Algebraic Topology Homework 4

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§ *Problems from 1.2*

EXERCISE 1.3 Let $p : \tilde{X} \longrightarrow X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\tilde{X}$ is compact Hausdorff if and only if X is compact Hausdorff.

Proof: First, a quick lemma.

Lemma 0.1. Suppose $f : X \longrightarrow Y$ is a local homeomorphism between topological spaces X and Y, i.e. That at every point $x \in X$ there exists some open neighborhood $U \subseteq X$ of x such that $f(U)$ is open and the restriction $f|_U$ is a homeomorphism onto $f(U)$. Then f is an open map.

Proof. Let $U \subseteq X$ be an open set. Around each point $x \in V$ we may find an open set U_x which maps homeomorphically to an open set $V_x = f(U_x)$ via f. Then $f(U_x \cap U)$ is open in V_x equipped with the subspace topology. This means $f(U_x \cap U) = V_x \cap A$ for some open set $A \subseteq X$, but then $f(U_x \cap U)$ is a finite intersection of opens in Y and hence itself open. We then have that

$$f(U) = \bigcup_{x \in U} f(U_x \cap U),$$

and therefore $f(U)$ is open. □

Let's take care of Hausdorffness first. Suppose X is Hausdorff and choose any two $x, y \in \tilde{X}$ with $x \neq y$. There are two cases to consider depending on whether or not x and y lie in the same fiber. If they do not, then by Hausdorffness on X we may find open neighborhoods $U \subseteq X$ for $p(x)$ and $V \subseteq X$ for $p(y)$ such that $U \cap V = \emptyset$. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open neighborhoods of x and y respectively since p is continuous. If instead $f(x) = f(y)$, take an open neighborhood $V \subseteq X$ of $f(x)$ which is evenly covered. Let U and U' be the open sets in the collection determined by $f^{-1}(V)$ which contain x and y respectively. Note that x and y are only contained in one such open set by the assumption that $f^{-1}(V)$ is a disjoint union of such sets. If $U = U'$, then the restriction $p|_U : U \longrightarrow V$ would not be injective and hence could not be a homeomorphism. But $p|_U$ is a homeomorphism, hence $y \notin U$ which implies U and U' are separating neighborhoods for x and y. In either case, if X is Hausdorff then $\tilde{X}$ is Hausdorff.

Now suppose that $\tilde{X}$ is Hausdorff. Pick two distinct points $x, y \in X$ and let U and V be evenly covered neighborhoods of x and y respectively. Choose $\tilde{x} \in f^{-1}(x), \tilde{y} \in f^{-1}(y)$, and let $\tilde{U}$ and $\tilde{V}$ be open sets mapping homeomorophically to U and V respectively via p such that $\tilde{x} \in \tilde{U}$ and $\tilde{y} \in \tilde{V}$. That is, pick $\tilde{x}$, $\tilde{y}$, $\tilde{U}$ and $\tilde{V}$ to be points/open sets in $\tilde{X}$ corresponding to x, y, U and V in X via p. Since $\tilde{X}$ is Hausdorff, we may find separating neighborhoods $\tilde{A}$ of $\tilde{x}$ and $\tilde{B}$ of $\tilde{y}$ such that $\tilde{A} \cap \tilde{B} = \emptyset$. Set $A = p(\tilde{U} \cap \tilde{A})$ and $B = p(\tilde{V} \cap \tilde{B})$. We have that $x \in A$ since $\tilde{x} \in \tilde{U} \cap \tilde{A}$ and likewise $y \in B$. Because p is a homeomorphism on $\tilde{U}$, it is an open map and A is therefore open in the subspace topology in U. This means there is some open set U' such that $A = U \cap U'$ in X, implying that A is open in X since finite intersections of opens are open. The set $B \subseteq X$ is open by similar reasoning. Finally, if there were some $z \in A \cap B$, then there must be some $\tilde{z} \in (\tilde{U} \cap \tilde{A}) \cap (\tilde{V} \cap \tilde{B})$ since $p|_{\tilde{U} \cap \tilde{V}} \longrightarrow U \cap V$ is a homeomorphism. There is no such $\tilde{z}$ since $\tilde{A}$ and $\tilde{B}$ were chosen to be separating neighborhoods in $\tilde{X}$, hence A and B are disjoint. This proves we can separate distinct points in X by open sets, and hence X is Hausdorff.

