3 First Properties of Schemes

Lemma 1. *a If B is a finitely generated A_f-algebra, then it is a finitely generated A-algebra.*

b Let $F : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes with associated ring homomorphism $\phi : B \to A$. Then the preimage of $\operatorname{Spec} B_f$ is $A_{\phi f}$.

c If B is a finitely generated A-algebra via $\phi : A \to B$, then for any element $f \in A$, the ring $B_{\phi f}$ is a finitely generated A_f algebra.

d Let $f_1, \ldots, f_n \in A$ be elements which generate a B-algebra A. If A_{f_i} is a finitely generated B-algebra for every i, then A is a finitely generated B-algebra.

e Let $f_1, \ldots, f_n \in B$ be elements which generate the unit ideal, and let A be a B-module. If A_{f_i} is a finitely generated B_{f_i}-module for every i, then A is a finitely generated B-module.

Proof. a Let $\{b_k\}$ be a finite set of elements of B such that $B = A_f[b_1, \ldots, b_n]$. Then $B = A[b_1, \ldots, b_n, \frac{1}{f}]$.

b Let $F : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes with associated ring homomorphism $\phi : B \to A$. Then the preimage of $\operatorname{Spec} B_f$ is $A_{\phi f}$.

c Obvious.

d If $\{\frac{a_{i1}}{f_i^{k_{i1}}}, \ldots, \frac{a_{in}}{f_i^{k_{ni}}}\}$ is a generating set for A_{f_i} over B then so is $\{a_{i1}, \ldots, a_{in}, \frac{1}{f_i}\}$ so we can assume that the generating sets are of this form. Let $S = \{a_{ij}, f_i\}$. We claim that $A = B[S]$. For an element $a \in A$, for each i we can write $a \in A_{f_i}$ as $a = \frac{p_i}{f_i^{k_i}}$ for some $k_i \in \mathbb{N}$ and $p_i \in B[a_{i1}, \ldots, a_{in}, f_i]$. Replacing p_i by $f_i^{\nu_i} p_i$ for suitable ν_i we can assume that all the k_i are the same, say $k \in \mathbb{N}$. Now by definition of the localization, writing a in this form means that for each i we have $f^{\ell_i}(f_i^k a - p_i) = 0$ for some ℓ_i. Again, we can replace p_i and k so that we have $(f_i^m a - p_i) = 0$ for each i. Now since the f_i generate A, the same is true of their Nth power for any N. So choosing $N = m$ we find that $f_1^m, \ldots, f_n^m$ generate A, and so we can write the unit as $1 = \sum_{i=1}^n g_i f_i^m$ for some $g_i \in A$. Coming back to our expressions with the p_i we now see that

$$0 = \sum_{i=1}^n g_i(f_i^m a - p_i) = \sum_{i=1}^n g_i f_i^m a - \sum_{i=1}^n g_i p_i = a - \sum_{i=1}^n g_i p_i$$

and so we have found an expression for a as $\sum_{i=1}^n g_i p_i \in B[S]$.

e This is essentially the same idea as in the previous part.

If $\{\frac{a_{i1}}{f_i^{k_{i1}}}, \ldots, \frac{a_{in}}{f_i^{k_{ni}}}\}$ is a generating set for A_{f_i} over B_{f_i} then so is $\{a_{i1}, \ldots, a_{in}\}$ so we can assume that the generating sets are of this form. Let $S = \{a_{ij}\}$.

We claim that $A = B[S]$. For an element $a \in A$, for each i we can write $a \in A_{f_i}$ as $a = \frac{\sum_{j=1}^n b_{ij}a_{ij}}{f_i^{k_i}}$ for some $k_i \in \mathbb{N}$ and $b_{ij} \in B$. As before, we can assume that all the k_i are the same, say $k \in \mathbb{N}$. Now by definition of the localization, writing a in this form means that for each i we have $f^{\ell_i}(f_i^k a - \sum_j b_{ij}a_{ij}) = 0$ for some ℓ_i. Again, we can replace the b_{ij} and k so that we have $(f_i^m a - \sum_j b_{ij}a_{ij}) = 0$ for each i. Now since the f_i generate B, the same is true of their Nth power for any N. So choosing $N = m$ we find that $f_1^m, \dots, f_n^m$ generate B, and so we can write the unit as $1 = \sum_{i=1}^n g_i f_i^m$ for some $g_i \in B$. Coming back to our expressions with the a_{ij} we now see that

$$0 = \sum_{i=1}^n g_i(f_i^m a - \sum_j b_{ij}a_{ij}) = \sum_{i=1}^n g_i f_i^m a - \sum_{i=1}^n g_i \sum_j b_{ij}a_{ij} = a - \sum_{i=1}^n g_i \sum_j b_{ij}a_{ij}$$

and so we have found an expression for a as $\sum_{i=1}^n g_i \sum_j b_{ij}a_{ij} \in B[S]$. □

Lemma 2. *Let* $\operatorname{Spec} A, \operatorname{Spec} B$ *be two open affine subsets of a scheme* X*. Then for every point* $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$ *there exists an open subset* U *with* $\mathfrak{p} \in U \subset \operatorname{Spec} A \cap \operatorname{Spec} B$ *such that* $U \cong \operatorname{Spec} A_f \cong \operatorname{Spec} B_g$ *for some* $f \in A$*,* $g \in B$*.*

Proof. The basic open affines form a basis for affine schemes and so since $\operatorname{Spec} A \cap \operatorname{Spec} B$ is open in $\operatorname{Spec} A$ it is a union of basic opens, one of which, say $\operatorname{Spec} A_{f'}$, contains $\mathfrak{p}$. This open will also be open in $\operatorname{Spec} B$ as well and so for the same reason there is some $g \in B$ such that $\mathfrak{p} \in \operatorname{Spec} B_g \subseteq \operatorname{Spec} A_{f'}$. In particular, the inclusion $\operatorname{Spec} A_{f'} \subseteq \operatorname{Spec} B$ gives us a ring homomorphism $B \xrightarrow{\phi} A_{f'}$. Now it can be checked that $A_f \cong B_g$ for some f and so we are done. □

Exercise 3.1. *Show that a morphism* $f : X \to Y$ *is locally of finite type if and only if for every open affine subset* $V = \operatorname{Spec} B$ *of* Y*,* $f^{-1}(V)$ *can be covered by open affine subsets* $U_j = \operatorname{Spec} A_j$*, where each* A_j *is a finitely generated* B*-algebra.*

Solution. ($\Leftarrow$) It is immediate from the definitions.

($\Rightarrow$) We use $F : X \to Y$ to denote the morphism of schemes. Let $V_i = \operatorname{Spec} B_i$ be a covering of Y by open affine subschemes such that $F^{-1}V_i$ is covered by open affines $\operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra. Each intersection $V_i \cap V$ is open in V_i and so is a union of basic open sets $\operatorname{Spec}(B_i)_{f_{ik}}$ of V_i since they form a base for the topology of $\operatorname{Spec} B_i$. Considering f_{ik} as an element of A_{ij} under the morphisms $B_i \to A_{ij}$, the preimage of $\operatorname{Spec}(B_i)_{f_{ik}}$ is $\operatorname{Spec}(A_{ij})_{f_{ik}}$, and the induced ring morphisms make each $(A_{ij})_{f_{ik}}$ a finitely generated $(B_i)_{f_{ik}}$-algebra.

So we can cover $\operatorname{Spec} B$ with open affines $\operatorname{Spec} C_i$ whose preimages are covered with open affines $\operatorname{Spec} D_{ij}$ such that each D_{ij} is a finitely generated C_i-algebra. Now given a point $\mathfrak{p}$ of $\operatorname{Spec} B$, it is contained in some $\operatorname{Spec} C_i$, but since these are open, there is a basic open affine $\operatorname{Spec} B_{g_\mathfrak{p}} \subseteq \operatorname{Spec} C_i$ that contains $\mathfrak{p}$. Associating $g_\mathfrak{p}$ with its image under the induced ring homomorphisms $B \to C_i$ and then $C_i \to D_{ij}$, it can be seen that $\operatorname{Spec}(C_i)_{g_\mathfrak{p}} \cong \operatorname{Spec} B_{g_\mathfrak{p}}$, the preimage of this is $\operatorname{Spec}(D_{ij})_{g_\mathfrak{p}}$, and $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra. The $\operatorname{Spec}(D_{ij})_{g_\mathfrak{p}}$ together cover the preimage of $\operatorname{Spec} B$, and since $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra, it follows that $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated B-algebra (add $g_\mathfrak{p}$ to the generating set). Hence, the preimage of $\operatorname{Spec} B$ can be covered by open affines $\operatorname{Spec} A_i$ such that each A_i is a finitely generated B algebra.

