has been studied in [1]. The aim of this essay is twofold.

(1) Understand the analysis in [1] and propose an alternative approach to the construction of self similar profiles following the perturbative approach developped for the heat equation in [2].

(2) Time permitting, understand and adapt the functional framework developped in [3] to prove the finite codimensional stability of these self similar profiles.

References

[1] Bizon, P.; Maison, D.; Waaserman, A., Self similar solutions of semilinear wave equations with a focusing non linearity, Non linearity **20** (2007), 2061-2074.

[2] Collot, C.; Raphaël, P.; Szeftel, J., *On the stability of self similar blow up for the energy super critical heat equation*, Mem. Amer. Math. Soc. 260 (2019), no. 1255.

[3] Merle, F.; Raphaël, P.; Rodnianski, I.; Szeftel, J., *On blow up for the energy super critical defocusing non linear Schrödinger equations*, arXiv:1912.11005, to appear in Invent. Math.

33. Spineless 4-manifolds ....
Professor J. Rasmussen

A *spine* of a manifold with boundary is an embedded closed submanifold whose inclusion into the larger manifold induces a homotopy equivalence. It is easy to construct a smooth 4-manifold homotopy equivalent to S^2 which has no smoothly embedded spine by attaching a single 2-handle to the 4-ball along an appropriate knot K. However any manifold constructed in this way admits a PL spine: that is, a simplicial complex homeomorphic to S^2 whose inclusion into the manifold induces a homotopy equivalence.

The aim of this essay is to understand a recent theorem of Levine and Lidman: there are 4-manifolds homotopy equivalent to S^2 which admit no PL spine. The main tool used in the proof is Heegaard Floer homology, in the form of Ozsváth and Szabó's d-invariants. The essay can treat the larger theory of Heegaard Floer homology as a black box, but within this context, it should give a relatively careful explanation of the d-invariants: how they are defined, their basic geometric properties, etc. It should then go on to discuss the proof of the theorem and the geometric constructions used in it.

Relevant Courses

Essential: Algebraic Topology

Useful: Differential Geometry, Knots, 4-Manifolds

References

[1] Juhasz, A., A survey of Heegaard Floer homology, A survey of Heegaard Floer homology, in *New ideas in low dimensional topology*, 237-296, World Scientific, 2015, `arXiv:1310.3418`.

[2] Gompf, R. and Stipsicz, A., *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, AMS, 1999.

[3] Levine, A. and Lidman, T., Simply connected, spineless 4-manifolds, *Forum Math. Sigma*, 7 (2019), `arXiv:1803.01765`

[4] Ozsváth, P. and Szabó, Z., Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary, *Advances in Math.* 173 (2003), 179-261, `arXiv:math/0110170`.

34. Soergel Bimodules and HOMFLY-PT homology ....
Professor J. Rasmussen

The HOMFLY-PT polynomial is an invariant of knots in S^3, which simultaneously generalizes the Alexander and Jones polynomials. This essay explores the HOMFLY-PT polynomial and its categorification, as studied by Khovanov, Rozansky, and Rouquier. Roughly speaking, the categorification is an invariant which assigns to a link L some graded homology groups; their graded Euler characteristic is the HOMFLY-PT polynomial of L.

One way to define the HOMFLY-PT polynomial of L is to apply the Ocneanu trace to the image of a braid representing L in the Hecke algebra. The essay should explain this definition. It should go on to explain (following Khovanov) how it can be categorified by replacing the Hecke algebra with the category of Soergel bimodules, and the Ocneanu trace with the Hochschild homology. Beyond this, there are many possibilities.

Relevant Courses

Essential:

Useful: Algebraic Topology, Representation Theory of Symmetric Groups, Knots

References

[1] Elias, B. *et al.*, *Introduction to Soergel Bimodules*, Springer Verlag, 2020.

[2] Jones, V., Hecke algebra representations of braid groups and link polynomials, *Annals of Mathematics* 122 (1986), 335-388.

[3] Khovanov, M., Triply-graded link homology and Hochschild homology of Soergel bimodules, *Int. J. Of Math.* 18 (2007) 869-885. `arXiv:math/0510265`

35. Stochastic Localization in Geometry and Combinatorics .... Dr. Julian Sahasrabudhe

Is there a $c > 0$ so that for every convex body $K \subseteq \mathbb{R}^d$ of volume 1, there exists a hyperplane H so that the $(d-1)$-volume of $C \cap H$ is at least c? This simple-sounding question, known as 'the Bourgain slicing problem', has remained poorly understood despite its importance and elegance. Recently, substantial progress was made on this conjecture by Y. Chen, who was working within a fascinating framework, introduced by Eldan, known as "stochastic localization".

The purpose of the first part of this essay is to give a description of this method and a detailed proof of Chen's breakthrough. It should include a gentle discussion of the technical tools involved (e.g. stochastic calculus) and a few (well chosen) examples to illustrate how the process runs. The second part of this essay would describe another application of stochastic localization, for which I suggest the combinatorial adaptation in [3].

I would also be happy if the essay took a slant to the combinatorial [3] and left the geometric [1], [3] more to the background. Although we should agree on these proportions beforehand.

Relevant Courses

Essential: Probability. Strong knowledge of basic linear algebra. Some idea what Fourier analysis is.

Useful: Familiarity with Brownian motion, although no deep knowledge is needed. The Eldan/Chen framework does involve Stochastic calculus, so some familiarity with that is ideal although far from necessary. Basics of the analysis of boolean functions.

References

[1] Y. Chen. An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture. *https://arxiv.org/abs/2011.13661*

[2] R. Eldan. Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *https://arxiv.org/abs/1203.0893*

[3] R. Eldan. R. Gross. Concentration on the Boolean hypercube via pathwise stochastic analysis. *https://arxiv.org/abs/1909.12067*

36. Modern changepoint analysis ....

Prof R. J. Samworth

One of the most commonly encountered issues with 'Big Data' is heterogeneity. When collecting vast quantities of data, it is usually unrealistic to expect that stylised, traditional statistical models of independent and identically distributed observations can adequately capture the complexity of the underlying data generating mechanism. Departures from such models may take many forms, including missing data, correlated errors and data combined from multiple sources, to mention just a few.

When data are collected over time, heterogeneity often manifests itself through non-stationarity, where the data generating mechanism varies with time. Perhaps the simplest form of non-stationarity assumes that population changes occur at a relatively small number of discrete time points. If correctly estimated, these 'changepoints' can be used to partition the original data set into shorter segments, which can then be analysed by using methods designed for stationary time series. Moreover, the locations of these changepoints are often themselves of significant practical interest.

Changepoint analysis is a classical statistical topic, dating back at least to the 1950s [1], but has undergone a renaissance over the last 5-10 years. This is partly due to the ubiquity of real-time monitoring of evolving processes that has become a characteristic feature of 21st century life (e.g. smart watches, defibrillators, Covid-19 case numbers, stock market movements), and partly due to the new statistical challenges associated with tracking many processes simultaneously, where we seek to borrow strength across the different series to identify changepoints.

This essay would synthesise, and potentially contribute to, recent developments in the field. These include, but are not limited to, both offline (retrospective) [2] and online (sequential) [3] changepoint detection and estimation; high-dimensional problems [4]; changes in mean, in slope (in regression problems) [5] or more general nonparametric changes [6]; multiple changes [7,8]; missing data [9]. Depending on the interests of the candidate, it could involve the proposal of a new method to address a contemporary changepoint challenge, a study of the theoretical justifications for different approaches or an empirical comparison of existing methods.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods

References

[1] Page, E. S. (1955) A test for a change in a parameter occurring at an unknown point. *Biometrika*, **42**, 523–527.

[2] Liu, H., Gao, C. and Samworth, R. J. (2021) Minimax rates in sparse, high-dimensional changepoint detection. *Ann. Statist.*, **49**, 1081–1112.

[3] Chen, Y., Wang, T. and Samworth, R. J. (2021+) High-dimensional, multiscale online changepoint detection. *J. Roy. Statist. Soc., Ser. B*, to appear.

[4] Wang, T. and Samworth, R. J. (2018) High dimensional change point estimation via sparse projection. *J. Roy. Statist. Soc., Ser. B*, **80**, 57–83.

[5] Fearnhead, P., Maidstone, R. and Letchford, A. (2019) Detecting changes in slope with an L_0 penalty. *J. Comput. Graph. Statist.*, **28**, 265–275.

[6] Padilla, O. M. H., Yu, Y., Wang D. and Rinaldo A. (2021) Optimal nonparametric change point detection and localization. *Electron. J. Statist.*, **15**, 1154–1201.

[7] Fryzlewicz, P. (2014) Wild binary segmentation for multiple change-point detection. *Ann. Statist.*, **42**, 2243–2281.

[8] Kovács, S., Li, H., Bühlmann, P. and Munk, A. (2020). Seeded Binary Segmentation: A general methodology for fast and optimal change point detection.
`https://arxiv.org/abs/2002.06633`.

[9] Follain, B., Wang, T. and Samworth, R. J. (2021) High-dimensional changepoint estimation with heterogeneous missingness. `https://arxiv.org/abs/2108.01525`.

37. Interpolation and double descent in high-dimensional statistics ....
Dr X. Bing, Dr O. Feng and Prof R. J. Samworth

Statisticians are well acquainted with bias-variance trade-offs, and the fact that overfitting can lead to large prediction error on test data. However, many modern classification and regression methods in Statistics and machine learning, such as neural networks, are based on heavily overparametrised models that may even interpolate the training data, yet still seem to exhibit excellent prediction performance in practice.

Over the last three years or so, researchers have sought to understand this phenomenon by analysing the prediction error of certain simple estimators as the number of parameters in the fit approaches, and then exceeds, the level at which interpolation of the training data is possible. As is well understood, before the interpolation threshold, the error decreases initially as we increase model complexity, but then increases as we start to suffer from overfitting. Much more surprisingly, however, beyond the interpolation threshold, we may encounter one (or more) further decreases in the prediction error curve. This is now known as the double descent phenomenon. References include [1], [2], [3], [4], [5], [6], [7], [8], [9].

This essay present a unified treatment of the research in this area. Candidates may wish to explore multiple descent phenomena empirically and/or theoretically in different settings to those that have previously been considered.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods

References

[1] Bartlett, P. L., Long, P. M., Lugosi, G. and Tsigler, A. (2020) Benign overfitting in linear regression. *PNAS*, **117**, 30063–30070.

[2] Belkin, M., Hsu, D., Ma, S. and Mandal, S. (2019) Reconciling modern machine-learning practice and the classical bias–variance trade-off. *PNAS*, **116**, 15849–15854.

[3] Belkin, M., Hsu, D., Xu, J. (2020), Two models of double descent for weak features. *SIAM J. Math. Data Science*, **2**, 1167–1180.

[4] Belkin, M, Rakhlin, A. and Tsybakov, A. B. (2019) Does data interpolation contradict statistical optimality? *AIStats*.

[5] Bunea, F., Strimas-Mackey, S. and Wegkamp, M. (2020) Interpolating predictors in high-dimensional factor regression. `arXiv:2002.02525`.

[6] Chinot, G., Löffler, M. and van de Geer, S. (2021) On the robustness of minimum norm interpolators and regularized empirical risk minimizers. `arxiv:2012.00807`.

[7] Li, Y. and Wei, Y. (2021) Minimum ℓ_1-norm interpolators: Precise asymptotics and multiple descent. `arXiv:2110.09502`.

[8] Liang, T. and Rakhlin, A. (2020) Just interpolate: Kernel 'ridgeless' regression can generalize. *Ann. Statist.*, **48**, 1329–1347.

[9] Hastie, T., Montanari, A., Rosset, S. and Tibshirani, R. J. (2021) Surprises in high-dimensional ridgeless least squares interpolation. *Ann. Statist.*, to appear.

38. Robustness and oracle inequalities ....
Dr O. Feng and Prof R. J. Samworth

Robustness and model misspecification are intimately linked ([1]). While great historical focus has been placed on consistency and rates of convergence under correct model specification, there is increasing recognition of the need to understand the performance of statistical procedures under as broad a class of data generating mechanisms as possible. Such analysis provides much deeper understanding, and aids the construction of more robust procedures.

Oracle inequalities compare the performance of a procedure to that of an 'oracle', with partial knowledge of the data generating mechanism. For instance, they may decompose an estimation error into a term reflecting how well we can approximate the truth using a sub-family of our model, and the error one would incur if it were known that the truth belonged to this sub-family. Such inequalities have now been established in the contexts of nonparametric shape-constrained inference ([2], [3], [4], [5], [6], [7]), where estimation may be possible without tuning parameters, and in high-dimensional penalised estimation ([8], [9], [10], [11], [12]), where a data-driven choice of tuning parameters may be compared with an oracle choice.

This essay should provide a synthesis of results in the area, including a description of the techniques of proof available. An ambitious candidate may seek to derive their own oracle inequality to provide robustness guarantees in a problem of their choice.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods, Nonparametric Inference under Shape Constraints

References

[1] Lerasle, M. (2019) Lecture Notes: Selected topics on robust statistical learning theory. `arXiv:1908.10761`.

[2] Zhang, C.-H. (2002) Risk bounds in isotonic regression. *Ann. Statist.*, **30**, 528–555.

[3] Bellec, P. C. (2018) Sharp oracle inequalities for least squares estimators in shape restricted regression. *Ann. Statist.*, **46**, 745–780.

[4] Han, Q., Wang, T., Chatterjee, S. and Samworth, R. J. (2019) Isotonic regression in general dimensions. *Ann. Statist.*, **47**, 2440–2471.

[5] Kim, A. K. H., Guntuboyina, A. and Samworth, R. J. (2018) Adaptation in log-concave density estimation. *Ann. Statist.*, **46**, 2279–2306.

[6] Feng, O., Guntuboyina, A., Kim, A. K. H. and Samworth, R. J. (2020) Adaptation in multivariate log-concave density estimation. *Ann. Statist.*, **49**, 129–153.

[7] Feng, O. Y., Chen Y., Han, Q., Carroll, R. J. and Samworth, R. J. (2021+) Nonparametric, tuning-free estimation of S-shaped functions. *J. Roy. Statist. Soc., Ser. B*, to appear.

[8] Bunea, F., Tsybakov, A. and Wegkamp, M. (2007) Sparsity oracle inequalities for the Lasso. *Electr. J. Statist.*, **1**, 169–194.

[9] Huang, J., Sun, T., Ying, Z., Yu, Y. and Zhang, C. H. (2013) Oracle inequalities for the lasso in the Cox model. *Ann. Statist.*, **41**, 1142–1165.

[10] van de Geer, S. A. and Bühlmann, P. (2009) On the conditions used to prove oracle results for the Lasso. *Electr. J. Statist.*, **3**, 1360–1392.

[11] Elsener, A. and van de Geer, S. (2019) Sharp oracle inequalities for stationary points of nonconvex penalized M-estimators. *IEEE Trans. Inf. Th.*, **65**, 1452–1472.

[12] Lecué, G. and Mitchell, C. (2012). Oracle inequalities for cross-validation type procedures. *Electr. J. Statist.*, **6**, 1803–1837.

39. The longest increasing subsequence in random permutations .... S. Sarkar

What is the maximal length of an increasing subsequence in a uniformly random permutation of the first n natural numbers? This became famous in the literature as the Ulam–Hammersley problem. In 1977, Vershik-Kerov and Logan-Shepp independently showed that this quantity is typically about $2\sqrt{n}$. The proof uses beautiful mathematics from different areas, including those of integer partitions and Young tableaux. A successful essay should give an account of these developments and include proofs of the important results. Twenty years after the works of Vershik-Kerov and Logan-Shepp, Baik, Deift, and Johansson wrote a seminal paper addressing the limiting distribution of the fluctuations of this quantity from the typical value. A successful essay should also provide a brief description of this and relevant results (overview of proofs will be sufficient, no detailed proof required).

Relevant Courses

Advanced probability, Combinatorics.

References

[1] D. Romik. The Surprising Mathematics of Longest Increasing Subsequences. *Cambridge University Press* 2015. (Relevant chapters: Chapter 1 and parts of Chapter 2)

[2] B. F. Logan and L. A. Shepp. A variational problem for random Young tableaux *Adv. Math.,* 1977.

[3] J. Baik, P. Deift and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.,* 1999.

40. Local sets of the two-dimensional Gaussian free field ....

L. Schoug

The two-dimensional Gaussian free field (GFF) is a random field satisfying a spatial Markov property which can be seen as the canonical Gaussian random surface model. It can be viewed as a two-dimensional time analog of Brownian motion. In [4], Schramm and Sheffield introduced the notion of local sets (sometimes called stopping sets, as they serve as the two-dimensional analog of stopping times for Brownian motion) of the 2D GFF, and proved that SLE_4 can be coupled with the 2D GFF as a local set and viewed as a level line. Following [4], it was proven that $\mathrm{SLE}_\kappa(\underline{\rho})$ processes can be coupled as local sets and viewed as flow lines ($\kappa < 4$) and level lines ($\kappa = 4$). In [3], [1] and [2] the theory of local sets was expanded on and the authors constructed new local sets with deep connections: the two-valued local sets and the first passage sets. The purpose of this essay is to review the theory of local sets, more precisely, the two-valued sets and the first passage sets. Moreover, a successful essay should contain proofs (or overviews of the proofs) of important results.

Relevant Courses

Essential: Advanced probability, Complex analysis.

Useful: Random planar geometry, Schramm-Loewner evolutions.

References

[1] Aru, Juhan; Lupu, Titus; Sepúlveda, Avelio. First passage sets of the 2D continuum Gaussian free field. Probab. Theory Related Fields 176 (2020), no. 3-4, 1303–1355.

[2] Aru, Juhan; Lupu, Titus; Sepúlveda, Avelio. The first passage sets of the 2D Gaussian free field: convergence and isomorphisms. Comm. Math. Phys. 375 (2020), no. 3, 1885–1929.

[3] Aru, Juhan; Sepúlveda, Avelio; Werner, Wendelin. On bounded-type thin local sets of the two-dimensional Gaussian free field. J. Inst. Math. Jussieu, 18 (2019), no. 3, 591–618.

[4] Schramm, Oded; Sheffield, Scott. A contour line of the continuum Gaussian free field. Probab. Theory Related Fields 157 (2013), no. 1-2, 47–80.

41. Fractal properties of Schramm-Loewner evolutions. ....

L. Schoug

Schramm-Loewner evolution (SLE) processes are random planar fractal curves which arise as the scaling limits of interfaces in many important critical statistical mechanics models in two dimensions. SLE and more generally, $\mathrm{SLE}_\kappa(\underline{\rho})$, have been intensely studied since their introduction and exhibit very interesting geometric behaviour. Their almost sure Hausdorff dimension was determined in [3] to be $\min(2, 1 + \kappa/8)$ and the Hausdorff dimension of the intersection of SLE_κ with $\mathbb{R}$ is almost surely $2 - 8/\kappa$ (see [2]). However, due to the roughness of these curves, the Hausdorff dimension of the curve itself or its intersection with the boundary are not enough to describe the dynamics of the curves. Instead, $\mathrm{SLE}_\kappa(\underline{\rho})$ curves exhibit multifractality, which in essence means that there are whole spectra of exponents which describe the different interactions of the curves see e.g. [1], [5], [6] and [7].Moreover, it turns out that the integral means spectrum – a quantity of substantial interest in complex analysis, which is typically very hard (if possible at all) to compute for deterministic domains – can be computed in the case of a random domain which is a complementary connected component of an SLE curve, see [4] and [5]. The purpose of this essay is to explore different notions of fractality for SLE and is very open in the sense that the student can choose which concepts to emphasize and which to exclude (this can be discussed beforehand by the advisor to get an impression of the different options). A successful essay should contain proofs or outlines of proofs of important results.

Relevant Courses

Essential: Complex analysis, Advanced probability, Stochastic calculus.

Useful: Schramm-Loewner evolutions, random planar geometry.

References

[1] Alberts, Tom; Binder, Ilia; Viklund, Fredrik. A dimension spectrum for SLE boundary collisions. Comm. Math. Phys. 343 (2016), no. 1, 273–298.

[2] Alberts, Tom; Sheffield, Scott. Hausdorff dimension of the SLE curve intersected with the real line. Electron. J. Probab. 13 (2008), no. 40, 1166–1188.

[3] Beffara, Vincent. The dimension of the SLE curves. Ann. Probab. 36 (2008), no. 4, 1421–1452.

[4] Beliaev, D.; Smirnov, S. Harmonic measure and SLE. Comm. Math. Phys. 290 (2009), no. 2, 577–595.

[5] Gwynne, Ewain; Miller, Jason; Sun, Xin. Almost sure multifractal spectrum of Schramm-Loewner evolution. Duke Math. J. 167 (2018), no. 6, 1099–1237.

[6] Johansson Viklund, Fredrik; Lawler, Gregory F. Almost sure multifractal spectrum for the tip of an SLE curve. Acta Math. 209 (2012), no. 2, 265–322.

[7] Schoug, Lukas. A multifractal boundary spectrum for $\mathrm{SLE}_\kappa(\rho)$. Probab. Theory Related Fields 178 (2020), no. 1-2, 173–233.

42. Invariance, causality and robustness ....
Professor R. D. Shah

The assumption that data are i.i.d. is ubiquitous in both the analysis and design of regression methods. However particularly when data have been collected over time, or in different environments, this assumption is rarely satisfied. Instead, the distribution of training and test data may be very different. As a consequence, the real world performance of statistical and machine learning methods can be substantially worse than expected. This is an issue of genuine practical importance, and there is now a rapidly growing body literature in both Statistics and Machine Learning attempting to address this.

One line of work that has attracted much attention involves exploiting causal connections between training and test data. The area is still in its infancy and there remain many unanswered questions.

This essay could take one of several directions. One approach might compare and contrast some of the different methods introduced relating to this problem, empirically and/or theoretically. Another direction might involve focusing on one or a few approaches, and investigating particular generalisations of the methods, such as considering nonlinear models where only linear models have been studied in existing work.

Relevant Courses

Useful: Modern Statistical Methods, Causal Inference

References

[1] N. Meinshausen & P. Bühlmann (2015) Maximin effects in inhomogeneous large-scale data *Annals of Statistics* 43(4):18001–1830

[2] D. Rothenhäusler, N. Meinshausen, P. Bühlmann, J. Peters (2021) Anchor regression: heterogeneous data meets causality *Journal of the Royal Statistical Society, Series B* 83(2) 215–247

[3] M. Arjovsky, L. Bottou, I. Gulrajani, D. Lopez-Paz (2019) Invariant risk minimization *arXiv preprint*

[4] E. Rosenfeld, P. Ravikumar, A. Risteski (2021) The Risks of Invariant Risk Minimization *ICLR 2021*

[5] R. Christiansen, N. Pfister, M. E. Jakobsen, N. Gnecco, J. Peters (2021) A causal framework for distribution generalization *arXiv preprint*

[6] N. Pfister, E. G. Williams, J. Peters, R. Aebersold, P. Bühlmann (2019) Stabilizing Variable Selection and Regression *arXiv preprint*

43. Displaceability of Lagrangian toric fibres ....

Dr J. Smith

Given a Lagrangian submanifold L in a symplectic manifold (X, ω), one can ask whether L can be displaced from itself via a Hamiltonian flow. Such questions arise naturally in the study of dynamical systems, but the solution turns out to be extremely subtle, leading to the development of Floer theory and remarkable connections to topology and algebraic geometry.

Toric manifolds provide an important testing ground for such questions, as they can be described combinatorially (and pictorially) via polytopes, and come with a natural supply of Lagrangian submanifolds: *toric fibres.* The aim of this essay is to describe recent progress on the displaceability question in this case.

You should begin by learning the basics of toric geometry, symplectic reduction, and moment polytopes as covered in [2], and how the method of probes [3] can be used to displace many fibres. Then understand the techniques used in [1] to prove non-displaceability. This paper has many beautiful examples, including a collide-and-scatter phenomenon, and you may wish to look for novel examples of your own. Depending on your interests you could then delve further into the details of Floer theory, or explore connections to mirror symmetry or the minimal model program.

Relevant Courses

Essential: Differential geometry, Symplectic geometry

Useful: Algebraic topology, Toric geometry (this may be helpful in providing some background but is certainly not essential; no knowledge of algebraic geometry is required for the essay)

References

[1] Miguel Abreu and Leonardo Macarini. *Remarks on Lagrangian intersections in toric manifolds.* Trans. Amer. Math. Soc. 365 (2013), 3851–3875. arXiv:1105.0640

[2] Ana Cannas da Silva. *Lectures on symplectic geometry.* Lecture Notes in Mathematics, 1764 (2001). https://people.math.ethz.ch/~acannas/Papers/lsg.pdf

[3] Dusa McDuff. *Displacing Lagrangian toric fibers via probes*, in *Low-dimensional and symplectic topology.* Proc. Sympos. Pure Math. 82 (2011), 131–160. arXiv:0904.1686

44. Mirror symmetry for $\mathbb{C}^*$ . . . . Dr J. Smith

Mirror symmetry is a vast web of results and conjectures connecting symplectic topology with algebraic geometry in a deep and surprising way. In its simplest form it asserts that certain spaces occur in mirror pairs X and $\check{X}$, such that the symplectic topology of one space is encoded in the algebraic geometry of the other, and vice versa. The goal of this essay is to acquaint yourself with some of the main ideas by studying the most fundamental example: where X and $\check{X}$ are both $\mathbb{C}^*$.

The core of the essay should be an exposition of the *wrapped Fukaya category* $\mathcal{W}(\mathbb{C}^*)$, and a proof of the homological mirror symmetry equivalence $\mathcal{W}(\mathbb{C}^*) \simeq D^b\mathrm{Coh}(\mathbb{C}^*)$, modulo a generation result for the wrapped category. This will require you to learn the basics of Lagrangian Floer theory, A_∞-algebra, and Fukaya categories as outlined in [2].

