Chapter 4

Riemannian Geometry

Now we can finally put to work the abstract notions discussed in the previous chapters. Loosely speaking, the Riemannian geometry studies the properties of surfaces (manifolds) "made of canvas". These are manifolds with an extra structure arising naturally in many instances. On such manifolds one can speak of the length of a curve, and the angle between two smooth curves. In particular, we will study the problem formulated in Chapter 1: why a plane (flat) canvas disk cannot be wrapped in an one-to-one fashinon around the unit sphere in $\mathbb{R}^3$. Answering this requires the notion of Riemann curvature which will be the central theme of this chapter.

4.1 Metric properties

4.1.1 *Definitions and examples*

To motivate our definition we will first try to formulate rigorously what do we mean by a "canvas surface".

A "canvas surface" can be deformed in many ways but with some limitations: it cannot be stretched as a rubber surface because the fibers of the canvas are flexible but not elastic. Alternatively, this means that the only operations we can perform are those which do not change the lengths of curves on the surface. Thus, one can think of "canvas surfaces" as those surfaces on which any "reasonable" curve has a well defined length.

Adapting a more constructive point of view, one can say that such surfaces are endowed with a clear procedure of measuring lengths of piecewise smooth curves.

Classical vector analysis describes one method of measuring lengths of smooth paths in $\mathbb{R}^3$. If $\gamma : [0, 1] \to \mathbb{R}^3$ is such a path, then its length is given by

$$\text{length}\,(\gamma) = \int_0^1 |\dot{\gamma}(t)| dt,$$

where $|\dot{\gamma}(t)|$ is the Euclidean length of the tangent vector $\dot{\gamma}(t)$.

We want to do the same thing on an abstract manifold, and we are clearly faced with one problem: how do we make sense of the length $|\dot{\gamma}(t)|$? Obviously, this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold. The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product (which can vary from point to point in a smooth way).

Definition 4.1.1. (a) A *Riemann manifold* is a pair (M, g) consisting of a smooth manifold M and a metric g on the tangent bundle, i.e., a smooth, symmetric positive definite $(0, 2)$–tensor field on M. The tensor g is called a *Riemann metric* on M.
(b) Two Riemann manifolds (M_i, g_i) $(i = 1, 2)$ are said to be *isometric* if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^* g_2 = g_1$. □

If (M, g) is a Riemann manifold then, for any $x \in M$, the restriction

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

is an inner product on the tangent space T_xM. We will frequently use the alternative notation $(\bullet, \bullet)_x = g_x(\bullet, \bullet)$. The length of a tangent vector $v \in T_xM$ is defined as usual,

$$|v|_x := g_x(v, v)^{1/2}.$$

If $\gamma : [a, b] \to M$ is a piecewise smooth path, then we define its length by

$$\boldsymbol{l}(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\gamma(t)} dt.$$

If we choose local coordinates $(x^1, \dots, x^n)$ on M, then we get a local description of g as

$$g = g_{ij} dx^i dx^j, \quad g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}).$$

Proposition 4.1.2. *Let M be a smooth manifold, and denote by $\mathcal{R}_M$ the set of Riemann metrics on M. Then $\mathcal{R}_M$ is a non-empty convex cone in the linear space of symmetric $(0, 2)$–tensors.*

Proof. The only thing that is not obvious is that $\mathcal{R}_M$ is non-empty. We will use again partitions of unity. Cover M by coordinate neighborhoods $(U_\alpha)_{\alpha \in \mathcal{A}}$. Let (x^i_α) be a collection of local coordinates on U_α. Using these local coordinates we can construct by hand the metric g_α on U_α by

$$g_\alpha = (dx^1_\alpha)^2 + \cdots + (dx^n_\alpha)^2.$$

Now, pick a partition of unity $\mathcal{B} \subset C_0^\infty(M)$ subordinated to the cover $(U_\alpha)_{\alpha \in \mathcal{A}}$, i.e., there exits a map $\phi : \mathcal{B} \to \mathcal{A}$ such that $\forall \beta \in \mathcal{B}$ $\operatorname{supp} \beta \subset U_{\phi(\beta)}$. Then define

$$g = \sum_{\beta \in \mathcal{B}} \beta g_{\phi(\beta)}.$$

The reader can check easily that g is well defined, and it is indeed a Riemann metric on M. □

Example 4.1.3. (The Euclidean space). The space $\mathbb{R}^n$ has a natural Riemann metric

$$g_0 = (dx^1)^2 + \cdots + (dx^n)^2.$$

The geometry of $(\mathbb{R}^n, g_0)$ is the classical Euclidean geometry. □

Example 4.1.4. (Induced metrics on submanifolds). Let (M, g) be a Riemann manifold and $S \subset M$ a submanifold. If $\imath : S \to M$ denotes the natural inclusion then we obtain by pull back a metric on S

$$g_S = \imath^* g = g|_S \,.$$

For example, any invertible symmetric $n \times n$ matrix defines a quadratic hypersurface in $\mathbb{R}^n$ by

$$\mathcal{H}_A = \big\{ x \in \mathbb{R}^n \; ; \; (Ax, x) = 1 \big\},$$

where $(\bullet, \bullet)$ denotes the Euclidean inner product on $\mathbb{R}^n$. $\mathcal{H}_A$ has a natural metric induced by the Euclidean metric on $\mathbb{R}^n$. For example, when $A = I_n$, then $\mathcal{H}_{I_n}$ is the unit sphere in $\mathbb{R}^n$, and the induced metric is called the *round metric* of S^{n-1}. □

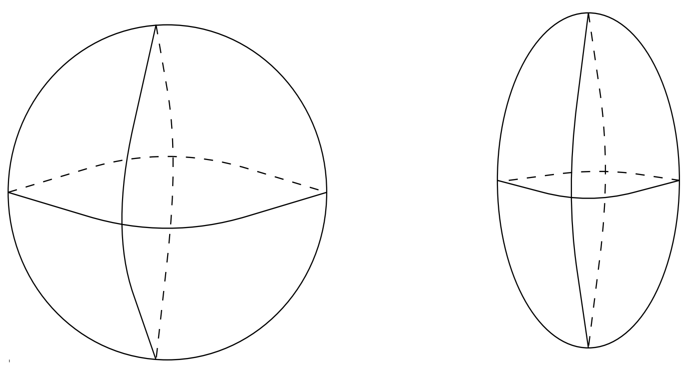

Fig. 4.1 *The unit sphere and an ellipsoid look "different".*

Remark 4.1.5. On any manifold there exist many Riemann metrics, and there is no natural way of selecting one of them. One can visualize a Riemann structure as defining a "shape" of the manifold. For example, the unit sphere $x^2 + y^2 + z^2 = 1$ is diffeomorphic to the ellipsoid $\frac{x^2}{1^2} + \frac{y^2}{2^2} + \frac{z^2}{3^2} = 1$, but they look "different " (see Figure 4.1). However, appearances may be deceiving. In Figure 4.2 it is illustrated the deformation of a sheet of paper to a half cylinder. They look different, but the

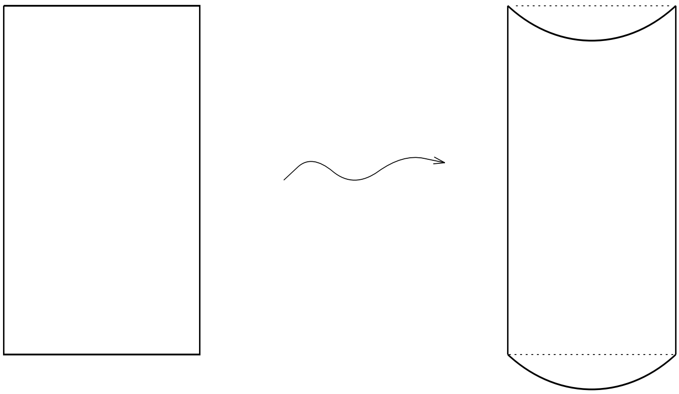

Fig. 4.2 *A plane sheet and a half cylinder are "not so different".*

metric structures are the same since we have not changed the lengths of curves on our sheet. The conclusion to be drawn from these two examples is that we have to be very careful when we use the attribute "different". □

Example 4.1.6. (The hyperbolic plane). The Poincaré model of the hyperbolic plane is the Riemann manifold $(\mathbf{D},\ g)$ where $\mathbf{D}$ is the unit open disk in the plane $\mathbb{R}^2$ and the metric g is given by

$$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2). \qquad \square$$

Exercise 4.1.7. Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{(u,v) \in \mathbb{R}^2\ ;\ v > 0\},$$

endowed with the metric

$$h = \frac{1}{4v^2}(du^2 + dv^2).$$

Show that the Cayley transform

$$z = x + \boldsymbol{i}y \mapsto w = -\boldsymbol{i}\frac{z + \boldsymbol{i}}{z - \boldsymbol{i}} = u + \boldsymbol{i}v$$

establishes an isometry $(\mathbf{D}, g) \cong (\mathcal{H}, h)$. □

Example 4.1.8. (Left invariant metrics on Lie groups). Consider a Lie group G, and denote by $\mathcal{L}_G$ its Lie algebra. Then any inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L}_G$ induces a Riemann metric $h = \langle \cdot, \cdot \rangle_g$ on G defined by

$$h_g(X,Y) = \langle X, Y \rangle_g = \langle (L_{g^{-1}})_* X, (L_{g^{-1}})_* Y \rangle,\ \ \forall g \in G,\ X, Y \in T_g G,$$

where $(L_{g^{-1}})_* : T_g G \to T_1 G$ is the differential at $g \in G$ of the left translation map $L_{g^{-1}}$. One checks easily that the correspondence

$$G \ni g \mapsto \langle \cdot, \cdot \rangle_g$$

is a smooth tensor field, and it is left invariant, i.e.,

$$L_g^* h = h \quad \forall g \in G.$$

If G is also compact, we can use the averaging technique of Subsection 3.4.2 to produce metrics which are both left and right invariant. □

4.1.2 *The Levi-Civita connection*

To continue our study of Riemann manifolds we will try to follow a close parallel with classical Euclidean geometry. The first question one may ask is whether there is a notion of "straight line" on a Riemann manifold.

In the Euclidean space $\mathbb{R}^3$ there are at least two ways to define a line segment.

(i) A line segment is the shortest path connecting two given points.
(ii) A line segment is a smooth path $\gamma : [0,1] \to \mathbb{R}^3$ satisfying

$$\ddot{\gamma}(t) = 0. \tag{4.1.1}$$

Since we have not said anything about calculus of variations which deals precisely with problems of type (i), we will use the second interpretation as our starting point. We will soon see however that both points of view yield the same conclusion.

Let us first reformulate (4.1.1). As we know, the tangent bundle of $\mathbb{R}^3$ is equipped with a natural trivialization, and as such, it has a natural trivial connection ∇^0 defined by

$$\nabla^0_i \partial_j = 0 \quad \forall i, j, \;\text{ where }\; \partial_i := \frac{\partial}{\partial x_i}, \;\; \nabla_i := \nabla_{\partial_i},$$

i.e., all the Christoffel symbols vanish. Moreover, if g_0 denotes the Euclidean metric, then

$$\left(\nabla^0_i g_0\right)(\partial_j, \partial_k) = \nabla^0_i \delta_{jk} - g_0(\nabla^0_i \partial_j, \partial_k) - g_0(\partial_j, \nabla^0_i \partial_k) = 0,$$

i.e., the connection is compatible with the metric. Condition (4.1.1) can be rephrased as

$$\nabla^0_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0, \tag{4.1.2}$$

so that the problem of defining "lines" in a Riemann manifold reduces to choosing a "natural" connection on the tangent bundle.

Of course, we would like this connection to be compatible with the metric, but even so, there are infinitely many connections to choose from. The following fundamental result will solve this dilemma.

Proposition 4.1.9. *Consider a Riemann manifold (M, g). Then there exists a* unique symmetric *connection ∇ on TM* compatible *with the metric g i.e.*

$$T(\nabla) = 0, \;\; \nabla g = 0.$$

The connection ∇ is usually called the Levi-Civita connection *associated to the metric g.*

Proof. *Uniqueness.* We will achieve this by producing an *explicit* description of a connection with the above two mproperties.

Let ∇ be such a connection, i.e.,

$$\nabla g = 0 \;\text{ and }\; \nabla_X Y - \nabla_Y X = [X, Y], \;\; \forall X, Y \in \operatorname{Vect}(M).$$

For any $X, Y, Z \in \mathrm{Vect}\,(M)$ we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

since $\nabla g = 0$. Using the symmetry of the connection we compute

$$Zg(X,Y) - Yg(Z,X) + Xg(Y,X) = g(\nabla_Z X, Y) - g(\nabla_Y Z, X) + g(\nabla_X Y, Z)$$

$$+g(X, \nabla_Z Y) - g(Z, \nabla_Y X) + g(Y, \nabla_X Z)$$

$$= g([Z,Y], X) + g([X,Y], Z) + g([Z,X], Y) + 2g(\nabla_X Z, Y).$$

We conclude that

$$g(\nabla_X Z, Y) = \frac{1}{2}\{Xg(Y,Z) - Yg(Z,X) + Zg(X,Y)$$

$$-g([X,Y], Z) + g([Y,Z], X) - g([Z,X], Y)\}. \tag{4.1.3}$$

The above equality establishes the uniqueness of ∇.

Using local coordinates $(x^1, \dots, x^n)$ on M we deduce from (4.1.3), with $X = \partial_i = \frac{\partial}{\partial x_i}$, $Y = \partial_k = \frac{\partial}{\partial x_k}$, $Z = \partial_j = \frac{\partial}{\partial x_j}$), that

$$g(\nabla_i \partial_j, \partial_k) = g_{k\ell}\Gamma^\ell_{ij} = \frac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right).$$

Above, the scalars Γ^ℓ_{ij} denote the *Christoffel symbols* of ∇ in these coordinates, i.e.,

$$\nabla_{\partial_i}\partial_j = \Gamma^\ell_{ij}\partial_\ell.$$

If $(g^{i\ell})$ denotes the inverse of $(g_{i\ell})$ we deduce

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right). \tag{4.1.4}$$

Existence. It boils down to showing that (4.1.3) indeed defines a connection with the required properties. The routine details are left to the reader. □

We can now define the notion of "straight line" on a Riemann manifold.

Definition 4.1.10. A *geodesic* on a Riemann manifold (M, g) is a smooth path

$$\gamma : (a,b) \to M,$$

satisfying

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0, \tag{4.1.5}$$

where ∇ is the Levi-Civita connection. □

Using local coordinates $(x^1, ..., x^n)$ with respect to which the Christoffel symbols are (Γ_{ij}^k), and the path γ is described by $\gamma(t) = (x^1(t), ..., x^n(t))$, we can rewrite the geodesic equation as a second order, nonlinear system of ordinary differential equations. Set

$$\frac{d}{dt} = \dot{\gamma}(t) = \dot{x}^i \partial_i.$$

Then,

$$\nabla_{\frac{d}{dt}} \dot{\gamma}(t) = \ddot{x}^i \partial_i + \dot{x}^i \nabla_{\frac{d}{dt}} \partial_i = \ddot{x}^i \partial_i + \dot{x}^i \dot{x}^j \nabla_j \partial_i$$
$$= \ddot{x}^k \partial_k + \Gamma_{ji}^k \dot{x}^i \dot{x}^j \partial_k \quad (\Gamma_{ij}^k = \Gamma_{ji}^k),$$

so that the geodesic equation is equivalent to

$$\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0 \quad \forall k = 1, ..., n. \tag{4.1.6}$$

Since the coefficients $\Gamma_{ij}^k = \Gamma_{ij}^k(x)$ depend smoothly upon x, we can use the classical Banach-Picard theorem on existence in initial value problems (see e.g. [4]). We deduce the following local existence result.

Proposition 4.1.11. *Let (M, g) be a Riemann manifold. For any compact subset $K \subset TM$ there exists $\varepsilon > 0$ such that for any $(x, X) \in K$ there exists a unique geodesic $\gamma = \gamma_{x,X} : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = X$.* □

One can think of a geodesic as defining a path in the tangent bundle $t \mapsto (\gamma(t), \dot{\gamma}(t))$. The above proposition shows that the geodesics define a local flow Φ on TM by

$$\Phi^t(x, X) = (\gamma(t), \dot{\gamma}(t)) \quad \gamma = \gamma_{x,X}.$$

Definition 4.1.12. The local flow defined above is called the *geodesic flow* of the Riemann manifold (M, g). When the geodesic glow is a global flow, i.e., any $\gamma_{x,X}$ is defined at each moment of time t for any $(x, X) \in TM$, then the Riemann manifold is called *geodesically complete.* □

The geodesic flow has some remarkable properties.

Proposition 4.1.13 (Conservation of energy). *If the path $\gamma(t)$ is a geodesic, then the length of $\dot{\gamma}(t)$ is independent of time.* □

Proof. We have

$$\frac{d}{dt}|\dot{\gamma}(t)|^2 = \frac{d}{dt} g(\dot{\gamma}(t), \dot{\gamma}(t)) = 2g(\nabla_{\dot{\gamma}(t)}, \dot{\gamma}(t)) = 0. \qquad \square$$

Thus, if we consider the sphere bundles

$$S_r(M) = \{X \in TM \; ; \; |X| = r\},$$

we deduce that $S_r(M)$ are invariant subsets of the geodesic flow.

Exercise 4.1.14. Describe the infinitesimal generator of the geodesic flow. □

Example 4.1.15. Let G be a connected Lie group, and let $\mathcal{L}_G$ be its Lie algebra. Any $X \in \mathcal{L}_G$ defines an endomorphism $\mathrm{ad}(X)$ of $\mathcal{L}_G$ by

$$\mathrm{ad}(X)Y := [X, Y].$$

The Jacobi identity implies that

$$\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)],$$

where the bracket in the right hand side is the usual commutator of two endomorphisms.

