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§ *Chapter 1*

EXERCISE 1. Let $(S, \mathcal{S})$ be a measurable space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$. Show that the following sets are also in $\mathcal{S}$.

(1) the set C_1 of all $x \in S$ such that $x \in A_n$ for at least 5 different values of n.
(2) the set C_2 of all $x \in S$ such that $x \in A_n$ for exactly 5 different values of n.
(3) the set C_3 of all $x \in S$ such that $x \in A_n$ for all but finitely many n ("finitely many" includes none).
(4) the set C_4 of all $x \in S$ such that $x \in A_n$ for at most finitely many values of n.

Proof: The following lemma will be useful throughout this problem.

Lemma 0.1. Let $X = \{A \subset \mathbb{N} \mid |A| < \infty\}$ be the collection of all finite subsets of X. X is countable.

Proof. Define a function $f : X \longrightarrow \{0,1\}^{\mathbb{N}}$ by $f(A) = b_A$ where $b_A = (b_0, b_1, ...)$ with $b_k = 1$ if $k \in A$ and 0 otherwise. If $f(A) = f(B)$, then for each $i \in A$, $(b_A)_i = 1 = (b_B)_i \implies i \in B$, and likewise for each $i \in B$, $(b_B)_i = 1 = (b_A)_i \implies i \in A$, so $A = B$ and f is injective. Now construct a function $g : X \longrightarrow \mathbb{N}$ by

$$g(A) = \sum_{i=0}^{\infty} f(A)_i 2^k.$$

Since each $A \in X$ is finite, $f(A)_i$ is well defined, and is injective by the uniqueness of binary expansions. $\square$

We now tackle the four parts of this question.

(1) Let $\mathcal{I} = \{I \subseteq \mathbb{N} \mid |I| = 5\}$. Define

$$B = \bigcup_{I\in\mathcal{I}} \bigcap_{i\in I} A_i.$$

Notice first that B is measurable. Each intersection $\bigcap_{i\in I} A_i$ is an intersection of five measurable sets, and is hence itself measurable. The collection $\mathcal{I}$ is a subset of X defined in the above lemma, and is hence countable.

I claim that $C_1 = B$. Indeed, if $x \in C_1$, then it is contained in A_n for at least five different values of n. Let $I = \{i_1, ..., i_5\}$ be five of these values. Then $x \in A_{i_1} \cap ... \cap A_{i_5}$ and hence $x \in B \implies C_1 \subseteq B$.

Now suppose that $x \in B$. Then $x \in \bigcap_{i\in I} A_i$ for some five element set $I \subseteq \mathbb{N}$. This means x is contained in A_n for at least five distinct values of n, and hence $x \in C_1 \implies B \subseteq C_1$. This proves that $C_1 = B$ and is therefore measurable.

(2) Let $\mathcal{I}$ be the same collection as in part (1). For each $I \in \mathcal{I}$, define

$$B_I = \bigcap_{i\in I} A_i \;\cap \bigcap_{i\in\mathbb{N}\setminus I} A_i^c.$$

Each B_I is a countable intersection of measurable sets and is hence measurable. Now define

$$B = \bigcap_{I \in \mathcal{I}} B_I.$$

The set B is measurable as it is a countable union of measurable sets. I claim that $C_2 = B$.

Suppose $x \in C_2$. Let $I \subseteq \mathbb{N}$ be the collection of the unique five natural numbers n for which $x \in A_n$. This means $x \notin A_n$ for all $n \in \mathbb{N} \setminus I$, or equivalently $x \in A_n^c$. This means $x \in B_I$ by the definition of B_I, and hence $x \in B$.

Conversely, if $x \in B$, then there is some five-element subset $I \subset \mathbb{N}$ such that $x \in B_I$. Since $x \in A_i$ for each $i \in I$, x is in A_n for five distinct values of n. Since $x \in A_i^c \iff x \notin A_i$ for all natural numbers i not in I, x is not in A_n for any other value of n. This implies that $x \in C_2$.

We have both inclusions, and hence $C_2 = B$ is measurable.

(3) Define

$$B = \bigcup_{n=0}^{\infty} \bigcap_{k \geq n}^{\infty} A_k.$$

This is a countable union and intersection of measurable sets and hence measurable.

Suppose $x \in C_3$. Then x avoids A_n for only finitely many values of n, hence there is some $N \in \mathbb{N}$ such that $x \in A_k$ for all $k > N$. This implies $x \in B$.

