M392C
Conformal Blocks,
Generalized Theta Functions,
and the Verlinde Formula

Chiara Damiolini

May 10, 2023

Essential information

Meeting time and place: T,Th 9:30-11:00 AM, PMA 9.166
Instructor: Chiara Damiolini (she/her)
PMA 12.130
chiara.damiolini@austin.utexas.edu
Office hours: Tuesdays 2-3PM 12.130
Textbook: *Conformal Blocks, Generalized Theta Functions and the Verlinde Formula* by S. Kumar [Kum22].

1 Introduction

The main focus of this course is to study conformal blocks from the perspective of representation theory and algebraic geometry. We will follows Kumar's book, in which each chapter contains a description of the references where the main results could be found in the literature.

In the first day of class I gave an overview of what we will see in the course and throughout the semester.

1.1 Possible project ideas

The final grade will be based on participation, a short paper and a presentation at the end of the semester (about 35 minutes). The presentation and the essay will be on topics which are connected to the course, also interpreted in a broad sense. In the next couple of weeks I will list on Canvas some ideas for your project, but I am more than happy to accept your suggestions.

2 Crash course in Lie algebras and their representations

I recall here what we need to know about the representation theory of simple Lie algebras. I suggest [Ser87] or [Hum78] for proofs or more details. A somewhat less standard reference (for mathematicians) is [DFMS97, Chapter XIII].

A Lie algebra over $\mathbb{C}$ is a vector space $\mathfrak{g}$ together with a bilinear operation $[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies

$$[a, a] = 0 \qquad \text{and} \qquad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \qquad \text{(Jacobi identity)}$$

for every $a, b, c \in \mathfrak{g}$. The form $[\,,\,]$ is called the Lie bracket. It follows that the form is anti-symmetric.

Example 2.0.1.
- Given any vector space V, we can define $[a, b] = 0$ for every $a, b \in V$. This is called an abelian Lie algebra.
- Given any $\mathbb{C}$-algebra $(R, +, \cdot)$, we can define a Lie bracket by $[a, b] = ab - ba$ for every $a, b \in R$. *Check that Jacobi holds.*
- Particular examples of the above are $\mathfrak{gl}_n$, $\mathfrak{sl}_n$ (traceless), $\mathfrak{so}_n$ (skew-symmetric);
- Given any $\mathbb{C}$-algebra A we can consider the $\mathbb{C}$-module of $\mathbb{C}$-linear derivations of A, i.e. $\mathbb{C}$-linear maps $D\colon A \to A$ such that $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. Again $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ defines a Lie bracket on the space of derivations. Observe that $D_1 \circ D_2$ is not a derivation! Special example is $\mathrm{Der}(\mathbb{C}[t]) = \mathbb{C}[t]d/dt$.

We now focus only on those finite dimensional Lie algebras which are simple. To define this notion, we first say that a Lie subalgebra I of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ if, for every $a \in I$ and $b \in \mathfrak{g}$, one has $[a, b] \in I$. Then we say that a finite dimensional algebra $\mathfrak{g}$ is *simple* if it is not abelian and does not contain any non trivial ideal. A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras.

Example 2.0.2. Note that the only Lie algebra which is abelian and has no non-trivial ideals is the one-dimensional abelian Lie algebra. This is not simple by definition.

The Lie algebra $\mathfrak{sl}_r$ is simple for every r, however $\mathfrak{gl}_r$ is not simple and one can show that $\mathfrak{sl}_r$ is a non-trivial ideal of $\mathfrak{gl}_r$.

Fact 2.0.3. *Simple Lie algebras are completely classified using Dynkin diagrams.*

2.1 Representation of simple Lie algebras

We are interested in the representation theory of simple Lie algebras of finite dimension. Throughout $\mathfrak{g}$ will be simple and finite dimensional.

Definition 2.1.1. A finite dimensional representation of $\mathfrak{g}$ is a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}_V$ for some finite dimensional vector space V. This means that a representation is a finite dimensional vector space V equipped with an action of $\mathfrak{g}$. A representation is called *simple* or *irreducible* if it does not contain non-trivial subrepresentations.

Example 2.1.2. An important representation is the *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}_{\dim \mathfrak{g}}$ which is defined by $\mathrm{ad}(a) := [a, -]$. Note that $\mathfrak{sl}_r$ is naturally equipped with another representation $\mathfrak{sl}_r \to \mathfrak{gl}_r$ given by inclusion. This is called the standard representation of $\mathfrak{sl}_r$.

Exercise 2.1.3. Observe that we can say that $\mathfrak{g}$ is simple if and only if the representation ad is simple and $\mathfrak{g}$ is non abelian.

A key theorem to the classification of representations of simple Lie algebras is the following theorem.

Theorem 2.1.4 (Weyl's theorem)**.** *Every finite dimensional representation of a semisimple Lie algebra is completely reducible, i.e. it is a direct sum of finitely many simple representations.*

We now introduce a form that will be essential not only to understand the representation theory of $\mathfrak{g}$, but also essential to the definition of conformal blocks.

Definition 2.1.5. The Killing form $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the bilinear and symmetric map defined by $\kappa(x, y) = \mathrm{trace}(\mathrm{ad}_x \circ \mathrm{ad}_y) = \mathrm{trace}([x, [y, -]])$.

Exercise 2.1.6. Check that $\kappa([a, b], c) + \kappa(b, [a, c]) = 0$.
Compute the Killing form on $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

Theorem 2.1.7 (Cartan criterion)**.** *A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if the the Killing form is non-degenerate, i.e. it induces an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$.*

We say that an element $a \in \mathfrak{g}$ is *semisimple* if ad_a is diagonalizable. Any Lie subalgebra whose elements are all semisimple is called a toral subalgebra. A maximal *toral* subalgebra is called a *Cartan subalgebra* and denoted $\mathfrak{h}$.

Exercise 2.1.8. Exhibit a Cartan subalgebra of $\mathfrak{sl}_n$.

Fact 2.1.9. *Every Cartan subalgebra is abelian and all Cartan algebras are conjugated. The dimension of any Cartan subalgebra is constant and is called the rank of $\mathfrak{g}$.*

Since $\mathfrak{h}$ is abelian and its elements are semisimple, the elements of $\mathfrak{h}$ are simultaneously diagonalizable. We can then write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{a \in \mathfrak{g} \text{ such that } [h, a] = \alpha(h)a\},$$

and we denote by Φ the subset of $\mathfrak{h}^* \setminus 0$ for which $\mathfrak{g}_\alpha \neq 0$. This is called the set of *roots* of $\mathfrak{g}$.

Fact 2.1.10. *One can show that $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\beta \neq -\alpha$ and also $\kappa(\mathfrak{g}_\alpha, \mathfrak{h}) = 0$ for every $\alpha \in \Phi$. This implies that the Killing form induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$.*

We will denote by t_α the element of $\mathfrak{h}$ associated to α via the Killing form, and by h_α the element $\dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$ which is called the *coroot* associated with α. This is the same as saying that h_α is the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$ because $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha)$.

Example 2.1.11. The main example to keep in mind is $\mathfrak{sl}_2$. This is three dimensional with elements H, X and Y and Lie bracket determined by $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. The (or better, *a*) Cartan algebra is given by $\mathfrak{h} = H\mathbb{C}$. We can check that $\kappa(H, H) = 8$. It follows that the element $H \in \mathfrak{h}$ corresponds, via the Killing form to the element of $\mathfrak{h}^*$ which sends H to 8. We will denote by 8 this character. Since we know that $[H, X] = 2X$, the element $X \in \mathfrak{g}_\alpha$, where $\alpha = 2$ (meaning $\alpha(H) = 2$). We deduce that $t_\alpha = H/4$ and so $h_\alpha = 2 \cdot H/4 \cdot (16/8) = H$. So we recover H as the coroot of $\alpha = 2$!

The above example not has told us what the roots of $\mathfrak{sl}_2$ are, and in particular we have seen that $\alpha(H_\alpha)$ is a positive integer. This is not characteristic only of $\mathfrak{sl}_2$, but in general, for every simple Lie algebra $\mathfrak{g}$ we have that the roots α have the properties that $\alpha(h_\alpha) = 2$. But we can say more, for every $\alpha, \beta \in \Phi$, we have that $\alpha(h_\beta)$ is an integer. *This follows from the fact that the space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a Lie algebra isomorphic to $\mathfrak{sl}_2$ and we know how to describe the representations of $\mathfrak{sl}_2$.*

Exercise 2.1.12. Now that we have seen the roots of $\mathfrak{sl}_2$, what are the roots of $\mathfrak{sl}_3$?

It then makes sense to state the following result.

Fact 2.1.13. *One can choose $r = \operatorname{rank}(\mathfrak{g})$-many linearly independent roots (call them simple roots) such that all the remaining roots are obtained by a linear combinations of the simple roots using only non negative integer coefficients, or only non positive integer coefficients.*

We choose such a basis of simple roots and denote them $\alpha_1, \ldots, \alpha_r$. From now on roots can and will be called positive (or negative) and will belong to Φ^+ (resp. Φ^-). The space $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$ called a *Borel* subalgebra of $\mathfrak{g}$.

Given any finite representation V of $\mathfrak{g}$, since the elements of $\mathfrak{h}$ are simultaneously diagonalizable we can decompose it as

$$V = \bigoplus_\lambda V_\lambda$$

for $\lambda \in \mathfrak{h}^*$. The elements of λ appearing in the decomposition above are called *weights* of V (so in particular we can say that a root is a weight for the adjoint representation). A non zero element of V is called a *maximal vector* (of weight λ) if $v \in v_\lambda$ and $\mathfrak{g}_\alpha(v) = 0$ for all positive roots α.

Fact 2.1.14. *For every finite representation V of $\mathfrak{g}$ there exists (at least) a maximal vector. A maximal vector is just an eigenvector for the action of the Borel subalgebra.*

If we assume that V is irreducible, then V must coincide with the space generated by the image of a maximal vector v under the action of $\mathfrak{g}$. In particular every maximal vector has the same weight, which is then called the maximal weight of the representation V.

Theorem 2.1.15. *The number $\lambda(h_{\alpha_i})$ are non-negative integers for all $i \in \{1, \dots, r\}$.*

This follows from the representation theory of $\mathfrak{sl}_2$. In fact, through the roots α_i, we can view V as a representation of $\mathfrak{sl}_2$. Then we use Weyl's theorem.

The elements $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_{\alpha_i}) \in \mathbb{Z}$ are called *weights* of $\mathfrak{g}$. A weight is *dominant* if further $\lambda(h_{\alpha_i}) \geq 0$. Using this terminology, we have seen that the maximal weight of an irreducible representation is a dominant weight of $\mathfrak{g}$. One can show that from every dominant weight λ one can construct an irreducible representation of maximal weight ϖ.

Theorem 2.1.16. *The set of finite dimensional and simple representations of $\mathfrak{g}$ are in bijection with the set of dominant weights.*

From now on, we will denote the set of finite dimensional simple representations of $\mathfrak{g}$ by P^+. The representation associated with the weight ϖ will be denoted V_ϖ.

3 Affine Lie algebras

Now that we understand the theory of simple Lie algebra, we can introduce *affine (Kac-Moody) Lie algebras*. Kac's book [Kac90] is the standard reference to understand the theory of these infinite dimensional Lie algebra.

3.1 The affine Lie algebra $\widehat{\mathfrak{g}}$

Throughout $\mathfrak{g}$ will denote a simple Lie algebra over $\mathbb{C}$ and the rank will be denoted r. We will assume that a Cartan $\mathfrak{h}$ and a basis of the root system Φ (or equivalently a Borel $\mathfrak{b}$) are chosen.

Since we have chosen a basis of positive roots, there exists a *highest root* θ, i.e. $\theta \in \Phi$ and for every $\alpha \in \Phi$ we have that $\theta - \alpha$ is a non-negative root. Call h_θ the associated coroot, which naturally lives in $\mathfrak{h}$.

Fact 3.1.1. *One can define the highest root θ as being the highest weight of the adjoint representation of $\mathfrak{g}$.*

Definition 3.1.2. We define $\langle | \rangle$ to be the scalar multiple of the Killing form such that $\langle h_\theta | h_\theta \rangle = 2$.

One can explicitly compute that $\kappa(\, , \,) = 2\check{h} \cdot \langle | \rangle$ for $\check{h}$ the *dual Coxeter number* of $\mathfrak{g}$.

Example 3.1.3. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have that $\theta = \alpha$ since this is the only positive root. It then follows that $h_\theta = H$. Since $\kappa(H, H) = 8$ and we want $\langle H|H \rangle = 2$, we have that $\langle | \rangle = \kappa(\, , \,)/4$ and the dual Coxeter number is then 2. One can show that the dual Coxeter number of $\mathfrak{sl}_r$ is always equal to r.

Definition 3.1.4. We define the loop Lie algebra to be the space $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$ with Lie bracket given by

$$[X \otimes f(t), Y \otimes g(t)] := [X, Y] \otimes f(t)g(t)$$

for all $X, Y \in \mathfrak{g}$ and $f(t), g(t) \in \mathbb{C}((t))$.

Remark 3.1.5. Since $\mathfrak{g}$ is finite dimensional, we may (and will) write $\mathfrak{g}((t))$ in place of $\mathfrak{g} \otimes \mathbb{C}((t))$ so that the element $X \otimes f(t)$ will simply be denoted $Xf(t)$ for $X \in \mathfrak{g}$ and $f(t) \in \mathbb{C}((t))$.

Definition 3.1.6. We define the *affine Lie algebra* $\widehat{\mathfrak{g}}$ as the Lie algebra whose underlying vector space is

$$\mathfrak{g}((t)) \oplus \mathbb{C}K,$$

where K is a place-holder variable, and the Lie bracket is given by

$$[Xf(t) + aK, Yg(t) + bK] = [X, Y]f(t)g(t) + \langle X|Y\rangle \mathrm{res}(g(t)df(t))K$$

for every $X, Y \in \mathfrak{g}$, $f, g \in \mathbb{C}((t))$ and $a, b \in \mathbb{C}$.

We note that the following holds:

- The algebra $\widehat{\mathfrak{g}}$ is a central extension of the loop algebra, i.e. it fits in an exact sequence of Lie algebras
$$0 \to \mathbb{C}K \to \widehat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$
and $\mathbb{C}K$ is central inside $\widehat{\mathfrak{g}}$, that is $[\mathbb{C}K, \widehat{\mathfrak{g}}] = 0$.
- This Lie algebra is invariant under the change of coordinate t. *This comes from the fact that the residue pairing does not depend on the chosen coordinate.*

Remark 3.1.7. In [Kac90], the author mainly studies central extensions of $\mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]$ instead of $\mathfrak{g}((t))$. We are interested in Laurent series, rather than Laurent polynomials, for the following reason. Consider a curve C over $\mathbb{C}$ and a point $P \in C$. Then when we expand a meromorphic function on C at the point P we obtain an element in $\mathbb{C}((t_P))$, where t_P is a local coordinate at the point P.

Question: How to classify central extension of $\mathfrak{g}((t))$?

Answer: One can show that every central extension of the loop algebra by $\mathbb{C}$ is obtained by rescaling the form $\langle\,|\,\rangle$ and actually that this central extension is universal. This can be found e.g. in the first two sections of [Gar80].

3.2 Integrable Representations of $\widehat{\mathfrak{g}}$

We describe here the class of representations of $\widehat{\mathfrak{g}}$ that will appear in the description of conformal blocks.

We have seen how, for a simple Lie algebra $\mathfrak{g}$, every irreducible representation of $\mathfrak{g}$ is uniquely determined by its maximal weight. Here an analogue description holds, where we are interested in describing highest-weight irreducible representations of $\widehat{\mathfrak{g}}$ where the element K acts by multiplication by a positive integer ℓ (this property goes under the name of *of level* ℓ). We will see how these representations are in bijection with the set

$$P_\ell^+ := \{\varpi \in P^+ \text{ such that } \varpi(h_\theta) \leq \ell\}.$$

This means that among all the simple representations of $\mathfrak{g}$, only those satisfying the level condition $\varpi(h_\theta) \leq \ell$ will give rise to the representations that we will be interested in.

Remark 3.2.1. Observe that while P^+ is infinite dimensional, P_ℓ^+ is a finite set.

Is is instructive to understand how, associated with such weights, we can construct an irreducible representation of $\widehat{\mathfrak{g}}$ of level ℓ. I will describe how to do this.

Definition 3.2.2. Given a Lie algebra L we define the *universal enveloping algebra* as the quotient of the tensor algebra $TL = \bigoplus_{i\geq 0} \otimes^i L$ (where $\otimes^0 L = \mathbb{C}$ and $\otimes^1 L = L$), by the ideal generated by

$$X \otimes Y - Y \otimes X - [X, Y]$$

for $X, Y \in L$.

In what follows we will denote by $X \circ Y$ the class of $X \otimes Y$ in UL. With this terminology, a representation of L is the same as a left module over the ring UL.

We start taking the representation V_λ associated with the weight $\lambda \in P_\ell^+$. To obtain a representation of $\widehat{\mathfrak{g}}$ it will be enough to construct a vector space on which $U\widehat{\mathfrak{g}}$ acts on the left.

Extend the action of $\mathfrak{g}$ to $\widehat{\mathfrak{g}}_{\geq 0}$. We first consider the Lie algebra

$$\widehat{\mathfrak{g}}_{\geq 0} := \mathfrak{g} \otimes \mathbb{C}[\![t]\!] \oplus \mathbb{C}K$$

which naturally is a Lie subalgebra of $\widehat{\mathfrak{g}}$ (and which is really a direct sum of Lie algebras since there are no poles). We now interpret V_λ as a module for $\widehat{\mathfrak{g}}_{\geq 0}$ by imposing the following rules

$$(Xf(t)) * (v) = f(0 = X(v) \qquad \text{and} \qquad aK * v = a\ell \cdot v$$

for all $X \in \mathfrak{g}$, $f(t) \in \mathbb{C}[\![t]\!]$ and $a \in \mathbb{C}$.