Assume that there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L}_G$ such that, for any $X \in \mathcal{L}_G$, the operator $\mathrm{ad}(X)$ is skew-adjoint, i.e.,

$$\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle. \tag{4.1.7}$$

We can now extend this inner product to a left invariant metric h on G. We want to describe its geodesics.

First, we have to determine the associated Levi-Civita connection. Using (4.1.3) we get

$$h(\nabla_X Z, Y) = \frac{1}{2}\{Xh(Y, Z) - Y(Z, X) + Zh(X, Y)$$
$$-h([X, Y], Z) + h([Y, Z], X) - h([Z, X], Y)\}.$$

If we take $X, Y, Z \in \mathcal{L}_G$, i.e., these vector fields are left invariant, then $h(Y, Z) = const.$, $h(Z, X) = const.$, $h(X, Y) = const.$ so that the first three terms in the above formula vanish. We obtain the following equality (at $1 \in G$)

$$\langle \nabla_X Z, Y \rangle = \frac{1}{2}\{-\langle [X, Y], Z \rangle + \langle [Y, Z], X \rangle - \langle [Z, X], Y \rangle\}.$$

Using the skew-symmetry of $\mathrm{ad}(X)$ and $\mathrm{ad}(Z)$ we deduce

$$\langle \nabla_X Z, Y \rangle = \frac{1}{2}\langle [X, Z], Y \rangle,$$

so that, at $1 \in G$, we have

$$\nabla_X Z = \frac{1}{2}[X, Z] \quad \forall X, Z \in \mathcal{L}_G. \tag{4.1.8}$$

This formula correctly defines a connection since any $X \in \mathrm{Vect}\,(G)$ can be written as a linear combination

$$X = \sum \alpha_i X_i \quad \alpha_i \in C^\infty(G) \quad X_i \in \mathcal{L}_G.$$

If $\gamma(t)$ is a geodesic, we can write $\dot{\gamma}(t) = \sum \gamma_i X_i$, so that

$$0 = \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = \sum_i \dot{\gamma}_i X_i + \frac{1}{2}\sum_{i,j} \gamma_i \gamma_j [X_i, X_j].$$

Since $[X_i, X_j] = -[X_j, X_i]$, we deduce $\dot{\gamma}_i = 0$, i.e.,

$$\dot{\gamma}(t) = \sum \gamma_i(0) X_i = X.$$

This means that γ is an integral curve of the left invariant vector field X so that the geodesics through the origin with initial direction $X \in T_1 G$ are

$$\gamma_X(t) = \exp(tX). \qquad \square$$

Exercise 4.1.16. Let G be a Lie group and h a bi-invariant metric on G. Prove that its restriction to $\mathcal{L}_G$ satisfies (4.1.7). In particular, on any compact Lie groups there exist metrics satisfying (4.1.7). □

Definition 4.1.17. Let $\mathcal{L}$ be a finite dimensional real Lie algebra. The *Killing pairing* or *form* is the bilinear map

$$\kappa : \mathcal{L} \times \mathcal{L} \to \mathbb{R},\ \kappa(x, y) := -\mathrm{tr}\,(\mathrm{ad}(x)\,\mathrm{ad}(y)) \quad x, y \in \mathcal{L}.$$

The Lie algebra $\mathcal{L}$ is said to be *semisimple* if the Killing pairing is a duality. A Lie group G is called *semisimple* if its Lie algebra is semisimple. □

Exercise 4.1.18. Prove that $SO(n)$, $SU(n)$ and $SL(n, \mathbb{R})$ are semisimple Lie groups, but $U(n)$ is not. □

Exercise 4.1.19. Let G be a *compact* Lie group. Prove that the Killing form is positive semi-definite[1] and satisfies (4.1.7).
Hint: Use Exercise 4.1.16. □

Exercise 4.1.20. Show that the parallel transport of X along $\exp(tY)$ is

$$(L_{\exp(\frac{t}{2}Y)})_*(R_{\exp(\frac{t}{2}Y)})_*X.$$ □

Example 4.1.21. (Geodesics on flat tori, and on $SU(2)$). The n-dimensional torus $T^n \cong S^1 \times \cdots \times S^1$ is an Abelian, compact Lie group. If $(\theta^1, ..., \theta^n)$ are the natural angular coordinates on T^n, then the flat metric is defined by

$$g = (d\theta^1)^2 + \cdots + (d\theta^n)^2.$$

The metric g on T^n is bi-invariant, and obviously, its restriction to the origin satisfies the skew-symmetry condition (4.1.7) since the bracket is 0. The geodesics through 1 will be the exponentials

$$\gamma_{\alpha_1,...,\alpha_n}(t) \quad t \mapsto (e^{\boldsymbol{i}\alpha_1 t}, ..., e^{\boldsymbol{i}\alpha_n t}) \quad \alpha_k \in \mathbb{R}.$$

If the numbers α_k are linearly dependent over $\mathbb{Q}$, then obviously $\gamma_{\alpha_1,...,\alpha_n}(t)$ is a closed curve. On the contrary, when the α's are linearly independent over $\mathbb{Q}$ then a classical result of Kronecker (see e.g. [42]) states that the image of $\gamma_{\alpha_1,...,\alpha_n}$ is dense in T^n!!! (see also Section 7.4 to come)

The special unitary group $SU(2)$ can also be identified with the group of unit quaternions

$$\left\{\, a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}\ ;\ a^2 + b^2 + c^2 + d^2 = 1 \,\right\},$$

[1]The converse of the above exercise is also true, i.e., any semisimple Lie group with positive definite Killing form is compact. This is known as Weyl's theorem. Its proof, which will be given later in the book, requires substantially more work.

so that $SU(2)$ is diffeomorphic with the unit sphere $S^3 \subset \mathbb{R}^4$. The round metric on S^3 is bi-invariant with respect to left and right (unit) quaternionic multiplication (verify this), and its restriction to $(1,0,0,0)$ satisfies (4.1.7). The geodesics of this metric are the 1-parameter subgroups of S^3, and we let the reader verify that these are in fact the great circles of S^3, i.e., the circles of maximal diameter on S^3. Thus, all the geodesics on S^3 are closed. □

4.1.3 *The exponential map and normal coordinates*

We have already seen that there are many differences between the classical Euclidean geometry and the general Riemannian geometry in the large. In particular, we have seen examples in which one of the basic axioms of Euclidean geometry no longer holds: two distinct geodesic (read lines) may intersect in more than one point. The global topology of the manifold is responsible for this "failure".

Locally however, things are not "as bad". Local Riemannian geometry is similar in many respects with the Euclidean geometry. For example, locally, all of the classical incidence axioms hold.

In this section we will define using the metric some special collections of local coordinates in which things are very close to being Euclidean.

Let (M, g) be a Riemann manifold and U an open coordinate neighborhood with coordinates $(x^1, ..., x^n)$. We will try to find a local change in coordinates $(x^i) \mapsto (y^j)$ in which the expression of the metric is as close as possible to the Euclidean metric $g_0 = \delta_{ij} dy^i dy^j$.

Let $q \in U$ be the point with coordinates $(0, ..., 0)$. Via a linear change in coordinates we may as well assume that

$$g_{ij}(q) = \delta_{ij}.$$

We can formulate this by saying that (g_{ij}) is Euclidean up to order zero.

We would like to "spread" the above equality to an entire neighborhood of q. To achieve this we try to find local coordinates (y^j) near q such that in these new coordinates the metric is Euclidean up to order one, i.e.,

$$g_{ij}(q) = \delta_{ij} \quad \frac{\partial g_{ij}}{\partial y^k}(q) = \frac{\partial \delta_{ij}}{\partial y^k}(q) = 0, \ \ \forall i, j, k.$$

We now describe a geometric way of producing such coordinates using the geodesic flow.

Denote as usual the geodesic from q with initial direction $X \in T_qM$ by $\gamma_{q,X}(t)$. Note the following simple fact.

$$\forall s > 0 \quad \gamma_{q,sX}(t) = \gamma_{q,X}(st).$$

Hence, there exists a small neighborhood V of $0 \in T_qM$ such that, for any $X \in V$, the geodesic $\gamma_{q,X}(t)$ is defined for all $|t| \leq 1$. We define the *exponential map* at q by

$$\exp_q : V \subset T_qM \to M, \ \ X \mapsto \gamma_{q,X}(1).$$

The tangent space T_qM is a Euclidean space, and we can define $\mathbf{D}_q(r) \subset T_qM$, the open "disk" of radius r centered at the origin. We have the following result.

Proposition 4.1.22. *Let (M, g) and $q \in M$ as above. Then there exists $r > 0$ such that the exponential map*

$$\exp_q : \mathbf{D}_q(r) \to M$$

is a diffeomorphism onto. The supremum of all radii r with this property is denoted by $\rho_M(q)$. □

Definition 4.1.23. The positive real number $\rho_M(q)$ is called the *injectivity radius* of M at q. The infimum

$$\rho_M = \inf_q \rho_M(q)$$

is called the *injectivity radius* of M. □

The proof of Proposition 4.1.22 relies on the following key fact.

Lemma 4.1.24. *The Fréchet differential at $0 \in T_qM$ of the exponential map*

$$D_0 \exp_q : T_qM \to T_{\exp_q(0)}M = T_qM$$

is the identity $T_qM \to T_qM$.

Proof. Consider $X \in T_qM$. It defines a line $t \mapsto tX$ in T_qM which is mapped via the exponential map to the geodesic $\gamma_{q,X}(t)$. By definition

$$(D_0 \exp_q)X = \dot{\gamma}_{q,X}(0) = X.$$ □

Proposition 4.1.22 follows immediately from the above lemma using the inverse function theorem. □

Now choose an orthonormal frame $(\boldsymbol{e}_1, ..., \boldsymbol{e}_n)$ of T_qM, and denote by $(\boldsymbol{x}^1, ..., \boldsymbol{x}^n)$ the resulting cartesian coordinates in T_qM. For $0 < r < \rho_M(q)$, any point $p \in \exp_q(\mathbf{D}_q(r))$ can be uniquely written as

$$p = \exp_q(\boldsymbol{x}^i \boldsymbol{e}_i),$$

so that the collection $(\boldsymbol{x}^1, ..., \boldsymbol{x}^n)$ provides a coordinatization of the open set $\exp_q(\mathbf{D}_q(r)) \subset M$. The coordinates thus obtained are called *normal coordinates* at q, the open set $\exp_q(\mathbf{D}_q(r))$ is called a *normal neighborhood*, and will be denoted by $\mathbf{B}_r(q)$ for reasons that will become apparent a little later.

Proposition 4.1.25. *Let $(\boldsymbol{x}^i)$ be normal coordinates at $q \in M$, and denote by $\boldsymbol{g}_{ij}$ the expression of the metric tensor in these coordinates. Then we have*

$$\boldsymbol{g}_{ij}(q) = \delta_{ij} \text{ and } \frac{\partial \boldsymbol{g}_{ij}}{\partial \boldsymbol{x}^k}(q) = 0 \quad \forall i, j, k.$$

Thus, the normal coordinates provide a first order contact between g and the Euclidean metric.

Proof. By construction, the vectors $\boldsymbol{e}_i = \frac{\partial}{\partial \boldsymbol{x}^i}$ form an orthonormal basis of T_qM and this proves the first equality. To prove the second equality we need the following auxiliary result.

Lemma 4.1.26. *In normal coordinates* $(\boldsymbol{x}_i)$ *(at q) the Christoffel symbols* Γ^i_{jk} *vanish at q.*

Proof. For any $(m^1, ..., m^n) \in \mathbb{R}^n$ the curve $\boldsymbol{x}^i = m^i t$ is the geodesic $t \mapsto \exp_q\left(\sum m^i \frac{\partial}{\partial \boldsymbol{x}^i}\right)$ so that

$$\Gamma^i_{jk}(\boldsymbol{x}(t))m^j m^k = 0.$$

In particular,

$$\Gamma^i_{jk}(0)m^j m^k = 0 \quad \forall m^j \in \mathbb{R}^n$$

from which we deduce the lemma. □

The result in the above lemma can be formulated as

$$g\left(\nabla_{\frac{\partial}{\partial \boldsymbol{x}^j}} \frac{\partial}{\partial \boldsymbol{x}^i}, \frac{\partial}{\partial \boldsymbol{x}^k}\right) = 0, \;\; \forall i, j, k$$

so that,

$$\nabla_{\frac{\partial}{\partial \boldsymbol{x}^j}} \frac{\partial}{\partial \boldsymbol{x}^i} = 0 \quad \text{at } q, \;\; \forall i, j. \tag{4.1.9}$$

Using $\nabla g = 0$ we deduce $\frac{\partial \boldsymbol{g}_{ij}}{\partial \boldsymbol{x}^k}(q) = (\frac{\partial}{\partial \boldsymbol{x}^k}\boldsymbol{g}_{ij}) \mid_q = 0$. □

The reader may ask whether we can go one step further, and find local coordinates which produce a second order contact with the Euclidean metric. At this step we are in for a big surprise. This thing is in general not possible and, in fact, there is a geometric way of measuring the "second order distance" between an arbitrary metric and the Euclidean metric. This is where the curvature of the Levi-Civita connection comes in, and we will devote an entire section to this subject.

4.1.4 *The length minimizing property of geodesics*

We defined geodesics via a second order equation imitating the second order equation defining lines in a Euclidean space. As we have already mentioned, this is not the unique way of extending the notion of Euclidean straight line to arbitrary Riemann manifolds.

One may try to look for curves of minimal length joining two given points. We will prove that the geodesics defined as in the previous subsection do just that, at least locally. We begin with a technical result which is interesting in its own. Let (M, g) be a Riemann manifold.

Lemma 4.1.27. *For each* $q \in M$ *there exists* $0 < r < \rho_M(q)$, *and* $\varepsilon > 0$ *such that,* $\forall m \in \mathbf{B}_r(q)$, *we have* $\varepsilon < \rho_M(m)$ *and* $\mathbf{B}_\varepsilon(m) \supset \mathbf{B}_r(q)$. *In particular, any two points of* $\mathbf{B}_r(q)$ *can be joined by a* unique *geodesic of length* $< \varepsilon$.

We must warn the reader that the above result does not guarantee that the postulated connecting geodesic lies entirely in $\mathbf{B}_r(q)$. This is a different ball game.

Proof. Using the smooth dependence upon initial data in ordinary differential equations we deduce that there exists an open neighborhood V of $(q,0) \in TM$ such that $\exp_m X$ is well defined for all $(m,X) \in V$. We get a smooth map

$$F : V \to M \times M \quad (m,X) \mapsto (m, \exp_m X).$$

We compute the differential of F at $(q,0)$. First, using normal coordinates $(\boldsymbol{x}^i)$ near q we get coordinates $(\boldsymbol{x}^i; \boldsymbol{X}^j)$ near $(q,0) \in TM$. The partial derivatives of F at $(q,0)$ are

$$D_{(q,0)}F(\frac{\partial}{\partial \boldsymbol{x}^i}) = \frac{\partial}{\partial \boldsymbol{x}^i} + \frac{\partial}{\partial \boldsymbol{X}^i}, \quad D_{(q,0)}F(\frac{\partial}{\partial \boldsymbol{X}^i}) = \frac{\partial}{\partial \boldsymbol{X}^i}.$$

Thus, the matrix defining $D_{(q,0)}F$ has the form

$$\begin{bmatrix} \mathbb{1} & 0 \\ * & \mathbb{1} \end{bmatrix},$$

and in particular, it is nonsingular.

It follows from the implicit function theorem that F maps some neighborhood V of $(q,0) \in TM$ diffeomorphically onto some neighborhood U of $(q,q) \in M \times M$. We can choose V to have the form $\{(m,X) \ ; \ |X|_m < \varepsilon, \quad m \in \mathbf{B}_\delta(q)\}$ for some sufficiently small ε and δ. Choose $0 < r < \min(\varepsilon, \rho_M(q))$ such that

$$m_1, m_2 \in \mathbf{B}_r(q) \Longrightarrow (m_1, m_2) \in U.$$

In particular, we deduce that, for any $m \in \mathbf{B}_r(q)$, the map $\exp_m : \mathbf{D}_\varepsilon(m) \subset T_mM \to M$ is a diffeomorphism onto its image, and

$$\mathbf{B}_\varepsilon(m) = \exp_m(\mathbf{D}_\varepsilon(m)) \supset \mathbf{B}_r(q).$$

Clearly, for any $m \in M$, the curve $t \mapsto \exp_m(tX)$ is a geodesic of length $< \varepsilon$ joining m to $\exp_m(X)$. It is the *unique geodesic* with this property since $F : V \to U$ is injective. □

We can now formulate the main result of this subsection.

Theorem 4.1.28. *Let q, r and ε as in the previous lemma, and consider the unique geodesic $\gamma : [0,1] \to M$ of length $< \varepsilon$ joining two points of $\mathbf{B}_r(q)$. If $\omega : [0,1] \to M$ is a piecewise smooth path with the same endpoints as γ then*

$$\int_0^1 |\dot{\gamma}(t)|dt \leq \int_0^1 |\dot{\omega}(t)|dt.$$

with equality if and only if $\omega([0,1]) = \gamma([0,1])$. Thus, γ is the shortest path joining its endpoints.

The proof relies on two lemmata. Let $m \in M$ be an arbitrary point, and assume $0 < R < \rho_M(m)$.

