Chapter 9

Projective bundles and their Chow rings

Keynote Questions

(a) Given eight general lines $L_1, \ldots, L_8 \subset \mathbb{P}^3$, how many plane conic curves in $\mathbb{P}^3$ meet all eight? (Answer on page 354.)

(b) Can a ruled surface (that is, a $\mathbb{P}^1$-bundle over a curve) contain more than one curve of negative self-intersection? (Answer on page 341.)

Many interesting varieties, such as scrolls, blow-ups of linear subspaces of projective spaces, and some natural parameter spaces for enumerative problems can be described as projective bundles over simpler varieties. In this chapter we will investigate such varieties and compute their Chow rings. This is a tremendously useful tool, and in particular will allow us to answer the first of the keynote questions above. It will also help us to describe the Chow ring of a blow-up, which we will do in Chapter 13.

9.1 Projective bundles and the tautological divisor class

Definition 9.1. A *projective bundle* over a scheme X is a map $\pi : Y \to X$ such that for any point $p \in X$ there is a Zariski open neighborhood $U \subset X$ of p in X with $Y_U := \pi^{-1}(U) \cong U \times \mathbb{P}^r$ as U-schemes; that is, there are commuting maps

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times \mathbb{P}^r \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

where $\pi_1 : U \times \mathbb{P}^r \to U$ is projection on the first factor.

https://doi.org/10.1017/CBO9781139062046.011 Published online by Cambridge University Press

One can make projective bundles from vector bundles as follows: First, if $\mathcal{E} \cong \mathcal{O}_X^{r+1}$ is a trivial vector bundle, then

$$X \times \mathbb{P}^r = \mathrm{Proj}(\mathcal{O}_X[x_0, \dots, x_r]) = \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^*),$$

and the structure map $\mathcal{O}_X \to \mathrm{Sym}\,\mathcal{E}^*$ corresponds to the projection $\pi : X \times \mathbb{P}^r \to X$. By definition, any vector bundle $\mathcal{E}$ becomes trivial on an open cover of X, so $\mathbb{P}\mathcal{E} := \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^*) \to X$ is a projective bundle, called the *projectivization of* $\mathcal{E}$. In fact, every projective bundle can be written as $\mathbb{P}\mathcal{E}$ for some vector bundle $\mathcal{E}$. Before we can prove this, we need to know a little more about projectivizations of vector bundles.[1]

From the local description of $\mathbb{P}\mathcal{E}$ as a product, it follows that the points of $\mathbb{P}\mathcal{E}$ correspond to pairs (x, ξ) with $x \in X$ and ξ a one-dimensional subspace $\xi \subset \mathcal{E}_x$ of the fiber $\mathcal{E}_x$ of $\mathcal{E}$. The bundle $\pi^*\mathcal{E}$ on $\mathbb{P}\mathcal{E}$ thus comes equipped with a *tautological subbundle* of rank 1, whose fiber at a point $(x, \xi) \in \mathbb{P}\mathcal{E}$ is the subspace $\xi \subset \mathcal{E}_x$ of the fiber $\mathcal{E}_x$ corresponding to the point $\xi \in \mathbb{P}\mathcal{E}_x$. This subbundle is denoted by $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$. On an open set $U \subset X$ where $\mathcal{E}$ becomes trivial, so that $\pi^{-1}U = U \times \mathbb{P}^r$, the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^r}(-1)$ from the second factor. We write $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) = \mathcal{H}om(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1), \mathcal{O}_{\mathbb{P}\mathcal{E}})$ for the dual bundle. Dualizing the inclusion of the tautological bundle, we get a surjection $\pi^*\mathcal{E}^* \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

We can get an idea of the relation between $\mathbb{P}\mathcal{E}$ and $\mathcal{E}$ from the case where $\mathcal{E}$ is a line bundle. In this case $\mathbb{P}\mathcal{E}$ is locally $X \times \mathbb{P}^0$, so the projection $\pi : \mathbb{P}\mathcal{E} \to X$ is an isomorphism. Identifying $\mathbb{P}\mathcal{E}$ with X via π, we see that $\pi^*(\mathcal{E}) = \mathcal{E}$, and moreover $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \mathcal{E}$.

From this example we see that the bundles $\mathcal{E}$ and $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ are not determined by the scheme $\mathbb{P}\mathcal{E}$ or even by the map $\pi : \mathbb{P}\mathcal{E} \to X$ — rather, the bundle $\mathcal{E}$ is an additional piece of data that determines the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$. We shall soon see that, in general, the projective bundle $\mathbb{P}\mathcal{E} \to X$ alone determines $\mathcal{E}$ up to tensor product with a line bundle (Proposition 9.4), and that the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ on $\mathbb{P}\mathcal{E}$ determines $\mathcal{E}$ completely (Proposition 9.3).

9.1.1 Example: rational normal scrolls

Before continuing with the general theory we pause to work out the case of projective bundles over $\mathbb{P}^1$. As we saw in Theorem 6.29, vector bundles on $\mathbb{P}^1$ are particularly simple: Each one is a direct sum of line bundles $\bigoplus_{i=0}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Write $\mathbb{P}^1 = \mathbb{P}V$, where V is a vector space of dimension 2, with homogeneous coordinates $s, t \in V^*$. Recall that for $a \geq 1$ the *rational normal curve* of degree a is the

[1] There is a conflicting definition that is also in use. Some sources, following Grothendieck, define the projectivization of $\mathcal{E}$ to be what we would call the projectivization of $\mathcal{E}^*$, that is, $\pi : \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}) \to X$. Its points correspond to 1-quotients of fibers of $\mathcal{E}$. We are following the classical tradition, which is also the convention adopted in Fulton [1984]. Grothendieck's convention is better adapted to the generalization from vector bundles to arbitrary coherent sheaves, which we will not use.

image of the morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^a, \quad (s,t) \mapsto (s^a, s^{a-1}t, \dots, t^a).$$

When $a = 0$ we take $\varphi : \mathbb{P}^1 \to \mathbb{P}^0$ to be the constant map. More invariantly, for $a \geq 0$, we can think of φ as the map $\mathbb{P}V \to \mathbb{P}^a = \mathbb{P}W^*$ given by the complete linear series

$$|\mathcal{O}_{\mathbb{P}^1}(a)| := \big(\mathcal{O}_{\mathbb{P}^1}(a), H^0(\mathcal{O}_{\mathbb{P}^1}(a))\big).$$

Fix a sequence of $r+1$ nonnegative integers $a_0, \dots, a_r$, and let $\mathcal{E} = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i)$. We will analyze the projective bundle $\mathbb{P}\mathcal{E}$ by mapping it to a projective space $\mathbb{P}^N$ using the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

Set $W_i = H^0(\mathcal{O}_{\mathbb{P}^1}(a_i)) = \operatorname{Sym}^i V^*$ and $W = H^0(\mathcal{E}^*) = \bigoplus W_i$, and write

$$N = \dim W - 1 = \sum (a_i + 1) - 1 = r + \sum a_i.$$

Inside $\mathbb{P}\mathcal{E}$, we consider the $r+1$ rational curves

$$C_i = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_i)) \cong \mathbb{P}^1.$$

There are natural maps

$$W = H^0(\mathcal{E}^*) \to H^0(\pi^*\mathcal{E}^*) \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)),$$

and from the commutative diagrams

$$\begin{tikzcd}
\bigoplus W_i = W \arrow[rr] \arrow[d, "\text{projection}"'] & & H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \arrow[d, "\text{restriction to } C_i"] \\
W_i = H^0(\mathcal{O}_{\mathbb{P}^1}(a_i)) \arrow[rr, "\cong"] & & H^0(\mathcal{O}_{\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a_i)}(1))
\end{tikzcd}$$

we see that $W \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ is a monomorphism and that its restriction to C_i is the complete linear series $|\mathcal{O}_{\mathbb{P}^1}(a_i)|$. Let $\varphi_i : \mathbb{P}^1 \to \mathbb{P}W_i^* \subset \mathbb{P}W^*$ be the corresponding morphism, which embeds C_i as the rational normal curve of degree a_i as above.

For each $p \in \mathbb{P}^1$, the restriction of the linear series $\mathcal{W} := (\mathcal{O}_{\mathbb{P}\mathcal{E}}(1), W)$ to the fiber $\mathbb{P}^r = \pi^{-1}(p)$ is a subseries of $|\mathcal{O}_{\mathbb{P}^r}(1)|$. Since the image contains the $r+1$ linearly independent points $\varphi_i(p)$, it is the complete linear series, and this fiber is mapped isomorphically to the $\mathbb{P}^r$ that is the linear span of the points $\varphi_i(p)$. Thus the linear series $\mathcal{W}$ is base point free, and defines a morphism $\varphi : \mathbb{P}\mathcal{E} \to \mathbb{P}^N$.

We define the *rational normal scroll*

$$S(a_0, \dots, a_r) \subset \mathbb{P}\Big(\bigoplus W_i\Big) = \mathbb{P}^N$$

to be the image $\varphi(\mathbb{P}\mathcal{E})$ of this morphism. It is the union of the r-dimensional planes spanned by $\varphi_0(p), \dots, \varphi_r(p)$ as p runs over $\mathbb{P}^1$:

$$S := S(a_0, \dots, a_r) = \bigcup_{p \in \mathbb{P}^1} \overline{\varphi_0(p), \dots, \varphi_r(p)}.$$

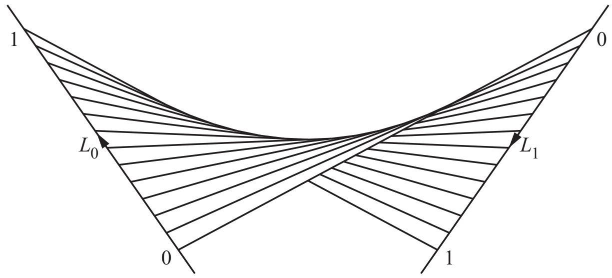

Figure 9.1 $S(1, 1)$, the union of lines joining corresponding points on the parametrized skew lines L_0 and L_1, is a nonsingular quadric in $\mathbb{P}^3$.

Since each $\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a_i)$ is embedded by the restriction of $\mathcal{W}$, and the distinct $\varphi_i(p)$ are linearly independent for every $p \in \mathbb{P}^1$, it is already clear that φ is set-theoretically an injection.

In the next section, we will show that $\mathcal{W}$ is the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$, and that when all $a_i > 0$ the map φ induces an isomorphism $\mathbb{P}\mathcal{E} \cong S$. The ideal of forms vanishing on a rational normal scroll is also easy to describe (Exercises 9.27–9.29).

We will also show that

$$\mathbb{P}\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)\right) \cong \mathbb{P}\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^1}(b_i)\right)$$

if and only if there is an integer b such that (after possibly reordering the indices) $b_i = a_i + b$ for all i; thus the description above can also be applied to describe the bundles $\mathbb{P}\left(\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)\right)$ even when some of the a_i are negative.

Some examples of this construction are already familiar. In the case $r = 0$, we have already noted that $S(a_0)$ is the rational normal curve of degree a_0 (or a point, if $a_0 = 0$). From the construction of $S(1, 1) \subset \mathbb{P}^3$ above as the union of lines joining corresponding points on two given disjoint lines, the images of φ_0 and φ_1, we see that $S(1, 1)$ is the nonsingular quadric in $\mathbb{P}^3$: the lines in the union are the lines in one of the two rulings, while the images of φ_0 and φ_1 are two of the lines in the other ruling (see Figure 9.1). Another instance is the scroll $S(1, 1, 1) \subset \mathbb{P}^5$, which is the *Segre threefold*, that is, the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$.

If $a_r = 0$, then from the construction we see that $S(a_0, \ldots, a_r)$ is a cone over $S(a_0, \ldots, a_{r-1})$, and similarly for the other a_i. This remark allows us to reduce most questions about scrolls to the case where all $a_i > 0$ for all i. For example, the quadric in $\mathbb{P}^3$ with an isolated singularity, that is, the cone over a nonsingular conic in $\mathbb{P}^2$, can be described as $S(2, 0)$ or $S(0, 2)$.

To describe the first example beyond these, the scroll $S(1, 2) \subset \mathbb{P}^4$, we choose an isomorphism between a line L and a nonsingular conic C lying in a plane disjoint from L. The scroll is then the union of the lines joining the points of L to the corresponding points of C.

There is much more to say about the geometry of rational normal scrolls, some of which will be deduced from the more general situation of projective bundles in the next sections, some in Exercises 9.27–9.29. For more information see Eisenbud and Harris [1987] or Harris [1995].

9.2 Maps to a projective bundle

One of our goals is to show that every projective bundle $\pi : Y \to X$ is the projectivization of a vector bundle $\mathcal{E}$ on X, as stated above. In fact, we will construct the bundle $\mathcal{E}$ from the geometry of π, as the dual of the direct image of a suitably chosen line bundle $\mathcal{L}$ on Y. To construct the isomorphism $Y \to \mathbb{P}\mathcal{E}$, we will use the following universal property, which generalizes the one for projective spaces:

Proposition 9.2 (Universal property of Proj)**.** *Given a vector bundle $\mathcal{E}$ on a scheme X, commutative diagrams of maps of schemes*

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \mathbb{P}\mathcal{E} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

are in natural one-to-one correspondence with line subbundles $\mathcal{L} \subset p^*\mathcal{E}$.

Proof: Given φ, we pull back the inclusion $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$ via φ and get

$$\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \varphi^*\pi^*\mathcal{E} = p^*\mathcal{E}.$$

Conversely, given $\mathcal{L} \subset p^*\mathcal{E}$, we may cover X by open sets on which $\mathcal{E}$ and $\mathcal{L}$ are trivial, and get a unique map over each of these using the universal property of ordinary projective space. By uniqueness, these maps glue together to give a map over all of X. □

To prepare for the next step we need at the least to know how to reconstruct $\mathcal{E}$ from a line bundle on $\mathbb{P}\mathcal{E}$. For future use, we will treat an easy generalization. Write $\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ for the m-th tensor power of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. Thus (for any integer m) the sheaf $\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ is the sheaf on $\mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^*)$ associated to the sheaf of $\mathrm{Sym}\,\mathcal{E}^*$-modules $(\mathrm{Sym}\,\mathcal{E}^*)(m)$ on X, obtained by shifting the grading of $\mathrm{Sym}\,\mathcal{E}^*$. For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}\mathcal{E}$ we write $\mathcal{F}(m)$ to denote $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$.

The surjection $\pi^*\mathcal{E}^* \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, restricted to the fiber over a point $(x, \xi) \in \mathbb{P}\mathcal{E}$, sends a linear form on $\mathcal{E}_x$ to its restriction to the subspace $\xi \subset \mathcal{E}_x$. Thus any global section σ of $\mathcal{E}^*$ gives rise to a global section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. The following result strengthens and extends this observation:

https://doi.org/10.1017/CBO9781139062046.011 Published online by Cambridge University Press

Proposition 9.3. *If $\pi : \mathbb{P}\mathcal{E} \to X$ is a projectivized vector bundle on X then for $m \geq 0$*

$$\pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(m) = \operatorname{Sym}^m \mathcal{E}^*,$$

and $R^i \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(m) = 0$ for $i > 0$.

