Toric Geometry:
Theorems and Definitions

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1 Dictionary

Toric geometry is concerned with the construction of varieties and schemes given by specifying semigroups and fans and other combinatorial objects. It is therefore useful to fix certain symbols.

- N: We define $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ and note that $N \cong \mathbb{Z}^n$.
- M : We define M to be the dual lattice of N, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$.
- $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$: We define $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}^n$.

2 What makes a toric variety?

2.1 Tori

2.2 Toric Varieties

2.3 Cones and Fans

Throughout this section, let $T \cong (\mathbb{C}^*)^n$ and $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$. Note that N is the collection of 1-parameter subgroups of T, or the set of cocharacters if you prefer that terminology. In addition, every variety is an integral separated scheme of finite type over $\operatorname{Spec}\mathbb{C}$ unless otherwise specified.

Definition 2.1. A *rational polyhedral cone* σ in N is a set $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ given by the positive span of some finite subset of $N_{\mathbb{R}}$, i.e. a set

$$\sigma = \mathrm{cone}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{k} c_i v_i \;\middle|\; c_i \in \mathbb{R}_{\geq 0} \right\}.$$

By rescaling the cone basis set, we may assume $v_i \in N$ for each $1 \leq i \leq k$, and from now on will do so.

Definition 2.2. Let $\sigma = \mathrm{cone}\{v_1, \ldots, v_k\}$ be a rational polyhedral cone. The *span* of σ is the smallest vector subspace V containing σ. We have that

$$V = \sigma + (-\sigma) = \{v_1, \ldots, v_k\} = \{\sigma\}.$$

The *dimension* of σ is the dimension of the span of σ. We say that σ is *full-dimensional* if $\dim\sigma = \dim N_{\mathbb{R}} = n$.

Definition 2.3. A rational polyhedral cone is said to be *strictly convex* if it doesn't contain a line, i.e. if it doesn't contain a one dimensional affine subspace of $N_{\mathbb{R}}$.

Unless otherwise specified, by "cone" we mean "strictly convex rational polyhedral cone".

Definition 2.4. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, the *dual cone* $\sigma\vee \subseteq M_{\mathbb{R}}$ is defined

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0,\ \forall v \in \sigma\}.$$

The pairing $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is simply the evaluation map $\langle m, u \rangle = m(u)$.

We further define the double dual $(\sigma^{\vee})^{\vee}$ by

$$(\sigma^{\vee})^{\vee} = \left\{ v \in N_{\mathbb{R}} \;\middle|\; \langle m, v \rangle \geq 0,\ \forall m \in \sigma^{\vee} \right\}$$

The following are fundamental facts regarding σ and $\sigma^{\vee}$.

Proposition 2.5. Let σ be a cone in N and $\sigma^{\vee}$ be its dual.

(a) $\sigma^{\vee}$ is a rational polyhedral cone in M (not necessarily strictly convex)

(b) $(\sigma^{\vee})^{\vee} = \sigma$

(c) σ is full-dimensional if and only if $\sigma^{\vee}$ is strictly convex

Definition 2.6. A *fan* Σ in N is a collection of cones in N such that

(i) if $\sigma \in \Sigma$ then every face of σ belongs to Σ

(ii) if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

We wish to construct varieties from cones and fans. Starting with a cone σ in N, we will associate to it an affine variety $X_\sigma = \operatorname{Spec} R_\sigma$. Given a fan Σ, we will construct a variety X_Σ by gluing together X_{σ_1} and X_{σ_2} along $X_{\sigma_1 \cap \sigma_2}$.

We focus first on building a variety X_σ from a cone σ in N. Here is our construction/definition.

Construction 2.7. Given a cone $\sigma \subseteq N_{\mathbb{R}}$ and its dual cone $\sigma^\vee \subseteq M_{\mathbb{R}}$, we define

$$S_\sigma := \sigma \cap M \tag{1}$$

to be the semigroup associated to σ. Note that some authors call this a monoid (we have inverses) and we think of it as an abelian group without inverses. We then consider the group algebra over $\mathbb{C}$ with basis S_σ:

$$\mathbb{C}[S_\sigma] = \left\{ \sum_{i=1}^{r} c_i \cdot z^{m_i} \;\middle|\; c_i \in \mathbb{C},\ m_i \in S_\sigma \subseteq M \right\}. \tag{2}$$

The addition on $\mathbb{C}[S_\sigma]$ is formal. The multiplication is defined $z^{m_i} \cdot z^{m_j} = z^{m_i + m_j}$ and is extended by distribution. E.g. we have that $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[t_1, \ldots, t_n]$ and $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$.

