Chapter 6

Lines on hypersurfaces

Keynote Questions

(a) Let $X \subset \mathbb{P}^4$ be a general quintic hypersurface. How many lines $L \subset \mathbb{P}^4$ does X contain? (Answer on page 228.)

(b) Let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of quartic surfaces. How many of the surfaces X_t contain a line? (Answer on page 233.)

(c) Let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^1}$ be a general pencil of cubic surfaces, and consider the locus $C \subset \mathbb{G}(1,3)$ of all lines $L \subset \mathbb{P}^3$ that are contained in some member of this family. What is the genus of C? What is the degree of the surface $S \subset \mathbb{P}^3$ swept out by these lines? (Answers on pages 233 and 233.)

(d) Can a smooth quartic hypersurface in $\mathbb{P}^4$ contain a two-parameter family of lines? (Answer on page 238.)

In this chapter we will study the schemes parametrizing lines (and planes of higher dimension) on a hypersurface. These are called *Fano schemes*. There are two phases to the treatment. It turns out that the enumerative content of the keynote questions above, and many others, can be answered through a single type of Chern class computation. But there is another side of the story, involving beautiful and important techniques for working with the tangent spaces of Hilbert schemes, of which Fano schemes are examples. These ideas will allow us to verify that the "numbers" we compute really correspond to the geometry that they are meant to reflect. We will go even beyond these techniques and explore a little of the local structure of the Fano scheme. There are many open questions in this area, and the chapter ends with an exploration of one of them.

6.1 What to expect

For what n and d should we expect a general hypersurface $X \subset \mathbb{P}^n$ of degree d to contain lines? What is the dimension of the family of lines we would expect it to contain? When the dimension is zero, how many lines will there be?

To answer these questions, we introduce in this chapter a fundamental object, the *Fano scheme* $F_k(X) \subset \mathbb{G}(k,n)$ parametrizing k-planes on X, and then study its geometry.

We will defer for a moment a discussion of the scheme structure on $F_k(X)$, and start by answering the first two of the questions above, since these have to do only with the underlying set $F_k(X) \subset \mathbb{G}(k,n)$. Even so, the answers may not be apparent at first, since (as we shall see) the equations on the Grassmannian of lines that describe the locus of lines L contained in a given hypersurface $X \subset \mathbb{P}^n$ may be complicated. But if we reverse the roles of L and X — that is, ask for the locus, in the space $\mathbb{P}^N$ of all hypersurfaces of degree d in $\mathbb{P}^n$, of the hypersurfaces X that contain a given line L — the equations are much simpler; in fact, given L, the locus of X that contain L is simply a linear subspace of $\mathbb{P}^N$.

To capitalize on this, we use an incidence correspondence: We set $N = \binom{n+d}{d} - 1$ and let $\mathbb{P}^N$ be the projective space parametrizing all hypersurfaces of degree d, and consider the variety $\Phi = \Phi(n,d,k)$ given by the formula

$$\Phi = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(k,n) \mid L \subset X\}.$$

(As the title of this chapter suggests, our primary focus will be on the case $k = 1$ of lines, but many of the constructions we make can be carried out just as readily for arbitrary-dimensional linear spaces, as here.)

That $\Phi \subset \mathbb{P}^N \times \mathbb{G}(k,n)$ is a closed subset may be seen by a number of elementary arguments (see for example Harris [1995, Chapter 6]); in any event, we will give explicit equations of Φ in the next section. The variety Φ will be quite useful in many ways; we call it the *universal Fano scheme* of k-planes on hypersurfaces of degree d in $\mathbb{P}^n$, since the fiber over any point $X \in \mathbb{P}^N$ is the Fano scheme $F_k(X) \subset \mathbb{G}(k,n)$ of k-planes on X. To start, we have:

Proposition 6.1. *Let $N = \binom{n+d}{d} - 1$. The universal Fano scheme $\Phi = \Phi(n,d,k) \subset \mathbb{P}^N \times \mathbb{G}(k, \mathbb{P}^n)$ is a smooth irreducible variety of dimension*

$$\dim \Phi(n,d,k) = N + (k+1)(n-k) - \binom{k+d}{k}.$$

Proof: As we said, the fibers of Φ over $\mathbb{G}(k,n)$ are readily described: For any plane $\Lambda \cong \mathbb{P}^k \subset \mathbb{P}^n$, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_\Lambda(d))$$

is a surjection, and the fiber of Φ over the point $\Lambda \in \mathbb{G}(k,n)$ is simply the projectivization of the kernel, that is, a projective space $\mathbb{P}^{N-\binom{k+d}{k}}$. $\square$

Proposition 6.1 gives us an "expected" answer to the questions raised at the beginning of this section, and also allows us to deduce the answer in some cases: We would expect the family of k-planes on a general hypersurface $X \subset \mathbb{P}^n$ of degree d — that is, the fiber of Φ over a general point $[X] \in \mathbb{P}^N$ — to have dimension $\varphi(n,d,k) := \dim \Phi - N$, and that in the case $\varphi < 0$ a general X will contain no k-planes. In fact, Proposition 6.1 immediately implies the second statement, and while it does not imply the first, it does imply a lower bound on the dimension of the family of such planes, should there be any. We collect these consequences in the following corollary:

Corollary 6.2. *(a) The dimension of any component of the family of k-planes on any hypersurface of degree d in $\mathbb{P}^n$ is at least*

$$\varphi(n,d,k) := (k+1)(n-k) - \binom{k+d}{k}.$$

(b) If $\varphi(n,d,k) < 0$, then the general hypersurface of degree d in $\mathbb{P}^n$ contains no k-planes.

(c) If $\varphi(n,d,k) \geq 0$ and the general hypersurface of degree d contains any k-planes, then every hypersurface of degree d contains k-planes, and every component of the family of k-planes on a general hypersurface of degree d has dimension exactly $\varphi(n,d,k)$.

Proof: Part (b) is immediate: If $\dim \Phi < N$, then Φ cannot dominate $\mathbb{P}^N$. For part (a), we observe that since a fiber of $\Phi(n,d,k)$ over $\mathbb{P}^N$ is cut out by N equations, the principal ideal theorem (Theorem 0.1) gives the desired lower bound. As for part (c), if the general hypersurface of degree d contains a k-plane, then $\Phi(n,d,k)$ dominates $\mathbb{P}^N$, and, since $\Phi(n,d,k)$ is projective, the map to $\mathbb{P}^N$ is surjective with general fiber of dimension $\dim \Phi - N = \varphi$. $\square$

We shall eventually show (Corollary 6.32 and Theorem 6.28) that if $\varphi(n,d,k) \geq 0$ then, except in some cases where $k > 1$ and $d = 2$, the general hypersurface actually does contain k-planes, so the results above apply. For example, if $d \leq 2n-3$, every hypersurface of degree d in $\mathbb{P}^n$ contains lines, and the family of lines on the general such hypersurface has dimension $\varphi(n,d,1) = 2n-3-d$.

Corollary 6.2 shows that the general surface in $\mathbb{P}^3$ of degree $d \geq 4$ contains no lines. But we can say more, using the same sort of incidence correspondence argument made above. For example, we will see in Exercise 6.64 that a general surface $S \subset \mathbb{P}^3$ of degree $d \geq 4$ containing a line contains only one. This implies that the locus $\Sigma \subset \mathbb{P}^N$ of surfaces that do contain a line has codimension $d-3$.

6.1.1 Definition of the Fano scheme

We begin by giving a direct definition of the Fano scheme $F_k(X)$ of k-planes on a scheme $X \subset \mathbb{P}^n$ that is local on $\mathbb{G}(k,n)$. We will return to the definition twice later in this chapter to give a global description and a universal property that justifies the idea that we are taking the “right” scheme structure. Note that it will suffice to give the definition of the Fano scheme $F_k(X)$ when X is a hypersurface in $\mathbb{P}^n$; for an arbitrary scheme $Y \subset \mathbb{P}^n$ we define

$$F_k(Y) = \bigcap_{\substack{Y \subset X \subset \mathbb{P}^n \\ X \text{ is a hypersurface}}} F_k(X).$$

To define $F_k(X)$ for a hypersurface X of degree d given by an equation $g = 0$, we use the idea that a plane L lies on X if and only if the restriction of g to L is zero. If we have a parametrization $\alpha : \mathbb{P}^k \to L$ of L, then we can pull back g via α; the condition $L \subset X$ is given by the vanishing of the coefficients of $\alpha^*(g)$.

In fact, we can give such parametrizations simultaneously for all planes $L \in \mathbb{G}(k,n)$ lying in an open set U of the open cover of $\mathbb{G}(k,\mathbb{P}^n)$ described in Section 3.2.2. Recall that such an open set U is defined as the set of all k-planes not meeting a fixed $(n-k-1)$-plane. If the latter is given by the vanishing of the first $k+1$ coordinates, then U may be identified with the affine space of $(k+1)\times(n-k)$ matrices: Any k-plane L belonging to U is the row space of a unique matrix of the form

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{0,k+1} & \cdots & a_{0,n+1} \\ 0 & 1 & \cdots & 0 & a_{1,k+1} & \cdots & a_{1,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{k,k+1} & \cdots & a_{k,n+1} \end{pmatrix}.$$

We can thus give a parametrization of L of the form

$$\mathbb{P}^k \ni (s_0,\ldots,s_k) \xrightarrow{\alpha} (s_0 \cdots s_k)A = \Big(s_0,\ldots,s_k, \sum_i a_{i,k+1}s_i,\ldots,\sum_i a_{i,n+1}s_i\Big),$$

where the $a_{i,j}$ are coordinates on $U \cong \mathbb{A}^{(k+1)(n-k)}$ and the s_i are homogeneous coordinates on our fixed source $\mathbb{P}^k$.

Now suppose that $X \subset \mathbb{P}^n$ is the hypersurface $V(g)$ given by the polynomial

$$g(z_0,\ldots,z_n) = \sum_{|\delta|=d} c_\delta z^\delta.$$

We substitute the $n+1$ coordinates of the parametrization α for the variables $z_0,\ldots,z_n$ of g, and arrive at a homogeneous polynomial of degree d in $s_0,\ldots,s_k$:

$$\alpha^*(g) = \sum_{|\delta|=d} e_\delta s^\delta.$$

The coefficients $\{e_\delta\}$ of this polynomial are polynomials in the coordinates $\{a_{i,j}\}$ on $U \subset \mathbb{G}(k,n)$, which we take as defining equations for $F_k(X)$.

We should check, of course, that the scheme structure defined in this way agrees on the overlap $U \cap V$ of two such open sets. This is straightforward, but we will skip it: In what follows we will see a remarkable intrinsic characterization of the Fano scheme that will imply it.

Finally, we return to the universal Fano scheme $\Phi \subset \mathbb{P}^N \times \mathbb{G}(k,n)$ discussed in the preceding section. We promised to give equations for Φ, and we have: The coefficients e_δ of the polynomial $\alpha^*(g)$ above may be viewed as polynomials in both sets of variables $\{a_{i,j}\}$ and $\{c_\delta\}$, and we take $\Phi \subset \mathbb{P}^N \times \mathbb{G}(k,n)$ to be the subscheme defined locally by these polynomials. Note that for any hypersurface $X \subset \mathbb{P}^n$ the Fano scheme $F_k(X)$ is the scheme-theoretic fiber of Φ over the point $X \in \mathbb{P}^N$.

In fact, there is a simpler way to characterize the scheme structure of Φ: it is reduced. This follows from the fact that the scheme-theoretic fibers of the projection $\Phi \to \mathbb{G}(k,n)$ are projective spaces (the e_δ are homogeneous linear in the variables c_δ). For the same reason, Φ is smooth.

This is very much *not* to say that the Fano scheme $F_k(X)$ is either smooth or reduced for a given X. It does imply that $F_k(X)$ is smooth and reduced for a *general* hypersurface $X \subset \mathbb{P}^n$, but we will see many examples of nonreduced and/or singular Fano schemes; part of the challenge of the subject is to figure out under what circumstances this may happen. As a first example, you may wish to consider the Fano scheme $F_1(Q)$ of lines on a quadric surface $Q \subset \mathbb{P}^3$; as you can see from the equations, $F_1(Q)$ is smooth if Q is smooth, but everywhere nonreduced if Q is singular. Apart from this being the first nontrivial example of such phenomena, what makes this interesting is that we will also be able to see this from two other viewpoints, without coordinates: once when we describe the class $[F_1(Q)] \in A(\mathbb{G}(1,3))$ in Section 6.2, and again at the end of Section 6.4.2 when we introduce the notion of first-order deformations.

Of course special hypersurfaces may well contain families of planes of dimension greater than $\varphi(n,d,k)$. We can easily give an upper bound on the possible dimension:

Proposition 6.3. *If $X \subset \mathbb{P}^n$ is an m-dimensional variety, then*

$$\dim F_k(X) \le (m-k)(k+1) = \dim \mathbb{G}(k,m),$$

with equality if and only if X is an m-plane.

Proof: We may assume without loss of generality that X is nondegenerate. Let $U \subset X^{k+1}$ be the open set consisting of $(k+1)$-tuples of linearly independent points, and let

$$\Gamma = \{((p_0,\dots,p_k),L) \in U \times F_k(X) \mid p_i \in L \text{ for all } i\}.$$

Via the projection $\Gamma \to U$, we see that $\dim \Gamma \le m(k+1)$. Since the fibers of the projection $\Gamma \to F_k(X)$ have dimension $k(k+1)$, we conclude that $\dim F_k(X) \le m(k+1) - k(k+1) = (m-k)(k+1)$, as required.

Equality of dimensions can hold only if the projection $\Gamma \to U$ is dominant, that is, if X contains the plane spanned by any $k+1$ general points of X, and this can happen only if X is a linear space. $\square$

In Section 6.8 we will discuss some open questions about these dimensions.