If time permits, there are various possible directions you could then follow. For example, you could discuss symplectic cohomology [5] and the closed–open and open–closed string maps in the context of this example, culminating in a proof of the generation result [1] used earlier in the essay. Or you could pursue modifications of the mirror symmetry statement by partially compactifying the algebro-geometric side and 'adding stops' on the symplectic side, leading to *partially* wrapped Fukaya categories [4, Section 2].

The obvious route into the essay is via a symplectic background, but the subject is heavily algebraic so you should be comfortable with rings, modules, exact sequences, etc, and be prepared to get familiar with more sophisticated concepts like A_∞-categories. If your background is in algebraic geometry then the essay could also be accessible to you if you're happy to spend some time learning the basics of symplectic geometry. In this field it is easy to get swamped by technicalities so please discuss your plans with me before you start. Auroux's Eilenberg lectures [3] give an excellent introduction to the subject and may help you orient yourself.

Relevant Courses

Essential: Symplectic geometry (or Algebraic geometry and willingness to learn some symplectic geometry), Algebraic topology

Useful: Algebraic geometry, Topics in symplectic topology

References

[1] Mohammed Abouzaid. *A geometric criterion for generating the Fukaya category.* Publ. Math. Inst. Hautes Études Sci. No. 112 (2010), 191–240. arXiv:1001.4593

[2] Denis Auroux. *A beginner's introduction to Fukaya categories.* Bolyai Soc. Math. Stud., 26 (2014), 85–136. arXiv:1301.7056

[3] Denis Auroux. *Eilenberg lectures 2016.* http://math.columbia.edu/~topology/Eilenberg_lectures_fall_2016

[4] Yankı Lekili and Alexander Polishchuk. *Auslander orders over nodal stacky curves and partially wrapped Fukaya categories.* J. Topol. 11 (2018), no. 3, 615–644. arXiv:1705.06023.

[5] Paul Seidel. *A biased view of symplectic cohomology.* CDM 2006, 211–253. arXiv:0704.2055.

45. Bordism categories and TFTs . . . . Dr. J. Steinebrunner

The d-dimensional bordism category $\mathbf{Bord}_d$ is a category where morphisms are represented by certain d-dimensional manifolds with boundary. To this category one can associate a classifying space $B(\mathbf{Bord}_d)$, which has been at the center of many recent developments in the field of Algebraic Topology of manifolds. The purpose of this essay is to study the bordism category and its classifying space through the lens of topological field theories (TFTs). (No prior physics knowledge is needed, as TFTs will be defined in purely mathematical terms.)

Depending on the student's interest one of two approaches to classifying invertible TFTs could be taken. One amounts to studying $B(\mathbf{Bord}_d)$ using a range of tools from Algebraic Topology. The other uses some (mild) category theory to relate TFTs to so-called cut-and-paste invariants of manifolds and computes them by a geometric argument.

Concretely, the essay should begin by defining $\mathbf{Bord}_d$ and then take one of two routes:

- The first option would have the goal of understanding the computation of $B(\mathbf{Bord}_d)$ in terms of the Madsen-Tillman spectrum $MTSO_d$, following [2, sections 3.2 and 3.3]. (The more expository [3] will be useful.) An application could be the computation of the fundamental group $\pi_1 B(\mathbf{Bord}_d)$ as in [1, appendix A], which classifies TFTs.

- The other option would focus on a direct proof that there is a 1-to-1 correspondence between invertible TFTs and cut-and-paste invariants of manifolds, as done in [5]. This can also be interpreted as a computation of the fundamental group $\pi_1 B(\mathbf{Bord}_d)$. Then these cut-and-paste invariants could be classified using elementary cobordisms and the Euler characteristic, following [4, section 4].

Relevant Courses

Essential: Part III Algebraic Topology.

Useful: While some basic familiarity with categories will be useful, no knowledge of category theory is required. (I.e. you don't have to have taken Part III Category Theory.)

References

[1] J. Ebert. *A vanishing theorem for characteristic classes of odd-dimensional manifold bundles.* J. Reine Angew. Math. 684 (2013), 1–29.

[2] S. Galatius, O. Randal-Williams. *Monoids of moduli spaces of manifolds.* Geom. Topol. 14 (2010), no. 3, 1243–1302.

[3] A. Hatcher. *A short exposition of the Madsen-Weiss theorem.* `http://pi.math.cornell.edu/~hatcher/Papers/MW.pdf`

[4] U. Karras, M. Kreck, W.D. Neumann, E. Ossa. *Cutting and pasting of manifolds; SK-groups.* Mathematics Lecture Series, No. 1. Publish or Perish, Inc., Boston, Mass., 1973. vii+70 pp.

[5] C. Rovi, M. Schoenbauer. *Relating Cut and Paste Invariants and TQFTs.* arXiv:1803.02939

46. Low Regularity Well-Posedness of Classical Yang–Mills Equations . . . . . . Dr G. Taujanskas

The classical Yang–Mills equations in four spacetime dimensions are a set of coupled nonlinear hyperbolic PDEs underpinning the Standard Model of Particle Physics. In order to understand the analytic structure of these equations, it is of interest to study the Cauchy problem for the Yang–Mills system from the point of view of low regularity initial data. In 1981 Eardley and Moncrief [1] discovered a remarkable physical space technique for performing certain lightcone estimates, proving that global solutions exist for initial data possessing just two derivatives. Some time later, Klainerman and Machedon [2, 3, 4] observed the celebrated *null structure* in the nonlinearities of the Yang–Mills equations, which allowed them to reduce the regularity of initial data to merely finite energy.

This essay will survey the existing literature on the subject of low regularity well-posedness of the Cauchy problem for the Yang–Mills equations, starting with [5] and focussing in particular on the main ideas of techniques discovered by Eardley–Moncrief [1] and Klainerman–Machedon [2, 3, 4]. An exceptional essay may go further to comment on more recent developments [6, 7].

Relevant Courses

Essential: Analysis of PDEs, General Relativity

Useful: Introduction to Non Linear Analysis, Differential Geometry

References

[1] D. M. Eardley and V. Moncrief, *The global existence of Yang–Mills–Higgs fields in 4-dimensional Minkowski space. I & II*, Comm. Math. Phys. 83 (1982), pp. 171-191 & 193-212.

[2] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Communications on Pure and Applied Mathematics, 46 (1993), pp. 1221-1268.

[3] S. Klainerman and M. Machedon, *On the Maxwell–Klein–Gordon equation with finite energy*, Duke Math. J., 74 (1994), pp. 19–44.

[4] S. Klainerman and M. Machedon, *Finite energy solutions of the Yang–Mills equations in* $\mathbb{R}^{3+1}$, Annals of Mathematics, 142 (1995), pp. 39–119.

[5] Y. Choquet-Bruhat and D. Christodoulou, *Existence of global solutions of the Yang–Mills, Higgs and spinor field equations in 3+1 dimensions*, Annales scientifiques de l'École Normale Supérieure, Ser. 4, 14 (1981), pp. 481–506.

[6] T. Tao, *Local well-posedness of the Yang–Mills equations in the temporal gauge below the energy norm*, Journal of Differential Equations, 189 (2003), pp. 366-382.

[7] M. Keel, T. Roy, and T. Tao, *Global well-posedness of the Maxwell–Klein–Gordon equation below the energy norm*, Discrete & Continuous Dynamical Systems, 30 (3) (2011), pp. 573–621.

47. Polynomial preserving processes ....

M. Tehranchi

A real-valued Markov process X is polynomial preserving if the function u defined by

$$u(t,x) = \mathbb{E}[f(X_t)|X_0 = x]$$

is a polynomial in x for all t whenever f is a polynomial. There is growing interest in modelling financial quantities with such processes since the computations involved in pricing certain derivative contracts are reasonably tractable.

This essay will survey the literature on polynomial preserving processes and related variants. Focus can be on the mathematical properties, such as characterisations of their generators, or can be on a particular application in finance, exploring their advantages and disadvantages compared to other modelling frameworks.

Relevant Courses

Essential: Advanced Financial Models, Stochastic Calculus & Applications

Useful: Advanced Probability

References

[1] S. Cheng and M. Tehranchi. Polynomial term structure models. *International Journal of Theoretical and Applied Finance.* **24**(2) (2021)

[2] Ch. Cuchiero, M. Keller-Ressel, and J. Teichmann. Polynomial processes and their applications to mathematical finance. *Finance and Stochastics* **16**: 711-740 (2012)

[3] D. Filipović and M. Larsson. Polynomial preserving diffusions and applications in finance. *Finance and Stochastic* **20**: 931–972 (2016)

48. Deforming Galois Representations ....

Professor J. A. Thorne

By a Galois representation, we typically mean a continuous homomorphism $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(A)$, where K is a number field and A is a topological ring. Deforming ρ means finding a ring B with ideal I such that $B/I \cong A$ and a representation $\rho_B : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(B)$ which specialises, modulo the ideal I, to ρ.

Galois deformation theory, introduced by Mazur in the 1980's, gives a systematic way to study deformations of Galois representations. Class field theory and Galois cohomology are essential tools. This theory has become an important part of the way we organise and understand Galois representations and has played a basic role in the proofs of significant results such as Fermat's Last Theorem and the Sato–Tate conjecture.

One possible route through this essay would be to understand Mazur's article [1] and to compute some explicit examples of Galois deformation rings, as in the work of Boston [2]. Another more ambitious path might be to focus on representations of the Galois groups of local fields, explaining how to define Galois deformation rings 'with conditions' as arising in p-adic Hodge theory and explaining the statement of the Breuil–Mézard conjecture [3]. Yet another path might be to follow Ramakrishna [4] in using Mazur's deformation theory, together with duality theorems in the Galois cohomology of number fields, to prove the existence of deformations of Galois representations with $\mathbb{F}_p$-coefficients to Galois representations with $\mathbb{Z}_p$-coefficients.

Relevant Courses

Essential: Local Fields, Commutative Algebra

Useful: Algebraic Geometry

References

[1] B. Mazur, Deforming Galois representations. *Galois groups over* $\mathbb{Q}$ (Berkeley, CA, 1987), pp. 385–437, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[2] N. Boston, Explicit deformation of Galois representations. *Invent. Math.* **103** (1991), no. 1, pp. 181–196.

[3] C. Breuil, A. Mézard, Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$. *Duke Math. J.* **115** (2002), no. 2, pp. 205–310.

[4] R. Ramakrishna, Lifting Galois representations. *Invent. Math.* **138** (1999), no. 3, pp. 537–562.

49. Galois representations for modular forms .... Professor J. A. Thorne

If $k \geq 12$ is an integer, then the Hecke operators T_p (p a prime number) give a family of commuting normal endomorphisms of the vector space $S_k(\mathrm{SL}_2(\mathbb{Z}))$ of cuspidal modular forms of weight k and level $\mathrm{SL}_2(\mathbb{Z})$. If $f(q) = \sum_{n=1}^{\infty} a_n q^n$ is a simultaneous eigenvector of the Hecke operators then the field $K_f = \mathbb{Q}(\{a_n\}_n)$ generated by the Fourier coefficients of f is a number field. Using étale cohomology, Deligne proved in the 1960's that associated to f there is a family of Galois representations $\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda})$, indexed by non-archimedean places λ of K_f, and with coefficients in the completion of K_f at the place λ.

The primary goal of this essay would be to explain Deligne's construction [1] of these representations, first summarising the necessary properties of the étale cohomology of algebraic varieties and then following Deligne's representation-theoretic analysis of the étale cohomology of modular curves. An alternative (and slightly more elementary) approach would be to consider the case of weight 2 modular forms of level $\Gamma_1(N)$, in which case the étale cohomology groups can be replaced by the ℓ-adic Tate modules of the Jacobians of modular curves (an approach described in [2] and in Conrad's appendix to [3]).

One could then continue on to some applications, for example discussing Deligne's reduction of the Ramanujan conjecture $|\tau(p)| \leq 2p^{11/2}$ to the Weil conjectures, or (following Serre and Swinnerton-Dyer [4]) the interpretation of Ramanujan's congruences for the τ function in terms of the images of the Galois representations associated to Ramanujan's modular form $\Delta(q) = \sum_{n=1}^{\infty} \tau(n)q^n$.

Relevant Courses

Essential: Modular Forms, Algebraic Geometry

Useful: Local Fields

References

[1] P. Deligne, Formes modulaires et représentations l-adiques. *Séminaire Bourbaki. Vol. 1968/69: Exposés 347—363*, Exp. No. 355, pp. 139–172, Lecture Notes in Math., 175, Springer, Berlin, 1971.

[2] F. Diamond, J. Shurman, A first course in modular forms. Graduate Texts in Mathematics, 228. Springer-Verlag, New York, 2005.

[3] K. A. Ribet, W. A. Stein, Lectures on Serre's conjectures. *Arithmetic algebraic geometry (Park City, UT, 1999)*, pp. 143–232, IAS/Park City Math. Ser., 9, Amer. Math. Soc., Providence, RI, 2001.

[4] H. P. F. Swinnerton-Dyer, On l-adic representations and congruences for coefficients of modular forms. *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972),* pp. 1–55. Lecture Notes in Math., Vol. 350, Springer, Berlin, 1973.

50. The Schinzel–Zassenhaus conjecture ....
Professor P. P. Varjú

Let $P(x)$ be a polynomial with integer coefficients with leading coefficient 1. If all the complex roots of P are contained in the unit disk, then it must be a divisor of the polynomial $x^n - 1$ for some n. This is a theorem of Kronecker.

Schinzel and Zassenhaus conjectured that when $P \nmid x^n - 1$ for any n, then P must have a root with absolute value at least $1 + c/\deg(P)$ for some absolute constant $c > 0$. This conjecture has been proved by Dimitrov [1] recently.

See the survey [2] for general background in this subject.

The purpose of this essay is to give a self-contained proof of the Schinzel-Zassenhaus conjecture.

Relevant Courses

None, but some knowledge of complex analysis and number fields is useful.

References

[1] V. Dimitrov, A proof of the Schinzel-Zassenhaus conjecture on polynomials, arXiv:1912.12545v1

[2] C. Smyth, The Mahler measure of algebraic numbers: a survey, arXiv:math/0701397v2

51. Primitive Ideals in Enveloping Algebras .... Dr S. J. Wadsley

Classifying all irreducible (not necessarily finite-dimensional) representations of a finite dimensional complex Lie algebra $\mathfrak{g}$ is a hopeless task. However, Dixmier [1] proposed studying a coarser classification — that of annihilators of irreducible representations, known as primitive ideals. This turns out to be both a reasonable and important task.

In this essay, you will focus mostly on the case $\mathfrak{g}$ is semisimple explaining the statements and proofs of the classification. Duflo was the first to prove in [2] that in this case every primitive ideal arises as the annihilator of an irreducible highest weight module. Joseph produced the first purely algebraic proof of this fact in [3]. See [4] for a survey by Joseph.

Relevant Courses

Essential: Finite Dimensional Lie and Associative Algebras.

References

[1] Dixmier, Jacques, Enveloping algebras, Graduate Studies in Mathematics, 11, American Mathemat ical Society, Providence, RI, 1996

[2] Duflo, Michel, Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'un e algèbre de Lie semi-simple, Ann. of Math. (2), 105, 1977, no. 1, 107–120,

[3] Joseph, A., Dixmier's problem for Verma and principal series submodules, J. London Math. So c. (2), 20, 1979, 2, 193–204

[4] Joseph, A. On the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra. Lie group representations, I, College Park, Md., 1982/1983, 30-76,

52. Kirillov's Orbit Method .... Dr S. J. Wadsley

Kirillov's Orbit Method

The Orbit method is a technique for understanding the irreducible unitary representations of a Lie group geometrically via the orbits of the action of the group on the dual of its Lie algebra known as its coadjoint orbits. This was first carried in the 1960s by Kirillov for nilpotent Lie groups though has found applications to more general Lie groups including p-adic Lie groups.

For this essay you might start by reading the survey [1] which is packed with suitable references and then pick one family of examples to study in more detail. For example you might consider the solvable case (eg as in [2]), the p-adic case (eg as in [3]) or the case of compact Lie groups (see various references in [1]). It should involve explaining the ideas, giving proofs of relevant results and illustrating both the ideas and proofs with examples.

Relevant Courses

Essential: Finite Dimensional Lie and Associative Algebras, Commutative Algebra

Useful: Symplectic Geometry

References

[1] Kirillov, Alexandre. "Merits and demerits of the orbit method." Bulletin of the American Mathematical Society 36.4 (1999): 433-488.

[2] Auslander, Louis, and Bertram Kostant. "Polarization and unitary representations of solvable Lie groups." Inventiones mathematicae 14.4 (1971): 255-354.

[3] Howe, Roger. "Kirillov theory for compact p-adic groups." Pacific Journal of Mathematics 73.2 (1977): 365-381.

53. Linear fields in anti-de Sitter spacetimes .... Dr Claude Warnick

Spacetimes with negative cosmological constant, known as anti-de Sitter spacetimes, attract much interest in theoretical physics owing to the conjectured AdS/CFT correspondence. A feature of these spacetimes is a lack of global hyperbolicity connected to the existence of a timelike conformal boundary. To study linear fields in these backgrounds, one is required to specify boundary conditions 'at infinity'. The goal of this essay is to study the mathematical questions this raises, and their consequences for physics, in particular of black holes. A good essay will include a discussion of the issue of boundary conditions for fields in anti-de Sitter, the consequences for well-posedness of the equations, and a review of results for linear fields on AdS black hole backgrounds.

Relevant Courses

Essential: General Relativity

Useful: Analysis of PDE; Black Holes

References

[1] Stability in Gauged Extended Supergravity, P Breitenlohner, D Freedman. Annals Phys. 144 (1982) 249.

[2] The massive wave equation in asymptotically AdS spacetimes, C Warnick. Commun.Math.Phys. 321 (2013) 85-111; arXiv: 1202.3445.

[3] On quasinormal modes of asymptotically anti-de Sitter black holes, C Warnick. Commun.Math.Phys. 333 (2015) 2, 959-1035; arXiv: 1306.5760.

[4] Asymptotic properties of linear field equations in anti-de Sitter space, G Holzegel, J Luk, J Smulevici, C Warnick. Commun.Math.Phys. 374 (2019) 2, 1125-1178; arXiv: 1502.04965.

54. Singular limits of hyperbolic PDE ....
Dr Claude Warnick

Many problems in physics involve limits which are singular from the point of view of the PDE underlying the problem. For example, we expect the incompressible Euler equations to arise as some limit of the equations governing a compressible fluid. These limits can be studied in a unified way within the theory of hyperbolic systems. The purpose of this essay is to give an overview of this theory, and some examples of applications.

Relevant Courses

Essential: Analysis of PDE

Useful: Introduction to nonlinear analysis

References

[1] S. Klainerman and A. Majda *"Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids,"*
Communications on Pure and Applied Mathematics 34 (1981), no. 4 481–524

[2] S. Schochet, *"Fast singular limits of hyperbolic PDEs,"* Journal of Differential Equations 114 (1994), no. 2 476–512.

55. The modularity theorem and the modular approach ....
Dr. H. Wiersema

A Diophantine equation is a polynomial equation where one is interested in integral solutions. The most famous one is $x^n + y^n = z^n$. This equation is the subject of Fermat's Last Theorem which states that there are no solutions in non-zero integers for integers $n > 2$. This was proven by Andrew Wiles, and a crucial ingredient to the proof was the modularity theorem.

A first goal of this essay would be to explore how the modularity theorem implies Fermat's Last Theorem. The essay could start with a short introduction to elliptic curves, modular forms and Galois representations. The next step would be to show how to obtain Galois representations from elliptic curves and feature results on Galois representations attached to modular forms. The essay should also include the modularity theorem and Ribet's level lowering theorem. Next the student should carefully study the Frey curve attached to a non-trivial solution of the Fermat equation and prove some of its main properties. After this, they should deduce Fermat's Last Theorem. Time permitting, the student could study more general versions of the modularity theorem.

There is plenty of literature available: a very accessible reference is [4], in more detail we have [3] and [6], and Chapter I of [1] should also be helpful.

Another route for the project is to roughly sketch how the modularity theorem implies Fermat's Last Theorem and then to explore how the modularity theorem can be used to study more general Diophantine equations. The starting point for this would be [5] and one could go on to study some of the methods in e.g. [2]. For this route there is also the possibility of doing explicit computations in for example MAGMA.

Relevant Courses

Essential: Number Fields

Useful: Local Fields, Elliptic Curves, Modular Forms

References

[1] Cornell, et al., *Modular Forms and Fermat's Last Theorem.* Springer, 1997.

[2] S. R. Dahmen, S. Siksek, *Perfect powers expressible as sums of two fifth or seventh powers*, Acta Arith. 164 (2014), no. 1, 65-100.

[3] F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics 228, Springer-Verlag, New York, 2005.

[4] Lozano-Robledo, A., et al. *Elliptic Curves, Modular Forms, and Their L-Functions.* American Mathematical Society ; Institute for Advanced Study, 2011.

[5] S. Siksek. *The modular approach to Diophantine equations*, Number Theory: Volume II: Analytic and Modern Tools (2007), 495-527.

[6] J. H. Silverman. *The Arithmetic of Elliptic Curves.* 2nd ed. New York: Springer, 2009. Print. Graduate Texts in Mathematics ; 106.

56. Rigidity theorems for hyperbolic groups ....
H. Wilton

A finitely generated group is called *word-hyperbolic* if triangles in its Cayley graph are uniformly thin. This condition defines a vast class of groups, first introduced by Gromov [5], and enables the geometric techniques developed by Thurston when studying hyperbolic 3-manifolds to be applied in a much wider setting.

The idea of the essay is to explore rigidity theorems for hyperbolic groups. A typical result is Paulin's theorem, which says that the outer automorphism group of a torsion-free hyperbolic group Γ is infinite if and only if Γ splits as an amalgamated free product or HNN extension over a cyclic subgroup [3, 6].

A successful essay, after describing some of the basic theory of hyperbolic groups [4, Chapters III.H and III.Γ], will explain the basic strategy used to prove theorems like Paulin's theorem (sometimes called the *Bestvina–Paulin method*): if rigidity fails (in this case, if the outer automorphism group is infinite), then one can apply a limiting argument to extract an action on an $\mathbb{R}$-tree [1]; one then applies Rips' classification of actions on $\mathbb{R}$-trees [2] to deduce a contradiction. More advanced essays will go into other results in the same vein, such as Rips–Sela's proof that rigid hyperbolic groups are co-Hopfian [7].

Relevant Courses

Essential: Part II Algebraic topology

References

[1] Mladen Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002. `http://www.math.utah.edu/~bestvina/eprints/handbook.ps`

[2] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Inventiones Mathematicae*, 121(2):287—321, 1995.

[3] M. R. Bridson and G. A. Swarup. On Hausdorff–Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)*, 40(3-4):267–289, 1994.

[4] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1999.

[5] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[6] Frédéric Paulin. Outer automorphisms of hyperbolic groups and small actions on $\mathbf{R}$-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.

[7] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geometric and Functional Analysis*, 4(3):337–371, 1994.

57. Outer space ....

H. Wilton

The action of a group Γ on itself by conjugation defines a natural map $\Gamma \to \mathrm{Aut}(\Gamma)$. Its image is the (normal) subgroup of *inner automorphisms*, and the quotient $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is called the *outer automorphism group* $\mathrm{Out}(\Gamma)$. When $\Gamma = F_n$, the non-abelian free group of rank n, the group $\mathrm{Out}(F_n)$ is especially complicated and interesting.

An important idea in geometric group theory is that one can study interesting groups by constructing nice spaces on which they act. For $\mathrm{Out}(F_n)$, Culler and Vogtmann [4] constructed a certain space of graphs, now known as 'Culler–Vogtmann Outer Space' and denoted by $\mathcal{CV}_n$. Topological properties of $\mathcal{CV}_n$ translate into group-theoretic properties of $\mathrm{Out}(F_n)$. For instance, Culler and Vogtmann showed that $\mathcal{CV}_n$ is contractible, from which it follows that $\mathrm{Out}(F_n)$ has finite cohomological dimension.

The idea of this essay is to describe the construction of $\mathcal{CV}_n$, to give a proof that it is contractible, and to deduce the corresponding results about $\mathrm{Out}(F_n)$. The original Culler–Vogtmann proof of contractibility is quite combinatorial, but a more transparent geometric proof was given by Skora — see [3] or [5]. The required results about cohomological dimension can be found in [2]. An excellent essay might go on to describe more general deformation spaces (along the lines of [3] or [5]), or to prove the existence of train tracks [1].

Relevant Courses

Essential: Part II Algebraic topology *Useful:* Part III Algebraic topology, Part III Mapping class groups

References

[1] Mladen Bestvina. A Bers-like proof of the existence of train tracks for free group automorphisms. `arXiv:1001.0325`, 2010.

[2] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[3] Matt Clay. Contractibility of deformation spaces of G-trees. *Algebr. Geom. Topol.*, 5:1481–1503 (electronic), 2005.

[4] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[5] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups, Geometry, and Dynamics*. Volume 1, Issue 2, 2007, pp. 135-181.

58. Morawetz estimates and general relativity ....

Dr. Z. Wyatt

Cast a stone into the centre of a still lake. After an initial splash, waves begin radiating out on the surface of the water. After some time, the centre calms and the amplitude of the waves decays to zero. In PDE theory, it is possible to prove estimates which capture such behaviour [7]. These are typically called Morawetz estimates (for PDEs on flat Minkowski spacetime) or integrated local energy decay (ILED) estimates (when the geometry is curved).

The goal of this essay is to motivate and prove some Morawetz/ILED estimates and to understand their application to PDE problems, in particular model PDE problems for the Einstein equations. The essay should begin with a presentation of a classical Morawetz estimate for wave-like equations in Minkowski (see e.g. [1, 5]). Once this is done, possible further topics include: (a) looking at applications of Morawetz inequalities to estimate solutions of the wave equation in the exterior of some reflecting body [6] (b) ILED estimates over compact regions $r \leq R$ for the wave equation on a Schwarzschild background including a discussion of the loss of derivatives due to the trapping effect of the photon sphere (see e.g. [2,4]), (c) the r^p-method of Dafermos and Rodnianski providing alternative ILED estimates in the region $r > R$ for the wave equation on a Schwarzschild background [3].