Finally, we define

$$X_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] \tag{3}$$

to be the affine toric variety associated to σ. Note that because we will eventually build toric varieties from fans whose affine pieces are given by pieces of the above form, we sometimes denote X_σ by U_σ instead.

It is still left to show that X_σ constructed in this way is in fact a toric variety.

Proposition 2.8. (Cox-Little-Scheck) If σ, S_σ, and X_σ are as in Construction (2.7) then X_σ is an affine toric variety.

Proof. See page 31 of Cox-Little Scheck. Fill it in later. □

One might ask, "Why do we define S_σ as a subset of the dual lattice M rather than the lattice N? Surely we could take $S_\sigma = \sigma \cap N$ and get an equally reasonable result."

COME BACK TO THE ABOVE QUESTION. CONSIDER REVERSE CONSTRUCTION – GIVEN AFFINE TORIC VARIETY $T \subseteq X$ CONSTRUCT A SEMIGROUP (HOWEVER ONE DOESN'T ALWAYS GET A CONE)

3 Smoothness of Affine Toric Varieties

The main goal of this section is a classification of smooth affine toric varieties associated to cones σ. This is Theorem (3.4). Before we proceed, however, we prove several useful lemmas. Throughout this section X_σ is an affine toric variety associated to a cone $\sigma \subseteq M_\mathbb{R}$.

Lemma 3.1. Let $\sigma = \text{cone}(v_1, \ldots, v_k) \subseteq N_\mathbb{R}$ be a cone. Suppose $\{v_1, \ldots, v_k\}$ forms some part of a $\mathbb{Z}$-basis for N. Then $X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma \leq n$.

Proof. Choose a basis $e_1, \ldots, e_n$ for N such that $v_i = e_i$ for $1 \leq i \leq k$ (that $\{v_1, \ldots, v_k\}$ is a $\mathbb{Z}$-basis for N exactly makes this possible). This implies that $S_\sigma = \sigma^\vee \cap M$ is generated by

$$e_1^*, \ldots, e_k^*, \pm e_{k+1}^*, \ldots, \pm e_n^* \in M.$$

To see this, it helps to note the e_i^* for $k+1 \leq i \leq n$ are exactly the basis vectors of M which are zero on σ. This means

$$\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, \ldots, t_k, t_{k+1}^\pm, \ldots, t_n^*] = \mathbb{C}[t_1, \ldots, t_n] \otimes_\mathbb{C} \mathbb{C}[t_{k+1}^\pm, \ldots, t_n^\pm].$$

□

Lemma 3.2. There exists a bijection correspondence

$$\begin{pmatrix} \text{closed points} \\ \text{of } X_\sigma \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{semigroup} \\ \text{homomorphisms} \\ S_\sigma \to \mathbb{C} \end{pmatrix}.$$

Proof. We have the following one-to-one correspondences:

$$\left\{ \begin{matrix} \text{closed points} \\ \text{in X} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{scheme maps} \\ \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[S_\sigma] \end{matrix} \right\} \leftrightarrow^{(*)} \left\{ \begin{matrix} \text{semigroup morphisms} \\ S_\sigma \to \mathbb{C} \end{matrix} \right\}.$$

Only $(*)$ is new. □

Definition 3.3. Define $x_\sigma \in X_\sigma$ to be the point corresponding to the semigroup map

$$S_\sigma \xrightarrow{x_\sigma} \mathbb{C},\ m \mapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases},$$

where

$$\sigma^\perp = \{m \in M_\mathbb{R} \mid \langle u, m \rangle = 0,\ \forall u \in \sigma\}.$$

We now proceed to Theorem (3.4).

Theorem 3.4. *Let $\sigma \subseteq N_\mathbb{R}$ be a cone and X_σ be the associated affine toric variety. The following are equivalent:*

(i) X_σ is smooth

(ii) σ is generated by a subset of a $\mathbb{Z}$-basis for N

(iii) $X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma$.

3.4. Lemma (3.1) gives us $(ii) \implies (iii)$. The fibre product of smooth schemes with smooth structure maps is again smooth, so $(iii) \implies (i)$ is clear. It is only left to prove $(i) \implies (ii)$. □