Now suppose $x \in B$. Then $x \in \bigcap_{k>n}^{\infty} A_k$ for some $n \in \mathbb{N}$, and is hence contained in all but at most finitely many A_n. This means $x \in C_3$.

We conclude that $C_3 = B$ and is hence measurable.

(4) Define

$$B = \bigcup_{n=0}^{\infty} \bigcap_{k \geq n}^{\infty} A_k^c.$$

As in part (3), the set B is measurable as it is a countable union and intersection of measurable sets.

If $x \in C_4$, then x is contained in at most finitely many A_n, or equivalently, that x is contained in all but finitely many complements A_n^c. There must then be some $N \in \mathbb{N}$ for which $x \in A_k$ for all $k > N$. This implies that $x \in B$.

If $x \in B$, then $x \in \bigcap_{k \geq n}^{\infty} A_n^c \iff x \notin \bigcup_{k \geq n}^{\infty} A_k$. Hence, x is contained in A_i for at most finitely many values of i, and consequently $x \in C_4$.

We conclude that $C_4 = B$ and is therefore measurable.

□

Exercise 2. (Atomic structure of algebras). A **partition** of a set S is a family $\mathcal{P}$ of non-empty subsets of S with the property that each $x \in S$ belongs to exactly one $A \in \mathcal{P}$.

(1) How many algebras are there on the set $S = \{1, 2, 3\}$?

(2) By constructing a bisection between the two families, show that the number of different algebras on a finite set S is equal to the number of different partitions of S. Note: the elements of the partition corresponding to an algebra are said to be its atoms.

(3) Does there exist an algebra with 754 elements?

Proof: (1) We can complete this simply by enumerating them, there are five:

$$\begin{aligned}
\mathcal{S}_1 &= \{\emptyset, S\} \\
\mathcal{S}_2 &= \{\emptyset, \{1\}, \{2,3\}, S\} \\
\mathcal{S}_3 &= \{\emptyset, \{2\}, \{1,3\}, S\} \\
\mathcal{S}_4 &= \{\emptyset, \{3\}, \{1,2\}, S\} \\
\mathcal{S}_5 &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S\}\,.
\end{aligned}$$

(2) We first prove the following lemma, which will be useful in the last part of this problem too:

Lemma 0.2. Let S be a finite set and $P = \{P_1, ..., P_n\}$ be a partition of S. Define

$$\mathcal{S} = \left\{ \bigcup_{i \in I} P_i \;\middle|\; I \subseteq \{1, ..., n\} \right\}$$

to be the collection of subsets of S obtained by taking unions of the P_i. Then $\mathcal{S} = \sigma(P)$.

Proof. We need only show that $\mathcal{S}$ is an algebra contained in $\sigma(P)$. It is clear that $\mathcal{S}$ is contained in $\sigma(P)$: algebras are closed under finite unions, hence $\bigcup_{i \in I} P_i$ is measurable for any $I \subseteq \{1, ..., n\}$. To see that $\mathcal{S}$ is an algebra, first notice that

$$\emptyset = \bigcup_{i \in \emptyset} P_i, \quad \text{and} \quad S = \bigcup_{i \in \{1,...,n\}} P_i$$

so $\mathcal{S}$ contains both $\emptyset$ and S. Next, if I_1 and I_2 are two subsets of $\{1, .., n\}$, then

$$\bigcup_{i \in I_1} P_i \,\cup\, \bigcup_{i \in I_2} P_i = \bigcup_{i \in I_1 \cup I_2} P_i,$$

so $\mathcal{S}$ is closed under finite unions. Finally,

$$\left(\bigcup_{i \in I} P_i \right)^c = S \setminus \left(\bigcup_{i \in I} P_i \right) = \bigcup_{i \in I^c} P_i$$

where $I^c = \{1, ..., n\} \setminus I$, so $\mathcal{S}$ is closed under complement. This completes the proof. $\square$