Relevant Courses

Essential: Analysis of PDEs

Useful: General Relativity

References

[1] Alinhac, "Hyperbolic Partial Differential Equations", Springer. Universitext (2009)

[2] Aretakis, "Lecture Notes on General Relativity" https://web.math.princeton.edu/~aretakis/columbiaGR.pdf

[3] Dafermos and Rodnianski, "A New physical-space approach to decay for the wave equation with applications to black hole spacetimes", XVIth International Congress on Mathematical Physics, World Scientific, London (2009)

[4] Dafermos and Rodnianski, “Lectures on black holes and linear waves” Clay Math.Proc. 17 (2013) 97-205

[5] Morawetz, “Time decay for the nonlinear Klein-Gordon equation,” Proc. R. Soc. Lond. A 306 (1968).

[6] Morawetz, “The limiting amplitude principle,” Comm. Pure Appl. Math. 15 (1962).

[7] Tao, “Morawetz inequalities”, https://celebratio.org/Morawetz_CS/article/666/

59. The BKL Conjecture .... Dr. Z. Wyatt

The FLRW cosmological models are built on the assumption of spatial homogeneity and isotropy. The prototypical anisotropic models, which are still spatially homogeneous, are given by the Kasner spacetimes. An important mathematical, and indeed philosophical, question is to understand the nature of cosmological spacetimes and their properties towards the initial singularity outside of such strict symmetry assumptions. In the early 1970s, Belinskiĭ, Khalatnikov and Lifshitz conjectured that, apart from exceptional cases, cosmological singularities should generically be spacelike and, as the singularity is approached, highly oscillatory in time. It was quickly realised that the singularity behaviour should also depend on the matter content of the spacetime, and singularities where the oscillatory behaviour is absent due to matter (typically a stiff perfect fluid) are referred to as quiescent singularities.

The goal of this essay is to present some concrete ideas related to the BKL conjecture, while ideally avoiding some of its more outlandish conjectures. The essay should begin with a survey of a few cosmological models, including the classification of spatially homogeneous but possibly anisotropic models according to the Bianchi groups (see e.g. [2, 5]). Once this is done, some further topics include: (a) generalised Kasner spacetimes and BKL-type limits suggesting oscillatory asymptotics (e.g. [3, 2§I.2, 5§5.8]) (b) oscillatory behaviour within the Mixmaster universe, i.e. Bianchi IX class (e.g. [4] and [8]) (c) quiescent singularities due to the presence of a perfect fluid or high-dimensions (e.g. [DH§I.4] and [6]) (d) stability within asymptotically velocity term dominated regimes, i.e. Kasner-like (e.g. [1], [7]).

Relevant Courses

Essential: General Relativity, basic knowledge of Lie groups/algebras

Useful: Cosmology

References

[1] Andersson and Rendall, “Quiescent cosmological singularities”, Commun.Math.Phys. 218 (2001)

[2] Belinski and Henneaux, “The Cosmological Singularity”, Cambridge Monographs on Mathematical Physics, (2017).

[3] Belinski, Khalatnikov and Lifshitz, ‘A general solution of the Einstein equations with a time singularity”, Adv. Phys. 31, 639 (1982).

[4] Berger, et al, “The Singularity in Generic Gravitational Collapse Is Spacelike, Local, and Oscillatory” Mod.Phys.Lett. A13 (1998) 1565-1574

[5] Choquet-Bruhat "General Relativity and the Einstein Equations", Oxford Mathematical Monographs, (2009).

[6] Demaret, Henneaux and Spindel, "Nonoscillatory behaviour in vacuum Kaluza-Klein cosmologies", Phys. Lett. B 164 (1985), no. 1-3, 27-30.

[7] Fournodavlos, Rodnianski and Speck, "Stable Big Bang formation for Einstein's equations: The complete sub-critical regime", https://arxiv.org/abs/2012.05888

[8] Ringström, "The Bianchi IX Attractor", Ann. Henri Poincaré 2 (2001), no. 3, 405–500.

60. Circle Packings and Random Geometry ....

Dr D. J. Yeo

A *circle packing* is a collection of closed discs in the plane, for which some pairs are tangent, and the remaining pairs are disjoint. We can define a planar graph G, whose vertices are the centres of the discs, in which two vertices are connected by an edge if the corresponding pair of circles are tangent. The famous circle packing theorem asserts that *every* finite planar graph can be constructed from a circle packing [5].

In the setting of infinite graphs, a result of He and Schramm [2] establishes that any circle packing of a particular infinite triangulation T is supported either on a disc or on the plane, but not both. Interestingly for probabilists, this dichotomy is related to recurrence and transience of simple random walk on T.

More recently, interest has focused on *random* planar graphs, such as the *Uniform Infinite Planar Triangulation*, and their limiting behaviour. There has been considerable progress in drawing relationships between circle packings, random walk properties [1,3], and spanning forest behaviour [4], of the UIPT and other discrete models of random geometry.

A successful essay will introduce the classical background, and report on one or more of these or other recent developments in this field.

Relevant Courses

Essential: Advanced Probability, Percolation and Related Topics

Useful: Schramm–Loewner Evolutions

References

[1] O. Angel, T. Hutchcroft, A. Nachmias, G. Ray, *Unimodular hyperbolic triangulations: circle packing and random walk.* Invent. Math. **206**(1), 229-268 (2016)

[2] Z.-X. He, O. Schramm, *Hyperbolic and parabolic packings.* Discret. Comput. Geom. **14**(2), 123-149 (1995)

[3] O. Gurel-Gurevich, A. Nachmias, *Recurrence of planar graph limits.* Ann. Math. (2) **177**(2), 761-781 (2013)

[4] T. Hutchcroft, A. Nachmias, *Uniform spanning forests of planar graphs.* Forum Math. Sigma 7, e29.

[5] P. Koebe, *Kontaktprobleme der konformen abbildung.* (1936)

[6] A. Nachmias, *Planar Maps, Random Walks and Circle Packing.* Springer Open Lecture Notes in Mathematics **2243**. (2019)

61. Random Parking ....

Dr D. J. Yeo

The *parking problem* is an interacting particle process based on a graph G, motivated by collisions of hashing functions in computer science [5]. In the mathematical setting, a number of agents perform nearest-neighbour walks on G, either simultaneously or in sequence, with each agent fixating (or *parking*) forever at the first vertex they encounter which isn't already occupied.

Randomness is often added through the initial locations of the agents, as well as i) random choice of walks [3,6]; or ii) a random underlying graph [2,4]. Questions might include: Does every agent find a parking space almost surely? Do finitely- or infinitely-many agents visit a typical vertex? Does the exact initial arrangement of the agents, or just the number / density affect the long-term behaviour?

A successful essay will describe at least one of these parking problems in a random setting. Ambitious students may consider including simulations, or extending existing methods to other types of graphs or trees (or explaining why it is challenging to do so), or relating to more general theory of interacting particle systems [1].

Relevant Courses

Essential: Advanced Probability,

Useful: Percolation and Related Topics,

References

[1] M. Cabezas, L. Rolla, V. Sidoravicius, *Recurrence and density decay for diffusion-limited annihilating systems.* Prob. Th. Rel. Fields **170**, 587-615 (2018)

[2] Q. Chen, C. Goldschmidt, *Parking on a random rooted plane tree.* Bernoulli **27**, 1 (2021)

[3] M. Damron, J. Gravner, M. Junge, H. Lyu, D. Sivakoff, *Parking on transitive unimodular graphs.* Ann. Appl. Probab. **29**, 2089-2113 (2019)

[4] C. Goldschmidt, M. Przykucki, *Parking on a random tree.* Combin. Probab. Comput. **28**(1), 23-45 (2019)

[5] A. Konheim, B. Weiss, *An occupancy discipline and applications.* SIAM J. Appl. Math **14**, 1266-1274 (1966)

[6] M. Przykucki, A. Roberts, A. Scott, *Parking on the integer*s. Pre-print `arXiv:1907.09437`.

62. Linkage Disequilibrium: Theory and Reality ....

Dr Q. Zhao

Linkage disequilibrium (LD) refers to the statistical association of alleles (variant forms of genes) at different loci in a given population. In general, LD should decrease as the genetic distance between the loci increases, and population geneticists have shown that, in theoretical models, the expected r^2 (R-squared) decays inverse proportionally to the recombination rate between the loci [1,2]. The rate of LD decay is crucial for many downstream analyses, but there have been few careful investigations into the rate of empirical LD decay over long genetic distances.

Statistically, the challenge is that small r^2 is very difficult to estimate and the sample r^2 is heavily biased.

This essay could review and critique the theoretical models for linkage disequilibrium and/or develop a statistical method to investigate the empirical patterns of LD decay using real data [3,4].

References

[1] Y. S. Song and J. S. Song, "Analytic computation of the expectation of the linkage disequilibrium coefficient," *Theoretical Population Biology*, vol. 71, p. 49–60, Feb 2007.

[2] R. Durrett. *Probability Models for DNA Sequence Evolution (2nd Edition)*. Springer, 2008.

[3] The 1000 Genomes Project Consortium. "A global reference for human genetic variation," *Nature* vol. 526, p. 68-74, Oct 2015.

[4] LDlink. `https://ldlink.nci.nih.gov`.

63. Inference after Model Selection . . . . Dr Q. Zhao

It is well known that ignoring model selection may jeopardise the validity of statistical inference, but what should be done? Several approaches to inference after model selection have been proposed in the last decade, including simultaneous inference [1], conditional inference [2-5], sample splitting combined with bootstrap [6], and sparsity-based inference [7]. This essay can review the theory for some of these approaches, discuss the different philosophies, or/and compare the empirical performance through numerical simulations.

Relevant Courses

Essential: Modern Statistical Methods.

References

[1] R. Berk and L. Brown and A. Buja and K. Zhang and L. Zhao. "Valid post-selection inference." *Annals of Statistiscs* 41 (2) 802-837, 2013.

[2] W. Fithian and D. Sun and J. E. Taylor. "Optimal inference after model selection". arXiv:1410.2597, 2014.

[3] J. D. Lee and D. L. Sun and Y. Sun and J. E. Taylor. "Exact post-selection inference, with application to the lasso." *Annals of Statistics* 44 (3) 907-927, 2016.

[4] X. Tian and J. E. Taylor. "Selective inference with a randomized response". *Annals of Statistics*, 46 (2), 679-710, 2018.

[5] D. Kivaranovic and H. Leeb. "On the length of post-model-selection confidence intervals conditional on polyhedral constraints". *Journal of the American Statistical Association*, 116 (534) 1-13, 2020.

[6] A. Rinaldo and L. Wasserman and M. G'Sell. "Bootstrapping and sample splitting for high-dimensional, assumption-lean inference". *Annals of Statistics* 47 (6) 3438-3469, 2019.

[7] A. Belloni and V. Chernozhukov and C. Hansen. "Inference on treatment effects after selection among high-dimensional controls". *Review of Economic Studies*, 81 (2), 608-650, 2014.

64. The Hodge–Tate decomposition for abelian varieties ....
Rong Zhou

For a smooth projective complex variety X, a fundamental result concerning its cohomology is the Hodge decomposition, which asserts there is a canonical isomorphism $\mathrm{H}^n(X,\mathbb{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^p(X,\Omega_X^q)$. In the late sixties, Tate showed that the p-adic étale cohomology of an abelian variety over $\mathbb{Q}_p$ with good reduction satisfies a similar decomposition, upon base changing to $\mathbb{C}_p$. The result, now known as the Hodge–Tate decomposition, and which was proved in general by Faltings, became the starting point of p-adic Hodge Theory, a subject which has seen far-reaching applications in modern number theory.

The aim of this essay is to give an account of Tate's proof of the Hodge–Tate decomposition for abelian varieties with good reduction. The proof reduces to the case of p-divisble groups, and involves a careful study of certain ramified extensions of $\mathbb{Q}_p$. The essay should begin by covering the basic theory of abelian varieties, including the associated Tate-modules and p-divisible groups. For the ambitious, you may wish discuss the case of general abelian varieties (i.e. without the assumption of good reduction), which is due to Raynaud; a different argument was also given later by Fontaine.

Relevant Courses

Essential: Local Fields

Useful: Elliptic Curves.

References

[1] J. Tate, p-divisible groups, *Proceedings of a Conference on Local Fields,* 1966.

[2] J-M. Fontaine, Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux. *Inventiones Mathematicae*, 65(3):379–409, 1982.

[3] O. Brinon and B. Conrad, CMI summer school notes on p-adic Hodge Theory, available at *https://math.stanford.edu/ conrad/papers/notes.pdf.*

[4] G. Faltings, p-adic Hodge Theory, *J. Amer. Math. Soc*, 1(1):255–299, 1988.

65. Complex Multiplication and Kronecker's Jugendtraum ....
Rong Zhou

For a general number field K, class field theory allows us to understand the structure of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, where K^{ab} is the maximal abelian extension of K. However, the question of how to find explicit generators for K^{ab} remains open and was stated by Kronecker as his *liebster Jugendtraum* or the *dearest dream of his youth.*

The solution is known in the case of the rational numbers by the Kronecker–Weber Theorem, which states that $\mathbb{Q}^{\mathrm{ab}}$ is generated by cyclotomic extensions. The aim of this essay is to understand the generalization of this result to quadratic imaginary fields. By considering roots of unity as $\exp(\frac{2\pi i}{n})$, we may interpret the Kronecker–Weber theorem as saying abelian extensions of $\mathbb{Q}$ are generated by special values of the complex analytic function exp. For quadratic imaginary fields K, the corresponding result says that K^{ab} is generated by the special values of certain modular and elliptic functions associated to elliptic curves with complex multiplication.

The essay should include a thorough discussion of the theory of CM elliptic curves, including the Main Theorem of Complex Multiplication, as well as the statements of global class field theory. Time permitting, you may also wish to discuss the generalizations of these results to abelian varieties with CM.

Relevant Courses

Essential: Local Fields

Useful: Elliptic Curves, Modular Forms.

References

[1] J. Silverman, The Arithmetic of Elliptic Curves, Graduate Texts in Mathematics, 106.

[2] J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Graduate Texts in Mathematics, 151.

[3] G. Shimura, Abelian Varieties with Complex Multiplication and Modular Functions, Princeton Mathematics Series.

66. Obstructions to embeddings in the coarse geometry of Banach spaces . . A. Zsák

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ has *compression modulus* ρ_f and *expansion modulus* ω_f defined as $\rho_f(t) = \inf\{d_Y(f(x), f(y)) : d_X(x, y) \geq t\}$ and $\omega_f(t) = \sup\{d_Y(f(x), f(y)) : d_X(x, y) \leq t\}$ which satisfy $\rho_f(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \omega_f(d_X(x, y))$ for all $x, y \in X$. We say that f is a *uniform embedding* if $\lim_{t\to 0^+} \omega_f(t) = 0$ and $\rho_f(t) > 0$ for all $t > 0$ (or, equivalently, if f is uniformly continuous, injective with uniformly continuous inverse). We say that f is a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$.

Coarse embeddings need not be continuous or injective. Yet they play an important rôle in the nonlinear theory of Banach spaces. For example, Yu established a remarkable connection between the Novikov and Baum–Connes conjectures on the one hand and coarse embeddings into Hilbert space on the other. Kasparov and Yu generalized this to coarse embeddings into uniformly convex Banach spaces. For a while it was not known if a coarse embedding into a uniformly convex Banach space implies a coarse embedding into Hilbert space until Johnson and Randrianarivony proved that ℓ_p does not coarsely embed into ℓ_2 for $p > 2$ [1]. Their method uses a coarse invariant involving negative definite kernels. Later Mendel and Naor gave a complete description of coarse embeddings between L_p spaces [2]. This was one of several applications of their seminal work on metric cotype. The only remaining case was showing that L_p does not coarsely embed into L_q when $p > q$ and $p > 2$. Their proof deals with general K-convex spaces. A simpler proof for uniformly convex spaces is given in [6].

There is strong evidence that ℓ_2 is the hardest space to coarsely embed into which raised the question whether ℓ_2 coarsely embeds into every infinite-dimensional Banach space. Baudier, Lancien and Schlumprecht gave a negative answer in [4]. The obstruction they used is a coarse invariant in form of a concentration inequality for Lipschitz functions. This was inspired by a concentration inequality of Kalton and Randrianarivony used in their study of the coarse geometry of $\ell_p \oplus \ell_q$ [3]. Later Baudier, Lancien, Motakis and Schlumprecht used the concentration inequality of [4] to give the first example of a coarsely rigid *unrestricted* class of infinite-dimensional Banach spaces [5].

A successful essay should begin with proofs of the result in [1] and its extension in [6]. It should then develop the concentration inequalities in [3], [4] and [5] with applications that should include proofs of the uniqueness of uniform structure on $\ell_p \oplus \ell_q$ for $1 < p < 2 < q < \infty$ [3], the coarse non-embeddability of ℓ_2 into Tsirelson's original space [4] and the construction of the new coarsely rigid class in [5].

Relevant Courses

Essential: The first part of Functional Analysis covering standard Banach space results.

References

[1] Johnson, W.B., Randrianarivony, N.L.: ℓ_p $(p > 2)$ does not coarsely embed into a Hilbert space. *Proc. Am. Math. Soc.* 134(4) (2006) 1045–1050.

[2] Mendel, M., Naor, A.: Metric cotype. *Ann. Math.* (2) 168(1), (2008) 247–298.

[3] Kalton, N. and N. Randrianarivony: The coarse Lipschitz geometry of $\ell_p \oplus \ell_q$. *Math. Ann.* 341 (2008), 223–237.

[4] F. Baudier, G. Lancien and Th. Schlumprecht: The coarse geometry of Tsirelson's space and applications. *J. Amer. Math. Soc.* 31(3) (2018), 699–717.

[5] F. Baudier, G. Lancien, P. Motakis and Th. Schlumprecht: A new coarsely rigid class of Banach spaces. *J. Inst. Math. Jussieu* (2020), 1–19.

[6] Eskenazis, Alexandros; Mendel, Manor; Naor, Assaf: Nonpositive curvature is not coarsely universal. *Invent. Math.* 217 (2019), no. 3, 833–886.

67. Solutions to Local Anomaly Cancellation .... Prof B C Allanach

The cancellation of local anomalies is often imposed on gauged quantum field theories. General solutions to anomaly cancellation with fixed field content often borrow techniques from algebraic geometry. The essay should explore solutions to anomaly cancellation conditions in one or more particle physics settings, for example in the minimal supersymmetric standard model, other Standard Model extensions or in $U(1)$ gauge theory with Weyl fermions.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Symmetries, Fields and Particles, Advanced Quantum Field Theory

Useful:

References

[1] Lie Algebras In Particle Physics: from Isospin To Unified Theories, Frontiers in Physics, Howard Georgi, Frontiers in Physics, 2nd ed, (1999)

[2] S. P. Martin, Adv. Ser. Direct. High Energy Phys. **21** (2010) 1 [Adv. Ser. Direct. High Energy Phys. **18** (1998) 1]

[3] B.C. Allanach, B.G. Gripaios, J. Tooby-Smith, Phys. Rev. Lett. 125 (2020), 161601.

[4] Diophantine Equations, L.J. Mordell, Academic Press, (1969).

68. Graph Neural Networks as PDEs .... Dr AI Aviles-Rivero and Prof. C-B Schönlieb

Recently, the connection between Partial Differential Equations (PDEs) and Convolutional Neural Networks (CNNs) has gained increasing attention, see [1,2,3]. The main idea behind existing works is to treat CNNs as PDE solvers and to learn a discrete representation of PDEs. A limitation of current techniques is that the PDE solution needs to be discretised before feeding it to the neural network, resulting into often non-optimal discretisation approaches and not supporting continuous-time representation. To tackle this, a few works have explored the inherent connection between Graph Neural Networks (GNNs)[4] and PDEs. GNNs have gained popularity due to the graph-structure representation that naturally arises in real-world problems. In particular, such representation offers the flexibility to work on unstructured grids e.g.[5,6,7,8] paving the way for more efficient computational solutions.

Following these recent developments in GNNs and PDEs, we propose two main paths for this essay. The first path is to investigate PDE principles for understanding existing GNN architectures, and to create a broadly family of GNNs for solving more flexible tasks as in [7,8]. The second option is to explore how to learn PDE solvers using efficient graph representations whilst allowing further flexibility such as allowing for continuous-time and free-form spatial domain e.g.[6]. We expect the essay writer to develop a rigorous mathematical underpinning for one of these cases, and to discuss some open questions that they find interesting.

Relevant Courses

Essential: None

Useful: Background knowledge in Machine Learning, Statistics and PDEs is useful, as is probability to the level of Part II Applied Probability. Some content from graph theory may also be useful.

References

[1] Long, Z., Lu, Y., Ma, X. and Dong, B. PDE-Net: Learning PDEs from data. ICML, 2018.

[2] Lu, Y., Zhong, A., Li, Q. and Dong, B., 2018, July. Beyond finite layer neural networks: Bridging deep architectures and numerical differential equations. ICML 2018.

[3] Ruthotto, L. and Haber, E. Deep neural networks motivated by partial differential equations. Journal of Mathematical Imaging and Vision, 2020.

[4] Wu, Z., Pan, S., Chen, F., Long, G., Zhang, C. and Philip, S.Y. A comprehensive survey on graph neural networks. IEEE transactions on neural networks and learning systems, 2020.

[5] Li, Z., Kovachki, N., Azizzadenesheli, K., Liu, B., Bhattacharya, K., Stuart, A. and Anandkumar, A., Multipole graph neural operator for parametric partial differential equations. NeurIPS, 2020.

[6] Iakovlev, V., Heinonen, M. and Lähdesmäki, H.. Learning continuous-time PDEs from sparse data with graph neural networks, ICLR 2021.

[7] Eliasof, M., Haber, E. and Treister, E. PDE-GCN: Novel Architectures for Graph Neural Networks Motivated by Partial Differential Equations, NeurIPS, 2021.

[8] Chamberlain, B.P., Rowbottom, J., Gorinova, M., Webb, S., Rossi, E. and Bronstein, M.M., 2021. GRAND: Graph Neural Diffusion, ICML 2021.

69. Mathematical Phyllotaxis .... Dr L. Ayton

The appearance of Fibonacci numbers in plant structures, such as sunflower spiral counts, has fascinated mathematicians for centuries. Almost all recent work has been variation on a Standard Model in which organs are treated as point nodes successively placed on a cylinder according to a given model. The model hypotheses represent long-known botanical principles that new nodes cannot be placed too close or too far away from the existing nodes, and locally lead to lattice-like structures. As a parameter of the model, like the diameter of the cylinder, is changed, the lattice can transition to another, more complex lattice, with a different spiral count. It can often be proved that these transitions move lattice counts to higher Fibonacci numbers. While mathematically compelling, empirical validation of the Standard Model is as yet weak. There is also no complete mathematical characterisation of the classes of models which will generate Fibonacci structure.

This essay will discuss the mathematics of lattices on cylinders and might review the similarity in bifurcation tree structure of spiral counts from two or more different models (eg Douady 1998 and other sources in the same volume). Students with a leaning towards pure mathematics could discuss number theoretic links, such as between the bifurcations of lattice count pairs and an elegant conformal mapping of lattice space which allows a compact description of the possible bifurcation structure (Atela et al 2002).

More applied students could discuss the ways in which spiral counts 'obvious' to the eye have been mathematically modelled, or compare the mathematical assumptions of node placement models to classic 19th century biological hypotheses such as Hofmeister's Rule, and more modern molecular understandings (Godin et al 2020). Recent work (Golé and Douady 2020) has explored the role of noise and non-lattice patterns, and this could potentially provide scope for new numerical simulation work or evaluation of quantitative data (Swinton 2016).

Relevant Courses

Essential: The Mathematical Phyllotaxis section of Topics in Mathematical Biology. The text associated with this course, *A Textbook of Mathematical Phyllotaxis*, is available on the course Moodle and will be useful but is mostly does not contain primary research and should not be the principal subject of the essay.

Useful: A background in bifurcation theory, such as in the Part II Dynamical Systems course, may help provide perspective. The Part II Mathematical Biology course is not directly relevant.

References

- Douady, Stephane. 1998. 'The Selection of Phyllotactic Patterns'. In Symmetry in Plants, edited by Roger V. Jean and D. Barabé, 335–58. World Scientific. Several other contributions in this volume will also be of interest.
- Atela, Pau, Christophe Golé, and Scott Hotton. 2002. 'A Dynamical System for Plant Pattern Formation: A Rigorous Analysis'. Journal of Nonlinear Science 12 (6): 641–76. https://doi.org/10.1007/s00332-002-0513-1.
- Godin, Christophe, Christophe Golé, and Stéphane Douady. 2020. 'Phyllotaxis as Geometric Canalization during Plant Development'. Development (Cambridge, England) 147 (19): dev165878. https://doi.org/10.1242/dev.165878.
- Golé, Christophe, and Stéphane Douady. 2020. 'Convergence in a Disk Stacking Model on the Cylinder'. Physica D: Nonlinear Phenomena 403 (February): 132278. https://doi.org/10.1016/j.physd
- Swinton, Jonathan, Erinma Ochu, and 'The MSI Turing's Sunflower Consortium'. 2016. 'Novel Fibonacci and Non-Fibonacci Structure in the Sunflower: Results of a Citizen Science Experiment'. Royal Society Open Science 3 (5): 160091. https://doi.org/10.1098/rsos.160091.

70. Quantum Time Crystals .... Professor B. Béri

We are all familiar with the concept of crystals: they are systems that spontaneously break spatial translation symmetry. "Time crystals" can be imagined analogously: systems that spontaneously break time translation symmetry [1]. While groundstates (or thermal states) spontaneously breaking continuous time translation symmetry cannot exist [2], time crystals are possible in driven systems where time translation symmetry is discrete [3,4]. The purpose of this essay is to give a coherent account of time crystals in quantum many-body systems. The essay should include, but not necessarily be limited to, discussions on: the subtleties involved in defining quantum time crystals (including the no-go theorem on continuous time translation symmetry breaking); how time crystals can persist, even without dissipation, instead of heating up to infinite temperature; why they might be well suited for noisy intermediate scale quantum devices [5]; and whether they have been realised in such devices in recent experiments [6].