We now move on to compactness. One direction is easy: from pointset topology we know that the image of a compact set under a continuous map is compact, hence $X = p(\tilde{X})$ is compact if $\tilde{X}$ is compact. To see this, take an open cover $\{U_\alpha\}_{\alpha \in A}$ of X. Then $\{p^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of $\tilde{X}$ and must have a finite subcover $\{p^{-1}(U_1), ..., p^{-1}(U_n)\}$. But then $\{pp^{-1}(U_1), ..., pp^{-1}(U_n)\} = \{U_1, ..., U_n\}$ is an open cover of X by the surjectivity of p.

Suppose now that X is compact and choose some open cover $\mathcal{U}$ of X. I first claim that for each $x \in X$, we may find an open neighborhood $V_x \subseteq X$ of x such that each lift of V_x is contained in U_α for some α. The idea is to shrink an evenly covered neighborhood of x until it satisfies the desired property. Indeed, since $f^{-1}(x) = \{\tilde{x}_1, ..., \tilde{x}_n\}$ is finite, we may find $U_1, ..., U_n$ such that $\tilde{x}_i \in U_i$. By the definition of a covering space, we can find an evenly covered neighborhood $V \subseteq X$ of x. Since each lift of V contains exactly one element of the fiber of x, there are exactly n-lifts of V, and we enumerate them $\tilde{V}_1, ..., \tilde{V}_n$. Define the intersection $\tilde{W}_i = \tilde{V}_i \cap U_i$. Each $\tilde{W}_i$ contains $\tilde{x}_i$, and is hence a nonempty open set of $\tilde{X}$. Furthermore, $W_i = p(\tilde{W}_i)$ is a homeomorphic to $\tilde{W}_i$ since the restriction of p to $\tilde{V}_i$ is a homeomorphism by definition. Now define $V_x = W_1 \cap ... \cap W_n$. We have that V_x is a neighborhood of x since $\tilde{x}_i \in \tilde{W}_i \implies x = p(\tilde{x}_i) \in W_i = p(\tilde{W}_i)$ for each i and each lift of V_x is entirely contained in $\tilde{W}_i \subseteq U_i$ for some $1 \leq i \leq n$, proving the claim.

Choosing such a V_x for each $x \in X$ gives us a cover $\mathcal{V}$ of X, and hence by compactness we may find a finite subcover $\mathcal{V}' = \{V_{x_1}, ..., V_{x_n}\}$. Because each V_x was constructed above as a subset of an evenly covered neighborhood of x, it to is an evenly covered neighborhood of x and hence has exactly n lifts. This implies that the lift $\mathcal{V}'$ to $\tilde{X}$ is also a finite cover. For each $1 \leq i \leq n$ and each lift V_{x_i}, there is some $U \in \mathcal{U}$ containing that lift, and since we have finitely many lifts for each V_{x_i} and finitely many V_{x_i} in the collection $\mathcal{V}'$, the lift of $\mathcal{V}'$ to $\tilde{X}$ has a refinement to a finite sub cover of $\mathcal{U}$. This proves that $\tilde{X}$ is compact. □

EXERCISE 1.4 Construct a simply-connected covering space of the space $X \subseteq \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Proof: Let $X \subseteq \mathbb{R}^3$ be the union of unit sphere S^2 in $\mathbb{R}^3$ with its diameter D connecting its north and south poles on the x-axis, i.e. the line segment between $(-1, 0, 0)$ and $(1, 0, 0)$. Now for $t \in \mathbb{R}$ let $T_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the homeomorphism of $\mathbb{R}^3$ given by translating by t along the x-axis, that is, $T_t(x, y, z) = (x + t, y, z)$.

I claim that

$$\tilde{X} = \bigcup_{n \in \mathbb{Z}} (T_{4n}(S^2) \cup T_{4n+1}(D)),$$

seen in Figure (1), is a simply-connected cover of X. By deformation retracting each $T_{4n+1}(D)$ to a point, we see that $\tilde{X}$ is homotopy equivalent to an infinite wedge of spheres, and hence simply connected. To see that it is a covering space of X, we need to construct a covering map. Define $p : \tilde{X} \longrightarrow X$ to be the inverse translation T_{-4n} on each sphere $T_{4n}(S^2)$ and the inverse translation $T_{-(4n+1)}$ followed by a reflection across the yz-plane on each line segment $T_{4n+1}(D)$. Then for each point $x \in X$ we can find an $\epsilon > 0$ small enough so that $p^{-1}(B_\epsilon(x) \cap X)$ is a union of disjoint homeomorphic copies of $B_\epsilon(x) \cap X$, each lift given by a translate or a reflection followed by a translate.