Lemma 4.1.29 (Gauss). *In $\mathbf{B}_R(m) \subset M$, the geodesics through m are orthogonal to the hypersurfaces*

$$\Sigma_\delta = \exp_q(S_\delta(0)) = \{ \exp_m(X); \ \ |X| = \delta \}, \ \ 0 < \delta < R.$$

Proof. Let $t \mapsto X(t)$, $0 \leq t \leq 1$ denote a smooth curve in T_mM such that $|X(t)|_m = 1$, i.e., $X(t)$ is a curve on the unit sphere $S_1(0) \subset T_mM$. We assume that the map $t \mapsto X(t) \in S_1(0)$ is an embedding, i.e., it is injective, and its differential is nowhere zero. We have to prove that the curve $t \mapsto \exp_m(\delta X(t))$ are orthogonal to the radial geodesics

$$s \mapsto \exp_m(sX(t)), \ \ 0 \leq s \leq R.$$

Consider the smooth map

$$f : [0, R] \times [0, 1] \to M, \ \ f(s, t) = \exp_m(sX(t)) \quad (s, t) \in (0, R) \times (0, 1).$$

If we use normal coordinates on $B_R(m)$ we can express f as the embedding

$$(0, R) \times (0, 1) \to T_mM, \ \ (s, t) \mapsto sX(t).$$

Set

$$\partial_s := f_* \left(\frac{\partial}{\partial s} \right) \in T_{f(s,t)}M.$$

Define ∂_t similarly. The objects ∂_s, and ∂_t are sections of the restriction of TM to the image of f. Using the normal coordinates on $\mathbf{B}_R(m)$ we can think of ∂_s as a vector field on a region in T_mM, and as such we have the equality $\partial_s = X(t)$. We have to show

$$\langle \partial_s, \partial_t \rangle = 0 \quad \forall (s, t),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined by g.

Using the metric compatibility of the Levi-Civita connection along the image of f we compute

$$\partial_s \langle \partial_s, \partial_t \rangle = \langle \nabla_{\partial_s} \partial_s, \partial_t \rangle + \langle \partial_s, \nabla_{\partial_s} \partial_t \rangle.$$

Since the curves $s \mapsto f(s, t = const.)$ are geodesics, we deduce

$$\nabla_{\partial_s} \partial_s = 0.$$

On the other hand, since $[\partial_s, \partial_t] = 0$, we deduce (using the symmetry of the Levi-Civita connection)

$$\langle \partial_s, \nabla_{\partial_s} \partial_t \rangle = \langle \partial_s, \nabla_{\partial_t} \partial_s \rangle = \frac{1}{2} \partial_t \, |\partial_s|^2 = 0,$$

since $|\partial_s| = |X(t)| = 1$. We conclude that the quantity $\langle \partial_s, \partial_t \rangle$ is independent of s. For $s = 0$ we have $f(0, t) = \exp_m(0)$ so that $\partial_t|_{s=0} = 0$, and therefore

$$\langle \partial_s, \partial_t \rangle = 0 \quad \forall (s, t),$$

as needed. □

Now consider any continuous, piecewise smooth curve

$$\omega : [a, b] \to \mathbf{B}_R(m) \setminus \{m\}.$$

Each $\omega(t)$ can be uniquely expressed in the form

$$\omega(t) = \exp_m(\rho(t)X(t)) \quad |X(t)| = 1 \quad 0 < |\rho(t)| < R.$$

Lemma 4.1.30. *The length of the curve $\omega(t)$ is $\geq |\rho(b)-\rho(a)|$. The equality holds if and only if $X(t) = const$ and $\dot{\rho}(t) \geq 0$. In other words, the shortest path joining two concentrical shells Σ_δ is a radial geodesic.*

Proof. Let $f(\rho, t) := \exp_m(\rho X(t))$, so that $\omega(t) = f(\rho(t), t)$. Then

$$\dot{\omega} = \frac{\partial f}{\partial \rho}\dot{\rho} + \frac{\partial f}{\partial t}.$$

Since the vectors $\frac{\partial f}{\partial \rho}$ and $\frac{\partial f}{\partial t}$ are orthogonal, and since

$$\left|\frac{\partial f}{\partial \rho}\right| = |X(t)| = 1$$

we get

$$|\dot{\omega}|^2 = |\dot{\rho}|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |\dot{\rho}|^2.$$

The equality holds if and only if $\frac{\partial f}{\partial t} = 0$, i.e. $\dot{X} = 0$. Thus

$$\int_a^b |\dot{\omega}| dt \geq \int_a^b |\dot{\rho}| dt \geq |\rho(b) - \rho(a)|.$$

Equality holds if and only if $\rho(t)$ is monotone, and $X(t)$ is constant. This completes the proof of the lemma. □

The proof of Theorem 4.1.28 is now immediate. Let $m_0, m_1 \in \mathbf{B}_r(q)$, and consider a geodesic $\gamma : [0, 1] \to M$ of length $< \varepsilon$ such that $\gamma(i) = m_i$, $i = 0, 1$. We can write

$$\gamma(t) = \exp_{m_0}(tX) \quad X \in \mathbf{D}_\varepsilon(m_0).$$

Set $R = |X|$. Consider any other piecewise smooth path $\omega : [a, b] \to M$ joining m_0 to m_1. For any $\delta > 0$ this path must contain a portion joining the shell $\Sigma_\delta(m_0)$ to the shell $\Sigma_R(m_0)$ and lying between them. By the previous lemma the length of this segment will be $\geq R - \delta$. Letting $\delta \to 0$ we deduce

$$\boldsymbol{l}(\omega) \geq R = \boldsymbol{l}(\gamma).$$

If $\omega([a, b])$ does not coincide with $\gamma([0, 1])$ then we obtain a strict inequality. □

Any Riemann manifold has a natural structure of metric space. More precisely, we set

$$d(p,q) = \inf\Big\{ \boldsymbol{l}(\omega);\ \ \omega : [0,1] \to M \ \text{ piecewise smooth path joining } p \text{ to } q \Big\}.$$

A piecewise smooth path ω connecting two points p, q such that $\boldsymbol{l}(\omega) = d(p,q)$ is said to be *minimal*. From Theorem 4.1.28 we deduce immediately the following consequence.

Corollary 4.1.31. *The image of any minimal path coincides with the image of a geodesic. In other words, any minimal path can be reparametrized such that it satisfies the geodesic equation.*

Exercise 4.1.32. Prove the above corollary. □

Theorem 4.1.28 also implies that any two nearby points can be joined by a unique minimal geodesic. In particular we have the following consequence.

Corollary 4.1.33. *Let $q \in M$. Then for all $r > 0$ sufficiently small*

$$\exp_q(\mathbf{D}_r(0))(= \mathbf{B}_r(q)\,) = \big\{ \, p \in M \quad d(p,q) < r \, \big\}. \tag{4.1.10}$$

Corollary 4.1.34. *For any $q \in M$ we have the equality*

$$\rho_M(q) = \sup\big\{ \, r;\ \ r \text{ satisfies } \ (4.1.10) \, \big\}.$$

Proof. The same argument used in the proof of Theorem 4.1.28 shows that for any $0 < r < \rho_M(q)$ the radial geodesics $\exp_q(tX)$ are minimal. □

Definition 4.1.35. A subset $U \subset M$ is said to be *convex* if any two points in U can be joined by a *unique minimal* geodesic which lies entirely *inside* U. □

Proposition 4.1.36. *For any $q \in M$ there exists $0 < R < \iota_M(q)$ such that for any $r < R$ the ball $\mathbf{B}_r(q)$ is convex.*

Proof. Choose ax $0 < \varepsilon < \frac{1}{2}\rho_M(q)$, and $0 < R < \varepsilon$ such that any two points m_0, m_1 in $B_R(q)$ can be joined by a unique minimal geodesic $[0,1] \ni t \mapsto \gamma_{m_0,m_1}(t)$ of length $< \varepsilon$, not necessarily contained in $\mathbf{B}_R(q)$. We will prove that $\forall m_0, m_1 \in \mathbf{B}_R(q)$ the map $t \mapsto d(q, \gamma_{m_0,m_1}(t))$ is convex and thus it achieves its maxima at the endpoints $t = 0, 1$. Note that

$$d(q, \gamma(t)) < R + \varepsilon < \rho_M(q).$$

The geodesic $\gamma_{m_0,m_1}(t)$ can be uniquely expressed as

$$\gamma_{m_0,m_1}(t) = \exp_q(r(t)X(t)) \quad X(t) \in T_qM \quad \text{with } r(t) = d(q, \gamma_{m_0,m_1}(t)).$$

It suffices to show $\frac{d^2}{dt^2}(r^2) \geq 0$ for $t \in [0,1]$ if $d(q,m_0)$ and $d(q,m_1)$ are sufficiently small.

At this moment it is convenient to use normal coordinates $(\boldsymbol{x}^i)$ near q. The geodesic γ_{m_0,m_1} takes the form $(\boldsymbol{x}^i(t))$, and we have

$$r^2 = (\boldsymbol{x}^1)^2 + \cdots + (\boldsymbol{x}^n)^2.$$

We compute easily

$$\frac{d^2}{dt^2}(r^2) = 2r^2(\ddot{\boldsymbol{x}}^1 + \cdots + \ddot{\boldsymbol{x}}^n) + |\dot{\boldsymbol{x}}|^2 \tag{4.1.11}$$

where $\dot{\boldsymbol{x}}(t) = \sum \dot{\boldsymbol{x}}^i \mathbf{e}_i \in T_qM$. The path γ satisfies the equation

$$\ddot{\boldsymbol{x}}^i + \Gamma^i_{jk}(\boldsymbol{x})\dot{\boldsymbol{x}}^j\dot{\boldsymbol{x}}^k = 0.$$

Since $\Gamma^i_{jk}(0) = 0$ (normal coordinates), we deduce that there exists a constant $C > 0$ (depending only on the magnitude of the second derivatives of the metric at q) such that

$$|\Gamma^i_{jk}(\boldsymbol{x})| \leq C|x|.$$

Using the geodesic equation we obtain

$$\ddot{\boldsymbol{x}}^i \geq -C|\boldsymbol{x}||\dot{\boldsymbol{x}}|^2.$$

We substitute the above inequality in (4.1.11) to get

$$\frac{d^2}{dt^2}(r^2) \geq 2|\dot{\boldsymbol{x}}|^2\left(1 - nC|\boldsymbol{x}|^3\right). \tag{4.1.12}$$

If we choose from the very beginning

$$R + \varepsilon \leq (nC)^{-1/3},$$

then, because along the geodesic we have $|x| \leq R+\varepsilon$, the right-hand side of (4.1.12) is nonnegative. This establishes the convexity of $t \mapsto r^2(t)$ and concludes the proof of the proposition. □

In the last result of this subsection we return to the concept of geodesic completeness. We will see that this can be described in terms of the metric space structure alone.

Theorem 4.1.37 (Hopf-Rinow). *Let M be a Riemann manifold and $q \in M$. The following assertions are equivalent:*
(a) $\exp_q$ is defined on all of T_qM.
(b) The closed and bounded (with respect to the metric structure) sets of M are compact.
(c) M is complete as a metric space.
(d) M is geodesically complete.
(e) There exists a sequence of compact sets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = M$ such that if $p_n \notin K_n$ then $d(q, p_n) \to \infty$.

Moreover, on a (geodesically) complete manifold any two points can be joined by a minimal geodesic. □

Remark 4.1.38. On a complete manifold there could exist points (sufficiently far apart) which can be joined by more than one minimal geodesic. Think for example of a manifold where there exist closed geodesic, e.g., the tori T^n. □

Exercise 4.1.39. Prove the Hopf-Rinow theorem. □

Exercise 4.1.40. Let (M, g) be a Riemann manifold and let (U_α) be an open cover consisting of *bounded geodesically convex open sets.* Set $d_\alpha = (\mathrm{diameter}\,(U_\alpha))^2$. Denote by g_α the metric on U_α defined by $g_\alpha = d_\alpha^{-1} g$ so that the diameter of U_α in the new metric is 1. Using a partition of unity (φ_i) subordinated to this cover we can form a new metric

$$\tilde{g} = \sum_i \varphi_i g_{\alpha(i)} \quad (\mathrm{supp}\,\varphi_i \subset U_{\alpha(i)}).$$

Prove that $\tilde{g}$ is a complete Riemann metric. Hence, on any manifold there exist complete Riemann metrics. □

4.1.5 *Calculus on Riemann manifolds*

The classical vector analysis extends nicely to Riemann manifolds. We devote this subsection to describing this more general "vector analysis".

Let (M, g) be an *oriented* Riemann manifold. We now have two structures at our disposal: a Riemann metric, and an orientation, and we will use both of them to construct a plethora of operations on tensors.

First, using the *metric* we can construct by duality the *lowering-the-indices* isomorphism $\mathfrak{L} : \mathrm{Vect}\,(M) \to \Omega^1(M)$.

Example 4.1.41. Let $M = \mathbb{R}^3$ with the Euclidean metric. A vector field V on M has the form

$$V = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Then

$$W = \mathfrak{L}V = Pdx + Qdy + Rdz.$$

If we think of V as a field of forces in the space, then W is the infinitesimal work of V. □

On a Riemann manifold there is an equivalent way of describing the exterior derivative.

Proposition 4.1.42. *Let*

$$\varepsilon : C^\infty(T^*M \otimes \Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$$

denote the exterior multiplication operator

$$\varepsilon(\alpha \otimes \beta) = \alpha \wedge \beta, \ \ \forall \alpha \in \Omega^1(M), \ \ \beta \in \Omega^k(M).$$

Then the exterior derivative d is related to the Levi-Civita on $\Lambda^k T^*M$ connection via the equality $d = \varepsilon \circ \nabla$.

Proof. We will use a strategy useful in many other situations. Our discussion about normal coordinates will payoff. Denote temporarily by D the operator $\varepsilon \circ \nabla$.

The equality $d = D$ is a local statement, and it suffices to prove it in any coordinate neighborhood. Choose $(\boldsymbol{x}^i)$ normal coordinates at an arbitrary point $p \in M$, and set $\partial_i := \frac{\partial}{\partial \boldsymbol{x}^i}$. Note that

$$D = \sum_i d\boldsymbol{x}^i \wedge \nabla_i, \quad \nabla_i = \nabla_{\partial_i}.$$

Let $\omega \in \Omega^k(M)$. Near p it can be written as

$$\omega = \sum_I \omega_I d\boldsymbol{x}^I,$$

where as usual, for any ordered multi-index I: $(1 \leq i_1 < \cdots < i_k \leq n)$, we set

$$d\boldsymbol{x}^I := d\boldsymbol{x}^{i_1} \wedge \cdots \wedge d\boldsymbol{x}^{i_k}.$$

In normal coordinates *at* p we have $(\nabla_i \partial_i)\,|_p = 0$ from which we get the equalities

$$(\nabla_i d\boldsymbol{x}^j)\,|_p\,(\partial_k) = -(d\boldsymbol{x}^j(\nabla_i \partial_k))\,|_p = 0.$$

Thus, *at* p,

$$D\omega = \sum_I d\boldsymbol{x}^i \wedge \nabla_i(\omega_I d\boldsymbol{x}^I)$$

$$= \sum_I d\boldsymbol{x}^i \wedge (\partial_j \omega_I d\boldsymbol{x}^I + \omega_I \nabla_i(d\boldsymbol{x}^I)) = \sum_I d\boldsymbol{x}^i \wedge \partial_j \omega_I = d\omega.$$

Since the point p was chosen arbitrarily this completes the proof of Proposition 4.1.42. □

Exercise 4.1.43. Show that for any k-form ω on the Riemann manifold (M, g) the exterior derivative $d\omega$ can be expressed by

$$d\omega(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \ldots, \hat{X}_i, \ldots, X_k),$$

for all $X_0, \ldots, X_k \in \operatorname{Vect}(M)$. ($\nabla$ denotes the Levi-Civita connection.) □

The Riemann metric defines a metric in any tensor bundle $\mathcal{T}^r_s(M)$ which we continue to denote by g. Thus, given two tensor fields T_1, T_2 of the same type (r, s) we can form their pointwise scalar product

$$M \ni p \mapsto g(T, S)_p = g_p(T_1(p), T_2(p)).$$

In particular, any such tensor has a pointwise norm

$$M \ni p \mapsto |T|_{g,p} = (T, T)_p^{1/2}.$$

Using the *orientation* we can construct (using the results in subsection 2.2.4) a natural *volume form* on M which we denote by dV_g, and we call it the *metric volume*. This is the positively oriented volume form of pointwise norm $\equiv 1$.

If $(x^1, ..., x^n)$ are local coordinates such that $dx^1 \wedge \cdots \wedge dx^n$ is positively oriented, then

$$dV_g = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n,$$

where $|g| := \det(g_{ij})$. In particular, we can integrate (compactly supported) functions on M by

$$\int_{(M,g)} f \stackrel{def}{=} \int_M f dv_g \quad \forall f \in C_0^\infty(M).$$

We have the following not so surprising result.

Proposition 4.1.44. $\nabla_X dV_g = 0$, $\forall X \in \mathrm{Vect}\,(M)$.

Proof. We have to show that for any $p \in M$

$$(\nabla_X dV_g)(e_1, ..., e_n) = 0, \tag{4.1.13}$$

where $e_1, ..., e_p$ is a basis of T_pM. Choose normal coordinates $(\boldsymbol{x}^i)$ near p. Set $\partial_i = \frac{\partial}{\partial \boldsymbol{x}^i}$, $\boldsymbol{g}_{ij} = g(\partial_i, \partial_k)$, and $e_i = \partial_i\,|_p$. Since the expression in (4.1.13) is linear in X, we may as well assume $X = \partial_k$, for some $k = 1, ..., n$. We compute

$$\begin{aligned}(\nabla_X dV_g)(e_1, ..., e_n) &= X(dV_g(\partial_1, ..., \partial_n))\,|_p \\ &\quad - \sum_i dv_g(e_1, ..., (\nabla_X \partial_i)\,|_p, ..., \partial_n).\end{aligned} \tag{4.1.14}$$

We consider each term separately. Note first that $dV_g(\partial_1, ..., \partial_n) = (\det(\boldsymbol{g}_{ij}))^{1/2}$, so that $X(\det(\boldsymbol{g}_{ij}))^{1/2}\,|_p = \partial_k(\det(\boldsymbol{g}_{ij}))^{1/2}\,|_p$ is a linear combination of products in which each product has a factor of the form $\partial_k \boldsymbol{g}_{ij}\,|_p$. Such a factor is zero since we are working in normal coordinates. Thus, the first term in(4.1.14) is zero. The other terms are zero as well since in normal coordinates at p we have the equality

$$\nabla_X \partial_i = \nabla_{\partial_k} \partial_i = 0.$$

Proposition 4.1.44 is proved. □

Once we have an orientation, we also have the *Hodge $*$-operator*

$$* : \Omega^k(M) \to \Omega^{n-k}(M),$$

uniquely determined by

$$\alpha \wedge *\beta = (\alpha, \beta) dV_g, \;\; \forall \alpha\; \beta \in \Omega^k(M). \tag{4.1.15}$$

In particular, $*1 = dV_g$.

