Lecture 12

O_K the Dedekind domain.

1 Lemma 10.4: Let $x \neq 0 \in O_K$. Then
$$(x) = \prod_{\substack{\mathfrak{p} \neq 0 \\ \text{prime ideal}}} \mathfrak{p}^{v_{\mathfrak{p}}(x)}$$

Proof: $x O_{K,(\mathfrak{p})} = (\mathfrak{p} O_{K,(\mathfrak{p})})^{v_{\mathfrak{p}}(x)}$ by definition of $v_{\mathfrak{p}}(x)$. Lemma follows from property localization:

$$I = J \iff I O_{K,(\mathfrak{p})} = J O_{K,(\mathfrak{p})} \ \forall \text{ prime ideals } \mathfrak{p} \ \square$$

Notation: O_K Dedekind domain, L/K finite separable extension, $\mathfrak{P} \subseteq O_L$, $\mathfrak{p} \subseteq O_K$ non zero prime ideals. We write $\mathfrak{P} \mid \mathfrak{p}$ if $\mathfrak{p} O_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$, $\mathfrak{P} \in \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$ $(e_i > 0)$.

Theorem 10.5: Let O_K be a Dedekind domain and L a finite separable extension of $K = \mathrm{Frac}(O_K)$. For $\mathfrak{p}$ a non-zero prime ideal of O_K, we write $\mathfrak{p} O_L = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$. Then the absolute values on L extending $| \ |_{\mathfrak{p}}$ (up to equivalence) are precisely $| \ |_{\mathfrak{P}_1}, \dots, | \ |_{\mathfrak{P}_r}$.

2 Proof: By Lemma 10.4 For any $x \in O_L$ and

Proof. By Lemma 10 for any $x \in \mathcal{O}_K$ and

$i = 1, \ldots, r$, we have $v_{\mathfrak{P}_i}(x) = e_i v_{\mathfrak{p}}(x)$.

Hence, up to equivalence, $|\cdot|_{\mathfrak{P}_i}$ extends $|\cdot|_{\mathfrak{p}}$.

Now suppose $|\cdot|$ is an abs. value on L extending $|\cdot|_{\mathfrak{p}}$. Then $|\cdot|$ is bounded on $\mathbb{Z}$ and hence $|\cdot|$ is non-archimedean.

Let $R = \{x \in L \mid |x| \leq 1\} \subseteq L$ be the valuation ring for L w.r.t. $|\cdot|$. Then $\mathcal{O}_K \subseteq R$, and since R is integrally closed (Lemma 6.8), we have $\mathcal{O}_L \subseteq R$. Set $\mathfrak{P} := \{x \in \mathcal{O}_L \mid |x| < 1\}$.

$$= \underbrace{\mathfrak{m}_R}_{\text{max. ideal in } R} \cap \mathcal{O}_L$$

$\Rightarrow$ $\mathfrak{P}$ a prime ideal in $\mathcal{O}_L$ ← non-zero since $\mathfrak{p} \subseteq \mathfrak{P}$

Then $\mathcal{O}_{L,(\mathfrak{P})} \subseteq R$, since $s \in \mathcal{O}_L \setminus \mathfrak{P} \Rightarrow |s| = 1$.

But $\mathcal{O}_{L,(\mathfrak{P})}$ is a DVR, hence a maximal subring of $L \Rightarrow \mathcal{O}_{L,(\mathfrak{P})} = R$.

3 Hence $|\ |$ is equiv. to $|\ |_{\mathfrak{P}}$.

Since $|\ |$ extends $|\cdot|_{\mathfrak{p}}$, $\mathfrak{P} \cap \mathcal{O}_K = \mathfrak{p}$

$$\Rightarrow \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r} \subseteq \mathfrak{P}$$

$$\Rightarrow \mathfrak{P} = \mathfrak{P}_i \text{ some } i. \qquad \square$$

Let K be number field. If $\sigma : K \to \mathbb{R}, \mathbb{C}$ is a real or complex embedding, then $x \mapsto |\sigma(x)|_\infty$ defines an abs. value on K (Ex. sheet 2) denoted by $|\cdot|_\sigma$

Corollary 10.6: Let K be a number field with ring of integers $\mathcal{O}_K$. Then any absolute value on K is equivalent to either

(i) $|\cdot|_\mathfrak{p}$ for some non-zero prime ideal of $\mathcal{O}_K$.

3 (ii) $|\cdot|_\sigma$ for some $\sigma : K \to \mathbb{R}, \mathbb{C}$

Proof: Case 1: $|\cdot|$ non-archimedean

Then $|\cdot||_\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime p by Ostrowski's theorem. Theorem 10.3 then implies $|\cdot|$ is equiv. to $|\cdot|_\mathfrak{p}$ for $\mathfrak{p}$ a prime of $\mathcal{O}_K$ dividing p.