Taking $m = 1$, we see that the map $\pi : \mathbb{P}\mathcal{E} \to X$, together with the tautological line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, determines $\mathcal{E}$.

Proof: Suppose that $\mathcal{E}$ has rank $r + 1$. Over an affine open set $U \subset X$ where $\mathcal{E}|_U \cong \mathcal{O}_U^{r+1}$, the natural maps $H^0(\pi^* \operatorname{Sym}^m \mathcal{E}|_U) \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)|_U)$ are isomorphisms, while $H^i(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)|_U) = 0$ for $i > 0$, so the proposition follows immediately from the definition of the direct image functors. □

Remark. Proposition 9.3 is a direct generalization of the standard computation of $H^0(\mathcal{O}_{\mathbb{P}^r}(m))$ — the case when X is a point. Though we will not make use of these facts, the rest of the computation of the cohomology of line bundles on a projective space, and Serre duality, also generalize, and one can show that

$$R^i \pi_* \mathcal{O}_{\mathbb{P}\mathcal{E}}(m) = \begin{cases} \operatorname{Sym}^m \mathcal{E}^* & \text{for } i = 0, \\ 0 & \text{for } 0 < i < r - 1, \\ \operatorname{Sym}^{-m-r-1} \mathcal{E} & \text{for } i = r. \end{cases}$$

(Here we adopt the convention that $\operatorname{Sym}^k \mathcal{E} = 0$ for $k < 0$.) As a part of our computation of the Chow ring of $\mathbb{P}\mathcal{E}$ in the next section, we will see that every line bundle on $\mathbb{P}\mathcal{E}$ has the form $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)$ for a unique line bundle $\mathcal{L}$ on X and integer m; that is, $\operatorname{Pic}(\mathbb{P}\mathcal{E}) \cong \operatorname{Pic} X \oplus \mathbb{Z}$. From the push-pull formula of Proposition B.7, we get a computation of the direct images of any line bundle:

$$R^i \pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(m)) = \mathcal{L} \otimes R^i \pi_*(\mathcal{O}_{\mathbb{P}\mathcal{E}}(m)).$$

See Dieudonné [1969, p. 308] for equivalent material, with references to EGA.

Serre duality also generalizes to a *relative duality*. For example, setting

$$\omega_{\mathbb{P}\mathcal{E}/B} = \textstyle\bigwedge^r \mathcal{E}(-r - 1)$$

we have $R^r \pi_*(\omega_{\mathbb{P}\mathcal{E}/B}) = \mathcal{O}_B$, and more generally

$$R^r \pi_*(\mathcal{M}) = \mathcal{H}om(\pi_*(\omega \otimes \mathcal{M}^{-1}), \mathcal{O}_B)$$

for any line bundle $\mathcal{M}$ on $\mathbb{P}\mathcal{E}$.

See Altman and Kleiman [1970], in particular Theorem 3.8, for most of this.

Supposing that $\mathcal{E}^*$ has a global section $\sigma \neq 0$, the proof of Proposition 9.3 shows that the corresponding section $\tilde{\sigma}$ of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ vanishes on the locus of pairs (x, ξ) such that σ_x vanishes on ξ; thus the divisor $(\tilde{\sigma})$ meets a general fiber of $\pi : \mathbb{P}\mathcal{E} \to X$ in a hyperplane. It will not in general meet *every* fiber of $\mathbb{P}\mathcal{E} \to X$ in a hyperplane, however;

the section σ of $\mathcal{E}^*$ may have zeros $x \in X$, and the divisor $(\tilde{\sigma}) \subset \mathbb{P}\mathcal{E}$ will contain the corresponding fibers $(\mathbb{P}\mathcal{E})_x = \pi^{-1}(x)$.

Using these ideas, we can characterize the schemes over X that are projective bundles:

Proposition 9.4. *Let $\pi : Y \to X$ be a smooth morphism of projective schemes whose (scheme-theoretic) fibers are all isomorphic to $\mathbb{P}^r$. The following are equivalent:*

(a) $Y = \mathbb{P}\mathcal{E}$ is the projectivization of a vector bundle $\mathcal{E}$ on X.

(b) $\pi : Y \to X$ is a projective bundle; that is, it is locally isomorphic to a product in the Zariski topology on X.

(c) There exists a line bundle $\mathcal{L}$ on Y whose restriction to each fiber $Y_x \cong \mathbb{P}^r$ of π is isomorphic to $\mathcal{O}_{\mathbb{P}^r}(1)$.

(d) There exists a Cartier divisor $D \subset Y$ intersecting a general fiber $Y_x \cong \mathbb{P}^r$ of π in a hyperplane.

Proof: Condition (a) clearly implies (b) and (c): The projectivization of a vector bundle is locally trivial in the Zariski topology, since a vector bundle is, and comes with the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$.

It is clear that (b) implies (d): Just take an isomorphism $\pi^{-1}(U) \cong U \times \mathbb{P}^r$ for any Zariski open $U \subset X$, choose a hyperplane $H \cong \mathbb{P}^{r-1} \subset \mathbb{P}^r$ and take D the closure in Y of $U \times H$.

Also, it is easy to see that (c) and (d) are equivalent: If D is a divisor as in (d), the line bundle $\mathcal{L} = \mathcal{O}_Y(D)$, restricted to a general fiber, is $\mathcal{O}_{\mathbb{P}^r}(1)$. By the constancy of the Euler characteristic of a sheaf in a flat family (Corollary B.12), the restriction of $\mathcal{L}$ to any fiber is $\mathcal{O}_{\mathbb{P}^r}(1)$.

Conversely, if $\mathcal{L}$ is a line bundle as in (c), tensoring with the pullback of an ample line bundle from X we can assume the existence of a nonzero global section of $\mathcal{L}$, whose zero locus will be the divisor of part (d).

To complete the argument we take $\mathcal{L}$ as in part (c), and we must prove that Y is as in part (a). For any $p \in X$ we have $H^1(\mathcal{L}_p) = H^1(\mathcal{O}_{\mathbb{P}^r}(1)) = 0$, so Theorem B.5 shows that $\mathcal{E} := \pi_*\mathcal{L}$ is a vector bundle whose fiber at p is $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$.

We claim that there is an isomorphism $\alpha : Y \to \mathbb{P}\mathcal{E}$ commuting with the projections to X. By Proposition 9.2 we can define the morphism α by giving a line bundle that is a subbundle of $\pi^*\mathcal{E}$, or equivalently a line bundle that is a homomorphic image of $\pi^*\mathcal{E}^* = \pi^*\pi_*\mathcal{L}$.

There is a natural map $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ coming from the definitions of π^* and π_*. Restricted to the fiber over a point p this map becomes the surjection $\mathcal{E}_p \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_p)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_p)}(1)$, so $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective. Let $\alpha : Y \to \mathbb{P}\mathcal{E}$ be the corresponding morphism.

The map α is an isomorphism on each fiber of π because it restricts to the map $\pi^{-1}(p) \cong \mathbb{P}^r \to \mathbb{P}^r$ given by the complete linear series $|\mathcal{O}_{\mathbb{P}^r}(1)|$. This shows that α is a set-theoretic isomorphism.

To prove that α is a scheme-theoretic isomorphism, we need to show that if α carries $y \in Y$ to a point $q \in \mathbb{P}\mathcal{E}$ then the map of local rings $\alpha^* : \mathcal{O}_{\mathbb{P}\mathcal{E},q} \to \mathcal{O}_{Y,y}$ is an isomorphism. Of course it is enough to prove this after completing both rings. Set $p = \pi(y)$. By smoothness, the completions of both local rings are isomorphic to $\widehat{\mathcal{O}}_{X,p}[\![z_0, \dots, z_r]\!]$. Since α commutes with the projections, it induces the identity modulo the maximal ideal of $\mathcal{O}_{X,p}$, and thus induces an isomorphism. □

We can also use Proposition 9.2 to see when two vector bundles give the same projective bundle:

Corollary 9.5. *Let X be a scheme. Two projective bundles $\pi : \mathbb{P}\mathcal{E} \to X$ and $\pi' : \mathbb{P}\mathcal{E}' \to X$ are isomorphic as X-schemes if and only if there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L} \otimes \mathcal{E}' = \mathcal{E}$. In this case the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ corresponds under the isomorphism to $\pi'^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1)$.*

Proof: Let $\mathcal{L}$ be a line bundle, and set $\mathcal{E}' = \mathcal{L} \otimes \mathcal{E}$. Tensoring the tautological subbundle $\mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to \pi'^*\mathcal{E}' = \pi'^*\mathcal{L} \otimes \pi'^*\mathcal{E}$ with $\pi'^*(\mathcal{L}^{-1})$, we get a subbundle $\pi'^*(\mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1) \to \pi'^*\mathcal{E}$. By Proposition 9.2 this determines a unique morphism of X-schemes $\varphi : \mathbb{P}\mathcal{E}' \to \mathbb{P}\mathcal{E}$ such that

$$\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \pi'^*(\mathcal{L}^{-1}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1).$$

The inverse map is defined similarly. The proof that they are inverse to each other is that the composite $\mathbb{P}\mathcal{E} \to \mathbb{P}(\mathcal{L} \otimes \mathcal{E}) \to \mathbb{P}\mathcal{E}$ corresponds to the original subbundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$.

Conversely, suppose that $\mathcal{E}'$ is a vector bundle on X, and let $\pi' : \mathbb{P}\mathcal{E}' \to X$ be the projection. If $\varphi : \mathbb{P}\mathcal{E}' \to \mathbb{P}\mathcal{E}$ is an isomorphism commuting with the projections to X, then since any isomorphism from $\mathbb{P}^n$ to itself preserves the bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ it follows that $\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ restricts on each fiber $\mathbb{P}(\mathcal{E}'_x) \cong \mathbb{P}^n$ to the bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. By Corollary B.6, $\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1) = \pi'^*(\mathcal{L}) \otimes \varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ for some line bundle $\mathcal{L}$ on X. Thus

$$\begin{aligned}
\mathcal{E}'^* = \pi'_*(\mathcal{O}_{\mathbb{P}\mathcal{E}'}(1)) &= \pi'_*(\pi'^*\mathcal{L} \otimes \varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \\
&= \mathcal{L} \otimes \pi'_*\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\
&= \mathcal{L} \otimes \pi'_*\varphi_*^{-1}\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\
&= \mathcal{L} \otimes \pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \\
&= \mathcal{L} \otimes \mathcal{E}^*,
\end{aligned}$$

and also $\varphi^*\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) = \pi^*(\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1)$, as claimed. □

9.3 Chow ring of a projective bundle

We now turn to the central problem of this chapter: to describe the Chow ring of a projective bundle $Y = \mathbb{P}\mathcal{E} \to X$. We will see that the Chow groups of Y depend only on the rank of $\mathcal{E}$, but the ring structure reflects the Chern classes of $\mathcal{E}$.

As we mentioned in Section 2.1.4, the Künneth theorem holds for the Chow ring of the product of any smooth variety with a projective space. Thus, if

$$Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^r}^{r+1}) = X \times \mathbb{P}^r$$

then

$$\begin{aligned} A(Y) &\cong A(X) \otimes_{\mathbb{Z}} A(\mathbb{P}^r) \\ &\cong A(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]/(\zeta^{r+1}) \\ &\cong A(X)[\zeta]/(\zeta^{r+1}), \end{aligned}$$

where ζ is the pullback of the hyperplane class on $\mathbb{P}^r$. In particular,

$$A(Y) = \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups. (Given that the pullback map $A(X) \to A(Y)$ is injective, here and in what follows we think of $A(X)$ as a subalgebra of $A(Y)$, suppressing the "π^*;" for example, when we write products of the form $\alpha\beta$ with $\alpha \in A(X)$ and $\beta \in A(Y)$, we mean $(\pi^*\alpha)\beta \in A(Y)$.)

The general case is not much more complicated:

Theorem 9.6. *Let $\mathcal{E}$ be a vector bundle of rank $r+1$ on a smooth projective scheme X, and let $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\mathbb{P}\mathcal{E})$. Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection. The map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is an injection of rings, and via this map*

$$A(\mathbb{P}\mathcal{E}) \cong A(X)[\zeta]/(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_{r+1}(\mathcal{E})).$$

In particular, the group homomorphism $A(X)^{\oplus r+1} \to A(\mathbb{P}\mathcal{E})$ given by $(\alpha_0, \ldots, \alpha_r) \mapsto \sum \zeta^i \pi^*(\alpha_i)$ is an isomorphism, so that

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups.

It is worth remarking that much of the statement of Theorem 9.6 remains true without the assumption that X is smooth: If $\mathcal{E}$ is a vector bundle of rank $r+1$ over an arbitrary scheme X and $\mathbb{P}\mathcal{E} = \operatorname{Proj}(\operatorname{Sym}\mathcal{E}^*)$ its associated projective bundle, then we have a well-defined line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on $\mathbb{P}\mathcal{E}$ such that $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ restricts to the

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hyperplane class on each fiber, and we can show that

$$A(\mathbb{P}\mathcal{E}) \cong \bigoplus_{i=0}^{r} \zeta^i A(X)$$

as groups, just as in the smooth case (see Fulton [1984, Chapter 3]). (Note that in this setting we do not have a ring structure on $A(X)$ or $A(\mathbb{P}\mathcal{E})$, but multiplication by the class ζ is still well-defined since it is the Chern class of a line bundle.)

It was one of the insights of Grothendieck [1958] that Theorem 9.6 could be inverted and used to *define* the Chern classes of $\mathcal{E}$ as the coefficients in the unique expression of ζ^{r+1} as a linear combination of the classes $1, \zeta, \dots, \zeta^r$ (or rather to prove the existence of classes satisfying the axioms of Theorem 5.3). The original definitions of Chern and Stiefel–Whitney classes in the 1930s came from topology. They did not mention degeneracy loci, but could be directly related to that characterization of the classes; as we have seen in Chapters 6 and 7, this is closer to the way Chern classes are thought of and used in practice. As a definition, however, it has the drawback of depending on the existence of global sections. (This is a problem only in the algebro-geometric context; in the continuous or $\mathcal{C}^\infty$ settings, thanks to partitions of unity there is never a shortage of sections.) While it is possible to define Chern classes for bundles with enough sections via degeneracy loci, and even (as we illustrate in Section 5.9.1) to prove basic properties such as the Whitney formula in that setting, in order to have a full toolkit of techniques for calculating Chern classes it is necessary to extend the definition to arbitrary bundles, and for this the Grothendieck–Serre definition is better.

We isolate part of the proof of Theorem 9.6 that will be useful elsewhere:

Lemma 9.7. *Let the hypotheses be as in Theorem 9.6. If $\alpha \in A(X)$, then*

$$\pi_*(\zeta^i \alpha) = \begin{cases} \alpha & \text{if } i = r, \\ 0 & \text{if } i < r. \end{cases}$$

Proof: By the push-pull formula (Proposition B.7), $\pi_*(\zeta^i \alpha) = \pi_*(\zeta^i)\alpha$. If $i < r$, then $\pi_*(\zeta^i)$ is zero for dimension reasons. If $i = r$, we see similarly that $\pi_*(\zeta^r)$ must be a multiple $m[X] \in A^0(X)$ of the fundamental class of X. Let η be the class of a point $x \in X$ and $f = \pi^*(\eta) = [\mathbb{P}\mathcal{E}_x]$ the class of the fiber $\mathbb{P}\mathcal{E}_x \cong \mathbb{P}^r$. Intersecting both sides of the equality $\pi_*(\zeta^r) = m[X]$ with η and taking degrees, we have

$$m = \deg(\pi_*(\zeta^r) \cdot \eta) = \deg(\zeta^r \cdot [\mathbb{P}^r]) = 1,$$

since the restriction of ζ to a fiber is the hyperplane class. □

In fact, we have encountered this construction before, in the proof of Lemma 5.12.