Exercise 3.2. *A morphism $f : X \to Y$ of schemes is* quasi-compact *if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

Lemma 3. *If a topological space has a finite cover consisting of quasi-compact open sets, it is quasi-compact.*

Proof. Suppose X is the topological space and $\{U_i\}_{i=1}^n$ the open cover with each U_i quasi-compact. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a cover for X. Then $\{V_j \cap U_i\}_{j\in J}$ is an open cover of U_i which has a subcover $\{V_j \cap U_i\}_{j\in J_i}$ where J_i is finite, since U_i is quasi-compact. Then $\cup_{i=1}^n \{V_j\}_{j\in J_i}$ is a finite subcover of $\mathcal{V}$. □

Solution. Let $\{\operatorname{Spec} B_i\}_{i\in I}$ be an open affine cover of Y such that the preimage $f^{-1} \operatorname{Spec} B_i$ of each $\operatorname{Spec} B_i$ is quasi-compact. Let $\operatorname{Spec} C \subseteq Y$ be an arbitrary open affine subset. Each intersection $\operatorname{Spec} B_i \cap \operatorname{Spec} C$ can be covered by opens that are basic in $\operatorname{Spec} B_i$ and since the $\operatorname{Spec} B_i$ form a cover for X, these opens, basic in the various $\operatorname{Spec} B_i$, cover $\operatorname{Spec} C$. Since $\operatorname{Spec} C$ is quasi-compact (Exercise II.2.13(b)), we can find a finite subcover $\{D(b_k)\}_{k=1}^n$ where for each k, $b_k \in B_{i_k}$ for some i_k. Now we cover each $f^{-1} \operatorname{Spec} B_i$ with open affine subschemes $\{\operatorname{Spec} A_{ij}\}_{j\in J_i}$. Since $f^{-1} \operatorname{Spec} B_i$ is quasi-compact, we can choose these in such a way that J_i is finite. The preimage of $D(b_k)$ in $\operatorname{Spec} A_{i_k j}$ is $\operatorname{Spec}(A_{i_k j})_{b_k}$, so we now have a finite cover $\cup_{k=1}^n \{\operatorname{Spec}(A_{i_k j})_{b_k}\}_{j\in J_{i_k}}$ of $f^{-1} \operatorname{Spec} C$ by open affines. Each open affine is quasi-compact (Exercise II.2.13(b)) and so applying the Lemma 3 we see that $f^{-1} \operatorname{Spec} C$ is quasi-compact.

Exercise 3.3.

- a *Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.*
- b *Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$ where each A_i is a finitely generated B-algebra.*
- c *Show also if f is of finite type, then for every open affine subset $V = \operatorname{Spec} B \subseteq Y$ and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.*

Solution. a We need only show that if f is of finite type then it is quasi-compact, the others follow immediately from the definitions. Since f is of finite type there is a cover of Y by open affines $\operatorname{Spec} B_i$ whose preimages are covered by finitely many open affines $\operatorname{Spec} A_{ij}$. We know from Exercise 2.13(b) that each $\operatorname{Spec} A_{ij}$ is quasi-compact. In general if a space can be covered by finitely many quasi-compact opens then it itself is quasi-compact [1], so we have found an open affine cover of Y whose preimages are quasi-compact. Hence, f is quasi-compact.

b Follows directly from Exercise 3.1, 3.2, and 3.3(a).

c Cover $f^{-1}(V)$ by affines $U_i = \operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra. We can cover each of the intersections $U_i \cap U$ with opens that are basic in both U and U_i by Lemma 2. Let $\operatorname{Spec} A_{f_i} = \operatorname{Spec}(A_i)_{g_i}$ be a cover of U by these basic opens, which we can choose to be finite since the morphism is quasi-compact. Since each A_i is a finitely generated B-algebra, $(A_i)_{g_i} = A_{f_i}$ is a finitely generated B algebra (Lemma 1), and therefore, since the $\operatorname{Spec} A_{f_i}$ form a finite cover of U, the ring A is a finitely generated B-algebra (Lemma 1).

Exercise 3.4. *Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.*

Solution. As usual, let $V_i = \operatorname{Spec} B_i$ be an affine cover of Y such that each preimage $f^{-1}V_i$ is affine $U_i = \operatorname{Spec} A_i$ and each A_i is a finitely generated B_i-module. We cover each intersection $U \cap U_i$ with opens $D(f_{ij}) = (B_i)_{f_{ij}}$ of U_i that are basic in both U and U_i and note that the preimage of $D(f_{ij})$ is $\operatorname{Spec}(A_i)_{f_{ij}}$ where f_{ij} is associated with its image in A_i. Since A_i is a finitely generated B_i-module, it follows that $(A_i)_{f_{ij}}$ is a finitely generated $(B_i)_{f_{ij}}$-module.

So now we have a cover of $V = \operatorname{Spec} B$ by opens $\operatorname{Spec} B_{g_i}$ that are basic in V and each of the preimages is affine $\operatorname{Spec} C_i$ and each C_i is a finitely generated B_{g_i}-module. We now use the affineness criterion from Exercise 2.17 as follows. Since $\operatorname{Spec} B$ is affine it is quasi-compact (Exercise 2.13(b)) there is a finite subcover $\operatorname{Spec} B_{g_1}, \dots, \operatorname{Spec} B_{g_n}$. Since this is a cover, the $g_1, \dots, g_n$ generated the unit ideal. This means their image in $\Gamma(U, \mathcal{O}_U)$ where $U = f^{-1} \operatorname{Spec} B$ also generate the unit ideal. Furthermore, the preimage of each $\operatorname{Spec} B_{g_i}$ is in fact U_{g_i} where we associated g_i with its image in $\Gamma(U, \mathcal{O}_U)$. So we can apply the criterion of Exercise 2.17(b) and find that U is affine.

Let $U = \operatorname{Spec} A$. To see that A is a finitely generated B-module we use Lemma 1.

Exercise 3.5. *Let $f : X \to Y$ be a morphism of schemes.*

[1] Let X be the space and U_i the finite cover. For any cover $\{V_i\}$ of X we get a cover $\{V_i \cap U_j\}$ for each U_j, which has a finite subcover by the assumption that the U_j are quasi-compact. The union of the V_i appearing in these finite subcovers will cover X since it covers a cover, and by construction it is finite.

a *Show that a finite morphism is quasi-finite.*

b *Show that a finite morphism is closed.*

c *Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.*

Solution. a Let $\mathfrak{p} \in Y$ be a point. By assumption of the morphism being finite there is an open affine scheme $\operatorname{Spec} B$ containing $\mathfrak{p}$ such that the preimage $f^{-1} \operatorname{Spec} B$ is affine, say $\operatorname{Spec} A$, and A is a finitely generated B-module, so we immediately reduce to the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. To show that the preimage of $\mathfrak{p}$ is finite is the same as showing that the fiber $\operatorname{Spec} A \otimes_B k(\mathfrak{p})$ has finitely many primes (Exercise II.3.10). Since A is a finitely generated B-module, it follows that $A \otimes_B k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-module, that is, a vector space of finite rank. Hence, there are a finite number of prime ideals.

b Note that a subset of a topological space is closed if and only if it is closed in every element of an open cover so we can reduce to the case where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and A is a finitely generated B-module, via say $\phi : B \to A$. So now we want to show that for every ideal $I \subset A$ there is an ideal $J \subset B$ such that $V(J) \subseteq \operatorname{Spec} B$ is the image of $V(I) \subseteq \operatorname{Spec} A$. We immediately have a candidate: $\phi^{-1} I$ so let $J = \phi^{-1} I$. For a point $\mathfrak{p} \in \operatorname{Spec} A$ we have $\mathfrak{p} \supseteq I \Rightarrow \phi^{-1}\mathfrak{p} \supseteq \phi^{-1} I$ so $fV(I) \subseteq V(J)$ and it remains to show that $V(I)$ is mapped surjectively onto $V(J)$. Replacing A and B by A/I and B/J, we just want to show that f is surjective. Given a point $\mathfrak{p} \in \operatorname{Spec} B$ the Going Up Theorem provides us with a point $\mathfrak{q} \in \operatorname{Spec} A$ that maps to $\mathfrak{p}$, and so we are done.

c
$$\operatorname{Spec} k[t, t^{-1}] \oplus k[t, (t-1)^{-1}] \to \operatorname{Spec} k[t]$$

Exercise 3.6. *Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.*

Solution. Let $U = \operatorname{Spec} A$ be an open affine subset of X. By definition, A is an integral domain and so (0) is a prime ideal. A closed subset $V(I)$ contains (0) if and only if (0) contains I and so we see that the closure of (0) is $V((0))$, i.e. all U. Hence, (0) is the generic point ξ of X. $\mathcal{O}_X(U)_{(0)} = \mathcal{O}_\xi$ is the fraction field of $\mathcal{O}_X(U)$.

Exercise 3.7. *Let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.*

Solution. Step 1: $k(X)$ is a finite field extension of $k(Y)$. Choose an open affine $\operatorname{Spec} B = V \subset Y$ and an open affine in its preimage $\operatorname{Spec} A = U \subset f^{-1}V$ such that A is a finitely generated B-algebra (by the finite type hypothesis). From the hypothesis that X is irreducible, it follows that U is irreducible, implying that A is integral.