Relevant Courses

Essential: None

Useful: Part II Quantum Information and Computation, Part III Statistical Field Theory

References

[1] A. Shapere and F. Wilczek, *"Classical Time Crystals"*, Phys. Rev. Lett. **109**, 160402; F. Wilczek *"Quantum Time Crystals"*, *ibid.*, 160401 (2012) [arXiv:1202.2537, 1202.2539].

[2] H. Watanabe and M. Oshikawa, *"Absence of Quantum Time Crystals"*, Phys. Rev. Lett. **114**, 251603 (2015) [arXiv:1410.2143].

[3] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, *"Phase Structure of Driven Quantum Systems"*, Phys. Rev. Lett. **116**, 250401 (2016) [arXiv:1508.03344]; D. V. Else,

B. Bauer, and C. Nayak, *"Floquet Time Crystals"*, Phys. Rev. Lett. **117**, 090402 (2016) [arXiv:1603.08001].

[4] For reviews, see D. V. Else, C. Monroe, C. Nayak, N. Y. Yao, *"Discrete Time Crystals"*, Annu. Rev. Cond. Mat. Phys. **11**, 467 (2020) [arXiv:1905.13232]; V. Khemani, R. Moessner, and S. L. Sondhi, *"A Brief History of Time Crystals"*, arXiv:1910.10745.

[5] M. Ippoliti *et al.*, *"Many-Body Physics in the NISQ Era: Quantum Programming a Discrete Time Crystal"*, PRX Quantum **2**, 030346 (2021).

[6] J. Randall *et al.*, *"Observation of a many-body-localized discrete time crystal with a programmable spin-based quantum simulator"*, arXiv:2107.00736; Google Quantum AI and collaborators, *"Observation of Time-Crystalline Eigenstate Order on a Quantum Processor"*, arXiv:2107.13571.

71. The Hadamard Condition in Quantum Field Theory on Curved Spacetime
Dr J.N. Butterfield

Quantum field theory in curved spacetime is a large subject, encompassing famous and important effects such as the Unruh and Hawking effects; cf. e.g. Parker and Toms (2009, especially Chapters 1-4). This essay focusses, not on these effects, but on some mathematical notions and results. They are central to the algebraic approach to the subject, and are of foundational interest, since they concern how one can formulate the theory without being able to invoke the symmetries of Minkowski spacetime (and so without preferred notions of vacuum and particle). The essay can specialize to the case of the Klein-Gordon field on a globally hyperbolic spacetime: which is the setting for which these notions and results have been most studied.

One central notion is the Hadamard condition on states: which says, roughly speaking, that the singularity structure of 2-point functions, as the two spacetime points get arbitrarily close, should be appropriately similar to that in flat spacetime. From the early 1990s, this condition became better understood through the use of microlocal analysis: beginning with Radzikowski's 1992 PhD proving Kay's conjecture (from 1988) that, roughly speaking, "local Hadamard implies global Hadamard": cf. Radzikowski (1996).

Over the next ten years, this led the way to a better understanding of the divergences of quantum field theory on curved spacetime, and to the definition of Wick polynomials as local and covariant operator-valued distributions; and more generally, to formulations of interacting quantum field theory on curved spacetime with a completeness and rigour matching those on Minkowski spacetime. The aim of the essay is to review these developments: for which I suggest taking the Hadamard condition as a guiding thread.

These developments are introduced, partly in historical terms, by Kay (2006) and Wald (2008). Hollands and Wald (2014) is a precise and pedagogic review. The state of the art in the early 1990s was expounded by Wald (1994, especially Chapter 4.6). Important original papers exploiting microlocal analysis include Hollands and Wald (2001, 2002, 2003). For orientation to these last three papers, note that: (i) Section 3 of (2001) makes precise the idea of local and covariant distributions, and its Sections 4, 5 discuss scaling and renormalization, which is the topic of (2003); (ii) (2002) is mostly technical, but its Sections 1, 2, 5 and appendices give overviews.

Relevant Courses

Essential: None

Useful: General relativity, Black holes, Philosophical aspects of quantum field theory, Philosophical aspects of QFT on curved spacetimes

References

Hollands., S and Wald, R. (2001), Local Wick polynomials and time ordered products of quantum fields in curved space-time, *Communications in Mathematical Physics* **223**, 289-326; arxiv gr-qc/0103074

Hollands., S and Wald, R. (2002), Existence of local covariant time ordered products of quantum fields in curved space-time, *Communications in Mathematical Physics*, **231**, 309-345; (DOI) 10.1007/s00220-002-0719-y ; arxiv gr-qc/0111108

Hollands., S and Wald, R. (2003), On the renormalization group in curved space- time, *Communications in Mathematical Physics*, **237**, 123, arxiv gr-qc/0209029

Hollands., S and Wald, R. (2014), Quantum Fields in curved spacetime; arxiv 1401.2026

Kay, B. (2006), Quantum field theory in curved spacetime, in *Encyclopedia of Mathematical Physics*, J.-P. Francoise, G. Naber and T.S. Tsou, eds.: Vol. 4, 202-212, Academic (Elsevier), arxiv: gr-qc/0601008

Radzikowski, M. (1996). Micro-local approach to the Hadamard condition in quantum field theory on curved spacetime, *Communications in Mathematical Physics*, **179**, 529-553.

Wald, R. (1994), *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press.

Wald, R. (2008), The history and present status of quantum field theory in curved spacetime, in Lehner C. et al. eds. *Einstein and the Changing Worldviews of Physics*, Einstein Studies Volume 12, Birkhauser; pp. 317-332. DOI 10.1007/978-0-8176-4940-1; arxiv: gr-qc/0608018

72. The hole argument and gauge equivalence in general relativity .... Dr. J. N. Butterfield

The hole argument, originally devised by Einstein in late 1913, as an argument *against* general covariance, forms a crossroads of the history, philosophy and indeed physics of general relativity.

After Einstein had in 1913 devised (with Grossmann) a theory (nowadays called ‘the Entwurf Theory’) satisfying all his desiderata for a relativistic theory of gravitation except general covariance, he came to doubt general covariance. He soon confirmed his doubts by inventing this argument, to the effect that *any* generally covariant theory would be radically indeterministic, since specifying all matter fields, and the metric field, throughout all of spacetime except a small region (the “hole”) could not determine the fields in the region. For general covariance means that, given one model i.e. solution of the theory, a “re-painting” of the matter and metric on the region’s set of points, by pushing forward with a diffeomorphism of the region, produces another model of the theory. And this model disagrees with the given model about which point has a given value of the matter and metric fields. Late in 1915, after Einstein had found the field equations of general relativity, which are of course generally covariant, he re-assessed the argument as showing only that we should not think of spacetime points as objects independent of their field values. For this history, and later debate about the meaning of general covariance, cf. [1].

This episode’s relevance to the interpretation of general relativity was advocated in papers by Stachel, Earman and Norton in the 1980s; since when, it has been centre-stage in philosophical debates about determinism, the comparison of spacetime points in different models ("which point is which?"), and 'absolute' vs. 'relational' conceptions of spacetime [2].

In physics, the main legacy of this episode has been, broadly speaking, to recognize 'gauge-freedom' about 'which spacetime point is which' in formulations of general relativity. This applies in particular to the initial-value problem: in short, one takes the future time-development from given initial data to be unique, only up to a diffeomorphism. Cf. the expositions in [3]. But there are subtleties. For in some contexts within general relativity, it is necessary to *not* identify spacetime points that are related by an isomorphism of models: that is, for the simple case of Lorentzian manifolds (M, g) (i.e. setting aside matter), points that are mapped, one to the other, by an isometry of the manifolds. In the jargon of the hole argument: in some contexts, we do *not* "drag points along" ("identify points across possible worlds") by the "hole-diffeomorphism".

One such context is when one spacetime is a limit of a sequence of others, in the sense developed by Geroch [4]. The idea is to consider a 1-parameter family of spacetimes M_λ $(\lambda > 0)$ (*not* a 1-parameter family of metrics on a given manifold), and then to make each M_λ a sub-manifold (a "leaf") of a 5-dimensional manifold $\mathcal{M}$, with λ parameterising the foliation; so that the limit spacetime is defined as, roughly, the boundary ∂M of $\mathcal{M}$. To write down limits, one uses a family of frames (tetrads) on the M_λ; and this invokes a congruence of curves in $\mathcal{M}$, each of which cuts each leaf once (so each curve is parameterised by λ). Thus points in different M_λ correspond other than by isomorphism (in particular, isometry).

Another such context concerns spacetimes that are asymptotically flat at spatial infinity. There are different ways to make this notion precise, but several of them agree in the verdict that two asymptotically flat spacetimes can be isomorphic but not gauge equivalent. That is: while a diffeomorphism that asymptotes to be a time-translation at spatial infinity is agreed to yield an isomorphic model, it only yields a gauge equivalent model (i.e. representing the same physical situation) if it asymptotes to a *zero* time-translation, i.e. the identity map, at spatial infinity: [5].

The purpose of the essay will be to review these debates; and if possible, to defend positions therein—especially about the central question, viz. under what conditions should one identify spacetime points related by an isomorphism of models.

Relevant Courses

Essential: None

Useful: General relativity, black holes

References

[1] J. Norton (1984) How Einstein found his Field Equations: 1912-1915, *Historical Studies in the Physical Sciences*, 14, 253-316; reprinted in Don Howard and John Stachel (eds.) *Einstein and the History of General Relativity*: Einstein Studies, Vol. 1 Boston: Birkhauser, 1989, pp. 101-159. Norton, General covariance and foundations of general relativity: eight decades of dispute, *Reports on Progress in Physics*, **56** (1993), 791-858.

[2] For an introduction, cf. J. Norton (2019), The hole argument, in *Stanford Encyclopedia of Philosophy* (internet), https://plato.stanford.edu/entries/spacetime-holearg/. For a detailed exposition of both the history and the philosophy, cf. J. Stachel (2014), The hole argument and some physical and philosophical implications. *Living Reviews in Relativity* 17, 1-66.

https://link.springer.com/article/10.12942/lrr-2014-1. The article which began the current discussion in philosophy is: J. Earman and J. Norton, (1987) What Price Spacetime Substantivalism *British Journal for the Philosophy of Science*, **38**, 515-525. One early response, discussing different definitions of determinism, is: J. Butterfield, The Hole Truth, *British Journal for the Philosophy of Science*, **40**, 1-28. A brief recent survey is: O. Pooley, The hole argument, arXiv:2009.09982; of which a more ample successor is: O. Pooley and J. Read, The mathematics and metaphysics of the hole argument, arXiv:2110.04036.

[3] R. Wald, *General Relativity*, Chap.10, University of Chicago Press, 1984; S. Hawking and G. Ellis, *The Large-scale Structure of Spacetime*, Chapter 7, Cambridge University Press; Klaas Landsman, *Foundations of General Relativity*, Chapter 8, Radboud University Press, freely downloadable at: www.radbouduniversitypress.nl; E. Gourgoulhon, 3+1 Formalism and Bases of Numerical Relativity, grqc/0703035, Section 8.

[4] R. Geroch, Limits of spacetimes, *Communications in Mathematical Physics*, **13** 1969, 180-193. Cf. Section 3 of: E. Curiel, the existence of spacetime structure, *British Journal of Philosophy of Science*, **69** 2018, 447-483.

[5] For introduction and references, cf. Section 4 of: G. Belot (2018), Fifty million Elvis fans can't be wrong *Nous* **54**, 946-981; http://philsci-archive.pitt.edu/12779/. Belot focusses mostly on the approach of: Ashtekar et al., (1991), The Covariant phase space of asymptotically flat gravitational fields, in M.Francavaglia ed., *Mechanics, Analysis and Geometry: 200 years after Lagrange*, 417-449.

73. Discrimination of quantum states and channels .... Professor N. Datta, R. Salzmann

Discriminating between two states of a quantum system is a fundamental constituent of many quantum information theoretic tasks and is studied rigorously in the field of binary quantum hypothesis testing. Most often one considers either the so-called *symmetric setting*, in which the two possible errors incurred in the task (the so-called type I and type II errors) are treated on an equal footing, or the so-called *asymmetric setting*, in which one minimizes the type II error probability under the constraint that the corresponding type I error probability is below a given threshold.

If multiple independent copies of the states are available, the corresponding optimal error probabilities in both settings are known to decay exponentially to zero as the number of copies tend to infinity. The corresponding exponential decay rates are given in terms of well-known quantum divergences, namely, the quantum Chernoff divergence [1,5] and the quantum relative entropy [3] respectively. These in turn impart and operational meanings to these divergences (see [7,4] for an overview).

An important variant of the above mentioned hypothesis testing task is the discrimination task of two quantum channels. Here, one differentiates between so-called *parallel strategies*, in which all copies of the corresponding channel are used in parallel, and the more general class of *adaptive strategies*. Recently, in the asymmetric setting of channel discrimination, the corresponding optimal exponential decay rates, both for the parallel and adaptive strategies, have been expressed in terms of the quantum relative entropy for channels [8, Theorem 3 & Theorem 6]. By establishing a chain rule for the relative entropy [2], it has then been shown that these two decay rates are in fact equal, implying that asymptotically there is no advantage of adaptive strategies over parallel ones. On the other hand, in the symmetric regime, using adaptive strategies can give an advantage over parallel strategies [6]. Finding entropic characterisations of the corresponding decay rates in the symmetric regime, however, still remains an open problem.

The aim of the essay is to review the results on binary quantum hypothesis testing for states and channels. If time permits, the essay could discuss an operational understanding/proof of why adaptive strategies do not offer an advantage over parallel ones in the asymmetric regime, and compare it with the symmetric regime. Moreover, trying to think of entropic characterisations for the optimal decay rates for symmetric channel discrimination would also be an interesting route to follow.

Relevant Courses

Essential: Part III Quantum Information Theory

References

[1] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete. Discriminating states: The quantum chernoff bound. Physical Review Letters, 98:160501, 2007.

[2] K. Fang, O. Fawzi, R. Renner and D. Sutter, Chain Rule for the Quantum Relative Entropy, Physical Review Letters, 124:100501, 2020.

[3] F. Hiai and D. Petz. The proper formula for relative entropy and its asymptotics in quantum probability. Communications in Mathematical Physics, 143(1):99–114, Dec 1991.

[4] S. Khatri, M. M. Wilde, Principles of Quantum Communication Theory: A Modern Approach, arXiv:2011.04672, 2020.

[5] M. Nussbaum and A. Szkola. The Chernoff lower bound for symmetric quantum hypothesis testing. The Annals of Statistics, 37(2):1040 – 1057, 2009.

[6] F. Salek, M. Hayashi, A. Winter, When are Adaptive atrategies in asymptotic quantum channel discrimination useful? arxiv:2011.06569, 2020.

[7] R. Salzmann and N. Datta, Interpolating between symmetric and asymmetric hypothesis testing, arXiv:2104.09553, 2021.

[8] X. Wang and M. M. Wilde, Resource theory of asymmetric distinguishability for quantum channels, Physical Review Research 1, 033169, 2019.

74. The family tree of quantum information theoretic protocols .... Professor N. Datta, Bjarne Bergh

An important problem in Quantum Information Theory is finding optimal protocols for various information processing tasks. These include for example data compression, information transmission, entanglement manipulation, state merging, channel simulation and many more. Initially, all these different protocols were considered to be unrelated and each one of them was studied individually, until it was realized that many of them could be related to each other by direct application of quantum teleportation and superdense coding. Eventually, a "family tree" of protocols was created where ancestral protocols can be used to perform the task of their children [1, 2, 3], starting from a single root protocol. This is naturally expressed in terms of a resource inequality formalism that expresses how quantum information theoretic resources can be transformed into each other.

Proving the optimality of many of these protocols originally employed a so-called *decoupling theorem*. While the idea of decoupling is very powerful, however, it needs to be tailored to separately to each information-theoretic task. Recently, it has been shown that decoupling can often be replaced by the usage of what is called the "convex-split lemma" and the "position-based-decoding" technique, which are usable much more as a black-box and have simplified many arguments [4,5].

Students taking this essay should first review the family tree structure of quantum information theoretic tasks in a comprehensive manner, showing how they are related. Secondly, you would explore how convex-split/position-based-decoding techniques can be used to construct optimal protocols. Ideally you would arrive at an overview detailing what is known and achievable using state-of-the-art techniques, but there is also the possibility of exploring new applications of these techniques.

Relevant Courses

Essential: Part III Quantum Information Theory

References

[1] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, 'The mother of all protocols: Restructuring quantum information's family tree', Proc. R. Soc. A., vol. 465, no. 2108, pp. 2537–2563, Aug. 2009, doi: 10.1098/rspa.2009.0202, arXiv: quant-ph/0606225.

[2] N. Datta and M.-H. Hsieh, 'The apex of the family tree of protocols: Optimal rates and resource inequalities', New J. Phys., vol. 13, no. 9, p. 093042, Sep. 2011, doi: 10.1088/1367-2630/13/9/093042, arXiv: 1103.1135.

[3] I. Devetak, A. W. Harrow, and A. Winter, "A Resource Framework for Quantum Shannon Theory," IEEE Trans. Inf. Theory, 54:4587–4618, 2008.

[4] A. Anshu, R. Jain, and N. A. Warsi, 'One shot entanglement assisted classical and quantum communication over noisy quantum channels: A hypothesis testing and convex split approach', IEEE Trans. Inform. Theory, vol. 65, no. 2, pp. 1287–1306, Feb. 2019, doi: 10.1109/TIT.2018.2851297, arXiv:1702.01940.

[5] Anshu, Anurag, Rahul Jain and Naqueeb Ahmad Warsi. "A Generalized Quantum Slepian–Wolf." IEEE Transactions on Information Theory 64 (2018): 1436-1453.

75. Twistor Transform .... M. Dunajski

One of the most remarkable achievements of Penrose's twistor program is the link it provides between solutions to certain linear and non-linear differential equations of mathematical physics and unconstrained holomorphic geometry of the twistor space. The essay would review the subject concentrating on linear massless fields four space-time dimensions. You should also explore one (or more!) of the following:

(a) Mathematical aspects of the construction such as isomorphisms between sheaf cohomology classes and massless fields.

(b) Connections with integrability.

(c) Generalisations to non-abelian gauge fields.

Relevant Courses

Solitons, Instantons and Geometry or Differential Geometry. A firm knowledge of basic complex analysis (e.g. IB Complex Methods, or Complex Analysis) is essential.

References

[1] Penrose, R. (1969), Solutions of the zero-rest-mass equations. J. Math. Phys. **10**, 38.

[2] Dunajski, M. (2009) Solitons, Instantons and Twistors. Oxford Graduate Texts in Mathematics **19**, Oxford University Press.

[3] Huggett, S. A. and Tod, K. P. (1994) An Introduction to Twistor Theory, LMS Student Texts 4 (second edition).

[4] Witten, E. (2003) Perturbative Gauge Theory As A String Theory In Twistor Space, hep-th/0312171.

[5] Atiyah, M., Dunajski, M. and Mason, L. (2017) Twistor theory at fifty: from contour integrals to twistor strings. `arXiv:1704.07464`. Proceedings of the Royal Society, **473**. 1

[6] Penrose, R. Rindler, W. (1986) *Spinors and Space–time* Vol I, II (Chapter 9). CUP.

76. Modelling building ventilation in the light of Covid-19 .... Dr D. Frank and Professor S. B. Dalziel

It is now generally accepted that airborne transmission plays a very important role for the SARS-CoV-2 coronavirus responsible for Covid-19. Tiny aerosol particles are emitted when someone breathes, speaks, coughs, etc., and these can remain suspended in the ambient air for a prolonged period of time. If the person is infected with Covid-19, then these aerosols can contain the virus and someone else breathing them in can lead to that person becoming infected. This potentially increases the risks of infection in enclosed indoor environments in which the existing ventilation flows are poorly configured or inadequate to remove the airborne contaminants in an effective manner. This route of infection is a major concern going forwards and remains one of the most challenging aspects of controlling transmission as most existing buildings have not been designed with airborne transmission in mind.

An indoor space can be ventilated naturally or ventilated mechanically by an air-conditioning system. The natural ventilation strategy relies on the existing air flows inside and outside the building that, for example, arise due to the heat emitted by sunlight, electrical appliances or the occupants, generating temperature differences between indoors and outdoors. In addition, the external wind can drive an airflow through the building. A significant advantage of the natural ventilation method is that it considerably improves the energy efficiency of buildings when compared with mechanical ventilation. In either case, the exact ventilation flow patterns developing in an indoor environment depend on a variety of factors such as the location of the air supply and exhaust vents, configuration of windows and doors, heating systems, building occupancy and outdoor conditions. How the ventilation flows combine with the exhaled respiratory droplets and aerosols is currently an open research question, as is which mitigation measures can be applied to control the spread of airborne contaminants.

An essay on this topic should start with a review of well-established mathematical models for ventilation. This review should be followed by a discussion of the implications of the ventilation flow patterns for the spread of airborne contaminants. The candidate may choose to highlight open questions in this research field or discuss possible mitigation strategies for the transmission of infectious aerosols (e.g. masks, negative pressure wards or aerodynamical sealing). If the candidate wishes, they may conduct a case study of a particular ventilation problem by means of CFD numerical simulations or using reduced model equations. The candidate may decide the direction of the essay depending on their interests, although there should be a mathematical flavour to the material presented. The references cited below offer a starting point for further reading.

Relevant Courses

Essential: Undergraduate Fluid Dynamics

Useful: Fluid Dynamics of the Environment

References

[1] T. Chenvidyakarn (2013). "Buoyancy Effects on Natural Ventilation". Cambridge: Cambridge University Press.

[2] P. F. Linden (1999). "The fluid mechanics of natural ventilation". *Annual Review of Fluid Mechanics* 31, 201-238

[3] R. K. Bhagat, M. S. D. Wykes, S. B. Dalziel, and P. F. Linden (2020). "Effects of ventilation on the indoor spread of COVID-19". *Journal of Fluid Mechanics* 903 F1, 1-18

77. Does the quantum state directly represent physical reality? .... Dr D. Pitalúa-García

The quantum state is the main mathematical object in quantum theory. However, its relation to physical reality is not completely understood. On the one hand, assuming that there cannot be instantaneous disturbance on a distant physical system, it was argued by Einstein, Podolsky and Rosen [1] that the quantum state does not provide a complete description of physical reality. The EPR paper led to Bell's theorem [2], stating that there are not locally causal hidden variable theories reproducing all the predictions of quantum theory. On the other hand, under some physical assumptions, for example that physical systems that are prepared independently have independent physical states, Pusey, Barrett and Rudolph [3] showed that the quantum state represents a state of physical reality. It is debatable whether the assumptions made in the PBR theorem are physically sensible. Different versions of the PBR theorem have been found, for example in Refs. [4,5], making different physical assumptions. By dropping some of the assumptions made in these theorems, physical models can be constructed in which the quantum state does not represent physical reality directly [6].

An ideal essay will review the literature on this research area, showing a clear understanding of the PBR theorem [3], and of its interconnection with the EPR argument [1] and with Bell's theorem [2]. The discussion of Refs. [4–6] is not expected to be in great detail, but should give a broader perspective on the PBR theorem.

Relevant Courses

Essential: None

Useful: Familiarity with the Postulates of Quantum Mechanics and the Dirac notation, for example, at the level of the Part II course Principles of Quantum Mechanics or the Part II course Quantum Information and Computation, would be useful.

References

[1] A. Einstein, B. Podolsky and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," Physical Review **47**, 777–780 (1935).

[2] J. S. Bell, "On the Einstein-Podolsky-Rosen Paradox," Physics **1**, 195–200 (1964).

[3] M. F. Pusey, J. Barrett and T. Rudolph, "On the reality of the quantum state," Nature Physics **8**, 475–478 (2012).

[4] J. Barrett, E. G. Cavalcanti, R. Lal and O. J. E. Maroney, "No ψ-epistemic model can fully explain the indistinguishability of quantum states," Physical Review Letters **112**, 250403 (2014).

[5] O. J. E. Maroney, "How statistical are quantum states?," arXiv:1207.6906v2 (2013).

[6] P. G. Lewis, D. Jennings, J. Barrett and T. Rudolph, "Distinct quantum states can be compatible with a single state of reality," Physical Review Letters **109**, 150404 (2012).

78. Matrix Quantum Mechanics .... Prof. S. A. Hartnoll

Take a large number N of quantum harmonic oscillators and couple them in a $U(N)$ invariant way. The resulting theory is called a matrix quantum mechanics. As $N \to \infty$ this theory turns out to have an emergent spatial dimension, and gives perhaps the simplest explicit example of emergent space and gravity.

This essay will start with a description of matrix quantum mechanics, its solution in terms of free fermions [1] (note that section 5 of this paper, about the quantum mechanical theory, is the most directly relevant, but builds on the earlier sections), and its relation to an emergent dimension as seen via the collective field approach [2] and via random triangulations of a string worldsheet [3].

The essay will then go on to discuss one or more advanced topics in matrix quantum mechanics. These could be, for example, the non-singlet sector of matrix quantum mechanics, matrix quantum mechanics on a Euclidean time circle and black holes, scattering and integrability in the spacetime theory, D-branes in two dimensional string theory, entanglement entropy in matrix quantum mechanics. The essay should focus on the matrix quantum mechanics of a single matrix (also called the "c=1" theory or "string theory in two dimensions"). It may, however, optionally include a brief discussion of the importance and difficulties of generalizing the theory to more than one matrix.

Relevant Courses

Essential: Quantum Field Theory.

Useful: General Relativity, Advanced Quantum Field Theory, String Theory, Gauge/Gravity Duality.

References

[1] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[2] S. R. Das and A. Jevicki, “String Field Theory and Physical Interpretation of $D = 1$ Strings,” Mod. Phys. Lett. A **5**, 1639-1650 (1990).