4 Case 2: $|\cdot|$ archimedean: Ex sheet 2. $\square$

§ <u>Completions</u>

$\mathcal{O}_K$ Dedekind domain, L/K finite separable

Let $\mathfrak{p}$ a prime of $\mathcal{O}_K$ and $\mathfrak{P}$ a prime of $\mathcal{O}_L$ s.t. $\mathfrak{P}$ divides $\mathfrak{p}$. We write $K_\mathfrak{p}$ and $L_\mathfrak{P}$ for the completions of K and L w.r.t. the absolute values defined by $|\cdot|_\mathfrak{p}$ and $|\cdot|_\mathfrak{P}$ respectively.

Lemma 10.7:

(i) The natural map $L \otimes_K K_\mathfrak{p} \to L_\mathfrak{P}$ is surjective.

(ii) $[L_\mathfrak{P} : K_\mathfrak{p}] \leq [L : K]$

Proof: Let $M := L K_\mathfrak{p} \subseteq L_\mathfrak{P}$. Then M is a finite extension of $K_\mathfrak{p}$ and $[M : K_\mathfrak{p}] \leq [L : K]$. Moreover M is complete $\overset{\text{Thm 6.1}}{}$ and since $L \subseteq M \subseteq L_\mathfrak{P}$, we have $M = L_\mathfrak{P}$. □

Lemma 10.8: (Chinese remainder theorem)

Let R be a ring. Let $I_1, \ldots, I_n \subseteq R$ be ideals s.t. $I_i + I_j = R$ $\forall i \neq j$. Then

(i) $\bigcap_{i=1}^n I_i = \prod_{i=1}^n I_i$ ($= I$ say).

(ii) $R/I \cong \prod_{i=1}^n R/I_i$.

Proof: Example sheet 2 □

Theorem 10.9: The natural map
$$L \otimes_K K_\mathfrak{p} \xrightarrow{\cong} \prod_{\mathfrak{P} | \mathfrak{p}} L_\mathfrak{P}$$ is an iso.

Proof: Write $L = K(\alpha)$ and let $f(x) \in K[x]$ be the minimal polynomial of α. Then we have
$$f(x) = f_1(x) \cdots f_r(x) \quad \text{in } K_\mathfrak{p}[x]$$
where $f_i(x) \in K_\mathfrak{p}[x]$ are distinct irreducible. (separable)

Since $L \cong K[x]/f(x)$, we have
$$L \otimes_K K_\mathfrak{p} \cong K_\mathfrak{p}[x]/f(x) \overset{CRT}{\cong} \prod_{i=1}^r K_\mathfrak{p}[x]/f_i(x)$$

Set $L_i := K_\mathfrak{p}[x]/f_i(x)$ a finite extension of $K_\mathfrak{p}$.

Then L_i contains both L and $K_\mathfrak{p}$ (use $K[x]/f(x) \hookrightarrow K_\mathfrak{p}[x]/f_i(x)$ injective since morphisms of fields). Moreover L is dense inside L_i.

6 Indeed since K is dense inside $K_\mathfrak{p}$, can approximate coefficients of an element of $K_\mathfrak{p}[x]/f_i(x)$ with an element of $K[x]/f(x)$.

The theorem follows from the following three claims.

(1) $L_i \cong L_\mathcal{P}$ for some prime $\mathcal{P}$ of $\mathcal{O}_L$ dividing $\mathfrak{p}$.

(2) Each $\mathcal{P}$ appears at most once.

(3) Each $\mathcal{P}$ appears at least once.

Proof of claims:

(1) Since $[L_i : K_\mathfrak{p}] < \infty$, there is a unique abs. value $|\cdot|$ on L_i extending $|\cdot|_\mathfrak{p}$. Theorem 10.5 $\Rightarrow |\cdot|_L$ equiv. to. $|\cdot|_\mathcal{P}$ for some $\mathcal{P}|\mathfrak{p}$. Since L is dense in L and L_i is complete, we have $L_i \cong L_\mathcal{P}$.

(2) Suppose $\varphi: L_i \cong L_j$ is an isomorphism preserving L and $K_\mathfrak{p}$, then

$$\varphi : K_\mathfrak{p}[x]/f_i(x) \xrightarrow{\sim} K_\mathfrak{p}[x]/f_j(x)$$

takes x to x and hence $f_i(x) = f_j(x)$ $\Rightarrow j = i$.

7 (3) By Lemma 10.7, the natural map $\pi_\mathcal{P} : L \otimes_K K_\mathfrak{p} \to L_\mathcal{P}$ is surjective for any $\mathcal{P}|\mathfrak{p}$. Since $L_\mathcal{P}$ is a field $\pi_\mathcal{P}$ factors through L_i for some i, and hence $L_i \cong L_\mathcal{P}$ by surjectivity of $\pi_\mathcal{P}$ (Lemma 10.7) $\square$

Eg. $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$, $f(x) = x^2+1$, Hensel $\Rightarrow \sqrt{-1} \in \mathbb{Q}_5$

hence (5) splits in $\mathbb{Q}(i)$, i.e. $5\mathcal{O}_L = \mathfrak{p}_1 \mathfrak{p}_2$.

$\square$