6.2 Fano schemes and Chern classes

To get global information about the Fano scheme of a hypersurface, we will express it as the zero locus of a section of a vector bundle on the Grassmannian. To understand the idea, suppose that $X \subset \mathbb{P}^n$ is the hypersurface $g = 0$, where g is a homogeneous form of degree d. As we have seen, the condition that X contain a particular k-dimensional linear space L is that g is sent to 0 by the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_L(d)).$$

To describe the scheme $F_k(X) \subset \mathbb{G}(k,n)$ of k-planes on X, we will realize the family of vector spaces $H^0(\mathcal{O}_L(d))$ (with varying k-planes L) as the fibers of a vector bundle in such a way that the images of g in these vector spaces are the values of a section σ_g of the bundle:

Proposition 6.4. *Let V be an $(n+1)$-dimensional vector space, and let $\mathcal{S} \subset V \otimes \mathcal{O}_{\mathbb{G}}$ be the tautological rank-$(k+1)$ subbundle on the Grassmannian $G = \mathbb{G}(k, \mathbb{P}V)$ of k-planes in $\mathbb{P}V \cong \mathbb{P}^n$. A form g of degree d on $\mathbb{P}V$ gives rise to a global section σ_g of $\operatorname{Sym}^d \mathcal{S}^*$ whose zero locus is $F_k(X)$, where X is the hypersurface $g = 0$.*

Thus, when $F_k(X)$ has expected codimension $\binom{k+d}{k} = \operatorname{rank}(\operatorname{Sym}^d \mathcal{S}^*)$ in G, we have

$$[F_k(X)] = c_{\binom{k+d}{k}}(\operatorname{Sym}^d \mathcal{S}^*) \in A(G).$$

Proof: The fiber of $\mathcal{S}$ over the point $[L] \in \mathbb{G}(k, \mathbb{P}V)$ representing the subspace $L \cong \mathbb{P}^k \subset \mathbb{P}V$ is the corresponding $(k+1)$-dimensional subspace of V. The fiber of the dual bundle $\mathcal{S}^*$ at $[L]$ is thus the space of linear forms on L, that is to say $H^0(\mathcal{O}_{\mathbb{P}L}(1))$, and the dual map $V^* \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{S}^*$ evaluated at a point $[L]$ takes a linear form $\varphi \in V^*$, thought of as a constant section of the trivial bundle $V^* \otimes \mathcal{O}_{\mathbb{G}}$, to the restriction of φ to L. The vector space of forms of degree d on $\mathbb{P}V$ is $\operatorname{Sym}^d V^* = H^0(\mathcal{O}_{\mathbb{P}V}(d))$, and the induced map on symmetric powers

$$\operatorname{Sym}^d V^* \to \operatorname{Sym}^d \mathcal{S}^*$$

evaluated at L takes a form g of degree d to its restriction to L, as required.

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Let $\sigma_g \in H^0(\operatorname{Sym}^d \mathcal{S}^*)$ be the global section of $\operatorname{Sym}^d \mathcal{S}^*$ that is the image of g. We claim that $F_k(X) \subset \mathbb{G}(k, \mathbb{P}V)$ is the zero locus of this section. It is enough to check this locally on an open covering of $\mathbb{G}(k, \mathbb{P}V)$, and we use the open covering by basic affine sets U described in Section 3.2.2. On such an open set, the bundle $\mathcal{S}$ is trivial, with the inclusion

$$\mathcal{S}|_U = \mathcal{O}_U^{k+1} \to V \otimes \mathcal{O}_U$$

given by the transpose of the matrix A. It follows that the dual map

$$A^* : H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_U = V^* \otimes \mathcal{O}_U \to \mathcal{S}^*|_U$$

is the restriction of linear forms from $\mathbb{P}V$, and its d-th symmetric power is the restriction of forms of degree d. Thus the value of σ_g at the point of U corresponding to a plane L is the restriction of g to L, or in other words the result of the substitution given in Section 6.1.1, as required. □

The fact that the class $[F_k(X)]$ depends only on d and n (assuming it has the expected dimension) has consequences by itself, even without calculating the actual class. For example, consider the lines on a quadric surface $Q \subset \mathbb{P}^3$. As we saw in Section 3.6.1, when Q is smooth the Fano scheme $F_1(Q)$ consists of two disjoint smooth conic curves in the Grassmannian $\mathbb{G}(1,3) \subset \mathbb{P}^5$. But what happens when Q is a cone over a smooth conic curve? Here the support of $F_1(Q)$ is a single conic curve in $\mathbb{G}(1,3)$, and we may deduce from this and Proposition 6.4 that $F_1(Q)$ *is everywhere nonreduced.*

6.2.1 Counting lines on cubics

We want to see how this works for the case of lines on a cubic surface $X \subset \mathbb{P}^3$. In the language above, we want to compute the class of the Fano scheme $F_1(X)$ in the Grassmannian $\mathbb{G} = \mathbb{G}(1,3)$. We saw in Section 5.6.2 that the Chern class of $\mathcal{S}^*$ is

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}.$$

Since $\mathcal{S}$ has rank 2, the rank of $\operatorname{Sym}^3 \mathcal{S}^*$ is 4, so we want to compute $c_4(\operatorname{Sym}^3 \mathcal{S}^*)$. To do this, we will apply the splitting principle (Section 5.4), which implies that to compute the Chern class we may pretend that $\mathcal{S}^*$ splits into a direct sum of two line bundles $\mathcal{L}$ and $\mathcal{M}$. Suppose that

$$c(\mathcal{L}) = 1 + \alpha \quad \text{and} \quad c(\mathcal{M}) = 1 + \beta.$$

By the Whitney formula,

$$c(\mathcal{S}^*) = (1+\alpha)(1+\beta),$$

so that

$$\alpha + \beta = \sigma_1 \quad \text{and} \quad \alpha \cdot \beta = \sigma_{1,1}.$$

If $\mathcal{S}^*$ were to split as above, then the bundle $\mathrm{Sym}^3 \mathcal{S}^*$ would split as well:

$$\mathrm{Sym}^3 \mathcal{S}^* = \mathcal{L}^3 \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^3,$$

so that we would have

$$c(\mathrm{Sym}^3 \mathcal{S}^*) = (1 + 3\alpha)(1 + 2\alpha + \beta)(1 + \alpha + 2\beta)(1 + 3\beta).$$

In particular, the top Chern class could be written

$$\begin{aligned} c_4(\mathrm{Sym}^3 \mathcal{S}^*) &= 3\alpha(2\alpha + \beta)(\alpha + 2\beta)3\beta \\ &= 9\alpha\beta(2\alpha^2 + 5\alpha\beta + 2\beta^2) \\ &= 9\alpha\beta(2(\alpha + \beta)^2 + \alpha\beta). \end{aligned}$$

Re-expressing this in terms of the Chern classes of $\mathcal{S}^*$ itself, we get

$$\begin{aligned} c_4(\mathrm{Sym}^3 \mathcal{S}^*) &= 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) \\ &= 27\sigma_{2,2}, \end{aligned}$$

so

$$\deg\big(c_4((\mathrm{Sym}^3 \mathcal{S}^*))\big) = 27;$$

by the splitting principle, these formulas hold even though $\mathcal{S}$ does not in fact split.

The whole Chern class of $\mathrm{Sym}^3 \mathcal{S}^*$ can also be computed by hand in this way, or with the following commands in *Macaulay2*:

```
loadPackage "Schubert2"
G = flagBundle({2,2}, VariableNames=>{s,q})
-- sets G to be the Grassmannian of 2-planes in 4-space,
-- and gives the names $s_i$ and $q_i$ to the Chern classes
-- of the sub and quotient bundles, respectively.
(S,Q)=G.Bundles
-- names the sub and quotient bundles on G
chern symmetricPower(3,dual S)
```

which returns the output

```
                     2                          2
o4 = 1 + 6q  + (21q  - 10q ) + 42q q  + 27q
           1       1      2       1 2      2

                    QQ[][s , s , q , q ]
                          1   2   1   2
o4 : --------------------------------------------
     (s  + q , s  + s q  + q , s q  + s q , s q )
       1    1   2    1 1    2   2 1    1 2   2 2
```

The answer on the first output line "o4" is written in terms of the Chern classes $q_i := c_i(Q)$, which generate the (rational) Chow ring of the Grassmannian, described on the second output line "o4."

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Since the class $c_4(\mathrm{Sym}^3 \mathcal{S}^*)$ is nonzero, we deduce that *every cubic surface must contain lines*, and thus that a general cubic surface contains only finitely many. Moreover, if a particular cubic surface $X \subset \mathbb{P}^3$ contains only finitely many lines, then the number of these lines, counted with the appropriate multiplicity (that is, the degree of the corresponding component of the zero scheme of σ_g), is 27. As we will soon see, the Fano scheme $F_1(X)$ of a smooth cubic surface X is necessarily of dimension zero and reduced, so the actual number of lines is always 27. In the next section we will develop a general technique that will allow us to prove this statement, and much more. We will also see, in Section 6.7, how to count lines in cases where X is singular.

6.3 Definition and existence of Hilbert schemes

It was Grothendieck's brilliant observation that the Grassmannian and the Fano scheme are special cases of a very general construction, the *Hilbert scheme*. Hilbert schemes are defined by a universal property that we will explain in this section, after making the property explicit for the Grassmannian and Fano schemes.

One of the useful properties of Hilbert schemes is a general formula for tangent spaces, which we will explain in the next section. For more remarks about Hilbert schemes in general, see Section 8.4.1.

6.3.1 A universal property of the Grassmannian

Recall from Theorem 3.4 that the Grassmannian $G = G(k+1, V) = \mathbb{G}(k, \mathbb{P}V)$ of $(k+1)$-planes in an $(n+1)$-dimensional vector space V, with its tautological subbundle $\mathcal{S} \subset V \otimes \mathcal{O}_G$, has the following universal property: Given any scheme B and any rank-$(k+1)$ subbundle $\mathcal{F}$ of the trivial bundle $V \otimes \mathcal{O}_B$, there is a unique morphism $\varphi : B \to G$ such that $\mathcal{F} = \varphi^*\mathcal{S}$. We could, of course, just as well express this in terms of a universal property of the quotient bundle $\mathcal{Q} = V \otimes \mathcal{O}_B/\mathcal{S}$, or of the $(n-k)$-subbundle $\mathcal{Q}^* \subset V^* \otimes \mathcal{O}_B$, which is the most convenient for what we will do in this section.

Similarly, the universal k-plane

$$\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}$$

is a universal family of k-planes in $\mathbb{P}V$ in the following sense: For any scheme B, we will say that a subscheme $\mathcal{L} \subset B \times \mathbb{P}V$ is a *flat family of k-planes in $\mathbb{P}V$* if the restriction $\pi : \mathcal{L} \to B$ of the projection $\pi_1 : B \times \mathbb{P}V \to B$ is flat, and the fibers over closed points of B are linearly embedded k-planes in $\mathbb{P}V$. We have then:

Proposition 6.5. *If $\pi : \mathcal{L} \subset B \times \mathbb{P}V \to B$ is a flat family of k-planes in $\mathbb{P}V$, then there is a unique map $\alpha : B \to G$ such that $\mathcal{L}$ is equal, as a subscheme of $B \times \mathbb{P}V$, to the pullback of the family $\mathcal{L}$ via α:*

$$\begin{tikzcd} \mathcal{L} \arrow[drr, "\pi"'] & = & B \times_G \Phi \arrow[r] \arrow[d] & \Phi \arrow[d, "\pi"] \\ & & B \arrow[r, "\alpha"'] & G \end{tikzcd}$$

Proof: We will prove the proposition by showing that the desired property can be reduced to the universal property of Theorem 3.4. Though the reduction may appear technical, it is really just an application of Theorem B.5, together with the remark that the ideal of any k-plane in $\mathbb{P}V$ is generated by $n-k$ independent linear forms.

To simplify the notation we denote the ideal sheaves of $\Phi \subset G \times \mathbb{P}V$ and $\mathcal{L} \subset B \times \mathbb{P}V$ by $\mathcal{I}$ and $\mathcal{J}$ respectively, and we write $\mathcal{J}(1)$ for $\mathcal{J} \otimes \mu^*\mathcal{O}_{\mathbb{P}V}(1)$, where μ denotes the projection onto $\mathbb{P}V$. We define $\mathcal{I}(1)$ similarly. For any scheme B, we denote the trivial bundle with fiber V^* and base B by $\mathcal{O}_B \otimes V^*$.

The proof consists of the following steps: We will begin by showing that $\pi_*\mathcal{J}(1)$ is a subbundle of $\mathcal{O}_B \otimes V^*$. Since Φ satisfies the same hypotheses as $\mathcal{L}$, the same reasoning will show that the sheaf $\pi_*\mathcal{I}(1)$ is a subbundle of $\mathcal{O}_G \otimes V^*$. We will see that this subbundle is equal to the subbundle $\mathcal{Q}^*$. It follows that there is a unique map $\alpha : B \to G$ such that

$$\alpha^*(\pi_*\mathcal{I}(1)) = \pi_*\mathcal{J}(1).$$

Finally, we will show that this last equation is equivalent to the equality

$$\mathcal{L} = B \times_\alpha \Phi$$

as families of k-planes in $\mathbb{P}V$, where $B \times_\alpha \mathcal{L}$ denotes the pullback $B \times_G \mathcal{L}$ defined using the map α.

The fact that $\pi_*(\mathcal{J}(1))$ is a bundle follows from Theorem B.5 and the remark that the restriction of $\pi_*(\mathcal{J}(1))$ to a fiber b is the $(n-k)$-dimensional linear space of forms vanishing on the k-plane $\mathcal{L}'_b \subset \{b\} \times \mathbb{P}V \cong \mathbb{P}V$. The natural map $\pi_*\mathcal{J}(1) \to \pi_*\mathcal{O}_{B\times\mathbb{P}V}(1) = \mathcal{O}_B \otimes V^*$ is an inclusion on fibers, so $\pi_*\mathcal{J}(1)$ is a subbundle, as claimed.

To identify $\pi_*\mathcal{I}(1)$ with $\mathcal{Q}^*$, we remark that both are subbundles of $\mathcal{O}_B \otimes V^*$, and at each point $b \in B$ their fibers are the same subspace — namely, the space of linear forms vanishing on $\mathcal{L}_b$. It now follows from the universal property of Theorem 3.4 that there is a unique morphism $\alpha : B \to G$ such that $\alpha^*\pi_*\mathcal{I}(1) = \pi_*\mathcal{J}(1)$ as subbundles of $\mathcal{O}_B \otimes V^*$.

We claim that this property of α implies the equality $\mathcal{L} = B \times_\alpha \Phi$. To prove this, it suffices to show that $\mathcal{J} = (\alpha \times 1)^*\mathcal{I}$, or equivalently $\mathcal{J}(1) = (\alpha \times 1)^*\mathcal{I}(1)$. If we restrict $\mathcal{L}$ to the fiber over $b \in B$, we get a subspace of $\mathbb{P}V$ whose ideal is generated by the linear forms it contains. For $b \in B$, Theorem B.5 identifies this space of linear forms with the fiber of $\pi_* \mathcal{J}(1)$ at b. Thus there is a surjection

$$\pi^* \pi_* \mathcal{J}(1) \to \mathcal{J}(1).$$

Similar remarks hold for $\mathcal{I}$. Thus the commutative diagram

$$\begin{tikzcd}[column sep=tiny] \pi^* \pi_* \mathcal{J}(1) = \pi^* \alpha^* \pi_* \mathcal{I}(1) \arrow[dr] & & (\alpha \times 1)^* \pi^* \pi_* \mathcal{I}(1) \arrow[dl] \arrow[ll, phantom, "="] \\ & \mathcal{O}_{B \times \mathbb{P}V} & \end{tikzcd}$$

shows that the ideal sheaves of $\mathcal{L}$ and $B \times_\alpha \Phi$ are equal.