Verma module construction. We then induce this representation to the whole $\widehat{\mathfrak{g}}$ as follows:

$$M_\lambda = U\widehat{\mathfrak{g}} \otimes_{U\widehat{\mathfrak{g}}_{\geq 0}} V_\lambda.$$

This makes sense because $U\widehat{\mathfrak{g}}$ is a right $U\widehat{\mathfrak{g}}_{\geq 0}$-module and V_λ is a left $U\widehat{\mathfrak{g}}_{\geq 0}$-module. We then have that M_λ is a left $U\widehat{\mathfrak{g}}$-module, where $U\widehat{\mathfrak{g}}$ is acting by multiplication on the left.

Irreducibility. The module that we have obtained is not irreducible! However, when $\lambda \in P_\ell^+$, this space contains a maximal $\widehat{\mathfrak{g}}$-submodule which we will call Z. This is generated by

$$\left(X_\theta t^{-1}\right)^{\ell - \lambda(h_\theta)+1} \otimes v_\lambda$$

for X_θ any element of $\mathfrak{g}_\theta$ and v_λ a maximal weight vector of V_λ. *We have that every Verma module will have a maximal ideal. Here the point is that when $\lambda(h_\theta) \leq \ell$, then there exists a proper ideal generated by the element above.* We then denote by $\mathcal{H}_\lambda$ the quotient of M_λ by this proper module.

The obtained module is not only irreducible, but it also *integrable*: a $\widehat{\mathfrak{g}}$-module M is integrable if and only if, for every nilpotent element $X \in \mathfrak{g}$ (e.g. $X \in \mathfrak{g}_\alpha$ for some root α) and for every $f(t) \in \mathbb{C}(\!(t)\!)$ there exists $N \in \mathbb{N}$ such that the element $(Xf(t))^N$ acts trivially on M.

Theorem 3.2.3. *All irreducible integrable representations of $\widehat{\mathfrak{g}}$ of level ℓ are constructed as above and if $\lambda \neq \mu$ then $\mathcal{H}_\lambda \neq \mathcal{H}_\mu$.*

Moreover $\mathcal{H}_\lambda$ satisfies also the following properties:

- As a vector space M_λ is isomorphic to $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}} V_\lambda$. Actually, this describes M_λ as a $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$-module.

- The element $\left(X_\theta t^{-1}\right)^{\ell-\lambda(h_\theta)+1} \otimes v_\lambda$ is annihilated by $\mathfrak{g} \otimes t\mathbb{C}[\![t]\!]$. Moreover this shows that the elements of Z have non negative degree, and actually at most $\ell - \lambda(h_\theta) + 1$.

- The subspace of $\mathcal{H}_\lambda$ killed by $\mathfrak{g} \otimes t\mathbb{C}[\![t]\!]$ is given by V_λ (in the description above, it's the image of $1 \otimes V_\lambda \subset M_\lambda$ in the quotient $\mathcal{H}_\lambda$). This doesn't contradict the above since this element is zero in $\mathcal{H}_\lambda$.

- The set of $\widehat{\mathfrak{g}}$-module endomorphisms of $\mathcal{H}_\lambda$ is given by $\mathbb{C}$ (i.e. it's only given by multiplication by a scalar).

Example 3.2.4. Let us analyze the case of $\mathfrak{g} = \mathfrak{sl}_2$. Here we have that all irreducible representations are given by non-negative integers $\lambda = \lambda(H)$, so that the representation V_λ has dimension $\lambda + 1$. It follows that P_ℓ^+ is in bijection with $\{0, 1, \ldots, \ell\}$ and so with the representations of dimension at most ℓ. It is illustrative to analyze the case $\ell = 1$. Here we only have two choices:

1. $\lambda = 0$: this corresponds to the trivial representation and the module Z is generated by $Xt^{-1} \circ Xt^{-1} \otimes v_0$, where all elements of $\mathfrak{sl}_2$ act trivially on v_0.

2. $\lambda = 1$: this corresponds to the standard representation $V_1 = \mathbb{C}^2$ and we denote v_1 its maximal weight vector (the other element is the image of v_1 under Y). In this situation Z is generated by $Xt^{-1} \otimes v_1$.

4 Spaces of covacua and conformal blocks

We have now all the ingredients to define the space of covacua (sometimes called space of coinvariants) and its dual, the space of conformal blocks.

4.1 Set up

Throughout this section, we are going to fix the following data:

- C a (possibly) nodal reduced and projective curve over $\mathbb{C}$;
- $P_1, \dots, P_n$ ordered, distinct, smooth points of C such that every component of C is marked by at least one of such points;
- $\mathfrak{g}$ a simple Lie algebra over $\mathbb{C}$;
- a positive integer ℓ and $\lambda_1, \dots, \lambda_n$ an ordered n-tuple of elements of P_ℓ^+.

This is the same as saying that we fix a sufficiently nice curve C on which we fix some points which are decorated by integrable representations of $\widehat{\mathfrak{g}}$ of the same level.

For future applications, we might want to also impose that the pointed curve $(C, P_\bullet)$ is Deligne-Mumford stable, that is there are only finitely many automorphisms of $(C, P_\bullet)$. It is not a condition that is necessary to the construction of conformal blocks.

Note that if a point P of C is smooth, then

$$\widehat{\mathcal{O}} := \varprojlim_n \frac{\mathcal{O}_C}{\mathfrak{m}_P^n} \cong \mathbb{C}[\![t]\!]$$

for some element $t \in \mathfrak{m}_P$. Note that this isomorphism depends on the choice of t. Similarly its fraction field $\widehat{\mathcal{K}}$ is non canonically isomorphic to $\mathbb{C}(\!(t)\!)$.

In particular for every $i \in [n]$ we can define the Lie algebra $\mathfrak{g} \otimes \widehat{\mathcal{K}}_i$ which is a coordinate free version of the loop algebra $\mathfrak{g}(\!(t_i)\!)$. We can also extend this algebra centrally and obtain $\widehat{\mathfrak{g}}_i := \mathfrak{g} \otimes \widehat{\mathcal{K}}_i \oplus \mathbb{C}K_i$ which can be seen as a coordinate independent version of $\widehat{\mathfrak{g}}$.

We now consider two Lie algebras associated with the above data:

- $\mathfrak{g}_{C \setminus P_\bullet} := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}(C \setminus P_\bullet)$. Moreover recall that $C \setminus P_\bullet$ is an affine curve, i.e. $C \setminus P_\bullet = \mathrm{Spec}(A)$ for A some $\mathbb{C}$-algebra. This means that this Lie algebra encode all the information about $C \setminus P_\bullet$.

- $\widehat{\mathfrak{g}}_{P_\bullet} := \bigoplus_{i=1}^n \mathfrak{g} \otimes \widehat{\mathcal{K}}_i \oplus \mathbb{C}K$ which is the quotient of the Lie algebra $\widehat{\mathfrak{g}}_1 \oplus \cdots \oplus \widehat{\mathfrak{g}}_n$ obtained identifying $K_1 = K_2 = \cdots = K_n$. Otherwise said, this is the central extension of $\bigoplus_{i=1}^n \mathfrak{g} \otimes \widehat{\mathcal{K}}_i$ by $\mathbb{C}k$ which has as Lie bracket

$$\left[(X_i f_i)_{i\in[n]}, (Y_i g_i)_{i\in[n]}\right] = ([X_i, Y_i] f_i g_i)_{i\in[n]} + \sum_{i=1}^n (X_i|Y_i)\mathrm{res}(g_i df_i).$$

Remark 4.1.1. A perhaps more natural way to define these algebras is to start with the sheaf of Lie algebras $\mathfrak{g}_C := \mathfrak{g} \otimes \mathcal{O}_C$ on C. Then $\mathfrak{g}_{C\setminus P_\bullet} = H^0(C \setminus P_\bullet, \mathfrak{g}_C)$, and $\mathfrak{g} \otimes \widehat{\mathcal{K}}_i = H^0(D^\times_{P_i}, \mathfrak{g}_C)$ where $D^\times_{P_i} = \mathrm{Spec}(\widehat{\mathcal{K}}_i)$.

Proposition 4.1.2. *The natural map of Lie algebras*

$$\mathfrak{g}_{C\setminus P_\bullet} \to \bigoplus_{i=1}^n \mathfrak{g} \otimes \widehat{\mathcal{K}}_i, \qquad X \otimes f \mapsto X \otimes (f_i)$$

induces a map of Lie algebras $\mathfrak{g}_{C\setminus P_\bullet} \to \widehat{\mathfrak{g}}_{P_\bullet}$.

Proof. What we have to prove is the compatibility with the Lie bracket, since the canonical map of vector spaces $\bigoplus_{i=1}^n \mathfrak{g} \otimes \widehat{\mathcal{K}}_i \to \widehat{\mathfrak{g}}_{P_\bullet}$ is not a map of Lie algebras. It is enough to prove that

$$\sum_{i=1}^n \mathrm{res}_{P_i}(g_i df_i) = 0$$

for every $f, g \in \mathcal{O}(C \setminus P_\bullet)$. But now recall the *Residue Theorem*: it states that the sum of the residues of a differential form on a smooth and projective curve is zero. When the curve C is smooth, then we are done since the global form gdf is meromorphic and can have poles only at the points $P_\bullet$. If C is not nodal, then we pass to its normalization and proceed as in the smooth case. □

From the representations λ_is we construct the (infinite-dimensional) vector space

$$\mathcal{H}_{\lambda_\bullet} := \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$$

which is naturally equipped with an action of $\widehat{\mathfrak{g}}_{P_\bullet}$, given explicitly by the formulas

$$(Xf_i)_{i\in[n]} * (v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes v_{i-1} \otimes (Xf_i) \circ v_i \otimes v_{i+1} \otimes \cdots \otimes v_n$$

and

$$K(v_1 \otimes \cdots \otimes v_n) = \ell(v_1 \otimes \cdots \otimes v_n).$$

Remark 4.1.3. We have seen in the previous chapter how to construct the $\widehat{\mathfrak{g}}$-representation $\mathcal{H}_\lambda$. In doing so we have used the $\widehat{\mathfrak{g}} = \mathfrak{g}(\!(t)\!)$, that is we have made a choice of a coordinate. We can now choose another coordinate ω of $\mathbb{C}(\!(t)\!)$, i.e. $\omega = \omega(t)$ with $\omega(0) = 0$ and $\omega'(0) \neq 0$ (this ensures that we can reverse the procedure and express t as a function of ω). This change of coordinates induces an isomorphism between $\mathfrak{g}(\!(t)\!)$ and $\mathfrak{g}(\!(\omega)\!)$ and this further induces an isomorphism between their central extensions since the residue pairing does not depend on the choice of a local parameter.

It is then natural to ask how $\mathcal{H}_\lambda$ is affected by this change of coordinates. The answer is that, up to a natural isomorphism, $\mathcal{H}_\lambda$ is not affected by this change (see Lemma 2.1.2 of the book). I will expand on this.

4.2 Definition and first properties

We are now ready to define the central elements of our course.

Definition 4.2.1. We define the space of covacua associated with the data above as the vector space

$$\mathbb{V}_C(P_\bullet, \lambda_\bullet) = \frac{\mathcal{H}_{\lambda_\bullet}}{(\mathfrak{g}_{C\setminus P}) \circ (\mathcal{H}_{\lambda_\bullet})}.$$

Its dual space

$$\mathbb{V}_C(P_\bullet, \lambda_\bullet)^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathbb{V}_C(P_\bullet, \lambda_\bullet), \mathbb{C})$$

is called the space of conformal blocks.

We will prove that this space is finite dimensional. Before showing this in general, we will consider an example.

Example 4.2.2. Let's take a very concrete example: $C = \mathbb{P}^1$ and only one point, which we can assume being $P = 0$. As always we fix $\mathfrak{g}$, ℓ and λ. Let z be a coordinate at zero so that $\mathfrak{g}_{C\setminus P} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}]$ and $\widehat{\mathfrak{g}}_P = \mathfrak{g}((z))$. We first of all see that every element of $\mathcal{H}_\lambda$ is a linear combination of elements of the type

$$X_1 z^{-n_1} \circ \cdots \circ X_r z^{-n_r} \otimes v_\lambda, \tag{4.1}$$

where $n_1 \geq \cdots \geq n_r \geq 0$ (with $r \geq 0$), $X_i \in \mathfrak{g}$ and v_λ a highest weight vector of V_λ. We observe that whenever $r \neq 0$, the element in (4.1) lies in the image of $\mathfrak{g}_{C\setminus P}$, and so it will be zero in the space of covacua. When r=0, the element (4.1) is the generator of $\mathcal{H}_\lambda$, i.e. the element $1 \otimes v_\lambda$.

- If $\lambda \neq 0$ (i.e. when V_λ is not the trivial representation), then we know that there exists $H \in \mathfrak{h}$ such that v_λ is an eigenvector for the action of $H \in \mathfrak{h}$ with non zero eigenvalue. In forumals $H(v_\lambda) = \lambda(H)v_\lambda$ and $\lambda(H) \neq 0$. Since further we have that $H \in \mathfrak{g}_{C\setminus P}$, we obtain that also $1 \otimes v_\lambda$ is zero in the space of covacua and so this is a zero dimensional vector space.
- When $\lambda = 0$ the above argument does not work because every element of $\mathfrak{g}$ will act trivially on v_0. This element is therefore never in the image of $\mathfrak{g}_{C\setminus P}$, and since all the other elements vanish, the space of covacua associated with $\lambda = 0$ over $\mathbb{P}^1$ is one dimensional.

We now prove that indeed the spaces of covacua (and hence by consequence the space of conformal blocks) are finite dimensional. We will use the following result.

Lemma 4.2.3. *Let $\mathcal{L}$ be a Lie algebra.*

- *Let $\mathcal{H}$ be an $\mathcal{L}$-module which is generated (as an $U\mathcal{L}$-module) by a finite dimensional vector space, which we will denote V.*
- *Assume that there exists a set of generators $\mathcal{E} = \{e_i\}$ of $\mathcal{L}$ whose element act locally nilpotently on $\mathcal{H}$ (i.e. for every $v \in \mathcal{H}$ and for every $e_i \in \mathcal{E}$ there exists $N \in \mathbb{N}$ such that $e_i^N(v) = 0$).*

- *Denote by* $\mathcal{L}^+ = \{X \in \mathcal{L}$ *such that* $X(v) = 0$ *for all* $v \in V\}$ *and let* $\mathcal{A}$ *be any Lie subalgebra of* $\mathcal{L}$ *such that the quotient* $\mathcal{L}/(\mathcal{A} + \mathcal{L}^+)$ *is a finite dimensional vector space.*

Under the above assumptions $\mathcal{H}/\mathcal{A}(\mathcal{H})$ *is a finite dimensional vector space.*

Theorem 4.2.4. *The space of covacua* $\mathbb{V}_C(P_\bullet, \lambda_\bullet)$ *and the space of conformal blocks* $\mathbb{V}_C(P_\bullet, \lambda_\bullet)^\dagger$ *is finite dimensional.*

Proof. For simplicity we will assume that C is marked by only one point. The proof proceeds similarly otherwise. Since the space of conformal blocks is the dual of the space of covacua, it will be enough to show that $\mathbb{V}_C(P, \lambda)$ is finite dimensional. We wish to apply Lemma 4.2.3 using

$$\mathcal{L} = \widehat{\mathfrak{g}}, \qquad \mathcal{H} = \mathcal{H}_\lambda, \qquad \mathcal{A} = \mathfrak{g}_{C \setminus P}.$$

We now check that the conditions of the lemma hold true.

- Indeed $\mathcal{H}_\lambda$ is generated by a finite dimensional vector space, V_λ.
- Note first that every element of $\widehat{\mathfrak{g}}$ of positive degree (i.e. belonging to $\mathfrak{g} \otimes \mathbb{C}t[\![t]\!]$) acts locally nilpotently on $\mathcal{H}$. Moreover, since every element of $\mathfrak{g}$ is generated by its nilpotent elements (i.e. those which are not in the Cartan). It follows that $\{X_\alpha t^{-n}\}_{\alpha \in \Phi, n \in \mathbb{Z}_{\geq 1}}$ together with a set of generators of $\mathfrak{g} \otimes \mathbb{C}t[\![t]\!] \oplus \mathbb{C}K$ will generate $\widehat{\mathfrak{g}}$.
- In this case $\mathcal{L}^+ \subseteq \mathfrak{g} \otimes \mathbb{C}t[\![t]\!] =: \mathfrak{g}^+$ (for instance we have that if $\lambda = 0$, then $\mathcal{L}^+$ also contains $\mathfrak{g}$) and we need to prove that the quotient $\mathfrak{g}_{C \setminus P} \setminus \widehat{\mathfrak{g}}/\mathcal{L}^+$ is finite dimensional. This is equivalent to proving that $\mathfrak{g}_{C \setminus P} \setminus \mathfrak{g}(\!(t)\!)/\mathfrak{g}[\![t]\!]$ is finite dimensional. Since $\mathfrak{g}$ is finite dimensional, this amounts to showing that $\mathcal{O}(C \setminus P) \setminus \mathbb{C}(\!(t)\!)/\mathbb{C}[\![t]\!]$ is finite dimensional. This is ensured by Riemann-Roch (or you can think about this quotient representing $H^1(C, \mathcal{O}_C)$ which we know being finite dimensional).

In class I might have considered only $X_\alpha t^n$ for $n \in \mathbb{Z}$, but note that this won't generate all the elements of $\mathfrak{g}(\!(t)\!)$, but only those in $\mathfrak{g}[t, t^{-1}]$.

We have checked all the conditions of Lemma 4.2.3 holds and so we can conclude that $\mathbb{V}_C(P, \lambda)$ is finite dimensional. □

We then conclude this section by proving that indeed Lemma 4.2.3 holds.