Proof of Theorem 9.6: Let $\psi : A(\mathbb{P}\mathcal{E}) \to \bigoplus_{i=0}^{r} A(X)\zeta^i$ be the map

$$\beta \mapsto \sum_i \pi_*(\zeta^{r-i}\beta)\zeta^i,$$

and let $\varphi : \bigoplus_{i=0}^{r} A(X) \to A(\mathbb{P}\mathcal{E})$ be the sum of the multiplications by powers of ζ:

$$\varphi : (\alpha_0, \ldots, \alpha_r) \mapsto \sum_i \zeta^i \alpha_i.$$

By Lemma 9.7, the composite $\psi\varphi$ is upper-triangular with ones on the diagonal; in particular, φ is a monomorphism.

To prove the additive part of Theorem 9.6, it now suffices to show that the subgroups $\zeta^i A(X)$ generate $A(\mathbb{P}\mathcal{E})$ additively. This is a relative version of the fact that the linear subspaces of a projective space generate its Chow ring, and the proof runs along the same lines. In the case of a single projective space, we used the technique of dynamic projection to degenerate a given subvariety $Z \subset \mathbb{P}^n$ to a multiple of a linear space; we do the same thing here, but in a family of projective spaces.

We start with a definition. If $Z \subset \mathbb{P}\mathcal{E}$ is a k-dimensional subvariety, we say that Z has *footprint* l if the image $W = \pi(Z)$ has dimension l, or equivalently if the general fiber of the map $\pi : Z \to W$ has dimension $k - l$.

Lemma 9.8. *If $Z \subset \mathbb{P}\mathcal{E}$ is a subvariety of dimension k and footprint l, then*

$$Z \sim Z' + \sum n_i B_i$$

for some subvarieties $B_i \subset \mathbb{P}\mathcal{E}$ such that:

(a) $[Z'] = \zeta^{r-k+l}\alpha$ for a class $\alpha \in A(X)$.
(b) Each B_i has footprint strictly less than l.

Applying the lemma repeatedly, we can express the class of an arbitrary subvariety as a sum of classes of the form $\zeta^i\alpha$, establishing the group isomorphism $A(\mathbb{P}\mathcal{E}) \cong \bigoplus \zeta^i A(X)$.

Proof of Lemma 9.8: By Corollary 9.5, replacing $\mathcal{E}$ with its tensor product with a line bundle $\mathcal{L}$ does not change $\mathbb{P}\mathcal{E}$, but has the effect of replacing the class ζ by $\zeta - \pi^* c_1(\mathcal{L})$. In particular, it does not affect the truth of our assertion, so we can assume from the outset that $\mathcal{E}^*$ is generated by global sections.

This done, we choose a point $x \in \pi(Z) \subset X$ and a general collection $\tau_0, \ldots, \tau_r$ of global sections of $\mathcal{E}^*$, making sure that the τ_i satisfy two conditions:

(a) $\tau_0(x), \ldots, \tau_r(x)$ are independent, that is, they span the fiber $\mathcal{E}_x$.
(b) The zero locus $(\tau_0(x) = \cdots = \tau_{k-l}(x) = 0) \subset \mathbb{P}\mathcal{E}_x$ is disjoint from the fiber $Z_x = Z \cap \mathbb{P}\mathcal{E}_x$ of Z over x.

These are both open conditions; let $U \subset X$ be the locus of $x \in X$ where they hold. Note in particular that, by the first condition, the bundle $\mathbb{P}\mathcal{E}$ is trivial over U, the sections $\tau_0, \ldots, \tau_r$ giving an isomorphism $\mathbb{P}\mathcal{E}_U \cong U \times \mathbb{P}^r$.

Now consider the one-parameter group of automorphisms A_t of $\mathbb{P}\mathcal{E}_U \cong U \times \mathbb{P}^r$ given, in terms of this trivialization, by the matrix

$$\begin{pmatrix} I_{k-l+1} & 0 \\ 0 & t \cdot I_{r-k+l} \end{pmatrix}.$$

Let $\widetilde{Z} = Z \cap \mathbb{P}\mathcal{E}_U$ be the preimage of U in Z (note that $\pi(Z) \cap U \neq \varnothing$, since $x \in \pi(Z)$); let Z_t be the closure of the image $A_t(\widetilde{Z})$ and let Z_0 be the limiting cycle, as $t \to 0$, of the subvarieties Z_t. In other words, let $\Phi^\circ \subset \mathbb{A}^1 \times \mathbb{P}\mathcal{E}$ be the incidence correspondence

$$\Phi^\circ = \{(t, p) \in \mathbb{A}^1 \times \mathbb{P}\mathcal{E} \mid t \neq 0 \text{ and } p \in A_t(\widetilde{Z})\};$$

let Φ be the closure of Φ° and let Z_0 be the fiber of Φ over $t = 0$.

What does Z_0 look like? Over the open subset $U \subset X$ the original cycle Z has been flattened to a multiple of the zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$. There is thus a unique component Z' of Z_0 dominating $W = \pi(Z)$, and it is the closure of the intersection of the common zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ with the preimage $\pi^{-1}(W \cap U)$.

Now, we have arranged for $\mathcal{E}^*$ to be generated by global sections, so that the linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ has no base locus. Since the τ_i are general sections of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, by Bertini the common zero locus $\tau_{k-l+1} = \cdots = \tau_r = 0$ of $r - k + l$ of them intersects the subvariety $\pi^{-1}(W)$ generically transversely, in a k-dimensional subvariety of $\mathbb{P}\mathcal{E}$ with class $[W] \cdot \zeta^{r-k+l}$; moreover, since this intersection is fibered over W with fibers $\mathbb{P}^{k-l}$, it is irreducible. In sum,

$$[Z'] = m[W] \cdot \zeta^{r-k+l}$$

for some multiplicity m.

To complete the proof we note that we do not need to know what happens over the complement of $U \cap W$ in W, because any component of Z_0 not dominating W necessarily has footprint smaller than l. $\square$

From this description of the Chow groups we see that we can write ζ^{r+1} as a linear combination of products of (pullbacks of) classes in $A(X)$ with lower powers of ζ — that is, ζ satisfies a monic polynomial f of degree $r + 1$ over $A(X)$. Thus the ring homomorphism $A(X)[\zeta] \to A(\mathbb{P}\mathcal{E})$ factors through the quotient $A(X)[\zeta]/(f)$. Since $A(X)[\zeta]/(f) \cong \bigoplus \zeta^i \mathbb{A}(X)$ as groups, it follows that the map $A(X)[\zeta]/(f) \to A(\mathbb{P}\mathcal{E})$ is an isomorphism of rings.

It remains to identify the polynomial f. Let $\mathcal{S} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$, and let $\mathcal{Q}$ be the cokernel of the natural inclusion $\mathcal{S} \to \pi^*\mathcal{E}$, a bundle of rank r. We have an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Identifying $A(X)$ with $\pi^*A(X)$ as before, we have

$$c(\mathcal{S}) \cdot c(\mathcal{Q}) = c(\mathcal{E})$$

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by the Whitney formula (Theorem 5.3).

We defined the class ζ to be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, which is the dual of $\mathcal{S}$; thus $c(\mathcal{S}) = 1 - \zeta$, and we can write this as

$$c(\mathcal{Q}) = c(\mathcal{E}) \cdot c(\mathcal{S})^{-1} = c(\mathcal{E})(1 + \zeta + \zeta^2 + \cdots).$$

Since $\mathcal{Q}$ is a vector bundle of rank r, we conclude that

$$0 = c_{r+1}(\mathcal{Q}) = \zeta^{r+1} + c_1(\mathcal{E})\zeta^r + c_2(\mathcal{E})\zeta^{r-1} + \cdots + c_r(\mathcal{E})\zeta + c_{r+1}(\mathcal{E}),$$

so the polynomial f is given by the formula in the theorem. □

If $\mathcal{L}$ is a line bundle on X then Corollary 9.5 shows that $\mathbb{P}\mathcal{E} \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$, but the class ζ is different in the two representations; the two classes differ by multiplication with the pullback of $\mathcal{L}$. The relation between the two resulting descriptions of the Chow ring is addressed in Exercises 9.30 and 9.31.

Using Theorem 9.6, we can immediately compute the degrees of rational normal scrolls, or, more generally, of any projectivized vector bundle $\mathbb{P}\mathcal{E}$ over a curve X, embedded by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$:

Corollary 9.9. *If $a_0, \dots, a_r$ are positive integers, then the degree of the rational normal scroll $S(a_0, \dots, a_r)$ is $\sum a_i$. More generally, if $\mathcal{E}$ is a vector bundle on a smooth curve X and the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ on $\mathbb{P}\mathcal{E}$ is very ample, then the degree of the image of $\mathbb{P}\mathcal{E}$ under the embedding given by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is $-\deg c_1(\mathcal{E})$.*

Note that degree and codimension of a scroll S satisfy the equation

$$\deg S = 1 + \operatorname{codim} S.$$

This is the minimal degree for any subvariety of projective space not contained in a hyperplane. The Veronese surface in $\mathbb{P}^5$, and any cone over it, also satisfy this equation, but these are the only "varieties of minimal degree." See Harris [1995, Theorem 19.19].

Proof: If the rank of $\mathcal{E}$ is $r + 1$ then the dimension of $\mathbb{P}\mathcal{E}$ is $r + 1$, so the degree of the image of $\mathbb{P}\mathcal{E}$ under the embedding given by $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ is $\deg \zeta^{r+1}$. Since X is one-dimensional, we have $c_i(\mathcal{E}) = 0$ for $i > 1$, so $\zeta^{r+1} = -c_1(\mathcal{E})$. If $X = \mathbb{P}^1$ and

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_r),$$

then $\deg c_1(\mathcal{E}) = -\sum a_i$ and $S(a_0, \dots, a_r)$ is the embedding of $\mathbb{P}\mathcal{E}$ by $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. □

9.3.1 The universal k-plane over $\mathbb{G}(k, n)$

In this section and the next, we will use Theorem 9.6 to give a description of the Chow ring of some varieties that arise often in algebraic geometry: the universal k-plane over the Grassmannian $\mathbb{G}(k, n)$ and the blow-up of $\mathbb{P}^n$ along a linear space.

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For the first of these, let $G = \mathbb{G}(k, \mathbb{P}V)$ be the Grassmannian parametrizing k-planes $\Lambda \subset \mathbb{P}V$ in the projectivization of an $(n+1)$-dimensional vector space V, and let Φ be the universal plane

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\},$$

initially introduced in Section 3.2.3. We can recognize Φ, via the projection $\pi : \Phi \to G$ on the first factor, as the projectivization $\mathbb{P}\mathcal{S}$ of the universal subbundle on G, and use Theorem 9.6 to describe $A(\Phi)$. We will use the notation introduced above: We will identify $A(G)$ with its image in $A(\Phi)$ via the pullback map π^*, and denote the first Chern class of the tautological bundle $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ by $\zeta \in A^1(\Phi)$.

Note that a linear form $l \in V^*$ on V gives rise to a section of $\mathcal{S}^*$ by restriction in turn to each subspace of V, hence to a section of $\pi^*\mathcal{S}^*$, and ultimately to a section of $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ via the surjection $\pi^*\mathcal{S}^* \to \mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ dual to the tautological inclusion $\mathcal{O}_{\mathbb{P}\mathcal{S}}(-1) \hookrightarrow \pi^*\mathcal{S}$. Simply put, if we think of $\Phi = \mathbb{P}\mathcal{S}$ as the variety of pairs $(\widetilde{\Lambda}, \xi)$ with $\widetilde{\Lambda} \subset V$ a $(k+1)$-dimensional subspace and $\xi \subset \widetilde{\Lambda}$ a one-dimensional subspace, then we can define a section σ_l of $\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)$ by setting

$$\sigma_l(\widetilde{\Lambda}, \xi) = l|_\xi.$$

In particular, we see that the zero locus of the section σ_l is just the locus of $(\widetilde{\Lambda}, \xi)$ such that ξ is contained in the hyperplane $\mathrm{Ker}(l) \subset V$, and hence *the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)) \in A^1(\Phi)$ is just the pullback of the hyperplane class on $\mathbb{P}V$ via the projection map $\eta : \Phi \to \mathbb{P}V$ on the second factor.*

Recalling the calculation of the Chern classes of the universal bundles on $\mathbb{G}(k, n)$ from Section 5.6.2 and applying Theorem 9.6, we conclude:

Proposition 9.10. *Let $G = \mathbb{G}(k, n)$ be the Grassmannian of k-planes in $\mathbb{P}^n$ and $\Phi \subset G \times \mathbb{P}^n$ the universal k-plane as above, with $\pi : \Phi \to G$ and $\eta : \Phi \to \mathbb{P}^n$ the projection maps. We have then*

$$A(\Phi) = A(G)[\zeta]/(\zeta^{k+1} - \sigma_1\zeta^k + \sigma_{1,1}\zeta^{k-1} + \cdots + (-1)^{k+1}\sigma_{1,1,\dots,1}),$$

where $\zeta \in A^1(\Phi)$ is the tautological class, or equivalently the pullback via η of the hyperplane class in $\mathbb{P}^n$.

The two special cases occurring most commonly are the cases $k = n-1$ of the universal hyperplane and the case $k = 1$ of the universal line. In the first case,

$$\Phi = \{(H, p) \in \mathbb{P}^{n*} \times \mathbb{P}^n \mid p \in H\},$$

and if we let ω be pullback to Φ of the hyperplane class in $\mathbb{P}^{n*}$, we have

$$A(\Phi) = \mathbb{Z}[\omega, \zeta]/(\omega^{n+1}, \zeta^{n+1}, \zeta^n - \omega\zeta^{n-1} + \cdots + (-1)^n\omega^n).$$

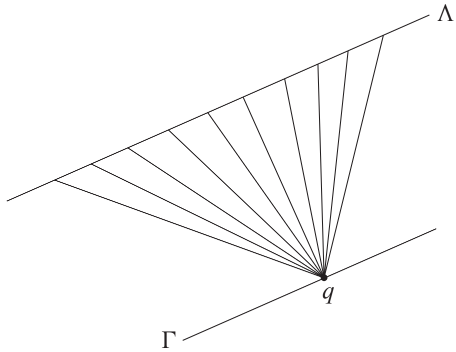

Figure 9.2 The fiber over a point under the projection of $\mathbb{P}^3$ from the line Λ.

We have written the ideal of relations in this way to emphasize the symmetry, but it is redundant: we could drop either ω^{n+1} or ζ^{n+1}. Note that when $a+b=\dim(\Phi)=2n-1$, we have

$$\deg(\omega^a\zeta^b)=\begin{cases}1 & \text{if } (a,b)=(n,n-1) \text{ or } (n-1,n),\\ 0 & \text{otherwise,}\end{cases}$$

which we could also see from the fact that $\Phi\subset\mathbb{P}^{n*}\times\mathbb{P}^n$ is a hypersurface of bidegree $(1,1)$.