Finally, we prove the uniqueness of α. Suppose that $\mathcal{L} = B \times_{\alpha'} \Phi$ for some morphism α'. We will show that $\alpha' = \alpha$ by showing that $\pi_* \mathcal{J}(1) = \alpha'^* \pi_* \mathcal{I}(1)$. But the hypothesis implies that $\mathcal{J}(1) = (\alpha' \times 1)^* \mathcal{I}(1)$. From the definition of the pushforward, we get a natural map

$$\alpha'^* \pi_* \mathcal{I}(1) \to \pi_* (\alpha' \times 1)^* \mathcal{I}(1) = \pi_* \mathcal{J}(1)$$

that is an isomorphism fiber-by-fiber, so we are done. □

6.3.2 A universal property of the Fano scheme

We realized the Fano scheme of a projective variety X as the subscheme of the Grassmannian consisting of planes lying in X, and as such it inherits a universal property:

Proposition 6.6. *If $X \subset \mathbb{P}^n$ is a subscheme, then the scheme $F_k(X)$ represents the functor of k-planes on X, in the sense that the correspondence above induces a one-to-one correspondence between morphisms of schemes $B \to F_k(X) \subset \mathbb{G}(k,n)$ and families of k-planes $\mathcal{L} \subset B \times X \subset B \times \mathbb{P}^n$ that are flat over B.*

Proof: This is a corollary of the statement for the Grassmannian: Suppose that $\mathbb{P}^n = \mathbb{P}V$ and X is defined by some homogeneous forms $g_i \in \operatorname{Sym}^{d_i} V^*$. Let $\mathcal{S}$ be the universal subbundle on $\mathbb{G}(k,n)$, so that the fiber of $\mathcal{S}^*$ at a point $[L] \in \mathbb{G}(k,n)$ is the space of linear forms on the corresponding k-plane $L \subset \mathbb{P}V$. Writing δ_{g_i} for the section of $\operatorname{Sym}^{d_i} \mathcal{S}^*$ that is the image of the form g_i, we see that g_i vanishes on L if and only if the sections σ_{g_i} vanish at the point $[L]$. □

6.3.3 The Hilbert scheme and its universal property

Grothendieck's idea was to ask, more generally: given any projective scheme X and a subscheme Y, "How Y can move within X?" More precisely and ambitiously: Can we describe all flat families $B \times X \supset \mathcal{Y} \to B$ including Y as a fiber? Is there a universal such family?

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When B is reduced, a family $\mathcal{Y}$ as above is flat if and only if the fibers all have the same Hilbert polynomial; in particular, any family over a reduced base whose fibers are all k-planes is automatically flat. (See for example Eisenbud and Harris [2000, Proposition III-56].) Grothendieck's idea was to define "the family of all subschemes" of X with Hilbert polynomial equal to $P_Y(d)$, the Hilbert polynomial of Y.

We might worry that this goes too far to be a generalization of the Fano scheme — could there be a subscheme of X that is not a k-plane but whose Hilbert polynomial is equal to that of a k-plane? The following result shows that all is well:

Proposition 6.7. *A subscheme $Y \subset \mathbb{P}^n$ is a linearly embedded k-plane $\mathbb{P}^k$ if and only if the Hilbert polynomial of Y is*

$$P(d) = \binom{d+k}{k} = \frac{(d+k)(d+k-1)\cdots(d+1)}{k(k-1)\cdots 1}.$$

Proof: Since the dimension of the d-th graded component of a polynomial ring on $k+1$ variables is $\binom{d+k}{k}$, the Hilbert polynomial of a linearly embedded k-plane is $P(d)$.

Conversely, suppose that Y has Hilbert polynomial P. From the degree and leading coefficient of P we see that Y is a scheme of dimension k and degree 1. Thus $L := Y_{\text{red}} \subset Y$ is a linearly embedded k-plane. This inclusion induces a surjection of homogeneous coordinate rings $S_Y \to S_L$, and the equality of Hilbert polynomials shows that it is an isomorphism in high degrees. Since the inclusions $L \subset Y \subset \mathbb{P}^n$ can be recovered as $\operatorname{Proj}(S_L) \subset \operatorname{Proj}(S_Y) \subset \operatorname{Proj}(S)$, where S is the homogeneous coordinate ring of $\mathbb{P}^n$, and since $\operatorname{Proj}(S_Y)$ depends only on the high degree part of S_Y, this shows $L \subset Y$ is actually an equality. $\square$

Here is the general definition and existence theorem for Hilbert schemes, showing that there is a unique "most natural" scheme structure:

Proposition–Definition 6.8. *Let $X \subset \mathbb{P}^n$ be a closed subscheme, and let $P(d)$ be a polynomial. There exists a unique scheme $\mathcal{H}_P(X)$, called the* Hilbert scheme *of X for the Hilbert polynomial P, with a flat family*

$$\mathcal{H}_P(X) \times X \supset \mathcal{Y} \xrightarrow{\pi} \mathcal{H}_P(X)$$

of subschemes of X, called the universal family *of subschemes of X with Hilbert polynomial P, having the following properties:*

- *The fibers of π all have Hilbert polynomial equal to $P(d)$.*
- *For any flat family*

$$B \times X \supset \mathcal{Y}' \xrightarrow{\pi} B$$

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whose fibers have Hilbert polynomial $P(d)$, there is a unique morphism $\alpha : B \to \mathcal{H}_P(X)$ such that $\mathcal{Y}'$ is equal to the pullback of $\mathcal{Y}$:

$$\begin{tikzcd}
\mathcal{Y}' \arrow[dr, "\pi"'] & = & B \times_{\mathcal{H}_P(X)} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi"] \\
& & B \arrow[r, "\alpha"'] & \mathcal{H}_P(X)
\end{tikzcd}$$

A compact way of stating the existence of the family $\mathcal{H}_P(X) \times X \supset \mathcal{Y} \to \mathcal{H}_P(X)$ and its universal property is to use the language of representable functors. Consider the contravariant functor from schemes to sets that is defined on objects by

$$F_{X,P} : B \mapsto \{\text{flat families } X \times B \supset \mathcal{Y}' \to B \text{ of subschemes of } X \subset \mathbb{P}^n \\ \text{whose fibers over closed points all have Hilbert polynomial } P\}$$

and that takes a map $B' \to B$ to the map of sets taking a flat family over B to its pullback to a family over B'. In this language, the existence theorem says that $F_{X,P}$ is *representable* by the scheme $\mathcal{H}_P(X)$, in the sense that

$$F_{X,P} \cong \mathrm{Mor}(-, \mathcal{H}_P(X))$$

as functors. The universal family in $F_{X,P}(\mathcal{H}_P(X))$ then corresponds to the identity map in $\mathrm{Mor}(\mathcal{H}_P(X), \mathcal{H}_P(X))$. See for example Eisenbud and Harris [2000, Chapter VI] for more about this idea.

Proof of uniqueness in Proposition 6.8: As with any object with a universal property, the uniqueness of a map $\pi : \mathcal{Y} \to \mathcal{H}_P(X)$ with the given properties is easy: Given another such map $\pi' : \mathcal{Y}' \to B$, the universal properties of the two produce maps $B \to \mathcal{H}_P(X)$ and $\mathcal{H}_P(X) \to B$ whose composition $\mathcal{H}_P(X) \to B \to \mathcal{H}_P(X)$ is the unique map guaranteed by the definition that corresponds to the family $\pi : \mathcal{Y} \to \mathcal{H}_P(X)$ itself—that is, the identity map—and similarly for the composite $B \to \mathcal{H}_P(X) \to B$. □

6.3.4 Sketch of the construction of the Hilbert scheme

The construction of $\mathcal{H}_P(X)$ and the universal family is also relatively easy to describe, though the proofs of the necessary facts are deeper. There are several approaches, all along the lines of Grothendieck's original idea (see Grothendieck [1966b]), but the following (from Bayer [1982]) is perhaps the most explicit.

We first treat the case when $X = \mathbb{P}^n$, since (as in the case of the Fano schemes) we shall see that the general case reduces to this. Let $S = \Bbbk[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. The Hilbert scheme $\mathcal{H}_P(X)$ is constructed as a subscheme of the Grassmannian of $P(d)$-dimensional subspaces of S_d, the space of homogeneous forms of degree d, for suitably large d. The possibility of doing so is provided by the following basic result from commutative algebra, which combines ideas of Macaulay and Gotzmann (see Green [1989] for a coherent account).

Theorem 6.9. *With notation as above, there is an integer $d_0(P)$ (explicitly computable from the coefficients of P) such that if $d \geq d_0(P)$ then the saturated homogeneous ideal I of any subscheme of X with Hilbert polynomial P is generated in degrees $\leq d$ and* $\dim(S_d/I_d) = P(d)$. *Further, a subspace $U \subset S_d$ of dimension $P(d)$ generates an ideal with Hilbert polynomial $P(d)$ if and only if*

$$\dim(S_{d+1}/S_1U) \geq P(d+1),$$

in which case

$$\dim(S_{d+1}/S_1U) = P(d+1).$$

Example 6.10. If X is any hypersurface of degree s in $\mathbb{P}^n$, then the Hilbert function of S_X is

$$\dim(S_X)_d = \binom{n+d}{n} - \binom{n+d-s}{n},$$

which is equal to a polynomial $P(d)$ of degree $n-1$ for all d such that $d \geq s$, as one can check immediately. Conversely, given any scheme $X \subset \mathbb{P}^n$ with this Hilbert polynomial, we see that $\dim X = n-1$, so X is a hypersurface, and the leading coefficient of the Hilbert polynomial tells us that $\deg X = s$. It follows that the saturated ideal of X is generated by a single form of degree s. In this case, every subspace $U \subset S_d$ generates an ideal with this Hilbert polynomial; the growth condition of the theorem is automatically satisfied.

Given Theorem 6.9, we choose $d \geq d_0(P)$, and take $\mathcal{H}_P(\mathbb{P}^n)$ to be the closed subscheme of the Grassmannian $G := G(\dim S_d - P(d), S_d)$ defined by determinantal equations saying that $\mathcal{H}_P(\mathbb{P}^n)$ consists of those $U \in G$ such that the vector space S_1U has the smallest possible dimension, which is $\dim S_{d+1} - P(d+1)$. Writing $\mathcal{S}$ for the universal subbundle of the trivial vector bundle $S_d \otimes \mathcal{O}_G$ on G, $\mathcal{H}_P(\mathbb{P}^n)$ is the subscheme defined by the condition that the composite map

$$S_1 \otimes \mathcal{S} \to S_1 \otimes S_d \otimes \mathcal{O}_G \to S_{d+1} \otimes \mathcal{O}_G$$

has corank $\geq P(d+1)$.

Further, we can construct the universal family $\mathcal{Y} \subset \mathbb{P}^n \times \mathcal{H}_P(\mathbb{P}^n)$ as follows. Let

$$\mathbb{P}^n \xleftarrow{\pi_1} \mathbb{P}^n \times G \xrightarrow{\pi_2} G$$

be the projection maps. There is a natural map $S_d \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n \times G}$, and composing this with the inclusion we get a map of sheaves

$$\pi_2^*U \otimes \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n \times G}.$$

Let $\widetilde{\mathcal{Y}}$ be the subscheme of $\mathbb{P}^n \times G$ defined by the image of this map, and let $\mathcal{Y} \to \mathcal{H}_P(\mathbb{P}^n)$ be the restriction to $\mathcal{H}_P(X) \subset G$ of the (non-)flat family given by the composite

$$\widetilde{\mathcal{Y}} \subset \mathbb{P}^n \times G \xrightarrow{\pi_2} G.$$

The universal property (which we will not prove) shows that these construction are independent of the choice of $d \geq d_0$ (up to canonical isomorphism).

So far we have only defined $\mathcal{H}_P(\mathbb{P}^n)$, but we can use this to construct $\mathcal{H}_P(X)$ for any $X \subset \mathbb{P}^n$. Let $I = I(X) \subset S$ be the ideal corresponding to X, and suppose that I is generated in degrees $\leq e$. Given the Hilbert polynomial P, we choose $d \geq \max\{d_0(P), e\}$. Then to define $\mathcal{H}_P(X)$ we simply add equations to $\mathcal{H}_P(\mathbb{P}^n)$ implying that $I_d \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-d)$ is contained in $U \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^n}(-d)$. This can be translated into a rank condition on a map of vector bundles, as before.

Example 6.11 (Example 6.10, continued)**.** The argument above shows that the Hilbert scheme of hypersurfaces of degree s in $\mathbb{P}^n$ is the projective space $\mathbb{P}^N$ of all homogeneous forms of degree s, and the universal family is the universal hypersurface

$$\mathcal{X} = \{(x, X) \in \mathbb{P}^n \times \mathbb{P}^N \mid x \in X\},$$

as one would hope.

Here is one way to understand the integer $d_0(P)$ that plays a central role in the construction. Recall that the set of monomials of given degree d can be ordered *lexicographically*, where

$$x^e := x_0^{e_0} \cdots x_n^{e_n} < x_0^{f_0} \cdots x_n^{f_n} =: x^f$$

if $e_i > f_i$ for the smallest i such that $e_i \neq f_i$ — informally put, if x^e involves more of the lowest-index variables than x^f. A monomial ideal $I \subset S$ is called *lexicographic* if, whenever $x^e < x^f$ are monomials of degree d and $x^f \in I$, then $x^e \in I$ too. It follows easily that the saturation of a lexicographic ideal is lexicographic.

Proposition 6.12. *Let* $S = \Bbbk[x_0, \dots, x_n]$.

(a) If I is any homogeneous ideal of S, then there is a lexicographic ideal J such that the Hilbert function of S/J is the same as that of S/I.
(b) If $P = P_I$ is the Hilbert polynomial S/I, then there is a unique saturated lexicographic ideal J_P with Hilbert polynomial P.

The integer $d_0(P)$ may be taken to be the maximal degree of a generator of J_P.

For example, if I is the principal ideal generated by a form of degree s as in the example above, this proposition gives $d_0 = s$. See Green [1989] for further information.

6.4 Tangent spaces to Fano and Hilbert schemes

In order to use the Chern class calculation of Section 6.2.1 to count the number of distinct lines on a cubic surface, we need to know when the Fano scheme is reduced. In the zero-dimensional case, this is the same as being smooth, and the question can thus be approached through a computation of Zariski tangent spaces. Happily, we can give a simple description of the Zariski tangent spaces of any Hilbert scheme.

We first state the main assertions for Fano schemes. They will allow us to deduce the exact number of lines on a general hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$, along with other geometric facts. We will then compute the tangent spaces in the general setting of Hilbert schemes (Theorem 6.21). In Section 6.7 below, we will show how to calculate the multiplicity of $F_1(X)$ at L by writing down explicit local equations for $F_1(X) \subset \mathbb{G}(1, n)$.

6.4.1 Normal bundles and the smoothness of the Fano scheme

We will make use of the universal property of Fano schemes to give a geometric condition for the smoothness of $F_k(X)$ at a given point. Recall that if $Y \subset X$ is a smooth subvariety of the smooth variety X, then the *normal bundle* $\mathcal{N}_{Y/X}$ of Y in X is the cokernel of the map of tangent bundles $\mathcal{N}_{Y/X} = \operatorname{coker}(\mathcal{T}_Y \to \mathcal{T}_X|_Y)$ induced by the inclusion of $Y \subset X$. Recall also that the Zariski tangent space of a scheme F at a point p is by definition $\operatorname{Hom}_{\mathcal{O}_p}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathcal{O}_p/\mathfrak{m}_p)$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_p$ of F at p.

