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LOCALIZATION IN EQUIVARIANT INTERSECTION THEORY AND THE BOTT RESIDUE FORMULA

By DAN EDIDIN and WILLIAM GRAHAM

Abstract. We prove the localization theorem for torus actions in equivariant intersection theory. Using the theorem we give another proof of the Bott residue formula for Chern numbers of bundles on smooth complete varieties. In addition, our techniques allow us to obtain residue formulas for bundles on a certain class of singular schemes which admit torus actions. This class is rather special, but it includes some interesting examples such as complete intersections and Schubert varieties.

1. Introduction. The purpose of this paper is to prove the localization theorem for torus actions in equivariant intersection theory. Using the theorem we give another proof of the Bott residue formula for Chern numbers of bundles on smooth complete varieties. In addition, our techniques allow us to obtain residue formulas for bundles on a certain class of singular schemes which admit torus actions. This class is rather special, but it includes some interesting examples such as complete intersections (cf. [BFQ]) and Schubert varieties.

Let T be a split torus acting on a scheme X . The T -equivariant Chow groups of X are a module over $R_T = \text{Sym}(\hat{T})$, where \hat{T} is the character group of T . The localization theorem states that up to R_T -torsion, the equivariant Chow groups of the fixed locus X^T are isomorphic to those of X . Such a theorem is a hallmark of any equivariant theory. The earliest version (for equivariant cohomology) is due to Borel [Bo]. Subsequently K -theory versions were proved by Segal [Se] (in topological K -theory), Quart [Qu] (for actions of a cyclic group), and Thomason [Th2] (for algebraic K -theory [Th2]).

For equivariant Chow groups, the localization isomorphism is given by the equivariant push-forward i_* induced by the inclusion of X^T to X . An interesting aspect of this theory is that the push-forward is naturally defined on the level of cycles, even in the singular case. The closest topological analogue of this is equivariant Borel-Moore homology (see [E-G3] for a definition), and a similar proof establishes localization in that theory.

For smooth spaces, the inverse to the equivariant push-forward can be written explicitly. It was realized independently by several authors ([I-N], [A-B], [B-V])

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that for compact spaces, the formula for the inverse implies the Bott residue formula. In this paper, we prove the Bott residue formula for actions of split tori on smooth complete varieties defined over an arbitrary field, also by computing $(i_*)^{-1}$ explicitly. Bott's residue formula has been applied recently in enumerative geometry (cf. [E-S], [Ko]) and there was interest in a purely intersection-theoretic proof. Another application of the explicit formula for $(i_*)^{-1}$ is given in [E-G2], where we prove (following Lerman [L]) a residue formula due to Kalkman.

An obvious problem, which should have applications to enumerative geometry (see e.g. [Ko]), is to extend the Bott residue formula to complete singular schemes. Such a formula can be derived when we have an explicit description of $(i_*)^{-1}[X]_T$, where $[X]_T$ denotes the equivariant fundamental class of the whole scheme, as follows: Let $n = \dim X$. If $[X]_T = i_*\alpha$ and $p(E)$ is a polynomial of weighted degree n in Chern classes of equivariant vector bundles on X , then $\deg(p(E) \cap [X])$ can be calculated as the residue of $\pi_*(i^*(p(E)) \cap \alpha)$ where π is the equivariant projection from the fixed locus of X to a point. This approach does not work for equivariant cohomology, because when X is singular there is no push-forward from $H_G^*(X) \rightarrow H_G^*(M)$. However, in K -theory, where such push-forwards exist, similar ideas were used by [BFQ] to obtain Lefschetz-Riemann-Roch formulas for the action of an automorphism of finite order.

The problem of computing $(i_*)^{-1}$ is difficult, but we can do it in a certain class of singular examples, in particular, if there is an equivariant embedding $X \xrightarrow{f} M$ into a smooth variety, and every component of X^T is a component of M^T . This condition is satisfied if $X \subset \mathbb{P}^r$ is an invariant subvariety where T acts linearly with distinct weights (and thus isolated fixed points) or if X is a Schubert variety in G/B . In this context we give a formula (Proposition 6) for $(i_*)^{-1}\alpha$ in terms of $f_*\alpha \in A_*^T(M)$. The case of Schubert varieties is worked out in detail in Section 5.2.

As a consequence it is possible to compute Chern numbers of bundles on X provided we know $f_*[X]_T \in A_*^T(M)$. Thus for example, if X is a T -invariant projective variety and T acts linearly with distinct weights on \mathbb{P}^n , then we can calculate Chern numbers, provided we know the equivariant fundamental class of X . Rather than write down a general formula, we illustrate this with an example: in Section 5 we use a residue calculation to show that

$$\int_Q c_1(\pi^*T_{\mathbb{P}^2})c_1(f^*T_{\mathbb{P}^3}) = 24$$

where $Q \xrightarrow{f} \mathbb{P}^3$ is a (singular) quadric cone, and $\pi: Q \rightarrow \mathbb{P}^2$ is the projection from a point not on Q . The methods can be applied in other examples.

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2. Review of equivariant Chow groups. In this section we review some of the equivariant intersection theory developed in [E-G3]. The key to the theory is the definition of equivariant Chow groups for actions of linear algebraic groups. All schemes are assumed to be of finite type defined over a field of arbitrary characteristic.

Let G be a g -dimensional group, X an n -dimensional scheme and V a representation of G of dimension l . Assume that there is an open set $U \subset V$ such that a principal bundle quotient $U \rightarrow U/G$ exists, and that $V - U$ has codimension more than $n - i$. Thus the group G acts freely on the product $X \times U$, and if any one of a number of mild hypotheses is satisfied then a quotient scheme $X_G = (X \times U)/G$ exists [E-G3]. In particular, if G is special—for example, if G is a split torus, the case of interest in this paper—a quotient scheme X_G exists.

Definition 1. Set $A_i^G(X) = A_{i+l-g}(X_G)$, where A_* is the usual Chow group. This definition is independent of the choice of V and U as long as $V - U$ has sufficiently high codimension.