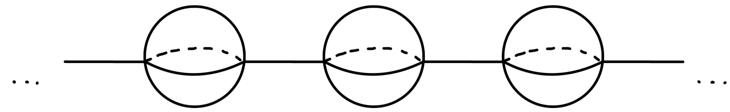

Figure 1: Universal cover of the sphere union a diameter

Now let X be the unit sphere S^2 in $\mathbb{R}^3$ union a circle S^1 which intersects S^2 at exactly two points. By homotoping these two points along the section of the great arc connecting them on the surface of S^2, we see that X is homotopy equivalent to $S^2 \vee S^1 \vee S^1$ (see Figure 2).

For its universal cover, we first take the Cayley graph Y from Example 1.45, the universal cover of $S^1 \vee S^1$ equipped with covering map $p_1 : Y \longrightarrow S^1 \vee S^1$. We then obtain a space $\tilde{X}$ by wedging a copy of S^2 at every point of $p_1^{-1}(x)$, where x denotes the basepoint of $S^1 \vee S^1$. The space $\tilde{X}$ is seen in Figure (3). The covering map $p : \tilde{X} \longrightarrow X$ sends each copy of S^2 to S^2 in X and is defined to be p_1 on Y. The space $\tilde{X}$ is simply connected since it is homotopy equivalent to the wedge of countably many spheres, again by the homotopy retracting each chord in Y to a point.

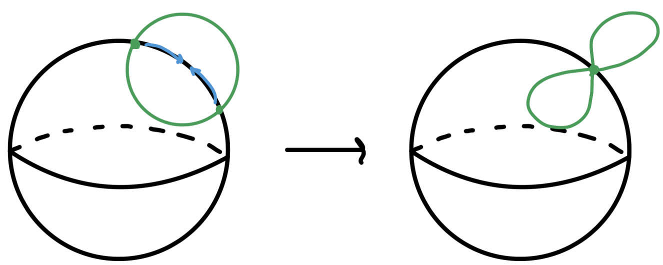

Figure 2: The sphere union a circle intersecting it twice is homotopy equivalent to $S^2 \vee S^1 \vee S^1$

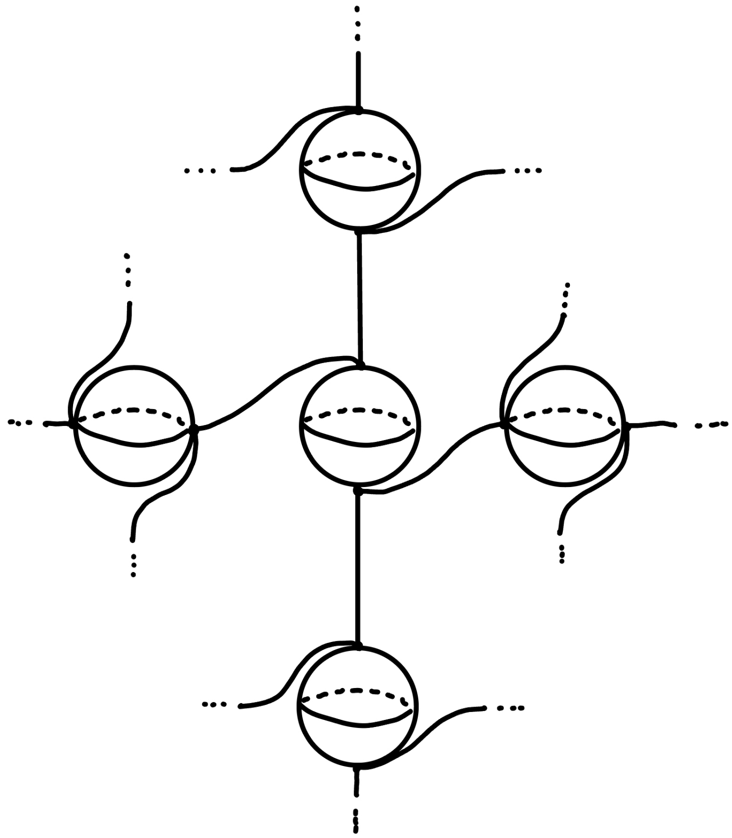

Figure 3: The universal cover $\tilde{X}$ of the sphere union a twice-intersecting circle.