The following theorem is a special case of a general result on Hilbert schemes, Theorem 6.21, which we will prove in the next section:

Theorem 6.13. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The Zariski tangent space of $F_k(X)$ at $[L]$ is $H^0(\mathcal{N}_{L/X})$.*

The result is intuitively plausible if we think of a section of $\mathcal{N}_{L/X}$ as providing a normal vector at each point in X, with a corresponding infinitesimal motion of X.

For a case that is easy to understand, take $k = 0$. The Hilbert scheme of points on a variety X is X itself, as one checks from the definition. The tangent space at a point $x \in X$ is thus the Zariski tangent space to X at x, and this is — identifying sheaves on the space $\{x\}$ with vector spaces — equal to $\operatorname{Hom}_{\mathcal{O}_X}(\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2, \mathcal{O}_x) = \mathcal{N}_{x/X}$. Before introducing the general machinery of the proof, we explain how the result can be used.

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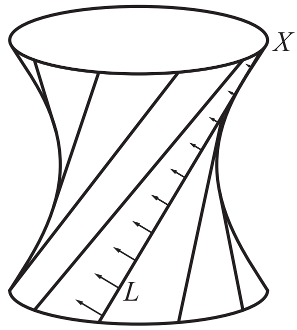

Figure 6.1 A tangent vector to the Fano scheme $F_1(X)$ at $[L]$ corresponds to a normal vector field along L in X.

Corollary 6.14. *Suppose that $L \subset X$ is a k-plane in a smooth variety $X \subset \mathbb{P}^n$, and let $[L] \in F_k(X)$ be the corresponding point. The dimension of $F_k(X)$ at $[L]$ is at most* $\dim H^0(\mathcal{N}_{L/X})$. *Moreover, $F_k(X)$ is smooth at $[L]$ if and only if equality holds.*

Proof of Corollary 6.14: By the principal ideal theorem, the dimension of the Zariski tangent space of a local ring is always at least the dimension of the ring, and equality holds if and only if the ring is regular. See Eisenbud [1995]. □

To apply Corollary 6.14, we need to be able to compute normal bundles, and this is often easy. For example, we have:

Proposition–Definition 6.15. *Suppose that $Y \subset X$ are schemes.*

(a) If X and Y are smooth varieties then $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$. For arbitrary schemes $Y \subset X$, we define $\mathcal{N}_{Y/X}$ by this formula.

(b) If $Y \subset X \subset W$ are schemes, and X is locally a complete intersection in W, then there is a left exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/W} \xrightarrow{\alpha} \mathcal{N}_{X/W}|_Y.$$

If all three schemes are smooth, then α is an epimorphism.

(c) If Y is a Cartier divisor on X then $\mathcal{N}_{Y/X} = \mathcal{O}_X(Y)$. More generally, if Y is the zero locus of a section of a bundle $\mathcal{E}$ of rank e on X, and Y has codimension e in X, then

$$\mathcal{N}_{Y/X} = \mathcal{E}|_Y.$$

Proof: (a) For any inclusion of subschemes $Y \subset X$, there is a right exact sequence involving the cotangent sheaves of X and Y:

$$\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

where d is the map taking the class of a (locally defined) function $f \in \mathcal{I}_{Y/X}$ to its differential $df \in \Omega_X|_Y$; see for example Eisenbud [1995, Proposition 16.12]. Since

X and Y are smooth, Y is locally a complete intersection in X, so $\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is a locally free sheaf on Y of rank equal to $\dim X - \dim Y = \operatorname{rank} \Omega_X|_Y - \operatorname{rank} \Omega_Y$. If the left-hand map d were not a monomorphism of sheaves, then the image of d would have strictly smaller rank, so the sequence could not be exact at $\Omega_X|_Y$. Thus d is a monomorphism, and we have an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X} \xrightarrow{d} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0$$

of bundles. Since Y is smooth, Ω_Y is locally free, so dualizing preserves exactness, and we get an exact sequence

$$0 \longleftarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y) \longleftarrow \mathcal{T}_X|_Y \longleftarrow \mathcal{T}_Y \longleftarrow 0,$$

where the right-hand map is the differential of the inclusion $Y \subset X$, proving that $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}^2_Y, \mathcal{O}_Y)$.

(b) From the inclusions $Y \subset X \subset W$, we derive an exact sequence of ideal sheaves

$$0 \longrightarrow \mathcal{I}_{X/W} \longrightarrow \mathcal{I}_{Y/W} \longrightarrow \mathcal{I}_{Y/X} \longrightarrow 0.$$

Applying the functor $\mathcal{H}om_{\mathcal{O}_W}(-, \mathcal{O}_Y)$ gives a left exact sequence

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/W} \longrightarrow \mathcal{H}om(\mathcal{I}_{Y/X}, \mathcal{O}_Y).$$

Since $\mathcal{H}om(\mathcal{I}_{Y/X}, \mathcal{O}_Y) \cong \mathcal{H}om(\mathcal{I}_{Y/X} \otimes \mathcal{O}_Y, \mathcal{O}_Y)$ and $\mathcal{I}_{Y/X} \otimes \mathcal{O}_Y \cong \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$, we get the desired sequence.

In the smooth case, we start with the exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_W|_X \longrightarrow \mathcal{N}_{X/W} \longrightarrow 0$$

that defines $\mathcal{N}_{X/W}$. We restrict to Y and factor out the subbundle T_Y from both $\mathcal{T}_X$ and $\mathcal{T}_W|_X$ to get the required exact sequence

$$0 \longrightarrow \mathcal{N}_{Y/X} \longrightarrow \mathcal{N}_{Y/W} \longrightarrow \mathcal{N}_{X/W}|_Y \longrightarrow 0.$$

(c) The first formula follows at once from part (a), since in that case $\mathcal{I}_{Y/X} = \mathcal{O}_X(-Y)$, and taking the dual of a bundle commutes with restriction.

For the second statement of part (c) we first give a geometric argument that works in the smooth case, and then a proof in general. Let Z be the total space of the bundle $\mathcal{E}$. The tangent bundle to Z restricted to the zero section $X \subset Z$ is $\mathcal{T}_X \oplus \mathcal{E}$.

Along the zero locus Y of σ, the derivative $D\sigma$ of σ is thus a map $\mathcal{T}_X|_Y \to \mathcal{T}_X|_Y \oplus \mathcal{E}_Y$. Since the component of $D\sigma$ that maps $\mathcal{T}_X|_Y$ to $\mathcal{E}_Y$ is zero along Y, the composite

$$\mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \xrightarrow{D\sigma} \mathcal{T}_X|_Y \oplus \mathcal{E}_Y \longrightarrow \mathcal{E}_Y$$

is zero. Locally at each point $y \in Y$, the image of $(\mathcal{T}_X)_y$ in $(\mathcal{T}_X)_y \oplus \mathcal{E}_y$ is the tangent space to $\sigma(X) \subset Z$. Since Y is smooth of codimension equal to the rank of $\mathcal{E}$, the

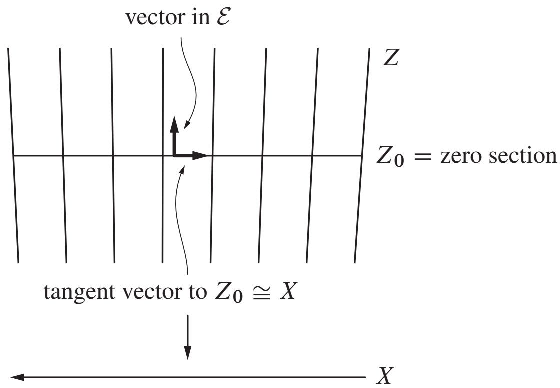

Figure 6.2 The tangent bundle to Z restricted to $Z_0 \cong X$ is $\mathcal{T}_X \oplus \mathcal{E}$.

manifold $\sigma(X)$ meets the zero locus $X \subset Z$ transversely. This means that $(\mathcal{T}_X)_y$ projects onto $\mathcal{E}_y$, and tells us that the composite map of bundles

$$\mathcal{T}_X|_Y \xrightarrow{D\sigma} \mathcal{T}_X|_Y \oplus \mathcal{E}_Y \longrightarrow \mathcal{E}_Y$$

is surjective. Considering the ranks, it follows that the sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{E}_Y \longrightarrow 0$$

is exact; that is, $\mathcal{N}_{Y/X} = \mathcal{E}_Y$.

With a more algebraic approach, we can avoid the hypothesis that X or Y is smooth. We may think of σ as defining the map $\mathcal{O}_X \to \mathcal{E}$ that sends $1 \in \mathcal{O}_X$ to $\sigma \in \mathcal{E}$. Dualizing, the statement that Y is the zero locus of σ means that the ideal sheaf $\mathcal{I}_{Y/X}$ is the image of the map $\sigma^* : \mathcal{E}^* \to \mathcal{O}_X$. Since the codimension of Y is e, we see that Y is locally a complete intersection. Thus the kernel of σ^* is generated by the Koszul relations; that is, the sequence

$$\cdots \longrightarrow \bigwedge^2 \mathcal{E}^* \xrightarrow{\kappa} \mathcal{E}^* \longrightarrow \mathcal{I}_{Y/X} \longrightarrow 0$$

is exact, where $\kappa(e \wedge f) = \sigma^*(e) f - \sigma^*(f) e$. Because the coefficients in the map κ lie in $\mathcal{I}_{Y/X}$, they become zero on tensoring with $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$, so we get the right exact sequence

$$\cdots \longrightarrow \bigwedge^2 \mathcal{E}^*|_Y \xrightarrow{0} \mathcal{E}^*|_Y \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X} \longrightarrow 0.$$

This shows that $\mathcal{E}^*|_Y \cong \mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$, whence $\mathcal{E}|_Y = \mathcal{E}^{**}|_Y = \mathcal{N}_{Y/X}$. □

In the special case where Y is a complete intersection of X with divisors on $\mathbb{P}^n$ of degrees d_i, the normal bundle is $\mathcal{N}_{Y/X} = \bigoplus \mathcal{O}_X(d_i)$, so the last statement of Proposition 6.15 takes a particularly simple form. We can make it even more explicit when both X and Y are complete intersections:

Corollary 6.16. *Suppose that $Y \subset X \subset \mathbb{P}^n$ are (not necessarily smooth) complete intersections of hypersurfaces with homogeneous ideals*

$$I_X = (g_1, \dots, g_s) \subset I_Y = (f_1, \dots, f_t), \quad g_i = \sum_j a_{i,j} f_j.$$

If $\deg f_i = \varphi_i$ *and* $\deg g_i = \gamma_i$, *then*

$$\mathcal{N}_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^{t} \mathcal{O}_Y(\varphi_i), \qquad \mathcal{N}_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i)$$

and $\mathcal{N}_{Y/X}$ is the kernel of the induced map $\alpha : \mathcal{N}_{Y/\mathbb{P}^n} \to \mathcal{N}_{X/\mathbb{P}^n}|_Y$ given by the matrix $(\overline{a}_{j,i})$, where $\overline{a}_{j,i}$ denotes the restriction of $a_{j,i}$ to Y.

Proof: The complete intersection X is the zero locus of the section $(g_1, \dots, g_s)$ of the bundle $\mathcal{O}_{\mathbb{P}^n}(\gamma_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(\gamma_s)$, and similarly for Y. Using the formula of part (c), we see that

$$\mathcal{N}_{X/\mathbb{P}^n} = \bigoplus_{i=1}^{s} \mathcal{O}_X(\gamma_i),$$

and similarly for Y. The identification of α follows at once from part (a). □

As an immediate application, we can finally show that there are exactly 27 distinct lines on every smooth cubic surface (pending, of course, the proof of Theorem 6.13):

Corollary 6.17. *Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$. If $F_1(X) \neq \varnothing$, then $F_1(X)$ is smooth and zero-dimensional. In particular, X contains at most finitely many lines, and if $d = 3$ then X contains exactly 27 distinct lines.*

See Corollary 6.27 for a strengthening.

Proof: Suppose $L \subset X$ is a line. As we saw in Section 2.4.2, the self-intersection number of L on X is negative, so the normal bundle $\mathcal{N}_{L/X}$ is a line bundle of negative degree. It follows that $\dim H^0(\mathcal{N}_{L/X}) = 0$, and Corollary 6.14 now implies that L is isolated and $F_1(X)$ is smooth at $[L]$.

In particular, in the case of the cubic surface the fact that the class of the Fano scheme is 27 points implies, with this result, that the Fano scheme actually consists of 27 reduced points. □

We will also be able to see Corollary 6.17 geometrically once we have introduced the notion of first-order deformation in Section 6.4.2

6.4.2 First-order deformations as tangents to the Hilbert scheme

The proof of Theorem 6.13 and its generalization involves the idea of a *first-order deformation* of a subscheme, which is the main content of this section. Suppose that

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Y is a closed subscheme of a scheme X, defined over the field $\Bbbk$. By a *deformation of* $Y \subset X$ *over a scheme* T *with distinguished point* $\operatorname{Spec}\Bbbk \in T$ we mean a subscheme $\mathcal{Y} \subset T \times X$, flat over T, whose fiber over the distinguished point $\operatorname{Spec}\Bbbk$ is equal to Y, that is, a diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\alpha"'] & \mathcal{Y} \arrow[r, hook] \arrow[d, "\beta"'] & T \times X \arrow[dl, "\text{projection}"] \\ \operatorname{Spec}\Bbbk \arrow[r, hook] & T & \end{tikzcd}$$

We think of the image of $\operatorname{Spec}\Bbbk \hookrightarrow T$ as a distinguished point of T, and we will denote it by $[Y]$.

A deformation is called *first-order* if its base T is the spectrum of a local ring of the form $R_m = \Bbbk[\epsilon_1, \dots, \epsilon_m]/(\epsilon_1, \dots, \epsilon_m)^2$ for some m. We set $T_m := \operatorname{Spec} R_m$. Note that this is a scheme with a unique closed point, which we shall denote by 0. We think of T_m as a first-order neighborhood of a point on a smooth m-dimensional variety.

It follows from the universal property of the Hilbert scheme that a first-order deformation of Y over T_m is the same thing as a morphism $T_m \to H$ sending 0 to $[Y]$.

In general, we will denote the set of morphisms of T_m into a $\Bbbk$-scheme Z sending 0 to a point $z \in Z$ by $\operatorname{Mor}_z(T_m, Z)$, so we have

$$\{\text{deformations of } Y \subset X \text{ over } T_m\} = \operatorname{Mor}_{[Y]}(T_m, H).$$

For simplicity we restrict ourselves for a while to the case $m = 1$, and consider deformations over T_1.