Remark. Because $X \times U \rightarrow X \times^G U$ is a principal G -bundle, cycles on $X \times^G U$ exactly correspond to G -invariant cycles on $X \times U$. Since we only consider cycles of codimension smaller than the dimension of $X \times (V - U)$, we may in fact view these as G -invariant cycles on $X \times V$. Thus every class in $A_i^G(X)$ is represented by a cycle in $Z_{i+l}(X \times V)^G$, where $Z_*(X \times V)^G$ indicates the group of cycles generated by invariant subvarieties. Conversely, any cycle in $Z_{i+l}(X \times V)^G$ determines an equivariant class in $A_i^G(X)$.

The properties of equivariant intersection Chow groups include the following.

(1) Functoriality for equivariant maps: proper pushforward, flat pullback, l.c.i pullback, etc.

(2) Chern classes of equivariant bundles operate on equivariant Chow groups.

(3) If X is smooth of dimension n , then we denote $A_{n-i}^G(X)$ as $A_G^i(X)$. In this case there is an intersection product $A_G^i(X) \times A_G^j(X) \rightarrow A_G^{i+j}(X)$, so the groups $\bigoplus_0^\infty A_G^i(X)$ form a graded ring which we call the equivariant Chow ring. Unlike the ordinary case $A_G^i(X)$ can be nonzero for any $i \geq 0$. (The existence of an intersection product follows from (1), since the diagonal $X \hookrightarrow X \times X$ is an equivariant regular embedding when X is smooth.)

(4) Of particular use for this paper is the equivariant self-intersection formula. If $Y \xrightarrow{i} X$ is a regularly embedded invariant subvariety of codimension d , then

$$i^* i_*(\alpha) = c_d^G(N_Y X) \cap \alpha$$

for any $\alpha \in A_*^G(Y)$.

2.1. Equivariant higher Chow groups. Let Y be a scheme. Denote by $A_i(Y, j)$ the higher Chow groups of Bloch [Bl] (indexed by dimension) or the groups $CH_{i, i-j}(X)$ defined in [Gi, Section 8]. Both theories agree with ordinary Chow groups when $j = 0$, and both extend the localization short exact sequence for ordinary Chow groups. However, in the case of Bloch's Chow groups the localization exact sequence has only been proved for quasi-projective varieties. The advantage of his groups is that they are naturally defined in terms of cycles on $X \times \Delta^j$ (where Δ^j is an algebraic j -simplex) and are rationally isomorphic to higher K -theory.

Both these theories can be extended to the equivariant setting. We define the higher Chow groups $A_i^G(X, j)$ as $A_{i+l-g}(X_G, j)$ for an appropriate mixed space X_G . Because of the quasi-projective hypothesis in Bloch's work, Bloch's equivariant higher Chow groups are only defined for (quasi)-projective varieties with linearized actions. However, Gillet's are defined for arbitrary schemes with a G -action. We will use two properties of the higher equivariant theories.

(a) If $E \rightarrow X$ is an equivariant vector bundle, then the equivariant Chern classes $c_i^G(E)$ operate on $A_*^G(X, j)$.

(b) If $U \subset X$ is an invariant open set, then there is a long exact sequence

$$\cdots \rightarrow A_i^G(U, 1) \rightarrow A_i^G(X - U) \rightarrow A_i^G(X) \rightarrow A_i^G(U) \rightarrow 0.$$

3. Localization. In this section we prove the main theorem of the paper, the localization theorem for equivariant Chow groups. For the remainder of the paper, all tori are assumed to be split, and the coefficients of all Chow groups are rational.

Let R_T denote the T -equivariant Chow ring of a point, and let \hat{T} be the character group of T .

PROPOSITION 1. [E-G1, Lemma 4] $R_T = \text{Sym}(\hat{T}) \simeq \mathbb{Q}[t_1, \dots, t_n]$, where n is the rank of T .

Remark. The identification $R_T = \text{Sym}(\hat{T})$ is given explicitly as follows. If $\lambda \in \hat{T}$ is a character, let k_λ be the corresponding 1-dimensional representation and let L_λ denote the line bundle $U \times^T k_\lambda \rightarrow U/T$. The map $\hat{T} \rightarrow R_T^1$ given by $\lambda \mapsto c_1(L_\lambda)$ extends to a ring isomorphism $\text{Sym}(\hat{T}) \rightarrow R_T$.

PROPOSITION 2. If T acts trivially on X , then $A_*^T(X) = A_*(X) \otimes R_T$.

Proof. If the action is trivial then $(U \times X)/T = U/T \times X$. The spaces U/T can be taken to be products of projective spaces, so $A_*(U/T \times X) = A_*(X) \otimes A_*(U/T)$. \square

If $T \xrightarrow{f} S$ is a homomorphism of tori, there is a pullback $\hat{S} \xrightarrow{f^*} \hat{T}$. This extends to a ring homomorphism $\text{Sym}(\hat{T}) \xrightarrow{f^*} \text{Sym}(\hat{S})$, or in other words, a map $f^*: R_S \rightarrow R_T$.

LEMMA 1. (cf. [A-B]) *In this situation, suppose there is a T -map $X \xrightarrow{\phi} S$. Then $t \cdot A_*^T(X, m) = 0$ for any $t = f^*s$ with $s \in R_S^+$.*

Proof of Lemma 1. Since A_S^* is generated in degree 1, we may assume that s has degree 1. After clearing denominators we may assume that $s = c_1(L_s)$ for some line bundle on a space U/S . The action of $t = f^*s$ on $A_*(X_T)$ is just given by $c_1(\pi_T^* f^* L_s)$ where π_T is the map $U \times^T X \rightarrow U/T$. To prove the lemma we will show that this bundle is trivial.