Proof of Lemma 4.2.3. Since the quotient $\mathcal{L}/(\mathcal{A} + \mathcal{L}^+)$ is finite, there exists finitely many elements $f_1, \ldots, f_n \in \mathcal{L}$ such $\mathcal{L} = \mathcal{A} + \mathcal{L}^+ \oplus \oplus_{i=1}^r \mathbb{C} f_i$. Each f_i can be expressed using finitely many elements of $\mathcal{E}$ and so we can write $\mathcal{L} = \mathcal{A} + \mathcal{L}^+ \oplus \oplus_{i=1}^n \mathbb{C} e_i$ where e_i are locally nilpotent generators. It then follows that

$$U(\mathcal{L}) = \sum_{(m_i) \in \mathbb{N}^n} U(\mathcal{A}) \cdot e_1^{m_1} \cdots e_n^{m_n} \cdot U(\mathcal{L}^+)$$

and so

$$\mathcal{H} = \sum_{(m_i) \in \mathbb{N}^n} U(\mathcal{A}) \cdot e_1^{m_1} \cdots e_n^{m_n} \cdot U(\mathcal{L}^+) V.$$

Since every element of $\mathcal{L}^+$ annihilates V, the only element of $U(\mathcal{L}^+)$ which will give a non trivial contribution is the identity, hence we have

$$\mathcal{H} = \sum_{(m_i)\in\mathbb{N}^n} U(\mathcal{A}) \cdot e_1^{m_1} \cdots e_n^{m_n} V.$$

Using the fact that every e_i acts locally nilpotently and the fact that V has finitely many elements, we see that each m_i is bounded above, and so the sum is finite, concluding the argument. $\square$

Remark 4.2.5. We can note that Lemma 4.2.3 holds also under weaker conditions, i.e. when there exists a set of generators $\mathcal{E}$ of $\mathcal{L}$ that act locally finitely on $\mathcal{H}$. This means that for every $v \in \mathcal{H}$ and $e \in \mathcal{E}$ the vector space spanned by $\{v, e(v), e^2(v), \dots\}$ is finite dimensional. See, e.g. [Sor96].

5 Propagation of vacua

In this section we explore one of the main properties of the spaces of covacua and conformal blocks. Throughout we will assume that $(C, P_\bullet)$ and $(\mathfrak{g}, \ell, \lambda_\bullet)$ are as in Section 4.1.

Theorem 5.0.1. *Let $Q \in C \setminus P$ be a smooth point of C, let V_0 be the trivial representation of $\mathfrak{g}$ and v_0 a non zero element of V_0. Then the map*

$$\mathcal{H}_{\lambda_\bullet} \to \mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}(V_0), \qquad v \mapsto v \otimes v_0$$

induces the isomorphism

$$\mathbb{V}_C(P_\bullet, \lambda) \cong \mathbb{V}_C(P_\bullet, Q; \lambda, 0).$$

This statement can be found—in its dual form—in [TUY89, Proposition 2.2.3].

Instead of providing the original proof of [TUY89], we will obtain Theorem 5.0.1 as a consequence of a more general statement which, as far as I know, was first observed in [Bea96]. Before stating this result, it will be convenient to introduce some notation. First of all, if an algebra $\mathcal{A}$ acts on a module $\mathcal{H}$, then the space of coinvariants $\mathcal{H}/\mathcal{A}(\mathcal{H})$ will be denoted $[\mathcal{H}]_{\mathcal{A}}$.

This is Beauville's result that will imply Theorem 5.0.1.

Theorem 5.0.2. *Let $(Q_1, \ldots, Q_s)$ be an s-tuple of smooth distinct points of $C \setminus P_\bullet$ and let $(\mu_1, \ldots, \mu_s)$ be an s-tuple of elements of P_ℓ^+. Define the action of $\mathfrak{g}_{C \setminus P_\bullet}$ on $V(\mu_\bullet) := V(\mu_1) \otimes \cdots \otimes V(\mu_s)$ as*

$$(Xf, v_1 \otimes \cdots \otimes v_s) = \sum_{j=1}^{s} v_1 \otimes \cdots \otimes v_{j-1} \otimes f(Q_j)X(v_j) \otimes v_{j+1} \otimes \cdots \otimes v_s,$$

for every $X \in \mathfrak{g}$, $f \in \mathcal{O}(C \setminus P_\bullet)$ and $v_j \in V(\mu_j)$. Then, the natural inclusion

$$\iota \colon V(\mu_\bullet) = V(\mu_1) \otimes \cdots \otimes V(\mu_s) \to \mathcal{H}(\mu_1) \otimes \cdots \otimes \mathcal{H}(\mu_s) = \mathcal{H}(\mu_\bullet)$$

induces the isomorphism

$$[\iota] \colon \mathcal{H}_{\lambda_\bullet} \otimes V(\mu_\bullet)]_{\mathfrak{g}_{C \setminus P_\bullet}} \cong [\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}(\mu_\bullet)]_{\mathfrak{g}_{C \setminus P_\bullet, Q_\bullet}}.$$

Proof. For simplicity we will assume that $s = 1$ and denote Q_1 by Q and μ_1 by μ. We first of all need to show that $[\iota]$ is indeed well defined. To do so it will be enough to show that the action of $\mathfrak{g}_{C \setminus P_\bullet}$ that we have defined on $V(\mu)$ coincides with the action of $\mathfrak{g}_{C \setminus P_\bullet} \subset \mathfrak{g}_{C \setminus P_\bullet, Q}$ on $\iota(V(\mu))$. This latter action was given by

$$Xf, v \mapsto Xf_Q \otimes v,$$

where f_Q represents the Laurent expansion of f at Q. Since f is regular at Q we have that $f_Q = \sum_{i\geq 0} a_i t_Q^i$. We now recall from Chapter 3 that V_λ is identified with the subspace of $\mathcal{H}_\lambda$ on which $\mathfrak{g}t[\![t]\!]$ acts trivially. Since $a_0 = f(Q)$, we conclude that the map is well defined.

The map is then well defined and since the map $\iota\colon V(\mu) \to \mathcal{H}_\mu$ factors through the Verma module M_μ, we can further see that $[\iota]$ also factors as

$$[\mathcal{H}_{\lambda_\bullet} \otimes V(\mu)]_{\mathfrak{g}_{C\setminus P_\bullet}} \xrightarrow{\phi} [\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}} \xrightarrow{\psi} [\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}.$$

We will show that both ϕ and ψ are isomorphisms.

Before we delve into the proofs of these two facts, it will be useful to make two observations:

(a) Since the curve $C \setminus P_\bullet$ is affine, there exists an element $f \in \mathcal{O}(C \setminus P_\bullet, Q)$ such that $f_Q = z^{-1}$ for a local coordinate z at Q (i.e. $D_Q \cong \mathrm{Spec}(\mathbb{C}[\![z]\!])$. By abuse of notation we will denote this function by z^{-1}. It then follows that $\mathcal{O}(C \setminus P_\bullet, Q) = \mathcal{O}(C \setminus P_\bullet) \oplus \mathbb{C}[z^{-1}]$ and hence $\mathfrak{g}_{C\setminus P_\bullet,Q} = \mathfrak{g}_{C\setminus P_\bullet} \oplus \mathbb{C}[z^{-1}]$.

(b) Given a Lie algebra $\mathcal{L}$ and two left $U\mathcal{L}$-modules M and N, the space of coinvariants $[M \otimes N]_\mathcal{L}$ is canonically isomorphic to the tensor product $M \otimes_{U\mathcal{L}} N$, where now we see M as being a right $U\mathcal{L}$-module. Indeed, we can see both spaces as being quotients of $M \otimes N$ by the relation $X(m) \otimes n + m \otimes X(n) = 0$ which we can translate as $-X(m) \otimes n = m \otimes X(n)$ (note that since there is a minus, the action on M is now on the right).

(c) We note that the Lie algebra $\mathfrak{g}_{C\setminus P_\bullet}$ acts on both $\mathcal{H}_{\lambda_\bullet}$ and on $V(\mu)$ and so the space of coinvariants can be described using the above remark. However, the action of $\mathfrak{g}_{C\setminus P_\bullet,Q}$ on $\mathcal{H}_{\lambda_\bullet} \otimes M_\mu$ (or on $\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu$) cannot be described as a combination of an action on $\mathcal{H}_{\lambda_\bullet}$ and an action on M_μ. We want to construct a Lie algebra $\mathcal{A}$ that acts on both $\mathcal{H}_{\lambda_\bullet}$ and M_μ and such that $[\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu]_\mathcal{A}$ is isomorphic to $[\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$. Let $\mathcal{A} := \mathfrak{g}_{C\setminus P_\bullet,Q} \oplus \mathbb{C}K_Q$ where the Lie bracket is given by

$$[Xf, Yg] = [X, Y]fg + K_Q(X|Y)\mathrm{res}(g_Q df_Q).$$

It follows that the element K_Q acts on $\mathcal{H}_\lambda$ by multiplication by $-\ell$, while it acts on M_μ by multiplication by ℓ. It follows that on the tensor product $\mathcal{H}_\lambda \otimes M_\mu$ the action of K_Q is trivial and so the action of $\mathcal{A}$ on $\mathcal{H}_\lambda \otimes M_\mu$ coincides with the action of $\mathfrak{g}_{C\setminus P_\bullet,Q}$ as wanted.

We are now ready to show that ϕ is an isomorphism.

$$\begin{aligned} [\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}} &= [\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_\mathcal{A} \\ \textit{using (b) and (c)} \qquad &= \mathcal{H}_{\lambda_\bullet} \otimes_{(U\mathcal{A})} M_\mu. \end{aligned}$$

Although M_μ is an $U\widehat{\mathfrak{g}}$-module, to understand the above tensor product we only need to see it as an $U\mathcal{A}$-module. Using the fact that $\mathcal{A} = \mathfrak{g}_{C\setminus P_\bullet,Q} \oplus \mathbb{C}K_Q$ we have that

$$M_\mu = U\mathcal{A} \otimes_{U(\mathfrak{g}_{C\setminus P_\bullet} \oplus \mathbb{C}K_Q)} V_\mu,$$

where we impose that K_Q acts by ℓ. Note that we have used the fact that $\mathfrak{g}_{C\setminus P_\bullet,Q} = \mathfrak{g}_{C\setminus P_\bullet} \oplus \mathfrak{g}z^{-1}[z^{-1}]$ to obtain such an expression. We then have

$$\begin{aligned} \mathcal{H}_{\lambda_\bullet} \otimes_{(U\mathcal{A})} M_\mu &= \mathcal{H}_{\lambda_\bullet} \otimes_{(U\mathcal{A})} U\mathcal{A} \otimes_{U(\mathfrak{g}_{C\setminus P_\bullet} \oplus \mathbb{C}K_Q)} V_\mu \\ &= \mathcal{H}_{\lambda_\bullet} \otimes_{U(\mathfrak{g}_{C\setminus P_\bullet} \oplus \mathbb{C}K_Q)} V_\mu \\ \textit{K_Q acts trivially} \qquad &= \mathcal{H}_{\lambda_\bullet} \otimes_{U(\mathfrak{g}_{C\setminus P_\bullet})} V_\mu \\ \textit{using (b)} \qquad &= [\mathcal{H}_{\lambda_\bullet} \otimes V_\mu]_{\mathfrak{g}_{C\setminus P_\bullet}}, \end{aligned}$$

and so we have shown that ϕ is an isomorphism.

To show that the map ψ is an isomorphism, we take the exact sequence

$$0 \to Z(\mu) \to M_\mu \to \mathcal{H}_\mu \to 0$$

and we apply to it the right-exact functor $[\mathcal{H}_{\lambda_\bullet} \otimes -]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$ obtaining the exact sequence

$$[\mathcal{H}_{\lambda_\bullet} \otimes Z(\mu)]_{\mathfrak{g}_{C\setminus P_\bullet,Q}} \longrightarrow [\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}} \xrightarrow{\psi} [\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}} \to 0.$$

We are then left to show that the image of $\mathcal{H}_{\lambda_\bullet} \otimes Z(\mu)$ inside $[\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$ vanishes. Recall that $Z(\mu)$ is generated by $(X_\theta z^{-1})^{\ell-\lambda(h_\theta)+1} v_\mu$, where z denotes a coordinate at Q. For simplicity we will use the notation $x := X_\theta z^{-1}$ and $k = \ell - \lambda(h_\theta) + 1 \geq 1$ in what follows.

Claim A. It is enough to show that for every $w \in \mathcal{H}_{\lambda_\bullet}$ the image of $w \otimes (x^k \otimes v_\mu)$ is zero in $[\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$.

Proof of Claim A. Every element of $\mathcal{H}_{\lambda_\bullet} \otimes Z(\mu)$ will be a linear combination of elements of the form $v \otimes (u \circ x^k \otimes v_\mu)$, with $v \in \mathcal{H}_{\lambda_\bullet}$ and $u \in U(\mathfrak{g}(\!(z)\!))$. Since $x^k \otimes v_\mu$ is annihilated by $\mathfrak{g}t[\![z]\!]$, we can further assume that $u \in U(\mathfrak{g}[z^{-1}])$. But note that $\mathfrak{g}[z^{-1}] \subset \mathcal{A}$ (defined in (c)), and since taking coinvariants with respect to $\mathfrak{g}_{C\setminus P_\bullet,Q}$ is the same as tensoring over $U(\mathcal{A})$, we see that inside $[\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$ the element $h \otimes u \circ x$ coincides with $-u \circ v \otimes x$, hence the claim.

We now use the integrability property of $\mathcal{H}_{\lambda_\bullet}$: For every $v \in \mathcal{H}_{\lambda_\bullet}$ and for every $Xf \in \mathfrak{g}_{C\setminus P_\bullet}$ with X nilpotent, there exists an $M \geq 0$ such that $(Xf)^N(v) = 0$ for every $N \geq M$. Observe also that in $[\mathcal{H}_{\lambda_\bullet} \otimes M_\mu]_{\mathfrak{g}_{C\setminus P_\bullet,Q}}$ we have the equality

$$(Xf)^N(v) \otimes u = -v \otimes (Xf_Q)^N(u),$$

for every $u \in M_\mu$. To finish the proof it will then suffice to show that there exists $Xf \in \mathfrak{g}_{C\setminus P_\bullet}$ such that for every $N \in \mathbb{N}$ we have $x^k \otimes v_\mu = (Xf)^N u$ for some $u \in M_\mu$.

Claim B. Let $f \in \mathcal{O}(C \setminus P_\bullet)u$ such that $f_Q(0) = 0$ and $f'_Q(0) = 1$. Set $y := X_{-\theta} f$. Then for every $p \geq k$ and for every $N \geq 0$ we have that x^p is a non zero multiple of $y^N x^{N+p} \otimes v_\mu$.

Proof of Claim B. We refer to the proof of Lemma 3.5 of [Bea96]. □

We have the following consequence

Corollary 5.0.3. *Assume that C is an irreducible curve and let $Q \in C$ be a smooth point disjoint from $P_\bullet$. Then the inclusion $\mathbb{C} = V(0) \subset \mathcal{H}(0)$ induces the isomorphism*

$$\mathbb{V}_C(P_\bullet, \lambda_\bullet) = [\mathcal{H}(0) \otimes V(\lambda_\bullet)]_{\mathfrak{g}_{C\setminus Q}}.$$

5.1 Consequences on $\mathbb{P}^1$

We will see here how Corollary 5.0.3 will allow us to give a finite dimensional presentation of the spaces of covacua (and conformal blocks) over the projective line.

Let $Q = \infty$ and $P_\bullet \in \mathbb{P}^1 \setminus \infty$. Let t be a coordinate at zero and z at infinity. The points P_i have are determined by $t = a_i$ for $a_i \in \mathbb{C}$.

Corollary 5.1.1. *The space of coinvariants $\mathbb{V}_{\mathbb{P}^1}(P_\bullet, \lambda_\bullet)$ is isomorphic to quotient of $V(\lambda_\bullet)$ by*

(a) $\mathfrak{g}V(\lambda_\bullet)$ and by

(b) the image of the operator $T^{\ell+1}$,

where $T\colon V(\lambda_\bullet) \to V(\lambda_\bullet)$ is the linear map defined by

$$T(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes a_i X_\theta(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n,$$

for every $v_i \in V(\lambda_i)$.

Proof. From the exact sequence

$$0 \to Z(0) \to M(0) \to \mathcal{H}(0) \to 0$$

we obtain the exact sequence

$$V(\lambda_\bullet) \otimes Z(0) \to [V(\lambda_\bullet) \otimes M(0)]_{\mathfrak{g}_{\mathbb{P}^1 \setminus \infty}} \to [V(\lambda_\bullet) \otimes V(\lambda_\bullet) \otimes]_{\mathfrak{g}_{\mathbb{P}^1 \setminus \infty}} \to 0.$$

Using Corollary 5.0.3 we can identify $\mathbb{V}_{\mathbb{P}^1}(P_\bullet, \lambda_\bullet)$ with $[V(\lambda_\bullet) \otimes \mathcal{H}(0)]_{\mathfrak{g}_{\mathbb{P}^1 \setminus \infty}}$. Note that $\mathfrak{g}_{\mathbb{P}^1 \setminus \infty} = \mathfrak{g}[t] = g[z^{-1}]$ can be decomposed as $\mathfrak{g} \oplus z^{-1}\mathfrak{g}[z^{-1}]$, and that $z^{-1}\mathfrak{g}[z^{-1}]$ is an ideal of $\mathfrak{g}[z^{-1}]$. It follows that

$$\begin{aligned}
[V(\lambda_\bullet) \otimes M(0)]_{\mathfrak{g}_{\mathbb{P}^1 \setminus \infty}} &= \Big[[V(\lambda_\bullet) \otimes M(0)]_{z^{-1}\mathfrak{g}[z^{-1}]}\Big]_{\mathfrak{g}} \\
&= \Big[V(\lambda_\bullet) \otimes_{Uz^{-1}\mathfrak{g}[z^{-1}]} M(0)\Big]_{\mathfrak{g}} \\
&= \Big[V(\lambda_\bullet) \otimes_{Uz^{-1}\mathfrak{g}[z^{-1}]} U(z^{-1}\mathfrak{g}[z^{-}1]) \otimes_{\mathbb{C}} \mathbb{C}v_0\Big]_{\mathfrak{g}} \\
&= [V(\lambda_\bullet) \otimes_{\mathbb{C}} \mathbb{C}v_0]_{\mathfrak{g}} = V(\lambda_\bullet)/\mathfrak{g}(V(\lambda_\bullet)),
\end{aligned}$$

where the last equality holds since $\mathfrak{g}$ acts trivially on v_0.