We now proceed to the problem. Let $\mathcal{P}$ denote the set of partitions of S and $\mathcal{A}$ the set of algebras on S. Define a function $f : \mathcal{P} \longrightarrow \mathcal{A}$ by sending a partition P to the algebra it generates, i.e. $f(P) = \sigma(P)$. To construct an inverse of f, I claim the following: given an algebra $\mathcal{S}$ on S, the collection A of atoms defined

$$A_{\mathcal{S}} = \{M_x \mid x \in S\}\,, \text{ where } M_x = \bigcap_{\substack{M \in \mathcal{S} \\ x \in M}} M$$

is a partition of S. Note that M_x is measurable in $\mathcal{S}$ since it is a finite intersection of measurable sets. First, notice that for every $x \in S$, $x \in M_x$, and hence $A_{\mathcal{S}}$ forms a cover of S. Now suppose there is some $y \in M_x$ such that $x \neq y$. Then for every measurable set $M \in \mathcal{S}$ which contains x, $y \in M$. If there were some measurable set N with $y \in N$ but $x \notin N$, then N^c would be a measurable set containing x but not y, which cannot happen. Thus, a measurable set contains x if and only if it contains y, and hence $M_x = M_y$. It follows that if $x \neq y$ and $M_x \cap M_y \neq \emptyset$, then there is some $z \in M_x \cap M_y \implies M_x = M_z = M_y$. Thus, any two sets M_x and M_y are either disjoint or equal. Since $A_{\mathcal{S}}$ covers S by pairwise disjoint sets, it is a partition.

Define $g : \mathcal{A} \longrightarrow \mathcal{P}$ by $\mathcal{S} \mapsto A_{\mathcal{S}}$. I claim that f and g are inverses. We first show that the algebra generated by $A_{\mathcal{S}}$ is exactly $\mathcal{S}$, i.e. that $\sigma(A_{\mathcal{S}}) = \mathcal{S}$. If $M \in \mathcal{S}$ is a measurable set, then for each $x \in M$, $M_x \subseteq M$. This implies that $M = \bigcup_{x \in M} M_x$. Since S is finite, this union is also finite, and hence M is measurable in $\sigma(\mathcal{A}_{\mathcal{S}})$. As $\sigma(\mathcal{A}_{\mathcal{S}})$ is the smallest algebra containing $\mathcal{A}_{\mathcal{S}}$, $\sigma(\mathcal{A}_{\mathcal{S}}) = \mathcal{S}$, and hence $f(g(\mathcal{S})) = \mathcal{S}$.

Now suppose we start with a partition P of S. By the Lemma, $\sigma(P)$ consists exclusively of unions of the P_i. Hence, if $x \in S$, P_i is the unique partition element containing x and M is a measurable set in $\sigma(P)$ containing x, then there is some $I \subseteq \{1, ..., n\}$ such that

$$M = \bigcup_{j \in I} P_j$$

and hence $j \in I$. This is a long way of saying that if M is a measurable set containing an element $x \in S$, then M also contains the unique P_i containing x. Hence, $P_i \subseteq M_x \subseteq P_i$, so $M_x = P_i$. This means $A_{\sigma(P)} = P$, so $g(f(P)) = P$.

Since $f(g(\mathcal{S})) = \mathcal{S}$ and $g(f(P)) = P$, f and g are inverses and we therefore have a correspondence

$$\{ \text{ partitions of } S \} \underset{P \longmapsto \sigma(P)}{\overset{\mathcal{A}_{\mathcal{S}} \longleftarrow\!\shortmid \mathcal{S}}{\rightleftarrows}} \{ \text{ algebras on } S \}.$$

(3) Short answer: No.

Longer answer: If S is a finite set and $\mathcal{S}$ is an algebra on S, then $\mathcal{S} = \sigma(P)$ for some partition $P = \{P_1, ..., P_n\}$ of S by part (2). I claim that $|\sigma(P)| = 2^{|P|}$. By the lemma proven at the start of part (2), every set $M \in \sigma(P)$ is of the form

$$M = \bigcup_{i \in I} P_i$$

for some $I \subseteq \{1, ..., n\}$. The map $f : \sigma(P) \longrightarrow \{0, 1\}^n$ defined

$$f\left(\bigcup_{i \in I} P_i\right) = (s_1, ..., s_n), \text{ where } s_j = \begin{cases} 0 & j \notin I \\ 1 & j \in I \end{cases}$$

is a bijection. There are 2^n elements in $\{0, 1\}^n$, so this proves the desired result. Since 754 is not a power of 2, there is no algebra with 754 elements.

$\square$

EXERCISE 3. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable if it is either monotone or convex.