Now A is finitely generated over B and therefore so is $k(B) \otimes_B A \cong B^{-1}A$. So by Noether's normalization lemma, there is an integer n and a morphism $k(B)[t_1, \ldots, t_n] \to B^{-1}A$ for which $B^{-1}A$ is integral over $k(B)[t_1, \ldots, t_n]$. Since it is integral, the induced morphism of affine schemes is surjective. But $\operatorname{Spec} B^{-1}A$ has the same underlying space as $f^{-1}(\eta) \cap U$ where η is the generic point of Y, and by assumption this is finite. Hence, since affine space always has infinitely many points and the Going-Up Theorem tells us that the morphism $\operatorname{Spec} B^{-1}A \to \operatorname{Spec} k(B)[t_1, \ldots, t_n]$ is surjective ($B^{-1}A$ is integral and integral over $k(B)[t_1, \ldots, t_n]$) we see that $n = 0$ and moreover, we have found that $B^{-1}A$ is integral over $k(B)$. Since it is also of finite type, this implies that it is finite over $k(B)$. With some work clearing denominators from elements of A, This implies that $k(B^{-1}A) = k(A)$ is finite over $k(B)$. That is, it is a finite field extension of $k(B)$.

Step 2: The case where X and Y are affine. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and consider a set of generators $\{a_i\}$ for A over B. Considered as an element of $k(A)$, each one satisfies some polynomial in $k(B)$ since it is a finite field extension. Clearing denominators we get a set of polynomials with coefficients in B. Let b be the product of the leading coefficients in these polynomials. Replacing B and A by B_b and A_b, all these leading coefficients become units, and so multiplying by their inverses, we can assume that the polynomials are monic. That is, A_b is finitely generated over B_b and there is a set of generators that all satisfy monic polynomials with coefficients in B_b. Hence, A_b is integral over B_b and therefore a finitely generated B_b-module.

Step 3: The general case. Now we return to the case where X and Y are not necassarily affine. Take an affine subset $V = \operatorname{Spec} B$ of X and cover $f^{-1}V$ with finitely many affine subsets $U_i = \operatorname{Spec} A_i$. By Step 2, for each i there is a dense open subset of V for which the restriction of f is finite. Taking the intersection of all of these gives a dense open subset V' of V such that $f^{-1}V' \cap U_i \to V'$ is finite for all i. Furthermore, a look at the previous step shows that V' is in fact a distinguished open of V. We want to shrink V' further so that $f^{-1}V'$ is affine. To start with, replace V with V' and similarly, replace U_i with $U_i \cap f^{-1}V'$. Since V' is a distinguished open of V, we still have an open affine subset of Y and the $U_i \cap f^{-1}V'$ (now written as U_i) form an affine cover of $f^{-1}V'$.

Let $U' \subseteq \cap U_i$ be an open subset that is a distinguished open in each of the U_i. So there are elements $a_i \in A_i$ such that $U' = \operatorname{Spec}(A_i)_{a_i}$ for each i. Since each A_i is finite over B, there are monic polynomials g_i with coefficients in B that the a_i satisfy. Take g_i of smallest possible degree so that the constant terms b_i are nonzero and define $b = \prod b_i$. Now the preimage of $\operatorname{Spec} B_b$ is $\operatorname{Spec}((A_i)_{a_i})_b$ (any i gives the same open) and $((A_i)_{a_i})_b$ is a finitely generated B_b module. so we are done.

Exercise 3.8. Normalization. *Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec} \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a* *normal integral scheme $\widetilde{X}$, and that there is a morphism $\widetilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism.*

To be done.

Solution.

Exercise 3.9. The Topological Space of a Product.

a *Let k be a field, and let $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$ and show that the underlying point set of the product is not the product of the underlying pointsets of the factors (even if k is algebraically closed).*

b *Let k be a field, let x and t be indeterminates over k. Then $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.*

Solution. a The stated product is the affine scheme of the ring $k[x] \otimes_k k[x]$ which is clearly isomorphic to $k[x, y]$ via (for example)

$$x \otimes 1 \mapsto x, \qquad 1 \otimes x \mapsto y$$

To see that the underlying point set of the product is not the product of the underlying point sets of the factors consider the point $(x - y) \subset k[x, y]$ (or equivalently $(x \otimes 1 - 1 \otimes x) \subset k[x] \otimes_k k[x]$). Each pair of points $((f), (g)) \subset sp\ \operatorname{Spec} k[x] \times sp\ \operatorname{Spec} k[y]$ (where f and g are irreducible or zero) gives a point $(f, g) \in \operatorname{Spec} k[x, y]$ which gets sent back to (f) and (g) via the projections. However, $(x - y)$ gets sent to (0) via both projections, yet $(0) \neq (x - y)$.

b Using greatest common denominators, every element of $k(s) \otimes_k k(t)$ can be written as

$$\frac{1}{c(s) \otimes d(t)} \left(\sum a_i(s) \otimes b_i(t) \right)$$

for some $a_i, c \in k[s], b_i, d \in k[t]$. So if

$$S = \{c(s)d(t) | c \in k[s], d \in k[t]\} \qquad \subset k[s, t]$$

then we can associate $k(s) \otimes_k k(t)$ with $S^{-1}k[s, t]$, the "holomorphic functions whose poles form horizontal and vertical lines in the plane". To see what this looks like geometrically, note that $S^{-1}k[s, t] = \varinjlim_{f \in S} k[s, t]_f$ and so $\operatorname{Spec} S^{-1}k[s, t]$ is the intersection (Spec is contravariant) of basic opens of $\mathbb{A}^2_k$ which are "complements of horizontal or vertical lines". More concretely, the points of $\operatorname{Spec} k(s) \otimes_k k(t)$ are the points of $\operatorname{Spec} k[s, t]$ that aren't in the preimage of one of the projections (excluding the generic point (0)). The topology and structure sheaf are the induced ones.

Exercise 3.10. Fibres of a Morphism.

a *If $f : X \to Y$ is a morphism, $y \in Y$ a point, show that $sp(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.*

b *Let $X = \operatorname{Spec} k[s,t]/(s-t^2)$ let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \mapsto s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed).*

Solution. a In the affine case we want to show that for a morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$ induced by $\phi : B \to A$, and a point $\mathfrak{p} \in \operatorname{Spec} B$, the preimage of $\mathfrak{p}$ is topologically homeomorphic to the space of $\operatorname{Spec} A \otimes_B (B_\mathfrak{p}/(\mathfrak{p}B_\mathfrak{p}))$. Consider the commutative diagram

$$\begin{tikzcd} X_y \arrow[d] & A \otimes_B k(\mathfrak{p}) & k(\mathfrak{p}) \arrow[l] \\ X & A \arrow[u, "\pi"] & B \arrow[l, "\phi"] \arrow[u] \end{tikzcd}$$

and a prime $\mathfrak{q}' \subset A \otimes_B k(\mathfrak{p})$. Along the top $\mathfrak{q}'$ gets pulled back to (0) and along the right, (0) gets pulled back to $\mathfrak{p}$ so along the left, $\mathfrak{q}'$ gets pulled back to $\mathfrak{q} \in f^{-1}\mathfrak{p}$ and so the morphism of topological spaces $\operatorname{Spec}(A \otimes_B k(\mathfrak{p})) \to \operatorname{Spec} A$ factors through $f^{-1}(y)$.

Injectivity. Suppose that $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$ get sent via π^{-1} to the same prime in A and consider an element of $\mathfrak{q}$. It can be represented by a sum $\sum_{i=1}^n a_i \otimes \frac{b_i}{c_i}$ where $a_i \in A, b_i, c_i \in B, c_i \notin \mathfrak{p}$. Since the tensor is over B, replacing a_i by $a_i b_i$ we can assume that $b_i = 1$. We have $\sum a_i \otimes \frac{1}{c_i} \in \mathfrak{q} \Rightarrow (\prod 1 \otimes c_j)\left(\sum a_i \otimes \frac{1}{c_i}\right) = \sum (a_i \prod_{j \neq i} c_j) \otimes 1) \in \mathfrak{q}$ and so $(a_i \prod_{j\neq i} c_j) \in \pi^{-1}\mathfrak{q} = \pi^{-1}\mathfrak{q}'$ which implies that $\sum (a_i \prod_{j \neq i} c_j) \otimes 1) \in \mathfrak{q}'$. Now multiplying by $1 \otimes \frac{1}{\prod c_i}$ we see that our original element $\sum a_i \otimes \frac{1}{c_i}$ is in $\mathfrak{q}'$. Therefore $\mathfrak{q} \subseteq \mathfrak{q}'$. By symmetry we also have $\mathfrak{q}' \subseteq \mathfrak{q}$ and so the morphism $X_y \to f^{-1}y$ is injective.