The identification of first-order deformations with morphisms from T_1 to H is the key to identifying the tangent space of H (and thus, in our case, of the Fano scheme). Indeed, for any closed $\Bbbk$-rational point z on any scheme Z we can identify the set $\operatorname{Mor}_z(T_1, Z)$ with the Zariski tangent space to Z at z. To describe the identification, recall that for any morphism $t : T_1 \to Z$ sending 0 to z we have a pullback map on functions, denoted $t^* : \mathcal{O}_{Z,z} \to R_1$. Restricting this map to $\mathfrak{m}_{Z,z}$, we get

$$t^*|_{\mathfrak{m}_{Z,z}} : \mathfrak{m}_{Z,z} \to \mathfrak{m}_{T,0} = \Bbbk\epsilon \cong \Bbbk.$$

Since t^* sends $\mathfrak{m}_{Z,z}^2$ to zero, we may identify $t^*|_{\mathfrak{m}_{Z,z}}$ with the induced map

$$t^*|_{\mathfrak{m}_{Z,z}} : \mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to \mathfrak{m}_{T,0} = \Bbbk\epsilon \cong \Bbbk.$$

Lemma 6.18. *Let $z \in Z$ be a $\Bbbk$-rational point on a $\Bbbk$-scheme. The map*

$$\operatorname{Mor}_z(T_1, Z) \to T_{z/Z} = \operatorname{Hom}_{\Bbbk}(\mathfrak{m}_{z/Z}/\mathfrak{m}_{z/Z}^2, \Bbbk)$$

sending a morphism t to the restriction of the pullback map on functions $t^*|_{\mathfrak{m}_{Z,z}}$ is bijective.

Proof: Giving a morphism $t : T_1 \to Z$ is equivalent to giving the local map of $\Bbbk$-algebras $t^* : \mathcal{O}_{Z,z} \to R_1$ that induces the identity map $\Bbbk \cong \mathcal{O}_{Z,z}/\mathfrak{m}_{Z,z} \to R_1/(\epsilon_1) = \Bbbk$. Thus t^* is determined by the induced map of vector spaces $\mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to (\epsilon_1) \subset R_1$.

Conversely, any map $\mathfrak{m}_{Z,z}/\mathfrak{m}_{Z,z}^2 \to (\epsilon)$ extends to a local algebra homomorphism $t^* : \mathcal{O}_{Z,z} \to \Bbbk[\epsilon]/(\epsilon^2) = R_1$. □

As we have explained, the universal property of the Hilbert scheme of $Y \subset X$ also allows us to identify $\mathrm{Mor}_{[Y]}(T_1, H)$ with the set of first-order deformations of $Y \subset X$ over T_1. Such deformations admit another very concrete description:

Theorem 6.19. *Suppose that $Y \subset X$ are schemes. There is a one-to-one correspondence between flat families of subschemes of X over the base T_m with central fiber Y and homomorphisms of $\mathcal{O}_Y$-modules $\mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^m$. In particular, flat families of deformations of Y in X over T_1 correspond to global sections of the normal sheaf of Y in X.*

We will use the following characterization of flatness over T_m:

Lemma 6.20. *If M is a (not necessarily finitely generated) module over the ring R_m, then M is flat if and only if the map*

$$M^m \xrightarrow{(\epsilon_1,\ldots,\epsilon_m)} M$$

induces an isomorphism $(M/(\epsilon_1,\ldots,\epsilon_m)M)^m \cong (\epsilon_1,\ldots,\epsilon_m)M$.

Proof: The general criterion of Eisenbud [1995, Proposition 6.1] says that M is flat if and only if the multiplication map $\mu_I : I \otimes_R M \to IM$ is an isomorphism for all ideals I. But every nontrivial ideal of R_m is a summand of $(\epsilon_1,\ldots,\epsilon_m) = (\epsilon)$, and, since $(R/(\epsilon))^m \cong (\epsilon)$, the map $\mu_{(\epsilon)}$ may be identified with the given map $(M/(\epsilon)M)^m \to (\epsilon)M$. □

Proof of Theorem 6.19: The problem is local, so we may assume that X and Y are affine. Since any homomorphism of sheaves $\mathcal{I}_Y \to \mathcal{O}_Y^m$ must annihilate $\mathcal{I}_Y^2$, we may identify a homomorphism

$$\varphi : \mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{(\varphi_1,\ldots,\varphi_m)} \mathcal{O}_Y^n$$

with the composition $\mathcal{I}_Y \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^m$. Let $\mathcal{I}_\varphi \subset \mathcal{O}_X \otimes R_m$ be the ideal

$$\mathcal{I}_\varphi := \Big\{ g + \sum\nolimits_j g_j\epsilon_j \;\Big|\; g \in \mathcal{I}_Y \text{ and } g_j \equiv \varphi_j(g) \bmod \mathcal{I}_Y \Big\},$$

and note that $\mathcal{I}_\varphi \supset \sum_j \epsilon_j \mathcal{I}_Y = (\epsilon)\mathcal{I}_Y$.

From $\mathcal{I}_\varphi$, we construct the family

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "\alpha"'] & Y_{T_m} \arrow[r, hook] \arrow[d, "\beta"'] & T_m \times X \arrow[dl, "\text{projection}"] \\
\operatorname{Spec}\Bbbk \arrow[r, hook] & T_m &
\end{tikzcd}$$

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where Y_{T_m} is defined by $\mathcal{I}_\varphi$. If we set all the $\epsilon_j = 0$, then $\mathcal{I}_\varphi$ becomes equal to $\mathcal{I}_Y$, so α is indeed the pullback of β.

We may identify $\mathcal{I}_\varphi/((\epsilon)\mathcal{I}_Y)$ with the graph of $\varphi : \mathcal{I}_Y \to \mathcal{O}_Y^m$ in

$$\begin{aligned}\mathcal{I}_Y \oplus \mathcal{O}_Y^m &\cong \mathcal{I}_Y \oplus \left(\bigoplus \mathcal{O}_Y \epsilon_j\right)\\ &\subset \mathcal{O}_X \oplus \left(\bigoplus \mathcal{O}_Y \epsilon_j\right)\\ &= \mathcal{O}_X[\epsilon]/((\epsilon)^2 + (\epsilon)\mathcal{I}_Y).\end{aligned}$$

Thus $\mathcal{I}_\varphi \cap (\epsilon)\mathcal{O}_X = (\epsilon)\mathcal{I}_Y$, and it follows that

$$\begin{aligned}(\epsilon)(\mathcal{O}_X/\mathcal{I}_\varphi) &= (\epsilon)\mathcal{O}_X/(\mathcal{I}_\varphi \cap (\epsilon)\mathcal{O}_X)\\ &= (\epsilon)\mathcal{O}_X/(\epsilon)\mathcal{I}_Y\\ &\cong (\mathcal{O}_X/\mathcal{I}_Y)^m \cong \mathcal{O}_Y^m.\end{aligned}$$

By Lemma 6.20, $\mathcal{O}_X/\mathcal{I}_\varphi$ is flat over R_m.

Conversely, given an R_m-algebra of the form

$$S := \mathcal{O}_X[\epsilon]/((\epsilon)^2 + \mathcal{I}),$$

the statement that Y is the pullback of $Y_{T_m} := \operatorname{Spec} S$ over the morphism $\operatorname{Spec} \Bbbk \subset T_m$ means that $\mathcal{I}$ is congruent to $\mathcal{I}_Y$ modulo (ϵ). Multiplying by (ϵ) and using that $(\epsilon)^2 = 0$, we see that $\mathcal{I} \supset (\epsilon)\mathcal{I}_Y$. If S is flat over R_m, then we must have $\mathcal{I} \cap (\epsilon) = (\epsilon)\mathcal{I}_Y$. Putting these facts together, we see that $\mathcal{I}/(\epsilon)\mathcal{I}_Y$ is the graph of a homomorphism $\mathcal{I}_Y \to (\epsilon)\mathcal{O}_X/(\epsilon)\mathcal{I}_Y \cong \mathcal{O}_Y^m$, and this is the inverse of the construction above. □

These results identify both the Zariski tangent space $T_{[Y],H}$ of the Hilbert scheme H of $Y \subset X$ at the point corresponding to Y, and the vector space of global sections of the normal sheaf, with the *set* of first-order deformations of Y in X, which we have already identified with the set $\operatorname{Mor}_{[Y]}(T_1, H)$. Since our goal is to compute the *dimension* of one of these two vector spaces in terms of the dimension of the dimension of the other, we must also ensure that the identification of sets preserves the vector space structure. This is the new content of the following result:

Theorem 6.21. *Suppose that $Y \subset X$ is a subscheme of a $\Bbbk$-scheme $X \subset \mathbb{P}^n$, and let H be the Hilbert scheme of Y. If $[Y] \in H$ denotes the point corresponding to Y, then*

$$T_{[Y]/H} \cong H^0(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2, \mathcal{O}_Y))$$

as vector spaces.

Theorem 6.13 is the special case where Y is a k-plane in X.

Proof of Theorem 6.21: We will show how to give the set of morphisms $T_1 \to H$ the structure of a vector space, and prove that this third structure is compatible with the bijections we have already given.

The rules for addition and scalar multiplication in the set $\mathrm{Mor}_{[Y]}(T_1, H)$ are similar, and the one for addition is more complicated, so will define addition, and check that it is compatible with the identifications of Lemma 6.18 and Theorem 6.19. We leave the analogous treatment of scalar multiplication to the reader.

As before, we set $R_m = \Bbbk[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^2$ and $T_m = \mathrm{Spec}\, R_m$ (we will only use the cases $m = 1$ and $m = 2$). A morphism of schemes $\Psi : T_2 \to H$ sending the closed point to $[Y]$ corresponds to a homomorphism $\psi : \mathfrak{m}_{H,[Y]} \to \Bbbk\epsilon_1 \oplus \Bbbk\epsilon_2$ or, equivalently, a pair of homomorphisms $\psi_1, \psi_2 : \mathfrak{m}_{H,[Y]} \to \Bbbk$, or a pair of morphisms $\Psi_1, \Psi_2 : T_1 \to H$ (in fancier language: T_2 is the coproduct of T_1 with itself in the category of pointed schemes). Moreover, there is an addition map

$$T_1 \xrightarrow{\text{(plus)}} T_2$$

that embeds T_1 as the closed subscheme with ideal $(\epsilon_1 - \epsilon_2) \subset R_2$. This map has the property that $\Psi \circ \text{(plus)} : T_1 \to H$ is the morphism corresponding to the sum $\psi_1 + \psi_2 : \mathfrak{m}_{H,[Y]} \to \Bbbk$.

Let $\mathcal{Y}_{\varphi_i}$ be the family obtained by pulling back the universal family along Ψ_i, and let $\varphi_i : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y$ be the homomorphism corresponding to this flat family. We have a pullback diagram

$$\begin{tikzcd}
\mathcal{Y}_{\varphi_1} \arrow[r] \arrow[d] & \mathcal{Y}_2 \arrow[d] & \mathcal{Y}_{\varphi_2} \arrow[l] \arrow[d] \\
T_1 \arrow[r] & T_2 & T_1 \arrow[l]
\end{tikzcd}$$

of flat families, where $\mathcal{Y}_2 \to T_2$ is the family obtained by pulling back along Ψ. To show that the addition law on the set $\mathrm{Mor}_{[Y]}(T_1, H)$ agrees with addition in the vector space $H^0(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2, \mathcal{O}_Y))$, it suffices to show that the pullback of $\mathcal{Y}_2$ along the map $\text{(plus)} : T_1 \to T_2$ is the family $\mathcal{Y}_{\varphi_1+\varphi_2}$.

Let $\varphi : \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{O}_Y^2$ be the homomorphism corresponding to $\mathcal{Y}_2$, so that the ideal of $\mathcal{Y}_2$ is the ideal $\mathcal{I}_\varphi$. If we compose φ with the map induced by the projection $R_2 \to R_1$ annihilating ϵ_2 we get the map φ_1, and similarly for ϵ_1 and φ_2. It follows that φ is in fact the map

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{\binom{\varphi_1}{\varphi_2}} \mathcal{O}_Y^2.$$

Thus if we pull back $\mathcal{Y}_2$ along the map (plus), that is, factor out $\epsilon_1 - \epsilon_2$ from the structure sheaf of $\mathcal{Y}_s$, the resulting algebra corresponds to the map $\varphi_1 + \varphi_2$, as required. □

Associated to any family

$$\mathcal{Y} \subset X \times B \xrightarrow{\pi} B$$

of subschemes of X is the *union of the schemes in the family*, defined to be the image of $\mathcal{Y}$ under the projection to X. In this spirit, if $Y \subset X$ are projective schemes, and B

is a subscheme of the Hilbert scheme of Y in X, then we define the *subscheme swept out by* B to be the union $Y' = Y'_B$ of the schemes in the restriction to B of the universal family over H.

We can now give a bound on the Zariski tangent spaces to Y' in the case where Y and X are smooth. Suppose that $p \in Y$ is a point of one of the schemes Y represented by points of B. The tangent space to Y' at p contains the tangent space to Y at p, so it is enough to bound the image of $T_p Y'$ in $T_p X / T_p Y$, which is the fiber at p of the normal bundle $(\mathcal{N}_{Y/X})_p$ of Y in X.

Intuitively, the amount the tangent space $T_p Y$ "moves" as Y moves in B is measured by the tangent space to B at $[Y]$, although some tangent vectors to B may produce trivial motions of $T_p Y$. Of course $T_{[Y]} B \subset T_{[Y]} H$, and by Theorem 6.21 the latter is $H^0(\mathcal{N}_{Y/X})$. Let $\varphi_{p,Y}$ be the evaluation map

$$\varphi_{p,Y} : H^0(\mathcal{N}_{Y/X}) \to (\mathcal{N}_{Y/X})_p = T_p X / T_p Y.$$

Proposition 6.22. *Let $Y \subset X$ be smooth projective schemes, and let $B \subset H$ be a closed subscheme of the Hilbert scheme of Y in X containing the point $[Y]$. If $p \in Y$ and Y' is the subscheme swept out by B, then*

$$T_p Y' / T_p Y \subset \varphi_{p,Y}(T_{[Y]} B).$$

This will follow directly from the following lemma:

Lemma 6.23. *Let $Z \subset Y$ be closed subschemes of a scheme X, and let $Z_\sigma, Y_\tau \subset$* $\operatorname{Spec} \Bbbk[\epsilon]/(\epsilon^2) \times X$ *be first-order deformations of Z and Y in X corresponding to the sections $\sigma \in H^0(\mathcal{N}_{Z/X})$ and $\tau \in H^0(\mathcal{N}_{Y/X})$. The scheme Z_σ is contained in Y_τ if and only if the images of σ and τ are equal under the maps*

$$\begin{array}{c} \sigma \in H^0(\mathcal{N}_{Z/X}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Z/X}, \mathcal{O}_Z) \\ \downarrow \\ \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Z) \\ \uparrow \\ \tau \in H^0(\mathcal{N}_{Y/X}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Y) \end{array}$$

induced by the inclusion $\mathcal{I}_{Y/X} \subset \mathcal{I}_{Z/X}$ and the projection $\mathcal{O}_Y \to \mathcal{O}_Z$. If Y and X are smooth, or more generally if $Y \subset X$ is locally a complete intersection, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Z) \cong \mathcal{N}_{Y/X}|_Z$, and thus $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_{Y/X}, \mathcal{O}_Z) \cong H^0(\mathcal{N}_{Y/X}|_Z)$.