Proof: Define the following collection of subsets of $\mathbb{R}$: $A = \{(-\infty, a) \mid a \in \mathbb{R}\}$. I claim that $\sigma(A) = \mathcal{B}(\mathbb{R})$, i.e. that A generates the Borel algebra on $\mathbb{R}$. To see this, it suffices to show that all open intervals are contained in $\sigma(A)$. Fix an open interval $(a, b) \subseteq \mathbb{R}$. Sigma algebras are closed under intersection, so $(-\infty, b) \cap (-\infty, a) = [a, b)$ is in $\sigma(A)$. To obtain an open set, take a countable sequence of these half-open intervals and union them:

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right).$$

Hence $(a, b) \in \sigma(A)$, and since $A \subseteq \mathcal{B}(\mathbb{R})$, we have $\sigma(A) = \mathcal{B}(\mathbb{R})$. We now move on to the problem at hand.

Suppose first that f is monotone increasing, so that $x \leq y \implies f(x) \leq f(y)$. By a theorem from class, it suffices to prove f is measurable on a generating set for the Borel algebra. Consider the interval $(-\infty, a)$ and set

$$b = \inf\{x \in \mathbb{R} \mid a \leq f(x)\}.$$

If $f^{-1}((-\infty, a)) = \emptyset$, then it is measurable. If $f^{-1}((-\infty, a))$ is not empty, then I claim that it is either $(-\infty, b)$ or $(-\infty, b]$. Indeed, if $x \in (-\infty, b)$ then $x < b \implies f(x) \leq f(b)$ by monotonicity. If $a \leq f(x)$, then we would have $f(b) \leq f(x)$ by the definition of b and hence $b = x$, which can't happen since $b \notin (-\infty, b)$. Thus $f(x) < a$ and $(-\infty, b) \subseteq f^{-1}((-\infty, a))$.

The reverse inclusion may not hold, as it is possible that $b \in f^{-1}((-\infty, a))$. If it is, then for all $x > b$ we have $f(b) \leq f(x) \implies a \leq f(x)$, and hence $x \notin f^{-1}((-\infty, a)) \implies f^{-1}((-\infty, a)) = (-\infty, b]$. If $b \notin f^{-1}((-\infty, a))$, then for all $x \geq b$ we have $a \leq f(b) \leq f(x) \implies x \notin f^{-1}((-\infty, a))$, in which case $f^{-1}((-\infty, a)) = (-\infty, b)$. In any case, $f^{-1}((-\infty, a))$ is measurable, and because f is measurable on a set which generates $\mathbb{B}(\mathbb{R})$, we conclude f is a measurable function.

Now suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone decreasing. The function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined $g(x) = -f(x)$ is then monotone increasing, and hence measurable by what we have already shown. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $h(x) = -x$, which is a polynomial function and hence continuous. We have that $f(x) = h \circ g(x) = -g(x)$, so f is a composition of measurable functions and hence itself measurable.

Now suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Fix $a \in \mathbb{R}$ and suppose we have two elements $x, y \in \mathbb{R}$ such that $x < y$ and $f(x), f(y) < a$. For every element z between x and y we may find a $t \in [0, 1]$ such that $z = (1 - t)x + ty$ $\big($specifically, $t = \frac{z-x}{y-x}\big)$ and hence

$$f(z) = f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y) < (1 - t)a + ta = a$$

by convexity. This means that for any two points $x, y \in f^{-1}((-\infty, a))$, the interval $[x, y]$ is contained in $f^{-1}((-\infty, a))$ too, and hence $f^{-1}((-\infty, a))$ is an interval (either open, closed or half open) and measurable.

If no such two points exist, then $f^{-1}((-\infty, a))$ is either a singleton or the emptyset, both of which are measurable. In any case, $f^{-1}((-\infty, a))$ is measurable for all $a \in \mathbb{R}$ and we conclude that f is a measurable function. $\square$

EXERCISE 4. One can obtain the product σ-algebra $\mathcal{S}$ on $\{-1, 1\}^{\mathbb{N}}$ as the Borel σ-algebra corresponding to a particular topology which makes $\{-1,1\}^{\mathbb{N}}$ compact. Here is how. Start by defining a mapping $d : \{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}} \to [0,\infty)$ by

$$d(s^1, s^2) = 2^{-i(s^1,s^2)}, \text{ where } i(s^1,s^2) = \inf\{i \in \mathbb{N} \mid s_i^1 \neq s_i^2\},$$

for $s^j = (s_1^j, s_2^j, ...)$, $j = 1, 2$.

(1) Show that d is a metric on $\{-1,1\}^{\mathbb{N}}$.

(2) Show that $\{-1,1\}^{\mathbb{N}}$ is compact under d.

(3) Show that each cylinder of $\{-1,1\}^{\mathbb{N}}$ is both open and closed under d.

(4) Show that each open ball is a cylinder.