The universal line will also come up a lot in the following chapters; in this case we have

$$A(\Phi)=A(\mathbb{G}(1,n))[\zeta]/(\zeta^2-\sigma_1\zeta+\sigma_{1,1}).$$

We will leave it to the reader to calculate the degrees of monomials $\sigma_1^a\sigma_{1,1}^b\zeta^c$ of top degree $a+2b+c=\dim(\Phi)=2n-1$ in Exercise 9.33.

9.3.2 The blow-up of $\mathbb{P}^n$ along a linear space

In Section 2.1.9 we saw how to describe the Chow ring of the blow-up of projective space at a point. We can now analyze much more generally and systematically the Chow ring of the blow-up $Z=\mathrm{Bl}_\Lambda\mathbb{P}^n$ of projective space $\mathbb{P}^n=\mathbb{P}V$ along any linear subspace $\Lambda\cong\mathbb{P}^{r-1}$. The key is to realize Z as the total space of a projective bundle.

To understand the picture, first recall that the blow-up is the graph of the rational map $\pi_\Lambda:\mathbb{P}^n\dashrightarrow\mathbb{P}^{n-r}$ given by projection from Λ. Thus $Z\subset\mathbb{P}^n\times\mathbb{P}^{n-r}$. We will show that the projection $Z\to\mathbb{P}^{n-r}$ to the second factor makes Z into a projective bundle. Certainly, each fiber of the projection is an r-dimensional projective space (see Figure 9.2). Concretely, if we choose an $(n-r)$-plane $\Gamma\subset\mathbb{P}^n$ disjoint from Λ, we can write

$$Z=\{(p,q)\in\mathbb{P}^n\times\Gamma\mid p\in\overline{\Lambda,q}\}.$$

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The fiber over a point $q \in \Gamma$ is thus the linear subspace $\overline{\Lambda, q} \cong \mathbb{P}^{n-r+1} \subset \mathbb{P}^n$. If we write $\mathbb{P}^n$ as $\mathbb{P}V$, then Λ corresponds to an r-dimensional linear subspace $V' \subset V$ and Γ corresponds to a complementary $(n-r+1)$-dimensional subspace W. The fiber of Z over $q \in \Gamma$ corresponds to the subspace spanned by V' and the one-dimensional subspace $\tilde{q}$ corresponding to q in W. Here V' is fixed, while the one-dimensional subspace varies over all such subspaces of W. This suggests that Z is the projectivization of the bundle $\mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus (V' \otimes \mathcal{O}_{\mathbb{P}^{n-r}})$, which we will now prove:

Proposition 9.11. *Let $V' \subset V$ be an r-dimensional subspace of an $(n+1)$-dimensional vector space V, and let*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus (V' \otimes \mathcal{O}_{\mathbb{P}^{n-r}}),$$

so that $\mathcal{E}$ is a vector bundle of rank $r+1$ on $\mathbb{P}^{n-r} = \mathbb{P}(V/V')$. The blow-up Z of $\mathbb{P}(V)$ along the $(r-1)$-dimensional subspace $\mathbb{P}(V')$, together with its projection to $\mathbb{P}^{n-r}$, is isomorphic to the projective bundle $\pi : \mathbb{P}\mathcal{E} \to \mathbb{P}^{n-r}$. Under this isomorphism, the blow-up map $Z \to \mathbb{P}^n$ corresponds to the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$.

Proof: Choose a complement $V/V' \cong W \subset V$ to V', so that $V = W \oplus V'$. With $\mathcal{E}$ as in the proposition, the natural inclusion $\mathcal{O}_{\mathbb{P}W}(-1) \subset (W \otimes \mathcal{O}_{\mathbb{P}W})$ induces an inclusion

$$\mathcal{E} \subset (W \otimes \mathcal{O}_{\mathbb{P}W}) \oplus (V' \otimes \mathcal{O}_{\mathbb{P}W}) = V \otimes \mathcal{O}_{\mathbb{P}W}.$$

The dual map, which is a surjection, induces an isomorphism $V^* \to H^0(\mathcal{E}^*) = V'^* \oplus W^*$. Thus $\mathcal{E}^*$ is generated by its global sections and the complete linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ corresponds to a map $\mathbb{P}\mathcal{E} \to \mathbb{P}V$.

The fiber of $\mathcal{E}$ over a point $q \in \mathbb{P}W$ is, as a subspace of V, equal to $V' \oplus \tilde{q}$, whose projectivization is the fiber over q of the blow-up Z of $\mathbb{P}V'$ in $\mathbb{P}V$. Thus, together with the projection map $\pi : \mathbb{P}\mathcal{E} \to \mathbb{P}W$, we get a closed immersion $\varphi : \mathbb{P}\mathcal{E} \to \mathbb{P}V \times \mathbb{P}W$ that maps the fiber of $\mathbb{P}\mathcal{E}$ isomorphically to Z. ☐

Corollary 9.12. *Let $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$ be the blow-up of an $(r-1)$-plane Λ in $\mathbb{P}^n$. Writing $\alpha, \zeta \in A^1(Z)$ for the pullbacks of the hyperplane classes on $\mathbb{P}^{n-r}$ and $\mathbb{P}^n$ respectively, we have*

$$A(Z) = \mathbb{Z}[\alpha, \zeta]/(\alpha^{n-r+1}, \zeta^{r+1} - \alpha\zeta^r).$$

With this notation the class of the exceptional divisor $E \subset Z$, the preimage of Λ in Z, is

$$[E] = \zeta - \alpha.$$

Proof: The Chern class of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-r}}(-1) \oplus (V' \otimes \mathcal{O}^r_{\mathbb{P}^{n-r}})$ is $1 - \alpha$, so the formula for $A(Z)$ follows at once from Theorem 9.6. Since ζ is the class of the preimage of a hyperplane $H \subset \mathbb{P}^n$ (which could contain Λ), and α is represented by the proper transform of a hyperplane containing Λ, we have $[E] = \zeta - \alpha$ as claimed. ☐

For example, in the case of the blow-up of the plane at a point we have

$$[E]^2 = (\zeta - \alpha)^2 = \zeta^2 - 2\alpha\zeta + \alpha^2 = -\zeta^2,$$

that is, minus the class of a point, as we already knew. But we can now compute $\deg[E]^n$ in general (Exercise 9.38).

9.3.3 Nested pairs of divisors on $\mathbb{P}^1$ revisited

We start by introducing two vector bundles that arise often in studying the geometry of rational curves; in particular, they will be a central object of study in Section 10.4.2.

To begin with, let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the projective space of polynomials of degree d in two variables modulo scalars — that is, divisors of degree d on $\mathbb{P}^1$. For any $e \geq d$, then, we can define a vector bundle $\mathcal{F}$ on $\mathbb{P}^d$ informally by associating to each divisor $D \in \mathbb{P}^d$ the vector space

$$\mathcal{F}_D = H^0(\mathcal{I}_D(e))$$

of polynomials of degree e on $\mathbb{P}^1$ vanishing on D. Similarly, we can define a bundle $\mathcal{E}$ on $\mathbb{P}^d$ informally by associating to each divisor $D \in \mathbb{P}^d$ the quotient vector space

$$\mathcal{E}_D = H^0(\mathcal{O}_{\mathbb{P}^1}(e))/H^0(\mathcal{I}_D(e)) = H^0(\mathcal{O}_D(e))$$

of polynomials of degree e modulo those vanishing on D. To define these bundles precisely, let $\mathcal{D} \subset \mathbb{P}^d \times \mathbb{P}^1$ be the universal divisor of degree d, that is

$$\mathcal{D} = \{(D, p) \in \mathbb{P}^d \times \mathbb{P}^1 \mid p \in D\},$$

and let $\mu : \mathbb{P}^d \times \mathbb{P}^1 \to \mathbb{P}^d$ and $\nu : \mathbb{P}^d \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection maps. We can then take

$$\mathcal{F} = \mu_*(\nu^*\mathcal{O}_{\mathbb{P}^1}(e) \otimes \mathcal{I}_{\mathcal{D}})$$

and

$$\mathcal{E} = \alpha_*(\nu^*\mathcal{O}_{\mathbb{P}^1}(e) \otimes \mathcal{O}_{\mathcal{D}});$$

an application of the theorem on cohomology and base change shows that these have the fibers indicated, and that the exact sequence of sheaves on $\mathbb{P}^d \times \mathbb{P}^1$

$$0 \longrightarrow \mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathcal{D}} \longrightarrow 0,$$

tensored with the line bundle $\nu^*\mathcal{O}_{\mathbb{P}^1}(e)$ and pushed forward to $\mathbb{P}^d$, gives the expected exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(e)) \otimes \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{9.1}$$

of bundles on $\mathbb{P}^d$.

Consider now the projectivization $\Phi = \mathbb{P}\mathcal{E}$ of the bundle $\mathcal{E}$. This is a variety we have encountered before, in Section 2.1.8: We can realize it as the subvariety

$$\Phi = \{(D, E) \in \mathbb{P}^d \times \mathbb{P}^e \mid E \geq D\}$$

of *nested pairs* of divisors of degrees d and e on $\mathbb{P}^1$. Moreover, under the inclusion of $\Phi = \mathbb{P}\mathcal{E}$ in $\mathbb{P}^d \times \mathbb{P}^e$, the pullback τ of the hyperplane class from $\mathbb{P}^e$ restricts to the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ on $\mathbb{P}\mathcal{E}$.

We can use this to describe the Chow ring of Φ, and correspondingly the Chern classes of $\mathcal{E}$. The key, as it was in Section 2.1.8, is to observe that $\Phi \cong \mathbb{P}^d \times \mathbb{P}^{e-d}$ abstractly, via the map

$$\alpha : \mathbb{P}^d \times \mathbb{P}^{e-d} \to \mathbb{P}^d \times \mathbb{P}^e, \quad (D, D') \mapsto (D, D + D').$$

Let σ, τ and μ be the pullbacks of the hyperplane classes on $\mathbb{P}^d$, $\mathbb{P}^e$ and $\mathbb{P}^{e-d}$, respectively. As we saw in Section 2.1.8, the pullback of the class τ to Φ is the sum $\sigma + \mu$. We can then rewrite the relation $\mu^{e-d+1} = 0$ in $A(\mathbb{P}\mathcal{E})$ as

$$0 = (\zeta - \sigma)^{e-d+1} = \sum (-1)^i \binom{e-d+1}{i} \sigma^i \zeta^{e-d+1-i},$$

and we conclude that

$$c_i(\mathcal{E}) = (-1)^i \binom{e-d+1}{i} \sigma^i.$$

To express this more compactly, we can write the total Chern class as

$$c(\mathcal{E}) = (1 - \sigma)^{e-d+1}.$$

In this form, it follows from the exact sequence (9.1) that

$$c(\mathcal{F}) = \frac{1}{(1-\sigma)^{e-d+1}} = \sum \binom{e-d+i}{i} \sigma^i,$$

so we have the Chern classes of $\mathcal{F}$ as well.

9.4 Projectivization of a subbundle

If $\mathcal{E}$ is a vector bundle on a smooth variety X and $\mathcal{F} \subset \mathcal{E}$ a subbundle then $\mathbb{P}\mathcal{F}$ is naturally a subvariety of $\mathbb{P}\mathcal{E}$, and we can ask for its class in the Chow ring $A(\mathbb{P}\mathcal{E})$. This will be a crucial element in understanding the Chow ring of a blow-up in general (Section 13.6); for now, it will allow us to answer Keynote Question (b).

Let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projection and let $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E}$ be the universal subbundle. A point $p \in \mathbb{P}\mathcal{E}$ lying over a point $x \in X$ corresponds to the one-dimensional space that is the fiber of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ at p. Thus $p \in \mathbb{P}\mathcal{F}$ if and only if this space is contained in the fiber of $\mathcal{F}$. In other words, $p \in \mathbb{P}\mathcal{F}$ if and only if the composite map

$$\varphi : \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \to \pi^*\mathcal{E} \to \pi^*(\mathcal{E}/\mathcal{F})$$

vanishes at p. We can view φ as a global section of the bundle

$$\mathcal{H}om(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1), \pi^*(\mathcal{E}/\mathcal{F})) \cong \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F}).$$

If we write everything in local coordinates then we see that $\mathbb{P}\mathcal{F}$ is scheme-theoretically defined by the vanishing of φ. Since the codimension of $\mathbb{P}\mathcal{F}$ is the same as the rank of $\mathcal{E}/\mathcal{F}$, it follows that $[\mathbb{P}\mathcal{F}] \in A(\mathbb{P}\mathcal{E})$ is given by a Chern class, which we can compute using the formula for the Chern class of the tensor product of a bundle with a line bundle (Proposition 5.17):

Proposition 9.13. *If X is a smooth projective variety and $\mathcal{F} \subset \mathcal{E}$ are vector bundles on X of ranks s and r respectively, then*

$$\begin{aligned}[\mathbb{P}\mathcal{F}] &= c_{r-s}(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})) \\ &= \zeta^{r-s} + \gamma_1 \zeta^{r-s-1} + \cdots + \gamma_{r-s} \in A^{r-s}(\mathbb{P}\mathcal{E}),\end{aligned}$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ and $\gamma_k = c_k(\mathcal{E}/\mathcal{F})$. Moreover, the normal bundle of $\mathbb{P}\mathcal{F}$ in $\mathbb{P}\mathcal{E}$ is $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*(\mathcal{E}/\mathcal{F})$.

This formula will be useful to us in many settings; for an immediate application, see Exercises 9.43 and 9.44.

An important reason to consider projectivized subbundles is suggested by the following characterization of sections. Giving a section — that is, a map $\alpha : X \to \mathbb{P}\mathcal{E}$ such that $\pi \circ \alpha$ is the identity — is the same as giving the image of the section; and we will therefore refer to the image as a section as well.

Proposition 9.14. *If $\mathcal{L} \subset \mathcal{E}$ is a line subbundle of a vector bundle $\mathcal{E}$ on a variety X, then $\mathbb{P}\mathcal{L} \subset \mathbb{P}\mathcal{E}$ is the image of a section $X \to \mathbb{P}\mathcal{E}$ of the projection $\mathbb{P}\mathcal{E} \to X$, and every section has this form.*

Informally: giving a section is the same as specifying point of $\mathbb{P}\mathcal{E}$ over each point of X, that is, giving a one-dimensional subspace of each fiber of $\mathcal{E}$.