See Figure 6.3.

Proof: The statement is local, so we can assume Z, Y and X are affine. We regard the global sections σ and τ as module homomorphisms $\mathcal{I}_{Z/X} \to \mathcal{O}_Z$ and $\mathcal{I}_{Y/X} \to \mathcal{O}_Y$.

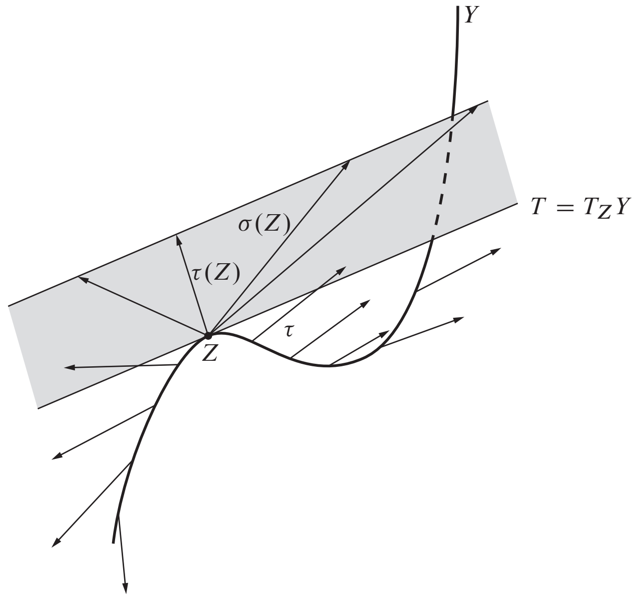

Figure 6.3 $\sigma(Z) \equiv \tau(Z)$ modulo the tangent line T to Y at Z, so the deformation of the point Z corresponding to σ keeps Z inside the deformation of Y corresponding to τ.

The schemes $Z_\sigma, Y_\tau \subset X \times T_1$ are given by the ideals

$$\mathcal{I}_\sigma = \{f + \epsilon f' \mid f \in \mathcal{I}_{Z/X} \text{ and } f' \equiv \sigma(f) \bmod \mathcal{I}_{Z/X}\}$$

and

$$\mathcal{I}_\tau = \{g + \epsilon g' \mid g \in \mathcal{I}_{Y/X} \text{ and } g' \equiv \tau(g) \bmod \mathcal{I}_{Y/X}\}$$

in $\mathcal{O}_X \otimes R_1 = \mathcal{O}_X \oplus \mathcal{O}_X \epsilon$.

Accordingly, we have $Z_\sigma \subset Y_\tau$—that is, $\mathcal{I}_\tau \subset \mathcal{I}_\sigma$—if and only if

$$\sigma(f) \equiv \tau(f) \bmod \mathcal{I}_{Z/X} \quad \text{for all } f \in \mathcal{I}_{Y/X},$$

which is the first statement of the lemma.

The second statement holds because, with the given hypothesis, $\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$ is a vector bundle, and thus

$$\begin{aligned} \mathcal{H}om(\mathcal{I}_{Y/X}, \mathcal{O}_Z) &= \mathcal{H}om(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Z) \\ &= \mathcal{H}om(\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}, \mathcal{O}_Y) \otimes_X \mathcal{O}_Z \\ &= \mathcal{N}_{Y/X}|_Z. \end{aligned}$$

□

Finally, we use the notion of first-order deformation to see Corollary 6.17 geometrically, via the Gauss map $\mathcal{G}_X : X \to \mathbb{P}^{3*}$ sending $p \in X$ to the tangent plane $\mathbb{T}_p X \subset \mathbb{P}^3$ (see Section 2.1.3). The restriction of $\mathcal{G}_X$ to a line $L \subset X \subset \mathbb{P}^3$ sends L to the dual line

$$L^\perp = \{H \in \mathbb{P}^{3*} \mid L \subset H\},$$

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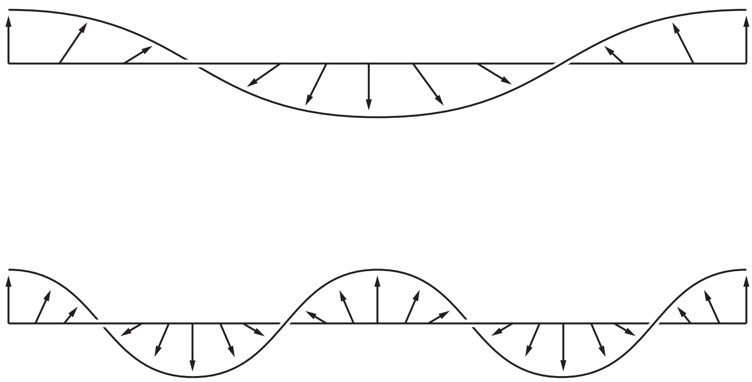

Figure 6.4 The tangent planes to a smooth quadric surface along a line wind once around the line, but in the case of a smooth cubic surface they wind around twice.

and this map, being given by the partial derivatives of the defining equation of X, has degree $d-1$. Thus, for example, as we travel along a line on a smooth quadric surface Q, the tangent planes to Q rotate once around the line; on a smooth cubic surface X, by contrast, they wind twice around the line (see Figure 6.4). But if $\widetilde{L}$ is a first-order deformation of L in $\mathbb{P}^3$, the direction of motion of a point $p \in L$ — that is, the 2-plane spanned by L and the normal vector $\sigma(p)$, where σ is the section of the normal bundle $\mathcal{N}_{L/\mathbb{P}^3}$ — is linear in p. It is thus impossible to find a first-order deformation of L on X, or on any smooth surface of higher degree.

Note that if X is singular at a point of L, the partial derivatives of the defining equation of X have a common zero along L, and so the degree of $\mathcal{G}_X : L \to L^\perp$ will be less than $d-1$. Thus, for example, the tangent planes to a quadric cone are constant along a line of its ruling, and if $L \subset X$ is a line on a cubic surface with an ordinary double point on L the Gauss map will have degree 1 on L. In this case, there *will* exist first-order deformations of L on X — as we will see shortly in Section 6.7

6.4.3 Normal bundles of k-planes on hypersurfaces

In order to apply the description of the tangent space $T_L F_k(X)$ to a Fano scheme $F_k(X)$ of k-planes on a hypersurface X at a point L, we need to know something about the normal bundle of L in X.

Suppose that $L \subset X \subset \mathbb{P}^n$ is a k-plane on a (not necessarily smooth) hypersurface X of degree d in $\mathbb{P}^n$. Choose coordinates so that the ideal of L is $I_L = (x_{k+1}, \dots, x_n)$, and let $I_X = (g) \subset I_L$. There is a unique expression

$$g = \sum_{i=k+1}^{n} x_i g_i(x_0, \dots, x_k) + h$$

with $h \in (x_{k+1}, \dots, x_n)^2$. Differentiating, we see that g_i, as a form on L, is the restriction to L of the derivative $\partial g/\partial x_i$.

Since the ideal of $L \subset \mathbb{P}^n$ is generated by $n-k$ linear forms, the normal bundle of L in $\mathbb{P}^n$ is $\mathcal{O}_L^{n-k}(1)$, and, similarly, the normal bundle of X in $\mathbb{P}^n$ is $\mathcal{O}_X(d)$. Thus the restriction $\mathcal{N}_{X/\mathbb{P}^n}|_L$ is $\mathcal{O}_L(d)$, and the left exact sequence of part (b) of Proposition 6.15 takes the form

$$0 \longrightarrow \mathcal{N}_{L/X} \longrightarrow \mathcal{O}_L^{n-k}(1) \xrightarrow{\alpha=(g_{k+1},\dots,g_n)} \mathcal{O}_L(d). \tag{6.1}$$

Proposition 6.24. *With notation as above, let*

$$\alpha = (g_{k+1}, \dots, g_n) : \mathcal{O}_L^{n-k}(1) \to \mathcal{O}_L(d).$$

(a) The map α is a surjection of sheaves if and only if the hypersurface X is smooth along L.
(b) The map α is surjective on global sections if and only if the point $[L]$ is a smooth point on $F_k(X)$ and the dimension of $F_k(X)$ at $[L]$ is equal to the "expected dimension" $(k+1)(n-k) - \binom{k+d}{k}$.
(c) The map α is injective on global sections if and only if the point $[L]$ is an isolated reduced (that is, smooth) point of $F_k(X)$.

Proof: (a) Since $L \subset X$, the derivatives of g along L are all zero, so X is smooth at a point $p \in L$ if and only if at least one of the normal derivatives $g_i = \partial g/\partial x_i$, for $i > k$, is nonzero at p. This is the condition that α is surjective as a map of sheaves.

(b) By Corollary 6.2, the dimension of $F_k(X)$ at any point is at least

$$D := (k+1)(n-k) - \binom{k+d}{k},$$

so $F_k(X)$ is smooth of dimension D at $[L]$ if and only if the tangent space $T_{[L]}F_k(X) = H^0\mathcal{N}_{L/X}$ has dimension D. Since

$$\dim H^0(\mathcal{O}_L^{n-k}(1)) = (k+1)(n-k) \quad \text{and} \quad \dim H^0(\mathcal{O}_L(d)) = \binom{d+k}{k},$$

we see from the exact sequence in (6.1) (before the proposition) that $\dim H^0\mathcal{N}_{L/X} = D$ if and only if α is surjective on global sections.

(c) The condition that $[L]$ is an isolated reduced point of $F_k(X)$ is the condition that $T_{[L]}F_k(X) = H^0\mathcal{N}_{L/X} = 0$, and by the argument of part (b) this happens if and only if α is injective on global sections. $\square$

We can unpack the conditions of Proposition 6.24 as follows: The condition of part (a) is equivalent to saying that the components g_i of the map α do not all vanish simultaneously at a point of L.

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Using the exact sequence (6.1), and assuming that X is smooth along L so that α is a surjection of sheaves, we see that the condition of part (b) that α is surjective on global sections is equivalent to the condition $H^1(\mathcal{N}_{L/X}) = 0$. On the other hand, the global sections $x_0, \dots, x_k$ of the i-th summand $\mathcal{O}_L(1) \subset \mathcal{O}_L^{n-1}(1)$ map by α to the sections $x_0 g_{k+i}, \dots, x_k g_{k+i}$, so the condition of surjectivity on sections is also equivalent to the condition that the ideal $(g_{k+1}, \dots, g_n)$ contains every form of degree d.

Similarly, it follows from the exact sequence that the condition of part (c) is equivalent to the condition $H^0(\mathcal{N}_{L/X}) = 0$. This means that there are no maps $\mathcal{O}_L$ to the kernel of α or, more concretely, that the g_i have no linear syzygies.

Although part (a) of Proposition 6.24 tells only about smoothness along L, we can do a little better: Bertini's theorem tells us that the general member of a linear series can only be singular along the base locus of the series, and it follows that the general X with a given map α is smooth except possibly along L. Thus if $\alpha : \mathcal{O}_L^{n-k}(1) \to \mathcal{O}_L(d)$ is any surjective map of sheaves, there is a smooth hypersurface X containing L such that $\mathcal{N}_{L/X} = \operatorname{Ker} \alpha$.

Example 6.25 (Cubic surfaces again)**.** The following gives another treatment of Corollary 6.17. In the case of a cubic surfaces $X \subset \mathbb{P}^3$, we have $n = d = 3$ and the expected dimension of $F_1(X)$ is $D = 0$. If we choose

$$g_2 = x_0^2, \quad g_3 = x_1^2$$

then the conditions in all three parts of Proposition 6.24 apply: g_2 and g_3 obviously have no common zeros in $\mathbb{P}^1$; because g_2 and g_3 are relatively prime quadratic forms, they have no linear syzygies; and since

$$(x_0, x_1)(x_0^2, x_1^2) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3) \subset \Bbbk[x_0, x_1, x_2, x_3],$$

the map α is surjective on global sections. Since the numbers of global sections of the source and target of α are equal, the map α is injective on global sections as well. We can see this directly, too: Because g_2, g_3 are relatively prime quadratic forms, the kernel of α is

$$\mathcal{O}_L(-1) \xrightarrow{\binom{-g_3}{g_2}} \mathcal{O}_L(1)^2,$$

so $\mathcal{N}_{L/X} = \mathcal{O}_L(-1)$, and we see again that $H^0(\mathcal{N}_{L/X}) = 0$.

From all this, we see that L will be an isolated smooth point of $F_1(X)$, where X is the hypersurface defined by the equation $x_0^2 x_2 + x_1^2 x_3 = 0$. Although this hypersurface is not smooth, Bertini's theorem, as above, shows that there are smooth cubics having the same map α. Since the rank of a linear transformation is upper-semicontinuous as the transformation varies, this will also be true for the general cubic surface containing a line. By Corollary 6.2, every cubic surface in $\mathbb{P}^3$ contains lines.

One special case of Proposition 6.24 shows that a smooth hypersurface of degree $d > 1$ cannot contain a plane of more than half its dimension:

Corollary 6.26. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d > 1$. If $L \subset X$ is a k-plane on X, and X is smooth along L, then*

$$k \leq \frac{n-1}{2}.$$

For example, there are no 2-planes on a smooth quadric hypersurface in $\mathbb{P}^4$ — even though the "expected dimension" $\varphi(4,2,2)$ is 0. This implies that all singular quadrics contain families of 2-planes of positive dimension — of course, it is easy to see this directly.

Proof: If $k > (n-1)/2$, then $k+1 > n-k$, so $n-k$ forms on $\mathbb{P}^k$ of strictly positive degree must have a common zero, and we can apply part (a) of Proposition 6.24. □

Remark. Corollary 6.26 is a special case of a corollary of the Lefschetz hyperplane theorem (see Appendix C), which tells us in this case that if $X \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth hypersurface and $Y \subset X$ is any subvariety of dimension $k > (n-1)/2$, then

$$\deg(X) \mid \deg(Y).$$

In the case of planes of the maximal dimension allowed by Corollary 6.26, Proposition 6.24 gives us particularly sharp information; note that this applies, in particular, to lines on surfaces in $\mathbb{P}^3$, and thus generalizes Corollary 6.17:

Corollary 6.27. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq 3$ containing a k-plane L with $k = (n-1)/2$. If X is smooth along L then $[L]$ is an isolated smooth point of the Fano scheme $F_k(X)$. If $n = d = 3$ — that is, if $X \subset \mathbb{P}^3$ is a cubic surface — then the converse is also true.*

If, in the setting of Proposition 6.24 we take an example where the g_i are general forms of degree $d-1$ in $k+1$ variables vanishing at some point of $\mathbb{P}^k$ with $d = 2, k > 1$ or $d > 3, k \geq 1$, then the g_i have no linear syzygies, so the corresponding $L \subset X$ will be a smooth point on the Fano scheme, though X is singular at a point of L. Thus the "converse" part of the corollary cannot be extended to these cases.