Example 4.1.45. To any vector field $F = P\partial_x + Q\partial_y + R\partial_z$ on $\mathbb{R}^3$ we associated its *infinitesimal work*

$$W_F = \mathfrak{L}(F) = Pdx + Qdy + Rdz.$$

The *infinitesimal energy flux* of F is the 2-form

$$\Phi_F = *W_F = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

The exterior derivative of W_F is the infinitesimal flux of the vector field $\boldsymbol{curl}\, F$

$$dW_F = (\partial_y R - \partial_z Q)dy \wedge dz + (\partial_z P - \partial_x R)dz \wedge dx + (\partial_x Q - \partial_y P)dx \wedge dy$$

$$= \Phi_{\boldsymbol{curl}\, F} = *W_{\boldsymbol{curl}\, F}.$$

The divergence of F is the scalar defined as

$$\boldsymbol{div}\, F = *d * W_F = *d\Phi_F$$

$$= * \{(\partial_x P + \partial_y Q + \partial_z R)dx \wedge dy \wedge dz\} = \partial_x P + \partial_y Q + \partial_z R.$$

If f is a function on $\mathbb{R}^3$, then we compute easily

$$*d * df = \partial_x^2 f + \partial_y^2 f + \partial_x^2 f = \Delta f. \qquad \square$$

Definition 4.1.46. (a) For any smooth function f on the Riemann manifold (M, g) we denote by $\boldsymbol{grad}\, f$, or $\boldsymbol{grad}_g\, f$, the vector field g-dual to the 1-form df. In other words

$$g(\boldsymbol{grad}\, f, X) = df(X) = X \cdot f \quad \forall X \in \mathrm{Vect}\,(M).$$

(b) If (M, g) is an *oriented* Riemann manifold, and $X \in \mathrm{Vect}\,(M)$, we denote by $\boldsymbol{div}\, X$, or $\boldsymbol{div}_g\, X$, the smooth function defined by the equality

$$L_X dV_g = (\boldsymbol{div}\, X) dV_g. \qquad \square$$

Exercise 4.1.47. Consider the unit sphere

$$S^2 = \left\{ (x, y, z) \in \mathbb{R}^3;\;\; x^2 + y^2 + z^2 = 1 \right\},$$

and denote by g the Riemann metric on S^2 induced by the Euclidean metric $g_0 = dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$.

(a) Express g in the spherical coordinates (r, θ, φ) defined as in Example 3.4.14.

(b) Denote by h the restriction to S^2 of the function $\hat{h}(x, y, z) = z$. Express $\boldsymbol{grad}^g\, f$ in spherical coordinates. $\square$

Proposition 4.1.48. *Let X be a vector field on the oriented Riemann manifold (M, g), and denote by α the 1-form dual to X. Then*

(a) $\boldsymbol{div}\, X = \mathrm{tr}\,(\nabla X)$, *where we view ∇X as an element of $C^\infty(\mathrm{End}\,(TM))$ via the identifications*

$$\nabla X \in \Omega^1(TM) \cong C^\infty(T^*M \otimes TM) \cong C^\infty(\mathrm{End}\,(TM)).$$

(b) $\boldsymbol{div}\, X = *d * \alpha$.

(c) If $(x^1, ..., x^n)$ are local coordinates such that $dx^1 \wedge \cdots \wedge dx^n$ is positively oriented, then

$$\boldsymbol{div}\, X = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i),$$

where $X = X^i \partial_i$.

The proof will rely on the following technical result which is interesting in its own. For simplicity, we will denote the inner products by $(\bullet, \bullet)$, instead of the more precise $g(\bullet, \bullet)$.

Lemma 4.1.49. *Denote by δ the operator*

$$\delta = *d* : \Omega^k(M) \to \Omega^{k-1}(M).$$

Let α be a $(k-1)$-form and β a k-form such that at least one of them is compactly supported. Then

$$\int_M (d\alpha, \beta) dV_g = (-1)^{\nu_{n,k}} \int_M (\alpha, \delta\beta) dV_g,$$

where $\nu_{n,k} = nk + n + 1$.

Proof. We have

$$\int_M (d\alpha, \beta) dV_g = \int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) + (-1)^k \int_M \alpha \wedge d * \beta.$$

The first integral in the right-hand side vanishes by the Stokes formula since $\alpha \wedge *\beta$ has compact support. Since

$$d * \beta \in \Omega^{n-k+1}(M) \text{ and } *^2 = (-1)^{(n-k+1)(k-1)} \text{ on } \Omega^{n-k+1}(M)$$

we deduce

$$\int_M (d\alpha, \beta) dV_g = (-1)^{k+(n-k+1)(n-k)} \int_M \alpha \wedge *\delta\beta.$$

This establishes the assertion in the lemma since

$$(n-k+1)(k-1) + k \equiv \nu_{n,k} \pmod 2.$$ □

Definition 4.1.50. Define $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$d^* := (-1)^{\nu_{n,k}} \delta = (-1)^{\nu_{n,k}} * d * .$$ □

Proof of the proposition. Set $\Omega := dV_g$, and let $(X_1, ..., X_n)$ be a local moving frame of TM in a neighborhood of some point. Then

$$(L_X \Omega)(X_1, ..., X_n) = X(\Omega(X_1, ..., X_n)) - \sum_i \Omega(X_1, ..., [X, X_i], ..., X_n). \tag{4.1.16}$$

Since $\nabla\Omega = 0$ we get

$$X \cdot (\Omega(X_1, ..., X_n)) = \sum_i \Omega(X_1, ..., \nabla_X X_i, ..., X_n).$$

Using the above equality in (4.1.16) we deduce from $\nabla_X Y - [X, Y] = \nabla_Y X$ that

$$(L_X \Omega)(X_1, ..., X_n) = \sum_i \Omega(X_1, ..., \nabla_{X_i} X, ..., X_n). \tag{4.1.17}$$

Over the neighborhood where the local moving frame is defined, we can find smooth functions f_i^j, such that

$$\nabla_{X_i} X = f_i^j X_j \Rightarrow \operatorname{tr}(\nabla X) = f_i^i.$$

Part (a) of the proposition follows after we substitute the above equality in (4.1.17).
Proof of (b) For any $f \in C_0^\infty(M)$ we have

$$L_X(f\omega) = (Xf)\Omega + f(\boldsymbol{div}\, X)\Omega.$$

On the other hand,

$$L_X(f\Omega) = (i_X d + d\, i_X)(f\Omega) = d\, i_X(f\Omega).$$

Hence

$$\{(Xf) + f(\boldsymbol{div}\, X)\}\, dV_g = d(i_X f\Omega).$$

Since the form $f\Omega$ is *compactly supported* we deduce from Stokes formula

$$\int_M d(i_X f\Omega) = 0.$$

We have thus proved that for any compactly supported function f we have the equality

$$-\int_M f(\boldsymbol{div}\, X)dV_g = \int_M (Xf)dV_g = \int_M df(X)dV_g$$

$$= \int_M (\boldsymbol{grad}\, f, X)dV_g = \int_M (df, \alpha)dV_g.$$

Using Lemma 4.1.49 we deduce

$$-\int_M f(\boldsymbol{div}\, X)dV_g = -\int_M f\delta\alpha dV_g \quad \forall f \in C_0^\infty(M).$$

This completes the proof of (b).
Proof of (c) We use the equality

$$L_X(\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n) = \boldsymbol{div}(X)(\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n).$$

The desired formula follows derivating in the left-hand side. One uses the fact that L_X is an even s-derivation and the equalities

$$L_X dx^i = \partial_i X^i dx^i \quad \text{(no summation)},$$

proved in Subsection 3.1.3. □

Exercise 4.1.51. Let (M, g) be a Riemann manifold and $X \in \mathrm{Vect}\,(M)$. Show that the following conditions on X are equivalent.
(a) $L_X g = 0$.
(b) $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathrm{Vect}\,(M)$.
(A vector field X satisfying the above equivalent conditions is called a *Killing* vector field.) □

Exercise 4.1.52. Consider a Killing vector field X on the oriented Riemann manifold (M, g), and denote by η the 1-form dual to X. Show that $\delta\eta = 0$, i.e., $\boldsymbol{div}(X) = 0$. □

Definition 4.1.53. Let (M, g) be an oriented Riemann manifold (possibly with boundary). For any k-forms α, β define

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle_M = \int_M (\alpha, \beta) dV_g = \int_M \alpha \wedge *\beta,$$

whenever the integrals in the right-hand side are finite. □

Let (M, g) be an oriented Riemann manifold with boundary ∂M. By definition, M is a closed subset of a boundary-less manifold $\tilde{M}$ of the same dimension. Along ∂M we have a vector bundle decomposition

$$(T\tilde{M}) \mid_{\partial M} = T(\partial M) \oplus \boldsymbol{n}$$

where $\boldsymbol{n} = (T\partial M)^{\perp}$ is the orthogonal complement of $T\partial M$ in $(TM) \mid_{\partial M}$. Since both M and ∂M are oriented manifolds it follows that $\boldsymbol{\nu}$ is a trivial line bundle. Indeed, over the boundary

$$\det TM = \det(T\partial M) \otimes \boldsymbol{n}$$

so that

$$\boldsymbol{n} \cong \det TM \otimes \det(T\partial M)^*.$$

In particular, $\boldsymbol{n}$ admits nowhere vanishing sections, and each such section defines an orientation in the fibers of $\boldsymbol{n}$.

An *outer normal* is a nowhere vanishing section σ of $\boldsymbol{n}$ such that, for each $x \in \partial M$, and any positively oriented $\omega_x \in \det T_x\partial M$, the product $\sigma_x \wedge \omega_x$ is a positively oriented element of $\det T_xM$. Since $\boldsymbol{n}$ carries a fiber metric, we can select a unique outer normal of pointwise length $\equiv 1$. This will be called the *unit outer normal*, and will be denoted by $\vec{\nu}$. Using partitions of unity we can extend $\vec{\nu}$ to a vector field defined on M.

Proposition 4.1.54 (Integration by parts). *Let (M, g) be a compact, oriented Riemann manifold with boundary, $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$. Then*

$$\int_M (d\alpha, \beta) dV_g = \int_{\partial M} (\alpha \wedge *\beta) \mid_{\partial M} + \int_M (\alpha, d^*\beta) dV_g$$

$$= \int_{\partial M} \alpha \mid_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta) \mid_{\partial M} + \int_M (\alpha, d^*\beta) dV_g$$

where $\hat{*}$ denotes the Hodge $*$-operator on ∂M with the induced metric $\hat{g}$ and orientation.

Using the $\langle \cdot, \cdot \rangle$ notation of Definition 4.1.53 we can rephrase the above equality as

$$\langle d\alpha, \beta \rangle_M = \langle \alpha, i_{\vec{\nu}}\beta \rangle_{\partial M} + \langle \alpha, d^*\beta \rangle_M.$$

Proof. As in the proof of Lemma 4.1.49 we have

$$(d\alpha, \beta)dV_g = d\alpha \wedge *\beta = d(\alpha \wedge *\beta) + (-1)^k \alpha \wedge d * \beta.$$

The first part of the proposition follows from Stokes formula arguing precisely as in Lemma 4.1.49. To prove the second part we have to check that

$$(\alpha \wedge *\beta)\,|_{\partial M} = \alpha\,|_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta)\,|_{\partial M}\ .$$

This is a local (even a pointwise) assertion so we may as well assume

$$M = \mathbf{H}^n_+ = \{(x^1, ..., x^n) \in \mathbb{R}^n\ ;\ x^1 \geq 0\},$$

and that the metric is the Euclidean metric. Note that $\vec{\nu} = -\partial_1$. Let I be an ordered $(k-1)$-index, and J be an ordered k-index. Denote by J^c the ordered $(n-k)$-index complementary to J so that (as sets) $J \cup J^c = \{1, ..., n\}$. By linearity, it suffices to consider only the cases $\alpha = dx^I$, $\beta = dx^J$. We have

$$*dx^J = \epsilon_J dx^{J^c} \quad (\epsilon_J = \pm 1) \tag{4.1.18}$$

and

$$i_{\vec{\nu}} dx^J = \begin{cases} 0 & ,\ 1 \notin J \\ -dx^{J'} & ,\ 1 \in J \end{cases},$$

where $J' = J \setminus \{1\}$. Note that, if $1 \notin J$, then $1 \in J^c$ so that

$$(\alpha \wedge *\beta)\ |_{\partial M} = 0 = \alpha\ |_{\partial M} \wedge \hat{*}(i_{\vec{\nu}}\beta)\ |_{\partial M},$$

and therefore, the only nontrivial situation left to be discussed is $1 \in J$. On the boundary we have the equality

$$\hat{*}(i_{\vec{\nu}} dx^J) = -\hat{*}(dx^{J'}) = -\epsilon'_J dx^{J^c} \quad (\epsilon'_J = \pm 1). \tag{4.1.19}$$

We have to compare the two signs ϵ_J and ϵ'_J. in (4.1.18) and (4.1.19). The sign ϵ_J is the signature of the permutation $J\vec{\cup}J^c$ of $\{1, ..., n\}$ obtained by writing the two increasing multi-indices one after the other, first J and then J^c. Similarly, since the boundary ∂M has the orientation $-dx^2 \wedge \cdots \wedge dx^n$, we deduce that the sign ϵ'_J is $(-1)\times$(the signature of the permutation $J'\vec{\cup}J^c$ of $\{2, ..., n\}$). Obviously

$$\operatorname{sign}(J\vec{\cup}J^c) = \operatorname{sign}(J'\vec{\cup}J^c),$$

so that $\epsilon_J = -\epsilon'_J$. The proposition now follows from (4.1.18) and (4.1.19). □

Corollary 4.1.55 (Gauss). *Let (M, g) be a compact, oriented Riemann manifold with boundary, and X a vector field on M. Then*

$$\int_M \boldsymbol{div}(X) dV_g = \int_{\partial M} (X, \vec{\nu}) dv_{g_\partial},$$

where $g_\partial = g\,|_{\partial M}$.

Proof. Denote by α the 1-form dual to X. We have

$$\int_M \boldsymbol{div}(X)dV_g = \int_M 1 \wedge *d*\alpha dV_g = \int_M (1, *d*\alpha)dV_g = -\int_M (1, d^*\alpha)dV_g$$

$$= \int_{\partial M} \alpha(\vec{\nu})dv_{g_\partial} = \int_{\partial M} (X, \vec{\nu})dv_{g_\partial}. \qquad \square$$

Remark 4.1.56. The compactness assumption on M can be replaced with an integrability condition on the forms α, β so that the previous results hold for non-compact manifolds as well provided all the integrals are finite. $\square$

Definition 4.1.57. Let (M, g) be an oriented Riemann manifold. The *geometric Laplacian* is the linear operator $\Delta_M : C^\infty(M) \to C^\infty(M)$ defined by

$$\Delta_M = d^*df = - * d * df = -\boldsymbol{div}(\boldsymbol{grad}\, f).$$

A smooth function f on M satisfying the equation $\Delta_M f = 0$ is called *harmonic.* $\square$

Using Proposition 4.1.48 we deduce that in local coordinates $(x^1, ..., x^n)$, the geometer's Laplacian takes the form

$$\Delta_M = -\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right),$$

where (g^{ij}) denotes as usual the matrix inverse to (g_{ij}). Note that when g is the Euclidean metric, then the geometers' Laplacian is

$$\Delta_0 = -(\partial_i^2 + \cdots + \partial_n^2),$$

which differs from the physicists' Laplacian by a sign.

Corollary 4.1.58 (Green). *Let (M, g) as in Proposition 4.1.54, and $f, g \in C^\infty(M)$. Then*

$$\langle f, \Delta_M g\rangle_M = \langle df, dg\rangle_M - \langle f, \frac{\partial g}{\partial\vec{\nu}}\rangle_{\partial M},$$

and

$$\langle f, \Delta_M g\rangle_M - \langle\Delta_M f, g\rangle_M = \langle\frac{\partial f}{\partial\vec{\nu}}, g\rangle_{\partial M} - \langle f, \frac{\partial g}{\partial\vec{\nu}}\rangle_{\partial M}.$$

Proof. The first equality follows immediately from the integration by parts formula (Proposition 4.1.54), with $\alpha = f$, and $\beta = dg$. The second identity is now obvious. $\square$

Exercise 4.1.59. (a) Prove that the only harmonic functions on a compact oriented Riemann manifold M are the constant ones.
(b) If $u, f \in C^\infty(M)$ are such that $\Delta_M u = f$ show that $\int_M f = 0$. $\square$

Exercise 4.1.60. Denote by $(u^1, \dots, u^n)$ the coordinates on the round sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ obtained via the stereographic projection from the south pole.
(a) Show that the round metric g_0 on S^n is given in these coordinates by

$$g_0 = \frac{4}{1+r^2}\{\, (du^1)^2 + \cdots + (du^n)^2 \,\},$$

where $r^2 = (u^1)^2 + \cdots + (u^n)^2$.
(b) Show that the n-dimensional "area" of S^n is

$$\boldsymbol{\sigma}_n = \int_{S^n} dv_{g_0} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})},$$

where Γ is Euler's Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt. \qquad \square$$

Hint: Use the "doubling formula"

$$\pi^{1/2}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma(s+1/2),$$

and the classical Beta integrals (see [39], or [101], Chapter XII)

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}dr = \frac{(\Gamma(n/2))^2}{2\Gamma(n)}. \qquad \square$$

Exercise 4.1.61. Consider the Killing form on $\underline{\mathrm{su}}(2)$ (the Lie algebra of $SU(2)$) defined by

$$\langle X, Y\rangle = -\mathrm{tr}\, X\cdot Y.$$

(a) Show that the Killing form defines a bi-invariant metric on $SU(2)$, and then compute the volume of the group with respect to this metric. The group $SU(2)$ is given the orientation defined by $e_1 \wedge e_2 \wedge e_3 \in \Lambda^3\underline{\mathrm{su}}(2)$, where $e_i \in \underline{\mathrm{su}}(2)$ are the *Pauli matrices*

$$e_1 = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \;\; e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \;\; e_3 = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$$

(b) Show that the trilinear form on $\underline{\mathrm{su}}(2)$ defined by

$$B(X,Y,Z) = \langle [X,Y], Z\rangle,$$

is skew-symmetric. In particular, $B \in \Lambda^3\underline{\mathrm{su}}(2)^*$.
(c) B has a unique extension as a left-invariant 3-form on $SU(2)$ known as the *Cartan form* on $SU(2)$ which we continue to denote by B. Compute $\int_{SU(2)} B$.
Hint: Use the natural diffeomorphism $SU(2) \cong S^3$, and the computations in the previous exercise. $\square$

4.2 The Riemann curvature

Roughly speaking, a Riemann metric on a manifold has the effect of “giving a shape” to the manifold. Thus, a very short (in diameter) manifold is different from a very long one, and a large (in volume) manifold is different from a small one. However, there is a lot more information encoded in the Riemann manifold than just its size. To recover it, we need to look deeper in the structure, and go beyond the first order approximations we have used so far.