First note that $L_s = U \times^S k$ for some action of S on the one-dimensional vector space k . The pullback bundle on X_T is the line bundle

$$U \times^T (X \times k) \rightarrow X_T$$

where T acts on k by the composition of $f: T \rightarrow S$ with the original S -action. Now define a map

$$\Phi: X_T \times k \rightarrow U \times^T (X \times k)$$

by the formula

$$\Phi(e, x, v) = (e, x, \phi(x) \cdot v)$$

(where $\phi(x) \cdot v$ indicates the original S action). This map is well defined since

$$\begin{aligned} \Phi(et, t^{-1}x, v) &= (et, t^{-1}x, \phi(t^{-1}x) \cdot v) \\ &= (et, t^{-1}x, t^{-1} \cdot (\phi(x) \cdot v)) \end{aligned}$$

as required. This map is easily seen to be an isomorphism with inverse $(e, x, v) \mapsto (e, x, \phi(x)^{-1} \cdot v)$. \square

PROPOSITION 3. *If T acts on X without fixed points, then there exists $r \in R_T^+$ such that $r \cdot A_*^T(X, m) = 0$. (Recall that $A_*^T(X, m)$ refers to T -equivariant higher Chow groups.)*

Before we prove Proposition 3, we state and prove a lemma.

LEMMA 2. *If X is a variety with an action of a torus T , then there is an open $U \subset X$ so that the stabilizer is constant for all points of U .*

Proof of Lemma 2. Let $\tilde{X} \rightarrow X$ be the normalization map. This map is T -equivariant and is an isomorphism over an open set. Thus we may assume X is normal. By Sumihiro's theorem, the T action on X is locally linearizable, so it suffices to prove the lemma when $X = V$ is a vector space and the action is diagonal.

If $V = k^n$, then let $U = (k^*)^n$. The n -dimensional torus \mathbf{G}_m^n acts transitively on U in the obvious way. This action commutes with the given action of T . Thus the stabilizer at each closed point of U is the same. \square

Proof of Proposition 3. Since $A_*^G(X) = A_*^G(X_{\text{red}})$ we may assume X is reduced. Working with each component separately, we may assume X is a variety. Let $X^0 \subset X$ be the $(G$ -invariant) locus of smooth points. By Sumihiro's theorem [Su], the action of a torus on a normal variety is locally linearizable (i.e., every point has an affine invariant neighborhood). Using this theorem it is easy to see that the set $X(T_1) \subset X^0$ of points with stabilizer T_1 can be given the structure of a locally closed subscheme of X . By Lemma 2 there is some T_1 such that $U = X(T_1)$ is open in X^0 , and thus in X .

The torus $T' = T/T_1$ acts without stabilizers, but the action of T' on U is not a priori proper. However, by [Th1, Proposition 4.10], we can replace U by a sufficiently small open set so that T' acts freely on U and a principal bundle quotient $U \rightarrow U/T$ exists. Shrinking U further, we can assume that this bundle is trivial, so there is a T map $U \rightarrow T'$. Hence, by the lemma, $t \cdot A_*^T(U) = 0$ for any $t \in A_T^*$ which is pulled back from $A_{T'}^*$.

Let $Z = X - U$. By induction on dimension, we may assume $p \cdot A_*^T Z = 0$ for some homogeneous polynomial $p \in R_T$. From the long exact sequence of higher Chow groups,

$$\dots A_*^T(Z, m) \rightarrow A_*^T(X, m) \rightarrow A_*^T(U, m) \rightarrow \dots$$

it follows that tp annihilates $A_*^T(X)$ where t is the pullback of a homogeneous element of degree 1 in R_S . \square

If X is a scheme with a T -action, we may put a closed subscheme structure on the locus X^T of points fixed by T [Iv]. Now $R_T = \text{Sym}(\hat{T})$ is a polynomial ring. Set $\mathcal{Q} = (R_T^+)^{-1} \cdot R_T$, where R_T^+ is the multiplicative system of homogeneous elements of positive degree.

THEOREM 1. (localization) *The map $i_*: A_*(X^T) \otimes \mathcal{Q} \rightarrow A_*^T(X) \otimes \mathcal{Q}$ is an isomorphism.*

Proof of Theorem 1. By Proposition 3, $A_*^T(X - X^T, m) \otimes \mathcal{Q} = 0$. Thus by the localization exact sequence $A_*^T(X^T) \otimes \mathcal{Q} = A_*^T(X) \otimes \mathcal{Q}$ as desired. \square

Remark. The strategy of the proof is similar to proofs in other equivariant theories, see for example [Hs, Chapter 3.2].

4. Explicit localization for smooth varieties. The localization theorem in equivariant cohomology has a more explicit version for smooth varieties because the fixed locus is regularly embedded. This yields an integration formula from which the Bott residue formula is easily deduced ([A-B], [B-V]). In this section we prove the analogous results for equivariant Chow groups of smooth varieties. Because equivariant Chow theory has formal properties similar to equivariant cohomology, the arguments are almost the same as in [A-B]. As before we assume that all tori are split.

Let F be a scheme with a trivial T -action. If $E \rightarrow F$ is a T -equivariant vector bundle on F , then E splits canonically into a direct sum of vector subbundles $\bigoplus_{\lambda \in \hat{T}} E_\lambda$, where E_λ consists of the subbundle of vectors in E on which T acts by the character λ . The equivariant Chern classes of an eigenbundle E_λ are given by the following lemma.

LEMMA 3. *Let F be a scheme with a trivial T -action, and let $E_\lambda \rightarrow F$ be a T -equivariant vector bundle of rank r such that the action of T on each vector in E_λ is given by the character λ . Then for any i ,*

$$c_i^T(E_\lambda) = \sum_{j \leq i} \binom{r-j}{i-j} c_j(E_\lambda) \lambda^{i-j}.$$

In particular the component of $c_i^T(E_\lambda)$ in R_T^r is given by λ^r .