Proof: By the universal property of $\pi : \mathbb{P}\mathcal{E} \to X$, giving a map $\alpha : X \to \mathbb{P}\mathcal{E}$ that “commutes with” the identity map $X \to X$ is the same as giving a line subbundle of $\mathcal{E}$. □

9.4.1 Ruled surfaces

Recall that a *ruled surface* is by definition the projectivization of a vector bundle of rank 2 over a smooth curve. We can now answer Keynote Question (b):

Proposition 9.15. *A ruled surface can contain at most one irreducible and reduced curve of negative self-intersection.*

Proof: Let X be a smooth curve, let $\pi : \mathbb{P}\mathcal{E} \to X$ be a ruled surface, and suppose that $C_1, C_2 \subset \mathbb{P}\mathcal{E}$ are two irreducible curves of strictly negative self-intersection. A fiber $\pi^{-1}(x)$ satisfies $[\pi^{-1}(x)]^2 = \pi^*([x]^2) = 0$, so the induced maps $\pi : C_i \to X$ are finite. Let $C_1' \to C_1$ be the normalization of C_1, and let $\alpha : C_1' \to C_1 \subset X$ be the corresponding map. Consider the pullback diagram

$$\begin{tikzcd} \mathbb{P}\alpha^*\mathcal{E} = C_1' \times_X \mathbb{P}\mathcal{E} \arrow[r, "\beta"] \arrow[d] & \mathbb{P}\mathcal{E} \arrow[d, "\pi"'] \\ C_1' \arrow[r, "\alpha"'] & X \end{tikzcd}$$

The preimage $\beta^{-1}(C_1) = C_1' \times_X C_1$ represents a cycle $m\Sigma_1 + D_1$, where Σ_1 is a section, D_1 has no component in common with Σ_1 and $m > 0$. Hence

$$\begin{aligned} m^2 \deg[\Sigma_1]^2 &= \deg[\Sigma_1][\beta^* C_1] - \deg[\Sigma_1][D_1] \\ &\leq \deg[\Sigma_1][\beta^* C_1] \\ &= \deg[\beta_* \Sigma_1][C_1] \\ &= \deg[C_1]^2, \end{aligned}$$

so $\deg[\Sigma_1]^2 < 0$.

Since a section pulls back to a section with the same self-intersection, we can repeat the process with a component of $\beta^{-1}C_2$ to obtain two sections Σ_1 and Σ_2 of negative self-intersection. We can analyze this case using Proposition 9.14. Suppose that $\Sigma_i = \mathbb{P}\mathcal{L}_i \subset \mathbb{P}\mathcal{E}$.

By Theorem 9.6, we have

$$A(\mathbb{P}\mathcal{E}) = A(X)[\zeta]/(\zeta^2 + c_1(\mathcal{E})\zeta),$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. Now $\deg(c_1(\mathcal{E})\zeta) = \deg \pi_*(c_1(\mathcal{E})\zeta) = \deg c_1(\mathcal{E})$ because ζ meets each fiber of π in degree 1. It then follows that $\deg \zeta^2 = -\deg c_1(\mathcal{E})$. By Proposition 9.13,

$$[\Sigma_i] = \zeta + c_1(\mathcal{E}) - c_1(\mathcal{L}_i),$$

so

$$0 > \deg[\Sigma_i]^2 = \deg \zeta^2 + 2 \deg c_1(\mathcal{E}) - 2 \deg \mathcal{L}_i.$$

Thus $2 \deg \mathcal{L}_i > \deg c_1(\mathcal{E})$. (Exercise 9.50 strengthens this conclusion slightly.)

Supposing now that $\Sigma_1 \neq \Sigma_2$, we get an exact sequence

$$0 \longrightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a sheaf with finite support; it follows that $\deg \mathcal{E} \geq \deg \mathcal{L}_1 + \deg \mathcal{L}_2 > \deg \mathcal{E}$, a contradiction. $\square$

By contrast, it is possible for a (nonruled) smooth projective surface to contain infinitely many irreducible curves of negative self-intersection; Exercises 9.45–9.47 show how to construct an example. It is an open problem (in characteristic 0) whether the self-intersections of irreducible curves on a surface S are bounded below, that is, whether a surface can contain a sequence $C_1, C_2, \ldots$ of irreducible curves with $\deg(C_n \cdot C_n) \to -\infty$. (In characteristic $p > 0$, János Kollár has shown us an example, described in Exercise 9.49.)

9.4.2 Self-intersection of the zero section

The total space of a vector bundle $\mathcal{E}$ on a scheme X may itself be considered as a scheme $\mathbb{A}\mathcal{E} := \operatorname{Spec}(\operatorname{Sym}\mathcal{E}^*)$ over X. For various purposes it is useful to have a compactification of $\mathbb{A}\mathcal{E}$, that is, a variety proper over X that includes $\mathbb{A}\mathcal{E}$ as an open subset, and we will describe the simplest such construction here.

It is natural to try to compactify each fiber by putting it inside a projective space of the same dimension, and we can do this globally by taking the projectivization of the direct sum $\mathcal{E} \oplus \mathcal{O}_X$; that is, we set

$$\overline{\mathcal{E}} := \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X).$$

Let r be the rank of $\mathcal{E}$. Since $c(\mathcal{E} \oplus \mathcal{O}_X) = c(\mathcal{E})$, we have

$$A(\overline{\mathcal{E}}) = A(X)[\zeta]/(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \cdots + c_r(\mathcal{E})\zeta).$$

In terms of coordinates, $\mathbb{A}\mathcal{E} \subset \overline{\mathcal{E}}$ is “the locus where the last coordinate is nonzero.” Its complement is the divisor $\mathbb{P}\mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$, which we therefore call the “hyperplane at infinity.” Since this is the locus where the section of $\mathcal{O}_{\overline{\mathcal{E}}}(1)$ corresponding to $1 \in \mathcal{O}_X \subset (\mathcal{E} \oplus \mathcal{O}_X)^*$ vanishes, we get

$$\zeta := c_1(\mathcal{O}_{\overline{\mathcal{E}}}(1)) = [\mathbb{P}\mathcal{E}].$$

(One can also see this from Proposition 9.13.)

The section $\mathbb{P}\mathcal{O}_X \subset \overline{\mathcal{E}}$ is the locus where all the coordinates in $\mathcal{E}^*$ vanish; it is thus the zero section of $\mathbb{A}\mathcal{E}$, which we will call Σ_0. By Proposition 9.13, we have $[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E})$. More generally, if τ is a global section of $\mathcal{E}$, then $(\tau, 1)$ is a nowhere-vanishing section of $\mathcal{E} \oplus \mathcal{O}_X$, and the line subbundle it generates corresponds to a section of $\overline{\mathcal{E}}$, which we will call Σ_τ. Using Proposition 9.13 or the family $\Sigma_{t\tau}$, which gives a rational equivalence between Σ_τ and Σ_0, we see that $[\Sigma_\tau] = [\Sigma_0]$. If τ vanishes in codimension r, then

$$\pi_*([\Sigma_0]^2) = \pi_*([[\Sigma_0][\Sigma_\tau]]) = [(\tau)_0] = c_r(\mathcal{E}).$$

We claim that this formula holds in general:

Proposition 9.16. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth variety X, and let $\pi : \overline{\mathcal{E}} = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \to X$ be the projection. Let $\iota : X \to \mathbb{A}(\mathcal{E}) \subset \overline{\mathcal{E}}$ be the zero section, with image $\Sigma_0 = \mathbb{P}(\mathcal{O}_X)$. We have*

$$\pi_*([\Sigma_0]^2) = c_r(\mathcal{E}),$$

and, for any class $\alpha \in A(X)$,

$$\iota^*\iota_*\alpha = \alpha c_r(\mathcal{E}).$$

Proof: By Proposition 9.13,

$$[\Sigma_0] = \zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E}).$$

Since Σ_0 is disjoint from the hyperplane at infinity $\mathbb{P}\mathcal{E} \subset \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$, which has class ζ, we get $[\Sigma_0]\zeta = 0$. (This also follows from the computation of $A(\overline{\mathcal{E}})$.) Thus

$$\begin{aligned}[\Sigma_0]^2 &= [\Sigma_0](\zeta^r + c_1(\mathcal{E})\zeta^{r-1} + \cdots + c_{r-1}(\mathcal{E})\zeta + c_r(\mathcal{E}))\\ &= [\Sigma_0]c_r(\mathcal{E}) \in A(\overline{\mathcal{E}}).\end{aligned}$$

From the push-pull formula we get $\pi_*([\Sigma_0]^2) = (\pi_*[\Sigma_0])c_r(\mathcal{E}) = c_r(\mathcal{E})$, proving the first assertion.

For the second assertion, we use the fact that π induces an isomorphism from Σ_0 to X, and thus $\iota^*\beta = \pi_*(\beta \cap [\Sigma_0])$ for any cycle β on $\overline{\mathcal{E}}$. Thus

$$\iota^*\iota_*\alpha = \iota^*(\iota_*\alpha[\Sigma_0]) = \pi_*(\iota_*\alpha[\Sigma_0]^2) = \alpha c_r(\mathcal{E}),$$

as required. □

See Theorem 13.7 for a generalization.

9.5 Brauer–Severi varieties

We defined a projective bundle to be a morphism $\pi : Y \to X$ that is isomorphic to a product with projective space over Zariski open subsets covering the target X. Interestingly, if we had weakened the condition to saying that π was a product locally in the étale, or analytic, topology on X, we would get in general a larger class of morphisms! In this section, we will illustrate the difference with an example of a morphism that satisfies the weaker condition but not the stronger.

We start with a definition: A *Brauer–Severi variety* over a variety X is a variety Y together with a proper, smooth map $\pi : Y \to X$ such that all the (scheme-theoretic) fibers of π are isomorphic to $\mathbb{P}^r$, for some fixed r. Thus any projective bundle $\pi : Y \to X$ is a Brauer–Severi variety. But, as we will see, the converse is false.

It is in fact the case that such a morphism π will be trivial locally in the étale (or, in case the ground field is $\mathbb{C}$, the analytic) topology, in the sense that every point $x \in X$ will have an étale or analytic neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{P}^r$. This is a

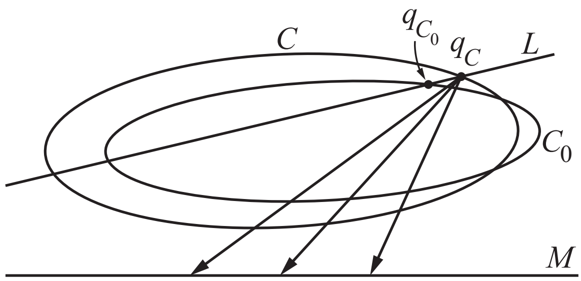

Figure 9.3 Local analytic triviality of the universal family of conics in the plane: $Y|_U \cong C \in U \times \mathbb{P}^1$ via projection from $q_C \in C \in U$.

consequence of the fact that $\mathbb{P}^r$ has no nontrivial deformations. But it may not be trivial locally in the Zariski topology. Here is an example:

Example 9.17. Let $\mathbb{P}^5$ be the space of conics in $\mathbb{P}^2 = \mathbb{P}V$, and consider the universal conic

$$\begin{array}{ccc} \Phi = \{(C, p) \in \mathbb{P}^5 \times \mathbb{P}^2 \mid p \in C\} & \xrightarrow{\pi_2} & \mathbb{P}^2 \\ \downarrow \pi_1 & & \\ \mathbb{P}^5 & & \end{array}$$

with its projections π_i to the two factors. We can realize Φ as the total space of a $\mathbb{P}^4$-bundle over $\mathbb{P}^2$ via π_2: Indeed, Φ is the projectivization of the rank-5 subbundle $\mathcal{E} \subset \mathrm{Sym}^2 V^*$ whose fiber $\mathcal{E}_p$ at a point p is the subspace of quadratic polynomials vanishing at p. (In particular, Φ is smooth.) In these terms, the tautological class $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in A^1(\Phi)$ is the pullback of the hyperplane class $\pi_1^*(\mathcal{O}_{\mathbb{P}^5}(1))$. By Theorem 9.6, the divisor class group $A^1(\Phi) \cong \mathbb{Z}^2$ is generated by the pullbacks of the hyperplane classes from $\mathbb{P}^2$ and $\mathbb{P}^5$. Note that these classes restrict to classes of degrees 2 and 0 on any fiber of π_1. Thus *the intersection of the fiber of π_1 with any divisor on Φ has even degree*.

We now consider the projection π_1. To obtain a map whose fibers are all isomorphic to $\mathbb{P}^1$, we let $X \subset \mathbb{P}^5$ be the open subset corresponding to smooth conics and let $\pi : Y = \Phi_X \to X$ be the restriction of π_1 to the preimage of X in Φ. By definition, the fibers of π are smooth conics, and in particular isomorphic to $\mathbb{P}^1$, so Φ_X is a Brauer–Severi variety over X.

We claim that $\pi : Y \to X$ is not a projective bundle. Indeed, if there were a nonempty Zariski open $U \subset X \subset \mathbb{P}^5$ such that $\pi : Y_U \to U$ were isomorphic to the projection to U of the product $U \times \mathbb{P}^1$, then we could take a section of Y_U and take its closure in Φ, obtaining a divisor in Φ meeting the general fiber of $\Phi \to \mathbb{P}^5$ in a reduced point. This contradicts the computation above. Thus $\pi : Y \to X$ is not a projective bundle.

If we work over the complex numbers, we can see directly that π is locally trivial in the analytic topology (and the same argument would work more generally for the étale topology). Let $C_0 \in X$ be a smooth conic. Choose lines $L, M \subset \mathbb{P}^2$ such that L is transverse to C_0 and $M \cap L \cap C_0 = \varnothing$. Over a sufficiently small analytic neighborhood U of $C_0 \in X$ we can solve analytically for a point $q_C \in C \cap L$. The restriction of Y to U is isomorphic to $U \times \mathbb{P}^1$ as U-schemes by the maps projecting a fiber C from q_C to M (see Figure 9.3).

The conclusion of this example may be interpreted as a theorem in polynomial algebra: It says that *there does not a exist a rational solution to the general quadratic polynomial*. In other words, there do not exist rational functions $X(a, b, c, d, e, f)$, $Y(a, b, c, d, e, f)$ and $Z(a, b, c, d, e, f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0.$$

This is a generalization of the statement that the roots of a quadratic polynomial in one variable are not expressible as rational functions of its coefficients, though much stronger: Polynomials in several variables have many more solutions than polynomials in one variable! The same is true of polynomials of any degree $d > 1$ in any number of variables (Exercise 9.51).

The set of Brauer–Severi varieties over a given variety X, modulo an equivalence relation that makes the projective bundles trivial, can be given the structure of a group, called the *Brauer group* of X. There is another avatar of this group, as the group of *Azumaya algebras* over $\mathcal{O}_X$ modulo those that are the endomorphism algebras of vector bundles. Understanding the Brauer groups of varieties is an important goal of arithmetic geometry. See for example Artin [1982] for more about Brauer–Severi varieties, and Grothendieck [1966a] or Serre [1979] for more on the Brauer group.

9.6 Chow ring of a Grassmannian bundle

Suppose that X is any smooth variety and $\mathcal{E}$ is a vector bundle of rank n on X. Generalizing the projective bundle associated to $\mathcal{E}$, we can form the *Grassmannian bundle* $G(k, \mathcal{E})$ of k-planes in the fibers of $\mathcal{E}$; that is,

$$G(k, \mathcal{E}) = \{(x, V) \mid x \in X, V \subset \mathcal{E}_x\} \xrightarrow{\pi} X.$$

(As with a single Grassmannian, we can realize $G(k, \mathcal{E})$ as a subvariety of the projectivization $\mathbb{P}(\bigwedge^k \mathcal{E})$.) There is a description of the Chow ring of $G(k, \mathcal{E})$ that extends both the description of the Chow ring of a projective bundle above and the description of the Chow ring of $G(k, n)$ given in Theorem 5.26; we will explain it here without proof.