The Riemann curvature tensor achieves just that. It is an object which is very rich in information about the “shape” of a manifold, and loosely speaking, provides a second order approximation to the geometry of the manifold. As Riemann himself observed, we do not need to go beyond this order of approximation to recover all the information.

In this section we introduce the reader to the Riemann curvature tensor and its associates. We will describe some special examples, and we will conclude with the Gauss-Bonnet theorem which shows that the local object which is the Riemann curvature has global effects.

☞ *Unless otherwise indicated, we will use Einstein's summation convention.*

4.2.1 *Definitions and properties*

Let (M, g) be a Riemann manifold, and denote by ∇ the Levi-Civita connection.

Definition 4.2.1. The *Riemann curvature* is the tensor $R = R(g)$, defined as

$$R(g) = F(\nabla),$$

where $F(\nabla)$ is the curvature of the Levi-Civita connection. □

The Riemann curvature is a tensor $R \in \Omega^2(\operatorname{End}(TM))$ explicitly defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

In local coordinates $(x^1, \dots, x^n)$ we have the description

$$R^{\ell}_{ijk}\partial_\ell = R(\partial_j, \partial_k)\partial_i.$$

In terms of the Christoffel symbols we have

$$R^{\ell}_{ijk} = \partial_j\Gamma^{\ell}_{ik} - \partial_k\Gamma^{\ell}_{ij} + \Gamma^{\ell}_{mj}\Gamma^{m}_{ik} - \Gamma^{\ell}_{mk}\Gamma^{m}_{ij}.$$

Lowering the indices we get a new tensor

$$R_{ijk\ell} := g_{im}R^{m}_{jk\ell} = \big(R(\partial_k, \partial_\ell)\partial_j, \partial_i \big) = \big(\partial_i, R(\partial_k, \partial_\ell)\partial_j \big).$$

Theorem 4.2.2 (The symmetries of the curvature tensor). *The Riemann curvature tensor R satisfies the following identities ($X, Y, Z, U, V \in \operatorname{Vect}(M)$).*

(a) $g(R(X,Y)U,V) = -g(R((Y,X),U,V)$.

(b) $g(R(X,Y)U,V) = -g(R(X,Y)V,U)$.

(c) (The 1st Bianchi identity)

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

(d) $g(R(X,Y)U,V) = g(R(U,V)X,Y)$.

(e) (The 2nd Bianchi identity)

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

In local coordinates the above identities have the form

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k},$$

$$R_{ijk\ell} = R_{k\ell ij},$$

$$R^i_{jk\ell} + R^i_{\ell jk} + R^i_{k\ell j} = 0,$$

$$(\nabla_i R)^j_{mk\ell} + (\nabla_\ell R)^j_{mik} + (\nabla_k R)^j_{m\ell i} = 0.$$

Proof. (a) It follows immediately from the definition of R as an $\mathrm{End}(TM)$-valued skew-symmetric bilinear map $(X,Y) \mapsto R(X,Y)$.

(b) We have to show that the symmetric bilinear form

$$Q(U,V) = g(R(X,Y)U,V) + g(R(X,Y)V,U)$$

is trivial. Thus, it suffices to check $Q(U,U) = 0$. We may as well assume that $[X,Y] = 0$, since (locally) X, Y can be written as linear combinations (over $C^\infty(M)$) of commuting vector fields. (E.g. $X = X^i\partial_i$). Then

$$Q(U,U) = g((\nabla_X\nabla_Y - \nabla_Y\nabla_X)U,U).$$

We compute

$$Y(Xg(U,U)) = 2Yg(\nabla_X U,U) = 2g(\nabla_Y\nabla_X U,U) + 2g(\nabla_X U,\nabla_Y U),$$

and similarly,

$$X(Yg(U,U)) = 2g(\nabla_X\nabla_Y U,U) + 2g(\nabla_X U,\nabla_Y U).$$

Subtracting the two equalities we deduce (b).

(c) As before, we can assume the vector fields X, Y, Z pairwise commute. The 1st Bianchi identity is then equivalent to

$$\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X = 0.$$

The identity now follows from the symmetry of the connection: $\nabla_X Y = \nabla_Y X$ etc.

(d) We will use the following algebraic lemma ([60], Chapter 5).

Lemma 4.2.3. *Let $R : E\times E\times E\times E \to \mathbb{R}$ be a quadrilinear map on a real vector space E. Define*

$$S(X_1,X_2,X_3,X_4) = R(X_1,X_2,X_3,X_4) + R(X_2,X_3,X_1,X_4) + R(X_3,X_1,X_2,X_4).$$

If R satisfies the symmetry conditions

$$R(X_1, X_2, X_3, X_4) = -R(X_2, X_1, X_3, X_4)$$

$$R(X_1, X_2, X_3, X_4) = -R(X_1, X_2, X_4, X_3),$$

then

$$\begin{aligned} R(X_1, X_2, X_3, X_4) &- R(X_3, X_4, X_1, X_2) \\ &= \frac{1}{2}\Big\{ S(X_1, X_2, X_3, X_4) - S(X_2, X_3, X_4, X_1) \\ &\qquad - S(X_3, X_4, X_1, X_2) + S(X_4, X_3, X_1, X_2) \Big\}. \end{aligned}$$

The proof of the lemma is a straightforward (but tedious) computation which is left to the reader. The Riemann curvature $R = g(R(X_1, X_2)X_3, X_4)$ satisfies the symmetries required in the lemma and moreover, the 1st Bianchi identity shows that the associated form S is identically zero. This concludes the proof of (d).
(e) This is the Bianchi identity we established for any linear connection (see Exercise 3.3.23). □

Exercise 4.2.4. Denote by $\mathcal{C}_n$ of n-dimensional curvature tensors, i.e., tensors $(R_{ijkl}) \in (\mathbb{R}^n)^{\otimes 4}$ satisfying the conditions,

$$R_{ijk\ell} = R_{k\ell ij} = -R_{jik\ell}, \;\; R_{ijk\ell} + R_{i\ell jk} + R_{ik\ell j} = 0, \;\; \forall i, j, k, \ell.$$

Prove that

$$\dim \mathcal{C}_n = \binom{\binom{n}{2} + 1}{2} - \binom{n}{4} = \frac{1}{2}\binom{n}{2}\left(\binom{n}{2} + 1\right) - \binom{n}{4}.$$

(**Hint:** Consult [13], Chapter 1, Section G.) □

The Riemann curvature tensor is the source of many important invariants associated to a Riemann manifold. We begin by presenting the simplest ones.

Definition 4.2.5. Let (M, g) be a Riemann manifold with curvature tensor R. Any two vector fields X, Y on M define an endomorphism of TM by

$$U \mapsto R(U, X)Y.$$

The *Ricci curvature* is the trace of this endomorphism, i.e.,

$$\mathrm{Ric}\,(X, Y) = \mathrm{tr}\,(U \mapsto R(U, X)Y).$$

We view it as a (0,2)-tensor $(X, Y) \mapsto \mathrm{Ric}\,(X, Y) \in C^\infty(M)$. □

If $(x^1, \dots, x^n)$ are local coordinates on M and the curvature R has the local expression $R = (R^\ell_{kij})$ then the Ricci curvature has the local description

$$\mathrm{Ric} = (\mathrm{Ric}_{ij}) = \sum_\ell R^\ell_{j\ell i}.$$

The symmetries of the Riemann curvature imply that Ric is a *symmetric* (0,2)-tensor (as the metric).

Definition 4.2.6. The *scalar curvature* s of a Riemann manifold is the trace of the Ricci tensor. In local coordinates, the scalar curvature is described by

$$s = g^{ij}\,\mathrm{Ric}_{ij} = g^{ij} R^{\ell}_{i\ell j}, \tag{4.2.1}$$

where (g^{ij}) is the inverse matrix of (g_{ij}). □

Let (M, g) be a Riemann manifold and $p \in M$. For any linearly independent $X, Y \in T_pM$ set

$$K_p(X,Y) = \frac{(R(X,Y)Y, X)}{|X \wedge Y|},$$

where $|X \wedge Y|$ denotes the Gramm determinant

$$|X \wedge Y| = \begin{vmatrix} (X,X) & (X,Y) \\ (Y,X) & (Y,Y) \end{vmatrix},$$

which is non-zero since X and Y are linearly independent. ($|X \wedge Y|^{1/2}$ measures the area of the parallelogram in T_pM spanned by X and Y.)

Remark 4.2.7. Given a metric g on a smooth manifold M, and a constant $\lambda > 0$, we obtained a new, rescaled metric $g_\lambda = \lambda^2 g$. A simple computation shows that the Christoffel symbols and Riemann tensor of g_λ are equal with the Christoffel symbols and the Riemann tensor of the metric g. In particular, this implies

$$\mathrm{Ric}_{g_\lambda} = \mathrm{Ric}_g\,.$$

However, the sectional curvatures are sensitive to metric rescaling.

For example, if g is the canonical metric on the 2-sphere of radius 1 in $\mathbb{R}^3$, then g_λ is the induced metric on the 2-sphere of radius λ in $\mathbb{R}^3$. Intuitively, the larger the constant λ, the less curved is the corresponding sphere.

In general, for any two linearly independent vectors $X, Y \in \mathrm{Vect}\,(M)$ we have

$$K_{g_\lambda}(X,Y) = \lambda^{-2} K_g(X,Y).$$

In particular, the scalar curvature changes by the same factor upon rescaling the metric.

If we think of the metric as a quantity measured in meter^2, then the sectional curvatures are measured in meter^{-2}. □

Exercise 4.2.8. Let $X, Y, Z, W \in T_pM$ such that $\mathrm{span}(X,Y) = \mathrm{span}(Z,W)$ is a 2-dimensional subspace of T_pM prove that $K_p(X,Y) = K_p(Z,W)$. □

According to the above exercise the quantity $K_p(X,Y)$ depends only upon the 2-plane in T_pM generated by X and Y. Thus K_p is in fact a function on $\mathbf{Gr}_2(p)$ the Grassmannian of 2-dimensional subspaces of T_pM.

Definition 4.2.9. The function $K_p : \mathbf{Gr}_2(p) \to \mathbb{R}$ defined above is called the *sectional curvature* of M at p. □

Exercise 4.2.10. Prove that

$$\mathbf{Gr}_2(M) = \text{disjoint union of } \ \mathbf{Gr}_2(p) \quad p \in M$$

can be organized as a smooth fiber bundle over M with standard fiber $\mathbf{Gr}_2(\mathbb{R}^n)$, $n = \dim M$, such that, if M is a Riemann manifold, $\mathbf{Gr}_2(M) \ni (p;\pi) \mapsto K_p(\pi)$ is a smooth map. □

4.2.2 *Examples*

Example 4.2.11. Consider again the situation discussed in Example 4.1.15. Thus, G is a Lie group, and $\langle \bullet, \bullet \rangle$ is a metric on the Lie algebra $\mathfrak{L}_G$ satisfying

$$\langle \mathrm{ad}(X)Y, Z\rangle = -\langle Y, \mathrm{ad}(X)Z\rangle.$$

In other words, $\langle \bullet, \bullet \rangle$ is the restriction of a bi-invariant metric $\mathfrak{m}$ on G. We have shown that the Levi-Civita connection of this metric is

$$\nabla_X Y = \frac{1}{2}[X,Y], \ \ \forall X, Y \in \mathfrak{L}_G.$$

We can now easily compute the curvature

$$R(X,Y)Z = \frac{1}{4}\left\{[X,[Y,Z]] - [Y,[X,Z]]\right\} - \frac{1}{2}[[X,Y],Z]$$

$$\text{(Jacobi identity)} = \frac{1}{4}[[X,Y],Z] + \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] = -\frac{1}{4}[[X,Y],Z].$$

We deduce

$$\langle R(X,Y)Z, W\rangle = -\frac{1}{4}\langle [[X,Y],Z], W\rangle = \frac{1}{4}\langle \mathrm{ad}(Z)[X,Y], W\rangle$$

$$= -\frac{1}{4}\langle [X,Y], ad(Z)W\rangle = -\frac{1}{4}\langle [X,Y],[Z,W]\rangle.$$

Now let $\pi \in \mathbf{Gr}_2(T_gG)$ be a 2-dimensional subspace of T_gG, for some $g \in G$. If (X,Y) is an orthonormal basis of π, viewed as left invariant vector fields on G, then the sectional curvature along π is

$$K_g(\pi) = \frac{1}{4}\Big\langle\, [X,Y],[X,Y]\,\Big\rangle \geq 0.$$

Denote the Killing form by $\kappa(X,Y) = -\mathrm{tr}\,\big(\mathrm{ad}(X)\ \mathrm{ad}(Y)\big)$. To compute the Ricci curvature we pick an orthonormal basis $E_1, \dots, E_n$ of $\mathfrak{L}_G$. For any $X = X^iE_i, Y = Y^jE_j \in \mathfrak{L}_G$ we have

$$\mathrm{Ric}\,(X,Y) = \frac{1}{4}\mathrm{tr}\,(Z \mapsto [[X,Z],Y])$$

$$= \frac{1}{4}\sum_i \langle [[X, E_i], Y], E_i)\rangle = -\frac{1}{4}\sum_i \langle \mathrm{ad}(Y)[X, E_i], E_i\rangle$$

$$= \frac{1}{4}\sum_i \langle [X, E_i], [Y, E_i]\rangle = \frac{1}{4}\sum_i \langle \mathrm{ad}(X)E_i, \mathrm{ad}(Y)E_i\rangle$$

$$= -\frac{1}{4}\sum_i \langle \mathrm{ad}(Y)\,\mathrm{ad}(X)E_i, E_i\rangle = -\frac{1}{4}\mathrm{tr}\,\big(\,\mathrm{ad}(Y)\,\mathrm{ad}(X)\,\big) = \frac{1}{4}\kappa(X, Y).$$

In particular, on a compact semisimple Lie group the Ricci curvature is a symmetric positive definite $(0, 2)$-tensor, and more precisely, it is a scalar multiple of the Killing metric.

We can now easily compute the scalar curvature. Using the same notations as above we get

$$s = \frac{1}{4}\sum_i \mathrm{Ric}\,(E_i, E_i) = \frac{1}{4}\sum \kappa(E_i, E_i).$$

In particular, if G is a compact semisimple group and the metric is given by the Killing form then the scalar curvature is

$$s(\kappa) = \frac{1}{4}\dim G. \qquad \square$$

Remark 4.2.12. Many problems in topology lead to a slightly more general situation than the one discussed in the above example namely to metrics on Lie groups which are only left invariant. Although the results are not as “crisp” as in the bi-invariant case many nice things do happen. For details we refer to [73]. □

Example 4.2.13. Let M be a 2-dimensional Riemann manifold (surface), and consider local coordinates on M, (x^1, x^2). Due to the symmetries of R,

$$R_{ijkl} = -R_{ijlk} = R_{klij},$$

we deduce that the only nontrivial component of the Riemann tensor is $R = R_{1212}$. The sectional curvature is simply a function on M

$$K = \frac{1}{|g|}R_{1212} = \frac{1}{2}s(g), \ \text{ where } |g| = \det(g_{ij}).$$

In this case, the scalar K is known as the *total curvature* or the *Gauss curvature* of the surface.