$\square$

EXERCISE 1.5 Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ inside the square. Show that for every covering space $\tilde{X} \longrightarrow X$ there is some neighborhood of the left edge of X that lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply-connected covering space.

Proof: Call the leftmost edge of X E. We first argue that if $\tilde{X}$ is a covering space of X then there is a neighborhood of E that lifts homeomorphically to $\tilde{X}$. By the definition of a covering space, for each point $x \in X$ there is some evenly covered neighborhood U_x and the set of all such neighborhoods forms a cover for X. Closed and bounded in $\mathbb{R}^2$ implies compact (Hein Borel baby) so there exists a finite subset $S \subseteq X$ such that $\{U_x\}_{x \in S}$ is a finite subcover of X. Define $T = \{x \in S \mid U_x \cap E \neq \emptyset\}$ and

$$U = \bigcup_{x \in T} U_x$$

to be the union of all these evenly covered neighborhoods which intersect E nontrivially. We argue that this set is an evenly covered neighborhood of E.

Because E is connected, there must be at least two distinct $x_1, x_2 \in T$ such that $U_{x_1} \cap U_{x_2} \neq \emptyset$. For any $y \in U_{x_1} \cap U_{x_2}$, let $V \subseteq U_{x_1} \cap U_{x_2}$ be an open neighborhood of y. Then V is an evenly covered neighborhood of y, and each of its disjoint copies in $p^{-1}(U_{x_1})$ must intersect the corresponding copy in $p^{-1}(U_{x_2})$. This implies that $U_{x_1} \cup U_{x_2}$ is an evenly mapped neighborhood for both x_1 and x_2. By redefining $U_{x_1} = U_{x_1} \cup U_{x_2}$ and removing x_2 from T, we reduce the cardinality of T by 1 and leave U unaffected, still covered by evenly covered sets. Since T is finite, repeating this process must eventually terminate with $|T| = 1$, leaving us with only U, an evenly covered set containing E. This proves that U is (one choice of) the desired neighborhood of E.

Since U is an open set in $X \subseteq \mathbb{R}^2$ equipped with the subspace topology, there must be an open set $V \subseteq \mathbb{R}^2$ such that $U = V \cap \mathbb{R}^2$. Let B denote the boundary of V, noting that it intersects E trivially because $E \subseteq V$. Since E is compact in $\mathbb{R}^2$ and $E \cap B = \emptyset$, the minimum distance r between E and B is attained by a pair of points in E and B and is positive. We can therefore find some $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. Since each point x in the vertical line $L = \{1/n\} \times [0, 1]$ of X is distance $1/n$ from E, $x \in V$. This implies that $L \subseteq V$, and hence $L \subseteq U$. Likewise, each point along the horizontal segments H_1 and H_2 connecting L to E is within distance $1/n$ of E and is hence also contained in U.

The union $E \cup H_1 \cup L \cup H_2$ lifts homeomorphically to $\tilde{X}$ as it is contained in U, but this means that $\tilde{X}$ contains a loop which is not nullhomotopic. We conclude that $\tilde{X}$ is not simply connected, and since $\tilde{X}$ was chosen to be an arbitrary cover space of X, that X has no universal cover.