(5) Show that $\{-1,1\}^{\mathbb{N}}$ is separable, i.e., it admits a countable dense subset.

(6) Conclude that $\mathcal{S}$ coincides with the Borel σ-algebra on $\{-1,1\}^{\mathbb{N}}$ under the metric d.

Proof: Throughout this problem, $S = \{-1,1\}^{\mathbb{N}}$ is the coin toss space and $\mathcal{S}$ is the product σ-algebra.

(1) First,

$$d(s^1,s^2) = 2^{-i(s^1,s^2)} = 0 \iff i(s^1,s^2)\infty \iff s^1 = s^2.$$

Second, for any two $s^1, s^2 \in S$, $i(s^1,s^2) = \inf\{i \in \mathbb{N} \mid s_i^1 \neq s_i^2\}) = i(s^2,s^1)$ so $d(s^1,s^2) = d(s^2,s^1)$ so symmetry is satisfied.

Finally, we show that d satisfies the triangle inequality. In fact, we show that d satisfies the ultrametric triangle inequality. Suppose we have $x, y, z \in S$ and set $i(x,y) = a$, $i(x,z) = b$ and $i(y,z) = c$. If $b \leq a$, then $d(x,y) \leq d(x,z) \leq \max\{d(x,z), d(y,z)\}$. If $b > a$, then $z_i = x_i$ for all $i \leq a$. Hence, $z_a = x_a \neq y_a$ and so $i(y,z) = a$. This implies $d(x,y) = d(y,z) \leq \max\{d(x,z), d(y,z)\}$.

Hence, d is an ultrametric on S.

(2) For metric spaces, compactness is equivalent to sequential compactness. We show that S satisfies the latter. Let $\{s^n\}_{n\in\mathbb{N}} \subset S$ be an arbitrary sequence in S. By the countable pigeonhole principle, we may choose $a_1 \in \{-1,1\}$ such that there exist infinitely many $i \in \mathbb{N}$ such that $s_1^i = a_1$. Now set $A_1 = \{s^i \mid s_1^i = a_i\}$ and note that $|A_1| = \infty$.

Suppose we have chosen $a_1, ..., a_n$ and constructed subsets $\{s^n\}_{n\in\mathbb{N}} \supset A_1 \supset ... \supset A_n$ such that for $1 \leq k \leq n$, $|A_k| = \infty$ and for all $s \in A_k$, $s_i = a_i$. Since A_n is an infinite set consisting of elements in S, there again is some a_{n+1} such that the set $A_n = \{s \in A_n \mid s_{n+1} = a_{n+1}\}$ satisfies $|A_{n+1}| = \infty$.

I claim that the element $a = (a_1, a_2, a_3, ...) \in S$ is a limit point of $\{s^n\}_{n\in\mathbb{N}}$. Choose $t^1 \in A_1$, and inductively choose $t^n \in A_n \setminus \{t^1, ..., t^{n-1}\}$. Then, by construction, $t_i^k = a_i$ for all $1 \leq i \leq k$, and hence

$$0 \leq \lim_{k\to\infty} d(t^k, a) \leq \lim_{k\to\infty} 2^{-k} = 0.$$

Since $\{t^k\}_{k\in\mathbb{N}} \subseteq \{s^n\}_{n\in\mathbb{N}}$ is a subsequence and $\{s^n\}_{n\in\mathbb{N}}$ was chosen arbitrarily, every sequence in S has a subsequence which converges to an element of S. Hence S is compact.

(3) We first describe the general form of a cylinder in S. The algebra on $\{-1,1\}$ is simply the power set $\{\emptyset, \{-1\}, \{1\}, \{-1,1\}\}$, so the set B in the definition of a cylinder is simply any subset $B \subseteq \{-1,1\}^k$. A cylinder with base B can then be specified fully with a multi-index $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ by

$$C_{B,\alpha} = \{s \in S \mid s_\alpha \in B\},$$

where for simplicity we denote by s_α the k-tuple $(s_{\alpha_1}, ..., s_{\alpha_k})$. We now wish to show that an arbitrary $C_{B,\alpha}$ is both open and closed, where $k = |\alpha|$ is the length of the multi-index. To see that it is open, it suffices to show that every point in $C_{B,\alpha}$ has an open neighborhood entirely contained in $C_{B,\alpha}$. Fix a point $s \in C_{B_\alpha}$ and choose

$$n > \max\{\alpha_1, ..., \alpha_k\}.$$

Then every point t in the open ball $B_{2^{-n}}(s)$ of radius 2^{-n} centered at s is a sequence matching s up until at least the nth component, i.e. $t_i = s_i$ for $i \leq n$. This implies that $t_\alpha = s_\alpha$ and hence $t \in C_{B,\alpha}$. We then have that $B_{2^{-n}}(s) \subseteq C_{B,\alpha}$, so $C_{B,\alpha}$ is open.