Proof. $E = E_\lambda$ is an equivariant line bundle with character λ , then $c_1^T(E)$ is calculated as $c_1(E \boxtimes 1_\lambda)$ on the mixed space $F_T \simeq F \otimes BT$. Thus $c_1^T(E) = c_1(E) + \lambda$. The general formula follows from the splitting principle (cf. [Fu1, Example 3.2.2]). \square

As noted above, $A_T^*(F) \supset A^*F \otimes R_T$. The lemma implies that $c_i^T(E)$ lies in the subring $A^*F \otimes R_T$. Because $A^N F = 0$ for $N > \dim F$, elements of $A^i F$, for $i > 0$, are nilpotent elements in the ring $A_T^*(F)$. Hence an element $\alpha \in A^d F \otimes R_T$ is invertible in $A_T^*(F)$ if its component in $A^0 F \otimes R_T^d \cong R_T^d$ is nonzero.

For the remainder of this section X will denote a smooth variety with a T action.

LEMMA 4. [Iv] *If X is smooth then the fixed locus X^T is also smooth.*

For each component F of the fixed locus X^T the normal bundle $N_F X$ is a T -equivariant vector bundle over F . Note that the action of T on $N_F X$ is nontrivial.

PROPOSITION 4. *If F is a component of X^T with codimension d then $c_d^T(N_F X)$ is invertible in $A_T^*(F) \otimes \mathcal{Q}$.*

Proof. By [Iv, Proof of Proposition 1.3], for each closed point $f \in F$, the tangent space $T_f F$ is equal to $(T_f X)^T$, so T acts with nonzero weights on the normal space $N_f = T_f X / T_f F$. Hence the characters λ_i occurring in the eigenbundle decomposition of $N_F X$ are all nonzero. By the preceding lemma, the component of $c_d^T(N_F X)$ in R_T^d is nonzero. Hence $c_d^T(N_F X)$ is invertible in $A_T^*(F) \otimes \mathcal{Q}$, as desired. \square

Using this result we can get, for X smooth, the following more explicit version of the localization theorem.

THEOREM 2. (Explicit localization) *Let X be a smooth variety with a torus action. Let $\alpha \in A_*^T(X) \otimes \mathcal{Q}$. Then*

$$\alpha = \sum_F i_{F*} \frac{i_F^* \alpha}{c_{d_F}^T(N_F X)},$$

where the sum is over the components F of X^T and d_F is the codimension of F in X .

Proof. By the surjectivity part of the localization theorem, we can write $\alpha = \sum_F i_{F*}(\beta_F)$. Therefore, $i_F^* \alpha = i_F^* i_{F*}(\beta_F)$ (the other components of X^T do not contribute); by the self-intersection formula, this is equal to $c_{d_F}^T(N_F X) \cdot \beta_F$. Hence $\beta_F = \frac{i_F^* \alpha}{c_{d_F}^T(N_F X)}$ as desired. \square

Remark. This formula is valid, using the virtual normal bundle, even if X is singular, provided that the embedding of the fixed locus in X is a local complete intersection morphism. Unfortunately, this condition is difficult to verify. However, if X is cut by a regular sequence in a smooth variety, and the fixed points are isolated, then the methods of [BFQ, Section 3] can be used to give an explicit localization formula. A similar remark applies to the Bott residue formula below.

If X is complete, then the projection $\pi_X: X \rightarrow pt$ induces push-forward maps $\pi_{X*}: A_*^T X \rightarrow R_T$ and $\pi_{X*}: A_*^T X \otimes \mathcal{Q} \rightarrow \mathcal{Q}$. There are similar maps with X replaced with any component F of X^T . Applying π_{X*} to both sides of the explicit localization theorem, and noting that $\pi_{X*} i_{F*} = \pi_{F*}$, we deduce the “integration formula” (cf. [A-B, Equation (3.8)]).

COROLLARY 1. (Integration formula) *Let X be smooth and complete, and let $\alpha \in A_*^T(X) \otimes \mathcal{Q}$. Then*

$$\pi_{X*}(\alpha) = \sum_{F \subset X^T} \pi_{F*} \left(\frac{i_F^* \alpha}{c_{d_F}^T(N_F X)} \right)$$

as elements of \mathcal{Q} .

Remark. If α is in the image of the natural map $A_*^T(X) \rightarrow A_*^T(X) \otimes \mathbb{Q}$ (which need not be injective), then the equation above holds in the subring R_T of \mathbb{Q} . The reason is that the left side actually lies in the subring R_T ; hence so does the right side. In the results that follow, we will have expressions of the form $z = \sum z_j$, where the z_j are degree zero elements of \mathbb{Q} whose sum z lies in the subring R_T . The pullback map from equivariant to ordinary Chow groups gives a map $i^*: R_T = A_*^T(pt) \rightarrow \mathbb{Q} = A_*(pt)$, which identifies the degree 0 part of R_T with \mathbb{Q} . Since $\sum z_j$ is a degree 0 element of R_T , it is identified via i^* with a rational number. Note that i^* cannot be applied to each z_j separately, but only to their sum. In the integration and residue formulas below we will identify the degree 0 part of R_T with \mathbb{Q} and suppress the map i^* .

The preceding corollary yields an integration formula for an element a of the ordinary Chow group A_0X , provided that a is the pullback of an element $\alpha \in A_0^T X$.

PROPOSITION 5. *Let $a \in A_0X$, and suppose that $a = i^*\alpha$ for $\alpha \in A_0^T X$. Then*

$$\deg(a) = \sum_F \pi_{F*} \left\{ \frac{i_F^* \alpha}{c_{d_F}^T(N_F X)} \right\}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X_T \\ \downarrow \pi_X & & \downarrow \pi_X^T \\ \text{pt} & \xrightarrow{i} & U/T. \end{array}$$

We have $\pi_{X*}(a) = \pi_{X*}i^*(\alpha) = i^*\pi_{X*}^T(\alpha)$. Applying the integration formula gives the result. \square

4.1. The Bott residue formula. Let E_1, \dots, E_s be a T -equivariant vector bundle on a complete, smooth n -dimensional variety X . Let $p(x_1^1, \dots, x_s^1, \dots, x_1^n, \dots, x_s^n)$ be a polynomial of weighted degree n , where x_j^i has weighted degree i . Let $p(E_1, \dots, E_s)$ denote the polynomial in the Chern classes of E_1, \dots, E_s obtained setting $x_j^i = c_i(E_j)$. The integration formula above will allow us to compute $\deg(p(E_1, \dots, E_s) \cap [X])$ in terms of the restriction of the E_i to X^T .