We now will see how, quotienting by the image of $V(\lambda_\bullet) \otimes Z(0)$ is equivalent to quotienting by the image of the operator $T^{\ell+1}$. Note that $Z(0)$ is generated, over $U(z^{-1}\mathfrak{g}[z^{-}1])$ by $(X_\theta z^{-1})^{\ell+1} \otimes v_0$ or equivalently by $(X_\theta t)^{\ell+1} \otimes v_0$. It follows that the image of every element of $V(\lambda_\bullet) \otimes Z(0)$ is a linear combination of elements of the type $[v \otimes (X_\theta t)^{\ell+1} \otimes v_0]$ (I used the brackets around the element to remember that we are inside $[V(\lambda_\bullet) \otimes M(0)]_{\mathfrak{g}[t]}$). But now note that

$$[v \otimes (X_\theta t)^{\ell+1} \otimes v_0 = (X_\theta t)^{\ell+1}(v) \otimes v_0] = [T^{\ell+1}(v) \otimes v_0],$$

which tells us that we further have to quotient $V(\lambda_\bullet)/\mathfrak{g}(V(\lambda_\bullet))$ by the image of the operator $T^{\ell+1}$ as claimed. Since there are no other relations, we are done. $\square$

We can infer a number of consequences from this result

(a) This is an alternative proof that the space of covacua is finite dimensional over $\mathbb{P}^1$. Moreover we can really bound the dimension by the product of the dimensions of the $V(\lambda_i)$s.

(b) This is the key idea that allowed [Fak12] to show that sheaves of coinvariants on $\overline{M}_{0,n}$ are globally generated. We will see in later classes how spaces of coinvariants indeed define sheaves and that the proof presented here can be extended to families of rational curves.

(c) Since $V(\lambda_\bullet)$ is a finite dimensional representation of $\mathfrak{g}$, and the element X_θ is nilpotent in $\mathfrak{g}$, the image of the operator $T^{(\ell+1)}$ will vanish for ℓ big enough. In this situation the space of covacua will be isomorphic to the quotient of $V(\lambda_\bullet)$ by the action of $\mathfrak{g}$. In [BGM15, BGM16], the authors discuss criteria for which $T^{(\ell+1)} = 0$.

(d) If we only have one representation, then we deduce that

$$\dim(\mathbb{V}_{\mathbb{P}^1}(\lambda)) = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}$$

First of all we observe that the image of $T^{\ell+1}$ is zero. Indeed we have that T acts as multiplication by X_θ and so $T^{\ell+1}$ coincides with the action of $(X_\theta)^{\ell+1}$ on V_λ. From the representation theory of V_λ we know that $X_\theta)^{\lambda(h_\theta)+1} = 0$, and since $\lambda(h_\theta) \leq 1$ we have that $T^{\ell+1}$ is necessarily trivial. Since V_λ is irreducible, then $\mathfrak{g}(V_\lambda)$ is either 0 or V_λ itself. The former situation happens only when $\mathfrak{g}$ acts trivially on V_λ, i.e. when $\lambda = 0$.

5.1.2. We now discuss what happens when we have two modules (V_λ and V_μ) attached at two points of $\mathbb{P}^1$. We first of all assume that the two points are given by $t = 0$ and $t = 1$. Under these assumptions we have that the action of $T^{\ell+1}$ is necessarily trivial, and so we are left to understand $[V_\lambda \otimes V_\mu]_{\mathfrak{g}}$, that is the quotient of $V_\lambda \otimes V_\mu$ by the action of $\mathfrak{g}$. We note that $V_\lambda \otimes V_\mu$ is not, in general, an irreducible representation of $\mathfrak{g}$, but can be decomposed as direct sum of irreducible components of $\mathfrak{g}$. Hence the dimension of the vector space $[V_\lambda \otimes V_\mu]_{\mathfrak{g}}$ will be equal to the number of trivial representations appearing in the decomposition of $V_\lambda \otimes V_\mu$ into irreducible $\mathfrak{g}$-representations. Otherwise said, this coincides with computing the subspace of $V_\lambda \otimes V_\mu$ on which $\mathfrak{g}$ acts trivially. To compute this number, we take a detour into dual representations. This concept will be used in the next chapter.

Definition 5.1.3. Let V be a $\mathfrak{g}$-representation. Then the dual vector space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is a $\mathfrak{g}$-representation by defining $(Xf)(v) := -f(X(v))$ for every $X \in \mathfrak{g}$, $f \in V^*$ and $v \in V$. This is called the dual representation of V.

Note that $(V^*)^*$ and V are naturally identified as $\mathfrak{g}$-representations.

Similarly, given two $\mathfrak{g}$-representations M and V, the space $\mathrm{Hom}_{\mathbb{C}}(V, M)$ becomes a representation of $\mathfrak{g}$ through the assignment

$$(Xf)(v) = X(f(v)) - f(X(v))$$

for every $f \in \mathrm{Hom}_{\mathbb{C}}(V, M)$ and $v \in V$. Note that $\mathrm{Hom}_{\mathbb{C}}(V, M) = V^* \otimes M$ and that the action just defined coincides with the usual action of $\mathfrak{g}$ on a tensor product of two $\mathfrak{g}$-representations.

The question is now: which is the biggest subspace of $\mathrm{Hom}_{\mathbb{C}}(V, M)$ on which $\mathfrak{g}$ act trivially? This consists of

$$\{f \in \mathrm{Hom}_{\mathbb{C}}(V, M) \text{ such that } X(f(v)) = f(X(v)) \text{ for all } v \in v\}.$$

Staring at this set, we recognize it as being given by $\mathrm{Hom}_{\mathfrak{g}}(V, M)$, i.e. the set of linear maps from V to M in the category of $\mathfrak{g}$-representations. If both V and M are simple, the Schur's Lemma ensures that

$$\mathrm{Hom}_{\mathfrak{g}}(V, M) = \begin{cases} \mathbb{C} & \text{if } V = M \\ 0 & \text{otherwise} \end{cases}.$$

Connecting all the arguments above we have then shown that

$$\mathbb{V}_{\mathbb{P}^1}(\lambda, \mu) = \begin{cases} \mathbb{C} & \text{if } V_{\lambda^*} = V_\mu \\ 0 & \text{otherwise} \end{cases}.$$

Remark 5.1.4. It might be useful to make two observations.

1. The relation $V_{\lambda^*} = V_\mu$ can be also expressed directly at the level of weights. In fact the *dual* of the weight λ can be abstractly defined as $\lambda^* := -w_0(\lambda)$, where w_0 is a particular element (the longest element) in the group of symmetries of the *root system* of $\mathfrak{g}$. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have that w_0 is given by $-\mathrm{Id}$ and so the dual of λ is λ itself. This is not surprising since V_λ and V_λ^* have the same dimension, and this is the only invariant for simple representations of $\mathfrak{sl}_2$.

2. If $\lambda \in P_\ell^+$, then also $\lambda^* \in P_\ell^+$ (i.e. the set P_ℓ^+ is closed under dualization). One can motivate this by noticing that $\lambda(h_\theta) = \tau(\lambda)h_{\tau(\theta)}$ for τ an isomorphism of the root system of $\mathfrak{g}$. One can then show that the map $\lambda \to \lambda^*$ described above is a map that preserves the set of positive roots, and so $h_{\theta^*} = h_\theta$, concluding the argument.

5.1.5. We can even go further and describe spaces of covacua associated with three representations. We did this in class.

I'll add the details here

6 Factorization Theorem

We are now exploring one of the main features of the spaces of covacua and conformal blocks which goes under the name of *factorization theorem*. This result allows us to express spaces of covacua on a nodal curves as a direct sum of appropriate spaces of coinvariants on the normalization of the curve itself. Before delving into the statement and proof of the theorem, let me mention some references

[TUY89] Here it's where the result first appeared. Some earlier works of Tsuchiya and Kanie had similar results on genus zero curves only. The proof uses the concept of correlation functions, concept that was also used to prove the propagation of vacua.

[Kum22] The first section of chapter three is dedicated to the proof of this theorem. The author does not use correlation functions.

[Loo13] This is a very different approach where a key input is Theorem 5.0.2.

I will combine the approaches of [Kum22] and [Loo13].

6.0.1. Throughout this section we will make use of the following notation:

- $(C, P_\bullet, \lambda_\bullet)$ are as in Section 4.1 and let Q be a node of C.
- We denote by $\eta\colon \widetilde{C} \to C$ the partial normalization of C at Q and denote by Q_+ and Q_- the two smooth points in $\eta^{-1}(Q)$. Note that it still makes sense to say that $\widetilde{C}$ is marked by the points $P_\bullet$ since η is an isomorphism between $\widetilde{C} \setminus \{Q_+, Q_-\}$ and $C \setminus Q$.

Remark 6.0.2. In this situation we have that $\mathcal{O}_C(C \setminus P_\bullet)$ can be described as the subring of $\mathcal{O}_{\widetilde{C}}(\widetilde{C} \setminus P_\bullet)$ consisting of those elements f such that $f(Q_+) = f(Q_-)$. Equivalently, this can be described using the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{I}_Q \arrow[r] \arrow[d, "="] & \mathcal{O}_C(C \setminus P_\bullet) \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathfrak{I}_Q \arrow[r] & \mathcal{O}_{\widetilde{C}}(\widetilde{C} \setminus P_\bullet) \arrow[r] & \mathbb{C} \oplus \mathbb{C} \arrow[r] & 0,
\end{tikzcd} \tag{6.1}$$

where $\mathfrak{I}_Q$ splits into$\mathfrak{I}_{Q_+}\mathfrak{I}_{Q_-}$ inside $\mathcal{O}_{\widetilde{C}}(\widetilde{C} \setminus P_\bullet)$.

A weak version of the factorization theorem can be expressed as the following statement. The vector spaces of covacua

$$\mathbb{V}_C(P_\bullet, \lambda_\bullet) \qquad \text{and} \qquad \bigoplus_{\mu \in P_\ell^+} \mathbb{V}_{\widetilde{C}}((P_\bullet, Q_+, Q_-), (\lambda_\bullet, \mu, \mu^*))$$

are isomorphic. We are actually going to exhibit an isomorphism between these spaces. We will need some preparation first.

6.0.3. As we have already observed in the previous chapter, given two representations M and V of $\mathfrak{g}$, the space $\mathrm{Hom}_{\mathbb{C}}(V, M)$ is naturally a $\mathfrak{g}$-representation. If $\mathfrak{g}_1$ and $\mathfrak{g}_1$ are two Lie algebras and V_1 and V_2 are representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, we can see $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ as a representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ by setting

$$(X, Y)(f)(v) = -f(X(v)) + Y(f(v)).$$

Furthermore, if $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$, then $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ can then be seen as a representation of $\mathfrak{g} \oplus \mathfrak{g}$, or as a representation of $\mathfrak{g}$ only. These are compatible by viewing $\mathfrak{g}$ as a sub Lie algebra of $\mathfrak{g} \oplus \mathfrak{g}$ via the diagonal embedding $\mathfrak{g} \ni X \mapsto (X, X) \in \mathfrak{g} \oplus \mathfrak{g}$.

6.0.4. For every $\mu \in P_\ell^+$, we denote by $\mathrm{Id}_\mu \in V_\mu \otimes V_{\mu*} = \mathrm{Hom}_{\mathbb{C}}(V_\mu, V_\mu)$ the identity element. Note that the action of $\mathfrak{g}$ on $V_\mu \otimes V_{\mu*}$ induces the trivial action of $\mathfrak{g}$ on Id_μ.

Let

$$\iota_\mu \colon V_\mu \otimes V_{\mu*} \to \mathcal{H}_\mu \otimes \mathcal{H}_{\mu*}$$

be the inclusion map, and define the map

$$F_\mu \colon \mathcal{H}_{\lambda_\bullet} \to \mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu*} \to \mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu \otimes \mathcal{H}_{\mu*}$$

by $F(h) = h \otimes \iota(\mu)(\mathrm{Id}_\mu)$ for every $h \in \mathcal{H}(\lambda)$.

Theorem 6.0.5. *The map $F \colon \mathcal{H}_{\lambda_\bullet} \to \bigoplus_{\mu \in P_\ell^+} \mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu \otimes \mathcal{H}_{\mu*}$ induces the isomorphism*

$$[F] \colon \mathbb{V}_C(P_\bullet, \lambda_\bullet) \to \bigoplus_{\mu \in P_\ell^+} \mathbb{V}_{\widetilde{C}}((P_\bullet, Q_+, Q_-), (\lambda_\bullet, \mu, \mu^*)).$$

Proof. We first of all observe that F arises as the composition of the maps

$$\mathcal{H}_{\lambda_\bullet} \xrightarrow{\oplus I_\mu} \bigoplus_{\mu \in P_\ell^+} \mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*} \xrightarrow{\oplus \iota_\mu} \bigoplus_{\mu \in P_\ell^+} \mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*},$$

where I_μ is the map $\otimes \mathrm{Id}_\mu$. Note also that Theorem 5.0.2 guarantees that

$$[\mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*}]_{\mathfrak{g}_{\widetilde{C} \setminus P_\bullet}} \xrightarrow{\oplus [\iota_\mu]} [\mathcal{H}_{\lambda_\bullet} \otimes \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}]_{\mathfrak{g}_{\widetilde{C} \setminus P_\bullet, Q_+, Q_-}}$$

is an isomorphism for every $\mu \in P_\ell^+$. This implies that the theorem is equivalent to the map

$$[\mathcal{H}_{\lambda_\bullet}]_{\mathfrak{g}_{C \setminus P_\bullet}} \xrightarrow{\oplus [I_\mu]} \bigoplus_{\mu \in P_\ell^+} [\mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*}]_{\mathfrak{g}_{\widetilde{C} \setminus P_\bullet}}$$

being an isomorphism. In what follows we will denote by I the map $\oplus I_\mu$ and $[I] = \oplus [I_\mu]$.

$[I]$ **is well defined.** It is enough to show that given $h \in \mathcal{H}_{\lambda_\bullet}$ and $Xf \in \mathfrak{g}_{C \setminus P_\bullet}$, we have that $I_\mu(Xf(h))$ lies in the image of the action of $\mathfrak{g}_{\widetilde{C} \setminus P_\bullet}$ on $\mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*}$.

Claim A. We claim that $I_\mu(Xf(h)) = Xf(I_\mu(h))$.

We first of all use Remark 6.0.2 to deduce that $\mathfrak{g}_{C\setminus P_\bullet}$ is a Lie subalgebra of $\mathfrak{g}_{\widetilde{C}\setminus P_\bullet}$, so it makes sense to write $Xf(I_\mu(h))$. By definition we have

$$Xf(I_\mu(h)) = Xf(h \otimes \mathrm{Id}_\mu) = Xf(h) \otimes \mathrm{Id}_\mu + h \otimes Xf(\mathrm{Id}_\mu) = I_\mu(Xf(h)) + h \otimes Xf(\mathrm{Id}_\mu).$$

We then need to prove that $Xf(\mathrm{Id}_\mu) = 0$. If we write $\mathrm{Id}_\mu = e_i \otimes \epsilon^i$, then

It should really be a sum over i, but we can use Einstein notation here

$$Xf(\mathrm{Id}_\mu) = f(Q_+)X(e_i)\otimes\epsilon^i + e_i\otimes f(Q_-)X(\epsilon^i) = f(Q)(X(e_i)\otimes\epsilon^i + e_i\otimes X(\epsilon^i)) = f(Q)X(\mathrm{Id}_\mu) = 0,$$

where again we have used Remark 6.0.2, and where the last equality holds because $\mathrm{Id}_\mu \in \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_\mu)$. Note that using (6.1), we could have deduced this result without writing Id_μ explicitly.

Surjectivity of $[I]$. We need to show that every element in $\mathcal{H}_{\lambda_\bullet} \otimes \bigoplus_{\mu\in P_\ell^+} V_\mu \otimes V_{\mu^*}$ is equivalent, up to an element lying in the image of $\mathfrak{g}_{\widetilde{C}\setminus P_\bullet}$, to an element of the form $I(h)$ for some h in $\mathcal{H}_{\lambda_\bullet}$. To do so, we observe that for every $Xf \in \mathfrak{g}_{\widetilde{C}\setminus P_\bullet}$ and $h\otimes v \in \mathcal{H}(\lambda_\bullet)\otimes\bigoplus_{\mu\in P_\ell^+} V_\mu\otimes V_{\mu^*}$ we have

$$Xf(h)\otimes v = -h\otimes Xf(v).$$

We would be done if we could show that every element of $\bigoplus_{\mu\in P_\ell^+} V_\mu \otimes V_{\mu^*}$ could be written as an element of the form $X_1f_1 \circ \cdots \circ X_rf_n(\sum \mathrm{Id}_\mu)$ for some $X_if_i \in \mathfrak{g}_{\widetilde{C}\setminus P_\bullet}$.

Claim B. It is enough to show that the map

$$\beta\colon U(\mathfrak{g}) \longrightarrow \bigoplus_{\mu\in P_\ell^+} V_\mu \otimes V_{\mu^*}$$

defined by $X \mapsto (X,0)(\sum \mathrm{Id}_\mu)$ is surjective (here we see $V_\mu \otimes V_{\mu^*}$ as a $\mathfrak{g}\oplus\mathfrak{g}$-representation as in 6.0.3).

This mean every element of $\bigoplus_{\mu\in P_\ell^+} V_\mu \otimes V_{\mu^*}$ can be written as an element of the form $X_1f_1 \circ \cdots \circ X_rf_n(\sum \mathrm{Id}_\mu)$ for some $X_if \in \mathfrak{g}_{\widetilde{C}\setminus P_\bullet}$ where we further require that $f(Q_-) = 0$ and $f(Q_+) = 1$. Note that the existence of f is guaranteed by Riemann-Roch.

We are then left to prove that the map β is surjective. This follows from the following observations.

- $V_\mu \otimes V_{\mu^*}$ is an irreducible $(\mathfrak{g}\oplus\mathfrak{g})$-module;
- The image of β is stable under the action of $\mathfrak{g}\oplus\mathfrak{g}$;
- The image of β non trivially intersects $V_\mu \otimes V_{\mu^*}$ for every μ.