Surjectivity. Let $\mathfrak{q} \in \operatorname{Spec} A$ be in the preimage of $\mathfrak{p}$ under f and consider the subset $\mathfrak{q}' = \{a \otimes \frac{1}{b} | a \in \mathfrak{q}, b \in B \backslash \mathfrak{p}\}$ of $A \otimes_B k(\mathfrak{p})$. With some elementary work it can be seen that this is an ideal, which is in fact prime, and that the preimage $\pi^{-1}\mathfrak{q}'$ is $\mathfrak{q}$.

Closedness. Let $I \subset A \otimes_B k(\mathfrak{p})$ be the radical ideal associated to a closed subset of $\operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$. Let $I' = \pi^{-1} I \subset A$, an ideal of A. For any prime ideal $\mathfrak{q} \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$, if $\mathfrak{q} \supseteq I$ then $\pi^{-1}\mathfrak{p} \supseteq \pi^{-1} I = I'$. Conversely, consider $\mathfrak{q} \in V(I') \cap f^{-1}\mathfrak{p}$ and its preimage $\mathfrak{q}' = \{a \otimes \frac{1}{b} | a \in \mathfrak{q}, b \in B \backslash \mathfrak{p}\} \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$. For any $a \otimes \frac{1}{b} \in I$ the element $(1 \otimes b)(a \otimes \frac{1}{b}) = a \otimes 1$ is also in I, and so $a \in I'$. Since $\mathfrak{q} \supseteq I'$ this implies that $a \in \mathfrak{a}$ and so $a \otimes \frac{1}{b} \in \mathfrak{q}'$. Hence $\mathfrak{q}' \supseteq I$. What we have shown is that a prime $\mathfrak{q}$ is in $V(I)$ if and only if $\pi^{-1}\mathfrak{q}$ is in $V(I') \cap f^{-1}\mathfrak{p}$. So the morphism $\operatorname{Spec}(A \otimes_B k(\mathfrak{p})) \to f^{-1}\mathfrak{p}$ is closed. Since it is also a continuous bijection, this proves that it is a homeomorphism.

b The ring of the fibre is the tensor $(k[s,t]/(s-t^2)) \otimes_k (k[s]/(s-a))$ which is isomorphic to the ring $k[s,t]/(s-t^2, s-a)$. Since $s = t^2 = a$ in this ring, every class can be represented uniquely by a polynomial of the form $a_0 + a_1 t$. Recalling that $t^2 = a$ it can be checked that if $a \neq 0$, a ring isomorphism $k[s,t]/(s-t^2, s-a) \cong k \oplus k$ is given by

$$(1,0) \leftrightarrow \frac{1}{2\sqrt{a}}t + \frac{1}{2} \qquad (0,1) \leftrightarrow -\frac{1}{2\sqrt{a}}t + \frac{1}{2}$$

and so the fibre has two points, both with residue field k. If $a = 0$, then $k[s,t]/(s-t^2, s-a) \cong k[t]/(t^2)$, which is a nonreduced one-point scheme. For the generic point, the ring of the fibre is $(k[s,t]/(s-t^2)) \otimes_k k(s) \cong k(t)[s]/(s-t^2)$ which is an extension of degree zero.

Exercise 3.11. Closed Subschemes.

a *Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.*

b *If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.*

c *Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'.*

d *Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.*

Solution. a *Step 1: X and X' are affine.* In this case by part (b) of this exercise is also affine and is infact $\operatorname{Spec} A/I$ for some suitable ideal of $A = \Gamma(X, \mathcal{O}_X)$. Then if $\Gamma(X', \mathcal{O}_{X'}) = B$ the morphism $Y \times_X X' \to X'$ is $\operatorname{Spec}(B \otimes_A (A/I)) \to \operatorname{Spec} B$ and since $B \otimes_A (A/I) \cong B/J$ where $J = \langle \phi I \rangle$ the ideal generated by the image of I, we see that the morphism $Y \times_X X' \to X'$ is a closed immersion.

Step 2: X is affine. Let $x \in X'$ be a point of X' and $\operatorname{Spec} A = U$ an open affine neighbourhood of X. As we have just seen, $Y \times_X U \to U$ is a closed immersion, and so since $Y \times_X U = (f')^{-1}U$ it follows that the morphism of stalks $(\mathcal{O}_{X'})_x \to (f'_* \mathcal{O}_{Y \times_X X'})_x$ is surjective. Furthermore, it shows that locally, f' is a homeomorphism onto a closed subset of X'. This is enough to conclude that f' is globally a homeomorphism onto a closed subset of X'. [2]

Step 3: X and X' general. Take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X and let $f^{-1}U_i = Y_i$ and $g^{-1}U_i = X'_i$ where $g : X' \to X$. From the previous step we know that the morphisms $Y_i \times_{U_i} X'_i \to X'_i$ are closed immersions. But these are the same as the morphisms $X'_i \times_X Y \to X'_i$ and so we have found an open cover of X' on which f' is a closed immersion. This is enough to conclude that $X' \times_X Y \to X'$ is a closed immersion (see footnote).

check this

To be done.

b Let $V_i = \operatorname{Spec} B_i$ be an open affine cover of Y. By definition of the induced topology, if the V_i are open in Y then there is some open $U_i \subseteq \operatorname{Spec} A$ such that $U_i \cap Y = V_i$. Since $\operatorname{Spec} A$ is affine the U_i are covered by basic open affines $D(a_{ij})$. Consider the composition $V_i = \operatorname{Spec} B_i \to Y \to \operatorname{Spec} A$. There is an induced ring homomorphism $\phi_i : A \to B_i$ and the preimage of $D(a_{ij})$ is $D(\phi_i a_{ij}) = \operatorname{Spec}(B_i)_{\phi_i a_{ij}} \subseteq \operatorname{Spec} B_i$. The complement $Y^c \subseteq \operatorname{Spec} A$ is open and therefore covered by basic open affines $D(g_i)$. Putting these two sets of basic opens together we get a cover of $\operatorname{Spec} A$ since every point in Y is covered as well as every point not in Y. Using the quasi-compactness of $\operatorname{Spec} A$ (since it is affine) we find a finite subcover $\{D(h_i)\}$ where $h_i = f_j$ or g_k for some j, k. As this is a cover, the h_i generate the unit ideal in A. That is, there are $k_i \in A$ such that $1 = \sum h_i k_i$. Under the ring homomorphism $A \to \Gamma(Y, \mathcal{O}_Y)$ unity is preserved and so the images of the h_i generate the unit ideal there also. But recall that $D(h_i) \subseteq Y$ were all affine, and so the criteria of Exercise I.2.17(b) is satisfied and we see that Y is affine. Now we use Exercise II.2.18(d) to find that $\phi : A \to B = \Gamma(Y, \mathcal{O}_Y)$ is surjective ($f^\#$ is surjective since $Y \to \operatorname{Spec} A$ is a closed embedding). Hence, $B \cong A/\ker\phi$ and Y is determined by the ideal $\ker\phi$.

finitely many?

check the structure sheaf isomorphism on stalks

c First assume that $X = \operatorname{Spec} A$ is affine, so $Y = \operatorname{Spec} A/I$ for some radical ideal I. As we have seen in the previous part, this implies that $Y' = \operatorname{Spec} B'$ is affine and is determined by an ideal $I' \subseteq A$. That is, $B' \cong A/I'$. Since Y' and Y share the same underlying closed set, $\sqrt{I} = \sqrt{I'}$. But I is already reduced and so $I = \sqrt{I'}$. Hence, the morphism $A \to A/I$ factors as $A \to A/I' \to A/I$. In fact, it factors uniquely. If X is not affine, we can take an open affine cover $\{\operatorname{Spec} A_i\}$. If $\{\operatorname{Spec} B_i\}$ and $\{\operatorname{Spec} B'_i\}$ are the retrictions of this cover to Y and Y' respectively, then as we have just seen, we obtain morphisms $g_i : \operatorname{Spec} B_i \to \operatorname{Spec} B'_i$ which factor

[2] Let Z be the image of f' in X'. To see that Z is closed it is enough to note that the closure of a set is the union of its closures on an open cover $\{U_i\}$ since $\overline{Z} \subseteq \bigcup \overline{U_i \cap Z} \subseteq \bigcup (U_i \cap \overline{Z}) = \overline{Z}$. To see that it is mapped homeomorphically we need just to show that it is closed (since we already know that it is bijective and continuous). But this follows from the same reasoning. That is, for a closed subset $Z \subseteq Y \times_X X'$ its image $f'(Z)$ is closed globally if and only if it is closed locally on an open cover. But this we have seen since locally $Y \times_X X'$ is mapped homeomorphically onto is image.