As a notational shorthand, write $p(E)$ for $p(E_1, \dots, E_s)$ and $p^T(E)$ for the corresponding polynomial in the T -equivariant Chern classes of E_1, \dots, E_s . Notice that $p(E) \cap [X] = i^*(p^T(E) \cap [X_T])$. We can therefore apply the preceding proposition to get the Bott residue formula.

THEOREM 3. (Bott residue formula) *Let E_1, \dots, E_r be a T -equivariant vector bundle on a complete, smooth n -dimensional variety. Then*

$$\deg(p(E) \cap [X]) = \sum_{F \subset X^T} \pi_{F*} \left(\frac{p^T(E|_F) \cap [F]_T}{c_{d_F}^T(N_F X)} \right).$$

Remark. Using techniques of algebraic de Rham homology, Hübl and Yekutieli [H-Y] proved a version of the Bott residue formula, in characteristic 0, for the action of any algebraic vector field with isolated fixed points.

By Lemma 3 the equivariant Chern classes $c_i^T(E_j|_F)$ and $c_{d_F}^T(N_F X)$ can be computed in terms of the characters of the torus occurring in the eigenbundle decompositions of $E_j|_F$ and $N_F X$ and the Chern classes of the eigenbundles. The above formula can then be readily converted (cf. [A-B]) to more familiar forms of the Bott residue formula not involving equivariant Chow groups. We omit the details. If the torus T is 1-dimensional, then degree zero elements of \mathcal{Q} are rational numbers, and the right-hand side of the formula is just a sum of rational numbers. This is the form of the Bott residue formula which is most familiar in practice.

5. Localization and residue formulas for singular varieties. In general, the problem of proving localization and residue formulas on singular varieties seems interesting and difficult. In this section we discuss what can be deduced from an equivariant embedding of a singular scheme X into a smooth M . The results are not very general, but (as we show) they can be applied in some interesting examples, for example, if X is a complete intersection in $M = \mathbb{P}^n$ and T acts on M with isolated fixed points, or if X is a Schubert variety in $M = G/B$.

The idea of using an embedding into a smooth variety to extract localization information is an old one. In the case of the action of an automorphism of finite order, the localization and Lefschetz Riemann-Roch formulas of [Qu], [BFQ] on quasi-projective varieties are obtained by a calculation on \mathbb{P}^n . Moreover, as in our case, the best formulas on singular varieties are obtained when the embedding into a smooth variety is well understood.

At least in principle, a localization theorem can be deduced if every component of X^T is a component of M^T . This holds, for example, if the action of T on M has isolated fixed points; or if X is a toric (resp. spherical) subvariety of a non-singular toric (resp. spherical) variety M . In particular, the condition holds if X is a Schubert variety and M is the flag variety. We have the following proposition.

PROPOSITION 6. *Let $f: X \rightarrow M$ be an equivariant embedding of X in a nonsingular variety M . Assume that every component of M^T which intersects X is contained in X . If F is a component of X^T , write i_F for the embedding of F in X , and j_F for the*

embedding of F in M . Then:

- (1) $f_*: A_*^T(X) \otimes \mathcal{Q} \rightarrow A_*^T(M) \otimes \mathcal{Q}$ is injective.
- (2) Let $\alpha \in A_*^T(X) \otimes \mathcal{Q}$. Then

$$\alpha = \sum_F i_{F*} \frac{j_{F*}^* \alpha}{c_{d_F}^T(N_F M)},$$

where the sum is over the components F of X^T and d_F is the codimension of F in M .

Proof. (1) Since the components of X^T are a subset of the components of M^T , $\oplus_{F \subset X^T} A_*^T(F)$ is an R_T -submodule of $\oplus_{F \subset M^T} A_*^T(F)$. By the localization theorem,

$$\sum_{F \subset X^T} j_{F*}(A_*^T(F)) \otimes \mathcal{Q} \simeq A_*^T(X) \otimes \mathcal{Q}$$

and

$$\sum_{F \subset M^T} i_{F*}(A_*^T(F)) \otimes \mathcal{Q} \simeq A_*^T(M) \otimes \mathcal{Q}.$$

Since $j_{F*} = f_* i_{F*}$, the result follows.

(2) By (1) it suffices to prove that

$$f_* \left(\alpha - \sum_F j_{F*} \frac{i_{F*}^* \alpha}{c_{d_F}^T(N_F M)} \right) = 0 \in A_*^T(X) \otimes \mathcal{Q}.$$

Since $j_{F*} = f_* i_{F*}$ the theorem follows from the explicit localization theorem applied to the class $f_* \alpha$ on the smooth variety M . \square

To obtain a residue formula that computes Chern numbers of bundles on X , we only need to know an expansion $[X]_T = \sum_{F \subset X^T} i_{F*}(\beta_F)$, where $\beta_F \in A_*^T(F)$. In this case we obtain the formula

$$\deg(p(E) \cap [X]) = \sum_{F \subset X^T} \pi_{F*} \left(\frac{p^T(i_F^*(E)) \cap \beta_F}{c_{d_F}^T(N_F M)} \right).$$

In the setting of Proposition 6, the classes β_F are given by $i_{F*}^* f_*[X]_T$. To obtain a useful residue formula, we need to make this expression more explicit. This is most easily done if we can express $f_*[X]_T$ in terms of Chern classes of naturally occurring equivariant bundles on M . The reason is that the pullback i_F^* of such Chern classes is often easy to compute, particularly if F is an isolated fixed point (cf. Lemma. 3). Indeed, this is why the Bott residue formula is a good calculational tool in the nonsingular case.