Proof of Corollary 6.27: If X is smooth along L, then by Proposition 6.24 the $k+1$ forms g_i of degree $d-1$ have no common zeros. It follows that they are a regular sequence, so all the relations among them are also of degree $d-1 \geq 2$, so, again by Proposition 6.24, $[L]$ is a smooth point of $F_k(X)$.

In the case of a cubic surface, g_2 and g_3 are quadratic forms in two variables. If they have a zero in common then they have a linear common factor, so they have a linear syzygy. □

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Despite the nonexistence of 2-planes on smooth quadric hypersurfaces $X \subset \mathbb{P}^4$ and other examples coming from Corollary 6.26, the situation becomes uniform for hypersurfaces of degree $d \geq 3$. The proof for the general case is quite complicated, and we only sketch it. In the next section we give a complete and independent treatment for the case of lines.

Theorem 6.28. *Set $\varphi = (k+1)(n-k) - \binom{k+d}{k}$.*

(a) If $k = 1$ or $d \geq 3$ and $\varphi \geq 0$, then every hypersurface of degree d contains k-planes, and the general hypersurface X of degree d in $\mathbb{P}^n$ has $\dim F_k(X) = \varphi$.

(b) If $\varphi \leq 0$ and X is a general hypersurface containing a given k-plane L, then L is an isolated smooth point of $F_k(X)$.

See Exercise 6.59 for an example that can be worked out directly.

Proof: (a) The first part follows from the second using Corollary 6.2. For the second part we use Proposition 6.24. We must show that, under the given hypotheses, a general $(n-k)$-dimensional vector space of forms of degree $d-1$ generates an ideal containing all the forms of degree d.

On the other hand, for part (b) we must show that a general $(n-k)$-dimensional vector space of forms of degree $d-1$ generates an ideal without linear syzygies.

These two statements together say that if $g_{k+1}, \dots, g_{k+n}$ is a general collection of $n-k$ forms of degree $d-1$ in $k+1$ variables, then the degree-d component of the ideal $(g_{k+1}, \dots, g_{k+n})$ has dimension equal to $\min\left\{(k+1)(n-k), \binom{k+d}{k}\right\}$. This is a special case of the formula for the maximal Hilbert function of a homogeneous ideal with generators in given degrees conjectured in Fröberg [1985]. This particular case of Fröberg's conjecture was proved in Hochster and Laksov [1987, Theorem 1]. $\square$

6.4.4 The case of lines

The case $k = 1$ of lines is special because, very much in contrast with the general situation, we can classify vector bundles on $\mathbb{P}^1$ completely. The following result is sometimes attributed to Grothendieck, although equivalent forms go back at least to the theory of matrix pencils of Kronecker and Weierstrass:

Theorem 6.29. *Any vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ is a direct sum of line bundles; that is,*

$$\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(e_i)$$

for some integers $e_1, \dots, e_r$.

The analogous statement is false for bundles on projective space $\mathbb{P}^n$ of dimension $n \geq 2$ (see for example Exercise 5.41).

Proof: We use the Riemann–Roch theorem for vector bundles on curves. Riemann–Roch theorems in general will be discussed in Chapter 14, where we will also discuss more aspects of the behavior of vector bundles on $\mathbb{P}^1$. The reader may wish to glance ahead or, since we will not make logical use of Theorem 6.29, defer reading this proof until then.

That said, we start with a basic observation: An exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow 0$$

on any variety X splits if and only if there exists a map $\beta : \mathcal{G} \to \mathcal{F}$ such that $\alpha \circ \beta = \mathrm{Id}_{\mathcal{G}}$. This will be the case whenever the map $\mathcal{H}om(\mathcal{G}, \mathcal{F}) \to \mathcal{H}om(\mathcal{G}, \mathcal{G})$ given by composition with α is surjective on global sections; from the exactness of the sequence

$$0 \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{E}) \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{G}) \longrightarrow 0,$$

this will in turn be the case whenever $H^1(\mathcal{H}om(\mathcal{G}, \mathcal{E})) = H^1(\mathcal{G}^* \otimes \mathcal{E}) = 0$.

Now suppose $\mathcal{E}$ is a vector bundle of rank 2 on $\mathbb{P}^1$, with first Chern class of degree d. By Riemann–Roch, we have

$$h^0(\mathcal{E}) \geq d + 2;$$

from this we may deduce the existence of a nonzero global section σ of $\mathcal{E}$ vanishing at $m \geq d/2$ points of $\mathbb{P}^1$, or equivalently of an inclusion of vector bundles $\mathcal{O}_{\mathbb{P}^1}(m) \hookrightarrow \mathcal{E}$ with $m \geq d/2$. We thus have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d-m) \longrightarrow 0,$$

and, since $2m - d \geq 0$, we have

$$H^1\big(\mathcal{H}om(\mathcal{O}_{\mathbb{P}^1}(d-m), \mathcal{O}_{\mathbb{P}^1}(m))\big) = H^1(\mathcal{O}_{\mathbb{P}^1}(2m-d)) = 0.$$

In this case, we conclude that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(d-m)$.

The case of a bundle $\mathcal{E}$ of general rank r follows by induction: If we let $\mathcal{L} \subset \mathcal{E}$ be a sub-line bundle of maximal degree m, we get a sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(m) \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \longrightarrow 0,$$

with $\mathcal{F}$ by induction a direct sum of line bundles $\mathcal{L}_i \cong \mathcal{O}_{\mathbb{P}^1}(e_i)$. Moreover, $e_i \leq m$ for all i: If $e_i > m$ for some i, then $\alpha^{-1}(\mathcal{L}_i)$ would be a bundle of rank 2 and degree $> 2m$; by the rank-2 case, this would contradict the maximality of m. Thus this sequence splits, and we are done. □

We remark in passing that vector bundles on higher-dimensional projective spaces $\mathbb{P}^n$ remain mysterious, even for $n = 2$, and open problems regarding them abound. To mention just one, it is unknown whether there exist vector bundles of rank 2 on $\mathbb{P}^n$, other than direct sums of line bundles, when $n \geq 6$. Interestingly, though, Theorem 6.29

provides a tool for the study of bundles on higher-dimensional projective spaces, via the notion of *jumping lines*, which we will discuss in Section 14.4

To return to our discussion of linear spaces on hypersurfaces, suppose that $X \subset \mathbb{P}^n$ is a hypersurface of degree d and $L \subset X$ a line. We choose coordinates so that L is defined by $x_2 = \cdots = x_n = 0$. As before, we write the equation of X in the form

$$\sum_{i=2}^{n} x_i g_i(x_0, x_1) + h,$$

with $h \in (x_2, \ldots, x_n)^2$, and we let α be the map $(g_2, \ldots, g_n) : \mathcal{O}_L^{n-1} \to \mathcal{O}_L(d)$. In this situation, the expected dimension of the Fano scheme $F_1(X)$ is $\varphi := 2n - 3 - d$. We will make use of this notation throughout this subsection.

We can say exactly what normal bundles of lines in hypersurfaces are possible. Since any vector bundle on $L \cong \mathbb{P}^1$ is a direct sum of line bundles, we may write $\mathcal{N}_{L/X} \cong \bigoplus_1^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i)$.

Proposition 6.30. *Suppose that $n \geq 3$ and $d \geq 1$. There exists a smooth hypersurface X in $\mathbb{P}^n$ of degree d, and a line $\mathbb{P}^1 \cong L \subset X$ such that $\mathcal{N}_{L/X} \cong \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^n}(e_i)$, if and only if*

$$e_i \leq 1 \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{n-2} e_i = n - 1 - d.$$

Proof: If the normal bundle is $\mathcal{N}_{L/X} \cong \bigoplus \mathcal{O}_{\mathbb{P}^1}(e_i)$, then, from the fact that there is an inclusion $\mathcal{N}_{L/X} \to \mathcal{N}_{L/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$, it follows that $e_i \leq 1$ for all i. Computing Chern classes from the exact sequence of sheaves on $\mathbb{P}^1$

$$0 \longrightarrow \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{n-1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0,$$

we get $\sum e_i = n - 1 - d$.

Conversely, suppose the e_i satisfy the given conditions. To simplify the notation, let $\mathcal{F} = \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^1}(e_i)$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1}^{n-1}(1)$. Let $\beta : \mathcal{F} \to \mathcal{G}$ be any map, and let α be the map $\mathcal{G} \to \mathcal{O}_{\mathbb{P}^1}(d)$ given by the matrix of $(n-2) \times (n-2)$ minors of the matrix of β, with appropriate signs.[1] The composition $\alpha\beta$ is zero because the i-th entry of the composite matrix is the Cauchy expansion of the determinant of a matrix obtained from β by repeating the i-th column.

[1] More formally and invariantly, α is the composite map

$$\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \bigwedge^{n-2} \mathcal{G}^* \xrightarrow{\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \bigwedge^{n-2} \beta^*} \mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \bigwedge^{n-2} \mathcal{F}^* \cong \mathcal{O}_{\mathbb{P}^1}(d),$$

where we have used an identification of $\mathcal{G}$ with $\mathcal{O}_{\mathbb{P}^1}(n-1) \otimes \bigwedge^{n-2} \mathcal{G}^*$ corresponding to a global section of $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-n+1) \otimes \bigwedge^{n-1} \mathcal{G}$.

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If we take β of the form

$$\beta = \begin{pmatrix} x_0^{1-e_1} & 0 & 0 & \cdots & 0 & 0 \\ x_1^{1-e_2} & x_0^{1-e_2} & 0 & \cdots & 0 & 0 \\ 0 & x_1^{1-e_3} & x_0^{1-e_3} & \cdots & \vdots & \vdots \\ 0 & 0 & x_1^{1-e_4} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & x_0^{1-e_{n-4}} & 0 \\ 0 & 0 & 0 & \cdots & x_1^{1-e_{n-3}} & x_0^{1-e_{n-3}} \\ 0 & 0 & 0 & \cdots & 0 & x_1^{1-e_{n-2}} \end{pmatrix},$$

then the top $(n-2) \times (n-2)$ minor will be x_0^{d-1} and the bottom $(n-2) \times (n-2)$ minor will be x_1^{d-1}. This shows that the map α will be an epimorphism of sheaves, so that the general such hypersurface X containing L will be smooth. By Eisenbud [1995, Theorem 20.9], the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\beta} \mathcal{G} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

is exact, so $\mathcal{N}_{L/X} \cong \mathcal{F}$. □

Corollary 6.31. *If $d \leq 2n-3$, then there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $F_1(X)$ is smooth of dimension $2n-3-d$ in a neighborhood of $[L]$.*

Proof: Using Proposition 6.30, we observe that, if $d \leq 2n-3$, we can choose all the e_i to be ≥ -1. With this choice, $\dim H^0(\mathcal{O}_{\mathbb{P}^1}(e_i)) = e_i + 1$ for all i and hence $\dim H^0(\mathcal{N}_{L/X}) = 2n-3-d$. Since $F_1(X)$ has dimension at least $2n-3-d$ everywhere, the result follows. □

Corollary 6.32. *If $d \leq 2n-3$, then every hypersurface of degree d in $\mathbb{P}^n$ contains a line.*

Proof: The universal Fano scheme $\Phi(n, d, 1)$ is irreducible of dimension $N-d+2n-3$. Moreover, Corollary 6.31 asserts that at some point $(X, L) \in \Phi$ the fiber dimension of the projection $\Phi(n, d, 1) \to \mathbb{P}^N$ is $2n-d-3$. It follows that this projection is surjective. □

We have seen above that the Fano scheme of any smooth cubic surface in $\mathbb{P}^3$ is reduced and of the correct dimension. We can now say something about the higher-dimensional case as well:

Corollary 6.33. *The Fano scheme of lines on any smooth hypersurface of degree $d \leq 3$ is smooth and of dimension $2n-3-d$. But if $n \geq 4$ and $d \geq 4$, then there exist smooth hypersurfaces of degree d in $\mathbb{P}^n$ whose Fano schemes are singular or of dimension $> 2n-3-d$.*

Proof: We follow the notation of Proposition 6.30. If $d \leq 3$, then for any $e_1, \ldots, e_{n-2}$ allowed by the conditions of the proposition we have that all the $e_i \geq -1$, and thus $h^0(\mathcal{N}_{L/X}) = \chi(\mathcal{N}_{L/X}) = 2n - 3$, proving that the Fano scheme is smooth and of expected dimension at L.

On the other hand, if $n \geq 4$ and $d \geq 4$ then we can take $e_1 = \cdots = e_{n-3} = 1$ and $e_{n-2} = 2 - d \leq -2$. In this case $h^0(\mathcal{N}_{L/X}) = 2n - 6 > 2n - 3 - d$, so the Fano scheme is singular or of "too large" dimension at L. $\square$

The first statement of Corollary 6.33 is an easy case of the conjecture of Debarre and de Jong, which we will discuss further in Section 6.8.

6.5 Lines on quintic threefolds and beyond

We can now answer the first of the keynote questions of this chapter: How many lines are contained in a general quintic threefold $X \subset \mathbb{P}^4$? More generally, we can now compute the number of distinct lines on a general hypersurface X of degree $d = 2n - 3$ in $\mathbb{P}^n$, the case in which the expected dimension of the family of lines is zero.

The set-up is the same as that for the lines on a cubic surface: The defining equation g of the hypersurface X gives a section σ_g of the bundle $\mathrm{Sym}^d\, \mathcal{S}^*$ on the Grassmannian $\mathbb{G}(1, n)$, the zero locus of σ_g is then the Fano scheme $F_1(X)$ of lines on X, and (assuming $F_1(X)$ has the expected dimension 0) the degree m of this scheme is the degree of the top Chern class $c_{d+1}(\mathrm{Sym}^d\, \mathcal{S}^*) \in A^{d+1}(\mathbb{G}(1, n))$. If we can show in addition that $H^0(\mathcal{N}_{L/X}) = 0$ for each line $L \subset X$, then it follows as in the previous section that the Fano scheme is zero-dimensional and reduced, so the actual number of distinct lines on X is exactly m.

To calculate the Chern class we could use the splitting principle. The computation is reasonable for $n = 4$, $d = 5$, the case of the quintic threefold, but becomes successively more complicated for larger n and d. *Schubert2* (in *Macaulay2*) instead deduces it from a Gröbner basis for the Chow ring. Here is a *Schubert2* script that computes the numbers for $n = 3, \ldots, 20$, along with its output:

```
loadPackage "Schubert2"
grassmannian = (m,n) -> flagBundle({m+1, n-m})
time for n from 3 to 20 do(
     G=grassmannian(1,n);
     (S,Q) = G.Bundles;
     d = 2*n-3;
     print integral chern symmetricPower(d, dual S))

27
2875
698005
305093061
210480374951
210776836330775
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289139638632755625
520764738758073845321
1192221463356102320754899
3381929766320534635615064019
116439626640205162647858259911 65
4783778650206319508831103239257812 5
```

The following result gives a geometric meaning to these numbers beyond the fact that they are degrees of certain Chern classes:

Theorem 6.34. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d \geq 1$, then the Fano scheme $F_1(X)$ of lines on X is reduced and has the expected dimension $2n - d - 3$.*

We now have the definitive answer to Keynote Question (a):

Corollary 6.35. *A general quintic threefold $X \subset \mathbb{P}^4$ contains exactly 2875 lines. More generally, the numbers in the Schubert2 output above are equal to the number of distinct lines on general hypersurfaces of degrees $3, 5, \ldots, 37$ and dimensions $2, 3, \ldots, 19$.*

We have seen that *every* smooth cubic surface has exactly 27 distinct lines. By contrast, the hypothesis of generality in the preceding corollary is really necessary for quintic threefolds: By Corollary 6.33, the Fano scheme of a smooth quintic threefold may be singular or positive-dimensional (we will see in Exercises 6.62 and 6.67 that both possibilities actually occur).

