Definitions and Theorems from Infinite Groups (Lent '22)

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§ *Free Groups and Presentations*

Definition 0.1 (Directly Finite)**.** We say that a ring R (not necessarily commutative) with unity is *directly finite* (D.F.) if $\forall a,b \in R$, $ab = 1 \implies ba = 1$.

Definition 0.2 (Group Algebra)**.** Let G be a group, K a field. The *group algebra*, denoted $K[G]$, is a K-algebra. As a set it consists of all finite linear combinations of elements in K and G :

$$K[G] = \left\{ \sum_{g\in G} \lambda_g \cdot g \;\middle|\; \lambda_g \in K, \lambda_g \neq 0 \text{ for only finitely many } g \right\}.$$

Addition is defined pointwise: $\lambda g + \lambda' g = (\lambda + \lambda')g$. Multiplication is defined

$$(\lambda g) \cdot (\mu h) = (\lambda \mu)(gh)$$

and extended by distribution.

Definition 0.3. labeldefn:directly-finite-group A group G is said to be *directly finite* (D.F.) if $K[G]$ is a directly finite ring for all fields K.

Example 0.4.

(i) if G is abelian then $K[G]$ is commutative, and therefore directly finite.

(ii) if G is finite then it is also directly finite.

Theorem 0.5 (Kaplansky)**.** *For any group G, the group algebra* $\mathbb{C}[G]$ *is directly finite.*

Theorem 0.6 (Elek, Szako ('01'))**.** *Every sofic group is directly finite.*

Proposition 0.7. Let G be a finite group. The map $\rho : G \longrightarrow \mathrm{Sym}(G)$ defined $\rho(g) : h \mapsto gh$ is an injective homomorphism. Moreover, for all $e \neq g \in G$, $\rho(g)$ has no fixed points.

Definition 0.8 (Hamming Distance)**.** Suppose $\sigma, \tau \in \mathrm{Sym}(n)$ are two permutations. The *Hamming distance* from σ to τ is

$$d_n(\sigma,\tau) = 1 - \frac{1}{n} \left|\{1 \leq i \leq n \mid \sigma(i) = \tau(i)\}\right|.$$

That is, it's a number between 0 and 1 and is equal to 0 if and only if $\sigma = \tau$.

Definition 0.9 (Sofic)**.** G is a sofic group if and only if $\forall A \subseteq G$, $\forall \varepsilon > 0$ there exists $n \in \mathbb{N}$ and a function $\phi : A \longrightarrow \mathrm{Sym}(n)$ such that

(i) for all $g, h \in A$, if $gh \in A$, then

$$d_n(\phi(gh), \phi(g)\phi(h)) \leq \varepsilon,$$

i.e. the distance is "small"

(ii) for all $e \neq g \in G$,

$$d_n(id_n, \phi(g)) \geq 1 - \varepsilon,$$

i.e. the distance is "large".

Such a function ϕ is a (A, ε)-representation.

Example 0.10. Every finite group is sofic.

Theorem 0.11. *Every abelian group is sofic.*

Lemma 0.12. $\mathbb{Z}$ is sofic.

Lemma 0.13. A group G is sofic if and only if every finitely generated subgroup of G is sofic.

Lemma 0.14. If G and H are sofic groups, then $G \times M$ is sofic.

Theorem 0.11. Given an abelian group G, we may assume it is finitely generated by lemma 0.13. By the structure theorem, we have

$$G \cong \mathbb{Z}^k \oplus \frac{\mathbb{Z}}{p_1^{n_1}} \oplus \ldots \oplus \frac{\mathbb{Z}}{p_i^{n_i}},$$

hence G is sofic by lemma 0.14. □

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Throughout this section, X is a set and $X^{-1} = \left\{ x^{-1} \,\middle|\, x \in X \right\}$ is the set of inverses of X.

Definition 1.1. A *word* in X is a finite sequence of symbols: $y_1 y_2 \ldots y_m$ with $y_i \in X \cup X^{-1}$. The empty word is a valid word. We denote the set of all words in X by $W(X)$.

Definition 1.2. *Concatenation of words* is a map $W(X) \times W(X) \longrightarrow W(X)$ defined

$$(y_1 \ldots y_m, z_1 \ldots z_n) \mapsto y_1 \ldots y_m z_1 \ldots z_n.$$

This map gives $W(X)$ the structure of a monoid where the empty word is the identity.

Definition 1.3. Given two words $w, v \in W(X)$, we say that $w \sim v$ if it is possible to pass from one word to the other by means of a finite sequence of the following two operations:

(a) insertion of an xx^{-1} or an $x^{-1}x$ for $x \in X$, as consecutive elements of a word;

(b) deletion of such an xx^{-1} or $x^{-1}x$.

The relation $\sim$ is an equivalence relation on $W(X)$ and we define the *free group on* X to be $\frac{W(X)}{\sim}$. The group operation on $F(X)$ is induced by concatenation.

Definition 1.4. A word $w \in W(X)$ is said to be *reduced* if there is no word v which can be obtained by operation (b) above; in other words, if w is the shortest word in the equivalence class $[x] \in F(X)$. For any other word $v \in W(X)$, there exists a unique reduced $w \in W(X)$ such that $[v] = [w]$.

Theorem 1.5. *Let X be a set, H a group, and $\phi : X \longrightarrow H$ a function. Then there exists a unique group homomorphism $\Phi : F(X) \longrightarrow H$ such that*

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, hook, "i"] & H \\ F(X) \arrow[ur, dashed, "\Phi"] & \end{tikzcd}$$

commutes.