$\operatorname{Spec} B_i \to \operatorname{Spec} A_i$. If $\mathfrak{p} \in \operatorname{Spec} A$ is a point in an intersection $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$, we take a basic open affine eighbourhood $D(f)$ of $\mathfrak{p}$ (contained in $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ (say with $f \in A_i$) and then we are still in the affine world, so we get a factoring $\operatorname{Spec}(B_i)_f \to \operatorname{Spec}(B_i')_f \to \operatorname{Spec}(A_i)_f$. Since this factoring was unique, this shows that the restriction of the g_i to the intersections agrees, and so the g_i give a well defined morphism $Y \to Y'$ which factors $Y \to X$. check the gluing

d

To be done.

Exercise 3.12. Closed Subschemes of $\operatorname{Proj} S$.

a *Let $\phi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of Exercise II.2.14 is equal to* $\operatorname{Proj} T$, *and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.*

b *If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X =$ $\operatorname{Proj} S$ defined as the mage of the closed immersion* $\operatorname{Proj} S/I \to X$. *Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.* To be done.

Solution. a

Exercise 3.13. Properties of Morphisms of Finite Type.

a *A closed immersion is a morphism of finite type.*

b *A quasi-compact open immersion is of finite type.*

c *A composition of two morphisms of finite type is of finite type.*

d *Morphisms of finite type are stable under base extension.*

e *If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.*

f *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and it f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.*

g *If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.*

Solution. a Take an open affine cover $U_i = \operatorname{Spec} A_i$ of the target scheme X. The restriction of $f : Y \to X$ to $f^{-1}U_i \to U_i$ is still a closed immersion and so it follows from Exercise II.3.11(b) that $f^{-1}U_i$ is affine, say $f^{-1}U_I = \operatorname{Spec} B_i$. Since the morphism of structure sheaves is surjective, the morphism $(A_i)_{\mathfrak{p}} \to (B_i)_{\phi^{-1}\mathfrak{p}}$ is surjective at every prime $\mathfrak{p} \in \operatorname{Spec} A_i$ and from this it follows that $A_i \to B_i$ is surjective. Hence, each B_i is a finitely generated A_i module.

b Let $i : U \to X$ be a quasi-compact open immersion. Since we already know that i quasi-compact, by Exercise II.3.3(a) we only need to show that it is locally of finite type. Let $\operatorname{Spec} A_i$ be an open affine cover of X. Then i restricts to open immersions $U_i \to \operatorname{Spec} A$. Each U_i is covered by basic open sets $D(f_{ij}) \cong \operatorname{Spec}(A_i)_{f_{ij}}$. Clearly, each $(A_i)_{f_{ij}}$ is a finitely generated A_i-algebra (generated by 1 and $\frac{1}{f_{ij}}$ for example) and so we have shown that i is locally of finite type.

c Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of composable morphisms of finite type. Let $W = \operatorname{Spec} C$ be an open affine subscheme of Z. By Exercise I.3.1 the preimage $g^{-1}W$ can be covered by finitely many open affine subschemes $\operatorname{Spec} B_i$ such that each B_i is a finitely generated C-algebra. Again by Exercise I.3.1, the preimage $f^{-1} \operatorname{Spec} B_i$ of each $\operatorname{Spec} B_i$ can be covered by finitely many open affine subschemes $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a finitely generated B_i-algebra. Hence, the preimage $(g \circ f)^{-1}W$ of W can be covered by finitely many open affine subschemes $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a finitely generated C-algebra, so by Exercise I.3.1 $g \circ f$ is a morphism of finite type.

d Consider a pullback square with f a morphism of finite type:

$$\begin{tikzcd}
X' \times_X Y \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

We want to show that $X' \times_X Y \to X'$ is a morphism of finite type. If X', X and Y are affine this is certainly true since $A \otimes_C B$ is a finitely generated B-algebra if A is a finitely generated C-algebra. If X' and X are both affine then it is true since a finite open affine cover $U_i \subseteq Y$ leads to a finite open affine cover $U_i \times_X X'$ of $Y \times_X X'$ and as we just noted, if the U_i are of finite type over X then the $U_i \times_X X'$ are of finite type over X'. Now suppose that just X is affine and let V_i be an open affine cover of Y'. Then each $V_i \times_X Y$ is of finite type over V_i and so since they cover X' and $V_i \times_X Y$ is the preimage of V_i we see that $X' \times_X Y$ is of finite type over X'.

So the only case left is when X is not affine. In this case, take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X and let $X'_i = g^{-1}U_i$ and $Y_i = f^{-1}U_i$. From the above work we see that $X'_i \times_{U_i} Y_i$ is of finite type over X'_i. But this is the same morphism as $X'_i \times_X Y \to X'_i$ and so we have found an open cover of X' on which f' is of finite type. This is enough to conclude that f' is of finite type.

e Let $\{\operatorname{Spec} C_i\}$ be an open cover of S. Since $X \xrightarrow{f} S$ (resp. $Y \xrightarrow{g} S$) is a scheme of finite type over S, the preimages $f^{-1} \operatorname{Spec} C_i$ (resp. $g^{-1} \operatorname{Spec} C_i$) can be covered by finitely many open affines, say $\{\operatorname{Spec} A_{ij}\}$ (resp. $\{\operatorname{Spec} B_{ik}\}$)

such that each A_{ij} (resp. B_{ik}) is a finitely generated C_i algebra. It can be seen from the construction of $X \times_S Y$ given in the text that $X \times_S Y$ is covered by the open affines $\operatorname{Spec}(A_{ij} \otimes_{C_i} B_{ik})$ for various i, j, k. Notice that for fixed i there are finitely many of these. Since A_{ij} and B_{ik} are finitely generated C_i algebras, it follows that $A_{ij} \otimes_{C_i} B_{ik}$ is a finitely generated C_i algebra (if $\{\alpha_\ell\} \subseteq A_{ij}$ and $\{\beta_m\} \subseteq B_{ik}$ are finitely generating sets then take $\{\alpha_\ell \otimes \beta_m\}$). So we have found an affine cover of S such that each of the preimages in $X \times_S Y$ satisfies the required property. Hence, $X \times_S Y \to S$ is a morphism of finite type.

f Since we are given that f is quasi-compact, by Exercise II.3.3(a) we just need to show that it is locally of finite type. Let $\mathcal{C} = \{\operatorname{Spec} C_i\}$ be an open affine cover of Z. Since gf is of finite type, the preimages $(gf)^{-1} \operatorname{Spec} C_i$ are covered by finitely many open affines $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a fintely generated C_i-algebra. Let $\{\operatorname{Spec} B_{ik}\}$ be an open affine cover of $g^{-1} \operatorname{Spec} C_i$ in Y. Then the preimage of each $\operatorname{Spec} B_{ik}$ is contained in $\cup_j \operatorname{Spec} A_{ij}$ so we can cover it with basic open affines coming from the $\operatorname{Spec} A_{ij}$. Stated differently, for each ik, the preimage $f^{-1} \operatorname{Spec} B_{ik}$ can be covered with affine schemes of the form $\operatorname{Spec}(A_{ij})_{a_{ik\ell}}$ for some j and some $a_{ik\ell} \in A_{ij}$. We then have a sequence of ring homomorphisms $C_i \to B_{ik} \to (A_{ij})_{a_{ik\ell}}$. The composition makes $(A_{ij})_{a_{ik\ell}}$ a finitely generated C_i-algebra (since A_{ij} is a finitely generated C_i algebra we can choose the generators for A_{ij} together with $\frac{1}{a_{ik\ell}}$) and hence, $(A_{ij})_{a_{ik\ell}}$ is a finitely generated B_{ik}-algebra (we can take the same generators as for over C_i as everything in C_i goes through B_{ik} anyway).

g Let $V_i = \operatorname{Spec} B_i$ be a finite affine cover of Y and $U_{ij} = \operatorname{Spec} A_{ij}$ be a finite affine cover of V_i such that each A_{ij} is a finitely generated B_i-algebra. Since each A_{ij} is a finitely generated B_i-algebra, and each B_i is noetherian (since Y is noetherian) it follows from Hilbert's Basis Theorem that the A_{ij} are noetherian. Hence, Y is locally noetherian.

To see that Y is quasi-compact, consider a finite open affine cover $\{U_i\}$ of X. Since f is of finite type it is quasi-compact (Exercise I.3.3(a)), and so the preimage $f^{-1}U$ of each U_i is quasi-compact (Exercise I.3.2). Now let $\{V_j\}_{j \in J}$ be an open cover of Y indexed by a set J. This gives an open cover of $f^{-1}U_i$ for each i, and since each of these is quasi-compact, there is a finite subcover, indexed by a finite set, say J_i. Now $\cup J_i$ is finite and the subcover indexed by it is still a cover so we have found a finite subcover of an arbitrary cover. Hence, Y is noetherian.

Exercise 3.14. *If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.*

Solution. We immediately reduce to the affine case, for if V is a closed subset containing every closed point, then for each U_i in an open affine cover, $V \cap U_i$ is a closed subset containing every closed point of U_i. So we can't have a proper closed subset containing every closed point globally, unless we can have them locally on affines.