Now consider the complement $D = S \setminus C_{B,\alpha}$. If D is empty, then $C_{B,\alpha} = S$ and is hence closed. Otherwise, there exists some $s \in D$ which necessarily satisfies $s_\alpha \notin B$. As before, choose $n > \max\{\alpha_1, ..., \alpha_k\}$. Any $t \in B_{2^-{n}}(s)$ then necessarily satisfies $t_\alpha = s_\alpha$, as above, and hence $B_{2^{-n}}(s) \subseteq D$. This shows that D is open, and hence $C_{B,\alpha}$ is closed. We conclude that cylinders are clopen in the metric topology on S induced by d.

(4) Consider an open ball $B_r(s)$ of radius $r \in [0,\infty)$ centered at a point $s \in S$, and set $n = \inf\{k \in \mathbb{N} \mid 2^{-k} \geq r\}$. By definition of n we have $B_{2^{-n}}(s) \supseteq B_r(s)$, but I claim that the reverse inclusion holds as well. Indeed, if $t \in B_{2^{-n}}(s)$, then $2^{-i(s,t)} < 2^{-n}$ which implies $i(s,t) < n$. We cannot have $i_2^{-i(s,t)} \geq r$ as it would contradict the minimality of n, hence $t \in B_r(s)$ and $B_{2^{-n}}(s) \subseteq B_r(s)$.

The rest of the argument is nearly immediate. A element $t \in S$ lies in $B_r(s) = B_{2^{-n}}(s)$ if and only if $s_i = t_i$ for all $1 \leq i < n$. If we set $\alpha = (1, ..., n)$ and $B = \{s_\alpha\}$, then we exactly get that $C_{B,\alpha} = B_{2^{-n}}(s) = B_r(s)$.

(5) Let $D = \{s \in S \mid s_i = 1 \text{ for only finitely many } i \in \mathbb{N}\}$. I claim this is a countable dense subset of S. To see that it is dense, choose any point $s \in S$. Let $t^n \in S$ be the element such that

$$t_i^n = \begin{cases} s_i & 1 \leq i \leq n \\ -1 & \text{otherwise} \end{cases}.$$

Then $t^n \in D$ and $d(t^n, s) \leq 2^{-n}$ so $\lim t^n = s$. This proves that D is dense in S.

To see it is countable, we construct an injection into the rational numbers $\mathbb{Q}$. We send $t \in D$ to a rational number r such that $0 \leq r < 2$ and whose nth digit is 0 if $t_n = -1$ and 1 if $t_n = 1$. More precisely, we define a map $f : D \longrightarrow \mathbb{Q}$ by

$$f(s) = \sum_{i=0}^{\infty} \frac{1}{10^i} \frac{1}{2}(1 + s_i).$$

The element $f(s)$ is indeed rational since it has a finite decimal expansion, and f is injective since $f(s) = f(t)$ if and only if $s_i = t_i$ for all $i \in \mathbb{N}$. Hence, D is a countable dense subset of S.

(6) We now argue that the Borel algebra on S equipped with the metric topology induced by d is equal to the product σ-algebra. Let D be the countable dense subset defined in part (5).

First, we show that that every open set $U \subseteq S$ is the countable union of measurable sets and is hence itself measurable. The intersection $U \cap D$ is countable and dense in U. By the openness of U, for each $t \in U \cap D$ we may find an open ball $B_{r_t}(t) \subseteq U$, and by the denseness of D,

$$U = \bigcup_{t \in U \cap D} B_{r_t}(t).$$

Since open balls in S are cylinders and cylinders are measurable, U is also measurable. This implies $\mathcal{B}(S) \subseteq \mathcal{S}$.

For the other inclusion, recall that $\mathcal{S}$ is generated as a σ-algebra by cylinders $C_{B,\alpha} \subseteq S$. Each of these sets is open by part (3), so $C_{B,\alpha}$ is measurable in the Borel σ-algebra on S. This implies that $\mathcal{S} \subseteq \mathcal{B}(S)$, and we conclude that the product σ-algebra on S is identical to the Borel sigma algebra induced by the metric topology.

$\square$