As in the projective bundle case, there is a *tautological subbundle* $\mathcal{S} \subset \pi^*\mathcal{E}$ defined on $G(k, \mathcal{E})$; this is a rank-k bundle whose fiber over a point (x, V) is the vector space

$V \subset \mathcal{E}_x$. Let $\mathcal{Q} = \pi^*(\mathcal{E})/\mathcal{S}$ be the *tautological quotient bundle*. As in the case of projective bundles, the Chow ring $A(G(k,\mathcal{E}))$ is generated as an $A(X)$-algebra by the Chern classes $c_i(\mathcal{S})$, and also by the classes $c_i(\mathcal{Q})$. To understand the relations they satisfy, consider the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi^*\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

By the Whitney formula

$$c(\mathcal{Q}) = \frac{c(\mathcal{E})}{c(\mathcal{S})}.$$

Since $\mathcal{Q}$ has rank $n-k$, the Chern classes $c_l(\mathcal{Q})$ vanish for $l > n-k$, and as in the projective bundle case (above) or the case of $G(k,n)$ (Theorem 5.26) this gives all the relations:

Theorem 9.18. *Let X be a smooth variety, and let $\mathcal{E}$ be a vector bundle of rank n on X. If $G = G(k,\mathcal{E}) \to X$ is the bundle of k-planes in the fibers of $\mathcal{E}$ then*

$$A(G) = A(X)[\zeta_1,\dots,\zeta_k] \Big/ \left(\left\{\frac{c(\mathcal{E})}{1-\zeta_1+\zeta_2+\cdots\pm\zeta_k}\right\}^l,\ l > \dim G - n + k\right),$$

where $\{\eta\}^l$ denotes the component of η of codimension *l and ζ_k has degree k.*

In fact, the same formula holds without the assumption that X is smooth, as long as one has developed the theory of Chern classes on singular varieties, as in Fulton [1984, Chapter 3]

One can go further and, fixing a sequence of ranks $0 < r_1 < \cdots < r_m < \operatorname{rank}\mathcal{E}$, consider the *flag bundle* $\mathbb{F}(r_1,\dots,r_m,\mathcal{E})$ whose fiber over a point of X is the set of all flags of subspaces of the given ranks in $\mathcal{E}$. There is again an analogous description of the Chow ring of this space. See Grayson et al. [2012] for this result and an interesting proof that is in some ways more explicit than the one we have given, even in the case of $A(G(k,n))$.

9.7 Conics in $\mathbb{P}^3$ meeting eight lines

The family of plane conics in $\mathbb{P}^3$ is naturally a projective bundle, and we will now use this fact, together with Theorem 9.6, to compute the number of such conics intersecting each of eight general lines $L_1,\dots,L_8 \subset \mathbb{P}^3$.

We start by checking that we should expect a finite number. There is a three-parameter family of planes in $\mathbb{P}^3$, and a five-parameter family of conics in each. Since two distinct planes intersect only in a line, the space of conics, whatever it is, should have dimension $3+5=8$.

Next, the locus D_L of conics meeting a given line $L \subset \mathbb{P}^3$ has codimension 1 in the space of conics: If $C \subset \mathbb{P}^3$ is the image of the map given by (F_0, F_1, F_2, F_3), the condition that C meet the line $Z_0 = Z_1 = 0$ is that F_0 and F_1 have a common zero. More geometrically: A one-parameter family of conics sweeps out a surface that meets L in a finite set, so a curve in the space of conics will intersect the locus of conics meeting L a finite number of times. It is reasonable, then, to ask whether there is only a finite number of conics that meet each of eight general lines and, if so, how many there are.

We will proceed as follows. First, as a parameter space for conics in $\mathbb{P}^3$, we will use a projective bundle $\mathcal{Q} \to \mathbb{P}^{3*}$, whose points correspond to pairs (H, C) with H a plane in $\mathbb{P}^3$ and C a conic in H; we will use the theory developed earlier in this chapter to calculate in its Chow ring. In particular, we will identify the class $\delta \in A(\mathcal{Q})$ of the cycle $D_L \subset \mathcal{Q}$ of conics meeting a given line L, and compute the number $\deg \delta^8$, our candidate for the number of conics meeting eight given general lines L_i.

To prove that this number is correct, we must show that the cycles D_{L_i} meet transversely, and this requires a tangent space calculation. To do this, we will show that our bundle $\mathcal{Q}$ is in fact isomorphic to the Hilbert scheme $\mathcal{H} = \mathcal{H}_{2m+1}(\mathbb{P}^3)$ of subschemes of $\mathbb{P}^3$ having Hilbert polynomial $p(m) = 2m + 1$. This will allow us to prove the necessary transversality by describing the tangent spaces to D_L in terms of the general description of the tangent spaces to Hilbert schemes from Theorem 6.21; this is a special case of an important general principle explained in Exercise 9.60.

9.7.1 The parameter space as projective bundle

Since the conics in a given plane naturally form a $\mathbb{P}^5$, and each conic is contained in a unique plane, it is plausible that the set of all conics in $\mathbb{P}^3$ is a $\mathbb{P}^5$-bundle over $\mathbb{P}^{3*}$, the projective space of planes in $\mathbb{P}^3$.

To make this structure explicit, consider the tautological exact sequence on $\mathbb{P}^{3*}$, which we may write as

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^4_{\mathbb{P}^{3*}} \xrightarrow{(x_0, x_1, x_2, x_3)} \mathcal{O}_{\mathbb{P}^{3*}}(1) \longrightarrow 0.$$

The projective bundle $\mathbb{P}\mathcal{S} \subset \mathbb{P}(\mathcal{O}^4_{\mathbb{P}^{3*}}) = \mathbb{P}^{3*} \times \mathbb{P}^3$ is the family of 2-planes in $\mathbb{P}^3$: the fiber of $\mathbb{P}\mathcal{S}$ over a point $a = (a_0, \ldots, a_3) \in \mathbb{P}^{3*}$ is the plane $H_a \subset \mathbb{P}^3$ defined by $\sum a_i x_i = 0$. The dual $\mathcal{S}^*_a$ is thus the space of linear forms on this plane, and, setting $\mathcal{E} := \operatorname{Sym}^2(\mathcal{S}^*)$, the fiber of $\mathbb{P}\mathcal{E}$ over the point a may be identified with the set of conics in H_a. We will therefore take $\mathcal{Q} = \mathbb{P}\mathcal{E}$ as our parameter space for conics in $\mathbb{P}^3$. Note that there is a *tautological family of conics in* $\mathbb{P}^3$

$$\mathcal{X} \subset \mathbb{P}\mathcal{E} \times_{\mathbb{P}^{3*}} \mathbb{P}\mathcal{S} \subset \mathbb{P}\mathcal{E} \times \mathbb{P}^3$$

whose points are pairs consisting of a conic in a 2-plane and a point on that conic, with projections both to $\mathbb{P}^{3*}$ and to $\mathbb{P}^3$.

https://doi.org/10.1017/CBO9781139062046.011 Published online by Cambridge University Press

From the dual of the exact sequence above, we derive an exact sequence

$$0 \longrightarrow \mathcal{O}^4_{\mathbb{P}^{3*}} \otimes \mathcal{O}_{\mathbb{P}^{3*}}(-1) \longrightarrow \mathrm{Sym}^2(\mathcal{O}^4_{\mathbb{P}^{3*}}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

If we denote the tautological class on $\mathbb{P}^{3*}$ by ω, then, taking into account that $\omega^4 = 0$, the Whitney formula (Theorem 5.3) yields

$$c(\mathcal{E}) = 1/(1-\omega)^4 = 1 + 4\omega + 10\omega^2 + 20\omega^3.$$

We can now apply Theorem 9.6 to describe the Chow ring of $\mathcal{Q}$. Letting $\zeta \in A^1(\mathcal{Q})$ be the first Chern class of the tautological quotient $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ of the pullback of $\mathcal{E}^*$ to $\mathcal{Q}$, we get

$$\begin{aligned} A(\mathcal{Q}) &= A(\mathbb{P}^{3*})[\zeta]/(\zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3) \\ &= \mathbb{Z}[\omega,\zeta]/(\omega^4, \zeta^6 + 4\omega\zeta^5 + 10\omega^2\zeta^4 + 20\omega^3\zeta^3). \end{aligned}$$

9.7.2 The class δ of the cycle of conics meeting a line

We next compute the class $\delta \in A^1(\mathcal{Q})$ of the divisor $D = D_L$ using the technique of undetermined coefficients. We know that $\delta = a\omega + b\zeta$ for some pair of integers a and b, and restricting to curves in $\mathcal{Q}$ gives us linear relations on a and b. Let $\Gamma \subset \mathcal{Q}$ be the curve corresponding to a general pencil $\{C_\lambda \subset H\}$ of conics in a general plane $H \subset \mathbb{P}^3$ and let $\Phi \subset \mathcal{Q}$ be the curve consisting of a general pencil of plane sections $\{H_\lambda \cap Q\}$ of a fixed quadric Q. We denote their classes in $A_1(\mathcal{Q})$ by γ and φ respectively.

We claim that the following table gives the intersection numbers between our divisor classes ω, ζ, δ, and the curves Γ, Φ:

	ω	ζ	δ
γ	0	1	1
φ	1	0	2

The calculation of the five intersection numbers other than $\zeta\varphi$ is easy, and we leave to the reader the pleasure of working them out (Exercise 9.54).

We can compute $\zeta\varphi$ as the degree of the restriction of the bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ to the curve Φ; equivalently, to show that $\zeta\varphi = 0$ we must show that $\mathcal{T} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ is trivial on Φ. To see this, recall that a point of $\mathcal{Q}$ is a pair (H, ξ), with H a plane in $\mathbb{P}^3$ and ξ a one-dimensional subspace of $H^0(\mathcal{O}_H(2))$; the fiber of $\mathcal{T}$ over the point (H, ξ) is the vector space ξ. Now, if $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ is the homogeneous quadratic polynomial defining Q, we see that the restrictions of F to the planes H_λ give an everywhere-nonzero section of $\mathcal{T}$ over Φ, proving that $\mathcal{T}|_\Phi$ is the trivial bundle, as required.

Given the intersection numbers in the table above, we conclude that

$$\delta = 2\omega + \zeta.$$

There is also a direct way to arrive at this class, which we will describe in Exercise 9.55.

9.7.3 The degree of δ^8

To compute δ^8, we need to know the degrees of the monomials $\omega^i \zeta^j$ of degree 8. To start with, we have $\omega^4 = 0$, and, since ω^3 is the class of a fiber of $\mathcal{Q} \to \mathbb{P}^{3*}$ and ζ restricts to the hyperplane class on this fiber, we have

$$\deg(\omega^3 \zeta^5) = 1.$$

To evaluate the next monomial $\omega^2 \zeta^6$, we use the relation

$$\zeta^6 = -4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3$$

of Theorem 9.6, which gives

$$\deg(\omega^2\zeta^6) = \deg \omega^2(-4\omega\zeta^5 - 10\omega^2\zeta^4 - 20\omega^3\zeta^3) = -4.$$

The same idea yields

$$\deg(\omega\zeta^7) = 6 \quad \text{and} \quad \deg \zeta^8 = -4.$$

Putting these together we obtain

$$\deg((2\omega + \zeta)^8) = \deg\left(\zeta^8 + 2\binom{8}{1}\omega\zeta^7 + 4\binom{8}{2}\omega^2\zeta^6 + 8\binom{8}{3}\omega^3\zeta^5\right) = 92.$$

Writing $\pi : \mathcal{Q} \to \mathbb{P}^3$ for the projection, the numbers $\deg(\omega^i\zeta^j)$ computed above may be interpreted (via the push-pull formula) as the degrees of the classes $\pi_*\zeta^k$, which are called *Segre classes* of the bundle $\mathcal{E}$. See Definition 10.1 and, for an alternative computation, Proposition 10.3.

9.7.4 The parameter space as Hilbert scheme

If $C \subset \Lambda$ is a smooth plane conic then the Hilbert polynomial of C is $p(m) = 2m + 1$. Let $\mathcal{H} := \mathcal{H}_{2m+1}$ be the Hilbert scheme of subschemes of $\mathbb{P}^3$ with this Hilbert polynomial, and let $\mathcal{C} \to \mathcal{H} \times \mathbb{P}^3$ be the universal family. We have already described the tautological family of plane conics $\mathcal{X} \to \mathcal{Q} \times \mathbb{P}^3$, and by the universal property of the Hilbert scheme there is a unique map $\psi : \mathcal{Q} \to \mathcal{H}$ such that $\mathcal{X} = (\psi \times 1)^*\mathcal{C}$.

Theorem 9.19. *$\mathcal{Q}$ with its universal family $\mathcal{X} \to \mathcal{Q} \times \mathbb{P}^3$ is isomorphic to $\mathcal{H}$ with its universal family $\mathcal{C} \to \mathcal{H} \times \mathbb{P}^3$ via the map ψ.*

We postpone the proof to develop a few necessary facts about subschemes C with Hilbert polynomial $p(m) = 2m + 1$. To show that C is really a conic, we first want to show that C is contained in a plane Λ — that is, there is a linear form vanishing on C. Since the number of independent linear forms on $\mathbb{P}^3$ is $4 = p(1) + 1$, it suffices to show that the value $h_C(1)$ of the Hilbert function of C — that is, the dimension $(S_C)_1$ of the degree-1 part of the homogeneous coordinate ring of C — is equal to $p(1)$.

Once this is established we must show that a nonzero quadratic form on Λ vanishes on C, and it suffices, for similar reasons as above, to show that $h_C(2) = \dim(S_C)_2 = 5 = p(2)$. In fact, we will prove that if $C \subset \mathbb{P}^3$ is any subscheme with Hilbert polynomial $p(m) = 2m + 1$, then the Hilbert function $h_C(m)$ of C is equal to $p(m)$ for all m. This is contained in the following result:

Proposition 9.20. *Let $C \subset \mathbb{P}^n$ be a subscheme, and let $\mathcal{I}_C$ be its ideal sheaf and $S_C = \Bbbk[x_0, \dots, x_n]/I$ its homogeneous coordinate ring.*

(a) If the Hilbert polynomial of S_C is $p_C(m) = 2m + 1$, then the Hilbert function of S_C is also equal to $2m + 1$.
(b) C is the complete intersection of a unique 2-plane and a (non-unique) quadric hypersurface.
(c) $H^1(\mathcal{I}_C(m)) = 0$ for all $m \geq 0$.

Proof: The form of the Hilbert polynomial implies that C has dimension 1 and degree 2. Thus a general plane section $\Gamma = \{x = 0\} \cap C$ is a subscheme of degree 2 in the plane, either two distinct points or one double point. In either case, the Hilbert function of Γ is $h_\Gamma(m) = 2$ for all $m \geq 1$. Writing S_C for the homogeneous coordinate ring of C, we have a surjective map $S_C \to S_\Gamma$ whose kernel contains xS_C, whence

$$h_C(m) - h_C(m-1) \geq h_\Gamma(m) = 2$$

for $m \geq 1$. Since $h_C(0) = 1$, it follows that $h_C(m) \geq 2m + 1$ for all $m \geq 0$, and that a strict inequality for any value of m implies the same for all larger values. Since $h_C(m) = p_C(m) = 2m + 1$ for large m, the inequality above must be an equality for all $m \geq 1$, proving the first statement.