So let $X = \operatorname{Spec} A$ be an affine scheme of finite type over a field. If we can show that the Jacobson radical is the same as the nilradical of A we are done, since the Jacobson radical corresponds to the smallest closed subset containing all the maximal points, and the nilradical corresponds to the closed set which is the whole space. But this is a statement of the Nullstellensatz.

An example of a scheme for which this is not true is $\operatorname{Spec} R$ for any local ring R of dimension greater than 0. The maximal ideal is unique and therefor equal to its own closure. Since the dimension is positive however, this is not the whole space.

Exercise 3.15. *Let X be a scheme of finite type over a field k (not necessarily algebraically closed).*

a *Show that the following three conditions are equivalent.*

   (a) $X \times_k \overline{k}$ is irreducible, where $\overline{k}$ denotes the algebraic closure of k.

   (b) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.

   (c) $X \times_k K$ is irreducible for every extension field K of k.

b *Show that the following three conditions are equivalent.*

   (a) $X \times_k \overline{k}$ is reduced.

   (b) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.

   (c) $X \times_k K$ is reduced for all extension fields K of k.

c *Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.*

To be done.

Solution. a

To be done.

b

c Consider

$$\operatorname{Spec} \mathbb{Q}[x, y]/(x^2 + y^2)$$

Since $x^2 + y^2$ is an irreducible polynomial in $\mathbb{Q}[x, y]$ the ideal $(x^2 + y^2)$ is prime and therefore the affine scheme is integral. However, after tensoring with $\mathbb{C}$ we get

$$\operatorname{Spec} \mathbb{C}[x, y]/((x - iy)(x + iy))$$

which is certainly not irreducible.

Now consider

$$\operatorname{Spec} \mathbb{Z}[x]/(x^2 - p)$$

for some prime p.In the integers since p is prime there is not solution to $x^2 = p$ and so $x^2 - p$ is irreducible implying that the affine scheme is integral. However after tensoring with $\mathbb{Z}/p$ we get

$$\operatorname{Spec}(\mathbb{Z}/p)[x]/(x^2)$$

which is certainly not reduced.

For an example over a field consider $\mathbb{F}_p(t)$ the function field over the field with p elements. We take our example to be

$$\operatorname{Spec} \mathbb{F}_p(t)[x]/(x^p - t)$$

which is integral as $x^p - t$ has no solutions in $\mathbb{F}_p(t)$. Tensoring with $\mathbb{F}_p(t^{\frac{1}{p}})$ however, our scheme becomes

$$\operatorname{Spec} \mathbb{F}_p(t^{\frac{1}{p}})[x]/(x - t^{\frac{1}{p}})^p$$

Exercise 3.16. Noetherian Induction. *Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y (in particular, $\mathscr{P}$ holds for the empty set). Then $\mathscr{P}$ holds for X.*

Solution. Let NP be the collection of closed subsets of X for which $\mathscr{P}$ does not hold. If NP is not empty, then since X is noetherian, it has a least element Z. If every proper closed subset of Z satisfies $\mathscr{P}$ then so does Z, however if there is a proper closed subset of Z that doesn't satisfy $\mathscr{P}$ then Z is not minimal. Hence, we have a contradiction and NP must be empty. So X satisfies $\mathscr{P}$.

Definition. A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point.

Exercise 3.17. Zariski spaces.

a *Show that if X is a noetherian scheme, then $sp(X)$ is a Zariski space.*

b *Show that any minimal nonempty closed subset of a Zariski space consists of one point. These are called closed points.*

c *Show that a Zariski space X satisfies T_0: given any two distinct points of X, there is an open set containing one but not the other.*

d *If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.*

e *Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_0 \in \overline{\{x_1\}}$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points.*

f *Let t be the functor on topological spaces introduced in the proof of (2.6): the points of $t(X)$ are the irreducible closed subsets of X and the closed subsets are the sets of the form $t(Y)$ where Y is a closed subset of X.*

If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore, X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

Solution. a We have already seen in the text (Caution 3.1.1) that $sp(X)$ is a noetherian topological space so we just need to show that each closed irreducible subset has a unique generic point. Note that for a closed irreducible subset Z of any topological space and an open subset U, either U contains the generic points of Z, or $U \cap Z = \varnothing$ (since if $\eta \notin U$ then U^c is a closed subset containing η and so $\overline{\{\eta\}} \subseteq U^c$ and therefore $U \cap Z = \varnothing$). So we can reduce to the affine case.

Suppose that X is affine. Then the irreducible closed subsets correspond to ideals I with the property that $\sqrt{I} = \sqrt{JK} \Rightarrow \sqrt{I} = \sqrt{J}$ or $\sqrt{K}$. We claim that ideals with this property are prime. To see this, suppose that $fg \in \sqrt{I}$. Then $\sqrt{I} = ((f)+\sqrt{I})((g)+\sqrt{I})$ and so either $\sqrt{I} = (f)+\sqrt{I}$ or $\sqrt{I} = g+\sqrt{I}$. Hence, either $f \in \sqrt{I}$ or $g \in \sqrt{I}$. It is straightforward that $\mathfrak{p}$ is a generic point for $V(\mathfrak{p})$ so we just need to show uniqueness. Suppose that $\mathfrak{p}, \mathfrak{q}$ are two generic points for a closed subset determind by an ideal I. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$.

b Let Z be a minimal nonempty closed subset. Since Z is minimal it is irreducible and therefore, by the previous part has a unique generic point η. For any point $x \in Z$, again since Z is minimal, we have $Z = \overline{\{x\}}$ and so $x = \eta$.

c Let x, y be the two distinct points and let $U = \overline{\{x\}}^c$. If $y \in U$ we are done. If not, then $y \in \overline{\{x\}}$. If $x \in \overline{\{y\}}$ then x and y are both generic points for the same closed irreducible subset, which contradicts the assuption that they were distinct. Hence, $x \in \overline{\{y\}}^c$.

d If $\eta \notin U$ then $\eta \in U^c$, a closed subset, and so $X = \overline{\{\eta\}} \subseteq U^c$. Therefore $U = \varnothing$.

e Let $X = \cup Z_i$ be the expression of X as the union of its irreducible closed subsets. In particular, the Z_i are the maximal irreducible closed subsets. Let η be the generic point of Z_i and x a point such that $\eta \in \overline{\{x\}}$. This implies that $Z_i \subseteq \overline{\{x\}}$ and so since the Z_i are maximal, $Z_i = \overline{\{x\}}$. Since the generic points of irreducible closed subsets are unique, this implies that $\eta = x$. So η is maximal. Conversely, suppose that η is maximal. η is in Z_i for some i. If η' is the unique generic point of Z_i then $\eta \in \overline{\{\eta'\}}$ and so since η is maximal, $\eta = \eta'$.

Let Z be a closed subset and $z \in Z$. Since $\overline{\{z\}}$ is the smallest closed subset containing z, and Z contains z, we have $\overline{\{z\}} \subseteq Z$.

f Since the lattice of closed subsets of $t(X)$ is the same as the lattice of closed subsets of X, we immediately have that $t(X)$ is noetherian. Now consider η, a closed irreducible subset of X, and its closure $\overline{\{\eta\}}$ in $t(X)$. This is the smallest closed subset of X containing η. Since η is itself a closed subset of X, we see that this is η. So if η' is a generic point for $\overline{\{\eta\}} \subseteq t(X)$ then $\overline{\{\eta\}} = \overline{\{\eta'\}}$, and so $\eta = \eta'$. Hence, each closed irreducible subset has a unique generic point.

If X is itself a Zariski space then there is a one-to-one correspondence between points and irreducible closed subsets. Hence, α isa bijection on the underlying sets. It is straightforward to see that its inverse is also continuous.

Exercise 3.18. Constructible sets. *Let X be a Zarisiki topological space. A constructible ubset of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every* ***open subset*** *is in $\mathfrak{F}$, (2) a* ***finite intersection*** *of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the* ***complement*** *of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.*

a *A subset of X is locally closed if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.*

b *Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.*

c *A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.*

d *If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.*

Solution. a Consider $\coprod_{i=1}^n Z_i \cap U_i \subseteq X$ where Z_i are closed subsets of X and U_i are open subsets of X. Note that (1) + (3) implies that all closed subsets of X are in $\mathfrak{F}$ and (2) + (3) implies that finite unions of elements of $\mathfrak{F}$ are in $\mathfrak{F}$. Hence, as long as the $Z_i \cap U_i$ are disjoint, $\coprod_{i=1}^n Z_i \cap U_i = \cup_{i=1}^n Z_i \cap U_i \in \mathfrak{F}$.