[3] I. R. Klebanov, “String theory in two-dimensions,” [arXiv:hep-th/9108019 [hep-th]].

79. Spontaneous Generation of Gravity Waves .... Professor Peter Haynes

Large-scale atmospheric and oceanic flows are typically close to geostrophic balance (Coriolis force balancing pressure gradient) and the ‘slow’ evolution of such flows is, at least qualitatively, described by the quasi-geostrophic equations which exclude ‘fast’ waves, i.e. internal inertia-gravity waves. The Rossby adjustment problem provides a canonical description of how a fluid evolves from an arbitrary initial condition, through the emission of inertia-gravity waves (or Poincaré waves in the shallow-water equation version), to a state of geostrophic balance. (It is convenient to use the general term ’gravity waves’ to describe these waves.)

The separation between ‘slow’ and ‘fast’ motion is not perfect. A flow that is initially geostrophically balanced and contains no gravity-wave motion may evolve continuously in time to reach a state where there is a significant amount of gravity wave activity. This phenomenon is observed in the real atmosphere and ocean and in mathematical models and is typically described as ’spontaneous generation’ of gravity waves.

Understanding of spontaneous generation has improved markedly over the last 20 years or so. Early numerical simulations appeared to show spontaneous generation but there were doubts over whether this apparent generation was a numerical artifact and would not appear in a real fluids and also over whether gravity waves had been completely excluded from the initial conditions. These doubts have now largely been resolved. Alongside this there has been progress in constructing relevant mathematical models and analysing their behaviour. Furthermore it is recognised that these spontaneously emitted waves may have important effects, e.g. triggering convection in the atmosphere or mixing events in the ocean.

It is suggested that reading for an essay on this topic starts with the reviews by Vanneste [1] and Plougonven and Zhang [2], the former putting more emphasis on a mathematical description of spontaneous generation, the latter more concerned with the phenomenon of spontaneous gravity-wave generation in the atmosphere. The essay itself should begin with a general review of spontaneous generation, defining clearly what it is, explaining why it is potentially important and theoretically challenging and summarising some of the key results on which current understanding is based. The essay could then move on to survey a particular sub-topic, for example simple mathematical models and accompanying theory (e.g. Vanneste and Yavneh [3]), or numerical simulation (e.g. Plougonven and Snyder [4]), or spontaneous generation in a particular flow configuration such as frontogenesis (e.g. Shakespeare and Taylor [5]).

Relevant Courses

Essential: Undergraduate course in fluid dynamics.

Useful: Fluid Dynamics of Climate, Fluid Dynamics of the Environment. (Neither is essential, but this essay topic does relate closely to parts of Fluid Dynamics of Climate and anyone considering this essay who is not taking that course is advised – and welcome - to discuss with the setter.)

References

[1] Vanneste, J., 2013. Balance and Spontaneous Wave Generation in Geophysical Flows. Annu. Rev. Fluid Mech. 2013. 45:147–72.

[2] Plougonven, R., Zhang, F., 2014. Internal gravity waves from atmospheric jets and fronts, Rev. Geophys., 52, 33–76.

[3] Vanneste, J, Yavneh, I., 2004. Exponentially small inertia-gravity waves and the breakdown of quasi-geostrophic balance. J. Atmos. Sci. 61, 211–23.

[4] Plougonven, R, Snyder, C., 2007. Inertia-gravity waves spontaneously excited by jets and fronts. Part I: Different baroclinic life cycles. J. Atmos. Sci. 64:2502–20.

[5] Shakespeare, C.J., Taylor, J.R., 2015: The spontaneous generation of inertia-gravity waves during frontogenesis forced by large strain: Numerical solutions. J. Fluid Mech., 772, 508

80. Computer Design of Quantum Experiments ....
Professor Adrian Kent

A quantum experiment can be characterized as an algorithm for producing and measuring a specified quantum state, using specified resources. Even for states that have a relatively simple mathematical description, the relevant algorithms can be quite complex, and efficient algorithms can be hard to find using human intuition alone. This has inspired work on computer programmes (e.g. [1,3]) that effectively design experiments and, in some cases, explain their working in terms accessible to human intution. These programmes have been applied, for example, to design and implement (e.g. [2]) experiments that produce entangled states with specific features as simply as possible and to develop theoretical insights (e.g. [3-5]).

An essay should review developments in this area, with reference to the underlying theory, explaining and assessing what has been achieved to date. In particular, the essay should discuss relevant aspects of the theory of entanglement and of experimental techniques used to generate entangled states. Although some related material is discussed in the Part III courses listed below, this will require some independent reading, guided by the references in the papers listed below.

Although it is not required, one possible direction would be to use the available software [6,7] to illustrate its functionality and limitations by carrying out independent computations.

Relevant Courses

Essential:

Useful:

Quantum Information Theory; Quantum Information, Foundations and Gravity; Quantum Computation

References

[1] Automated Search for new Quantum Experiments, M. Krenn et al., PRL 116, 090405 (2016)

[2] Observation of nonlocal quantum interference between the origins of a four-photon state in a silicon chip, L.-T. Feng et al., arxiv:2103.14277

[3] Conceptual Understanding through Efficient Automated Design of Quantum Optical Experiments, M. Krenn et al., Phys. Rev. X 11, 031044 (2021)

[4] Quantum Experiments and Graphs: Multiparty States as Coherent Superpositions of Perfect Matchings, M. Krenn et al., PRL 119, 240403 (2017)

[5] Quantum experiments and graphs II: Quantum interference, computation, and state generation, X. Gu et al., PNAS 116 10 4147–4155

[6] https://github.com/XuemeiGu/MelvinPython

[7] https://github.com/aspuru-guzik-group/Theseus

81. Functional analysis of neural networks: approximation rates for vector-valued neural networks ....

Dr Y. Korolev

Universal approximation properties of various types of neural networks have been known since the late 1980's. But it has also been shown that the approximation rates in terms of the number of neurons scale exponentially with the dimension of the input space. However, certain types of functions can be approximated with dimension-independent Monte-Carlo rates [1]. The functional-analytic study of the spaces of such functions has recently become an active area of research [2-5].

The existence of dimension-independent rates suggests that such rates can also be obtained in the infinite-dimensional setting, that is, when both the input space and output space are infinite-dimensional Hilbert or Banach spaces. The study of neural networks as nonlinear operators between infinite-dimensional spaces has recently gained attention [6,7].

The goal of this essay is to provide a comprehensive review of existing literature on approximation rates for neural networks (and more generally, functional analysis of neural networks), with a particular emphasis on results related to neural networks acting between infinite-dimensional spaces, and exploring directions for future work in this area. There is a fair degree of flexibility in this essay. A good review of the state of the art is given in [8].

Relevant Courses

Essential: Functional Analysis or equivalent (Banach space theory)

Useful: Probability and Measure, Convex Analysis (neither is a requirement, and you may prefer to take a different viewpoint)

References

[1] A. Barron. "Universal approximation bounds for superpositions of a sigmoidal function". In: IEEE Transactions on Information Theory 39.3 (1993), pp. 930–945.

[2] F. Bach. "Breaking the Curse of Dimensionality with Convex Neural Networks". In: Journal of Machine Learning Research 18.19 (2017), pp. 1–53.

[3] W. E, C. Ma, and L. Wu. "Barron Spaces and the Compositional Function Spaces for Neural Network Models". arxiv:1906.08039. 2019.

[4] W. E and S. Wojtowytsch. "Representation formulas and pointwise properties for Barron functions". arxiv:2006.05982. 2020.

[5] F. Bartolucci, E. De Vito, L. Rosasco, and S. Vigogna. "Understanding neural networks with reproducing kernel Banach spaces". arXiv: 2109.09710. 2021.

[6] N. H. Nelsen and A. M. Stuart. "The Random Feature Model for Input-Output Maps between Banach Spaces". arXiv:2005.10224. 2020.

[7] Y. Korolev. "Two-layer neural networks with values in a Banach space". arXiv:2105.02095. 2021.

[8] W. E, C. Ma, S. Wojtowytsch, and L. Wu. "Towards a Mathematical Understanding of Neural Network-Based Machine Learning: what we know and what we don't". arxiv:2009.10713. 2020.

82. Convective instabilities in galaxy clusters ....
Dr H. N. Latter

Galaxy clusters are gravitationally bound astrophysical structures comprising hundreds, even thousands, of galaxies. Well known examples 'near' us include the Virgo, Coma, and Hercules clusters. A characteristic feature of these structures is the extremely hot ($\sim 10^7$ K) ionised gas that permeates the space between the galaxies. This gas, referred to as the intracluster medium (ICM), is weakly collisional, and as a consequence the conduction of heat and momentum is anisotropic, aligning itself with the local magnetic field. The anisotropy of the heat transport, in particular, is associated with two unusual 'convective' instabilities: the magnetothermal instability and the heat-flux buoyancy instability (MTI and HBI) (see references [1], [2], and [3]). Currently, researchers are testing how these (and the disordered flows they initiate) influence the global structure and properties of galaxy clusters.

In this essay you should discuss the basic physics of weakly collisional and magnetised plasma, and then review the linear theory of the MTI and HBI, paying attention to how the familiar convective stability results are altered by the anisotropic heat flux. You may then survey the nonlinear simulations ([4],[5], [6]), their potential role in conundrums such as the 'cooling flow problem' ([7]), or you could have a look at 'micro-instabilities' caused by the plasma's pressure anisotropy ([8]).

Relevant Courses

Essential: Astrophysical Fluid Dynamics

Useful: Stellar structure and evolution

References

[1] Balbus, 2000, Astrophysical Journal, 534, 420.

[2] Balbus 2001, Astrophysical Journal, 562, 909.

[3] Quataert, 2008, Astrophysical Journal, 673, 758.

[4] Parrish and Stone, 2007, Astrophysical Journal, 664, 135.

[5] Parrish et al., 2009, Astrophyical Journal, 703, 96.

[6] Kunz et al., 2012, Astrophysical Journal, 754, 122.

[7] Peterson and Fabian, 2006, Physics Reports, 427, 1.

[8] Schekochihin et al., 2005, Astrophysical Journal, 629, 139.

83. Structure formation in Saturn's rings ....

Dr H. N. Latter

The rings of Saturn are perhaps the most familiar and beautiful objects in astrophysics; they also exhibit some of its most puzzling phenomena ([1]). Complex patterns and waves have been carved into the disk by a myriad of processes: viscous and gravitational instabilities, impacts with micrometeoroids, and the gravitational influence of external and internal moons, to name but a few. The patterns extend over a vast range of lengthscales (from 100 metres to 100 kilometres) and are only partially understood. For a review of their observations see [2], for a review of their theory see [3], and for a general review of planetary rings see [4].

Your essay could either (a) survey the observations of these structures and the relevant physics in each case, or (b) concentrate on just one class of structure and go into some mathematical/physical detail. Specific topics that could be discussed include:

(i) How to model the rings. The kinetic theory of cold and dense granular flow.

(ii) Gravitational instability and canted self-gravity wakes in the A and B-rings.

(iii) Viscous overstability and periodic microstructure on 100m scales in the A and B-rings.

(iv) The ballistic transport process: sharp inner ring edges, and 100-1000km structure in the C-ring and inner B-ring.

(v) Spiral density wave launching in the A-ring by external satellites.

(vi) Embedded 100m moonlets in the A-ring ('propellers').

(vii) The bizarre dynamics and structures of the braided F-ring.

Relevant Courses

Essential: Astrophysical fluid dynamics, Dynamics of astrophysical disks

References

[1] Cassini Imaging Lab, http://www.ciclops.org/ir_index_main/Cassini

[2] Colwell et al., 2009. In: Dougherty, Esposito, Krimigis (Eds.), *Saturn from Cassini-Huygens*, Springer, p375.

[3] Schmidt et al., 2009. In: Dougherty, Esposito, Krimigis (Eds.), *Saturn from Cassini-Huygens*, Springer, p413.

[4] Tiscareno, 2011. In: Kalas and French (Eds.), *Planets, Stars and Stellar Systems*, Springer. (arXiv:1112.3305)

84. Data assimilation and super-resolution reconstruction of experimental turbulent flows ....

Dr A. Lefauve and Prof S. B. Dalziel

Data assimilation [1] is routinely used in meteorology and oceanography to provide a mathematical framework to complete and improve sparse experimental observations with computational fluid dynamics (i.e. numerical solutions of the Navier-Stokes equations).

Two 'classical' assimilation techniques have been applied to fluid dynamics [2]: variational methods, which are based on the use of the optimal control theory to minimise the error between observations and numerical solutions, and Kalman filter methods, which are based on a Bayesian formulation of the problem to propagate in time the statistics of the state vector. In the past few years, many new 'machine learning' techniques have been developed [e.g. 3,4], based on convolutional neural networks, to increase the spatial resolution ('super-resolution') and temporal resolution ('in-betweening') of experimental measurements (see figure 1).

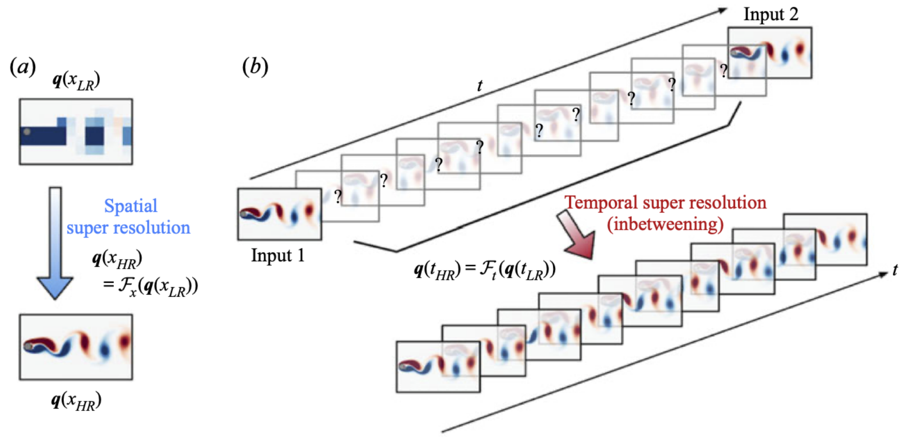

Figure 1: Schematics of the reconstruction, deducing the most likely high-resolution state vector from a low-resolution input (reproduced from [4]).

This essay is motivated by our desire to apply assimilation and reconstruction techniques to the cutting-edge experimental measurements of stratified turbulent flows performed in the G. K. Batchelor Laboratory in DAMTP [5]. This essay should start by surveying the various classical and machine learning data assimilation techniques in the fluid dynamics literature, focusing on (i) their mathematical foundations and properties, (ii) respective pros and cons, (iii) practical algorithms, (iv) expected computational cost, (v) potential volume of training data needed, and (vi) applicability to our specific experimental datasets.

At least two challenges may require further developments beyond existing methods. First, our datasets not only include three components of velocity in three dimensions, but also the density field, which is governed by an advection-diffusion equation coupled to the Navier-Stokes equations; second, our three-dimensional measurements are not strictly instantaneous due to the scanning of the laser sheet, i.e. successive two-dimensional slices are acquired with time lags [5]. The essay should discuss these challenges and possible resolutions to them.

There is also potential to carry out some computations, or develop new mathematics, but it is not required.

Relevant Courses

Essential: Undergraduate fluid dynamics (Part II or equivalent), and some familiarity with statistics and numerical solutions of PDEs

Useful: Mathematics of Machine Learning (II), Statistics (IB or II), Numerical Analysis (II), Optimisation (IB), Laboratory Demonstrations in Fluid Dynamics (III)

References

[1] Lewis, J. M., Lakshivarahan, S. & Dhall S. *Dynamic Data Assimilation: A Least Squares Approach.* Cambridge University Press (2006)

[2] Mons, V., Chassaign, J-C., Gomez, T. & Sagaut, P. Reconstruction of unsteady viscous flows using data assimilation schemes. *Journal of Computational Physics*, vol. 316, 255-280 (2016)

[3] Liu, B., Tang, J., Huang, H. & Lu, X-Y. Deep learning methods for super-resolution reconstruction of turbulent flows. *Physics of Fluids*, vol. 32, 025105 (2020)

[4] Fukami, K., Fukagata, K. & Taira, K. Machine-learning-based spatio-temporal super resolution reconstruction of turbulent flows. *Journal of Fluid Mechanics*, vol. 909, A9 (2021)

[5] Partridge, J. L., Lefauve, A. & Dalziel, S. B. A versatile scanning method for volumetric measurements of velocity and density fields. *Measurement Science and Technology*, vol. 30, 055203 (2019)

85. The Matter-Antimatter Asymmetry of the Universe .... Dr. F. Muia

The abundance of baryonic matter in the Universe is determined to be roughly 5% of the total energy density through *Cosmic Microwave Background* measurements. While it is natural to assume a baryon-symmetric initial condition for the early Universe, cosmic ray observations show no evidence for primordial antimatter, implying that a matter-antimatter asymmetry has to be present already at the time of recombination, and even earlier, during Big Bang Nucleosynthesis. This is quantified to be $(n_B - n_{\bar{B}})/n_\gamma = n_B/n_\gamma \simeq 6 \times 10^{-10}$, where n_B, $n_{\bar{B}}$, n_γ are the present number densities of baryons, antibaryons and photons respectively.

In order to generate a baryon asymmetry in the early Universe, three conditions - called the *Sakharov's conditions* have to be satisfied, namely the violation of the baryon number conservation, the violation of C and CP and the departure from thermal equilibrium. There is a number of mechanisms that satisfy these requirements and could lead to the generation of the matter-antimatter asymmetry in the early Universe, or *baryogenesis*. Notable examples are *i)* leptogenesis, that is strictly related to the generation of neutrino masses, *ii)* electroweak baryogenesis, that has been severely constrained by LHC results, *iii)* Affleck-Dine baryogenesis, that makes use of the dynamics of flat directions in the scalar potential of supersymmetric theories. The goal of this essay is to review one or more of these mechanisms, emphasizing their possible signatures in terms of gravitational wave production.

Relevant Courses

Essential: Cosmology; Quantum Field Theory.

Useful: Symmetries, Particles and Fields; Supersymmetry; Physics Beyond the Standard Model.

References

[1] Introduction to the Theory of the Early Universe - Hot Big Bang; D. S. Gorbunov and V. A. Rubakov.

[2] On the origin of matter in the Universe; P. Di Bari; arXiv:2107.13750 and references therein.

[3] Baryogenesis from the weak scale to the grand unification scale; D. Bodeker, W. Buchmuller; arXiv:2009.07294 and references therein.

[4] A mini review on Affleck-Dine baryogenesis; A. Mazumdar and R. Allahverdi; DOI: 10.1088/1367-2630/14/12/125013.

86. Cosmological Phase Transitions and Gravitational Wave Production ... Dr. F. Muia

The Standard Model of cosmology describes the evolution of the Universe as it cools down from a very hot and dense state, whose precise conditions depend on the inflationary model used, to the current value of the *Cosmic Microwave Background* temperature of $T_{\text{CMB}} \sim 2.7\,\text{K}$. Despite there is no direct obervation implying that the temperature of the Universe has ever been larger than that required for *Big Bang Nucleosynthesis* to work, i.e. $T_{\text{BBN}} \simeq 3\,\text{MeV}$, it is likely that the Universe had a much larger temperature in the past, in order to accommodate the generation of dark matter and matter-antimatter asymmetry. If that is the case, it is possible that the Universe underwent a few cosmological phase transitions, due to a mismatch between the properties of the ground state of the theory at vanishing and non-vanishing temperature. If the phase transition is of first order, the scalar potential features a barrier between two of its minima and the phase transition occurs through the nucleation of bubbles of the new vacuum inside the old vacuum. The bubbles expand, collide and eventually the new vacuum permeates the entire Universe. This process leads to the generation of gravitational waves in the early Universe, through three different sources: *i)* bubble collisions, *ii)* sound waves and *iii)* magnetohydrodynamics turbulence.

On top of these sources, which are intrinsically related to the dynamics of the phase transition, depending on the topology of the vacuum before and after the transition, topological defects might form, such as domain walls, cosmic strings, monopoles and textures. The dynamics of topological defects is another source of gravitational waves in the early Universe.

In this essay we will review one of the following topics:

a) the possible phase transitions patterns that lead to gravitational wave production in the early Universe due to points *i)* $-$ *iii)* above, in theories beyond the Standard Model such as supersymmetric theories and grand unified theories;

b) the possible phase transitions patterns that lead to gravitational wave production in the early Universe due to the formation of topological defects, in theories beyond the Standard Model such as supersymmetric theories and grand unified theories.

Relevant Courses

Essential: Cosmology; Quantum Field Theory.

Useful: Symmetries, Particles and Fields; Supersymmetry; Physics Beyond the Standard Model.

References

[1] Introduction to the Theory of the Early Universe - Hot Big Bang; D. S. Gorbunov and V. A. Rubakov.

[2] Cosmological Backgrounds of Gravitational Waves; C. Caprini and D. Figueroa; arXiv:1801.04268, Sec.s 2, 3, 4, 8, 9 and references therein.

[3] Review of cosmic phase transitions: their significance and experimental signatures; A. Mazumdar and G. White; arXiv:1811.01948 and references therein.

[4] Cosmic Strings; M. B. Hindmarsh and T. W. B. Kibble; hep-th/9411342.

87. Warped Astrophysical Discs .... Professor G. Ogilvie

In a spherically symmetric gravitational potential, orbital motion is possible in any plane containing the centre of the potential. A warped astrophysical disc is a fluid flow dominated by orbital motion, in which the orbital plane varies continuously with radius and possibly with time. Both the shape of the disc and its mass distribution evolve as a result of internal torques that transport angular momentum within the disc, as well as any external torques applied to the disc.

Astrophysical discs are expected to be warped whenever a misalignment occurs in the system, as in the classic problem in which a black hole is fed with gas having an orbital angular momentum that is not parallel to the spin of the black hole; indeed, the shape adopted by the disc in this situation is a problem of considerable interest. It is also possible for a warp to arise through instability of an aligned system, as in the case of accretion on to a magnetized star.

Linear and nonlinear theories of the dynamics of warped discs have been discussed since the 1970s [1], and increasing use is being made of global numerical simulations (e.g. [2]). The aim of this essay would be to review the subject concisely, with an emphasis on recent developments of a theoretical, computational or observational nature.

An introduction to the subject and a selection of useful references can be found in Sections 1 and 2 of reference [3]. Use of the ADS archive `adsabs.harvard.edu` is highly recommended. Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

Essential: Dynamics of Astrophysical Discs

Useful: Astrophysical Fluid Dynamics

References

[1] Bardeen, J. M. & Petterson, J. A. (1975). The Lense–Thirring effect and accretion disks around Kerr black holes. *Astrophys. J. Lett.* **195**, L65–L67

[2] Lodato, G. & Price, D. J. (2010). On the diffusive propagation of warps in thin accretion discs. *Mon. Not. R. Astron. Soc.* **405**, 1212–1226 (`arxiv.org/abs/1002.2973`)

[3] Ogilvie, G. I. & Latter, H. N. (2013). Local and global dynamics of warped astrophysical discs. *Mon. Not. R. Astron. Soc.* **433**, 2403–2419 (`arxiv.org/abs/1303.0263`)

88. Wave Attractors in Rotating and Stratified Fluids .... Professor G. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics which might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.
2. The relation, if any, between the propagation of rays within a container and the existence of invisicid normal modes.
3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.
4. The roles of nonlinearity and instability in wave attractors.
5. The relevance of wave attractors to tidal dissipation in astrophysical systems.

Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

Useful: Astrophysical Fluid Dynamics

References

[1] Maas, L. R. M. & Lam, F.-P. A. (1995). *J. Fluid Mech.* **300**, 1

[2] Maas, L. R. M., Benielli, D., Sommeria, J. & Lam, F.-P. A. (1997). *Nature* **388**, 557

[3] Rieutord, M., Georgeot, B. & Valdettaro, L. (2001). *J. Fluid Mech.* **435**, 103

[4] Ogilvie, G. I. (2005). *J. Fluid Mech.* **543**, 19

[5] Jouve, L. & Ogilvie, G. I. (2014). *J. Fluid Mech.* **745**, 223

89. The String Landscape vs The Swampland .... Professor Fernando Quevedo

Despite its uniqueness string theory seems to have an enormous set of vacua known as the landscape. In particular, although fully realistic string models have not been found, the efforts to try to find four-dimensional universes with some properties similar to ours usually leads to a huge number of solutions. Their potential predictive power is then much limited and a series of 'swampland' conjectures have been proposed in the past few years that aim to limit this vastness. The idea of the swampland is that not every low-energy effective field theory can be completed in the ultra violet to be a proper vacuum of a full-fledge quantum gravity theory (string or otherwise). The conjectures range from well established results such as the absence of global symmetries and that gravity seems to be the weakest force, to more speculative ones referring to the existence of de Sitter solutions, cosmological inflation, etc. The aim of the essay is to review and assess critically some of these conjectures.

Relevant Courses

Essential: Quantum Field Theory, General Relativity, The Standard Model *Useful:* Supersymmetry, String Theory.

References

[1] T. D. Brennan, F. Carta and C. Vafa, "The String Landscape, the Swampland, and the Missing Corner," PoS **TASI2017** (2017), 015 doi:10.22323/1.305.0015 [arXiv:1711.00864 [hep-th]].

[2] E. Palti, "The Swampland: Introduction and Review," Fortsch. Phys. **67** (2019) no.6, 1900037 doi:10.1002/prop.201900037 [arXiv:1903.06239 [hep-th]].

[3] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications," [arXiv:2102.01111 [hep-th]].