Injectivity. Instead of showing that the map $[I]$ is injective directly, we will construct an inverse map (up to constant). Let $b_\mu\colon V_\mu \otimes V_{\mu*} \to \mathbb{C}$ be the trace map or, equivalently the evaluation map. Observe that, up to scalars, this is the only map $V_\mu \otimes V_{\mu*} \to \mathbb{C}$ which is $\mathfrak{g}$-equivariant, i.e. satisfying $b(X(v\otimes\alpha)) = 0$ (since $\mathfrak{g}$ acts trivially on $\mathbb{C}$).

The map $b := \oplus \mathrm{Id}_{\mathcal{H}_{\lambda_\bullet}} \otimes b_\mu$ defines a map

$$\bigoplus_{\mu \in P_\ell^+} \mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu*} \longrightarrow \mathcal{H}_\lambda$$

and we aim to show that this defines a map between the coinvariants

$$[b] := \sum [\mathrm{Id}_{\mathcal{H}_{\lambda_\bullet}} \otimes b_\mu] \colon \bigoplus_{\mu \in P_\ell^+} [\mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*}]_{\mathfrak{g}_{\widetilde{C} \setminus P_\bullet}} \longrightarrow [\mathcal{H}_\lambda]_{\mathfrak{g}_{C \setminus P_\bullet}}$$

and that realizes the inverse of $[I]$.

In order to show that it is well defined, we proceed in steps and, from (6.1) we deduce the following diagram of Lie algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \otimes \mathfrak{I}_Q \arrow[r] \arrow[d, "="] & \mathfrak{g}_{C \setminus P_\bullet} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, "\Delta"] & 0 \\ 0 \arrow[r] & \mathfrak{g} \otimes \mathfrak{I}_Q \arrow[r] & \mathfrak{g}_{\widetilde{C} \setminus P_\bullet} \arrow[r] & \mathfrak{g} \oplus \mathfrak{g} \arrow[r] & 0, \end{tikzcd} \tag{6.2}$$

where $\mathfrak{g} \otimes \mathfrak{I}$ is an ideal of both $\mathfrak{g}_{C \setminus P_\bullet}$ and $\mathfrak{g}_{\widetilde{C} \setminus P_\bullet}$.

Claim C. The map $[b]_{\mathfrak{g} \otimes \mathfrak{I}_Q}$ is well defined.

The Lie algebra $\mathfrak{g} \otimes \mathfrak{I}_Q$ acts trivially on $V_\mu \otimes V_{\mu^*}$ for every μ, so we have that

$$\bigoplus_{\mu \in P_\ell^+} [\mathcal{H}_{\lambda_\bullet} \otimes V_\mu \otimes V_{\mu^*}]_{\mathfrak{g} \otimes \mathfrak{I}_Q} = \bigoplus_{\mu \in P_\ell^+} [\mathcal{H}_{\lambda_\bullet}]_{\mathfrak{g} \otimes \mathfrak{I}_Q} \otimes V_\mu \otimes V_{\mu^*}$$

concluding the proof of the claim.

We can now use Lemma 6.0.6 with $M^* = [\mathcal{H}_{\lambda_\bullet}]_{\mathfrak{g} \otimes \mathfrak{I}_Q}$ and $N = V_\mu \otimes V_{\mu^*}$, to conclude that the map $[\mathrm{Id}_{\mathcal{H}_{\lambda_\bullet}} \otimes b_\mu]$ is well defined for every μ, and so also $[b]$ is well defined.

To conclude we are left to show that the composition $[b] \circ [I]$ is injective. Since

$$[b][I](h) = [b]\Big(\sum_{\mu \in P_\ell^+} h \otimes \mathrm{Id}_\mu\Big) = \sum_\mu h(\dim V_\mu) = \Big(\sum_{\mu \in P_\ell^+} \dim_\mu\Big) h$$

and $(\sum_{\mu \in P_\ell^+} \dim_\mu) \neq 0$ we are done. □

Lemma 6.0.6. *Let M and N be two $\mathfrak{g} \oplus \mathfrak{g}$-representations and via $\delta\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, they can be seen as representations for $\mathfrak{g}$ as well. Let $b \colon N \to \mathbb{C}$ be a $\mathfrak{g}$-equivariant map. View $\mathrm{Hom}_{\mathbb{C}}(M, N)$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module and $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ as a $\mathfrak{g}$-module. Then the map*

$$b \circ - \colon \mathrm{Hom}_{\mathbb{C}}(M, N) \to \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$$

induces a well defined map

$$[b \circ -] \colon [\mathrm{Hom}_{\mathbb{C}}(M, N)]_{\mathfrak{g} \oplus \mathfrak{g}} \to [\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})]_{\mathfrak{g}}.$$

Proof. I leave the proof to the reader. □

Exercise 6.0.7. Compute the dimension of $\mathbb{V}_C(P, 0)$ where C is the nodal curve of equation $zy^2 = x^2(x - z)$ inside $\mathbb{P}^2$.

Add computations on $\mathbb{P}^1$ with 4 marked points and note that we always have the same dimension

7 Sheaves of covacua

We have seen how the dimension of the space of coinvariants attached to a rational curve marked by 4 points (2 of them marked trivially) is constant. We will see that this is not a coincidence, but that indeed the dimension of the spaces of coinvariants only depend on the genus of the underlying curve and the representations input. In order to do so, it is convenient to describe spaces of covacua in *families*. More precisely, given a family of curves over a base scheme B, we will be construct here a sheaf over B whose fibers coincide with the spaces of covacua previously defined. Before delving into the definition and properties, I suggest [ACGH85, ACG11] to know more about moduli of curves.

7.0.1. Throughout we will consider the data $(\mathcal{C} \to B, P_\bullet)$ satisfying the following conditions:

(1) $\pi\colon \mathcal{C} \to B$ is a proper, flat morphism over a smooth and irreducible scheme S such that every fiber is a reduced, projective curve with at worst nodal singularities;

(2) The map π has n disjoint sections $P_1, \ldots, P_n$ landing on the smooth locus of π.

(3) Assume that the restricion of π to the open subset $\mathcal{C} \setminus \sqcup P_i(B)$ is an affine map.

(4) Assume that for every i there exists an isomorphism of $\mathcal{O}_B$-modules

$$\widehat{\mathcal{O}}_{\mathcal{C},P_i(B)} \cong \mathcal{O}_B[\![t_i]\!],$$

where $\widehat{\mathcal{O}}_{\mathcal{C},P_i(B)}$ is the completion of $\mathcal{O}_{\mathcal{C}}$ along the closed subscheme $P_i(B)$.

We note that given $(\mathcal{C} \to B, P_\bullet)$ satisfying only (1) and (2), it is not true that also (3) and (4) hold true. What is true, however, is that there exists a covering $B' \to B$ such that the pullback of $(\mathcal{C} \to B, P_\bullet)$ to B' will satisfy (3) and (4).

Remark 7.0.2. Note that since π is flat, then all the curves will have the same (arithmetic) genus.

7.0.3. Let $(\mathcal{C} \to B, P_\bullet)$ as above and fix also $(\mathfrak{g}, \ell, \lambda_\bullet)$ with $\lambda_i \in P_\ell^+$. We construct the sheaf of covacua over B associated with these data following the following steps: first we construct the analogue of $\mathcal{H}(\lambda_\bullet)$ and then a Lie algebra analogue to $\mathfrak{g}_{C\setminus P}$ acting on it.

7.0.4. We define $\mathcal{H}_B(\lambda_\bullet)$ to be the constant sheaf over B given by $\mathcal{O}_B \otimes_{\mathbb{C}} \mathcal{H}(\lambda_\bullet)$. Equivalently we can define $\mathcal{H}_B(\lambda_\bullet)$ as being the representation of

$$\widehat{\mathfrak{g}}_{P_\bullet} := \bigoplus_{i=1}^n \mathfrak{g} \otimes \mathcal{O}_B(\!(t_i)\!) \oplus \mathcal{O}_B C,$$

which is induced by $\bigotimes V_\lambda$ and on which the central element C acts by the scalar ℓ.

Remark 7.0.5. Given R a $\mathbb{C}-algebra$, it is not true, in general that $R((t)) = \mathbb{C}((t)) \otimes_{\mathbb{C}} R$ or $R[\![t]\!] = \mathbb{C}[\![t]\!] \otimes_{\mathbb{C}} R$.

7.0.6. We define $\mathfrak{g}_{\mathcal{C}\setminus P_\bullet}$ to be $\mathfrak{g} \otimes \pi_* j_* j^* \mathcal{O}_{\mathcal{C}}$, where j is the open map $\mathcal{C} \setminus \sqcup P_i(B) \to \mathcal{C}$. Note that if $B = \operatorname{Spec}(R)$, then $\mathcal{C} \setminus \sqcup P_i(B)$ will be equal to $\operatorname{Spec}(A)$ for some R-algebra A, and so $\mathfrak{g}_{\mathcal{C}\setminus P_\bullet} = \mathfrak{g} \otimes A$, which we now see as an R-module rather than as an A-module. Note that there is a canonical map $\mathfrak{g}_{\mathcal{C}\setminus P_\bullet} \to \widehat{\mathfrak{g}}_{P_\bullet}$ and so there is a well defined map of sheaves

$$\theta \colon \mathfrak{g}_{\mathcal{C}\setminus P_\bullet} \otimes \mathcal{H}_B(\lambda_\bullet) \to \mathcal{H}_B(\lambda_\bullet).$$

Definition 7.0.7. We define the sheaf of covacua associated with $(\mathcal{C} \to B, P_\bullet)$ and $(\mathfrak{g}, \ell, \lambda_\bullet)$ to be the sheaf of $\mathcal{O}_B$-modules given by

$$\mathbb{V}_{\mathcal{C}\to B}(P_\bullet, \lambda_\bullet) = \operatorname{coker}(\theta) = \frac{\mathcal{H}_B(\lambda_\bullet)}{\mathfrak{g}_{\mathcal{C}\setminus P_\bullet}(\mathcal{H}_B(\lambda_\bullet))}.$$

The sheaf of conformal blocks is defined to the the dual $\mathcal{O}_B$-module of $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet, \lambda_\bullet)$.

Proposition 7.0.8. *The sheaf of covacua $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet, \lambda_\bullet)$ is a coherent sheaf over B.*

Proof. This is a local statement, so we can assume that $B = \operatorname{Spec}(R)$. The argument is similar to that of Theorem 4.2.4, in the sense that this will follow from Lemma 4.2.3. In order to check that we are in the assumption of that lemma, we are only left to check that $\mathcal{O}_{\mathcal{C}}(\mathcal{C}\setminus P_\bullet) + \bigoplus_{i=1}^n R[\![t_i]\!]$ has finite codimension inside $\bigoplus_{i=1}^n R((t_i))$. This follows from the fact that the map π restricted to $\mathcal{C}\setminus P_\bullet$ is affine, which implies that $\bigoplus_{i=1}^n R((t_i))/(\mathcal{O}_{\mathcal{C}}(\mathcal{C}\setminus P_\bullet) + \bigoplus_{i=1}^n R[\![t_i]\!])$ is isomorphic to $R^1\pi_*(\mathcal{O}_{\mathcal{C}})$ (see also [Loo13]). □

7.1 The stack of genus g curves

An object heavily studied in geometry is $\overline{\mathcal{M}}_{g,n}$, which can be seen as the *space* which parametrizes family of pointed curves. This means that a map $B \to \overline{\mathcal{M}}_{g,n}$ is equivalent to the data of $(\pi \colon \mathcal{C} \to B, P_1, \dots, P_n)$ such that:

(1) $\pi \colon \mathcal{C} \to B$ is a proper, flat morphism over a smooth and irreducible scheme S such that every fiber is a reduced, projective curve with at worst nodal singularities;

(2) The map π has n disjoint sections $P_1, \dots, P_n$ landing on the smooth locus of π.

(S) Every fiber has finitely many automorphisms.

Condition (S) is called a stability condition and, although very important, we will not focus on it in this course. This implies, but not equivalent, to the fact that if $g = 0$, then $n \geq 3$ and if $g = 1$ then $n \geq 1$.

7.1.1. The reason why I write *space* is because we $\overline{\mathcal{M}}_{g,n}$ is not a scheme, but a stack. However, this is what's called a smooth Deligne-Mumford stack, which means that, locally étale, it behaves like a smooth scheme. More explicitly, one can show that there exists a smooth scheme U and an étale and surjective map $U \to \overline{\mathcal{M}}_{g,n}$. The dimension of U is $3g - 3 + n$ and we call this the dimension of $\overline{\mathcal{M}}_{g,n}$.

7.1.2. The *subspace* of $\overline{\mathcal{M}}_{g,n}$ parametrizing only smooth curves is denoted by $\mathcal{M}_{g,n}$. It corresponds to an open subscheme of U and indeed we will see that $\overline{\mathcal{M}}_{g,n}$ is a space which is stratified by the number of nodes that curves are allowed to have.

7.1.1 Sheaves of covacua over $\overline{\mathcal{M}}_{g,n}$

In order to define a sheaf $\mathbb{V}$ over a stack like $\overline{\mathcal{M}}_{g,n}$, it is enough to give

(a) For every $b\colon B \to \overline{\mathcal{M}}_{g,n}$ a sheaf $\mathbb{V}(b)$ over B;

(b) An isomorphism $\gamma^*\mathbb{V}(b) \cong \beta^*\mathbb{V}(c)$, where

$$\begin{tikzcd} B \times_{\overline{\mathcal{M}}_{g,n}} C \arrow[r, "\gamma"] \arrow[d, "\beta"] & B \arrow[d, "b"] \\ C \arrow[r, "c"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

satisfying a cocycle condition.

This should remind you of how one defines a sheaf on a scheme from sheaves on an open cover and gluing data.

7.1.3. If we want to define the sheaf of covacua on $\overline{\mathcal{M}}_{g,n}$ associated with $(\mathfrak{g}, \ell, \lambda_1, \dots, \lambda_n)$, we then first need to define a sheaf on B for every map $b\colon B \to \overline{\mathcal{M}}_{g,n}$. As described above, such a map corresponds to the data of $(\pi\colon \mathcal{C} \to B, P_1, \dots, P_n)$ satisfying (1) and (2) above. We saw in 7.0.1 that in order to define the sheaf of covacua for families of curves, we also need conditions (3) ($C \setminus P_\bullet(B)$ is affine over B) and (4) (existence of coordinates at $P_\bullet$). As already noticed, although it is not true that these conditions hold, there exists an étale map $B' \to B$ such that

(III) There exists sections $Q_1, \dots, Q_m of \pi'\colon C' \to B$ (where $C' = C \times_B B'$) which are disjoint from each other and from $P'_\bullet$ and such that $C' \setminus (P_\bullet, Q_\star)$ is affine over B'.

(IV) There exists coordinates at $P_\bullet$ and $Q_\star$.

We then define the sheaf $\mathbb{V}_{C'/B'}(\lambda_\bullet; P_\bullet)$ over B' as being given by

$$\mathbb{V}_{C'/B'}(P_\bullet, Q_\star; \lambda_\bullet, 0_\star).$$

7.1.4. This defines a sheaf over every B' which is independent of the choice of the coordinates made at the marked points. A priori this definition depends on the choice of the points $Q_\star$, but propagation of vacua provides a canonical isomorphism between the sheaves that are obtained with different choices of the extra points Qs. This means that this construction indeed defines a sheaf over B.

7.1.5. To show that indeed this defines a sheaf over $\overline{\mathcal{M}}_{g,n}$, one needs to repeat this construction for every scheme B and for every map $B \to \overline{\mathcal{M}}_{g,n}$ and exhibit isomorphisms as in condition (b). This is done similarly to what we have just discussed.

Definition 7.1.6. For every $(\mathfrak{g}, \ell, \lambda_\bullet)$, we denote by $\mathbb{V}_g(\lambda_\bullet)$ the sheaf of covacua on $\overline{\mathcal{M}}_{g,n}$ which we have just constructed.

8 Flat projective connection

We show in this section that the sheaf of covacua $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet,\lambda)$ attached to a family of smooth curves $\mathcal{C}$ over a smooth base B can be seen as a representation of an algebra $\hat{T}$ which surjects onto the tangent bundle $\mathcal{T}_B$ of B. Actually one can prove that $\hat{T}$ acts on $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet,\lambda)$ through a central extension of $\mathcal{T}_B$, defining in this way a projectively flat connection on the sheaf of covacua. The main upshot is to combine the Leibniz property of the action of $\hat{T}$ and the coherence of the sheaves of vacua $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet,\lambda)$ to show that $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet,\lambda)$ is a locally free sheaf over B. We will begin with some definitions.

8.1 What is a connection?

I will recall here what we mean by a connection. Let R be a smooth $\mathbb{C}$-algebra. Let $\mathrm{Der}_{R/\mathbb{C}}$ denote the R-module of $\mathbb{C}$-linear derivations of R, that is

$$\mathrm{Der}_{R/\mathbb{C}} := \{\theta \in \mathrm{Hom}_{\mathbb{C}}(R,R) \text{ such that } \theta(ab) = \theta(a)b + a\theta(b) \text{ for all } a,b\in R\}$$

We denote by $\Omega_{R/\mathbb{C}}$ the module of Kähler differetials of R over $\mathbb{C}$. Recall that there is a universal derivation $d\colon R \to \Omega_{R/\mathbb{C}}$.

Let M be an R-module.

Definition 8.1.1. A *connection* on M is a $\mathbb{C}$-linear map $\nabla\colon M \to M\otimes_R \Omega_{R/\mathbb{C}}$ such that $\nabla(am) = m\otimes da + a\nabla(m)$ for all $m\in M$ and $a\in R$. Equivalently a*connection* on M is an R-linear map $\nabla\colon \mathrm{Der}_{R/\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(M,M)$ such that $\nabla_\theta(am) = \theta(a)m + a\nabla_\theta(m)$ for every $m\in M$, $a\in R$ and $\theta\in\mathrm{Der}_{R/\mathbb{C}}$.