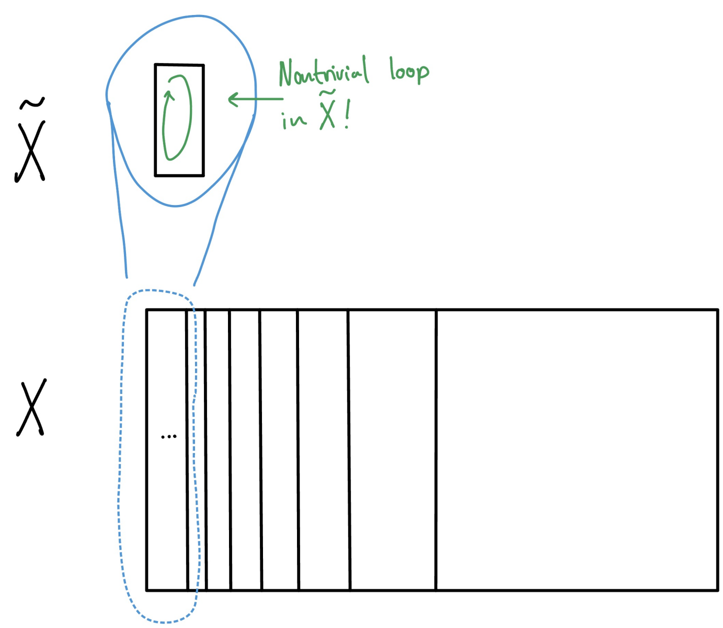

Figure 4: $\tilde{X}$ has a nontrivial loop.

$\square$

Exercise 1.9 Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \longrightarrow S^1$ is nullhomotopic. [Use the covering space $\mathbb{R} \longrightarrow S^1$].

Proof: Note first that there is no finite subgroup of the additive group $\mathbb{Z}$. Indeed, if $g \in \mathbb{Z}$ were of finite order $k \geq 1$, then $k \cdot g = 0 \implies g = 0$ since $\mathbb{Z}$ is an integral domain. The continuous map $f : X \longrightarrow S^1$ induces a morphism $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(S^1, s_0)$, and since $\pi_1(X, x_0)$ is finite by assumption, this induced map must be the trivial map. This implies that $f_*(\pi_1(X, x_0)) = \langle 0 \rangle \subseteq p_*(\pi_1(\mathbb{R}, r_0))$, where $p : \mathbb{R} \longrightarrow S^1$ is the covering map, and hence by the lifting criterion there exists a lift $\tilde{f} : X \longrightarrow \mathbb{R}$ of f sending x_0 to the arbitrarily chosen basepoint $r_0 \in \mathbb{R}$.

Since $\mathbb{R}$ deformation retracts to to r_0, $\tilde{f} : X \longrightarrow \mathbb{R}$ is nullhomotopic to the constant map $g : X \longrightarrow \mathbb{R}$ defined $g(x) = r_0$. Let $\tilde{F} : X \times [0,1] \longrightarrow \mathbb{R}$ be the homotopy taking $\tilde{f}$ to g, i.e. a homotopy such that $\tilde{F}(x,0) = \tilde{f}(x)$ and $\tilde{F}(x,1) = r_0$. Both $\tilde{F}$ and p are continuous, hence the composition $p \circ \tilde{F}$ is continuous. But this is a homotopy taking f to the constant map sending everything in X to s_0, since $f = p \circ \tilde{f} = p \circ \tilde{F}(x,0)$ and $p \circ \tilde{F}(x,1) = p(r_0) = s_0$. We conclude that $f : X \longrightarrow S^1$ is nullhomotopic. $\square$

EXERCISE 1.12 Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2 and $(ab)^4$, and prove that this covering space is indeed the correct one.

Proof: The covering space corresponding to the subgroup normally generated by a^2, b^2 and $(ab)^4$ is seen in (5). Call this space $\tilde{X}$, set $X = S^1 \vee S^1$, and let $p : \tilde{X} \to X$ be the covering map obtained by identifying all the a edges and all the b edges in $\tilde{X}$. The a and b loops in $\tilde{X}$ correspond to a^2 and b^2 respectively, while traversing the outer edges of the graph gives a loop corresponding to $(ab)^4$. To check that $\tilde{X}$ does indeed correspond to the subgroups *normally* generated by a^2, b^2 and $(ab)^4$, it suffices to check that $\tilde{X}$ is a normal cover by Proposition 1.39. We therefore need to check that the group of deck transformations acts transitively on the fiber $p^{-1}(x)$ of the basepoint $x \in X$.