90. Wave propagation in random media .... Dr Orsola Rath Spivack

The propagation of a scalar wave ϕ in any medium is described by the wave equation

$$\nabla^2\phi(x,t) - \frac{1}{c^2}\frac{\partial^2\phi(x,t)}{\partial t^2} = F(x,t), \quad x \in \mathbb{R}^3, t \in \mathbb{R} \tag{1}$$

with or without a source term $F(x,t)$. In this essay, you should only consider the case $F(x,t) = 0$, or possibly a source localised in the plane $z = 0$: $F(x,t) = \delta(z)\exp(-i\omega t)$. For time-harmonic waves $\phi(x,t) = \psi(x)\exp(-i\omega t)$ and $F(x,t) = f(x)\exp(-i\omega t)$, equation (1) reduces to the time-independent Helmholtz equation, which in the case $F = 0$ is

$$\nabla^2\psi + k^2\psi = 0 \ , \tag{2}$$

where $k = \frac{\omega}{c} = k_0 n(x)$, and $n(x)$ is the refractive index of the medium.

Most realistic media vary in a complicated way, often with fast variations on a spatial scale which is small compared with the propagation distance, as well as slower variations on a much larger scale, that can often be ignored. The spatial variations of the medium are described by the refractive index, and the fast variations are usually random in nature. In many cases, such as propagation through turbulent atmosphere, or the ocean, or biological tissue, this can be described by a refractive index $n(x) = 1 + w(x)$, where $w(x)$ is a zero-mean stationary random process. When the index of refraction is a random process, the wave field is itself a random process and we are interested in how the statistics of the random medium affects the statistics of the wave field, i.e. the so-called 'moments' of the field. Finding solutions for these quantities is sa difficult problem. When the wave varies slowly in the direction of propagation, the paraxial approximation applies. It turns out that, in this case, it is possible to derive and solve equations for some of the key moments of the field, including the mean intensity.

This essay should focus on deriving the equations for the key moments of the field, providing analytical solutions where possible, and explaiining their range of validity and their basic physical significance. Then you could choose to focus on numerical implementations of propagation in random media, particularly for quantities where an exact solution is not available, commenting on the numerical model and issues of efficiencies or comparison to experimental results (from the literature: no numerical implementation of your own is required). Or you could choose to focus on properties of existence, uniqueness, and continuity of solutions for some cases. Or you could choose to focus on one (or two) particular applications to physical problems. The choice should depend on your interests and background.

A few possible references are given below, but more specific references depending on the choice of focus will be provided.

Relevant Courses

Essential: Knowledge of the wave equation and basic concepts in wave propagation, from any course.

References

[1] F. D. Tappert, The Parabolic Approximation Method, in J. B. Keller, J. S. Papadakis, eds., Wave Propagation and Underwater Acoustics, Lecture Notes in Physics, vol. 70, Springer, New York, 1977

[2] B. J. Uscinski, Elements of Wave Propagation in Random Media, McGraw Hill, 1977

[3] A. Ishimaru, Wave propagation and Scattering in Random Media. IEEE Press, 1997

[4] E. Jakeman and K. D. Ridley, Modeling Fluctuations in Scattered Waves, CRC Press, 2006

[5] J. Garnier and K. Sølna, Scintillation in the white-noise paraxial regime, Comm. Part. Differ. Equat. 39 (2014), 626-650

[6] F. Bailly, J.-F. Clouet, and J.-P. Fouque, Parabolic and white noise approximation for waves in random media, SIAM J. Appl. Math. 56 (1996), 1445-1470

91. Advantages, limitations and challenges in photoacoustic imaging .... Dr. O. Rath Spivack

Hybrid imaging techniques have attracted considerable attention in the past few years as a promising emerging medical imaging tool [1]. These techniques combine a high contrast modality with a high resolution modality, and have the potential for delivering the advantages of both techniques. Examples include ultrasound elastography, magnetic resonance elastography, photoacoustic tomography, magnetic resonance electric impedance tomography, and others. They are mostly not yet established as standard imaging techniques in a clinical setting, and are still the object of much theoretical investigation.

This essay should concentrate on photoacoustic tomography, describe the physical effect on which it is based, and explain the equations used to model the wave propagation and the inverse problem [2,3].

An overview should be given of the type of inverse problems that need solving, and of the range of validity of the models used [4,5]. Then, the essay could focus on the theoretical results for stability and uniqueness of solutions and the conditions needed [6], or on the validity of the approximations in applications, and perhaps the merits of different numerical methods for the solution [7,8]. The choice should depend on your interests and background.

Finally, the essay should give a view of current challenges and possible avenues for addressing them, especially in relation to the focus chosen in the main part of the essay.

Relevant Courses

Essential: Basic knowledge of PDEs, and particularly the wave equation, from any undergraduate course.

Useful: The Part III courses 'Inverse problems', or 'Numerical Soluitions of Differential Equations', further background in PDEs, or in wave propagation in random media, or in numerical techniques such as Finite Element Methods may be useful, depending on the choice for the focus of this essay.

References

[1] Ammari H An Introduction to Mathematics of Emerging Biomedical Imaging, Springer (2008)

[2] Cox B, Laufer JG, Arridge SR, Beard PC Quantitative spectroscopic photoacoustic imaging: a review *Journal of Biomedical Optics* 17(6), 061202 (2012)

[3] Xu M and Wang LV Photoacoustic imaging in biomedicine *Rev. Sci. Instruments*, 77, 041101 (2006)

[4] Arridge SR & Schotland, JC Optical tomography: forward and inverse problems *Inverse Problems* 25, 123010 (2009)

[5] Ammari H, Bossy E, Jugnon V, Kang H Mathematical Modeling in Photoacoustic Imaging of Small Absorbers *SIAM Review* 52(4) 677–695 (2010)

[6] Bal G, Jollivet A, Jugnon V Inverse transport theory of photoacoustics *Inverse Problems* 26, 025011 (2010)

[7] Martí-López L, Bouza-Domínguez J, Hebden JC, Arridge SR, Martínez-Celorio RA Validity conditions for the radiative transfer equation *J. Opt. Soc. Am.* A 20(11), 2046-2056 (2003)

[8] Kim AD and Ishimaru A Optical diffusion of continuous-wave, pulsed, and density waves in scattering media and comparisons with radiative transfer *Applied Optics* 37(22) 5313-5319 (1998)

92. Perturbations of a Kerr black hole .... Harvey Reall

This essay concerns linear gravitational perturbations of a Kerr black hole. Remarkably, such perturbations can be described by a single wave equation: the Teukolsky equation [1]. The essay should start by explaining the derivation of this equation [1,2], and how a metric perturbation can be reconstructed from a solution of the "adjoint" equation [3]. It should then explain the proof of "mode stability" of a Kerr black hole, i.e., the statement that there do not exist solutions of the Teukolsky equation with time dependence $e^{-i\omega t}$ where $\mathrm{Im}(\omega) > 0$ [4,5]. The essay should be written at a level that is accessible for another Part III student who has attended the Black Holes course.

Relevant Courses

Essential: Black Holes

Useful:

References

[1] S. Teukolsky, Astrophys. J. **185** (1973) 635-647

[2] J.M. Stewart and M. Walker, Proc. Roy. Soc. Lond. **A341** (1974), 49-74

[3] R.M. Wald, Phys. Rev. Lett. **41** (1978), 203-206

[4] B.F. Whiting, J. Math. Phys. **30** (1989), 1301

[5] M. Casals and R. Teixeira da Costa, arXiv:2105.13329

93. AdS Euclidean Wormholes .... Jorge E. Santos

Euclidean wormholes are fascinating classical solutions of general relativity. These illusive geometries have been a constant source of inspiration in science-fiction, but for many years have been regarded by theoretical physicist as the poor cousin of Euclidean instanton solutions of the Einstein equation. This has changed drastically in the last decade, and wormholes have found many applications in theoretical physics, with particular emphasis on holography.

The essay should be roughly divided into two parts. In the first part, the essay should review what an AdS Euclidean wormhole is, and how it connects to the concept of *baby Universe*. Furthermore, it should explain why Euclidean wormholes with anti de-Sitter asymptotics are hard to come by in light of the Witten-Yau theorem [1]. In the second part, the essay should review a topic in holography where Euclidean wormholes play a central role. These include, but are not limited to, the factorisation problem [2,3,4] and the semi-classical description of the Page curve [5].

The essay should be written in a language accessible to other Part III students taking similar courses.

Relevant Courses

Essential: General Relativity and Black Holes

Useful: Quantum field theory, advanced quantum field theory, String theory and Gauge/ Gravity duality.

References

[1] E. Witten and S.-T. Yau, *Connectedness of the boundary in the AdS / CFT correspondence*, Adv. Theor. Math. Phys. 3 (1999) 1635–1655.

[2] S. W. Hawking, *Wormholes in Space-Time*, Phys. Rev. D37 (1988) 904–910.

[3] J. M. Maldacena and L. Maoz, *Wormholes in AdS*, JHEP 02 (2004) 053.

[4] D. Marolf and J. E. Santos, *AdS Euclidean wormholes*, Class. Quant. Grav. **38** (2021) no.22, 224002.

[5] D. Marolf and H. Maxfield, *Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information*, arXiv:2002.08950.

94. Mapping fermions to qubits ....
Dr S. Strelchuk

Simulating fermionic systems is one of the most promising applications of quantum computers. In order to simulate fermionic particles, one needs to construct a mapping that converts fermionic interactions to the state of qubits. The key characteristic of a good mapping is its ability to translate local fermionic interactions into qubit interactions with a high degree of locality. The efficiency of mapping has strong implications both on the number of qubits as well as the gate cost of simulations.

The first known mapping was produced almost a century ago by Jordan and Wigner [1]. In recent years, there have been several new approaches that gave rise to new, efficient mappings bringing the quest of simulating fermionic quantum systems on near-term devices closer to reality. They include Bravyi-Kitaev map [2,3], Derby-Klassen [4] map and fermionic enumeration schemes [5].

This essay should discuss these approaches and their applications.

Relevant Courses

Useful:

Quantum Information Theory (M24)

Quantum Computation (L24)

References

Nielsen M., "The Fermionic canonical commutation relations and the Jordan-Wigner transform", `https://michaelnielsen.org/blog/archive/notes/fermions_and_jordan_wigner.pdf`

Bravyi, S. B. Kitaev, A. Y. "Fermionic Quantum Computation". Ann. Phys. 2002, 298, 210–226

Tranter, A., Love, P. J., Mintert, F. and Coveney, P. V. "A comparison of the Bravyi–Kitaev and Jordan–Wigner transformations for the quantum simulation of quantum chemistry". Journal of chemical theory and computation, 14(11), 5617–5630.

Derby, C., Klassen, J., Bausch, J. and Cubitt, T. "Compact fermion to qubit mappings". Physical Review B, 104(3), 035118.

Chiew, M. and Strelchuk, S. "Optimal fermion-qubit mappings". arXiv preprint arXiv:2110.12792.

95. Symmetry protected quantum computation ....
Dr. S. Strelchuk

Permutational quantum computing (PQC) is a simple computational model that operates by first performing Schur transform on the input state followed by a polynomial number of permutation gates [1]. It is a natural quantum computational model conjectured to capture nonclassical aspects of quantum computation. The backing for this conjecture came from several computational problems that are solvable on the PQC machine in polynomial time, but the only known classical algorithm required exponential runtime.

PQC was found to be classically efficiently simulatable [2,3], and authors in [3] proposed a new, enhanced model called Symmetry Protected Quantum Computation (SPQC). Measurements in SPQC are of a special kind: given any two qubits one measures whether they are in a singlet (total spin 0) or triplet (total spin 1) state. Similarly to PQC, it is conjectured to capture many non-classical phenomena. Supplemented with single-qubit gates X and Z SPQC is capable of universal quantum computation.

This essay will provide a coherent account of the SPQC and discuss its relation to PQC.

Relevant Courses

Useful:

Quantum Information Theory (M24)

Quantum Computation (L24)

References

[1] Jordan S. P. "Permutational Quantum Computing", Quantum Inf. Comput., 10, 470–497 (2010)

[2] Havlicek V., Strelchuk S., "Quantum Schur Sampling Circuits can be Strongly Simulated", Phys. Rev. Lett. 121, 060505 (2018)

[3] Havlicek V., Strelchuk S., "Classical algorithm for quantum SU(2) Schur sampling", Phys. Rev. A 99, 062336 (2019)

[4] Freedman, M. H., Hastings, M. B., Zini, M. S. "Symmetry Protected Quantum Computation". arXiv:2105.04649. (2021)

96. Scattering and Resonances from Lattice QCD .... C. E. Thomas

In recent years experiments have observed a number of puzzling hadrons that do not appear to fit with a simple picture of a hadron as either a meson (quark-antiquark state) or a baryon (three-quark state). These observations have generated a lot of interest and hadrons are currently the subject of many theoretical and experimental investigations.

The interactions between quarks and gluons are described by Quantum Chromodynamics (QCD) and lattice QCD is a non-perturbative technique that enables first-principles computations of the properties of hadrons using numerical methods. However, the vast majority of hadrons are not stable and instead are strongly-decaying resonances. It is not possible to directly compute scattering amplitudes in finite-volume Euclidean-time lattice QCD calculations, but a relationship between the discrete spectrum of energies in a finite volume and infinite-volume scattering amplitudes enables these to be determined indirectly. The mass, width and other properties of any resonance present can then be extracted.

The essay should discuss the connection between the finite-volume spectrum and scattering amplitudes, illustrating this by considering examples in one or more simple models, and then discuss applications to hadron scattering and resonances in QCD. It should not describe in any detail lattice QCD itself or how finite-volume energies are computed.

Relevant Courses

Essential: Part II Applications of Quantum Mechanics (or another course that covers basic scattering theory); Quantum Field Theory

Useful: Symmetries, Fields and Particles; Advanced Quantum Field Theory; The Standard Model

References

[1] Briceño, Dudek, Ross, "Scattering processes and resonances from lattice QCD", Rev. Mod. Phys. 90, 2 (2018) [arXiv:1706.06223]

[2] Lüscher, "Two-particle states on a torus and their relation to the scattering matrix", Nucl. Phys. B 354, 531 (1991).

[3] Rummukainen, Gottlieb, "Resonance scattering phase shifts on a nonrest frame lattice", Nucl. Phys. B 450, 397 (1995) [arXiv:hep-lat/9503028].

[4] Kim, Sachrajda, Sharpe, "Finite-volume effects for two-hadron states in moving frames", Nucl. Phys. B 727, 218 (2005) [arXiv:hep-lat/0507006].

[For arXiv papers see https://arxiv.org/]

97. Vanishing Viscosity Limit in Domains with Wall-boundaries: The Kato Criterion . . . .

Edriss S. Titi

It is commonly accepted that turbulent flows are governed by the Navier-Stokes equations, and that they occur when the dimensionless Reynolds number is very large. Therefore, it is natural to ask what are the governing equations as the Reynolds number tends to infinity, or as the viscosity tends to zero in the Navier-Stokes equations. In 1972, T. Kato [3] (see also [4]) proved that if the solutions of the Euler equations of ideal incompressible flows in $\mathbb{R}^3$ are regular enough, in the time interval $[0, T]$, then the strong solutions of the Navier-Stokes equations, with the same initial data, converge to the corresponding solution of the Euler equations in a subinterval of time. P. Constantin [2] established similar result in the case of periodic boundary conditions, i.e. in the torus $\mathbb{T}^3$. Later on, N. Masmoudi [8] improved these results. Notably, the above results are established in the absence of wall-boundaries and boundary effect. To investigate the boundary effect for flows confined in domains with wall-boundaries, T. Kato [5] established a sufficient condition criterion for the convergence of the Navier-Stokes solutions to the Euler ones as the viscosity tends to zero. Kato's criterion involves a delicate analysis of the boundary-layer behaviour of the solutions of the Navier-Stokes equations. In section 4 of [1] the authors provide a survey of the above results and provide a slight improvement of the Kato criterion. In [6-7] the authors investigated similar problems for an inviscid regularization model of the Euler equations and for a second-grade viscoelastic model.

Interested students can choose one of the following essays/projects:

Essay 1: This essay will be about the Kato criterion [5] and its improvement as described in section 4 of [1].

Essay 2: This essay will be about the Kato criterion [5] and the results reported in [6-7]. In particular, a comparison analysis is an essential part of this essay.

Relevant Courses

Essential: Analysis of Partial Differential Equations. Introduction to Nonlinear Analysis.

Useful: Mathematical Analysis of the Incompressible Navier-Stokes Equations. Some knowledge of Fluid Mechanics.

References

[1] C. Bardos and E.S. Titi, *Mathematics and turbulence: where do we stand?*, Journal of Turbulence, **14(2)** (2013), 42–76.

[2] P. Constantin, Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations, *Commun. Math. Phys.*, **104(2)** (1986), 311–326.

[3] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *J. Funct. Anal.*, **9** (1972), 296–305.

[4] T. Kato Quasi-linear equations of evolution, with applications to partial differential equations. In: Spectral Theory and Differential Equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), Lecture Notes in Math., Vol. 448. Berlin: Springer, 1975, pp. 25–70

[5] T. Kato, Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary, *Seminar on Nonlinear Partial Differential Equations* (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., Vol. 2, Springer, New York, 1984, pp. 85–98.

[6] M. Lopes Filho, H. Nussenzveig Lopes, E.S. Titi and A. Zang, Convergence of the 2D Euler-α to Euler equations in the Dirichlet case: indifference to boundary layers, *Physica D*, **292-293** (2015), 51–61.

[7] M. Lopes Filho, H. Nussenzveig Lopes, E.S. Titi and A. Zang, On the approximation of 2D Euler equations by second-grade fluid model with Dirichlet boundary conditions, *Journal of Mathematical Fluid Mechanics*, **17(2)** (2015), 327–340.

[8] N. Masmoudi, Remarks about the inviscid limit of the Navier–Stokes system, *Commun. Math. Phys.*, **270** (2007), 777–788.

98. Gauged Linear Sigma Models ....

David Tong

Geometry becomes blurry and interesting when viewed through the eyes of a 2d quantum field theory. The appropriate quantum field theories are called, for historic reasons, *sigma models*.

A "non-linear" sigma model is a quantum field theory in which the fields are coordinates on some manifold M known as the *target space*. A "linear" sigma model achieves the same effect at low-energies through judicious use of gauge interactions. This means that 2d gauge theories can give us a novel perspective on geometry.

The primary purpose of this essay is to review the construction of supersymmetric gauged linear sigma models, due originally to Witten. The simplest such sigma models have target space $M = \mathbf{CP}^N$. These are interesting in their own right because the massless fields of the classical theory develop a mass due to quantum interactions. The story becomes richer when one realises a way to carve out a Calabi-Yau manifold $X \subset \mathbf{CP}^N$ in which case the low-energy dynamics is a superconformal field theory.

The essay will describe how these different phases are realised in some simple, supersymmetric gauge theories. There are a number of more advanced topics that the essay could explore, including ideas associated to mirror symmetry.

Relevant Courses

Essential: QFT, AQFT, GR, and Supersymmetry

Useful: String Theory

References

The original paper is by Edward Witten, "Phases of $N = 2$ Theories in Two Dimensions", https://arxiv.org/abs/hep-th/9301042hep-th/9301042, Nucl.Phys.B 403 (1993) 159.

Some background material on 2d gauge theories and sigma models, including a description of the non-supersymmetric $\mathbf{CP}^N$ sigma model, can be found in Chapter 7 of David Tong's, http://www.damtp.cam.ac.uk/user/tong/gaugetheory.htmlLectures on Gauge Theory.

A review of supersymmetric gauged linear models, with emphasis on the underlying geometry, can be found in Parts 2 and 3 of the "Mirror Symmetry" book. These sections are written by Kentaro Hori. The book can be downloaded directly from the http://claymath.org/library/monographs/cmim01c.pdfClay Mathematics Institute.

99. On the stability of the Standard Model vacuum .... M. Ubiali

The Higgs boson and the top quark masses are crucial parameters to determine whether the Standard Model (SM) ground state is absolutely stable, unstable or metastable.

The purpose of this essay is twofold. In the first half, the essay will describe and reproduce the complete one-loop calculation of the tunnelling probability of the SM Universe at zero temperature, improved by means of two-loop renormalization-group equations, starting from the derivation provided in Ref. [1]. Optionally, the essay can further explore the implication of the analytical inclusion of gravitational corrections to the vacuum decay rate [2].

In the second half, the essay can either focus on how considerations on the Universe stability make us infer new particle-physics properties at small distances [3], or, by assuming the validity of the SM up to an arbitrary high-energy scale, find out what information on the early stages of the Universe can be extracted from a precise measurement of the Higgs boson mass [4].

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory. *Optional:* Cosmology.

References

[1] G. Isidori, G. Ridolfi and A. Strumia, *"On the metastability of the standard model vacuum,"* Nucl. Phys. B **609** (2001), 387-409 doi:10.1016/S0550-3213(01)00302-9 [arXiv:hep-ph/0104016 [hep-ph]].

[2] G. Isidori, V. S. Rychkov, A. Strumia and N. Tetradis, *"Gravitational corrections to standard model vacuum decay,"* Phys. Rev. D **77** (2008), 025034 doi:10.1103/PhysRevD.77.025034 [arXiv:0712.0242 [hep-ph]].

[3] C. P. Burgess, V. Di Clemente and J. R. Espinosa, *“Effective operators and vacuum instability as heralds of new physics,”* JHEP **01** (2002), 041 doi:10.1088/1126-6708/2002/01/041 [arXiv:hep-ph/0201160 [hep-ph]].

[4] J. R. Espinosa, G. F. Giudice and A. Riotto, *“Cosmological implications of the Higgs mass measurement,”* JCAP **05** (2008), 002 doi:10.1088/1475-7516/2008/05/002 [arXiv:0710.2484 [hep-ph]].

(and references therein)

100. Hamiltonian formulation of lattice gauge theories ....
Professor M. B. Wingate

The path integral formulation of quantum field theories has been the most practical way to carry out calculations in gauge theories. Lorentz and gauge symmetries are manifest in the Lagrangian. Feynman rules for weak-coupling perturbation theory are readily derived. By discretizing space and *imaginary* time, lattice gauge theories can be solved numerically, even at strong coupling, using stochastic sampling of path integrals.

Hamiltonian approaches to gauge theories pose several challenges. Lorentz symmetry is explicitly broken, gauge-fixing is required, and, even after discretizing space and time (real or imaginary), the Hilbert space must be truncated.

Recently, two developments have motivated people to tackle these challenges: the construction of quantum computing devices and ideas from quantum information theory about reducing the size of the Hilbert space needed in classical algorithms. Having practical computational methods for Hamiltonian-formulated gauge theories would enable calculations requiring *real* time or which otherwise do not allow stochastic sampling.

A successful essay with this title will explain how lattice gauge theories can be formulated in the Hamiltonian approach. It will then proceed to discuss some recent ideas about approaches for computation on classical or quantum computers.

Relevant Courses

Essential: Quantum Field Theory

Useful: Advanced Quantum Field Theory; Symmetries, Particles and Fields

References

[1] D. Tong, *Gauge Theory*, `http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html`.

[2] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).

[3] Z. Davoudi, I. Raychowdhury, A. Shaw, Phys. Rev. D **104**, 074505 (2021) [arXiv:2009.11802].

101. Global Regularity and Long-time Behavior of the Three-dimensional Ladyzhenskaya-Smagorinsky Eddy-viscosity Turbulence Model ....
Edriss S. Titi

Global regularity of the three-dimensional Navier-Stokes equations is one the most challenging problems in mathematical analysis, and is one of the millennium problems. Already in 1967, Ladyzhenskaya suggested a nonlinear viscous modification of the Navier-Stokes equations and established global regularity for this model [3-5] (see also the book [1]). In his investigation of atmospheric dynamics Smagorinsky [7] suggested in 1963 a similar system of equations, as an eddy-viscosity closure model of turbulence for the Reynolds equations.

In [5] Ladyzhenskaya studied the long-time behavior of her modified mathematical model of the Navier-Stokes. In addition to establishing the global regularity, she also shows the existence of a compact global attractor and the finite-number of determining modes (see, e.g., [6], [8] and references therein about global attractors for various evolution equations).

A general notion of determining functionals for the two-dimensional Navier-Stokes equations, capitalizing on previous work about determining modes, nodes and local averages is studied in [2].

This essay is about the mathematical analysis reported in [5]. Moreover, about using the ideas and notions developed in [2] to extend the result concerning finite number of determining modes, for the Ladyzhenskaya-Smagorinsky model, to general determining functionals/parameters/degrees of freedom for this model.

Relevant Courses

Essential: Analysis of Partial Differential Equations. Introduction to Nonlinear Analysis.

Useful: Mathematical Analysis of the Incompressible Navier-Stokes Equations. Some knowledge of Fluid Mechanics.

References

[1] L.C. Berselli, T. Iliescu and W.J. Layton,*Mathematics of Large Eddy Simulation of Turbulent Flows.* Scientific Computation, Springer.

[2] B. Cockburn, D. Jones and E.S. Titi, *Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems*, Mathematics of Computation, **66** (1997), 1073–1087.

[3] O.A. Ladyzhenskaya, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. In O.A. Ladyzhenskaya, editor, Proc. of the Steklov Institute of Mathematics, **102** (1967), Boundary Value Problems of Mathematical Physics. V, Providence, Rhode Island, 1970. AMS.

[4] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow.* Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach Science Publishers, New York, 1969.

[5] O. A. Ladyzhenskaya, Attractors for the modifications of the three-dimensional Navier-Stokes equations, *Phil. Trans. R. Soc. Lond. A*, **346** (1994), 173–190.

[6] J. C. Robinson, *Infinite-dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics.

[7] J. Smagorinsky, General circulation experiments with the primitive equations, *Mon. Weather Rev.*, **91** (1963), 99–164.

[8] Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Ed, Applied Math Sci. 68, Springer-Verlag, 1997.

102. Stationary boson stars in general relativity .... U. Sperhake

Light bosonic particles are one of the most promising candidates for the dark-matter content of the universe. Quite remarkably, complex-valued scalar or vector fields can form genuinely stationary configurations where the dispersive character of the matter field is exactly compensated by its own gravitation. These configurations are known as *boson stars*. Even though no evidence for for their existence has been found so far, interest in this class of stars has been reinvigorated by the rise of gravitational-wave observations as a new channel to studying the universe.