The second statement follows. From $h_C(1) = 3$, we see that C is be contained in a unique plane Λ. From $h_C(2) = 5$, we see that C lies on five linearly independent quadrics; since at most four of these can contain Λ, we see that C lies on a quadric $Q \subset \mathbb{P}^3$ not containing Λ. The subscheme $C' := \Lambda \cap Q$ also has Hilbert function $2m + 1$, and since $C \subset C'$ they are equal.

To prove the last statement, we use the long exact sequence in cohomology

$$H^0(\mathcal{I}_C(m)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(m)) \longrightarrow H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{I}_C(m)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^n}(m)).$$

Since the last term is zero and the cokernel of the map $H^0(\mathcal{I}_C(m)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(m))$ is the component of degree m in S_C, it suffices to show that $h^0(\mathcal{O}_C(m)) = 2m + 1$. But as C is defined in the plane by a quadratic hypersurface, we have also a sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(m-2)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow H^0(\mathcal{O}_C(m)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^2}(m-2)),$$

and, since the twists of $\mathcal{O}_{\mathbb{P}^2}$ have no intermediate cohomology, we get

$$h^0(\mathcal{O}_C(m)) = h^0(\mathcal{O}_{\mathbb{P}^2}(m)) - h^0(\mathcal{O}_{\mathbb{P}^2}(m-2)) = \binom{m+2}{2} - \binom{m}{2} = 2m+1,$$

as required. □

Proof of Theorem 9.19: By Proposition 9.20, the fibers of $\mathcal{C} \subset \mathcal{H} \times \mathbb{P}^3$ over closed points of $\mathcal{H}$ are precisely the distinct conics in $\mathbb{P}^3$. Since this is also true for $\mathcal{X} \subset \mathcal{Q} \times \mathbb{P}^3$, the map $\psi : \mathcal{Q} \to \mathcal{H}$ is bijective on closed points.

Since $\mathcal{Q}$ is smooth, it now suffices to prove that $\mathcal{H}$ is smooth. From the bijectivity of ψ, we see that $\dim \mathcal{H} = \dim \mathcal{Q} = 8$, so it suffices, in fact, to prove that the tangent space to $\mathcal{H}$ at each point $[C]$ has dimension 8. By Theorem 6.21, there is an isomorphism $T_{[C]/\mathcal{H}} \cong H^0(\mathcal{N}_{C/\mathbb{P}^3})$. Using Proposition 9.20 again, we know that C is a complete intersection of a linear form and a quadric. Thus $\mathcal{N}_{C/\mathbb{P}^3} = (\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2))|_C$, and the dimension of the tangent space is $h^0(\mathcal{O}_C(1)) + h^0(\mathcal{O}_C(2))$.

By Proposition 9.20, $H^1(\mathcal{I}_{C/\mathbb{P}^3}(m)) = 0$ for all m, so the desired value is the sum of the values of the Hilbert function of C at 1 and at 2. Putting this together, we get

$$\dim T_{[C]/\mathcal{H}} = (2 \cdot 1 + 1) + (2 \cdot 2 + 1) = 8$$

as required. □

9.7.5 Tangent spaces to incidence cycles

To prove that the D_{L_i} intersect transversely we need to compute their tangent spaces at the points of intersection. This task is made easier by the fact that, for general L_i, the intersection of the D_{L_i} takes place in the locus U of smooth conics, as we shall now prove:

Lemma 9.21. *For a general choice of lines $L_1, \dots, L_8 \subset \mathbb{P}^3$, no singular conic meets all eight.*

Proof: The family of singular conics has dimension 7, and the family of lines meeting a line, or a singular conic, has dimension 3. Thus the family consisting of 8-tuples of lines meeting a singular conic has dimension $7 + 3 \cdot 8 = 31$, while the family of 8-tuples of lines has dimension $8 \cdot 4 = 32$. □

Next we describe the tangent spaces to the cycles D_L at points in U. Again, we use the computation of the tangent space to $\mathcal{Q} \cong \mathcal{H}$ at a point $[C]$ corresponding to a conic C as $T_{[C]/\mathcal{H}} = H^0(\mathcal{N}_{C/\mathbb{P}^3})$.

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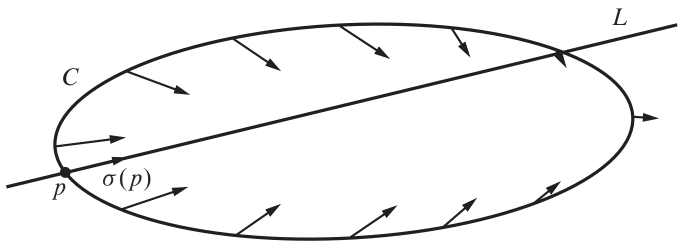

Figure 9.4 If C is a conic meeting a line L at a point p, then a deformation of C corresponding to a normal section σ remains in D_L if and only if $\sigma(p)$ is tangent to L.

Proposition 9.22. *Let $L \subset \mathbb{P}^3$ be a line and $D_L \subset \mathcal{H}$ the locus of conics meeting L. If $C \subset \mathbb{P}^3$ is a smooth plane conic such that $C \cap L = \{p\}$ is a single reduced point, then D_L is smooth at $[C]$, and its tangent space at $[C]$ is the space of sections of the normal bundle whose value at p lies in the normal direction spanned by L; that is,*

$$T_{[C]}D_L = \left\{\sigma \in H^0(\mathcal{N}_{C/\mathbb{P}^3}) \;\middle|\; \sigma(p) \in \frac{T_pL + T_pC}{T_pC}\right\}.$$

See Figure 9.4 for an illustration.

Proof: We prove Proposition 9.22 by introducing an incidence correspondence: For $L \subset \mathbb{P}^3$ a line, we let

$$\Phi_L = \{(p, C) \in L \times \mathcal{H} \mid p \in C\}.$$

The image of Φ_L under the projection π_2 to the second factor is the cycle $D_L \subset \mathcal{H}$ of conics meeting L. By Lemma 6.23, the tangent space to Φ_L at the point (p, C) is

$$T_{(p,C)}\Phi_L = \{(v, \sigma) \in T_pL \times H^0(\mathcal{N}_{C/\mathbb{P}^3}) \mid \sigma(p) \equiv v \bmod T_pC\}.$$

In particular, Φ_L will be smooth at (p, C), and the projection π_2 will carry its tangent space injectively to the space of sections $\sigma \in H^0(\mathcal{N}_{C/\mathbb{P}^3})$ such that $\sigma(p) \in (T_pL + T_pC)/T_pC$. Since the map π_2 is one-to-one over p, it follows that D_L is smooth at $[C]$ with this tangent space. □

This argument also applies to Hilbert schemes in a more general context; see Exercise 9.60.

Corollary 9.23. *Let C be a smooth conic in $\mathbb{P}^3$. If $L_1, \ldots, L_8$ are general lines meeting C at general points, then the cycles $D_{L_1}, \ldots, D_{L_8} \subset \mathcal{Q} \cong \mathcal{H}$ meet transversely at $[C]$.*

Proof: By Proposition 9.22, it suffices to show that the eight linear conditions specifying that a global section of the normal bundle of C lie in specified one-dimensional subspaces at eight points of C are independent, for a general choice of the points and the subspaces. Since the rank of the normal bundle is 2, this is a special case of Lemma 9.24, proved below. □

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Lemma 9.24. *Let $\mathcal{E}$ be a vector bundle on a projective variety X, and let $V \subset H^0(\mathcal{E})$ be a vector space of global sections. If $p_1, \dots, p_k \in X$ are general points and $V_i \subset E_{p_i}$ a general linear subspace of codimension 1 in the fiber $\mathcal{E}_{p_i}$ of $\mathcal{E}$ at p_i, then the subspace $W = \{\sigma \in V \mid \sigma(p_i) \in V_i\}$ has dimension*

$$\dim W = \max\{0, \dim(V) - k\}.$$

The obvious analog of this result fails if we allow $\operatorname{codim} V_i > 1$; see Exercise 9.53.

Proof: Proceeding inductively, it suffices to show the case $k = 1$, and note that if the general section in V had value in every hyperplane $V_i \subset \mathcal{E}_p$ at a dense set of points $p \in X$ then $V = 0$. □

9.7.6 Proof of transversality

Proposition 9.25. *If $L_1, \dots, L_8 \subset \mathbb{P}^3$ are eight general lines, then the cycles $D_{L_i} \subset \mathcal{Q}$ intersect transversely.*

Proof: To start, we introduce the incidence correspondence

$$\Sigma = \{(L_1, \dots, L_8; C) \in \mathbb{G}(1,3)^8 \times \mathcal{Q} \mid C \cap L_i \neq \varnothing \text{ for all } i\}.$$

Since the locus of lines $L \subset \mathbb{P}^3$ meeting a given smooth conic C is an irreducible hypersurface in the Grassmannian $\mathbb{G}(1,3)$, we see via projection to $\mathcal{Q}$ that Σ is irreducible of dimension 32.

Now, let $\Sigma_0 \subset \Sigma$ be the locus of $(L_1, \dots, L_8; C)$ such that the cycles D_{L_i} fail to intersect transversely at $[C]$; this is a closed subset of Σ. By Corollary 9.23, $\Sigma_0 \neq \Sigma$, so $\dim \Sigma_0 < 32$. It follows that Σ_0 does not dominate $\mathbb{G}(1,3)^8$, so for a general point $(L_1, \dots, L_8) \in \mathbb{G}(1,3)^8$ the cycles D_{L_i} are transverse at every point of their intersection. □

In sum, we have proved:

Theorem 9.26. *There are exactly 92 distinct plane conics in $\mathbb{P}^3$ meeting eight general lines, and each of them is smooth.*

As with any enumerative formula that applies to the general form of a problem, the computation still tells us something in the case of eight arbitrary lines. For one thing, it says that if $L_1, \dots, L_8 \subset \mathbb{P}^3$ are *any* eight lines, there will be at least one conic meeting all eight (here we have to include degenerate conics as well as smooth), and, if we assume that the number of conics meeting all eight (again including degenerate ones) is finite, then, assigning to each such conic C a multiplicity (equal to the scheme-theoretic degree of the component of the intersection $\bigcap D_{L_i} \subset \mathcal{H}$ supported at $[C]$, since the cycles D_L are Cohen–Macaulay), the total number of conics will be 92. In particular, as long as the number is finite, there cannot be *more* than 92 distinct conics meeting all eight lines.

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In Exercises 9.56–9.68 we will look at some other problems involving conics in $\mathbb{P}^3$, including some problems involving calculations in $A(\mathcal{H})$, some other applications of the techniques we have developed here and some problems that require other parameter spaces for conics.

9.8 Exercises

Exercise 9.27. Choosing coordinates $x_0, x_1, \ldots, x_a$ on $\mathbb{P}^a$ corresponding to the monomials $s^a, s^{a-1}, \ldots, t^a$, show that the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{a-1} \\ x_1 & x_2 & \cdots & x_a \end{pmatrix}$$

vanish identically on the rational normal curve $S(a)$. By working in local coordinates, show that the ideal I generated by the minors defines the curve scheme-theoretically. Find a set of monomials forming a basis for the ring $\Bbbk[x_0, x_1, \ldots, x_a]/I$, and show that in degree d it has dimension $ad + 1$. By comparing this with the Hilbert function of $\mathbb{P}^1$, prove that I is the saturated ideal of the rational normal curve.

Exercise 9.28. In order to do the same as we did in the previous exercise for surface scrolls, prove that the Hilbert polynomial $f_S(d)$ of the surface scroll $S(a, b) \subset \mathbb{P}^{a+b+1}$ satisfies

$$f_S(d) \geq (a + b)\binom{d+1}{2} + d + 1.$$

Exercise 9.29. Let $x_0, \ldots, x_{a+b+1}$ be coordinates in $\mathbb{P}^{a+b+1}$. Prove that the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{a-1} & x_{a+1} & x_{a+2} & \cdots & x_{a+b} \\ x_1 & x_2 & \cdots & x_a & x_{a+2} & x_{a+3} & \cdots & x_{a+b+1} \end{pmatrix}$$

vanish on a surface scroll $S(a, b)$. As in Exercise 9.27, show that the ideal I generated by the minors defines the surface scheme-theoretically. Then, using Exercise 9.28, prove that I is the saturated ideal of the surface scroll.

Exercise 9.30. Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle on X and $\mathbb{P}\mathcal{E} \to X$ its projectivization. Let $\mathcal{L}$ be any line bundle on X; as we have seen, there is a natural isomorphism $\mathbb{P}\mathcal{E} \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$, such that

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{L})}(1) \cong \mathcal{O}_{\mathbb{P}\mathcal{E}}(1) \otimes \pi^*\mathcal{L}^*.$$

Using the results of Section 5.5.1, show that the two descriptions of the Chow ring of $\mathbb{P}\mathcal{E} = \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ agree.

Exercise 9.31. Let $\pi : Y \to X$ be a projective bundle.

(a) Show that the direct sum decomposition of the group $A(X)$ given in Theorem 9.6 depends on the choice of vector bundle $\mathcal{E}$ with $Y \cong \mathbb{P}\mathcal{E}$.
(b) Show that if we define group homomorphisms $\psi_i : A(Y) \to A(X)^{\oplus i+1}$ by

$$\psi_i : \alpha \mapsto (\pi_*(\alpha), \pi_*(\zeta\alpha), \ldots, \pi_*(\zeta^i\alpha)),$$

then the filtration of $A(Y)$ given by

$$A(Y) \supset \operatorname{Ker}(\psi_0) \supset \operatorname{Ker}(\psi_1) \supset \cdots \supset \operatorname{Ker}(\psi_{r-1}) \supset \operatorname{Ker}(\psi_r) = 0$$

is independent of the choice of $\mathcal{E}$.
Hint: Give a geometric characterization of the cycles in each subspace of $A(Y)$.

Exercise 9.32. In Example 9.17, we used intersection theory to show that there does not exist a rational solution to the general quadratic polynomial; that is, there do not exist rational functions $X(a, \ldots, f)$, $Y(a, \ldots, f)$ and $Z(a, \ldots, f)$ such that

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \equiv 0.$$

To gain some appreciation of the usefulness of intersection theory, give an elementary proof of this assertion.

Exercise 9.33. Let

$$\Phi = \{(L, p) \in \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L\}$$

be the universal line in $\mathbb{P}^n$, and let σ_1, $\sigma_{1,1}$ and ζ be the pullbacks of the Schubert classes $\sigma_1 \in A^1(\mathbb{G}(1,n))$, $\sigma_{1,1} \in A^2(\mathbb{G}(1,n))$ and the hyperplane class $\zeta \in A^1(\mathbb{P}^n)$ respectively. Find the degree of all monomials $\sigma_1^a \sigma_{1,1}^b \zeta^c$ of top degree $a + 2b + c = \dim(\Phi) = 2n - 1$.

