Toric Geometry: Example Sheet 3

Isaac Martin

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EXERCISE 1. The Picard group of a toric variety is always a finitely generated abelian group, and the Picard rank of a toric variety X is the rank of $\operatorname{Pic} X$. Give examples to show that the Picard rank of smooth toric surfaces is unbounded, i.e. for any integer N there is a smooth toric surface with Picard rank larger that N.

Proof: This is likely *not* how one is supposed to solve this problem, but here it goes. From the get go, we invoke [CLS11, Proposition 4.2.6.] to note that, in the case that X_Σ is smooth, every Weil divisor is Cartier. Since $Pic(X_\Sigma) \hookrightarrow \operatorname{Cl}(X_\Sigma)$, this tells us that $\operatorname{Pic}(X_\Sigma) \cong \operatorname{Cl}(X_\Sigma)$ if and only if X_Σ is smooth. Our goal, then, is to construct a fan Σ such that $\operatorname{Cl}(X_\Sigma)$ has rank equal to an arbitrary integer and X_Σ is smooth, in which case we immediately have that the Picard rank of X_Σ is N by [CLS11, Proposition 4.2.6.]. For $N = \mathbb{Z}^n$, we aim to construct a toric variety X_Σ with Picard rank n, as this suffices to solve the problem.

Let Σ be a fan in $N \cong \mathbb{Z}^n$ such that $\Sigma(1) = \{\tau_1, ..., \tau_{2n}\}$ where

$$\tau_i = \begin{cases} e_i & \text{if } 1 \leq i \leq n \\ -e_{i-n} & \text{if } n < i \leq 2n \end{cases}.$$

It is actually enough to let Σ be the collection consisting of the origin in N and these rays. The primitive generators of these rays (which we use interchangeably with the rays themselves when there is no confusion) all individually form a subset of a $\mathbb{Z}$-basis for N and hence their corresponding cones are smooth, and we require only the rays for the subsequent calculation. By [CLS11, Theorem 4.1.3.], we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} \bigoplus_{i=1}^{2n} \mathbb{Z} \cdot D_{\tau_i} \longrightarrow \operatorname{Cl}(X_\Sigma) \longrightarrow 0,$$

and it is short exact since the $\mathbb{R}$-span of $\Sigma(1)$ is all of $N_\mathbb{R}$. To understand the map φ it suffices to understand its action on the standard basis of M, $m_j \in M$ where $m_j(e_k) = \delta_{jk}$ for $1 \leq j, k \leq n$. We have that

$$\varphi(m_j) = \sum_{i=1}^{2n} m_j(\tau_i) D_{\tau_i} = D_{\tau_j} - D_{\tau_{j+n}}.$$

As it is a map of Abelian groups, we understand φ to be given by a $2n \times n$ matrix, and when $n = 3$ it is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Up to automorphism of $\bigoplus_{i=1}^{2n} \mathbb{Z} \cdot D_{\tau_i} \cong \mathbb{Z}^{2n}$, $\varphi(M) \cong \mathbb{Z}^n$ (see this from the Smith normal form of A). We then have that

$$\operatorname{Cl}(X_\Sigma) \cong \operatorname{coker} \varphi \cong \mathbb{Z}^{2n} / \mathbb{Z}^n \cong \mathbb{Z}^n.$$

Since Σ is a smooth fan, X_Σ is a smooth scheme and $\text{Pic}(X_\Sigma) \cong \text{Cl}(X_\Sigma) \cong \mathbb{Z}^n$. We can therefore obtain a toric variety of Picard rank n for any positive integer n.

It might be the case that "$\text{Cl}(X_\Sigma)$ is torsion free" implies "X_Σ is smooth", in which case it is redundant to discuss the smoothness of the fan, but I'm not sure that this is true for non-affine schemes.

□

(Alternate solution, included primarily for author's benefit. Feel free to ignore.) We may be able to do better than this in the following sense: for fixed $n \in \mathbb{N}$ and arbitrary $a \in \mathbb{N}$, does there exists a fan Σ in N such that $\text{rank}\,\text{Pic}(X_\sigma) \geq a$?

It is sufficient to construct Σ such that $\text{Pic}(X_\Sigma)$ has rank a. To do this, as before, we attempt to pick rays $\tau_1, \ldots, \tau_{a+n}$ such that the corresponding fan Σ is smooth. We then get for free that $\text{Pic}(X_\Sigma) \cong \text{Cl}(X_\Sigma)$ is torsion free, and hence

$$\text{rank}\,\text{Pic}(X_\Sigma) = \text{rank} \oplus_{\tau \in \Sigma(1)} \mathbb{Z} \cdot D_\tau \; - \; \text{rank}\,\varphi(M) = a.$$

In the case that $n = 2$, let $\tau_1 = (1,0), \tau_2 = (0,1)$, and $\tau_i = (-q_{i-2}, p_{i-2})$ for $3 \leq i \leq a+2$. Here, p_j is the j^{th} prime number ordered by magnitude for $1 \leq j \leq a$. We choose $q_1 = 1$ and q_j to be the minimal prime number such that $q_j \neq p_j$ and

$$\frac{p_j}{q_j} < \frac{p_{j-1}}{q_{j-1}}$$

for $2 \leq j \leq a$. This condition on q_j is chosen purely for aesthetic purposes so that τ_i is the i^{th} ray moving counterclockwise from τ_1.

E.g. $\tau_3 = (-1,2), \tau_3 = (-2,3), \tau_4 = (-3,5), \tau_5 = (-11,7)$ etc.