Although in algebraic geometry it is more common to use the first definition above, we will instead use the second one. Note that $\mathrm{Der}_{R/\mathbb{C}}$ is naturally a Lie algebra, with $[\theta_1,\theta_2] = \theta_1\theta_2 - \theta_2\theta_1$. Similarly, also $\mathrm{Hom}_{\mathbb{C}}(M,M)$ is a Lie algebra through the commutator.

Definition 8.1.2. A connection ∇ on M is *flat* if ∇ is a map of Lie algebras. Equivalently $\nabla_{[\theta_1,\theta_2]} - [\nabla_{\theta_1},\nabla_{\theta_2}]$ (which is called the curvature of ∇) is the zero map

Let S be a smooth scheme over $\mathbb{C}$ and $\mathcal{F}$ a a quasi-coherent $\mathcal{O}_S$-module. Recall that if $U = \mathrm{Spec}(R)$ is an open subscheme of S, then $\mathcal{T}_S(U) = \mathrm{Der}_{R/\mathbb{C}}$.

Definition 8.1.3. A *connection* on $\mathcal{F}$ is a map $\nabla\colon \mathcal{T}_S\otimes\mathcal{F}\to\mathcal{F}$ which is locally a connection as in Definition 8.1.1. Equivalently it is the data of connections $\nabla^i\colon \mathrm{Der}_{R_i/\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_i,\mathcal{F}_i)$, for an open covering $\{U_i = \mathrm{Spec}(R_i)\}$ of S, such that $\nabla^i|_{U_{ij}} = \nabla^j|_{U_{ij}}$.

When the curvature of every ∇^i are zero, then the connection is called flat, so that giving a flat connection on $\mathcal{F}$ is the same as making $\mathcal{F}$ a D-module.

Definition 8.1.4. A *projective connection* on $\mathcal{F}$ is the data of connections $\nabla^i \colon \mathrm{Der}_{R_i/\mathbb{C}} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_i, \mathcal{F}_i)$, for an open covering $\{U_i = \mathrm{Spec}(R_i)\}$ of S, such that for every $p \in U_{ij}$ and $\theta \in \mathcal{T}_{S,p}$ one has $\nabla^i(\theta) - \nabla^j(\theta) = k(p)\mathrm{Id}_{\mathcal{F}_p}$ for some $k(p) \in \mathbb{C}$.

Definition 8.1.5. A *flat projective connection* is a projective connection such that the curvature of ∇^i is a scalar operators for every i.

Equivalently, a *flat projective connection* on $\mathcal{F}$ is the data of a central extension of Lie algebras

$$0 \to L \to \Theta \to \mathcal{T}_S \to 0$$

with L a line bundle on S together with an action of Θ on $\mathcal{F}$ (e.g. we see $\mathcal{F}$ as being a representation of the Lie algebra Θ).

8.2 Sugawara representation

In order to construct $\hat{T}$ and its action on $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet, \lambda)$, we will need some preparation. Throughout, everytime R is a $\mathbb{C}$-algebra, we will assume that $\mathrm{Spec}(R)$ is a smooth scheme.

8.2.1. We first of all complete the universal enveloping algebra $U\widehat{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$. We define

$$\bar{U}\widehat{\mathfrak{g}} = \varprojlim \frac{U\widehat{\mathfrak{g}}}{U\widehat{\mathfrak{g}} \circ \mathfrak{g}t^n[\![t]\!]}.$$

Note that the element $\sum_{i\geq 0} Xt^{-i} \circ Yt^i$ did not belong to $U\widehat{\mathfrak{g}}$, but does live in $\bar{U}\widehat{\mathfrak{g}}$. Similarly, we can do this construction starting with $U\widehat{\mathfrak{g}}_R$, where $\widehat{\mathfrak{g}}_R = \widehat{\mathfrak{g}}\hat{\otimes}R = \mathfrak{g} \otimes R(\!(t)\!) \oplus RK$.

Note that $\bar{U}\widehat{\mathfrak{g}}$ naturally acts on $\mathcal{H}(\lambda)$ by left multiplication and similarly $\bar{U}\widehat{\mathfrak{g}}_R$ acts on $R \otimes \mathcal{H}(\lambda) = \mathcal{H}_R(\lambda)$.

8.2.2. The Virasoro algebra Vir_R (over R) is the central extension

$$0 \to cR \to \mathrm{Vir}_R \to R(\!(t)\!)d/dt \to 0$$

with the bracket given by

$$[L_p, L_q] = (p-q)L_{p+q} + c\frac{p^3-p}{12}\delta_{p+q,0},$$

where we have used the notation $L_p := -t^{p+1}d/dt$.

We now want to see Vir_R inside $\bar{U}\widehat{\mathfrak{g}}_R$—or more precisely inside $\bar{U}\widehat{\mathfrak{g}}_R[(K+\check{h})^{-1}]$. Before constructing such an injection, we will need some notation

- We denote by $\{X_i\}$ and $\{X^j\}$ two basis of $\mathfrak{g}$ such that $(X_j|X^k) = \delta_{j,k}$. One can (and we will) actually choose such elements so that $X_i = X^i$ for every i. The element $\sum X_j \circ X_j \in U\mathfrak{g}$ is called the Casimir element of $\mathfrak{g}$ and acts on every irreducible representation of $\mathfrak{g}$ by a scalar. By definition, it acts on the adjoint representation of $\mathfrak{g}$ by the scalar $2\check{h}$.

- We define
$$: Xt^a \circ Yt^b := \begin{cases} Xt^a \circ Yt^b & \text{if } a \leq b \\ Yt^b \circ Xt^a & \text{if } a > b \end{cases}$$
for every $X, Y \in \mathfrak{g}$ and $a, b \in \mathbb{Z}$. This allows us to concisely move to the right the "most positive element" and to the left the "most negative element".

We now have all the terminology needed to state the following result.

Theorem 8.2.3. *The map* $T\colon \mathrm{Vir}_R \longrightarrow \bar{U}\mathfrak{g}_R[(K+\check{h})^{-1}]$ *given by*
$$L_p \mapsto \frac{1}{2(K+\check{h})} \sum_{i\in\mathbb{Z}} \sum_{j=1}^{\dim(\mathfrak{g})} : X_j t^{-i} \circ X_j t^{p+i} :$$
and
$$c \mapsto \frac{K \dim(\mathfrak{g})}{2(K+\check{h})}$$
is a map of Lie algebras, i.e. $T[\theta_1, \theta_2] = T(\theta_1) \circ T(\theta_2) - T(\theta_2) \circ T(\theta_1)$ *for every* $\theta_1, \theta_2 \in \mathrm{Vir}_R$.

8.2.4. Note that in $\bar{U}\widehat{\mathfrak{g}}_R$ we need to impose that the element $K + \check{h}$ is actually invertible. However, in the appliations that we are going to use, K will always be identified with a positive integer ℓ (the level), so that $\ell + \check{h}$ will automatically be a non zero element of $\mathbb{Z}$, and so invertible. The main application of the above theorem is the following.

Corollary 8.2.5. *We have defined an action of the Virasoro algebra* Vir_R *on* $\mathcal{H}_R(\lambda)$*. Similarly* $\bigoplus \mathrm{Vir}_R$ *acts on* $\mathcal{H}_R(\lambda_\bullet)$*.*

Remark 8.2.6. Note that the action of Vir_R on $\mathcal{H}_R(\lambda)$ is R-linear, i.e. it is induced by the action of $\mathrm{Vir}_{\mathbb{C}}$ on $\mathcal{H}(\lambda)$.

Note that what's discussed so far didn't use any geometry, so we cannot expect that this action will descend to an action on the space of vacua.

8.3 Some geometry

In order to define $\hat{T}$, we will relate the tangent bundle of $\mathcal{C}$, with the tangent bundle of B. In order to do this, let's set up some notation and terminology.

1. Throughout $(\pi\colon \mathcal{C} \to B, P)$ is a family of smooth curves over a smooth base scheme $B = \mathrm{Spec}(R)$, which is marked satisfying all the assumptions of 7.0.1 with $n = 1$ (this last assumption is only for simplicity).

2. Denote by $D_P^\times$ the formal punctured neighbourhood of $P(B)$ inside $\mathcal{C}$, which is isomorphic to $\mathrm{Spec}(L) = \mathrm{Spec}(R(\!(t)\!))$.

3. We will use the notation $\mathcal{C}^\circ = \mathcal{C} \setminus P(B)$ and $\pi^\circ\colon \mathcal{C}^\circ \to B$ the restriction of π to $\mathcal{C}$, which is affine by assumptions. Hence $\mathcal{C}^\circ = \mathrm{Spec}(A)$ for A a smooth R-algebra.

We will use the following notation:

A-modules	R-modules
$\mathcal{T}_A = \mathrm{Der}_{\mathbb{C}}(A)$: tangent bundle of $\mathcal{C}^\circ$ (relative to $\mathbb{C}$)	$\theta_A := \pi^\circ_* \mathcal{T}_A$
	$\theta_{A,R}$: R-module of derivations of A which are $\mathbb{C}$-linear and which restrict to derivations $R \to R$.
$\mathcal{T}_{A/R} = \mathrm{Der}_R(A)$: tangent bundle of $\mathcal{C}^\circ$ relative to B	$\theta_{A/R} := \pi^\circ_* \mathcal{T}_{A/R}$

L-modules	R-modules
$\mathcal{T}_L$: tangent bundle of $D_P^\times$ (relative to $\mathbb{C}$)	$\theta_L := \pi^\circ_* \mathcal{T}_L$
	$\theta_{L,R}$: R-module of derivations of L which are $\mathbb{C}$-linear and which restrict to derivations $R \to R$.
$\mathcal{T}_{L/R} = R(\!(t)\!)d/dt$: tangent bundle of $D_P^\times$ relative to B	$\theta_{L/R} := \pi^\circ_* \mathcal{T}_{L/R}$

Lemma 8.3.1. *The map π° induces an exact sequence of Lie algebras over R*

$$0 \to \theta_{A/R} \to \theta_{A,R} \to \mathrm{Der}_{\mathbb{C}}(R) \to 0$$

Proof. Since the map π° is smooth we have the short exact sequence of A-modules

$$0 \to \mathcal{T}_{A/R} \to \mathcal{T}_A \to \pi^{\circ *}\mathrm{Der}_{\mathbb{C}}(R) \to 0.$$

Since π° is affine, we can push the above s.e.s. obtaining another s.e.s. of R-modules

$$0 \to \theta_{A/R} \to \theta_{A,R} \to \pi^\circ_* \pi^{\circ *}\mathrm{Der}_{\mathbb{C}}(R) \to 0.$$

Pulling back along the inclusion $\mathrm{Der}_{\mathbb{C}}(R) \to \pi^\circ_* \pi^{\circ *}\mathrm{Der}_{\mathbb{C}}(R)$ we obtain the exact sequence

$$0 \to \theta_{A/R} \to \theta_{A,R} \to \mathrm{Der}_{\mathbb{C}}(R) \to 0$$

as claimed. □

8.3.2. The upshot is that for every $\partial \in \mathrm{Der}_{\mathbb{C}}(R)$, we can choose (at least) one element $\theta \in \theta_{A,R}$ such that $\overline{\theta} = \partial$.

Lemma 8.3.3. *The exact sequence*

$$0 \to \theta_{L/R} \to \theta_{L,R} \to \mathrm{Der}_{\mathbb{C}}(R) \to 0$$

splits (not canonically).

Proof. It is enough to define a section of the surjection $p\colon \theta_{L,R} \to \mathrm{Der}_{\mathbb{C}}(R)$. We define the section $\sigma_t\colon \mathrm{Der}_{\mathbb{C}}(R) \to \theta_{L,R}$ by the assignments

$$\sigma_t(\partial)(a) = \partial(a) \text{ for every } a \in R \text{ and } \sigma_t(\partial)(t) = 0.$$

This clearly is a section of p and it is non canonical since it depends on the choice of a local coordinate t at P. □

8.3.4. In view of the above lemma, we will write every element $\theta \in \theta_{L,R}$ as $\theta^h + \theta^v$ with $\theta^h \in \mathrm{Der}_{\mathbb{C}}(R)$ and $\theta^v \in \theta_{L,R}$ characterized by the property that $\theta^h(t) = 0$. This implies that the central extension $\mathrm{Vir}_R \to \theta_{L/R}$ induces a central extension $\mathsf{VIR}_R \to \theta_{L,R}$. We have seen how Vir_R is identified with a subspace of $\bar{U}\widehat{\mathfrak{g}}_R$, and hence identified with a set of endomorphisms of $\mathcal{H}_R(\lambda)$. We can give a similar description of VIR_R: as vector space, VIR_R is identified with $\mathrm{Vir}_R \oplus \mathrm{Der}_{\mathbb{C}}(R)$, where Vir_R acts on $\mathcal{H}_R(\lambda)$ as in Corollary 8.2.5, and $\mathrm{Der}_{\mathbb{C}}(R)$ acts on $\mathcal{H}_R(\lambda)$ by derivations. That is for every $\partial \in \mathrm{Der}_{\mathbb{C}}(R)$ and $a \otimes h \in R \otimes \mathcal{H}(\lambda)$ we have $\partial(a \otimes h) = \partial(a) \otimes h$.

8.3.5. Note also that there are maps $\theta_{A/R} \to \theta_{L/R}$ and $\theta_{A,R} \to \theta_{L,R}$ compatible with the exact sequences described in the above lemmas.

Definition 8.3.6. We define $\hat{T}$ to be the Lie subalgebra of VIR_R generated by the image of $\theta_{A,R}$ inside $\theta_{L,R}$.

Note that since we have identified VIR_R with an algebra of endomorphisms of $\mathcal{H}_R(\lambda)$, also $\hat{T}$ has a natural action on $\mathcal{H}_R(\lambda)$.

Theorem 8.3.7. *The action of $\hat{T}$ on $\mathcal{H}_R(\lambda)$*

(a) satisfies the condition

$$\theta(a \otimes h) = \overline{\theta}(a) \otimes h + a \otimes h$$

for every $a \in R$, $h \in \mathcal{H}(\lambda)$ and $\theta \in \hat{(T)}$; and

(b) induces an action on $\mathbb{V}_{\mathcal{C}\to B}(\lambda, P)$.

Proof. *To be added* □

Remark 8.3.8. The above result naturally generalizes to the case in which the curve $\mathcal{C} \to B$ is marked by more points $P_1, \dots, P_n$. In this situation we have that $\oplus_{i=1}^n \mathrm{Vir}_R$ is a central extension of $\oplus_{i=1}^n (\theta_{L,R})_i$ and similarly $\oplus_{i=1}^n \mathsf{VIR}_R$ is a central extension of $\oplus_{i=1}^n (\theta_{L/R})_i$. We can then realize $\hat{T}$ to be the Lie subalgebra of $\oplus_{i=1}^n \mathsf{VIR}_R$ which is generated by the image of $\theta_{A,R}$ in $\oplus_{i=1}^n (\theta_{L/R})_i$.

One of the consequences of Theorem 8.3.7 is the following result.

Corollary 8.3.9. *Under the assumptions that $\mathcal{C} \to B$ is a family of smooth curves over a smooth base scheme B, the sheaf of covacua $\mathbb{V}_{\mathcal{C}\to B}(P_\bullet, \lambda_\bullet)$ is a locally free sheaf over B. Hence the sheaf $\mathbb{V}_g(\lambda_\bullet)$ is locally free over $\mathcal{M}_{g,n}$.*

Proof. *In the handwritten notes from the class over Zoom* □

9 Sheaves of covacua are locally free

We show here that the sheaf of covacua $\mathbb{V}_g(\lambda_\bullet)$ is locally free over the whole $\overline{\mathcal{M}}_{g,n}$, extending the result from the interior $\mathcal{M}_{g,n}$ to the whole space. We have already seen that factorization allows us to describe the spaces of covacua on a nodal curve in terms of analogue spaces of covacua on the normalization of the curve. In this section we will see that we can actually refine this result by showing that the sheaf of vacua associated with a smooth and infinitesimal deformation $\mathcal{C}$ of a nodal curve C is isomorphic to a direct sum of trivial deformations of sheaves of covacua over the normalization of C. In order to state this result more precisely, we will need to introduce some notation

9.1 Construction of a smoothing deformation

Let $(C, P^\bullet)$ be a nodal curve and let Q be one of its nodes. Denote by $\widetilde{C}$ the normalization of C at Q, so that $\widetilde{C}$ is now marked by points $P_\bullet$, Q_+ and Q_-. We will fix coordinates at all these points. We now explain how to obtain a family of curves $\mathcal{C}$ over $\Delta = \mathrm{Spec}(\mathbb{C}[\![\tau]\!])$ such that the special fiber $\mathcal{C}|_0 = C$ and the generic fiber $\mathcal{C}|_\eta$ has one less node than C.

9.1.1. The idea which is for instance explained in [Loo13], is to find a deformation $\mathcal{C}$ of C which replaces the formal neighborhood $\mathbb{C}[\![t_+, t_-]\!]/t_+t_-$ of the nodal point Q with the $\mathbb{C}[\![\tau]\!]$-algebra $\mathbb{C}[\![t_+, t_-, \tau]\!]/t_+t_- = \tau$. This can be achieved with the following geometric construction. We first normalize the curve C at Q and we blow up the trivial deformation $\widetilde{C}[\![\tau]\!]$ of $\widetilde{C}$ at the points Q_+ and Q_- and note that the formal coordinate rings at $Q_\pm$ in the strict transform are of the form $\mathbb{C}[\![t_\pm, \tau/t_\pm]\!]$. We then obtain the neighborhood $\mathbb{C}[\![t_+, t_-, \tau]\!]/t_+t_- = \tau$ by identifying t_+ with τ/t_-.

9.1.2. We now give a construction of the family $\mathcal{C}$ by constructing compatible families

$$(\pi_n \colon \mathcal{C}^n \longrightarrow \mathrm{Spec}(\mathbb{C}[\tau]_n), P^n_\bullet)$$

where $\mathbb{C}[\tau]_n := \mathbb{C}[\tau]/(\tau^{n+1})$ for $n \in \mathbb{N}$. As these are infinitesimal deformations, we only need to change the structure sheaf, while the underlying topological space does not change.