Although the conditions to obtain localization and residue formulas are rather strong, they are satisfied in some interesting cases. We will consider in detail two examples: complete intersections in projective spaces, and Schubert varieties in G/B . For complete intersections some intrinsic formulas can be deduced using the virtual normal bundle (see the remark after Theorem 2). In this section our point of view for complete intersections is different. We do not use the virtual normal bundle, but instead use the fact that if $X \xrightarrow{f} M$ is a complete intersection, it is easy to calculate $f_*[X]_T \in A_*^T(M)$. As an example of our methods we do a localization and residue calculation on a singular quadric in \mathbb{P}^3 .

As a final remark, note that to compute Chern numbers of bundles on X which are pulled back from M , it suffices to know $f_*[X]_T$, for then we can apply residue formulas on M . Information about the fixed locus in X is irrelevant. The interesting case is when the bundles are not pulled back from M ; see the example of the singular quadric below.

5.1. Complete intersections in projective space. For simplicity we consider the case where the dimension of T is 1. If T acts on a vector space V with weights a_0, \dots, a_n then $A_T^*(\mathbb{P}(V)) = \mathbb{Z}[h, t] / \prod (h + a_i t)$. We are interested in complete intersections X in $\mathbb{P}(V)$ where the functions f_i defining X are, up to scalars, preserved by the T -action, i.e., $t \cdot f_i = t^{a_i} f_i$. In this case we say f_i has weight a_i .

The following lemma is immediate.

LEMMA 5. *Suppose X is a hypersurface in $\mathbb{P}(V)$ defined by a homogeneous polynomial f of degree d and weight a . Then $[X]_T = dh + at \in A_T^*(\mathbb{P}(V))$. Hence if X is a complete intersection in $\mathbb{P}(V)$ defined by homogeneous polynomials f_i of degree d_i and weight a_i , then $[X]_T = \prod (d_i h + a_i t)$.*

If T acts on V with distinct weights, then T has isolated fixed points on $M = \mathbb{P}(V)$, and (trivially) every component of X_T is a component of M^T ; so by the preceding discussion there is a useful residue formula. In particular using a little linear algebra we can easily obtain a formula for $[X]_T$ in terms of the fixed points in X and the weights of the action. We omit the details to avoid a notational quagmire, but the ideas are illustrated in the example of the singular quadric.

5.2. Schubert varieties in G/B . In this section, we work over an algebraically closed field. For simplicity, we take Chow groups to have rational coefficients, and let $R = R_T \otimes \mathbb{Q}$ denote the rational equivariant Chow ring of a point.

Let G be a reductive group and B a Borel subgroup, and $\mathcal{B} = G/B$ the flag variety. In the discussion below, the smooth variety \mathcal{B} will play the role of M , and the Schubert variety X_w the role of X .

Let $T \subset B$ be a maximal torus. T acts on \mathcal{B} with finitely many fixed points,

indexed by $w \in W$; denote the corresponding point by p_w . More precisely, if we let w denote both an element of the Weyl group $W = N(T)/T$ and a representative in $N(T)$, then p_w is the coset wB . The flag variety is a disjoint union of the B -orbits $X_w^0 = B \cdot p_w$. The B -orbit X_w^0 is called a Schubert cell and its closure X_w a Schubert variety. If e denotes the identity in W and w_0 the longest element of W , then X_e is a point and $X_{w_0} = \mathcal{B}$.

We have $X_w = \cup_{u \leq w} X_u^0$. The T -equivariant Chow group of X_w is a free R_T -module with basis $[X_u]_T$, for $u \leq w$.

Let $j_u: p_u \hookrightarrow \mathcal{B}$. Fix $w \in W$ and let $f: X_w \hookrightarrow \mathcal{B}$. For $u \leq w$ let $i_u: p_u \hookrightarrow X_w$. If $v \leq w$ let $[X_v]_T$ denote the equivariant fundamental class of X_v in $A_*^T(X_w)$. We want to make explicit the localization theorem for the variety X_w (which is singular in general), i.e., to compute $[X_v]_T$ in terms of classes $i_{u*}\beta_u$.

The (rational) equivariant Chow groups $A_T^*(\mathcal{B})$ can be described as follows. We consider two maps $\rho_1, \rho_2: R \rightarrow A_T^*(\mathcal{B})$. The map ρ_1 is the usual map $R \rightarrow A_T^*(\mathcal{B})$ given by equivariant pullback from a point. The definition of ρ_2 is as follows. For each character $\lambda \in \hat{T}$ set $\rho_2(\lambda) = c_1^T(M_\lambda)$ where M_λ is the line bundle $G \times^B k_\lambda \rightarrow \mathcal{B}$; extend ρ_2 to an algebra map $R \rightarrow A_T^*(\mathcal{B})$. The map $R \otimes_{R^W} R \rightarrow A_T^*(\mathcal{B})$ taking $r_1 \otimes r_2$ to $\rho_1(r_1)\rho_2(r_2)$ is an isomorphism (see e.g. [Br]).

We adopt the convention that the Lie algebra of B contains the positive root vectors. We can identify $T_{p_w}(\mathcal{B})$ with $\mathfrak{g}/(Ad w)\mathfrak{b}$. This is a representation of T corresponding to the T -equivariant normal bundle of the fixed point p_w . We identify $A_T^*(p_w) \cong R$. From our description of $T_{p_w}(\mathcal{B})$, we see that (if n denotes the dimension of \mathcal{B}) $c_n^T(N_{p_w}\mathcal{B})$ is the product of the roots in $\mathfrak{g}/(Ad w)\mathfrak{b}$, which is easily seen to give

$$c_n^T(N_{p_w}\mathcal{B}) = c_w := (-1)^n(-1)^w \prod_{\alpha > 0} \alpha,$$

where n is the number of roots $\alpha > 0$.

