Algebraic Topology Homework 9

Isaac Martin

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§ *Problems from 2.1*

EXERCISE 22. Prove by induction on dimension the following facts about the homology of finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$

(c) If X has k n-cells, then $H_n(X)$ is generated by at most k-elements.

§ *Problems from 2.2*

EXERCISE 5. Show that any two reflections of S^n across different n-dimensional hyperplanes are homotopic, in fact homotopic through reflections. [The linear algebra formula for a reflection in terms of inner products may be helpful.]

Proof: The linear algebra formula Hatcher alludes to is reflection in the direction of u given by $f_u(x) = x - 2u \cdot \frac{x \cdot u}{u^2}$. It is a map on $\mathbb{R}^{n+1}$ which reflects a point x across the hyper plane through the origin whose normal vector is $0 \neq u \in \mathbb{R}^{n+1}$. Notice that for any vector $0 \neq u$, the reflection f_u in the direction of u

(1) negates u

$$f_u(u) = u - 2u\frac{u \cdot u}{u^2} = u - 2u = -u,$$

(2) fixes the hyper plane $x \cdot u = 0$

$$x \cdot u = 0 \implies f_u(x) = x - 2u\frac{x \cdot u}{u^2} = x - 0 = x,$$

(3) is an involution

$$\begin{aligned} f_u\left(f_u(x)\right) &= \left(x - 2u\frac{x \cdot u}{u^2}\right) - 2u\frac{\left(x - 2u\frac{x \cdot u}{u^2}\right) \cdot u}{u^2} \\ &= x - 2u\frac{x \cdot u}{u^2} - 2u\frac{x \cdot u}{u^2} + 2u\frac{2u^2\frac{x \cdot u}{u^2}}{u^2} \\ &= x - 4u\frac{x \cdot u}{u^2} + 4\frac{x \cdot u}{u^2} \\ &= x \end{aligned}$$

(4) and is a norm-preserving isometry (is "norm preserving" redundant?)

$$\begin{aligned}\|f_u(x)\|^2 &= \left(x - 2u\frac{x\cdot u}{u^2}\right)\cdot\left(x - 2u\frac{x\cdot u}{u^2}\right)\\ &= x^2 - 4x\cdot u\frac{x\cdot u}{u^2} + \frac{4(x\cdot u)^2}{u^2}\\ &= x^2 = \|x\|^2,\end{aligned}$$

which should be enough to convince us that this is indeed a reflection. Because f_u is norm preserving, it is a continuous map which maps sends S^n to S^n, and for notational convenience we will redefine f_u to be the restriction $f_u|_{S^n} : S^n \longrightarrow S^n$.

Consider some other vector $0 \neq v \in \mathbb{R}^{n+1}$ and suppose that the line between v and u does not contain the origin. Let $\gamma : I \longrightarrow \mathbb{R}^{n+1}$ be the linear interpolation from u to v, i.e. the map $\gamma(t) = u\cdot t - (1-t)\cdot v$. Then the map $F : S^n \times I \longrightarrow S^n$ defined $F_t(x) = f_{\gamma(t)}(x)$ is continuous and satisfies $F_0(x) = f_v(x)$ and $F_1(x) = f_u(x)$; hence, it is a homotopy between f_u and f_v comprised itself entirely of reflection maps.

If the line between v and u does contain the origin, then choose some other nonzero point $w \in \mathbb{R}^{n+1}$ which is not on the linear subspace spanned by u and v. By what we have already shown, $f_u \simeq f_w$ and $f_v \simeq f_w$, and since homotopy equivalence is an equivalence relation, $f_u \simeq f_v$. □

Exercise 7. For an invertible linear transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ show that the induced map on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$ is id or $-\,\mathrm{id}$ according to whether the determinant of f is positive or negative. [Use Gaussian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with $\pm_1$'s on the diagonal.]

Exercise 8. A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \longrightarrow \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \longrightarrow S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Exercise 12. Show that the quotient map $S^1 \times S^1 \longrightarrow S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \longrightarrow S^1 \vee S^1$ is nullhomotopic.

§ ***Problems from 2.B***

Exercise 1.

Exercise 2.