Let $\tilde{x}$ be the orange point in the top left of the figure, near 11 o'clock. To move $\tilde{x}$ clockwise around $\tilde{X}$, we apply the deck transformations corresponding to a and b interchangably, starting with a. For example, applying a takes $\tilde{x}$ to the blue point near 1 o'clock in the figure, ab takes $\tilde{x}$ to the orange point near 2 o'clock and so on. This procedure allows us to take $\tilde{x}$ to any other point in $p^{-1}(x)$, and reversing this process can take any point in $p^{-1}(x)$ to $\tilde{x}$. By passing through $\tilde{x}$ we can take any point in $p^{-1}(x)$ to any other point, hence the action of deck transformations on $\tilde{X}$ is transitive and $\tilde{X}$ is a normal cover.

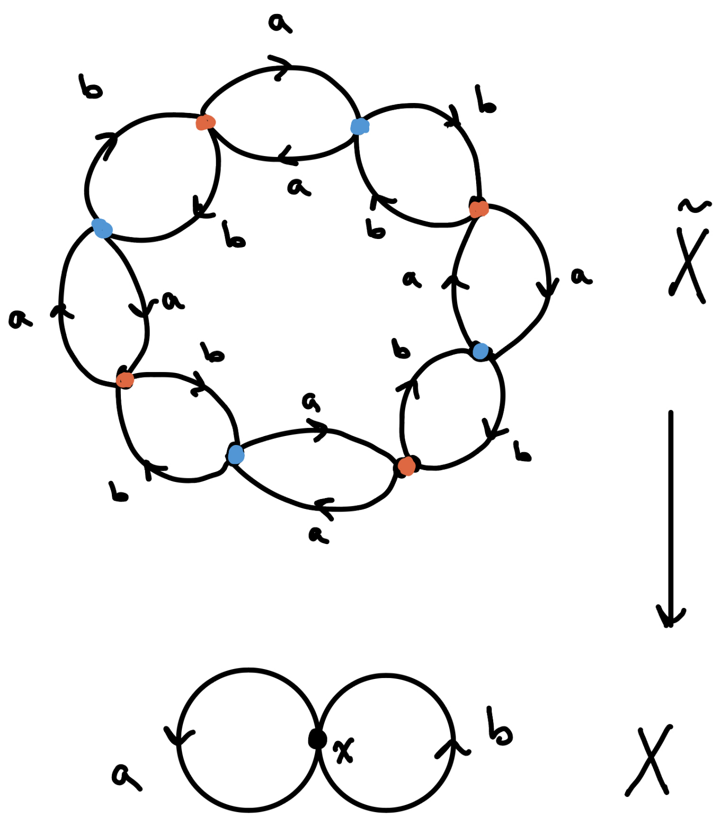

Figure 5: Graph corresponding to $\langle a^2, b^2, (ab)^4 \rangle$.

$\square$

EXERCISE 1.15 Let $p : \tilde{X} \longrightarrow X$ be a simply-connected covering space of X and let $A \subseteq X$ be a path-connected, locally path-connected subspace, with $\tilde{A} \subseteq \tilde{X}$ a path-component of $p^{-1}(A)$. Show that $p : \tilde{A} \longrightarrow A$ is the covering space corresponding to the kernel of the map $\pi_1(A) \longrightarrow \pi_1(X)$.

Proof: Let $p' = p|_{\tilde{A}}$ be the restriction of of p to $\tilde{A}$ with codomain A, $\iota : A \longrightarrow X$ the inclusion of A into X and $\tilde{\iota} : \tilde{A} \longrightarrow \tilde{X}$ the inclusion of $\tilde{A}$ into $\tilde{X}$. Note that we have $\iota \circ r = p \circ \tilde{\iota}$ by definition, and that the induced maps on fundamental groups satisfies the same relation.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\iota}} & \tilde{X} \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{\iota} & X \end{array}$$

Since X and A are both path connected, we can choose x_0 to be the basepoint of each without this choice affecting their respective fundamental groups. Let $\tilde{x}_0$ be the basepoint for $\tilde{A}$ and $\tilde{X}$ in the same way, such that $r(\tilde{x}_0) = x_0$. With this notation in place, functoriality of the fundamental group gives us a corresponding commutative diagram:

$$\begin{array}{ccc} \pi_1(\tilde{A}, \tilde{x}_0) & \xrightarrow{\tilde{\iota}_*} & \pi_1(\tilde{X}, \tilde{x}_0) \\ \Big\downarrow r_* & & \Big\downarrow p_* \\ \pi_1(A, x_0) & \xrightarrow{\iota_*} & \pi_1(X, x_0) \end{array}.$$

I first claim that the map r is a covering space of A. For each $x \in A$, take an evenly covered neighborhood U of x in X. Then $U \cap A$ is also an evenly covered neighborhood. Taking the preimage in $\tilde{X}$ under p, intersecting with $\tilde{A}$ and then taking the image under r then gives us an evenly covered neighborhood of x in A whose preimage is entirely contained in $\tilde{A}$.