To obtain a localization formula we also need to know the maps $j_u^*: A_T^*(\mathcal{B}) \rightarrow A_T^*(p_u)$, where $j_u: p_u \hookrightarrow \mathcal{B}$ is the inclusion. We have identified $A_T^*(\mathcal{B}) = R \otimes_S R$ and $A_T^*(p_w) = R$. Thus, we may view j_u^* as a map $R \otimes_S R \rightarrow R$. There is a natural action of $W \times W$ on $R \otimes_S R$. Let $m: R \otimes_S R \rightarrow R$ denote the multiplication map.

LEMMA 6. *For $u \in W$, the map $j_u^*: R \otimes_S R \rightarrow R$ equals the composition $m \circ (1 \times u)$.*

Proof. It suffices to show that $j_u^*\rho_1(\lambda) = \lambda$ and $j_u^*\rho_2(\lambda) = u\lambda$. Now, $j_u^*\rho_1$ is just the equivariant pullback by the map $p_u \rightarrow pt$. Since this equivariant pullback is how we identify $A_T^*(p_u) = A_T^*(pt) = R$, with these identifications, $j_u^*\rho_1$ is the identity map, $j_u^*\rho_1(\lambda) = \lambda$. Also, by definition $j_u^*\rho_2(\lambda) = c_1^T(M_\lambda|_{p_u})$. As a representation of T , $M_\lambda|_{p_u} \cong k_{u\lambda}$, so $c_1^T(M_\lambda|_{p_u}) = u\lambda$, as desired. \square

If $F \in R \otimes_S R$ is a polynomial set $F(u) = j_u^* F \in R$. Recall that we have fixed w and let $f: X_w \rightarrow \mathcal{B}$ denote the inclusion; for $v \leq w$, $[X_v]_T$ denotes a class in $A_*^T(X_w)$. By work of Fulton and Pragacz-Ratajski, for G classical, it is known how to express $f_*[X_v]_T \in A_T^*(\mathcal{B})$ in terms of the isomorphism $A_T^*(\mathcal{B}) \cong R \otimes_{R^W} R$. More precisely, Fulton and Pragacz-Ratajski ([Fu2], [P-R]) define elements in $R \otimes_{\mathbb{Q}} R$ which project to $f_*[X_v]_T$ in $R \otimes_{R^W} R$. Let F_u denote either the polynomial defined by Fulton or that defined by Pragacz-Ratajski. Using these polynomials we can get an explicit localization formula for Schubert varieties.

PROPOSITION 7. *With notation as above, the class $[X_v]_T$ in $A_*^T(X_w) \otimes \mathbb{Q}$ is given by*

$$[X_v]_T = (-1)^n \frac{1}{\prod_{\alpha > 0} \alpha} \sum_{u \leq v} (-1)^u i_{u*} (F_v(u) \cap [p_u]_T).$$

Proof. This is an immediate consequence of the preceding discussion and Proposition 6. \square

Remark. Taking $w = w_0$, so $X_w = \mathcal{B}$, the above formula is an explicit inverse to the formula of [Br, Section 6.5, Proposition (ii)]. This shows $\frac{f_w(u)}{\prod_{\alpha > 0} \alpha}$ is Brion's equivariant multiplicity of X_w at the fixed point p_u , and also links Brion's proposition to [G, Theorem 1.1]. Again, for $X_w = \mathcal{B}$, the corresponding results in equivariant cohomology were proved by Arabia [Ar] and Kostant-Kumar [KK].

5.3. A singular quadric in \mathbb{P}^3 . In this section we consider the example of the singular quadric $Q \xrightarrow{f} \mathbb{P}^3$ defined by the equation $x_0x_1 + x_2^2 = 0$ (note that we allow the characteristic to be 2). Let $\mathbb{P}^2 \subset \mathbb{P}^3$ be the hyperplane defined by the equation $x_2 = 0$ and let $\pi: Q \rightarrow \mathbb{P}^2$ be the projection from $(0, 0, 1, 0)$. As a sample of the kinds of the residue calculations that are possible, we prove the following proposition.

PROPOSITION 8.

$$\int_Q c_1(\pi^* T_{\mathbb{P}^2}) c_1(f^* T_{\mathbb{P}^3}) = 24.$$

Proof. We will prove this by considering the following torus action. Let $T = \mathbb{G}_m$ act on \mathbb{P}^3 with weights $(1, -1, 0, a)$, where $a \notin \{0, -1, 1\}$. The quadric is invariant under this action, so T acts on Q . Since $(0, 0, 1, 0)$ is a fixed point, π is an equivariant map where T acts on \mathbb{P}^2 with weights $(1, -1, a)$. Thus $c_1^T(\pi^* T_{\mathbb{P}^2}) c_1^T(f^* T_{\mathbb{P}^3}) \cap [Q]_T$ defines an element of $A_*^T(Q) \otimes \mathbb{Q}$ which we will express as a residue in terms of the fixed points for the action of T on Q . To

do this we need to express $[Q]_T$ in terms of the fixed points. By Proposition 6 this can be done if we know $f_*[Q]_T \in A_*^T(\mathbb{P}^3)$. Since Q is a quadric of weighted degree 0 with respect to the T -action, $f_*[Q]_T = 2h \in A_*^T(\mathbb{P}^3)$.

Since everything can be done explicitly, we will calculate more than we need and determine the entire R_T -module $A_*^T(Q)$ in terms of the fixed points.

Explicit localization on the singular quadric. The quadric has a decomposition into affine cells with one cell in dimensions 0, 1 and 2. The open cell is $Q_0 = \{(1, -x^2, x, y) | (x, y) \in k^2\}$. In dimension 1 the cell is $l_0 = \{(0, 1, 0, x) | x \in k\}$, and in dimension 0, the cell is the singular point $p_s = \{(0, 0, 0, 1)\}$. (In characteristic 2, this point is not an isolated singular point.) Thus, $A_i(Q) = \mathbb{Z}$ for $i = 0, 1, 2$ with generators $[Q]$, $[l]$ and $[p_s]$. Moreover, these cells are T -invariant, so their equivariant fundamental classes form a basis for $A_*^T(Q)$ as an $R_T = \mathbb{Z}[t]$ module. Let \mathbb{I} , L , and P_s denote the corresponding equivariant fundamental classes $[Q]_T$, $[l]_T$, and $[p_s]_T$.