Exercise 9.34. Consider the flag variety $\mathbb{F}$ of pairs consisting of a point $p \in \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$ containing p; that is,

$$\mathbb{F} = \{(p, L) \in \mathbb{P}^3 \times \mathbb{G}(1,3) \mid p \in L \subset \mathbb{P}^3\}.$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^1$-bundle over $\mathbb{G}(1, 3)$, or as a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Exercise 9.35. By Theorem 9.6, the Chow ring of the product $\mathbb{P}^3 \times \mathbb{G}(1, 3)$ is just the tensor product of their Chow rings; that is

$$A(\mathbb{P}^3 \times \mathbb{G}(1,3)) = A(\mathbb{G}(1,3))[\zeta]/(\zeta^4).$$

In these terms, find the class of the flag variety $\mathbb{F} \subset \mathbb{P}^3 \times \mathbb{G}(1, 3)$ of Exercise 9.34.

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Exercise 9.36. Generalizing the preceding problem, let

$$\mathbb{F}(0, k, r) = \{(p, \Lambda) \in \mathbb{P}^r \times \mathbb{G}(1, r) \mid p \in \Lambda\}.$$

Find the class of $\mathbb{F}(0, 1, r) \subset \mathbb{P}^r \times \mathbb{G}(1, r)$.

Exercise 9.37. Generalizing Exercise 9.35 in a different direction, let

$$\Phi_r = \{(L, M) \in \mathbb{G}(1, r) \times \mathbb{G}(1, r) \mid L \cap M \neq \varnothing\}.$$

Given that by Theorem 9.18 we have

$$A(\mathbb{G}(1, r) \times \mathbb{G}(1, r)) \cong A(\mathbb{G}(1, r)) \otimes A(\mathbb{G}(1, r)),$$

find the class of Φ_r in $A(\mathbb{G}(1, r) \times \mathbb{G}(1, r))$ for:

(a) $r = 3$.
(b) $r = 4$.
(c) General r.

Exercise 9.38. Let Z be the blow-up of $\mathbb{P}^n$ along an $(r-1)$-plane, and let $E \subset Z$ be the exceptional divisor. Find the degree of the top power $[E]^n \in A(Z)$.

Exercise 9.39. Again let $Z = \mathrm{Bl}_\Lambda \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along an $(r-1)$-plane Λ. In terms of the description of the Chow ring of Z given in Corollary 9.12, find the classes of the following:

(a) The proper transform of a linear space $\mathbb{P}^s$ containing Λ, for each $s > r$.
(b) The proper transform of a linear space $\mathbb{P}^s$ in general position with respect to Λ (that is, disjoint from Λ if $s \leq n - r$, and transverse to Λ if $s > n - r$).
(c) In general, the proper transform of a linear space $\mathbb{P}^s$ intersecting Λ in an l-plane.

Exercise 9.40. Let $Z = \mathrm{Bl}_L \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line. In terms of the description of the Chow ring of Z given in Corollary 9.12, find the classes of the proper transform of a smooth surface $S \subset \mathbb{P}^3$ of degree d containing L.

Exercise 9.41. Now let $Z = \mathrm{Bl}_L \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ along a line, and let $S \subset \mathbb{P}^4$ be a smooth surface of degree d containing L. Show by example that the class of the proper transform of S in Z is not determined by this data. For example, try taking $S = S(1, 2) \subset \mathbb{P}^4$ a cubic scroll, with L either

(a) a line of the ruling of S, or
(b) the *directrix* of S, that is, the unique curve of negative self-intersection,

and observe that you get different answers.

Exercise 9.42. Let $Z = \mathrm{Bl}_\Lambda \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along an $(r-1)$-plane Λ; that is, if we consider the subspace $\mathbb{P}^{n-r} \subset \mathbb{G}(r,n)$ of r-planes containing Λ, we have

$$Z = \{(p,\Gamma) \in \mathbb{P}^n \times \mathbb{P}^{n-r} \mid p \in \Gamma\}.$$

Using the description of the Chow ring of Z given in Corollary 9.12, find the class of $Z \subset \mathbb{P}^n \times \mathbb{P}^{n-r}$.

Exercise 9.43. Let C be a smooth curve, $\mathcal{E}$ a vector bundle of rank r on C and $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$ two subbundles of complementary ranks s and $r-s$ such that for general $p \in C$ the fibers $\mathcal{F}_p$ and $\mathcal{G}_p$ are complementary in $\mathcal{E}_p$. In terms of the Chern classes of the three bundles, describe the locus of $p \in C$ where $\mathcal{F}_p \cap \mathcal{G}_p \neq 0$:

(a) By using the result of Proposition 9.13 to calculate the class of the intersection $\mathbb{P}\mathcal{F} \cap \mathbb{P}\mathcal{G}$ in $\mathbb{P}\mathcal{E}$.
(b) By considering the bundle map $\mathcal{F} \oplus \mathcal{G} \to \mathcal{E}$.

Exercise 9.44. To generalize the preceding problem: Let X be a smooth projective variety of any dimension, $\mathcal{E}$ a vector bundle of rank r on X and $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$ subbundles of ranks a and b with $a + b \leq r$. Describe the locus of $p \in C$ where $\mathcal{F}_p \cap \mathcal{G}_p \neq 0$, assuming this locus has the expected codimension $r + 1 - a - b$.

We will see how to generalize this calculation substantially using the *Porteous formula* of Chapter 12; see Exercise 12.11.

The following three exercises show one way to construct a surface with infinitely many reduced and irreducible curves of negative self-intersection.

Exercise 9.45. Let F and G be two general polynomials of degree 3 in $\mathbb{P}^2$, and let $\{C_t = V(t_0F + t_1G)\}_{t \in \mathbb{P}^1}$ be the associated pencil of curves; let $p_1, p_2, \dots, p_9$ be the base points of this pencil. Show that for very general $t \in \mathbb{P}^1$ (that is, for all but countably many t) the line bundle $\mathcal{O}_{C_t}(p_1 - p_2)$ is not torsion in $\mathrm{Pic}(C_t) = A^1(C_t)$.

Exercise 9.46. Now let S be the blow-up of the plane at the points $p_1, \dots, p_9$ — that is, the graph of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by (F, G) — and let $E_1, \dots, E_9$ be the exceptional divisors. Show that there is a biregular automorphism $\varphi : S \to S$ that commutes with the projection $S \to \mathbb{P}^1$ and carries E_1 to E_2.

Exercise 9.47. Using the result of Exercise 9.45, show that the automorphism φ of Exercise 9.46 has infinite order, and deduce that the surface S contains infinitely many irreducible curves of negative self-intersection.

Exercise 9.48. An amusing enumerative problem: In the circumstances of the preceding exercises, for how many $t \in \mathbb{P}^1$ will $\mathcal{O}_{C_t}(p_1 - p_2)$ be torsion of order 2 — that is, for how many t will $\mathcal{O}_{C_t}(2p_1) \cong \mathcal{O}_{C_t}(2p_2)$?

Exercise 9.49. Let C be a smooth curve of genus $g \geq 2$ over a field of characteristic $p > 0$; let $\varphi : C \to C$ be the Frobenius morphism. If $\Gamma_n \subset C \times C$ is the graph of φ^n and $\gamma_n = [\Gamma_n] \in A^1(C \times C)$ its class, show that the self-intersection $\deg(\gamma_n^2)$ goes to $-\infty$ as $n \to \infty$.

Exercise 9.50. Show that if $\mathcal{E}$ is a vector bundle of rank 2 and degree e on a smooth projective curve X, and $\mathcal{L}$ and $\mathcal{M}$ sub-line bundles of degrees a and b corresponding to sections of $\mathbb{P}\mathcal{E}$ with classes σ and τ, then

$$\sigma\tau = e - a - b \quad \text{and} \quad \sigma^2 + \tau^2 = 2e - 2a - 2b.$$

In particular, if $\mathcal{L}$ and $\mathcal{M}$ are distinct then $\deg \sigma^2 + \deg \tau^2 \geq 0$, with equality holding if and only if $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$.

Exercise 9.51. Let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$. Using the analysis of Example 9.17 as a template, show that for $d > 1$ the universal hypersurface

$$\Phi_{d,n} = \{(X, p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in X\} \to \mathbb{P}^N$$

admits no rational section.

Exercise 9.52. Consider the flag variety $\mathbb{F}$ of pairs consisting of a point $p \in \mathbb{P}^4$ and a 2-plane $\Lambda \subset \mathbb{P}^4$ containing p; that is,

$$\mathbb{F} = \{(p, L) \mid p \in \Lambda \subset \mathbb{P}^4\} \subset \mathbb{P}^4 \times \mathbb{G}(2, 4).$$

$\mathbb{F}$ may be viewed as a $\mathbb{P}^2$-bundle over $\mathbb{G}(2, 4)$, or as a $\mathbb{G}(1, 3)$-bundle over $\mathbb{P}^4$. Calculate the Chow ring $A(\mathbb{F})$ via each map, and show that the two descriptions agree.

Exercise 9.53. Show that the analog of Lemma 9.24 is false if we allow the V_i to have codimension >1: in other words, if $V_i \subset \mathcal{E}_{p_i}$ is a general linear subspace of codimension m_i, then the corresponding subspace $W \subset H^0(\mathcal{E})$ need not have dimension $\max\{0, h^0(\mathcal{E}) - \sum m_i\}$.
Hint: Consider a bundle whose sections all lie in a proper subbundle.

Exercise 9.54. Calculate the remaining five intersection numbers in the table of intersection numbers on page 349 of Section 9.7.2.

Exercise 9.55. To find the class $\delta = [D_L] \in A^1(\mathcal{H})$ of the cycle of conics meeting a line directly, restrict to the open subset $U \subset \mathcal{H}$ of pairs $(H, \xi) \in \mathcal{H}$ such that H does not contain L (since the complement of this open subset of $\mathcal{H}$ has codimension 2, any relation among divisor classes that holds in U will hold in $\mathcal{H}$). Show that we have a map $\alpha : U \to L$ sending a pair (H, ξ) to the point $p = H \cap L$, and that in U the divisor D_L is the zero locus of the map of line bundles

$$T \to \alpha^* \mathcal{O}_L(2)$$

sending a quadric $Q \in \xi$ to $Q(p)$.

https://doi.org/10.1017/CBO9781139062046.011 Published online by Cambridge University Press

Exercise 9.56. Let $\Delta \subset \mathcal{H}$ be the locus of singular conics.

(a) Show that Δ is an irreducible divisor in $\mathcal{H}$.
(b) Express the class $\delta \in A^1(\mathcal{H})$ as a linear combination of ω and ζ.
(c) Use this to calculate the number of singular conics meeting each of seven general lines in $\mathbb{P}^3$.
(d) Verify your answer to the last part by calculating this number directly.

Exercise 9.57. Let $p \in \mathbb{P}^3$ be a point and $F_p \subset \mathcal{H}$ the locus of conics containing the point p. Show that F_p is six-dimensional, and find its class in $A^2(\mathcal{H})$.

Exercise 9.58. Use the result of the preceding exercise to find the number of conics passing through a point p and meeting each of six general lines in $\mathbb{P}^3$, the number of conics passing through two points p, q and meeting each of four general lines in $\mathbb{P}^3$, and the number of conics passing through three points p, q, r and meeting each of two general lines in $\mathbb{P}^3$. Verify your answers to the last two parts by direct examination.

Exercise 9.59. Find the class in $A^3(\mathcal{H})$ of the locus of double lines (note that this is five-dimensional, not four!).

Exercise 9.60. Suppose that $X \subset \mathbb{P}^n$ is a subscheme of pure dimension l and $\mathcal{H}$ a component of the Hilbert scheme parametrizing subschemes of $\mathbb{P}^n$ of pure dimension $k < n - l$ in $\mathbb{P}^n$; let $[Y] \in \mathcal{H}$ be a smooth point corresponding to a subscheme $Y \subset \mathbb{P}^n$ such that $Y \cap X = \{p\}$ is a single reduced point, and suppose moreover that p is a smooth point of both X and Y. Finally, let $\Sigma_X \subset \mathcal{H}$ be the locus of subschemes meeting X.

Use the technique of Proposition 9.22 to show that $\Sigma_X \subset \mathcal{H}$ is smooth at $[Y]$, of the expected codimension $n - k - l$, with tangent space

$$T_{[Y]}\Sigma_X = \left\{ \sigma \in H^0(\mathcal{N}_{Y/\mathbb{P}^n}) \;\middle|\; \sigma(p) \in \frac{T_p X + T_p Y}{T_p Y} \right\}.$$

The next few problems deal with an example of a phenomenon encountered in the preceding chapter: the possibility that the cycles in our parameter space corresponding to the conditions imposed in fact do not meet transversely, or even properly.

Exercise 9.61. Let $H \subset \mathbb{P}^3$ be a plane, and let $\mathcal{E}_H \subset \mathcal{H}$ be the closure of the locus of smooth conics $C \subset \mathbb{P}^3$ tangent to H. Show that this is a divisor, and find its class $[\mathcal{E}_H] \in A^1(\mathcal{H})$.

Exercise 9.62. Find the number of smooth conics in $\mathbb{P}^3$ meeting each of seven general lines $L_1, \dots, L_7 \subset \mathbb{P}^3$ and tangent to a general plane $H \subset \mathbb{P}^3$. More generally, for $k = 1, 2$ and 3 find the number of smooth conics in $\mathbb{P}^3$ meeting each of $8 - k$ general lines $L_1, \dots, L_{8-k} \subset \mathbb{P}^3$ and tangent to k general planes $H_1, \dots, H_k \subset \mathbb{P}^4$.

Exercise 9.63. For $k \geq 4$, why do the methods developed here not work to calculate the number of smooth conics in $\mathbb{P}^3$ meeting each of $8-k$ general lines $L_1, \ldots, L_{8-k} \subset \mathbb{P}^3$ and tangent to k general planes $H_1, \ldots, H_k \subset \mathbb{P}^3$? What can you do to find these numbers? (In fact, we have seen one way to deal with this in Chapter 8.)

Next, some problems involving conics in $\mathbb{P}^4$:

Exercise 9.64. Now let $\mathcal{K}$ be the space of conics in $\mathbb{P}^4$ (again, defined to be complete intersections of two hyperplanes and a quadric). Use the description of $\mathcal{K}$ as a $\mathbb{P}^5$-bundle over the Grassmannian $\mathbb{G}(2, 4)$ to determine its Chow ring.

Exercise 9.65. In terms of your answer to the preceding problem, find the class of the locus D_Λ of conics meeting a 2-plane Λ, and of the locus $\mathcal{E}_L$ of conics meeting a line $L \subset \mathbb{P}^4$.

Exercise 9.66. Find the expected number of conics in $\mathbb{P}^4$ meeting each of 11 general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$.

Exercise 9.67. Prove that your answer to the preceding problem is in fact the actual number of conics by showing that for general 2-planes $\Lambda_1, \ldots, \Lambda_{11} \subset \mathbb{P}^4$ the corresponding cycles D_{Λ_i} intersect transversely.

Finally, here is a challenge problem:

Exercise 9.68. Let $\{S_t \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ be a general pencil of quartic surfaces (that is, take A and B general homogeneous quartic polynomials, and set $S_t = V(t_0 A + t_1 B) \subset \mathbb{P}^3$). How many of the surfaces S_t contain a conic?