Now we want to show $p : \tilde{A} \longrightarrow A$ is the covering space of A corresponding to the kernel of the map $\pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$, or more precisely, that $r_*(\pi_1(\tilde{A}, \tilde{x}_0)) = \ker \iota_*$. Because $\tilde{X}$ is simply-connected, $\pi_1(\tilde{X}, \tilde{x}_0) = 0$. The composition $p_* \circ \tilde{\iota}_*$ is therefore trivial, implying $\iota_* \circ r_* = 0$ too by the commutativity of the above diagram. This implies that $r_*(\pi_1(\tilde{A}, x_0)) \subseteq \ker \iota_*$.

For the other inclusion, suppose we start with an element $[\gamma] \in \ker \iota_*$. This lifts to a unique path α in $\tilde{A}$ with initial endpoint $\tilde{x}_0$, and the composition of α with $\tilde{\iota}$ gives a path in $\tilde{X}$. The commutativity of the first diagram means that $\tilde{\iota} \circ \alpha$ is a lift of $\iota \circ \gamma$, but this is homo topic to the trivial path at x_0 in X since $[\gamma] \in \ker \iota_*$, hence $\tilde{\iota} \circ \alpha$ is actually a loop based at x_0. This means α is a loop in $\tilde{A}$ and so $[\alpha]$ is an element $\pi_1(\tilde{A}, \tilde{x}_0)$. We know $[\gamma] = [r \circ \alpha]$ already since α was defined to be a lift of γ, hence $[\gamma] = r_*([\alpha])$. This gives us the other inclusion, and we are done. $\square$

Exercise 1.23 Show that if a group G acts freely and properly discontinuously on a Hausdorff space X, then the action is a covering space action. (Here "properly discontinuously" means that each $x \in X$ has a neighborhood U such that $\{g \in G \mid U \cap g(U) \neq \emptyset\}$ is finite.) In particular, a free action of a finite group on a Hausdorff space is a covering space action.

Proof: Suppose that G is a group which acts freely and properly discontinuously on a Hausdorff space X. For an open set $U \subseteq X$, define

$$S_U = \{g \in G \mid U \cap g(U) \neq \emptyset\}.$$

By the "properly discontinuous" hypothesis, we know that for each $x \in X$ we may find some open neighborhood U_x such that S_{U_x} is finite. We need only show that we can refine our choice of U_x to guarantee S_{U_x} is empty, as this is the definition of a covering space action.

Fix $x \in X$ and U_x as above, and enumerate S_{U_x} as $\{g_1, \ldots, g_n\}$ where $g_1 = e$. Because the action of G is free, $g_i x = g_j x$ only if $i = j$. Using the Hausdorffness of X we can therefore find an open neighborhood V_i of $g_i x$ for each $1 \leq i \leq n$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Define $U = U_x \cap \bigcap_{i=1}^n g_i^{-1}(V_i)$. Each set $g^{-1}(V_i)$ contains $g_i^{-1} g_i x = x$ and is open, so U is open and nonempty. Consider $U \cap g(U)$ for an arbitrary $g \in G$. If $g \notin S_{U_x}$ then $U \cap g(U) \subseteq U_x \cap g(U_x) = \emptyset$, and if $g = g_i$ for some $1 < i \leq n$, then

$$U \cap g(U) \subseteq g_1(g_1^{-1}(V_1)) \cap g_i(g_i^{-1}(V_i)) = V_1 \cap V_i = \emptyset.$$

Thus, U is an open neighborhood of x such that $U \cap g(U) = \emptyset$ whenever $e \neq g \in G$, and hence the action of G on X is a covering space action.

In particular, this means that every free action of a finite group on a Hausdorff space is a covering space action, since the action of a finite group on a topological space is automatically what Hatcher calls a "properly discontinuous" group action. $\square$