There are three fixed points $p_s = (0, 0, 0, 1)$, $p = (1, 0, 0, 0)$ and $p' = (0, 1, 0, 0)$. These points have equivariant fundamental classes in $A_*^T(Q^T)$ which we denote by P_s , P , and P' . By abuse of notation we will not distinguish between P_s and $i_*(P_s)$.

Both $A_*^T(Q^T)$ and $A_*^T(Q)$ are free R_T -modules of rank 3, with respective ordered bases $\{P_s, P, P'\}$ and $\{P_s, L, \mathbb{I}\}$. The map i_* is a linear transformation of these R_T -modules, and we will compute its matrix with respect to these ordered bases. This matrix can be easily inverted, provided we invert t , and so we obtain $(i_*)^{-1}$.

The equivariant Chow ring of \mathbb{P}^3 is given by

$$A_*^T(\mathbb{P}^3) = \mathbb{Z}[t, h]/(h - t)(h + t)h(h + at)$$

so $A_*^T(\mathbb{P}^3)$ is free of rank 4 over R_T , with basis $\{1, h, h^2, h^3\}$.

To compute i_*P_s , i_*P , and i_*P' , we take advantage of the fact that the push-forward $f_*: A_*^T(Q) \rightarrow A_*^T(\mathbb{P}^3)$ is injective. Moreover it is straightforward to calculate the push-forward to \mathbb{P}^3 of all the classes in our story. To simplify the notation, we will use f_* to denote either of the maps $A_*^T(Q) \rightarrow A_*^T(\mathbb{P}^3)$ or $A_*^T(Q^T) \rightarrow A_*^T(\mathbb{P}^3)$. We find:

$$f_*(\mathbb{I}) = 2h$$

$$f_*(L) = (h - t)h$$

$$f_*(P_s) = h^3 - ht^2$$

$$f_*(P) = h^3 + (a - 1)h^2t - aht^2$$

$$f_*(P') = h^3 + (a + 1)h^2t + aht^2.$$

This implies that

$$\begin{aligned} i_*(P_s) &= P_s \\ i_*(P) &= (a-1)tL + P_s \\ i_*(P') &= (a+1)t^2\mathbb{I} + (a+1)tL + P_s. \end{aligned}$$

So the matrix for i_*^T is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & (a-1)t & (a+1)t \\ 0 & 0 & (a+1)t^2 \end{pmatrix}.$$

Inverting this matrix we obtain

$$\begin{pmatrix} 1 & \frac{1}{t(-a)} & \frac{2}{t^2(a^2-1)} \\ 0 & \frac{1}{t(a-1)} & \frac{1}{t^2(1-a)} \\ 0 & 0 & \frac{1}{t^2(a+1)} \end{pmatrix}.$$

Thus we can write (after supressing the $(i_*)^{-1}$ notation)

$$\begin{aligned} P_s &= P_s \\ L &= \frac{1}{t(a-1)}(-P_s + P) \\ \mathbb{I} &= \frac{1}{t^2(a^2-1)}(2P_s - (a+1)P + (a-1)P') \end{aligned}$$

Calculation of Chern numbers. We now return to the task of computing $c_1(\pi^*T_{\mathbb{P}^2})c_1(f^*T_{\mathbb{P}^3}) \cap \mathbb{I}$. To simplify notation, set $\alpha_1 := c_1(\pi^*T_{\mathbb{P}^2})$ and $\alpha_2 := c_1(f^*T_{\mathbb{P}^3})$ and $\alpha := \alpha_1\alpha_2$. By the calculations above,

$$\alpha_1\alpha_2 \cap \mathbb{I} = i_*(i^*\alpha_1 i^*\alpha_1 \cap \frac{t^{-2}}{a^2-1}(2P_s - (a+1)P + (a-1)P')).$$

To compute the class explicitly we must compute the restrictions of α_1 and α_2 to each of the fixed points P_s , P and P' .

The tangent space to P_s in \mathbb{P}^3 has weights $(1-a, -1-a, -a)$. Thus $\alpha_2|_{P_s} = (1-a)t - (1+a)t - at = -3at$. To compute $\alpha_1|_{P_s}$ observe that P_s is the inverse image of the fixed point $(0, 0, 1) \in \mathbb{P}^2$. Since $T_{\mathbb{P}^2}$ has weights $(1-a, -1-a)$ at this point, $c_1(\pi^*T_{\mathbb{P}^2})|_{P_s} = (1-a)t + (-1-a)t = -2t$.

The restrictions to the other two fixed points can be calculated similarly. In particular

$$\begin{aligned}\alpha_1|_P &= (a-3)t & \alpha_1|_{P'} &= (a+3)t \\ \alpha_2|_P &= (a-4)t & \alpha_2|_{P'} &= (a+4)t\end{aligned}$$

Thus,

$$\begin{aligned}\alpha \cap \mathbb{I} &= \frac{12a^2}{a^2-1}P_s - \frac{(a-3)(a-4)(a+1)}{a^2-1}P \\ &+ \frac{(a+3)(a+4)(a-1)}{a^2-1}P' \in A_*^T(Q) \otimes \mathbb{Q}.\end{aligned}$$

Thus,

$$\begin{aligned}\deg(c_1(\pi^*T_{\mathbb{P}^2})c_1(f^*T_{\mathbb{P}^3}) \cap [Q]) &= \frac{12a^2}{a^2-1} - \frac{(a-3)(a-4)(a+1)}{a^2-1} \\ &+ \frac{(a+3)(a+4)(a-1)}{a^2-1} \\ &= 24.\end{aligned}$$

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