In spite of their conceptual simplicity, single boson stars can reveal a wide range of interesting properties and, in the form of binary systems, may even emit gravitational-wave signals detectable by present and future gravitational-wave observatories like LIGO, Virgo or LISA. This essay focuses on non-rotating and rotating single scalar-field boson-stars and their modelling with analytic and numerical means. In spherical symmetry, the resulting ground-state solutions form a one-parameter family of stars characterised by their central scalar-field amplitude. In axial symmetry, the differential equations admit solutions that describe rotating boson stars, but only for quantised values of the angular momentum. Dynamical numerical simulations furthermore provide evidence that rotating boson stars composed of scalar fields are unstable whereas their vector-field cousins are stable

This essay should discuss the equations governing stationary boson stars composed of scalar fields in spherical and in axial symmetry. The essay should furthermore summarise the main properties of these boson stars and how these are deduced. Introductory reviews on this topic are given in Refs. [1, 2], the opening 3 chapters of [3] and the introduction in [4]. More details about the calculation of spherically symmetric scalar boson stars are also given in Secs. 2.1 and 2.3 of [4]. The equations for rotating boson stars have been derived in the slow-rotation limit in [5], but the first calculation of of rotating solutions has only been achieved in the fully non-linear framework a few years later [6]. The numerical simulation of dynamical boson stars is a young field of research and only relevant in the context of this essay by its observation that rotating scalar boson stars appear to be unstable [7, 8]. Possible extensions of this essay beyond a literature review may consider a more detailed discussion of the slow-rotation limit in Ref. [5], a more in-depth discussion of the rotating solutions of Schunck and Mielke [6] or an attempt at constructing numerical solutions along the lines of Sec. 2.3 in Ref. [4] (it is not required that these solution succeed as long as a clear description of the methods and problems encountered is given).

Relevant Courses

Essential: General Relativity

Useful: Black Holes, Cosmology

References

References

[1] E. W. Mielke and F. E. Schunck. Boson stars: Early history and recent prospects. In *Recent developments in theoretical and experimental general relativity, gravitation, and relativistic field theories. Proceedings, 8th Marcel Grossmann meeting, MG8, Jerusalem, Israel, June 22-27, 1997. Pts. A, B*, pages 1607–1626, 1997.

[2] Eckehard W. Mielke and Franz E. Schunck. Boson stars: Alternatives to primordial black holes? *Nucl. Phys. B*, 564:185–203, 2000.

[3] Luca Visinelli. Boson Stars and Oscillatons: A Review. 9 2021.

[4] T. Helfer, U. Sperhake, R. Croft, M. Radia, B.-X. Ge, and E. A. Lim. Malaise and remedy of binary boson-star initial data. 8 2021.

[5] Y. Kobayashi, M. Kasai, and T. Futamase. Does a boson star rotate? *Phys. Rev. D*, 50:7721–7724, 1994.

[6] F. E. Schunck and E. W. Mielke. Rotating boson star as an effective mass torus in general relativity. *Phys. Lett.*, A249:389–394, 1998.

[7] N. Sanchis-Gual, F. Di Giovanni, M. Zilhão, C. Herdeiro, P. Cerdá-Durán, J. A. Font, and E. Radu. Nonlinear Dynamics of Spinning Bosonic Stars: Formation and Stability. *Phys. Rev. Lett.*, 123(22):221101, 2019.

[8] F. Di Giovanni, N. Sanchis-Gual, P. Cerdá-Durán, M. Zilhão, C. Herdeiro, J. A. Font, and E. Radu. Dynamical bar-mode instability in spinning bosonic stars. *Phys. Rev. D*, 102(12):124009, 2020.

103. Direct-adjoint-looping stability methods ....
C. P. Caulfield

It is of great practical importance to understand how systems undergo the transition to disorder. It is commonly believed that 'normal' mode flow instabilities play a central role in such transition processes, and the conventional argument is that the 'most unstable' normal mode will dominate the nonlinear evolution of a system and hence lead the flow to transition. However, in many situations, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. [1] A particularly attractive method to consider such transient growth problems is the so-called 'direct-adjoint looping' method, which can be generalised to consider fully nonlinear perturbations, where the developing perturbations can reach a sufficiently large amplitude to nontrivially modify the 'base system'. [2,3] This method is particularly well-suited to consider generalised stability problems, where the measure which is being extremised is not necessarily the 'energy' of the developing perturbation. [4] Indeed, there are several interesting mathematical issues about the most appropriate measures to use, and this essay could approach the general issue of perturbation 'growth' in a range of environmentally, industrially or even biologically important flows from a variety of mathematical and computational directions.

Relevant Courses

Useful:

Courses in continuum mechanics.

References

[1] P. J. Schmid 2007. Non-modal stability theory. *Ann. Rev. Fluid Mech.* **39** 129-162.

[2] R. R. Kerswell, C. C. T. Pringle & A. P. Willis 2014. An optimization approach for analysing nonlinear stability with transition to turbulence in fluids as an exemplar. *Rep. Prog. Phys.* **77** 085901.

[3] R. R. Kerswell 2018. Nonlinear nonmodal stability theory. *Ann. Rev. Fluid Mech.* **50** 319-345.

[4] D. P. G. Foures, C. P. Caulfield & P. J. Schmid 2014. Optimal mixing in two-dimensional plane Poiseuille flow at finite Péclet number. *J. Fluid Mech.* **748** 241-277.

104. Instability and perturbation growth in stratified shear flows .... C. P. Caulfield

Stratified shear flows, where both the fluid density and the velocity vary with height, are extremely common both in the environment and in industrial contexts. It is of great practical importance to understand how such flows undergo the transition to turbulence, as turbulence typically hugely increases mixing, transport and dissipation within such flows. It is commonly believed that 'normal' mode flow instabilities play a central role in this transition process, and the conventional argument is that the 'most unstable' normal mode will dominate the nonlinear evolution of the flow, and hence lead the flow to transition. However, the underlying linearized operator is non-normal, and so it is possible for substantial transient growth of perturbations to occur. Although this has been widely studied in unstratified flows, [1,2] the transient behaviour of stratified flows has been much less-studied. Also, stratified shear flows are prone to multiple, qualitatively different primary and secondary instabilities (particularly when the density distribution develops sharp interfaces [3]) and it appears that the transition to turbulence is typically associated with **secondary** instabilities which only develop once the primary instability has saturated [4]. There are also several interesting mathematical issues about the 'optimal' measures of perturbation growth to use, as the potential energy as well as the kinetic energy of the perturbation varies in a stratified flow [5], and this essay could approach the general issue of perturbation growth in stratified shear flows from a variety of mathematical and computational directions.

Relevant Courses

Useful:

Fluid dynamics of climate.

Fluid dynamics of the environment.

References

[1] P. J. Schmid 2007. Non-modal stability theory. *Ann. Rev. Fluid Mech.* **39** 129-162.

[2] R. R. Kerswell 2018. Nonlinear nonmodal stability theory. *Ann. Rev. Fluid Mech.* **50** 319-345.

[3] H. Salehipour, C. P. Caulfield & W. R. Peltier 2016. Turbulent mixing due to the Holmboe wave instability at high Reynolds number. *J. Fluid Mech.* **258** 255-285.

[4] C. P. Caulfield 2021. Layering, instabilities and mixing in turbulent stratified flows. *Ann. Rev. Fluid Mech.* **35** 135-167.

[5] A. Kaminski, J. R. Taylor & C. P. Caulfield 2014. Transient growth in strongly stratified shear layers. *J. Fluid Mech.* **758** R4, 12 pages.

105. Electromagnetic Duality ....
N. Dorey

The apparent symmetry of the vacuum Maxwell equations under an interchange of electric and magnetic fields is broken in nature by the apparent absence of magnetic charges. In some non-abelian gauge theories, this is remedied by the appearance of classical solutions corresponding to magnetic monopoles. However, these objects appear quite distinct from the elementary quanta of the theory carrying electric charge so there is still no obvious symmetry. Despite this, Montonen and Olive, made the bold conjecture that gauge theories can exhibit an exact symmetry between electric and magnetic degrees of freedom, now known as *electromagnetic duality.* A particularly striking feature of this symmetry, is that the gauge coupling of the dual theories are inversely related: when one description is weakly coupled the other is strongly coupled. This explains why electromagnetic duality is hard to test and also why it is of great interest: it potentially provides a dual weakly coupled description of strongly coupled QFT.

Although the extent to which these ideas apply to the gauge theories of the Standard Model is still far from clear, much progress has been made in the context of supersymmetric gauge theory [2]. In the maximally supersymmetric case [3], $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, there is abundant evidence on an exact electric-magnetic duality closely related to the original proposal of Montonen and Olive. Versions of electric-magnetic duality also play a key role in understanding the strong coupling behaviour of theories with less supersymmetry. This essay will investigate some of these developments. A good starting point for further reading is the review [4].

Relevant Courses

Essential: QFT.

Useful: Supersymmetry, Advanced QFT, Symmetries, fields and particles, Solitons, instantons and geometry.

References

[1] C. Montonen and D. I. Olive, Phys. Lett. B **72** (1977), 117-120

[2] E. Witten and D. I. Olive, Phys. Lett. B **78** (1978), 97-101

[3] H. Osborn, “Topological Charges for N=4 Supersymmetric Gauge Theories and Monopoles of Spin 1,” Phys. Lett. B **83** (1979), 321-326

[4] J. M. Figueroa-O’Farrill, ”Electromagnetic duality for children.” Lecture notes (1998). `https://ncatlab.org/nlab/files/FigueroaElectromagneticDuality.pdf`

106. Synchronisation of Eukaryotic Flagella ....
Professor R.E. Goldstein

Many eukaryotic microorganisms swim by beating flexible flagella in a characteristic breast-stroke manner. Whether it is the two opposing flagella on certain unicellular organisms or the multitudes of them on multicellular organisms, it is a nearly ubiquitous observation that the flagella exhibit long periods of near-perfect synchrony. This synchronisation may take the form of both frequency and phase locking, or frequency locking with long-wavelength phase modulations in the form of “metachronal waves”. In the past decade there has been a flurry of experimental and theoretical activity that has uncovered the dominant mechanisms underlying this synchronisation, namely hydrodynamical coupling through the fluid surrounding the organisms and elastic couplings between the bases of the flagella inside the cells. This essay would ideally be both a review of the important theoretical and experimental observations regarding synchronisation and an exploration of the possibility that metachronal waves arise from spatially-varying coupling between distant flagella in a form that arises from unsteady Stokes flows.

Relevant Courses

Essential: Slow Viscous Flow or Biological Physics

Useful:

References

[1] Taylor, G. I., Analysis of the swimming of microscopic organisms. *Proc. R. Soc. Lond. A* **209**, 447–461 (1951).

[2] Niedermayer, T., Eckhardt, B. & Lenz, P., Synchronization, phase locking, and metachronal wave formation in ciliary chains. *Chaos* **18**, 037128 (2008).

[2] Polin, M., Tuval, I., Drescher, K., Gollub, J.P. & Goldstein, R.E. *Chlamydomonas* swims with two 'gears' in a eukaryotic version of run-and-tumble locomotion. *Science* **325**,487–490 (2009).

[4] Brumley, D.R., Wan, K.Y., Polin, M. & Goldstein, R.E. Flagellar synchronization through direct hydrodynamic interactions. *eLife* **3**, e02750 (2014).

[5] Brumley, D.R., Polin, M., Pedley, T.J. & Goldstein, R.E. Metachronal waves in the flagellar beating of *Volvox* and their hydrodynamic origin. *J. R. Soc. Interface* **12**, 20141358 (2015).

[6] Brout, N., *et al.*. Direct measurement of unsteady microscale Stokes flow using optically driven microspheres. *Phys. Rev. Fluids* **6**, 053102 (2021).

107. C-theorems ....

Dr A. C. Wall

35 years ago, Alexander Zamolodchikov proved the existence of a quantity C in 2 dimensional field theories, which monotonically decreases under renormalization group flow, is stationary at fixed points, and agrees with the central charge c there. This c-theorem is interesting, because it restricts which field theories can flow to other field theories, even at strong coupling where calculations may be difficult.

Recently, there has been work extending monotonicity results to higher dimensions, for example there is an F-theorem in 3 dimensions and an a-theorem in 4 dimensions. Another interesting development is alternative proofs of c-theorems which rely on properties of the vacuum entanglement entropy.

In this essay you will critically review the current status of c-theorems in various dimensions. Your essay should carefully examine two or more proofs in detail, paying careful attention to all limitations (e.g. can you find counterexamples when some of the assumptions are not met?). It should also explain at least one nontrivial application of monotonicity to specific field theories.

Relevant Courses

Essential: Advanced Quantum Field Theory *or* Statistical Field Theory

Useful: String Theory, Quantum Information Theory

References

[1] A. B. Zamolodchikov (1986), " 'Irreversibility' of the Flux of the Renormalization Group in a 2-D Field Theory", JETP Lett. 43: 730–732.

[2] H. Casini, M. Huerta "A c-theorem for the entanglement entropy", J. Phys. A40:7031-7036,2007, arxiv:cond-mat/0610375.

[3] T. Grover, "Chiral Symmetry Breaking, Deconfinement and Entanglement Monotonicity", Phys. Rev. Lett. 112, 151601 (2014), arXiv:1211.1392.

[4] Z. Komargodski, A. Schwimmer, "On Renormalization Group Flows in Four Dimensions", JHEP12(2011)099, arXiv:1107.3987.

[5] H. Casini, E. Teste, G. Torroba, "Markov property of the CFT vacuum and the a-theorem", Phys. Rev. Lett. 118, 261602 (2017), arXiv:1704.01870.

108. The central limit theorem in infinite dimensions ....

Professor R. Nickl

The classical central limit theorem asserts that for i.i.d. mean zero random vectors $X_1, \dots, X_N$ in a finite-dimensional normed linear space $B \simeq \mathbb{R}^d$, the rescaled partial sums $N^{-1/2} \sum_{i=1}^N X_i$ converge in distribution to a normal random vector if and only if $E\|X_1\|_B^2 < \infty$. Strikingly, when B is instead an infinite-dimensional Banach space, the latter moment condition is not sufficient any longer, but rather the geometry of B starts to play an essential role [3]. By duality arguments for the Banach space B, the problem can be equivalently posed as the *uniform central limit theorem* for empirical measures $P_N = N^{-1} \sum_{i=1}^N \delta_{X_i}$, where one considers weak convergence of abstract function class $\mathcal{F}$ indexed stochastic processes

$$f \mapsto \nu_N(f) := \sqrt{N}(P_N - P) = \frac{1}{\sqrt{N}} \sum_{i=1}^N (f(X_i) - Ef(X_1)), \quad f \in \mathcal{F},$$

towards some Gaussian process $(\mathbb{G}(f) : f \in \mathcal{F})$, see [2,3,4]. The techniques underlying proofs of such infinite-dimensional (or functional) central limit theorems have been at the heart of key developments in *concentration of measure theory* and *high-dimensional probability*, with a wide range of applications in mathematical statistics and machine learning [3,4]. This essay will explore this material in some detail, possibly (but not necessarily) with an eye on applications of such results in contemporary data science.

Relevant Courses

Essential: Probability & Measure and Linear analysis from Part II.

Useful: Principles of Statistics and Mathematics of Machine Learning from Part II

References

[1] Richard M Dudley, *Uniform central limit theorems*, 2nd edition, Cambridge University Press 2014

[2] Evarist Giné and Richard Nickl, *Mathematical foundations of infinite-dimensional statistical models*, Cambridge University Press 2016.

[3] Michel Ledoux and Michel Talagrand; *Probability in Banach spaces.* Springer 1991

[4] Aad van der Vaart and Jon Wellner, *Weak convergence and empirical processes*, Springer 1996.

109. Integrable Systems from Four Dimensional Chern-Simons Theory ..... David Skinner

Integrable systems are non-linear systems that are nonetheless tractable. There are a bewildering array of examples, each of which serves as a starting point for perturbation towards more realistic systems. However, 'nearby' interacting systems often lose integrability completely, so it's interesting to ask what makes a theory integrable in the first place.

Over the past few years, Costello, Witten & Yamazaki have shown that many integrable systems (both classical and quantum) can be obtained from a four-dimensional QFT based on a variant of Chern-Simons theory. This is a gauge theory, but with an action and properties that are very different from those of Yang-Mills theory. CWY show that the observables of this theory are completely equivalent to various integrable systems, where the details of which specific system one obtains depends on choices (such as the choice of gauge group, the boundary conditions on the fields, and the presence of line and surface defect operators) one makes in setting up the four-dimensional theory.

Relevant Courses

Essential: Quantum Field Theory and Advanced Quantum Field Theory are both essential.

Useful: Prior knowledge of some integrable systems (such as from the Part II Integrable Systems course or similar) may be helpful. Some familiarity with both Statistical Field Theory and with Representation Theory may also be useful.

References

[1] Costello, K., Witten, E. & Yamazaki, M., *Gauge Theory and Integrability I*, ICCM Not. **6** (2018), 46-119.

[2] Costello, K., Witten, E. & Yamazaki, M., *Gauge Theory and Integrability II*, ICCM Not. **6** (2018), 120-146.

[3] Costello, K. & Yamazaki, M., *Gauge Theory and Integrability III*, 1908.02289 [`hep-th`].

[4] Baxter, R.J., *Exactly Solved Models in Statistical Mechanics*, Acad. Press. (1989).

[5] Dorey, P. *Exact S-Matrices*, `hep-th/9810026`.

[6] Lamers, J., *A Pedagogical Introduction to Quantum Integrability*, PoS. Modave2014 (2015) 001-074.

110. Amplitude constraints on effective field theories ....
S. Melville

The fundamental physical principles of unitarity, causality and locality underpin all modern quantum field theory techniques. The goal of this essay is to translate these foundational (physically intuitive) notions into corresponding (mathematically rigorous) properties obeyed by scattering amplitudes. These properties are powerful because they can be used to systematically rule out effective field theories—in particular, many models can produce reasonable-looking phenomenology on large scales, but through their scattering amplitudes we can infer a violation of unitarity, causality or locality on short scales. Leveraging these basic principles to constrain low-energy models is nowadays known as “positivity bounds”, or “S-matrix bootstrap” [1-4].

After reviewing these amplitude constraints, a successful essay would then apply these to *either* (i) chiral perturbation theory, thereby bounding the scattering lengths of pions from first principles [5-7], *or* (ii) Einstein-Maxwell theory for photons coupled to gravity, thereby “proving” the weak gravity conjecture [8-10].

Relevant Courses

Essential: Quantum Field Theory

Useful: Symmetries, Fields and Particles; Advanced Quantum Field Theory

References

S Matrix constraints:

[1] “S-matrix theory of strong interactions”,
G. F. Chew 1961
Benjamin, New York (Available from: https://archive.org/details/smatrixtheoryofs00chew)

[2] “The Analytic S Matrix”,
R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne 1966
Cambridge University Press (Google-Books-ID: VWTnlTyjwjMC)

[3] “Causality, analyticity and an IR obstruction to UV completion”,
A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi 2006
JHEP 06 (2010) 014 (arXiv:hep-th/0602178)

[4] “Causality, Unitarity and Symmetry in Effective Field Theory”
T. Trott 2020
JHEP 07 (2021) 143 (arXiv:2011.10058)

Chiral perturbation theory:

[5] ‘Effective Field Theory with Nambu-Goldstone Modes”,
A. Pich 2018
Les Houches lecture notes (arXiv:1804.05664, though see also hep-ph/9806303 for long version)

[6] “Dispersion Relation Bounds for $\pi\pi$ Scattering”,
A.V. Manohar and V. Mateu 2008
Phys. Rev., D77:094019 (arXiv:0801.3222)

[7] ‘Generalized positivity bounds on chiral perturbation theory”,
Y.-J. Wang, F.-K. Guo, C. Zhang and S.-Y. Zhou 2020
JHEP 07 (2020) 214 (arXiv:2004.03992)

Weak Gravity Conjecture:

[8] "The String landscape, black holes and gravity as the weakest force",
N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa 2006
JHEP 06 (2007) 060 (arXiv:hep-th/0601001)

[9] "Infrared Consistency and the Weak Gravity Conjecture",
C. Cheung and G. N. Remmen 2014
JHEP 12 (2014) 087

[10] "Causality, Unitarity, and the Weak Gravity Conjecture",
N. Arkani-Hamed, Y.-t. Huang, J.-Y. Liu and G. N. Remmen 2021
(arXiv:2109.13937)

111. Type theory for homotopy theory and higher categories .... Dr J. Vicary

Homotopy type theory [1] is a modern approach to synthetic topology, using methods from computer science to give term representations of geometrical entities. The approach is promising as a new foundation for formalization of mathematics, and gives a new way to reason about homotopical notions. Other type theoretical approaches to homotopy theory and higher category theory have also emerged recently [2, 3]. There is plenty of fertile ground here for an essay, which could summarize the basic technical ideas in one or more of these approaches, explaining the main technical innovations, and making clear how they give a language for homotopy theory. A distinction-level essay could go on to look more closely at some more advanced aspects, where there may also be the potential for original results.

References

[1] The Univalent Foundations Program, "Homotopy Type Theory: Univalent Foundations of Mathematics", `https://homotopytypetheory.org/book/`

[2] E. Finster and S. Mimram, "A Type Theoretical Definition of Weak ω-Categories", 2017, 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS),
`https://arxiv.org/abs/1706.02866`

[3] G. Brunerie, "On the homotopy groups of spheres in homotopy type theory," 2016,
`https://arxiv.org/abs/1606.05916`

112. Hilbert's 16th problem and biochemical reaction networks .... Dr T. Plesa and Dr M. Bruna

A *reaction network* is a set of rules (called reactions) according to which a given set of elements (called species) interact [1,2]. For example, the network $\{X_1 \to X_2, X_1 + X_2 \to X_3\}$ involves a set of three species $\{X_1, X_2, X_3\}$ interacting according to two reactions: the first one describes a conversion of the species X_1 into the species X_2, while the second one describes an interaction between X_1 and X_2 which produces a new species X_3. Reaction networks are suitable for modelling a variety of chemical and biological phenomena, particularly processes in biochemistry - a branch of science that studies molecular architecture of living systems.

Under suitable conditions, the temporal evolution of molecular concentrations of the species from a given biochemical network can be described by a system of first-order autonomous ordinary differential equations (ODEs) with polynomial right-hand sides, called *polynomial ODEs.*

A question of great importance is: given a biochemical network, what are the long-time solutions of the corresponding ODEs? In particular, a species concentration may converge to a time-independent solution (equilibrium) or, more exotically, to a time-dependent one, such as a periodic solution (cycle). Of special interest are isolated periodic solutions, known as *limit cycles*. Biochemically, limit cycles play critical roles in a wide range of processes key to survival, such as circadian clocks which synchronize physiology of living organisms with periodic changes of day and night [3].

Analysis of limit cycles in polynomial ODEs is in general exceptionally difficult. In 1900, David Hilbert posed 23 mathematical challenges, 16th of which includes the following seemingly simple question: what is the maximum number and relative location of limit cycles in *two-dimensional* (two-variable) polynomial ODE systems of degree n [4]? After more than 100 years, Hilbert's 16th problem remains unsolved for every $n \geq 2$ [5]. In context of biochemistry, this problem is further complicated by the fact that, as opposed to general polynomial ODEs, the dependent variables in the biochemical ODEs must be non-negative and bounded, as they represent molecular concentrations. An approach to address Hilbert's 16th problem is to design specific example polynomial ODEs which display a given number of limit cycles. Using this approach, specific two-dimensional quadratic ODEs displaying four limit cycles have been designed [6]; in biochemical context, two-dimensional cubic ODEs with two coexisting stable limit cycles have been put forward [7].

There is a considerable degree of flexibility in this essay. One option is to review the literature on general polynomial ODEs with multiple limit cycles, and highlight how it relates to biochemical reaction networks. Focus could also be placed on the methods which allow one to map non-biochemical polynomial ODEs to biochemical ones; in this case, there is an opportunity for designing novel biochemical networks with multiple limit cycles. One could also focus on investigating how stochastic effects (noise) influence the dynamics of such exotic biochemical systems.

Relevant Courses

Useful (but not essential): Mathematical Biology, Dynamical Systems.

References

[1] Feinberg, M., 2019. Foundations of Chemical Reaction Network Theory. 10.1007/978-3-030-03858-8.

[2] Tóth, J., Nagy, A. L., Papp, D., 2018. Reaction Kinetics: Exercises, Programs and Theorems: Mathematica for Deterministic and Stochastic Kinetics. Germany: Springer New York.

[3] Vilar, J. M. G., Kueh, H. Y., Barkai, N., Leibler, S. Mechanisms of noise-resistance in genetic oscillators. *Proceedings of the National Academy of Sciences, USA*: 5988–5992, 2002.

[4] Hilbert, D., 1902. Mathematical problems. *Mary Newton Transl., Bull. Amer. Math. Soc.*, 8: 437–479.

[5] Ilyashenko, Y. Centennial History of Hilbert's 16th Problem. *Bulletin of the American Mathematical Society* 39: 301–354, 2002.

[6] Kuznetsov, N.V., Kuznetsova, O.A., Leonov, G.A. Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system. *Differ. Equ. Dyn. Syst.*, 2012.

[7] Plesa, T., Vejchodský, T., Erban, R. 2017. Test models for statistical inference: Two-dimensional reaction systems displaying limit cycle bifurcations and bistability. *Stochastic Dynamical Systems, Multiscale Modeling, Asymptotics and Numerical Methods for Computational Cellular Biology*, Springer International Publishing.

113. Generative Adversarial Networks in Biomedical Imaging .... Dr P. W. Sweeney

In recent years, generative adversarial networks (GANs) [1] have started a revolution in deep learning, in part, due to their unsupervised nature of learning and cross-disciplinary applicability. For example, in particle physics [2], image enhancement [3] or more nefarious applications [4]. At their core, GANs are a zero-sum game between neural networks which dynamically and automatically discover and learn patterns in input data. In doing so, the model generates new synthetic data which could have plausibly been drawn from the original dataset. As such, GANs lend themselves to biomedical image analysis where deep learning approaches are hindered by their limited access to large annotated datasets [5].