In particular, if M is oriented, and the form $dx^1 \wedge dx^2$ defines the orientation, we can construct a 2-form

$$\varepsilon(g) = \frac{1}{2\pi}K\,dv_g = \frac{1}{4\pi}s(g)dV_g = \frac{1}{2\pi\sqrt{|g|}}R_{1212}dx^1 \wedge dx^2.$$

The 2-form $\varepsilon(g)$ is called the *Euler form associated to the metric g.* We want to emphasize that this form is defined *only* when M is *oriented.*

We can rewrite this using the pfaffian construction of Subsection 2.2.4. The curvature R is a 2-form with coefficients in the bundle of skew-symmetric endomorphisms of TM so we can write

$$R = A \otimes dV_g, \quad A = \frac{1}{\sqrt{|g|}} \begin{bmatrix} 0 & R_{1212} \\ R_{2112} & 0 \end{bmatrix}$$

Assume for simplicity that (x^1, x^2) are normal coordinates at a point $q \in M$. Thus at q, $|g| = 1$ since ∂_1, ∂_2 is an orthonormal basis of T_qM. Hence, at q, $dV_g = dx^1 \wedge dx^2$, and

$$\varepsilon(g) = \frac{1}{2\pi} g\big(R(\partial_1, \partial_2 \big)\partial_2, \partial_1) dx^1 \wedge dx^2 = \frac{1}{2\pi} R_{1212} dx^1 \wedge dx^2.$$

Hence we can write

$$\varepsilon(g) = \frac{1}{2\pi} \boldsymbol{Pf}_g(-A) dv_g =: \frac{1}{2\pi} \boldsymbol{Pf}_g(-R).$$

The Euler form has a very nice interpretation in terms of holonomy. Assume as before that (x^1, x^2) are normal coordinates at q, and consider the square $S_t = [0, \sqrt{t}] \times [0, \sqrt{t}]$ in the (x^1, x^2) plane. Denote the (counterclockwise) holonomy along ∂S_t by $\mathcal{T}_t$. This is an orthogonal transformation of T_qM, and with respect to the orthogonal basis (∂_1, ∂_2) of T_qM, it has a matrix description as

$$\mathcal{T}_t = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}.$$

The result in Subsection 3.3.4 can be rephrased as

$$\sin\theta(t) = -tg(R(\partial_1, \partial_2)\partial_2, \partial_1) + O(t^2),$$

so that

$$R_{1212} = \dot{\theta}(0).$$

Hence R_{1212} is simply the infinitesimal angle measuring the infinitesimal rotation suffered by ∂_1 along S_t. We can think of the Euler form as a "density" of holonomy since it measures the holonomy per elementary parallelogram. □

4.2.3 *Cartan's moving frame method*

This method was introduced by Élie Cartan at the beginning of the 20th century. Cartan's insight was that the local properties of a manifold equipped with a geometric structure can be very well understood if one knows how the frames of the tangent bundle (compatible with the geometric structure) vary from one point of the manifold to another. We will begin our discussion with the model case of $\mathbb{R}^n$. Throughout this subsection we will use Einstein's convention.

Example 4.2.14. Consider an *orthonormal moving frame* on $\mathbb{R}^n$, $X_\alpha = X^i_\alpha \partial_i$, $\alpha = 1, ..., n$, where $(x^1, \dots, x^n)$ are the usual Cartesian coordinates, and $\partial_i := \frac{\partial}{\partial x^i}$. Denote by (θ^α) the *dual coframe*, i.e., the moving frame of $T^*\mathbb{R}^n$ defined by

$$\theta^\alpha(X_\beta) = \delta^\alpha_\beta.$$

The 1-forms θ^α measure the infinitesimal displacement of a point P with respect to the frame (X_α). Note that the TM-valued 1-form $\theta = \theta^\alpha X_\alpha$ is the differential of the identity map $\mathbb{1} : \mathbb{R}^n \to \mathbb{R}^n$ expressed using the given moving frame.

Introduce the 1-forms ω^α_β defined by

$$dX_\beta = \omega^\alpha_\beta X_\alpha, \tag{4.2.2}$$

where we set

$$dX_\alpha := \left(\frac{\partial X^i_\alpha}{\partial x^j} \right) dx^j \otimes \partial_i.$$

We can form the matrix valued 1-form $\omega = (\omega^\alpha_\beta)$ which measures the infinitesimal rotation suffered by the moving frame (X_α) following the infinitesimal displacement $x \mapsto x + dx$. In particular, $\omega = (\omega^\alpha_\beta)$ is a skew-symmetric matrix since

$$0 = d\langle X_\alpha, X_\beta \rangle = \langle \omega \cdot X_\alpha, X_\beta \rangle + \langle X_\alpha, \omega \cdot X_\beta \rangle.$$

Since $\theta = d\mathbb{1}$ we deduce

$$0 = d^2\mathbb{1} = d\theta = d\theta^\alpha \otimes X_\alpha - \theta^\beta \otimes dX_\beta = (d\theta^\alpha - \theta^\beta \wedge \omega^\alpha_\beta) \otimes X_\alpha,$$

and we can rewrite this as

$$d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta, \text{ or } d\theta = -\omega \wedge \theta. \tag{4.2.3}$$

Above, in the last equality we interpret ω as an $n \times n$ matrix whose entries are 1-forms, and θ as a column matrix, or an $n \times 1$ matrix whose entries are 1-forms.

Using the equality $d^2 X_\beta = 0$ in (4.2.2) we deduce

$$d\omega^\alpha_\beta = -\omega^\alpha_\gamma \wedge \omega^\gamma_\beta, \text{ or equivalently, } d\omega = -\omega \wedge \omega. \tag{4.2.4}$$

The equations (4.2.3)–(4.2.4) are called *the structural equations* of the Euclidean space. The significance of these structural equations will become evident in a little while. □

We now try to perform the same computations on an arbitrary Riemann manifold (M, g), $\dim M = n$. We choose a local orthonormal moving frame $(X_\alpha)_{1 \le \alpha \le n}$, and we construct similarly its dual coframe $(\theta^\alpha)_{1 \le \alpha \le n}$. Unfortunately, there is no natural way to define dX_α to produce the forms ω^α_β entering the structural equations. We will find them using a different (dual) search strategy.

Proposition 4.2.15 (E. Cartan). *There exists a collection of 1-forms* $(\omega^\alpha_\beta)_{1 \le \alpha, \beta \le n}$ uniquely defined *by the requirements*

(a) $\omega^\alpha_\beta = -\omega^\beta_\alpha$, $\forall \alpha, \beta$.

(b) $d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta$, $\forall\alpha$.

Proof. Since the collection of two forms $(\theta^\alpha \wedge \theta^\beta)_{1\leq\alpha<\beta\leq n}$ defines a local frame of $\Lambda^2 T^*\mathbb{R}^n$, there exist functions $g^\alpha_{\beta\gamma}$, uniquely determined by the conditions

$$d\theta^\alpha = \frac{1}{2} g^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma, \;\; g^\alpha_{\beta\gamma} = -g^\alpha_{\gamma\beta}.$$

Uniqueness. Suppose that there exist forms ω^α_β satisfy the conditions (a) and (b) above. Then there exist functions $f^\alpha_{\beta\gamma}$ such that

$$\omega^\alpha_\beta = f^\alpha_{\beta\gamma}\theta^\gamma.$$

Then the condition (a) is equivalent to
(a1) $f^\alpha_{\beta\gamma} = -f^\beta_{\alpha\gamma}$,
while (b) gives
(b1) $f^\alpha_{\beta\gamma} - f^\alpha_{\gamma\beta} = g^\alpha_{\beta\gamma}$.

The above two relations uniquely determine the f's in terms of the g's via a cyclic permutation of the indices α, β, γ

$$f^\alpha_{\beta\gamma} = \frac{1}{2}(g^\alpha_{\beta\gamma} + g^\beta_{\gamma\alpha} - g^\gamma_{\alpha\beta}). \tag{4.2.5}$$

Existence. Define $\omega^\alpha_\beta = f^\alpha_{\beta\gamma}\theta^\gamma$, where the f's are given by (4.2.5). We let the reader check that the forms ω^α_β satisfy both (a) and (b). □

The reader may now ask why go through all this trouble. What have we gained by constructing the forms ω, and after all, what is their significance?

To answer these questions, consider the Levi-Civita connection ∇. Define $\hat{\omega}^\alpha_\beta$ by

$$\nabla X_\beta = \hat{\omega}^\alpha_\beta X_\alpha.$$

Hence

$$\nabla_{X_\gamma} X_\beta = \hat{\omega}^\alpha_\beta(X_\gamma)X_\alpha.$$

Since ∇ is compatible with the Riemann metric, we deduce in standard manner that $\hat{\omega}^\alpha_\beta = -\hat{\omega}^\beta_\alpha$.

The differential of θ^α can be computed in terms of the Levi-Civita connection (see Subsection 4.1.5), and we have

$$d\theta^\alpha(X_\beta, X_\gamma) = X_\beta\theta^\alpha(X_\gamma) - X_\gamma\theta^\alpha(X_\beta) - \theta^\alpha(\nabla_{X_\beta}X_\gamma) + \theta^\alpha(\nabla_{X_\gamma}X_\beta)$$

$$(\text{use } \theta^\alpha(\mathrm{X}_\beta) = \delta^\alpha_\beta = \text{ const}) = -\theta^\alpha(\hat{\omega}^\delta_\gamma(X_\beta)X_\delta) + \theta^\alpha(\hat{\omega}^\delta_\beta(X_\gamma)X_\delta)$$

$$= \hat{\omega}^\alpha_\beta(X_\gamma) - \hat{\omega}^\alpha_\gamma(X_\beta) = (\theta^\beta \wedge \hat{\omega}^\alpha_\beta)(X_\beta, X_\gamma).$$

Thus the $\hat{\omega}$'s satisfy both conditions (a) and (b) of Proposition 4.2.15 so that we must have

$$\hat{\omega}^\alpha_\beta = \omega^\alpha_\beta.$$

In other words, the matrix valued 1-form (ω^α_β) is the 1-form associated to the Levi-Civita connection in this local moving frame. In particular, using the computation in Example 3.3.12 we deduce that the 2-form

$$\Omega = (d\omega + \omega \wedge \omega)$$

is the Riemannian curvature of g. The *Cartan structural equations* of a Riemann manifold take the form

$$d\theta = -\omega \wedge \theta, \;\; d\omega + \omega \wedge \omega = \Omega. \tag{4.2.6}$$

Comparing these with the Euclidean structural equations we deduce another interpretation of the Riemann curvature: it measures "the distance" between the given Riemann metric and the Euclidean one". We refer to [92] for more details on this aspect of the Riemann tensor.

The technique of orthonormal frames is extremely versatile in concrete computations.

Example 4.2.16. We will use the moving frame method to compute the curvature of the *hyperbolic plane*, i.e., the upper half space

$$\mathbf{H}_+ = \{(x, y)\,;\; y > 0\}$$

endowed with the metric $g = y^{-2}(dx^2 + dy^2)$.

The pair $(y\partial_x, y\partial_y)$ is an orthonormal moving frame, and $(\theta^x = \frac{1}{y}dx, \theta^y = \frac{1}{y}dy)$ is its dual coframe. We compute easily

$$d\theta^x = d(\frac{1}{y}dx) = \frac{1}{y^2}dx \wedge dy = (\frac{1}{y}dx) \wedge \theta^y,$$

$$d\theta^y = d(\frac{1}{y}dy) = 0 = (-\frac{1}{y}dx) \wedge \theta^x.$$

Thus the connection 1-form in this local moving frame is

$$\omega = \begin{bmatrix} 0 & -\frac{1}{y} \\ \frac{1}{y} & 0 \end{bmatrix} dx.$$

Note that $\omega \wedge \omega = 0$. Using the structural equations (4.2.6) we deduce that the Riemann curvature is

$$\Omega = d\omega = \begin{bmatrix} 0 & \frac{1}{y^2} \\ -\frac{1}{y^2} & 0 \end{bmatrix} dy \wedge dx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta^x \wedge \theta^y.$$

The Gauss curvature is

$$K = \frac{1}{|g|} g(\Omega(\partial_x, \partial_y)\partial_y, \partial_x) = y^4(-\frac{1}{y^4}) = -1. \qquad \square$$

Exercise 4.2.17. Suppose (M, g) is a Riemann manifold, and $u \in C^\infty(M)$. Define a new metric $g_u := e^{2f}g$. Using the moving frames method, describe the scalar curvature of g_u in terms of u and the scalar curvature of g. $\square$

4.2.4 *The geometry of submanifolds*

We now want to apply Cartan's method of moving frames to discuss the local geometry of submanifolds of a Riemann manifold.

Let (M, g) be a Riemann manifold of dimension m, and S a k-dimensional submanifold in M. The restriction of g to S induces a Riemann metric g_S on S. We want to analyze the relationship between the Riemann geometry of M (assumed to be known) and the geometry of S with the induced metric.

Denote by ∇^M (respectively ∇^S) the Levi-Civita connection of (M, g) (respectively of (S, g_S). The metric g produces an orthogonal splitting of vector bundles

$$(TM)\mid_S \cong TS \oplus N_S S.$$

The N_S is called the normal bundle of $S \hookrightarrow M$, and it is the orthogonal complement of TS in $(TM)\mid_S$. Thus, a section of $(TM)\mid_S$, that is a vector field X of M along S, splits into two components: a tangential component X^τ, and a normal component, X^ν.

Now choose a local orthonormal moving frame $(X_1, ..., X_k; X_{k+1}, ..., X_m)$ such that the first k vectors $(X_1, ..., X_k)$ are tangent to S. Denote the dual coframe by $(\theta^\alpha)_{1\leq\alpha\leq m}$. Note that

$$\theta^\alpha\mid_S = 0 \quad \text{for } \alpha > k.$$

Denote by (μ^α_β), $(1 \leq \alpha, \beta \leq m)$ the connection 1-forms associated to ∇^M by this frame, and let σ^α_β, $(1 \leq \alpha, \beta \leq k)$ be the connection 1-forms of ∇^S corresponding to the frame $(X_1, \dots, X_k)$. We will analyze the structural equations of M restricted to $S \hookrightarrow M$.

$$d\theta^\alpha = \theta^\beta \wedge \mu^\alpha_\beta \quad 1 \leq \alpha, \beta \leq m. \tag{4.2.7}$$

We distinguish two situations.

A. $1 \leq \alpha \leq k$. Since $\theta^\beta \mid_S = 0$ for $\beta > k$ the equality (4.2.7) yields

$$d\theta^\alpha = \sum_{\beta=1}^{k} \theta^\beta \wedge \mu^\alpha_\beta, \quad \mu^\alpha_\beta = -\mu^\beta_\alpha \quad 1 \leq \alpha, \beta \leq k.$$

The uniqueness part of Proposition 4.2.15 implies that along S

$$\sigma^\alpha_\beta = \mu^\alpha_\beta \quad 1 \leq \alpha, \beta \leq k.$$

This can be equivalently rephrased as

$$\nabla^S_X Y = (\nabla^M_X Y)^\tau \quad \forall X, Y \in \mathrm{Vect}\,(S). \tag{4.2.8}$$

B. $k < \alpha \leq m$. We deduce

$$0 = \sum_{\beta=1}^{k} \theta^\beta \wedge \mu^\alpha_\beta.$$

At this point we want to use the following elementary result.

Exercise 4.2.18 (Cartan Lemma). *Let V be a d-dimensional real vector space and consider p linearly independent elements $\omega_1, \dots, \omega_p \in \Lambda^1 V$, $p \leq d$. If $\theta_1, \dots, \theta_p \in \Lambda^1 V$ are such that*

$$\sum_{i=1}^{p} \theta_i \wedge \omega_i = 0,$$

then there exist scalars A_{ij}, $1 \leq i, j \leq p$ such that $A_{ij} = A_{ji}$ and

$$\theta_i = \sum_{j=1}^{p} A_{ij} \omega_j. \qquad \square$$

Using Cartan lemma we can find smooth functions $f^\lambda_{\beta\gamma}$, $\lambda > k$, $1 \leq \beta, \gamma \leq k$ satisfying

$$f^\lambda_{\beta\gamma} = f^\lambda_{\gamma\beta}, \ \text{ and } \ \mu^\lambda_\beta = f^\lambda_{\beta\gamma} \theta^\gamma.$$

Now form

$$\mathcal{N} = f^\lambda_{\beta\gamma} \theta^\beta \otimes \theta^\gamma \otimes X_\lambda.$$

We can view $\mathcal{N}$ as a symmetric bilinear map

$$\operatorname{Vect}(S) \times \operatorname{Vect}(S) \to C^\infty(N_S).$$

If $U, V \in \operatorname{Vect}(S)$

$$U = U^\beta X_\beta = \theta^\beta(U) X_\beta \quad 1 \leq \beta \leq k,$$

and

$$V = V^\gamma X_\gamma = \theta^\gamma(V) X_\gamma \quad 1 \leq \gamma \leq k,$$

then

$$\mathcal{N}(U, V) = \sum_{\lambda > k} \left\{ \sum_\beta \left(\sum_\gamma f^\lambda_{\beta\gamma} \theta^\gamma(V) \right) \theta^\beta(U) \right\} X^\lambda$$

$$= \sum_{\lambda > k} \left(\sum_\beta \mu^\lambda_\beta(V) U^\beta \right) X_\lambda.$$

The last term is precisely the normal component of $\nabla^M_V U$. We have thus proved the following equality, so that we have established

$$\left(\nabla^M_V U\right)^\nu = \mathcal{N}(U, V) = \mathcal{N}(V, U) = \left(\nabla^M_U V\right)^\nu. \tag{4.2.9}$$

The map $\mathcal{N}$ is called the *2nd fundamental form*[2] of $S \hookrightarrow M$.

There is an alternative way of looking at $\mathcal{N}$. Choose

$$U, V \in \operatorname{Vect}(S), \quad N \in C^\infty(N_S).$$

[2]The *first* fundamental form is the induced metric.

If we write $g(\bullet,\bullet)=\langle \bullet,\bullet\rangle$, then

$$\langle \mathcal{N}(U,V),N\rangle=\langle \big(\nabla^M_U V\big)^\nu ,N\rangle=\langle \nabla^M_U V,N\rangle$$

$$=\nabla^M_U\big(V,N\rangle-\langle V,\nabla^M_U N\rangle=-\langle V,\big(\nabla^M_U N\big)^\tau\rangle.$$

We have thus established

$$-\langle V,\big(\nabla^M_U N\big)^\tau\rangle=\langle \mathcal{N}(U,V),N\rangle=\langle \mathcal{N}(V,U),N\rangle=-\langle U,\big(\nabla^M_V N\big)^\tau\rangle. \quad (4.2.10)$$

The 2nd fundamental form can be used to determine a relationship between the curvature of M and that of S. More precisely we have the following celebrated result.