We choose the τ_i in this way so that they may completed to a $\mathbb{Z}$-basis for $N \cong \mathbb{Z}^2$. Indeed, τ_1 and τ_2 clearly make up part of a $\mathbb{Z}^2$-basis, and I claim this is true for τ_i when $3 \leq i \leq a+2$ as well. It suffices to show that there exists another $(x,y) \in \mathbb{Z}^2$ such that

$$\det \begin{pmatrix} p_{i-2} & q_{i-2} \\ x & y \end{pmatrix} = p_{i-2}y - q_{i-2}x = 1.$$

This is true by the Fundamental theorem of arithmetic since p_{i-2} and q_{i-2} are relatively prime by construction. Hence the fan Σ consisting solely of these rays and the origin is a smooth fan.

Using the same notation as in the above solution, the map φ is then given by the $((2+a) \times 2)$-matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ -2 & 3 \\ -3 & 5 \\ \vdots & \vdots \end{pmatrix}$$

which has Smith normal form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{pmatrix}$$

Which has image isomorphic to $\mathbb{Z}^2$. This means that $\mathrm{Pic}(X_\Sigma) = \mathrm{Cl}(X_\Sigma) = \mathbb{Z}^{a+2}/\varphi(M) = \mathbb{Z}^a$ as desired. $\square$

PROBLEM 4 Find a toric variety whose class group contains both torsion and non-torsion elements.

Proof: The process of calculating X_Σ is really quite straightforward, but we outline the steps in greater detail than perhaps necessary for the benefit of the author when exam season begins.

Let $n = 3$ and let Σ be a fan whose rays $\Sigma(1) = \{\tau_1, \tau_2, \tau_3\}$ are $\tau_1 = (d, 1, 0)$, $\tau_2 = (0, -1, 0)$ and $\tau_3 = (0, 1, 0)$. Denote by D_i the toric divisor D_{τ_i} for $i = 1, 2, 3$. As discussed in class and in [CLS11, Theorem 4.1.3.], we have an exact sequence

$$M \xrightarrow{\varphi} \bigoplus_{i=1}^{3} \mathbb{Z} \cdot D_i \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

and we note that this sequence is not short exact since $|\Sigma|$ is contained in the hyperplane generated by $(1, 0, 0)$ and $(0, 1, 0)$ in $N_\mathbb{R}$. The map $M \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z} \cdot D_i$ is quite explicit; we simply send $m \in M$ to $\sum_{i=1}^{3} m(v_i) D_i$ where v_i denote the ith primitive lattice generator of the sublattice in N spanned by $\Sigma(1)$. It suffices to understand the map on the canonical basis of M, the elements $m_1 = (1, 0, 0)$, $m_2 = (0, 1, 0)$ and $m_3 = (0, 0, 1)$. The images of these elements is given below:

$$\begin{aligned}
m_1 &\mapsto \langle m_1, (d, 1, 0)\rangle \cdot D_1 + \langle m_1, (0, -1, 0)\rangle \cdot D_2 + \langle m_1, (0, 1, 0)\rangle \cdot D_3 = d \cdot D_1 \\
m_2 &\mapsto \langle m_2, (d, 1, 0)\rangle \cdot D_1 + \langle m_2, (0, -1, 0)\rangle \cdot D_2 + \langle m_2, (0, 1, 0)\rangle \cdot D_3 = -D_1 + D_2 \\
m_3 &\mapsto \langle m_3, (d, 1, 0)\rangle \cdot D_1 + \langle m_3, (0, -1, 0)\rangle \cdot D_2 + \langle m_3, (0, 1, 0)\rangle \cdot D_3 = 0
\end{aligned}$$

As $M \cong \mathbb{Z}^3$ and $\bigoplus_{i=1}^{3} \mathbb{Z} \cdot D_i \cong \mathbb{Z}^3$, the above maps make it clear that φ is multiplication by the matrix

$$A = \begin{pmatrix} d & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the latter matrix is simply A in Smith normal form. Exactness of the above sequence means that $\mathrm{Cl}(X_\Sigma) \cong \operatorname{coker} \varphi = \mathbb{Z}^3/\mathrm{im}(A)$, and the Smith normal form of A makes it clear that

$$\mathrm{Cl}(X_\Sigma) = \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.$$

Notice that φ's failure to be injective was crucial here. If it were injective, then the image of each m_i would have given a copy of $d_i\mathbb{Z}$ for some nonzero $d_i \in \mathbb{Z}$, giving us only torsion. It therefore made sense to look for rays contained in a hyperplane of $N_\mathbb{R}$.

Alternatively, we could have chosen a fan with more rays than the rank of M, e.g. for $M \cong \mathbb{Z}^2$ a fan Σ such that $\Sigma(1) = \{(-4,1),(0,1),(2,-1)\}$. In this case we have injectivity of φ, but because $\operatorname{rank} M < \operatorname{rank} \bigoplus \mathbb{Z} \cdot D_{\tau_i}$, $\operatorname{coker} \varphi$ must contain a torsion-free submodule by default. It is up to us, therefore, to choose rays which produce torsion in $\mathrm{Cl}(X_\Sigma)$. Our above choices do the trick; one can check that $\varphi(M) \cong \mathbb{Z} \oplus 2\mathbb{Z} \subseteq \mathbb{Z}^3$ so we end up with $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.A

Final note: I find this process rather miraculous. I proved a (very) small result the involving torsion divisors of certain affine varieties in prime characteristic for my undergraduate thesis, a project which, if nothing else, taught me how intractable $\mathrm{Cl}(X)$ is for even rather simplistic schemes. I would've had a much easier time generating examples had I known a bit of toric geometry back then. $\square$