Let $\mathfrak{F}'$ be the collection of subsets of X that can be written as a finite disjoint union of locally closed subsets. We have just shown that $\mathfrak{F}' \subset \mathfrak{F}$, so by definition, if $\mathfrak{F}'$ satisfies (1), (2), and (3) then $\mathfrak{F}' = \mathfrak{F}$. We immediately have that (1) is satisfied since $U \cap X = U$ and X is closed. If $\coprod_{i=1}^n Z_i \cap U_i$ and $\coprod_{i=1}^n Z_i' \cap U_i'$ are two elements of $\mathfrak{F}'$ then their intersection is

$$\left(\coprod_{i=1}^n Z_i \cap U_i\right) \cap \left(\coprod_{i=1}^n Z_i' \cap U_i'\right) = \coprod_{i,j=1}^n (Z_i \cap Z_j') \cap (U_i \cap U_j')$$

which is in $\mathfrak{F}'$ so (2) is satisfied. We show (3) by induction on n. Let $\mathfrak{F}_n' \subset \mathfrak{F}$ be the collection of subsets of X that can be written as a finite disjoint union of n locally closed ubsets. Note that $\cup_n \mathfrak{F}_n' = \mathfrak{F}'$ and that we have already shown that, an intersection an element of an element of $\mathfrak{F}_n'$ and an element of $\mathfrak{F}_m'$ is in $\mathfrak{F}'$. Let $S \in \mathfrak{F}_1'$. So $S = U \cap Z$. Then its complement is

$$S^c = (U \cap Z)^c = U^c \cup Z^c = U^c \coprod (Z^c \cap U)$$

which is in $\mathfrak{F}'$. Now let $S \in \mathfrak{F}'_n$ and suppose that for all $i < n$, complements of memebers of $\mathfrak{F}'_i$ are in $\mathfrak{F}'$. We can write S as $S = S_{n-1} \coprod S_1$ for some $S_{n-1} \in \mathfrak{F}'_{n-1}$ and $S_1 \in \mathfrak{F}'_1$. The complement of S is then $S^c_{n-1} \cap S^c_1$. We know that S^c_{n-1} and S^c_1 are in $\mathfrak{F}'$ by inductive hypothesis and we know that their intersection is in $\mathfrak{F}'$ by (2) which we proved above. Hence, S^c is in $\mathfrak{F}'$ and we are done.

b Let $S \in \mathfrak{F}$. If if the generic point η is in S then $\overline{S} \supseteq \overline{\{\eta\}} = X$ so S is dense.

  For the converse, we use the fact that for an irreducible Zariski space, every nonempty open subset contains the generic point (Exercise 3.17(d)). Suppose $S = \coprod_{i=1}^n Z_i \cap U_i$ is dense, that is, its closure is X. The closure $\overline{S}$ is the smallest closed subset that contains S so any closed subets, for example $\cup_i Z_i$ that contains S, contains the closure. Hence, $\cup_i Z_i \supseteq \overline{S} = X$. But X is irreducible and so $Z_i = X$ for some i. So up to reindexing, $S = U_n \coprod \left(\coprod_{i=1}^{n-1} Z_i \cap U_i \right)$. Since every nonempty open subset contains the generic point, S contains the generic point.

c It is immediate the closed (resp. open) subsets are constructible and stable under specialization (resp. generization). Suppose that $S = \coprod_{i=1}^n Z_i \cap U_i$ is a constructible set stable under specialization and let x be the generic point of an irreducible component of Z_i that intersects U_i nontrivially. Since S is closed under specialization, S contains every point in the closure of $\{x\}$. So S contains every point of every irreducible component of each Z_i. That is, $S \supseteq \cup Z_i$. Now consider a point $x \in S$. It is contained in so Z_i and so $S \subseteq \cup Z_i$. Hence $S = \cup Z_i$ is closed.

  Now suppose S is a constructible set, stable under generization. Then S^c is a closed set, stable under specialization and therefore closed, so S is open.

d
$$f^{-1}\left(\coprod_{i=1}^n Z_i \cap U_i\right) = \coprod_{i=1}^n f^{-1}(Z_i \cap U_i) = \coprod_{i=1}^n f^{-1}(Z_i) \cap f^{-1}(U_i)$$

  Since f is continuous, $f^{-1}Z_i$ is closed and $f^{-1}U_i$ is open, hence, the primage of a constructible set is constructible.

Exercise 3.19. *Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particuler, $f(X)$, which need not be either open or closed, is a constructible subset of Y.*

a Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral, noetherian schemes, and f is a dominant morphism.

b *In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\phi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\phi(a) \neq 0$, then ϕ extends to a homomorphism ϕ' of B into K, such that $\phi'(b) \neq 0$.*

c *Use noetherian induction to complete the proof.*

d *Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be open of closed.*

Solution. a If $S \subseteq X$ is a constructible set then we can restrict the morphism to $f|_S : S \to Y$. So it is enough to show that $f(X)$ itself is constructible. If $\{V_i\}$ is an affine cover of Y and $\{U_{ij}\}$ is an affine cover for each $f^{-1}(V_i)$ then if $f(U_{ij})$ is constructible for each i, j then $f(X) = \cup f(U_{ij})$ is constructible, so we can assume that X and Y are affine. Similarly, if $\{V_i\}$ are the irreducible components of Y and $\{U_{ij}\}$ the irreducible components of $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible for each i, j then $f(X) = \cup f(U_{ij})$ is constructible, so we can assume that X and Y are irreducible. Reducing a scheme (or ring) doesn't change the topology, so we can assume that X and Y are reduced. Putting these last two together, we can assume that X and Y are integral.

The last thing is to show that we can assume f is dominant. Suppose that $f(X)$ is constructible for every dominant morphism. We have an induced morphism $f' : X \to \overline{f(X)} = C$ from X into the closure of its image C. Then f' is certainly dominant, so $f'(X)$ is constructible in C. This means it can be written as $\coprod U_i \cap Z_i$ a disjoint union of locally closed subsets. Since C is closed in Y, each Z_i is still closed in Y. The subsets U_i on the otherhand, can be obtained as $U_i = V_i \cap C$ for some open subsets V_i of Y, by the definition of the induced topology on C. We now have, $f(X) = \coprod U_i \cap Z_i = \coprod V_i \cap (C \cap Z_i)$, which is constructible.

b If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine integral noetherian schemes, and f is a dominant morphism, then $f : X \to Y$ is induced by a morphism $\phi : A \to B$. Since A is integral it has a generic point $\eta = (0)$ and since f is dominant, η is in the image of f. That is, there is some $\mathfrak{p} \subseteq B$ such that $\phi^{-1}\mathfrak{p} = (0)$. Since every element of B is contained in a prime ideal, in particular this means that ϕ is injective. By the assumption that f is finite type, we have that B is a finitely generated A-algebra. We now use the following lemma, with $b = 1$ to find an element a of A with the stated properties. We claim that $D(a) \subseteq f(\operatorname{Spec} B)$. To see this, suppose that $\mathfrak{p} \in D(a)$. So $a \neq \mathfrak{p}$ and the image of a under the composition $\phi : A \to A/\mathfrak{p} \to Frac(A/\mathfrak{p}) \to \overline{Frac(A/\mathfrak{p})}$ is nonzero. This means that we can lift ϕ to a homomorphism $\phi' : B \to K$ in which 1 is not zero. This means the kernel of ϕ' is a proper prime ideal $\mathfrak{q}$ of B. We now have

$A \cap \mathfrak{q} = A \cap \ker \phi' = \ker \phi = \mathfrak{p}$ and so $\mathfrak{q}$ gets sent to $\mathfrak{p}$ under $X \to Y$. Hence, $D(f)$ is contained in the image of f.

Lemma 4. Let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\phi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\phi(a) \neq 0$, then ϕ extends to a homomorphism ϕ' of B into K, such that $\phi'(b) \neq 0$.

Proof. First suppose that B is generated over A by one element. Then either $B \cong A[x]$ or $B \cong A[x]/(f(x))$ where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is an irreducible polynomial of degree, say n (irreducible since B is integral).

In the first case, given $b = g(x) = b_0 + b_1 x + \cdots + b_d x^d$ we choose $a = b_d$. Then for a homomorphism $\phi : A \to K$ into an algebraically closed field, we get a nonzero (since $\phi(b_d) \neq 0$) polynomial $\phi(g)(x) \in K[x]$ which has d roots. Since K is algebraically closed, we choose an element α that is not a root, and define $\phi' : B \to K$ by sending x to α.