Let U be an open subset of C and $n \in \mathbb{N}_0$. If U does not contain Q we set $\mathcal{O}_{\mathcal{C}^n}(U) := \mathcal{O}_C(U)[\tau]/\tau^{n+1}$. Otherwise, if $Q \in U$, we set

$$\mathcal{O}_{\mathcal{C}^n}(U) := \ker\left(\frac{\mathbb{C}[\![t_+, t_-]\!][\tau]}{t_+t_- = \tau, \tau^{n+1}} \oplus \mathcal{O}_{\mathcal{C}^n}(U \setminus \{Q\}) \xrightarrow{\alpha_n - \beta_n} \frac{\mathbb{C}(\!(t_+)\!)[\tau]}{\tau^{n+1}} \oplus \frac{\mathbb{C}(\!(t_-)\!)[\tau]}{\tau^{n+1}}\right)$$

where

$$\alpha_n \colon \frac{\mathbb{C}[\![t_+,t_-]\!][\tau]}{t_+t_- = \tau, \tau^{n+1}} \longrightarrow \frac{\mathbb{C}(\!(t_+)\!)[\tau]}{\tau^{n+1}} \oplus \frac{\mathbb{C}(\!(t_-)\!)[\tau]}{\tau^{n+1}}$$

is the $\mathbb{C}[\tau]_n$-linear morphism given by $t_+ \mapsto (t_+, (t_-)^{-1}\tau)$ and $t_- \mapsto ((t_+)^{-1}\tau, t_-)$, and

$$\beta_n \colon \mathcal{O}_{\mathcal{C}^n}(U \setminus \{Q\}) \longrightarrow \frac{\mathbb{C}(\!(t_+)\!)[\tau]}{\tau^{n+1}} \oplus \frac{\mathbb{C}(\!(t_-)\!)[\tau]}{\tau^{n+1}}$$

sends $\psi \in \mathcal{O}_{\mathcal{C}^n}(U \setminus \{Q\})$ to (ψ_+, ψ_-) where $\psi_\pm$ is the expansion of ψ at the point $Q_\pm$ using the identifications $\mathcal{O}_{\mathcal{C}^n}(U \setminus \{Q\}) = \mathcal{O}_C(U \setminus \{Q\})[\tau]/\tau^{n+1} = \mathcal{O}_{\widetilde{C}}(U \setminus \{Q_+, Q_-\})[\tau]/\tau^{n+1}$.

Remark 9.1.3. Observe that the completion of $\mathcal{O}_{\mathcal{C}^n}$ at the point Q is isomorphic to $\mathbb{C}[\![t_+,t_-]\!][\tau]/(t_+t_- = \tau, \tau^{n+1}) = \mathbb{C}[\![t_+,t_-]\!]/(t_+t_-)^{n+1}$. In fact note that once we take the completion of $\mathcal{O}_{\mathcal{C}^n}(U \setminus Q)$ at the point Q we obtain exactly $\frac{\mathbb{C}(\!(t_+)\!)[\tau]}{\tau^{n+1}} \oplus \frac{\mathbb{C}(\!(t_-)\!)[\tau]}{\tau^{n+1}}$, and so the map β_n becomes the identity. The kernel of $\alpha_n - \beta_n$ is then identified with $\mathbb{C}[\![t_+,t_-]\!][\tau]/(t_+t_- = \tau, \tau^{n+1})$ as claimed.

Note moreover that for all $n \in \mathbb{N}$ there are natural maps $g^n \colon \mathcal{C}^{n-1} \to \mathcal{C}^n$ induced by the identity on topological spaces and by the projection $\mathbb{C}[\tau]_n \to \mathbb{C}[\tau]_{n-1}$ on the structure sheaves. Since the points $P_\bullet$ are disjoint from Q, we can set $\mathcal{P}^n_\bullet$ to be the trivial deformation of $P_\bullet$. The maps g^n are compatible with the sections $\mathcal{P}^n_\bullet$ and one can further show that the family $\mathcal{C}^n$ is a curve over $\mathrm{Spec}(\mathbb{C}[\tau]_n)$ deforming C.

9.1.4. By taking the direct limit of this family of deformations we obtain the formal scheme $\mathcal{C}^\infty$ over $\mathrm{Spf}(\mathbb{C}[\![\tau]\!])$. To prove that this is algebraizable, we can invoke Grothendieck's existence theorem ([Gro63, Théorème 5.4.5]) so that we are left to prove that the family $(\mathcal{C}^n)_n$ is equipped with a compatible family of very ample line bundles. This is true because given any smooth point P of C and m sufficiently big, we know that $\mathcal{O}_C(mP)$ is a very ample line bundle on C. Since P lies in the smooth locus of C these line bundles extend naturally to very ample line bundles on $\mathcal{C}^n$ providing the wanted family of very ample line bundles.

9.1.5. We will call the deformation $(\mathcal{C} \to \Delta, \mathcal{P}_\bullet)$ of $(C, P_\bullet)$ a *canonical smoothing* of $(C, P_\bullet)$ and it indeed has central fiber isomorphic to $(C, P_\bullet)$ and generic fiber which has one less node than C.

Remark 9.1.6. Observe that by construction the formal neighbourhood of $\mathcal{C}$ at Q is $\mathbb{C}[\![t_+,t_-,\tau]\!]/t_+t_- = \tau$ as asserted in 9.1.1. Moreover we have that $\widehat{\mathcal{O}}_{\mathcal{C}}(\mathcal{C} \setminus \mathcal{P}_\bullet)$ consists of elements

$$(f,g) \in \frac{\mathbb{C}[\![t_+,t_-,\tau]\!]}{t_+t_- = \tau} \oplus \mathcal{O}_{\widetilde{\mathcal{C}}}(\widetilde{\mathcal{C}} \setminus \mathcal{P}_\bullet, Q_\pm)$$

such that $\alpha(f) = \beta(g)$, where α and β are the natural maps to the punctured disks in $\widetilde{\mathcal{C}}$ around Q_+ and Q_- (i.e. to $\mathbb{C}(\!(t_+)\!)[\![\tau]\!] \oplus \mathbb{C}(\!(t_-)\!)[\![\tau]\!]$) obtained by taking the limits of α_n and β_n as $n \to \infty$. In particular we have that the expansion of g inside $\mathbb{C}(\!(t_+)\!)[\![\tau]\!] \oplus \mathbb{C}(\!(t_-)\!)[\![\tau]\!]$ will be an element of the form

$$\sum_{i,j\geq 0} g_{ij}(t_+^{i-j}\tau^i, t_-^{j-i}\tau^j)$$

for some coefficients $g_{ij} \in \mathbb{C}$.

9.2 Finer factorization

We now want to relate $\mathbb{V}_{\mathcal{C}/\Delta}(\lambda_\bullet, \mathcal{P}_\bullet)$ to $\mathbb{V}_C(\lambda_\bullet, P_\bullet)$. To do so, we will first of all consider another family of curves over Δ, that is the trivial family $\widetilde{\mathcal{C}} := \widetilde{C} \times \Delta$ which is naturally marked by $P_\bullet \times \Delta$ (and $Q_\pm \times \Delta$), which we will still denote by $P_\bullet$ (and $Q_\pm$).

The imprecise statement is that there exists an isomorphism

$$\mathbb{V}_{\mathcal{C}/\Delta}(\mathcal{P}_\bullet; \lambda) \cong \bigoplus_{\mu \in P_\ell^+} \mathbb{V}_{\widetilde{\mathcal{C}}/\Delta}(P_\bullet, Q_\pm; \lambda_\bullet, \mu, \mu^*) \tag{9.1}$$

of sheaves over Δ. Moreover, the latter is isomorphic to

$$\bigoplus_{\mu \in P_\ell^+} \mathbb{V}_{\widetilde{C}}(P_\bullet, Q_\pm; \lambda_\bullet, \mu, \mu^*)[\![\tau]\!] = \mathbb{V}_C(P_\bullet, \lambda_\bullet)[\![\tau]\!]$$

where the last equality follows from factorization.

9.2.1. We now realize the isomorphism (9.1) as being induced from a $\mathbb{C}[\![\tau]\!]$-linear map

$$\mathcal{H}_{\lambda_\bullet}[\![\tau]\!] \to \mathcal{H}_{\lambda_\bullet[\![\tau]\!]} \otimes_{\mathbb{C}[\![\tau]\!]} \left(\bigoplus_\mu \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}[\![\tau]\!] \right)$$

which has the shape $h \mapsto h \otimes \sum_{\mu \in P_\ell^+} \epsilon_\mu$ for some $\epsilon_\mu \in \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}[\![\tau]\!]$ such that $\epsilon_\mu(\tau = 0)$ is a non zero multiple of Id_μ.

9.2.2. Note that for every $\mu \in P_\ell^+$, the space $\mathcal{H}_\mu$ can be decomposed as $\bigoplus_{d \in \mathbb{N}} \mathcal{H}_\mu(d)$, where $\mathcal{H}_\mu(0) = V_\mu$ and $\mathcal{H}_\mu(d)$ consists of linear combination of elements of the form $X_1 t^{-n_1} \dots X_r t^{-n_r} v$ for $X_i \in \mathfrak{g}$, $n_i \in \mathbb{N}$, $\sum n_i = d$ and $v \in V_\mu$. We can then see that although $\mathcal{H}_\mu$ is infinite dimensional, all its graded pieces $\mathcal{H}_\mu(d)$ are finite dimensional.

Lemma 9.2.3. *There is a unique non degenerate bilinear form* $(\ ; \ ) \colon \mathcal{H}_\mu \times \mathcal{H}_{\mu^*} \to \mathbb{C}$ *satisfying the following conditions*

- $(X t^n v; w) + (v; X t^{-n} w) = 0$ *for every* $n \in \mathbb{Z}$ *and* $X \in \mathfrak{g}$, $v \in \mathcal{H}_\mu$ *and* $w \in \mathcal{H}_{\mu^*}$.
- $(v; \phi) = \phi(v)$ *for every* $v \in V_\mu$ *and* $\phi \in V_{\mu^*}$.

Proof. This can be shown by induction on the degree of the elements of $\mathcal{H}_\mu$ and $\mathcal{H}_{\mu^*}$. □

Remark 9.2.4. Charlie observed that the map $(\ ; \ )$ can also be interpreted as the natural map realizing the isomorphism between the space of covacua $\mathbb{V}_{\mathbb{P}^1}(0, \infty; \mu, \mu^*)$ and $\mathbb{C}$.

The conditions defining the pairing $(\ ; \ )$ imply that the spaces $\mathcal{H}_\mu(d)$ and $\mathcal{H}_{\mu^*}(d)$ are dual to each other and that $(a; b) = 0$ whenever a and b have different degree. This guarantees that we can choose bases $\{e_i(d)\}$ of $\mathcal{H}_\mu(d)$ and $\{e^i(d)\}$ of $\mathcal{H}_{\mu^*}(d)$ such that $(e_i(d), e^j(d)) = \delta_{ij}$. We define $\epsilon_\mu(d) := \sum_i e_i(d) \otimes e^i(d)$ and set

$$\epsilon_\mu := \sum_{d \geq 0} \epsilon_\mu(d) \tau^d.$$

Note that if we evaluate ϵ_μ at $\tau = 0$ we obtain $\epsilon_\mu(0)$ which, up to a scalar is indeed Id_μ. We then define the map

$$\epsilon\colon \mathcal{H}_{\lambda_\bullet}[\![\tau]\!] \to \mathcal{H}_{\lambda_\bullet}[\![\tau]\!] \otimes_{\mathbb{C}[\![\tau]\!]} \left(\bigoplus_\mu \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}[\![\tau]\!]\right), \qquad h \mapsto h \oplus \sum_\mu \epsilon_\mu.$$

Lemma 9.2.5. *If the map ϵ induces a map between spaces of covacua, which we call $[\epsilon]$, then $[\epsilon]$ is an isomorphism.*

Proof. By construction we have the commutative diagram

$$\begin{tikzcd}
{[\mathcal{H}_{\lambda_\bullet}[\![\tau]\!]]_{\mathfrak{g}_{\mathcal{C}\setminus\mathcal{P}_\bullet}}} \arrow[r, "{[\epsilon]}"] \arrow[d] & {[\mathcal{H}_{\lambda_\bullet}[\![\tau]\!] \otimes_{\mathbb{C}[\![\tau]\!]} \left(\bigoplus_\mu \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}[\![\tau]\!]\right)]_{\mathfrak{g}_{\widetilde{\mathcal{C}}\setminus\mathcal{P}_\bullet, Q_\pm}}} \arrow[d] \\
{[\mathcal{H}_{\lambda_\bullet}]_{\mathfrak{g}_{\mathcal{C}\setminus P_\bullet}}} \arrow[r, "{[\epsilon]_0}"] & {[\mathcal{H}_{\lambda_\bullet} \otimes \left(\bigoplus_\mu \mathcal{H}_\mu \otimes \mathcal{H}_{\mu^*}\right)]_{\mathfrak{g}_{\widetilde{C}\setminus P_\bullet, Q_\pm}}}
\end{tikzcd}$$

where the vertical maps are obtained by evaluating τ at 0. Note that $[\epsilon]_0$ is, up to non zero scalars, the same map $[F]$ from Theorem 6.0.5, and so it is an isomorphism. Since spaces of covacua are finite dimensional, we can use Nakayama Lemma to conclude that $[\epsilon]$ is an isomorphism in the first place. Indeed, we first see that $[\epsilon]$ must be surjective. Furthermore, the target of $[\epsilon]$ is a free $\mathbb{C}[\![\tau]\!]$-module, and so the surjective map $[\epsilon]$ admits a section σ. It is enough to show that $\ker(\sigma) = 0$ and Nakayama Lemma ensures that this is true. □

Lemma 9.2.6. *The map ϵ indeed defines a map between the spaces of covacua.*

Proof. We are left to show that for every $Xf \in \mathfrak{g}_{\mathcal{C}\setminus\mathcal{P}_\bullet}$ and $h \in \mathcal{H}_{\lambda_\bullet}$, the element $Xf(h) \otimes \sum \epsilon_\mu$ is indeed in the image of the action of $\mathfrak{g}_{\widetilde{\mathcal{C}}\setminus\mathcal{P}_\bullet, Q_\pm}$. Note that, from the description that we have given of $\mathcal{O}_\mathcal{C}$, we have that $\mathcal{O}_\mathcal{C}(\mathcal{C}\setminus\mathcal{P}_\bullet)$ is a subring of $\mathcal{O}_{\widetilde{\mathcal{C}}}(\widetilde{\mathcal{C}}\setminus\mathcal{P}_\bullet, Q_\pm)$. The map $[\epsilon]$ is then well defined if $\mathfrak{g}_{\mathcal{C}\setminus\mathcal{P}_\bullet}$ acts trivially on each ϵ_μ, since we would have

$$Xf(h \otimes \sum \epsilon_\mu) = Xf(h) \otimes \sum \epsilon_\mu + \sum_\mu h \otimes Xf(\epsilon_\mu) = Xf(h) \otimes \sum \epsilon_\mu.$$

The action of $\mathfrak{g}_{\mathcal{C}\setminus\mathcal{P}_\bullet}$ on ϵ_μ is given by the natural map $\mathfrak{g}_{\mathcal{C}\setminus\mathcal{P}_\bullet} \to \mathfrak{g}(\mathbb{C}(\!(t_+)\!)[\![\tau]\!] \oplus \mathbb{C}(\!(t_-)\!)[\![\tau]\!])$ (described in Remark 9.1.6). From that explicit description, we deduce that it is enough to show that the element $(Xt^{i-j}\tau^i, Xt^{j-i}\tau^j)$ acts trivially on ϵ_μ for every μ. This is equivalent to

$$\begin{aligned}
0 = (Xt^{i-j}\tau^i, Xt^{j-i}\tau^j)(\epsilon_\mu) &= (Xt^{i-j}\tau^i \otimes \mathrm{Id})(\epsilon_\mu) + (\mathrm{Id} \otimes Xt^{j-i}\tau^j)(\epsilon_\mu) \\
&= \sum_d (Xt^{i-j} \otimes \mathrm{Id})(\epsilon_\mu(d))\tau^{i+d} + \sum_d (\mathrm{Id} \otimes Xt^{j-i})(\epsilon_\mu(d))\tau^{j+d}
\end{aligned}$$

which, by setting $i - j = n$, amounts to check that

$$\sum_n (Xt^n \otimes \mathrm{Id})(\epsilon_\mu(d+n)) + (\mathrm{Id} \otimes Xt^{-n})(\epsilon_\mu(d)) = 0$$

holds true for every $n \in \mathbb{Z}$. One can check that this holds true by using the non-degeneracy of the pairing (;) and the fact that ϵ_μ is defined using orthonormal bases with respect to that pairing. □

9.3 Local freeness

We now have all the ingredients to show that $\mathbb{V}_g(\lambda_\bullet)$ is a locally free sheaf over $\overline{\mathcal{M}}_{g,n}$. To do this, we will use the following fact: $\overline{\mathcal{M}}_{g,n}$ admits a stratification by opens

$$\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n}^{(0)} \subset \overline{\mathcal{M}}_{g,n}^{(1)} \subset \cdots \subset \overline{\mathcal{M}}_{g,n}^{(3g+n-4)} \subset \overline{\mathcal{M}}_{g,n}^{(3g+n-3)} = \overline{\mathcal{M}}_{g,n},$$

where $\overline{\mathcal{M}}_{g,n}^{(k)}$ parametrizes those curves which have at most k nodes. In order to show that $\mathbb{V}_g(\lambda_\bullet)$ is locally free over $\overline{\mathcal{M}}_{g,n}$, it will be enough to show that being locally free on $\overline{\mathcal{M}}_{g,n}^{(k)}$ implies that it is locally free on $\overline{\mathcal{M}}_{g,n}^{(k+1)}$. We indeed know, from the previous chapter, that $\mathbb{V}_g(\lambda_\bullet)$ is locally free on $\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n}^{(0)}$, so we only are left to show that we can run an induction argument.