***Theorema Egregium* (Gauss).** Let R^M (resp. R^S) denote the Riemann curvature of (M,g) (resp. $(S,g|_S)$. Then for any $X,Y,Z,T\in \operatorname{Vect}(S)$ we have

$$\begin{aligned}\langle R^M(X,Y)Z,T\rangle &=\langle R^S(X,Y)Z,T\rangle\\ &\quad+\langle \mathcal{N}(X,Z),\mathcal{N}(Y,T)\rangle-\langle \mathcal{N}(X,T),\mathcal{N}(Y,Z)\rangle.\end{aligned} \quad (4.2.11)$$

Proof. Note that

$$\nabla^M_X Y=\nabla^S_X Y+\mathcal{N}(X,Y).$$

We have

$$R^M(X,Y)X=[\nabla^M_X,\nabla^M_Y]Z-\nabla^M_{[X,Y]}Z$$

$$=\nabla^M_X\big(\nabla^S_Y Z+\mathcal{N}(Y,Z)\big)-\nabla^M_Y\big(\nabla^S_X Z+\mathcal{N}(X,Z)\big)-\nabla^S_{[X,Y]}Z-\mathcal{N}([X,Y],Z).$$

Take the inner product with T of both sides above. Since $\mathcal{N}(\bullet,\bullet)$ is N_S-valued, we deduce using (4.2.8)-(4.2.10)

$$\langle R^M(X,Y)Z,T\rangle=\langle \nabla^M_X\nabla^S_Y Z,T\rangle+\langle \nabla^M_X\mathcal{N}(Y,Z),T\rangle$$

$$-\langle \nabla^M_Y\nabla^S_X Z,T\rangle-\langle \nabla^M_Y\mathcal{N}(X,Z),T\rangle-\langle \nabla^S_{[X,Y]}Z,T\rangle$$

$$=\langle [\nabla^S_X,\nabla^S_Y]Z,T\rangle-\langle \mathcal{N}(Y,Z),\mathcal{N}(X,T)\rangle+\langle \mathcal{N}(X,Z),\mathcal{N}(Y,T)\rangle-\langle \nabla^S_{[X,Y]}Z,T\rangle.$$

This is precisely the equality (4.2.11). □

The above result is especially interesting when S is a transversally oriented hypersurface, i.e., S is a codimension 1 submanifold such that the normal bundle N_S is trivial[3]. Pick an orthonormal frame $\boldsymbol{n}$ of N_S, i.e., a length 1 section of N_S, and choose an orthonormal moving frame $(X_1,...,X_{m-1})$ of TS.

Then $(X_1,...,X_{m-1},\boldsymbol{n})$ is an orthonormal moving frame of $(TM)|_S$, and the second fundamental form is completely described by

$$\mathcal{N}_{\boldsymbol{n}}(X,Y):=\langle \mathcal{N}(X,Y),\boldsymbol{n}\rangle.$$

[3]Locally, all hypersurfaces are transversally oriented since N_S is locally trivial by definition.

$\mathcal{N}_{\boldsymbol{n}}$ is a bona-fide symmetric bilinear form, and moreover, according to (4.2.10) we have

$$\mathcal{N}_{\boldsymbol{n}}(X,Y) = -\langle \nabla^M_X \boldsymbol{n}, Y \rangle = -\langle \nabla^M_Y \boldsymbol{n}, X \rangle.$$

In this case, Gauss formula becomes

$$\langle R^S(X,Y)Z,T \rangle = \langle R^M(X,Y)Z,T \rangle - \begin{vmatrix} \mathcal{N}_{\boldsymbol{n}}(X,Z) & \mathcal{N}_{\boldsymbol{n}}(X,T) \\ \mathcal{N}_{\boldsymbol{n}}(Y,Z) & \mathcal{N}_{\boldsymbol{n}}(Y,T) \end{vmatrix}.$$

Let us further specialize, and assume $M = \mathbb{R}^m$. Then

$$\langle R^S(X,Y)Z,T \rangle = \begin{vmatrix} \mathcal{N}_{\boldsymbol{n}}(X,T) & \mathcal{N}_{\boldsymbol{n}}(X,Z) \\ \mathcal{N}_{\boldsymbol{n}}(Y,T) & \mathcal{N}_{\boldsymbol{n}}(Y,Z) \end{vmatrix}. \tag{4.2.12}$$

In particular, the sectional curvature along the plane spanned by X, Y is

$$\langle R^S(X,Y)Y,X \rangle = \mathcal{N}_{\boldsymbol{n}}(X,X)\cdot\mathcal{N}_{\boldsymbol{n}}(Y,Y) - |\mathcal{N}_{\boldsymbol{n}}(X,Y)|^2.$$

This is a truly remarkable result. On the right-hand side we have an extrinsic term (it depends on the "space surrounding S"), while in the left-hand side we have a purely intrinsic term (which is defined entirely in terms of the internal geometry of S). Historically, the extrinsic term was discovered first (by Gauss), and very much to Gauss surprise (?!?) one does not need to look outside S to compute it. This marked the beginning of a new era in geometry. It changed dramatically the way people looked at manifolds and thus it fully deserves the name of The Golden (egregium) Theorem of Geometry.

We can now explain rigorously why we cannot wrap a plane canvas around the sphere. Notice that, when we deform a plane canvas, the only thing that changes is the *extrinsic geometry*, while the *intrinsic geometry* is not changed since the lengths of the "fibers" stays the same. Thus, any intrinsic quantity is invariant under "bending". In particular, no matter how we deform the plane canvas we will always get a surface with Gauss curvature 0 which cannot be wrapped on a surface of constant *positive* curvature! Gauss himself called the total curvature a"bending invariant".

Example 4.2.19. (Quadrics in $\mathbb{R}^3$). Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a selfadjoint, invertible linear operator with at least one positive eigenvalue. This implies the quadric

$$Q_A = \{u \in \mathbb{R}^3 \,;\, \langle Au, u \rangle = 1\},$$

is nonempty and smooth (use implicit function theorem to check this). Let $u_0 \in Q_A$. Then

$$T_{u_0}Q_A = \{x \in \mathbb{R}^3 \,;\, \langle Au_0, x \rangle = 0\} = (Au_0)^{\perp}.$$

Q_A is a transversally oriented hypersurface in $\mathbb{R}^3$ since the map $Q_A \ni u \mapsto Au$ defines a nowhere vanishing section of the normal bundle. Set $\boldsymbol{n} = \frac{1}{|Au|}Au$.

Consider an orthonormal frame (e_0, e_1, e_2) of $\mathbb{R}^3$ such that $e_0 = \boldsymbol{n}(u_0)$. Denote the Cartesian coordinates in $\mathbb{R}^3$ with respect to this frame by (x^0, x^1, x^2), and set $\partial_i := \frac{\partial}{\partial x_i}$. Extend (e_1, e_2) to a local moving frame of TQ_A near u_0.

The second fundamental form of Q_A at u_0 is

$$\mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_j) = \langle \partial_i \boldsymbol{n}, \partial_j \rangle|_{u_0}.$$

We compute

$$\partial_i \boldsymbol{n} = \partial_i \left(\frac{Au}{|Au|} \right) = \partial_i(\langle Au, Au \rangle^{-1/2}) Au + \frac{1}{|Au|} A\partial_i u$$

$$= -\frac{\langle \partial_i Au, Au \rangle}{|Au|^{3/2}} Au + \frac{1}{|Au|} \partial_i Au.$$

Hence

$$\mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_j)|_{u_0} = \frac{1}{|Au_0|} \langle A\partial_i u, e_j \rangle|_{u_0}$$

$$= \frac{1}{|Au_0|} \langle \partial_i u, Ae_j \rangle|_{u_0} = \frac{1}{|Au_0|} \langle e_i, Ae_j \rangle. \tag{4.2.13}$$

We can now compute the Gaussian curvature at u_0.

$$K_{u_0} = \frac{1}{|Au_0|^2} \begin{vmatrix} \langle Ae_1, e_1 \rangle & \langle Ae_1, e_2 \rangle \\ \langle Ae_2, e_1 \rangle & \langle Ae_2, e_2 \rangle \end{vmatrix}.$$

In particular, when $A = r^{-2}I$ so that Q_A is the round sphere of radius r we deduce

$$K_u = \frac{1}{r^2} \quad \forall |u| = r.$$

Thus, the round sphere has constant positive curvature. □

Example 4.2.20. (Gauss). Let Σ be a transversally oriented, compact surface in $\mathbb{R}^3$, e.g., a connected sum of a finite number of tori. Note that the Whitney sum $N_\Sigma \oplus T\Sigma$ is the trivial bundle $\underline{\mathbb{R}}^3_\Sigma$. We orient N_Σ such that

$$\text{orientation}\, N_\Sigma \wedge \text{orientation}\;\; T\Sigma = \text{orientation}\, \mathbb{R}^3.$$

Let $\boldsymbol{n}$ be the unit section of N_Σ defining the above orientation. We obtain in this way a map

$$\mathcal{G} : \Sigma \to S^2 = \{u \in \mathbb{R}^3 \, ; \, |u| = 1\}, \quad \Sigma \ni x \mapsto \boldsymbol{n}(x) \in S^2.$$

The map $\mathcal{G}$ is called the *Gauss map* of $\Sigma \hookrightarrow S^2$. It really depends on how Σ is embedded in $\mathbb{R}^3$ so it is an *extrinsic object*. Denote by $\mathcal{N}_{\boldsymbol{n}}$ the second fundamental form of $\Sigma \hookrightarrow \mathbb{R}^3$, and let (x^1, x^2) be normal coordinates at $q \in \Sigma$ such that

$$\text{orientation}\, T_q\Sigma = \partial_1 \wedge \partial_2.$$

Consider the Euler form ε_Σ on Σ with the metric induced by the Euclidean metric in $\mathbb{R}^3$. Then, taking into account our orientation conventions, we have

$$2\pi \varepsilon_\Sigma(\partial_1, \partial_2) = R_{1212} = \begin{vmatrix} \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_2) \end{vmatrix}. \tag{4.2.14}$$

Now notice that

$$\partial_i \boldsymbol{n} = -\mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_1)\partial_1 - \mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_2)\partial_2.$$

We can think of $\boldsymbol{n}, \partial_1 |_q$ and $\partial_2 |_q$ as defining a (positively oriented) frame of $\mathbb{R}^3$. The last equality can be rephrased by saying that the derivative of the Gauss map

$$\mathcal{G}_* : T_q\Sigma \to T_{\boldsymbol{n}(q)} S^2$$

acts according to

$$\partial_i \mapsto -\mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_1)\partial_1 - \mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_2)\partial_2.$$

In particular, we deduce

$$\mathcal{G}_* \text{ preserves (reverses) orientations} \Longleftrightarrow R_{1212} > 0 \ (< 0), \tag{4.2.15}$$

because the orientability issue is decided by the sign of the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_1) & -\mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) \\ 0 & -\mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_1) & -\mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_2) \end{vmatrix}.$$

At q, $\partial_1 \perp \partial_2$ so that,

$$\left\langle \partial_i \boldsymbol{n}, \partial_j \boldsymbol{n} \right\rangle = \mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_1)\mathcal{N}_{\boldsymbol{n}}(\partial_j, \partial_1) + \mathcal{N}_{\boldsymbol{n}}(\partial_i, \partial_2)\mathcal{N}_{\boldsymbol{n}}(\partial_j, \partial_2).$$

We can rephrase this coherently as an equality of matrices

$$\begin{bmatrix} \langle \partial_1 \boldsymbol{n}, \partial_1 \boldsymbol{n} \rangle & \langle \partial_1 \boldsymbol{n}, \partial_2 \boldsymbol{n} \rangle \\ \langle \partial_2 \boldsymbol{n}, \partial_1 \boldsymbol{n} \rangle & \langle \partial_2 \boldsymbol{n}, \partial_2 \boldsymbol{n} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_2) \end{bmatrix} \times \begin{bmatrix} \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_2) \end{bmatrix}^t.$$

Hence

$$\begin{vmatrix} \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_1) & \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) \\ \mathcal{N}_{\boldsymbol{n}}(\partial_1, \partial_2) & \mathcal{N}_{\boldsymbol{n}}(\partial_2, \partial_2) \end{vmatrix}^2 = \begin{vmatrix} \langle \partial_1 \boldsymbol{n}, \partial_1 \boldsymbol{n} \rangle & \langle \partial_1 \boldsymbol{n}, \partial_2 \boldsymbol{n} \rangle \\ \langle \partial_1 \boldsymbol{n}, \partial_2 \boldsymbol{n} \rangle & \langle \partial_2 \boldsymbol{n}, \partial_2 \boldsymbol{n} \rangle \end{vmatrix}. \tag{4.2.16}$$

If we denote by dv_0 the metric volume form on S^2 induced by the restriction of the Euclidean metric on $\mathbb{R}^3$, we see that (4.2.14) and (4.2.16) put together yield

$$2\pi |\varepsilon_\Sigma(\partial_1, \partial_2)| = |dv_0(\partial_1 \boldsymbol{n}, \partial_2 \boldsymbol{n})| = |dv_0(\mathcal{G}_*(\partial_1), \mathcal{G}_*(\partial_2))|.$$

Using (4.2.15) we get

$$\varepsilon_\Sigma = \frac{1}{2\pi} \mathcal{G}_\Sigma^* dv_0 = \frac{1}{2\pi} \mathcal{G}_\Sigma^* \varepsilon_{S^2}. \tag{4.2.17}$$

This is one form of the celebrated Gauss-Bonnet theorem. We will have more to say about it in the next subsection.

Note that the last equality offers yet another interpretation of the Gauss curvature. From this point of view the curvature is a *"distortion factor"*. The Gauss map "stretches" an infinitesimal parallelogram to some infinitesimal region on the unit sphere. The Gauss curvature describes by what factor the area of this parallelogram was changed. In Chapter 9 we will investigate in greater detail the Gauss map of *arbitrary* submanifolds of a Euclidean space. □

4.2.5 *The Gauss-Bonnet theorem for oriented surfaces*

We conclude this chapter with one of the most beautiful results in geometry. Its meaning reaches deep inside the structure of a manifold and can be viewed as the origin of many fertile ideas.

Recall one of the questions we formulated at the beginning of our study: explain unambiguously why a sphere is “different” from a torus. This may sound like forcing our way in through an open door since everybody can “see” they are different. Unfortunately this is not a conclusive explanation since we can see only 3-dimensional things and possibly there are many ways to deform a surface outside our tight 3D Universe.

The elements of Riemann geometry we discussed so far will allow us to produce an invariant powerful enough to distinguish a sphere from a torus. But it will do more than that.

Theorem 4.2.21. (Gauss-Bonnet Theorem. Preliminary version.) *Let S be a compact oriented surface without boundary. If g_0 and g_1 are two Riemann metrics on S and $\varepsilon_{g_i}(S)$ $(i = 0, 1)$ are the corresponding Euler forms then*

$$\int_S \varepsilon_{g_0}(S) = \int_S \varepsilon_{g_1}(S).$$

Hence the quantity $\int_S \varepsilon_g(S)$ is independent of the Riemann metric g so that it really depends only on the topology of S!!!

The idea behind the proof is very natural. Denote by g_t the metric $g_t = g_0 + t(g_1 - g_0)$. We will show

$$\frac{d}{dt}\int_S \varepsilon_{g_t} = 0 \quad \forall t \in [0, 1].$$

It is convenient to consider a more general problem.

Definition 4.2.22. Let M be a compact oriented manifold. For any Riemann metric g on E define

$$\mathcal{E}_M(M, g) = \int_M s(g)dV_g,$$

where $s(g)$ denotes the scalar curvature of (M, g). The functional $g \mapsto \mathcal{E}(g)$ is called the *Hilbert-Einstein functional.* □

We have the following remarkable result.

Lemma 4.2.23. *Let M be a compact oriented manifold without boundary and $g^t = (g^t_{ij})$ be a 1-parameter family of Riemann metrics on M depending smoothly upon $t \in \mathbb{R}$. Then*

$$\frac{d}{dt}\mathcal{E}(g^t) = -\int_M \Big\langle \operatorname{Ric}_{g^t} - \frac{1}{2}s(g^t)g^t \, , \, \dot{g}^t \Big\rangle_t dV_{g^t}, \quad \forall t.$$

In the above formula $\langle \cdot, \cdot \rangle_t$ denotes the inner product induced by g^t on the space of symmetric (0,2)-tensors while the dot denotes the t-derivative.

Definition 4.2.24. A Riemann manifold (M, g) of dimension n is said to be *Einstein* if the metric g satisfies *Einstein's equation*

$$\mathrm{Ric}_g = \frac{s(x)}{n} g,$$

where $s(x)$ denotes the scalar curvature.

Example 4.2.25. Observe that if the Riemann metric g satisfies the condition

$$\mathrm{Ric}_g(x) = \lambda(x) g(x) \tag{4.2.18}$$

for some smooth function $\lambda \in C^\infty(M)$, then by taking the traces on both sides of the above equality we deduce

$$\lambda(x) = \frac{s(g)(x)}{n}, \quad n := \dim M.$$

Thus, the Riemann manifold is Einstein if and only if it satisfies (4.2.18).

Using the computations in Example 4.2.11 we deduce that a certain constant multiple of the Killing metric on a compact semisimple Lie group is an Einstein metric.

We refer to [13] for an in-depth study of the Einstein manifolds. □

The $(0, 2)$-tensor

$$\mathcal{E}_{ij} := R_{ij}(x) - \frac{1}{2} s(x) g_{ij}(x)$$

is called the *Einstein tensor* of (M, g).