Now suppose that $B \cong A[x]/(f(x))$. Let $b \in B$ and let $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in A[x]$ be a representative for b with $m < n$. Choose $a = b_m \in A$. Now given a morphim $\phi : A \to K$, we obtain polynomials $\phi(f)(x), \phi(g)(x) \in K[x]$ which since K is algebraically closed can be written as $a_n \prod_{i=1}^{n} (x - \alpha_i)$ and $b_m \prod_{i=1}^{m} (x - \beta_j)$ for some $\alpha_i, \beta_j \in K$. Note that the α_i are all distinct. Choosing an $\alpha_i \notin \{\beta_1, \ldots, \beta_m\}$ we get a morphism $\phi' : B \to K$ defined by $x \mapsto \alpha_i$ which extends ϕ. Now the image of b is $b_m \prod_{i=1}^{m} (\alpha_i - \beta_j)$ which is nonzero by our choice of α_i and the fact that $\phi(a) = \phi(b_m) \neq 0$. justify this

Now suppose that we have a B generated by n elements over A. So $B \cong A[x_1, \ldots, x_n]/\mathfrak{p}$ for some prime idea $\mathfrak{p}$. Let $\psi : A[x_1, \ldots, x_{n-1}] \to A[x_1, \ldots, x_n]$ denote the inclusion. It can be shown that $A' \stackrel{def}{=} A[x_1, \ldots, x_{n-1}]/\psi^{-1}\mathfrak{p}$ is a noetherian integral domain and $A' \subset B$ satisfies the assumptions of the lemma. So we have reduced to the case where B is generated by one element, which we have already proven. ∎ justify this

c We need to show that given a closed subset $Z \subseteq Y$ of Y, if $Z' \cap f(X)$ is constructible for every proper closed subset of Z, then $Z \cap f(X)$ is constructible. The result that $f(X)$ is constructible will then follow by Noetherian induction.

So suppose that $Z' \cap f(X)$ is constructible for every closed proper subset of a closed subset $Z \subseteq Y$. To be done.

d Consider the morphism $\operatorname{Spec} k[t, t^{-1}, (t-1)^{-1}] \to \operatorname{Spec} k[x, y]$ determined by $y \mapsto 0, x \mapsto t$. It is a composition

$$\mathbb{A}^1_k - \{0,1\} \to \mathbb{A}^1_k - \{0\} \to \mathbb{A}^2_k$$

where the image of the second morphism is the hyperbola $xy = 1$, a closed subset of $\mathbb{A}^2_k$. The closure of the image of the composition is $xy = 1$ but the image is missing the point $(1, 1)$.

Exercise 3.20. Dimension. *Let X be an integral scheme of finite type over a field k (not necassarily algebraically closed). Use appropriate results from Section I.1 to prove the following:*

a For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$.

b Let $K(X)$ be the function field of X. Then $\dim X = tr.d.\ K(X)/k$.

c If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} : P \in Y\}$.

d If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

e If U is a nonempty open subset of X, then $\dim U = \dim X$.

f If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

Lemma 5. *Let P be a point of X. Then there is an inclusion reversing bijection between irreducible subsets of X containing P and prime ideals of $\mathcal{O}_{X,P}$.*

Proof. Let $U = \operatorname{Spec} B$ be an open affine subset of X containing P and let $\mathfrak{p}$ be the prime ideal of B corresponding to P. So we have an isomorphism $\mathcal{O}_{X,P} \cong B_{\mathfrak{p}}$. We will use bijections

$$\begin{aligned} \{Z \subseteq X : Z \text{ cl. irr. and } P \in Z\} &\leftrightarrow \{Z \subseteq U : Z \text{ cl. irr. and } P \in Z\} \\ &\leftrightarrow \{I \subseteq B : I \text{ prime and } \mathfrak{p} \supseteq I\} \\ &\leftrightarrow \{I \subseteq B_{\mathfrak{p}} : I \text{ prime }\} \end{aligned}$$

The only one of these that is not immediately a bijection is the first one.

If Z is a closed irreducible subset of X containing P, then $U \cap Z$ is a nonempty closed subset of U. We can write $Z = (U^c \cap Z) \cup (\overline{U \cap Z})$. Since we assumed that Z is irreducible and intersects U we have $Z = \overline{U \cap Z}$. To see that $Z \cap U$ is irreducible, suppose we write it as $U \cap Z = Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subseteq U$. Since the Z_i are closed in U we have $Z_i = U \cap \overline{Z_i}$ where $\overline{Z_i}$ is their closure in X. Now $Z = \overline{U \cap Z} = \overline{Z_1 \cup Z_2} = \overline{Z_1} \cup \overline{Z_2}$ and so $Z = \overline{Z_i}$ for $i = 1$ or 2. Say 1. Then $Z \cap U = Z_1$ and so $Z \cap U$ is irreducible.

Conversely, if Z is a closed irreducible subset of U, then consider $\overline{Z}$. Writing it as $\overline{Z} = Z_1 \cup Z_2$ we get $Z = U \cap \overline{Z} = (U \cap Z_1) \cup (U \cap Z_2)$ and so either $Z = U \cap Z_1$ or $Z = U \cap Z_2$. Say $Z = U \cap Z_1$. Then $\overline{Z} \subseteq Z_1$ and so since $\overline{Z} = Z_1 \cup Z_2$ we find that $Z = Z_1$. So Z is irreducible. □

Solution. a Via the lemma, any chain of distinct prime ideals of $\mathcal{O}_P$ gives a chain of distinct closed irreducibles of X containing P, so $\dim \mathcal{O}_P \leq \dim X$. In particular, since any maximal chain of distinct closed irreducible subset of X ends in a closed point, say Q, we have equality for at least one point Q.

Now for P again an arbitrary point, the fraction field of $\mathcal{O}_P$ is the same as the function field of X (since it is integral) and so by Theorem I.1.8A(a), we have

$$\dim \mathcal{O}_P = tr.d.\ K(\mathcal{O}_P)/k$$
$$= tr.d.K(X)/k = tr.d.\ K(\mathcal{O}_Q)/k = \dim \mathcal{O}_Q = \dim X$$

b This is contained in the proof of the previous part.

c By definition, $\operatorname{codim}(Y, X)$ is the infimum of the codimension of the closed irreducible subsets of Y and so we can assume that Y is irreducible. In the case where Y is irreducible, it has a unique generic point η, and $\mathcal{O}_{\eta,X} \supseteq \mathcal{O}_{P,X}$ for any $P \in Y$. This implies that $\dim \mathcal{O}_{\eta,X} \leq \dim \mathcal{O}_{P,X}$ for any P and hence, $\inf\{\dim \mathcal{O}_{P,X} : P \in Y\} = \dim \mathcal{O}_{\eta,X}$. Now the result follows from the lemma.

d Suppose that X, Y are affine and Y is irreducible. Then Y corresponds to a prime ideal $\mathfrak{p}$ in $B = \Gamma(X, \mathcal{O}_X)$. We have the following immediate equalities: $\dim Y =$ height $\mathfrak{p}$, $\dim B/\mathfrak{p} = \operatorname{codim}(Y, X), \dim X = \dim B$. From these and Theorem I.1.8A(b) the result follows.

Now suppose X and Y are not necassarily affine, but Y is still irreducible. Then it has a generic point η, and choosing an affine neighbourhood $U = \operatorname{Spec} A$ of η we get a new pair, U and $Y' = U \cap Y$ which are affine, and therefore satisfy

$$\dim Y' + \operatorname{codim}(Y', U) = \dim U$$

To be done.

e They have the same function fields, and so the equality follows from the second part of this exercise.

To be done.

f

Exercise 3.21. *Let R be a discrete valuation ring containing its residue field k. Let $X = \operatorname{Spec} R[t]$ be the affine line over $\operatorname{Spec} R$. Show that statements (a), (d), (e) of Exercise 3.20 are false for X.*

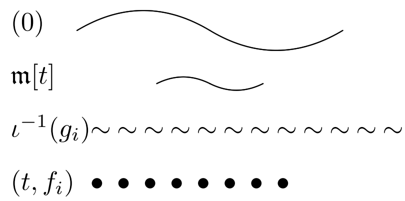
Solution. First we describe X. To list the points, we separate them into two groups by considering the preimages of the closed and generic points under $R[t] \to R$. Topologically, these are isomorphic to $\operatorname{Spec} k[t]$ and $\operatorname{Spec} K[t]$ and so the points of $\operatorname{Spec} R[t]$ are of four kinds. To describe these we name the morphisms $\pi : R[t] \to k[t]$ and $\iota : R[t] \to K[t]$. Then a point of $\operatorname{Spec} R[t]$ is one of

a $\iota^{-1}(0) = (0)$,

b $\iota^{-1}(f) = (f)$ for a polynomial $f \in R[t]$ irreducible in $K[t]$, and therefore also in $R[t]$ (note that by clearing denominators of coefficients, every polynomial in $K[t]$ can be written as a product of a unit in K and a polynomial in $R[t]$).

c $\pi^{-1}(0) = \mathfrak{m}[t]$,

d $\pi^{-1}(\overline{f}) = \mathfrak{m}[t] + (f)$ for a polynomial $f \in R[t]$ which is irreducible module $\mathfrak{m}[t]$, and therefore also irreducible $R[t]$.



$\boxed{\text{To be done.}}$

(a) A closed point of the form $\iota^{-1}\mathfrak{p}$. $\boxed{\text{To be done.}}$

(d) $\boxed{\text{To be done.}}$

(e)

Exercise 3.22. Dimension of the Fibres of a Morphism.

$\boxed{\text{To be done.}}$

Solution.

Exercise 3.23. *If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in Exercises I.3.15 and I.3.16, and it t is the functor of II.2.6 then $t(V \times W) = t(V) \times_k t(W)$.*

$\boxed{\text{To be done.}}$

Solution.