Since being locally free is a local property, we can replace $\overline{\mathcal{M}}_{g,n}$ by its atlas U, or even by an ètale covering $B \to U$ on which the universal pointed curve $(C_B \to B, P_\bullet)$ satisfies the conditions of 7.0.1. We will then run the induction argument here since the stratification of $\overline{\mathcal{M}}_{g,n}$ induces the stratification

$$B^{(0)} \subset B^{(1)} \subset \cdots \subset B^{(3g+n-4)} \subset B^{(3g+n-3)} = B.$$

Let $x \in B^{(k)} \setminus B^{(k-1)}$ be a point which corresponds to a curve $C \to \mathrm{Spec}(\mathbb{C})$ which has exactly k nodes. Using the smoothing construction of the previous section, we can find a family of curves $(\mathcal{C} \to \Delta, \mathcal{P}_\bullet)$ such that $\mathcal{C}|_0 = C$ while $\mathcal{C}|_\eta$ has $k-1$ nodes. This means that the curve $\mathcal{C} \to \Delta$ corresponds to a map $\Delta \to B^{(k)}$ whose closed point goes to $B^{(k)}$, but which generically lands in $B^{(k-1)}$. Using the isomorphism (9.1) and the fact that $\mathbb{V}_B(\lambda_\bullet)$ is locally free when restricted to $B^{(k-1)}$, we deduce that $\mathbb{V}_B(\lambda_\bullet)|_\Delta$ is locally free over Δ. This proves that $\mathbb{V}_B(\lambda_\bullet)$ is locally free on a neighbourhood of the point $x \in B^{(k)}$, and so that we can inductively show that $\mathbb{V}_B(\lambda_\bullet)$ is locally free. This proves the following

Theorem 9.3.1. *The sheaf of covacua $\mathbb{V}_g(\lambda_\bullet)$ is a locally free sheaf over $\overline{\mathcal{M}}_{g,n}$.*

Since $\overline{\mathcal{M}}_{g,n}$ is connected we obtain the following corollary.

Corollary 9.3.2. *The rank of $\mathbb{V}_g(\lambda_\bullet)$ is constant.*

10 Verlinde formula

We now want to give a closed formula to compute the rank of $\mathbb{V}_g(\lambda_\bullet)$ for every $\mathfrak{g}$ and $\lambda_\bullet$. To do so, we will use the formalism of fusion rings and translate the problem into an algebraic/combinatorial count. Throughout we will use a lot of *black boxes*.

10.1 The Fusion Ring

Let A be a finite set with an involution $a \mapsto a^*$. We will denote by $\mathbb{N}[A]$ the monoid generated by A, that is $\mathbb{N}[A] = \bigoplus_{a\in A} \mathbb{N}a$. The involution $*$ extends by linearity to an involution of $\mathbb{N}[A]$.

Definition 10.1.1. We define a (genus zero) non-degenerate fusion rule to be a map $\mathcal{F}\colon \mathbb{N}[A] \to \mathbb{N}$ satisfying the following properties:

(f1) $\mathcal{F}(0) = 0$;

(f2) there exists $a \in A$ such that $\mathcal{F}(a) \geq 1$;

(f3) $\mathcal{F}(x) = \mathcal{F}(x*)$ for every x $\in \mathbb{N}[A]$;

(f4) For every $x, y \in \mathbb{N}[A]$ the equality $\mathcal{F}(x + y) = \sum_{a\in A} \mathcal{F}(x + a)\mathcal{F}(y + a*)$ holds;

(f5) For every $a \in A$ there exists $\bar{a} \in A$ such that $\mathcal{F}(a + \bar{a}) = 1$.

Associated with the fusion rule $\mathcal{F} = \mathcal{F}_0$ we can associate higher genus fusion rules $\mathcal{F}_g$ for every $g \in \mathbb{N}$ by inductively define

$$\mathcal{F}_g(x) = \sum_{a\in A} \mathcal{F}_{g-1}(x + a + a*).$$

Example 10.1.2. One can show that if $A = P_\ell^+$ and $*$ is the involution sending $\lambda \to \lambda$, then the map

$$\mathcal{F}_g(\sum_{j=1}^{n} n_i\lambda_i) := \mathrm{rank}\mathbb{V}_g(\underbrace{\lambda_1, \ldots, \lambda_1}_{n_1 \text{ times}}, \lambda_2, \ldots, \lambda_{r-1}, \underbrace{\lambda_r, \ldots \lambda_r}_{n_r \text{ times}})$$

defines a fusion rules. This will be the main example we are going to consider. Note that except for (f3), we have already verified all the required conditions.

Proposition 10.1.3. *The monoid $(\mathbb{Z}[A], +, \cdot)$ is naturally a commutative ring unit $1 \in A$ and with the product induced by $a \cdot b = \sum_{c\in A} \mathcal{F}(a + b + c*)c$ for every $a, b \in A$. This is called the fusion ring associated with $\mathcal{F}$. Moreover the linear map $t\colon \mathbb{Z}[A] \to \mathbb{Z}$ defined by $t(\prod_{a\in A} a^{n_a}) = \mathcal{F}(\sum_a n_a a)$ satisfies the condition $t(a \cdot b) = \delta_{a,b*}$.*

The proof of this proposition is not complicated and it is an easy exercise that follows directly from the axioms of a non-degenerate fusion rule. Moreover one has that there exists exactly one element of A, which turns out to be the identity of $\mathbb{Z}[A]$ and so we denote it 1, such that $\mathcal{F}(1) = 1$. For all the other elements $a \in A \setminus \{1\}$ we have $\mathcal{F}(a) = 0$.

Example 10.1.4. In the case $A = P_\ell^+$ the unit is played by the representation $[V_0]$. Moreover we can write the product explicitly as $[V_\lambda] \cdot [V_\mu] = \sum_\nu \mathrm{rank}\mathbb{V}_0(\lambda, \mu, \nu^*)[V_\nu]$. So in the case $\mathfrak{g} = \mathfrak{sl}_2$, where we identify P_ℓ^+ with $\{0, 1, \ldots, \ell\}$ we have that

$$[V_a] \cdot [V_b] = \sum_{c \in S} [V_c],$$

where $S = \{c \in \{0, 1, \ldots, \ell\}$ such that $a + b + c = 2m$ and $a, b, c \leq m$ for some integer $m \leq \ell\}$

Note that to every element $x \in \mathbb{Z}[A]$ we can associate its trace $\mathrm{Tr}(x)$, that is the trace of the multiplication by x as an endomorphism of $\mathbb{Z}[A]$. We now define the element $\Omega := \sum_{a \in} aa*$— called the Casimir element of $\mathbb{Z}[A]$—and which plays a role in the following result.

Lemma 10.1.5. *For every $x \in \mathbb{Z}[A]$ one has $\mathrm{Tr}(x) = t(x \cdot \Omega)$. Moreover by induction we can show that*

$$\mathcal{F}(a_1 + \cdots + a_n) = t(a_1 \cdots a_n \cdot \Omega^{g-1}).$$

We can actually say more about the structure of the ring $\mathbb{Z}[A]$ (and its tensor with $\mathbb{C}$).

Proposition 10.1.6. *The $\mathbb{C}$-algebra $\mathbb{C}[A]$ is reduced, hence we can identify $\mathbb{C}[A]$ with the space of maps $\mathbb{C}^{S_A} = S_A \to \mathbb{C}$ where S_A is the set of $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[A] \to \mathbb{C}$. Under this isomorphism, the involution $*$ becomes complex conjugation.*

If we combine Lemma 10.1.5 and Proposition 10.1.6 we deduce that

$$\mathcal{F}_g(a_1 + \cdots + a_n) = \sum_{\chi \in S_A} \chi(a_1) \cdots \chi(a_n) \chi(\Omega)^{g-1} = \sum_{\chi \in S_A} \chi(a_1) \cdots \chi(a_n) (\sum_{a \in A} |\chi(a)|^1)^{g-1}$$

which tells us that the problem of computing $\mathcal{F}_g(x)$ is translated into the problem of computing characters of $\mathbb{C}[A]$. We will see how to do this in our case of interest, i.e. when $A = P_\ell^+$.

10.2 Characters of the Fusion ring

In what follows we will denote the fusion ring $\mathbb{Z}[P_\ell^+]$ by $R_\ell[\mathfrak{g}]$ and denote its product by $\underset{\ell}{\cdot}$. We will show that all the characters of $R_\ell[\mathfrak{g}]$ arise from certain characters of the representation ring $R[\mathfrak{g}]$. We recall that $R[\mathfrak{g}]$ is defined as $\oplus_{\lambda \in P^+} \mathbb{Z}[V_\lambda]$ (where we recall P^+ denotes the space of weights of $\mathfrak{g}$, which is in bijection with the finite dimensional simple representations of $\mathfrak{g}$). The monoid $R[\mathfrak{g}]$ is naturally a commutative ring with unit V_0 and with product given by $[V_\lambda] \otimes [V_\mu] = \sum n_\nu [V_\nu]$ whenever the decomposition of the representation $V_\lambda \otimes V_\mu$ into simple factors is of the form $\oplus_{\nu \in P} V_\nu^{n_\nu}$.

Example 10.2.1. Although $R_\ell[\mathfrak{g}] \subset R[\mathfrak{g}]$, the inclusion is not a ring homomorphism. For instance we can see that if $\ell = 1$ and $\mathfrak{g} = \mathfrak{sl}_2$, then $[V_1] \otimes [V_1] = [V_2] + [V_0]$, while $V_1 \underset{\ell}{\cdot} V_1 = V_0$.

Surprisingly, one can realize $R_\ell[\mathfrak{g}]$ as a quotient of $R[\mathfrak{g}]$! We will now construct a surjective map $\pi\colon R[\mathfrak{g}] \to R_\ell[\mathfrak{g}]$, but we will omit the proof that this is indeed a ring homomorphism [Faltings, Kumar, Beauville]. In order to define π, we will need to introduce some notation.

Denote by P the lattice of weights of $\mathfrak{g}$ and let $P_\mathbb{R} = P \otimes_\mathbb{Z} \mathbb{R}$ be the r-dimensional vector space generated by P (where $r = \operatorname{rank}\mathfrak{g} = \dim(\mathfrak{h})$). This vector space is naturally acted by the *affine Weyl group* W_ℓ which is generated by the Weyl group W of reflections and by translations by $(\ell + \check{h})\alpha$, where $\check{h}$ is the dual coxeter number of $\mathfrak{g}$ and α is any long root of $\mathfrak{g}$. For every element $w \in W_\ell$ we denote by $\epsilon(w)$ the determinant of the element $\overline{w} \in W$ (so that is $\epsilon(w)$ is either $+1$ or -1).

The space $P_\mathbb{R}$ can then be subdivided into fundamental domains under the action of W_ℓ called alcoves. The complements of the interior of the alcoves in $P_\mathbb{R}$ is the union of the hyperplanes

$$H_{\alpha,n} = \{\lambda \in P_\mathbb{R} \text{ such that } \langle \lambda | \alpha \rangle = n(\ell + \check{h})\}$$

where α runs over the set of long roots of $\mathfrak{g}$ and $n \in \mathbb{Z}$. The fundamental alcove $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\lambda \in P_\mathbb{R} \text{ such that } \lambda(h_\alpha) \geq 0 \text{ for every } \alpha > 0 \text{ and } \lambda(h_\theta) \leq \ell + \check{h}\}.$$

Following the standard notation, we set $\rho = (\sum_{\alpha>0} \alpha)/2$ and observe that $\rho(h_\alpha) = 1$ for every $\alpha > 0$ and also $\rho(h_\theta) = \check{h} - 1$. This implies the following result.

Lemma 10.2.2. *The set $\mathcal{A}^\circ \cap P$ is in bijection with P_ℓ^+ via the map $\mathcal{A}^\circ \cap P \ni \lambda \mapsto \lambda - \rho \in P_\ell^+$.*

Definition 10.2.3. We define $\pi\colon R[\mathfrak{g}] \to R_\ell[\mathfrak{g}]$ to be

$$\pi([V_\lambda]) = \begin{cases} 0 & \text{if } \lambda + \rho \in H_{\alpha,n} \text{ for some } \alpha \text{ and } n \\ \epsilon(w)[V_\mu] & \text{otherwise,} \end{cases}$$

where μ and w are the unique elements in P_ℓ^+ and W_ℓ respectively such that $\lambda + \rho = w(\mu + \rho)$.

Theorem 10.2.4. *The map π is a ring homomorphism.*

Proof. We omit the proof, but refer to [Kum22, Proof of Theorem 4.2.8] and to [Fal94]. □

Instead or reporting the proof, we will see why this is true in the case of $\mathfrak{g} = \mathfrak{sl}_2$, so that $\check{h} = 2$. In this situation we explicitly describe the map π using modular arithmetic. More precisely, for every $a \in \mathbb{Z}$, denote by $\bar{a} \in \{0, \ldots, 2\ell+1\}$ be its class in $\mathbb{Z}/(2\ell+2)\mathbb{Z}$ that for ever

$$\pi([V_a]) = \begin{cases} [V_{\bar{a}}] & \text{if } 0 \leq \bar{a} \leq \ell \\ -[V_{\bar{a}}] & \text{if } \ell + 2 \leq \bar{a} \leq 2\ell + 2 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover we note that both $R[\mathfrak{g}]$ and $R_\ell[\mathfrak{g}]$ are generated by V_1 and so to show that π is a ring homomorphism, it is enough to check that $\pi([V_a] \otimes [V_1]) = \pi([V_a]) \ldots [V_1]$. From Clebsh-Gordan we have that $[V_a] \otimes [V_1] = [V_{a+1}] + [V_{a-1}]$ and so, using the above description of π and Example 10.1.4 we can indeed conclude that the map π is a ring homomorphism.

10.2.1 Characters

We now want to understand which characters of $R[\mathfrak{g}]$ factors through characters of $R_\ell[\mathfrak{g}]$. We first of all introduce the group G which is the unique simply connected group having $\mathfrak{g}$ as its Lie algebra. For $\mathfrak{sl}_r$ this group G will be SL_r. It contains a maximal torus $T = (\mathbb{C}^\times)^r$ whole Lie algebra is given by the Cartan subalgebra $\mathfrak{h} = \mathbb{C}^r$. Moreover we have that the exponential map $\mathfrak{h} \to T$ is surjective map and that every representation of $\mathfrak{g}$ naturally induces a representation of G (there is an equivalence of category between the cateogory of $\mathfrak{g}$-representations and of G-representations). For every element $t \in T$, we can then associate the character $\mathrm{ch}_t \colon R[\mathfrak{g}] \to \mathbb{C}$ which associates to $[V_\lambda]$ the trace of the operator t acting on V_λ (now seen as the group representation). Note that if we write $t = e^h$ for some $h \in \mathfrak{h}$, then this consists of the trace of the operator e^h. But we know that the action of $\mathfrak{h}$ is diagonalizable on V_λ (this was the key observation to obtain that every simple representation is of the form V_λ for some weight λ), and so we have that $\mathrm{ch}_{e^h}([V_\lambda]) = \sum e^x$ where the sum runs over all the eigenvalues of h on V_λ.

Example 10.2.5. When $\mathfrak{g} = \mathfrak{sl}_2$, we have that $\mathrm{ch}_{e^{aH}}(V_\lambda) = e^{a\lambda} + e^{a\lambda - 2a} + \cdots + e^{-a\lambda}$ for every $\lambda \geq 0$ and $a \in \mathbb{C}$. Note moreover that when a is not an integer multiple of $i\pi$, we can rewrite the above expression as

$$\mathrm{ch}_{e^{aH}}(V_\lambda) = \frac{\sinh((\lambda+1)a}{\sinh(a)}.$$

The above example tells us that, except for some exceptional values of $t \in T$, we can give an explicit formula for the character ch_t. This is note true only for $\mathfrak{sl}_2$, but for every simple Lie algebra $\mathfrak{g}$ and we now formulate the precise statement (Weyl's character formula).

Bibliography

[ACG11] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Bea96] A. Beauville. Conformal blocks, fusion rules and the Verlinde formula. In *Proceedings of the Hirzebruch 65 conference on algebraic geometry, Bar- Ilan University, Ramat Gan, Israel, May 2-7, 1993*, pages 75–96. Ramat-Gan: Bar-Ilan University, 1996.

[BGM15] P. Belkale, A. Gibney, and S. Mukhopadhyay. Vanishing and identities of conformal blocks divisors. *Algebr. Geom.*, 2(1):62–90, 2015.

[BGM16] P. Belkale, A. Gibney, and S. Mukhopadhyay. Nonvanishing of conformal blocks divisors on $\overline{M}_{0,n}$. *Transform. Groups*, 21(2):329–353, 2016.

[DFMS97] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal field theory.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[Fak12] N. Fakhruddin. Chern classes of conformal blocks. In *Compact moduli spaces and vector bundles*, volume 564 of *Contemp. Math.*, pages 145–176. Amer. Math. Soc., Providence, RI, 2012.

[Fal94] G. Faltings. A proof for the Verlinde formula. *J. Algebraic Geom.*, 3(2):347–374, 1994.

[Gar80] H. Garland. The arithmetic theory of loop groups. *Inst. Hautes Études Sci. Publ. Math.*, (52):5–136, 1980.

[Gro63] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[Hum78] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Kac90] V. G. Kac. *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[Kum22] S. Kumar. *Conformal blocks, generalized theta functions and the Verlinde formula*, volume 42 of *New Math. Monogr.* Cambridge: Cambridge University Press, 2022.

[Loo13] E. Looijenga. From WZW models to modular functors. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 427–466. Int. Press, Somerville, MA, 2013.

[Ser87] J.-P. Serre. *Complex semisimple Lie algebras.* Springer-Verlag, New York, 1987. Translated from the French by G. A. Jones.

[Sor96] C. Sorger. La formule de Verlinde. *Astérisque*, (237):Exp. No. 794, 3, 87–114, 1996. Séminaire Bourbaki, Vol. 1994/95.

[TUY89] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. Integrable systems in quantum field theory and statistical mechanics, Proc. Symp., Kyoto/Jap. and Kyuzeso/Jap. 1988, Adv. Stud. Pure Math. 19, 459-566 (1989)., 1989.