The essay should provide a rigorous understanding of the mathematics behind existing, popular GAN architectures [6,7]. The essay writer should then explore their application in biomedical imaging, for example, in image-to-image translation, semantic segmentation and data synthesis tasks; and discuss any problems regarding their applicability, in addition to any open questions which they find interesting.

Relevant Courses

Essential: None.

Useful: Background knowledge in Machine Learning and Statistics is useful but not essential.

References

[1] Goodfellow, I. J. (2014). Generative Adversarial Networks. arXiv:1406.2661.

[2] ATLAS, Collaboration. (2018). Deep generative models for fast shower simulation in ATLAS.

[3] Wang, X. et al. (2019). ESRGAN: Enhanced Super-Resolution Generative Adversarial Networks. ECCV 2018: Computer Vision – ECCV 2018 Workshops, pp 63-79.

[4] Doyle, M. (May 16, 2019). John Beasley lives on Saddlehorse Drive in Evansville. Or does he?. Courier and Press.

[5] Skandarani, Y. (2021). GANs for Medical Image Synthesis: An Empirical Study. arXiv:2105.05318.

[6] Stanczuk, J. (2021). Wasserstein GANs Work Because They Fail (to Approximate the Wasserstein Distance). arXiv:2103.01678.

[7] Wang, Y. (2020). A Mathematical Introduction to Generative Adversarial Nets. arXiv:2009.00169.

114. Unconditional quantum advantage .... Dr. S. Strelchuk

In a circuit model, demonstrating quantum computational advantage relied on assuming one or several complexity-theoretic conjectures. Recently, there has been a demonstration of the first rigorous separation between the computational power of the so-called 'shallow' (constant-depth) circuits and their constant-depth classical counterparts [1]. It was shown that shallow quantum circuits can solve 2D Hidden Linear Function problem with unit probability whereas the analogous classical circuits compute the answer with probability at most $\frac{7}{8}$. This was further strengthened in [2] to a similar separation in the average-case setting.

Furthermore, quantum advantage was shown to hold even when quantum circuits are subject to noise [2,3] while classical circuits remain noiseless. This time, the problem being solved is a Magic Square problem, which is related to the so-called Mermin-Peres Magic Square.

This essay should discuss unconditional quantum advantage for noiseless [1,2] and, optionally, noisy [3,4] quantum circuits.

Relevant Courses

Useful:

Quantum Information Theory (M24)

Quantum Computation (L24)

References

[1] Bravyi, S., Gosset, D., and König, R. (2018). Quantum advantage with shallow circuits. Science, 362(6412), 308-311.

[2] Bravyi, S., Gosset, D., Koenig, R., and Tomamichel, M. (2020). Quantum advantage with noisy shallow circuits. Nature Physics, 16(10), 1040-1045.

[3] Atsuya Hasegawa, François Le Gall, "Quantum Advantage with Shallow Circuits under Arbitrary Corruption", https://arxiv.org/abs/2105.00603

[4] Le Gall, F. (2019). Average-case quantum advantage with shallow circuits. In 34th Computational Complexity Conference (CCC 2019). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik. https://arxiv.org/abs/1810.12792

115. Field Theories of Run-and-Tumble motion .... Dr J Pausch

Run-and-Tumble motion is the stochastic process that describes the movement of many bacteria such as *E.coli*. It consists of a repeated and random switching between two types of motion: ballistic movement along a straight line and a random change of orientation. As bacteria might sometimes be trapped or they might follow a gradient, Run-and-Tumble motion is also studied in potentials – the easiest being the harmonic potential [2]. Furthermore, these bacteria often appear in groups in which their movement might be inhibited. In such settings, their collective behaviour can show phase separation [1].

A good essay on this topic will survey the relevant literature and highlight the interesting observables and the reasons why they are of interest.

Relevant Courses

Potentially Useful: Theoretical Physics of Soft Condensed Matter, Statistical Field Theory, Non-equilibrium Statistical Field Theory, Stochastic Processes in Theoretical Physics

References

[1] Statistical Mechanics of Interacting Run-and-Tumble Bacteria, J Tailleur and M E Cates, Phys. Rev. Lett. **100**, 218103, May 2008.

[2] Run-and-tumble motion: field theory and entropy production, Rosalba Garcia-Millan and Gunnar Pruessner, J. Stat. Mech.: Theory Exp. **2021** 063203, June 2021.

[3] Field Theory of Free Run and Tumble Particles in d Dimensions, Ziluo Zhang and Gunnar Pruessner, arXiv:2106.07383, 2021.

116. Coherence and string diagrams for monoidal categories .... P.T. Johnstone

Coherence theory begins with the proof by Saunders Mac Lane [1] that a finite list of commutative diagrams suffices to ensure the commutativity of all possible diagrams which can be drawn using the associativity and unit isomorphisms in a monoidal category. Subsequently similar results were established for more complicated structures such as closed categories [2,3]; however, the original proofs of these results, using essentially syntactic techniques, became increasingly complex and hard to understand. The introduction of the essentially geometrical calculus of string diagrams (borrowed from the Feynman diagrams of theoretical physics) made it possible to give much more conceptual proofs of these theorems; they also exposed unexpected connections between this area of category theory and knot theory. Good survey articles include [4] and [5].

Relevant Courses

Essential: Category Theory

References

[1] S. Mac Lane, Natural associativity and commutativity. Rice University Studies 49 (1963), 28-46.

[2] G.M. Kelly and S. Mac Lane, Coherence for closed categories, J. Pure Appl. Algebra 1 (1971), 97–140.

[3] G.M. Kelly, M. Laplaza, G. Lewis and S. Mac Lane, Coherence in categories, Lecture Notes in Math. vol. 281 (Springer–Verlag, 1972).

[4] A. Joyal and R.H. Street, Braided tensor categories, Adv. Math. 102 (1993), 20-78.

[5] P. Selinger, A survey of graphical languages for monoidal categories, in: Bob Coecke (ed.) New Structures for Physics, Lecture Notes in Physics, vol 813 (Springer–Verlag, 2010).

117. Derived Category, Stability Conditions, K3 Surfaces. .... I. Grojnowski

The aim of this essay is to give an introduction to derived categories and homological algebra, as it is applied in algebraic geometry and/or in representation theory.

As well as learning some abstract machinery, you will be gaining proficiency in either computing cohomology of coherent sheaves (geometry), or in representation theory (quivers, symplectic reflection algebras, finite groups).

There are various ways into this essay, for those with and without algebraic geometry.

If you know some algebraic geometry, then an ambitious aim for this essay would be to understand the papers of Bayer and Macri computing the nef and movable cones for moduli spaces of sheaves on a K3 surface.

Begin with Fourier-Mukai transform, which is an involution on the derived category of an elliptic curve. Use this to describe all vector bundles on an elliptic curve. More generally, do this for abelian varieties. A nice consequence is the Torelli theorem.

Then back up, and learn classical Koszul duality, the derived equivalence between modules for SV and $\wedge V$; as well as Beilinson's theorem describing all sheaves on P^n.

You may then learn the theorem that every quasi-compact quasi-seperated scheme is 'derived affine' (Bondal and van den Bergh, after Thomasan and Neeman).

After this, try some explicit examples of flops by blowing up and down; this is due to Bondal-Orlov. You can continue in this way, and learn some of the derived category approach to birational geometry. Some lovely classical computations can be studied in this language; see for example the papers of Kuznetsov.

This is already enough for an essay, but if you're a glutton for excess:

Now read Bridgeland's papers on stability conditions. He constructs a complex variety attached to an abelian or derived category, each point of which parameterizes a t-structure and a notion of stability condition. You should compute these spaces for a line bundle over P^2, at least.

You are now in a position to read the papers of Bridgeland, and Bayer and Macri, which compute this for a K3 surface, and relate it to the minimal model program.

If you don't know any algebraic geometry, this is still a good essay— derived equivalences are at the heart of representation theory— but the examples you should work with are representations of groups, or quivers, or Hecke algebras. One very rich source of examples are the 'cluster algebras' of Fomin-Zelevinsky; Keller's survey paper (listed below) is a fun introduction.

If you are interested in this essay, come talk to me and we'll find a way into the subject that works best with your background.

References

[1] For the Basics of Homological Algebra, the textbooks of Weibel, and Gelfand-Manin.

[2] Bridgeland, Spaces of Stability Conditions, `arxiv/0611510`.

[3] Bridgeland, Derived Categories of Coherent Sheaves, `arxiv/0602129`.

[4] Bondal, Orlov, Derived Categories of Coherent Sheaves, `arxiv/0206295`.

[5] Keller, Cluster Algebras, Quiver Representations and Triangulated Categories, `arxiv/0807.1960`.

[6] Bayer, Macri, MMP for Moduli of Sheaves on K3s via Wall Crossing, `arXiv/1301.6968`.

[7] Bayer, Macri, Projectivity and Birational Geometry of Bridgeland Moduli Spaces `arxiv/1203.4613`.

[8] Bridgeland, Stability Conditions on K3 Surfaces, `arxiv/0307164`.

118. Random walk on dynamical percolation .... **Perla Sousi**

Let $G = (V, E)$ be a finite connected graph and let $p \in (0, 1)$. Assign independent Poisson processes of parameter μ to every edge $e \in E$. At the time of the Poisson process, the state of the edge e refreshes to open with probability p and closed with probability $1 - p$. Let X be a continuous time random walk that moves as follows: it stays at the current vertex for an exponential time of parameter 1 and then chooses one neighbour (in the graph G) equally likely. If the edge connecting the two vertices is open, then the walk makes the jump, otherwise it stays in place.

How long does it take for the walk to reach equilibrium? What is the maximum expected hitting time and the cover time? In recent works, estimates on these quantities have been established, but the bounds are not sharp and in the case of $G = \mathbb{Z}_n^d$ they do not cover the whole supercritical regime.

A successful essay will contain background on the problem, give an account of the current state of the art and include proofs of some results.

Relevant Courses

Essential: Advanced Probability, Mixing Times of Markov Chains

References

[1] Y. Peres, A. Stauffer and J. Steif. Random walks on dynamical percolation: mixing times, mean squared displacement and hitting times. Probab. Theory Related Fields 162 (2015), no. 3-4, 487-530.

[2] Y. Peres, P. Sousi and J. Steif. Mixing time for random walk on supercritical dynamical percolation. Probab. Theory Related Fields 162 (2020), no. 3-4, 809-849.

[3] J. Hermon and P. Sousi. A comparison principle for random walk on dynamical percolation. Ann. Probab. 48 (2020), no. 6, 2952-2987.

119. Evolution of Antibiotic Resistance in Heterogeneous Environments ... **Dr Nuno M Oliveira**

The process in which bacterial populations evolve resistance to antibiotics poses an important threat to public health. Recent mathematical models and *in vitro* experiments [1,2] suggest that spatial heterogeneity accelerates the evolution of resistance. These mathematical models take the form of continuous time Markov processes, with space represented by discrete compartments [3,4] or continuum [5,6]. While stochastic nature of models is needed to properly capture the rare mutation events, it is instructive to compare these models with older deterministic ”island” models [7]. ”Island” models neglect mutations completely and concentrate on competitive ecological interactions only. A good essay will survey the stochastic models, where the student compares and contrasts these models to their deterministic counterpart, highlighting the advantages and disadvantages of each for modelling the evolution of antibiotic resistance.

Relevant Courses

Essential: Mathematical Biology, Dynamical Systems

Useful: Stochastic Processes in Theoretical Physics and Biology

Potentially useful: Biological Physics and Fluid Dynamics, Topics in Mathematical Biology, Theoretical Physics of Soft Condensed Matter, Non-equilibrium Statistical Field Theory

References

[1] Qiucen Zhang, Guillaume Lambert, David Liao, Hyunsung Kim, Kristelle Robin, Chih-kuan Tung, Nader Pourmand, and Robert H Austin. Acceleration of emergence of bacterial antibiotic resistance in connected microenvironments. Science, 333(6050):1764–1767, 2011.

[2] Michael Baym, Tami D Lieberman, Eric D Kelsic, Remy Chait, Rotem Gross, Idan Yelin, and Roy Kishony. Spatiotemporal microbial evolution on antibiotic landscapes. Science, 353(6304): 1147–1151, 2016.

[3] Rutger Hermsen and Terence Hwa. Sources and sinks: a stochastic model of evolution in hetero-geneous environments. Physical review letters, 105(24):248104, 2010.

[4] Rutger Hermsen, J Barrett Deris, and Terence Hwa. On the rapidity of antibiotic resistance evolution facilitated by a concentration gradient. Proceedings of the National Academy of Sciences, 109(27):10775–10780, 2012.

[5] R Hermsen. The adaptation rate of a quantitative trait in an environmental gradient. Physical biology, 13(6):065003, 2016.

[6] Harrison Steel and Antonis Papachristodoulou. The effect of spatiotemporal antibiotic inhomo-geneities on the evolution of resistance. Journal of theoretical biology, 486:110077, 2020.

[7] Thomas Lenormand. Gene flow and the limits to natural selection. Trends in Ecology & Evolution, 17(4):183–189, 2002.

120. Universal Portfolios ....

Professor I. Kontoyiannis

This essay will explore and describe the connections between information-theoretic tools and the theory of *universal portfolios* in finance. This is a collection of ideas, mathematical techniques, and algorithms for selecting an optimal portfolio for investment in general discrete-time markets. Various notions of optimality are considered, and the methods introduced lead to very interesting probabilistic analyses. The essay is expected to describe the basic theory as outlined in Chapter 16 of the textbook [1] and developed in the first papers [2–5]. More ambitious essays will explore the more recent literature, in consultation with the essay setter.

Relevant Courses

Essential: Basic probability

Useful: Coding and cryptography

References

[1] Cover, T.M. and Thomas, J.A. *Elements of information theory*, second edition, J. Wiley & Sons, New York, 2012.

[2] Bell, R.M., and Cover, T.M. "Competitive optimality of logarithmic investment." *Mathematics of Operations Research*, **5**, no. 2, pp. 161-166, May 1980.

[3] Cover, T.M. "An algorithm for maximizing expected log investment return." *IEEE Transactions on Information Theory*, **30**, no. 2, pp. 369-373, 1984.

[4] Cover, T.M., and Gluss, D.H. "Empirical Bayes stock market portfolios." *Advances in Applied Mathematics*, **7**, no 2 pp. 170-181, 1986.

[5] Bell, R., and Cover, T.M. "Game-theoretic optimal portfolios." *Management Science*, **34**, no. 6, pp. 724-733.

121. t-Intersecting Families .... Professor I. Leader

This essay focuses on two very deep and interesting results about families that are t-intersecting as opposed to just intersecting.

The first is the 'complete intersection theorem' of Ahlswede and Khachatrian, which solves the conjecture of Frankl by finding the greatest possible size of a t-intersecting family of r-sets from a ground set of size n. This is, of course, answered by the second Erdos-Ko-Rado theorem when n is very large, but that result gives no information at all about what happens for normal-sized n.

The second is about a setting where there is some more structure: permutations. We say that a family of permutations is *intersecting* if for any two of them there is some point that they both map to the same place. It is an easy exercise to show that such a family has size at most $(n-1)!$. But what happens for t-intersecting? The famous conjecture on the answer was the Deza-Frankl conjecture, and this was resolved by an amazing paper of Ellis, Friedgut and Pilpel.

The essay would cover the content of the papers below. Note that the second paper does use a little representation theory, but what is needed is nicely summarised at the start of the paper, so that prior familiarity with representation theory is not vital at all. Note also that the first paper, while short, is very densely written.

Relevant Courses

Essential: Combinatorics

References

[1] R. Ahlswede and L.H. Khachatrian, The complete intersection theorem for systems of finite sets, *European J. Comb.* **18** (1997), 125-136.

[2] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. Amer. Math. Soc.* **24** (2011), 649-682.

122. The stability and instability of Cauchy horizons in general relativity . . Prof. M. Dafermos

The term *Cauchy horizon* was coined by Stephen Hawking in the late 1960s [1]. These horizons delimit the boundary of the region of spacetime which is uniquely determined from initial data. Cauchy horizons indeed occur in the celebrated Reissner–Nordström and Kerr spacetimes in the interior of their black hole regions. This means that the future of observers falling into these black holes can not be predicted from initial data. This "failure of determinism" has important repercussions for the theory.

An attractive way out of the problems posed by Cauchy horizons was conjectured by Penrose [2]: According to his *strong cosmic censorship conjecture*, the region of spacetime uniquely determined from initial data is generically *inextendible*. Cauchy horizons would thus be unstable.

Penrose's conjecture has inspired a lot of work over the years. It is now understood that the instability properties of Cauchy horizons are subtle, and in fact, Cauchy horizons have both stability and instability properties [3, 4, 5, 6, 7, 8]. The purpose of this essay is to survey some part of this work, restricted to spherically symmetric models, where the most complete results are available. The essay could choose to focus more on the C^0 stability of Cauchy horizons for the Einstein–Maxwell–real scalar field system under spherical symmetry [5, 6, 8] or on the C^2 instability [8], but should include at least some discussion of both for context.

Relevant Courses

Essential:

General Relativity, Black holes

Useful:

Analysis of Partial Differential Equations

References

[1] S. W. Hawking. *The occurrence of singularities in cosmology. III. Causality and singularities.* Proc. Roy. Soc. A., 300(1461):187–201, 1967.

[2] R. Penrose. *Gravitational collapse.* In C. Dewitt-Morette, editor, Gravitational Radiation and Gravitational Collapse, volume 64 of IAU Symposium, pages 82–91. Springer, 1974.

[3] J. M. McNamara. Behaviour of scalar perturbations of a Reissner–Nordström black hole inside the event horizon. Proc. R. Soc. A., 364:121–134, 1978.

[4] E. Poisson and W. Israel. Internal structure of black holes. Phys. Rev. D, 41:1796?1809, 1990.

[5] M. Dafermos. Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations. Ann. of Math., 158(3):875–928, 2003.

[6] M. Dafermos. The interior of charged black holes and the problem of uniqueness in general relativity. Comm. Pure Appl. Math., 58(4):445–504, 2005.

[7] M. Dafermos and J. Luk. The interior of dynamical vacuum black holes I: The C^0 Stability of the Kerr Cauchy horizon. arXiv:1710.01722, preprint, 2017.

[8] J. Luk and S.-J. Oh. Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region. Ann. of Math. (2), 190(1):1–111, 2019.

123. Pseudo-finite Model Theory . . . . Professor J. M. E. Hyland

The established techniques of Model Theory do not apply when attention is restricted to finite structures and researchers in Finite Model Theory consider questions and use techniques with no evident analogues in Model Theory proper. However it proves fruitful to change perspective and consider also structures all of whose first-order properties are finitely satisfiable. It turns out that one can then make effective use of usual model theoretic ideas. The resulting area of Pseudo-finite Model Theory is of interest in its own right and has applications inter alia to Combinatorics and Computer Science. An essay could focus on the subject itself or on a chosen application area.

Candidates are advised to start with the surveys in the References. They can easily be found online. Pillay's in particular gives references for a number of directions possible for an essay. Further sources are readily available. Those contemplating the essay may contact Martin Hyland at `m.hyland@dpmms.cam.ac.uk`.

Relevant Courses

Essential: None

Useful: Logic and Computability, Model Theory (Reading Course).

References

[1] A. Pillay and friends. Pseudofinite Model Theory. 2015, available online.

[2] J. Väänänen. Pseudo-finite model theory. Matematica Contemporanea, **24**, 2003, 169-183, available online.

124. Machine Learning for Classification of Astronomical Time Series . . . . . . K. S. Mandel

The night sky is replete with astronomical sources that change in brightness over time, including variable stars, gravitational lensing events, and stellar explosions, such as supernovae and kilonovae. The Vera Rubin Observatory (VRO) is a new 8-meter telescope being constructed in Chile that will begin a 10-year Legacy Survey of Space and Time (LSST) in 2024. It will regularly scan the sky and record brightness time series (light curves) of millions of time-varying sources in multiple colours of light, revolutionising our understanding of these astrophysical phenomena. Astronomers and data scientists have engineered a variety of machine learning algorithms to automatically sift through the massive data streams and classify the variable and transient sources underlying the time series data. The Photometric LSST Astronomical Time Series Classification Challenge (PLAsTiCC) is an open data challenge to the community to develop and apply new methods to classify simulated astronomical time-series data in preparation for observations from LSST. This data challenge poses the question: how well can we classify objects in the sky that vary in brightness from realistic simulated LSST time-series data, with all its observational and statistical challenges?

This essay is meant to review the relevant scientific motivations, the statistical challenges involved, some of the algorithms that have been developed, their pros and cons, and the classification metrics by which their performance are evaluated. The student will have the opportunity to implement the method(s) of his or her choice or invention on the datasets available from the PLAsTiCC challenge posted on Kaggle (https://www.kaggle.com/c/PLAsTiCC-2018) and evaluate their performance, either for the general challenge, or for more focused scientific goals. Originality and creativity are encouraged.

Relevant Courses

Essential: Astrostatistics

References

[1] The PLAsTiCC Team, et al. *The Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC): Data set.* 2018, https://arxiv.org/abs/1810.00001.

[2] Hložek, R., et al. *Results of the Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC).* 2020, https://arxiv.org/abs/2012.12392.

[2] Malz, A., et al. *The Photometric LSST Astronomical Time-Series Classification Challenge (PLAsTiCC): Selection of a Performance Metric for Classifiation Probabilities Balancing Diverse Science Goals.* 2019, The Astronomical Journal, 158, 171.

[3] Narayan, G., et al. *Machine-learning-based Brokers for Real-time Classification of the LSST Alert Stream.* 2018, The Astrophysical Journal Supplement, 236, 9.

[4] Lochner, M., et al. *Photometric Supernova Classification with Machine Learning.* 2016, The Astrophysical Journal Supplement, 225, 31.

[5] Richards, J., et al. *Semi-supervised learning for photometric supernova classification.* 2012, Monthly Notices of the Royal Astronomical Society, 419, 1121.

[6] Kessler, R., et al. *Results from the Supernova Photometric Classification Challenge.* 2010, Publications of the Astronomical Society of the Pacific, 122, 1415.

125. Clouds in Exoplanetary Atmospheres .... Professor N. Madhusudhan

Clouds are one of the main factors obscuring observable features in transmission spectra of the atmospheres of extrasolar planets. Clouds are thought to be ubiquitous in planetary atmospheres and can be particularly challenging for molecular detections in temperate low-mass exoplanets, including bio-signatures in Earth-like exoplanets. Accurately modelling the physics of these clouds is therefore important for robust atmospheric characterisation of exoplanets. Microphysics-based models, which describe clouds by examining the processes undergone by their constituent particles (principally nucleation, condensation, and evaporation), have shown success in modelling the atmospheres of solar system planets and of hot Jupiters.

This essay will describe the theory and governing equations of cloud microphysics. It will survey and compare the existing cloud models in the literature and assess their suitability for modelling Earth-like exoplanet atmospheres and fitting such models to observations. Emphasis will be placed on the tradeoff between accuracy and computational complexity, and on developing a framework in which to understand the accuracy required to gain a first order understanding of the relevant physics, for use as a first level of analysis before committing greater computational resources.

Relevant Courses

Essential: Extrasolar Planets - Atmospheres and Interiors

Useful: Inverse Problems, Analysis of Partial Differential Equations

References

[1] Ackerman, A. and Marley, M. "Precipitating Condensation Clouds in Sub- stellar Atmospheres", The Astrophysical Journal 556.2 (Aug. 2001), pp. 872–884. doi: 10.1086/321540. arXiv: astro-ph/0103423 [astro-ph]

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126. Symmetries and quantum error correction .... Dr. S. Strelchuk

To realize full capabilities of quantum computers it is vital to be able to protect against errors that occur during the computation. Quantum error correction codes are designed to address this problem. However, achieving full protection against arbitary errors comes at a price: It was shown that there exists no quantum error correcting code that is capable to transversely implement a universal gate set [1,2].

There are a number of ways to circumvent this limitation: using concatenated stabilizer codes [3], quantum state distillation [4] and ideas from holographic theories of quantum gravity [5]. The essay should discuss Eastin-Knill theorem and its approximate version as well ways to circumvent the limitation.

Relevant Courses

Useful:

Quantum Information Theory (M24)

Quantum Computation (L24)

References

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127. Cartan geometries in elasticity and soft matter physics .... Dr R. Adhikari

Elie Cartan provided an approach to differential geometry that gives primacy to the notion of invariance under group action. The computational apparatus in this approach is the moving frame, expressed as an exterior differential system on a suitable Lie group. The integrability conditions of the system define the torsion and curvature of the geometry. The application of such geometries to the theory of topological defects in crystals was recognised by Kondo, Bilby, Bullough, Smith and Korner. Less recognised is the fact that such geometries are applicable to many problems in elasticity and soft matter, in particular, to the coordinate-free description of sub-manifolds of homogeneous spaces. Examples of the latter includes filaments and sheets in three-dimensional Euclidean space and filaments on the surface of a sphere. The elastic description of materials with microstructure, such as Cosserat solids, is another area where Cartan geometries, with their emphasis on connections rather than metrics, provide a coherent perspective. Further, Cartan geometries suggest structure-preserving discretisations of the governing equations that are invaluable in numerical work.

This essay is an invitation to survey the modern literature on Cartan geometries in a language accessible to the continuum mechanics and soft matter audiences, to situate the classical work of Kondo and others within the modern framework, and to explore applications to moving manifolds and microstructured continuua.

Relevant Courses

Essential: Differential Geometry, Theoretical Physics of Soft Matter

Useful: General Relativity, Numerical Analysis

References

[1] Cartan for beginners: from differential geometry via moving frames and exterior differential systems. T. Ivey and J. Landsberg

[2] Cartan's generalisation of Klein's Erlangen program. R. W. Sharpe

[3] From Frenet to Cartan: the method of moving frames. J. Clelland

[4] Gauge theory and defects in solids: D. Edelen and D. Lagoudas