Exercise 4.2.26. Consider a 3-dimensional Riemann manifold (M, g). Show that

$$R_{ijk\ell} = \mathcal{E}_{ik} g_{j\ell} - \mathcal{E}_{i\ell} g_{jk} + \mathcal{E}_{j\ell} g_{ik} - \mathcal{E}_{jk} g_{i\ell} + \frac{s}{2}(g_{i\ell} g_{jk} - g_{ik} g_{j\ell}).$$

In particular, this shows that on a Riemann 3-manifold the full Riemann tensor is completely determined by the Einstein tensor. □

Exercise 4.2.27. (Schouten-Struik, [87]**).** Prove that the scalar curvature of an Einstein manifold of dimension ≥ 3 is constant.
Hint: Use the 2nd Bianchi identity. □

Notice that when (S, g) is a compact oriented Riemann surface two very nice things happen.
(i) (S, g) is Einstein (recall that only R_{1212} is nontrivial).
(ii) $\mathcal{E}(g) = 2 \int_S \varepsilon_g$.

Theorem 4.2.21 is thus an immediate consequence of Lemma 4.2.23.

Proof of the lemma We will produce a very explicit description of the integrand

$$\frac{d}{dt}\Big(s(g^t) dV_{g^t} \Big) = \Big(\frac{d}{dt} s(g^t) \Big) dV_{g^t} + s(g^t) \Big(\frac{d}{dt} dV_{g^t} \Big) \tag{4.2.19}$$

of $\frac{d}{dt}\mathcal{E}(g^t)$. We will adopt a "roll up your sleeves, and just do it" strategy reminiscent to the good old days of the tensor calculus frenzy. By this we mean that we will work in a nicely chosen collection of local coordinates, and we will keep track of the zillion indices we will encounter. As we will see, the computations are not as hopeless as they may seem to be.

We will study the integrand (4.2.19) at $t = 0$. The general case is entirely analogous. For typographical reasons we will be forced to introduce new notations. Thus, $\hat{g}$ will denote (g^t) for $t = 0$, while g^t will be denoted simply by g. A hat over a quantity means we think of that quantity at $t = 0$, while a dot means differentiation with respect to t at $t = 0$.

Let q be an arbitrary point on S, and denote by $(x^1, \dots, x^n)$ a collection of $\hat{g}$-normal coordinates at q. Denote by ∇ the Levi-Civita connection of g and let Γ^i_{jk} denote its Christoffel symbols in the coordinates (x^i).

Many nice things happen *at* q, and we list a few of them which will be used later.

$$\hat{g}_{ij} = \hat{g}^{ij} = \delta_{ij}, \;\; \partial_k \hat{g}_{ij} = 0. \tag{4.2.20}$$

$$\hat{\nabla}_i \partial_j = 0, \;\; \hat{\Gamma}^i_{jk} = 0. \tag{4.2.21}$$

If $\alpha = \alpha_i dx^i$ is a 1-form then, *at* q,

$$\delta_{\hat{g}} \alpha = \sum_i \partial_i \alpha_i. \;\text{ where }\; \delta = *d*\,. \tag{4.2.22}$$

In particular, for any smooth function u we have

$$(\Delta_{M,\hat{g}} u)(q) = -\sum_i {\partial_i}^2 u. \tag{4.2.23}$$

Set

$$h = (h_{ij}) := (\dot{g}) = (\dot{g}_{ij}).$$

The tensor h is a symmetric $(0,2)$-tensor. Its $\hat{g}$-trace is the scalar

$$\operatorname{tr}_{\hat{g}} h = \hat{g}^{ij} h_{ij} = \operatorname{tr} \mathcal{L}^{-1}(h),$$

where $\mathcal{L}$ is the lowering the indices operator defined by $\hat{g}$. In particular, *at* q

$$\operatorname{tr}_{\hat{g}} h = \sum_i h_{ii}. \tag{4.2.24}$$

The curvature of g is given by

$$R^\ell_{ikj} = -R^\ell_{ijk} = \partial_k \Gamma^\ell_{ij} - \partial_j \Gamma^\ell_{ik} + \Gamma^\ell_{mk}\Gamma^m_{ij} - \Gamma^\ell_{mj}\Gamma^m_{ik}.$$

The Ricci tensor is

$$R_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik}.$$

Finally, the scalar curvature is

$$s = \operatorname{tr}_g R_{ij} = g^{ij} R_{ij} = g^{ij}\left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{mk}\Gamma^m_{ij} - \Gamma^k_{mj}\Gamma^m_{ik}\right).$$

Differentiating s at $t=0$, and then evaluating *at* q we obtain

$$\begin{aligned}\dot{s} &= \dot{g}^{ij}\left(\partial_k\hat{\Gamma}^k_{ij}-\partial_j\hat{\Gamma}^k_{ik}\right)+\delta^{ij}\left(\partial_k\dot{\Gamma}^k_{ij}-\partial_j\dot{\Gamma}^k_{ik}\right)\\ &= \dot{g}^{ij}\hat{R}_{ij}+\sum_i\left(\partial_k\dot{\Gamma}^k_{ii}-\partial_i\dot{\Gamma}^k_{ik}\right).\end{aligned}\tag{4.2.25}$$

The term $\dot{g}^{ij}$ can be computed by derivating the equality $g^{ik}g_{jk}=\delta^i_k$ at $t=0$. We get

$$\dot{g}^{ik}\hat{g}_{jk}+\hat{g}^{ik}h_{jk}=0,$$

so that

$$\dot{g}^{ij}=-h_{ij}.\tag{4.2.26}$$

To evaluate the derivatives $\dot{\Gamma}$'s we use the known formulæ

$$\Gamma^m_{ij}=\frac{1}{2}g^{km}\left(\partial_i g_{jk}-\partial_k g_{ij}+\partial_j g_{ik}\right),$$

which, upon differentiation at $t=0$, yield

$$\begin{aligned}\dot{\Gamma}^m_{ij} &= \frac{1}{2}\left(\partial_i\hat{g}_{jk}-\partial_k\hat{g}_{ij}+\partial_j\hat{g}_{ik}\right)+\frac{1}{2}\hat{g}^{km}\left(\partial_i h_{jk}-\partial_k h_{ij}+\partial_j h_{ik}\right)\\ &= \frac{1}{2}\left(\partial_i h_{jm}-\partial_m h_{ij}+\partial_j h_{im}\right).\end{aligned}\tag{4.2.27}$$

We substitute (4.2.26) -(4.2.27) in (4.2.25), and we get, *at* q

$$\begin{aligned}\dot{s} &= -\sum_{i,j}h_{ij}\hat{R}_{ij}+\frac{1}{2}\sum_{i,k}(\partial_k\partial_i h_{ik}-\partial_k{}^2h_{ii}+\partial_k\partial_i h_{ik})-\frac{1}{2}\sum_{i,k}(\partial_i{}^2h_{kk}-\partial_i\partial_k h_{ik})\\ &= -\sum_{i,j}h_{ij}\hat{R}_{ij}-\sum_{i,k}\partial_i{}^2h_{kk}+\sum_{i,k}\partial_i\partial_k h_{ik}\\ &= -\langle\widehat{\mathrm{Ric}},\dot{g}\rangle_{\hat{g}}+\Delta_{M,\hat{g}}\mathrm{tr}_{\hat{g}}\,\dot{g}+\sum_{i,k}\partial_i\partial_k h_{ik}.\end{aligned}\tag{4.2.28}$$

To get a coordinate free description of the last term note that, *at* q,

$$(\hat{\nabla}_k h)(\partial_i,\partial_m)=\partial_k h_{im}.$$

The total covariant derivative $\hat{\nabla}h$ is a $(0,3)$-tensor. Using the $\hat{g}$-trace we can construct a $(0,1)$-tensor, i.e., a 1-form

$$\mathrm{tr}_{\hat{g}}(\hat{\nabla}h)=\mathrm{tr}(\mathcal{L}_{\hat{g}}^{-1}\hat{\nabla}h),$$

where $\mathcal{L}_{\hat{g}}^{-1}$ is the raising the indices operator defined by $\hat{g}$. In the local coordinates (x^i) we have

$$\mathrm{tr}_{\hat{g}}(\hat{\nabla}h)=\hat{g}^{ij}(\hat{\nabla}_i h)_{jk}dx^k.$$

Using (4.2.20), and (4.2.22) we deduce that the last term in (4.2.28) can be rewritten (at q) as

$$\delta\mathrm{tr}_{\hat{g}}\,(\hat{\nabla}h)=\delta\mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g}).$$

We have thus established that

$$\dot{s} = -\langle \widehat{\mathrm{Ric}}, \dot{g}\rangle_{\hat{g}} + \Delta_{M,\hat{g}} \mathrm{tr}_{\hat{g}}\, \dot{g} + \delta \mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g}). \tag{4.2.29}$$

The second term of the integrand (4.2.19) is a lot easier to compute.

$$dV_g = \pm\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n,$$

so that

$$d\dot{V}_g = \pm\frac{1}{2}|\hat{g}|^{-1/2}\frac{d}{dt}|g|dx^1 \wedge \cdots \wedge dx^n.$$

At q the metric is Euclidean, $\hat{g}_{ij} = \delta_{ij}$, and

$$\frac{d}{dt}|g| = \sum_i \dot{g}_{ii} = |\hat{g}| \cdot \mathrm{tr}_{\hat{g}}(\dot{g}) = \langle \hat{g}, \dot{g}\rangle_{\hat{g}}|\hat{g}|.$$

Hence

$$\dot{\mathcal{E}}(g) = \int_M \Big\langle \Big(\frac{1}{2}s(\hat{g})\hat{g} - \mathrm{Ric}_{\hat{g}} \Big) , \dot{g} \Big\rangle_{\hat{g}} dV_{\hat{g}} + \int_M \Big(\Delta_{M,\hat{g}}\, \mathrm{tr}_{\hat{g}}\, \dot{g} + \delta \mathrm{tr}_{\hat{g}}(\hat{\nabla}\dot{g}) \Big) dV_{\hat{g}}.$$

Green's formula shows the last two terms vanish and the proof of the Lemma is concluded. □

Definition 4.2.28. Let S be a compact, oriented surface without boundary. We define its *Euler characteristic* as the number

$$\chi(S) = \frac{1}{2\pi}\int_S \varepsilon(g),$$

where g is an arbitrary Riemann metric on S. The number

$$g(S) = \frac{1}{2}(2 - \chi(S))$$

is called the *genus* of the surface. □

Remark 4.2.29. According to the theorem we have just proved, the Euler characteristic is independent of the metric used to define it. Hence, the Euler characteristic is a *topological invariant* of the surface. The reason for this terminology will become apparent when we discuss DeRham cohomology, a $\mathbb{Z}$-graded vector space naturally associated to a surface whose Euler characteristic coincides with the number defined above. So far we have no idea whether $\chi(S)$ is even an integer. □

Proposition 4.2.30.

$$\chi(S^2) = 2 \;\text{ and }\; \chi(T^2) = 0.$$

Proof. To compute $\chi(S^2)$ we use the round metric g_0 for which $K = 1$ so that

$$\chi(S^2) = \frac{1}{2\pi}\int_{S^2} dv_{g_0} = \frac{1}{2\pi}\mathrm{area}_{g_0}(S^2) = 2.$$

To compute the Euler characteristic of the torus we think of it as an Abelian Lie group with a bi-invariant metric. Since the Lie bracket is trivial we deduce from the computations in Subsection 4.2.2 that its curvature is zero. This concludes the proof of the proposition. □

Proposition 4.2.31. *If S_i (i=1,2) are two compact oriented surfaces without boundary then*

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Thus upon iteration we get

$$\chi(S_1\#\cdots\#S_k) = \sum_{i=1}^{k}\chi(S_i) - 2(k-1),$$

for any compact oriented surfaces $S_1, \dots, S_k$. In terms of genera, the last equality can be rephrased as

$$g(S_1\#\cdots\#S_k) = \sum_{i=1}^{k} g(S_i).$$

In the proof of this proposition we will use special metrics on connected sums of surfaces which require a preliminary analytical discussion.

Consider $f : (-4, 4) \to (0, \infty)$ a smooth, even function such that
(i) $f(x) = 1$ for $|x| \leq 2$.
(ii) $f(x) = \sqrt{1-(x+3)^2}$ for $x \in [-4, -3.5]$.
(iii) $f(x) = \sqrt{1-(x-3)^2}$ for $x \in [3.5, 4]$.
(iv) f is non-decreasing on $[-4, 0]$.
One such function is shown in Figure 4.3.

Denote by S_f the surface inside $\mathbb{R}^3$ obtained by rotating the graph of f about the x-axis. Because of properties (i)-(iv), S_f is a smooth surface diffeomorphic[4] to S^2. We denote by g the metric on S_f induced by the Euclidean metric in $\mathbb{R}^3$. Since S_f is diffeomorphic to a sphere

$$\int_{S_f} K_g dV_g = 2\pi\chi(S^2) = 4\pi.$$

Set

$$S_f^{\pm} := S_f \cap \{\pm x > 0\}, \;\; S_f^{\pm 1} := S_f \cap \{\pm x > 1\}$$

[4]One such diffeomorphism can be explicitly constructed projecting along radii starting at the origin.

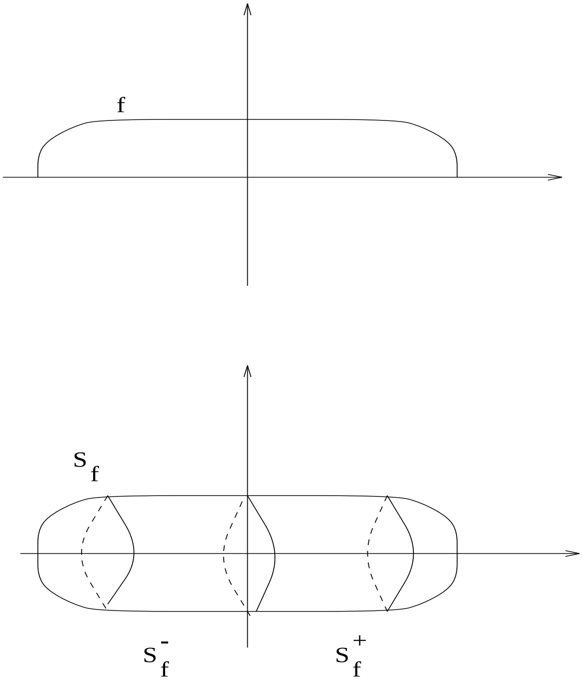

Fig. 4.3 *Generating a hot-dog-shaped surface*

Since f is even we deduce

$$\int_{S_f^{\pm}} K_g dv_g = \frac{1}{2}\int_{S_f} K_g dv_g = 2\pi. \tag{4.2.30}$$

On the other hand, on the neck $C = \{|x| \leq 2\}$ the metric g is locally Euclidean $g = dx^2 + d\theta^2$, so that over this region $K_g = 0$. Hence

$$\int_C K_g dv_g = 0. \tag{4.2.31}$$

Proof of Proposition 4.2.31 Let $D_i \subset S_i$ $(i = 1, 2)$ be a local coordinate neighborhood diffeomorphic with a disk in the plane. Pick a metric g_i on S_i such that (D_1, g_1) is *isometric* with S_f^+ and (D_2, g_2) is isometric to S_f^-. The connected sum $S_1 \# S_2$ is obtained by chopping off the regions S_f^1 from D_1 and S_f^{-1} from D_2 and (isometrically) identifying the remaining cylinders $S_f^{\pm} \cap \{|x| \leq 1\} = C$ and call O the overlap created by gluing (see Figure 4.4). Denote the metric thus obtained

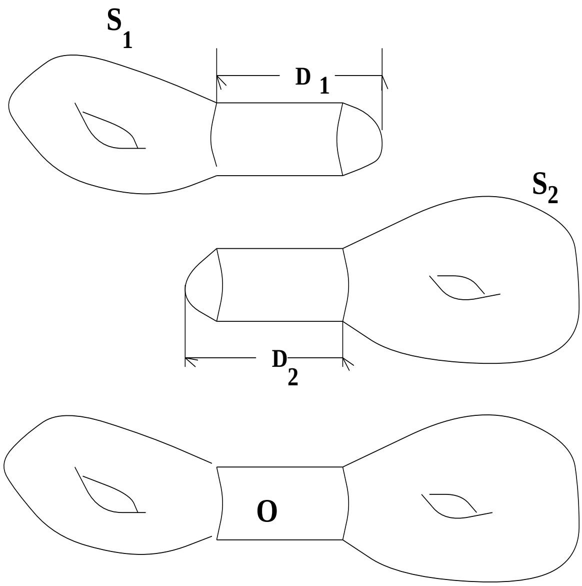

Fig. 4.4 *Special metric on a connected sum*

on $S_1 \# S_2$ by $\hat{g}$. We can now compute

$$\begin{aligned}
\chi(S_1 \# S_2) &= \frac{1}{2\pi}\int_{S_1\#S_2} K_{\hat{g}} dv_{\hat{g}} \\
&= \frac{1}{2\pi}\int_{S_1\setminus D_1} K_{g_1} dV_{g_1} + \frac{1}{2\pi}\int_{S_2\setminus D_2} K_{g_2} dV_{g_2} + \frac{1}{2\pi}\int_O K_g\, dV_g \\
&\overset{(4.2.31)}{=} \frac{1}{2\pi}\int_{S_1} K_{g_1} dV_{g_1} - \frac{1}{2\pi}\int_{D_1} K_g dV_g \\
&+ \frac{1}{2\pi}\int_{S_2} K_{g_2} dV_{g_2} - \frac{1}{2\pi}\int_{D_2} K_g dV_g \overset{(4.2.30)}{=} \chi(S_1) + \chi(S_2) - 2.
\end{aligned}$$

This completes the proof of the proposition. □

Corollary 4.2.32 (Gauss-Bonnet). *Let Σ_g denote the connected sum of g-tori. (By definition $\Sigma_0 = S^2$.) Then*

$$\chi(\Sigma_g) = 2 - 2g \ \text{ and } \ g(\Sigma_g) = g.$$

In particular, a sphere is not diffeomorphic to a torus.

Remark 4.2.33. It is a classical result that the only compact oriented surfaces are the connected sums of g-tori (see [68]), so that the genus of a compact oriented surface is a complete topological invariant. □

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