The 2875 lines on a quintic threefold have played a significant role in algebraic geometry, and even show up in physics. For example, the Lefschetz hyperplane theorem (see for example Milnor [1963]) implies that all 2875 are homologous to each other, but one can show that they are linearly independent in the group of cycles modulo algebraic equivalence (Ceresa and Collino [1983]). On the other hand, the number of rational curves of degree d on a general quintic threefold, of which the 2875 lines are the first example, is one of the first predictions of *mirror symmetry* (see for example Cox and Katz [1999]).

Proof of Theorem 6.34: We already know that for general X of degree $d > 2n - 3$ the Fano scheme $F_1(X)$ is empty, so we henceforward assume that $d \leq 2n - 3$. We have seen in Corollary 6.31 that there exists a pair (X, L) with $X \subset \mathbb{P}^n$ a smooth hypersurface of degree d and $L \subset X$ a line such that $\dim T_L F_1(X) = 2n - 3 - d$; that is, $F_1(X)$ is smooth of the expected dimension in a neighborhood of L. We now use an incidence correspondence to deduce that, for general X, the lines $L \in F_1(X)$ with this property form an open dense subset of $F_1(X)$. In particular, if $d = 2n - 3$ then X contains just a finite number of lines, every one of which is a reduced point of $F_1(X)$.

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Let $\mathbb{P}^N$ be the projective space of forms of degree d in $n+1$ variables, whose points we think of as hypersurfaces in $\mathbb{P}^n$. Consider the projection maps from the universal Fano scheme $\Phi := \Phi(n, d, 1)$:

$$\begin{tikzcd} & \Phi = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1, n) \mid L \subset X\} \arrow[dl, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathbb{P}^N & & \mathbb{G}(1, n) \end{tikzcd}$$

so that the fiber of φ over the point X of $\mathbb{P}^N$ is the Fano scheme $F_1(X)$ of X. As we have seen in Proposition 6.1, Φ is smooth and irreducible of dimension $N + 2n - 3 - d$. It follows that the fiber of φ through any point of Φ has dimension $\geq 2n - 3 - d$.

The set of points of $\mathbb{P}^N$ where the fiber dimension of φ is equal to $2n-3-d$ is open; within that, the set U of points where the fiber is smooth is also open. Corollary 6.31 shows that this open set is nonempty; given this, it follows that if X is a general hypersurface of degree d, then any component of $F_1(X)$ is generically reduced of dimension $N + 2n - 3 - d$. Since $F_1(X)$ is defined by the vanishing of a section of a bundle of rank $d + 1$, it is locally a complete intersection. Thus $F_1(X)$ cannot have embedded components, and the fact that it is generically reduced implies that it is reduced. □

6.6 The universal Fano scheme and the geometry of families of lines

In Keynote Question (c) we asked: What is the degree of the surface S in $\mathbb{P}^3$ swept out by the lines on a cubic surface as the cubic surface moves in a general pencil? What is the genus of the curve $C \subset \mathbb{G}(1, 3)$ consisting of the points corresponding to lines on the various elements of the pencil of cubic surfaces? We can answer such questions by giving a "global" view of the universal Fano scheme as the zero locus of a section of a vector bundle, just as we have done for Fano schemes of individual hypersurfaces.

We will compute the degree of S as the number of times S intersects a general line. The task of computing this number is made easier by the fact that a general point of the surface lies on only one of the lines in question (reason: a general point that lies on two lines would have to lie on lines from different surfaces in the pencil, and thus would lie in the base locus of the pencil, contradicting the assumption that it was a general point). Thus the degree of the surface is the same as the degree of the curve $C \subset \mathbb{G}(1, 3)$ in the Plücker embedding (see Section 4.2.3 for a more general statement).

Let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$. The incidence correspondence

$$\Phi = \Phi(n, d, 1) = \{(X, L) \in \mathbb{P}^N \times \mathbb{G}(1, n) \mid L \subset X\},$$

which we call the *universal* or *relative* Fano scheme of lines on such hypersurfaces, was introduced in Section 6.1. We can learn about its global geometry by realizing it as the zero locus of a section of a bundle, just as in the case of the Fano scheme of a given hypersurface.

We have seen that the maps of vector spaces

$$\{\text{polynomials of degree } d \text{ on } \mathbb{P}^n\} \to \{\text{polynomials of degree } d \text{ on } L\}$$

for different $L \in \mathbb{G}(1, n)$ fit together to form a bundle map

$$V \otimes \mathcal{O}_{\mathbb{G}(1,n)} \to \operatorname{Sym}^d \mathcal{S}^*$$

on the Grassmannian $\mathbb{G}(1, n)$, where $V = H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ is the vector space of all polynomials of degree d. Likewise, the inclusions

$$\langle f \rangle \hookrightarrow V$$

fit together to form a map of vector bundles on $\mathbb{P}V \cong \mathbb{P}^{19}$

$$\mathcal{T} = \mathcal{O}_{\mathbb{P}^{19}}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^{19}},$$

where $\mathcal{T}$ is the universal subbundle on $\mathbb{P}^{19}$.

We will put these two constructions together to understand not only $\Phi(n, d, 1)$, but also its restriction to a general linear space of forms $M \subset \mathbb{P}^N$. We denote the restriction of the universal Fano scheme to M by $\Phi(n, d, 1)|_M$.

Theorem 6.36. *The universal Fano scheme $\Phi(n, d, 1)|_M$ of lines on a general m-dimensional linear family $M = \mathbb{P}^m$ of hypersurfaces of degree d in $\mathbb{P}^n$ is reduced and of codimension $d + 1$ in the $(2n - 2 + m)$-dimensional space $\mathbb{P}^m \times \mathbb{G}(1, n)$. It is the zero locus of a section of the rank-$(d + 1)$ vector bundle $\mathcal{E} = \pi_2^* \operatorname{Sym}^d \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ on that space, so its class is $c_{d+1}(\mathcal{E})$.*

Proof: The fact that $\Phi(n, d, 1)|_M$ is reduced and of the expected dimension follows from Bertini's theorem and the corresponding statement for Φ (Proposition 6.1). To characterize $\Phi(n, d, 1)|_M$ as the zero locus of a section of a vector bundle, it likewise suffices to treat the case $M = \mathbb{P}(\operatorname{Sym}^d V^*)$, the space of all forms of degree d, so that $m = N := \dim V^* - 1$.

Consider the product of $\mathbb{P} \operatorname{Sym}^d V^*$ and the Grassmannian $\mathbb{G}(1, n)$, and its projections

$$\mathbb{P}(\operatorname{Sym}^d V^*) \xleftarrow{\pi_1} \mathbb{P}(\operatorname{Sym}^d V^*) \times \mathbb{G}(1, n) \xrightarrow{\pi_2} \mathbb{G}(1, n).$$

On the product, we have maps

$$\pi_1^* \mathcal{O}_{\mathbb{P}\,\mathrm{Sym}^d\, V^*}(-1) \longrightarrow \pi_1^* \mathrm{Sym}^d\, V^* \cong \pi_2^* \mathrm{Sym}^d\, V^* \longrightarrow \pi_2^* \mathrm{Sym}^d\, \mathcal{S}^*.$$

Restricted to the fiber over the point of $\mathbb{P}\,\mathrm{Sym}^d\, V^*$ corresponding to f, the composite map takes a generator of $\pi_1^* \mathcal{O}_{\mathbb{P}\,\mathrm{Sym}^d\, V^*}(-1)|_{\langle f \rangle}$ to σ_f. Thus the zero locus of the composite map is the incidence correspondence $\Phi(n, d, 1)$.

Let σ be the corresponding global section of the bundle

$$\begin{aligned}\mathcal{E} &:= \mathcal{H}om(\pi_1^* \mathcal{O}_{\mathbb{P}\,\mathrm{Sym}^d\, V^*}(-1),\ \pi_2^* \mathrm{Sym}^d\, \mathcal{S}^*)\\ &\cong \pi_2^* \mathrm{Sym}^d\, \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^N}(1).\end{aligned}$$

The zero locus of the composite map is the same as the zero locus of σ. Moreover, if we restrict to an open subset of the Grassmannian over which the universal subbundle $\mathcal{S}$ is trivial, then the vanishing of σ is given by the local equations we originally used to define the scheme structure on Φ. □

Theorem 6.36 allows us to calculate the class of Φ in the Chow ring of $\mathbb{P}^N \times \mathbb{G}(1, n)$, which immediately gives the answers to Keynote Question (c). To express this, we will use the symbol ζ for the pullback to $\mathbb{P}^{19} \times \mathbb{G}(1, 3)$ of the hyperplane class on the space $\mathbb{P}^{19}$ of cubic surfaces (and for the pullback to $\mathbb{P}^{34} \times \mathbb{G}(1, 3)$ of the hyperplane class on the space $\mathbb{P}^{34}$ of quartic surfaces), and the symbols $\sigma_{i,j}$ for the pullbacks to $\mathbb{P}^{19} \times \mathbb{G}(1, 3)$ and $\mathbb{P}^{34} \times \mathbb{G}(1, 3)$ of the corresponding classes in $A(\mathbb{G}(1, 3))$.

Corollary 6.37. *The class of the universal Fano scheme* $\Phi(3, 3, 1)$ *of lines on cubic surfaces in* $\mathbb{P}^3$ *is*

$$\begin{aligned}[\Phi(3, 3, 1)] &= c_4(\pi_2^* \mathrm{Sym}^3\, \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{19}}(1))\\ &= 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4,\end{aligned}$$

while the class of the universal Fano scheme $\Phi(3, 4, 1)$ *of lines on quartic surfaces in* $\mathbb{P}^3$ *is*

$$\begin{aligned}[\Phi(3, 4, 1)] &= c_5(\pi_2^* \mathrm{Sym}^4\, \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{34}}(1))\\ &= 320\sigma_{2,2}\zeta + 220\sigma_{2,1}\zeta^2 + (30\sigma_2 + 55\sigma_{1,1})\zeta^3 + 10\sigma_1\zeta^4 + \zeta^5.\end{aligned}$$

If C is the curve of lines on a general pencil of cubic surfaces, then the degree of C is 42 and the genus of C is 70. The number of quartic surfaces in a general pencil that contain a line is 320.

Restricting to a point in $\mathbb{P}^{19}$, we see again that a general cubic surface X will contain

$$[\Phi(3, 3, 1)] \cdot \zeta^{19} = 27$$

lines.

Proof of Corollary 6.37: The identifications of $[\Phi(3,3,1)]$ and $[\Phi(3,4,1)]$ with the given Chern classes is part of Theorem 6.36.

For the explicit computations of the Chern classes one can use the splitting principle or appeal to *Schubert2*. Here is the computation, via the splitting principle, for the case of $\Phi(3,3,1)$, the fourth Chern class of the bundle $\mathcal{E}$ on $\mathbb{P}^{19} \times \mathbb{G}(1,3)$:

Formally factoring the Chern class of $\pi_2^* \mathcal{S}^*$ as

$$c(\pi_2^* \mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1} = (1+\alpha)(1+\beta),$$

we can write

$$\begin{aligned} &c(\pi_1^* \mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \pi_2^* \operatorname{Sym}^3 \mathcal{S}^*) \\ &\qquad = (1+3\alpha+\zeta)(1+2\alpha+\beta+\zeta)(1+\alpha+2\beta+\zeta)(1+3\beta+\zeta), \end{aligned}$$

and in particular the top Chern class is given by

$$\begin{aligned} c_4(\pi_1^* \mathcal{O}_{\mathbb{P}^{19}}(1) \otimes \pi_2^* \operatorname{Sym}^3 \mathcal{S}^*) &= (3\alpha+\zeta)(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta)(3\beta+\zeta) \\ &\in A^4(\mathbb{P}^{19} \times \mathbb{G}(1,3)). \end{aligned}$$

Evaluating, we first have

$$(3\alpha+\zeta)(3\beta+\zeta) = 9\sigma_{1,1} + 3\sigma_1\zeta + \zeta^2,$$

and then

$$(2\alpha+\beta+\zeta)(\alpha+2\beta+\zeta) = 2\sigma_1^2 + \sigma_{1,1} + 3\sigma_1\zeta + \zeta^2.$$

Multiplying out, we have

$$[\Phi] = 27\sigma_{2,2} + 42\sigma_{2,1}\zeta + (11\sigma_2 + 21\sigma_{1,1})\zeta^2 + 6\sigma_1\zeta^3 + \zeta^4.$$

Here is the corresponding *Schubert2* code:

```
n=3
d=3
m=19

P = flagBundle({1,m}, VariableNames=>{z,q1})
(Z,Q1)=P.Bundles
V = abstractSheaf(P,Rank =>n+1)
G = flagBundle({2,n-1},V,VariableNames=>{s,q})
(S,Q) = G.Bundles
p = G.StructureMap
ZG = p^*(dual Z)
chern_4 (ZG**symmetricPower_d dual S)
```

Replacing the line "`d = 3`" with "`d = 4`," we get the corresponding result for $\Phi(3,4,1)$.

From the computation of $[\Phi(3,3,1)]$, we see that the number of lines on members of a general pencil of cubics meeting a given line is

$$[\Phi] \cdot \sigma_1 \cdot \zeta^{18} = 42,$$

from which we deduce that the degree of C, which is equal to the degree of the surface swept out by the lines on our pencil of cubics, is 42. For the genus $g(C)$ of C, we use part (c) of Proposition 6.15 to conclude that the normal bundle of C is the bundle $\mathcal{E}|_C$, where $\mathcal{E}$ is the restriction to $\mathbb{P}^1 \times \mathbb{G}(1,3)$ of the bundle $\pi_2^* \operatorname{Sym}^3 \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)$, whose section defines $\Phi(3,3,1)_{\mathbb{P}^1}$, as in Corollary 6.37. From the exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_{\mathbb{P}^1 \times \mathbb{G}(1,3)}|_C \longrightarrow \mathcal{N}_{C/\mathbb{P}^1 \times \mathbb{G}(1,3)} \longrightarrow 0,$$

we deduce that the degree of $\mathcal{T}_C$, which is $2 - 2g(C)$, is

$$\begin{aligned} \deg \mathcal{T}_C &= \deg c_1(\mathcal{T}_C) = \deg([C] c_1(\mathcal{T}_{\mathbb{P}^1 \times \mathbb{G}(1,3)})) - \deg c_1(\mathcal{N}_{C/\mathbb{P}^1 \times \mathbb{G}(1,3)}) \\ &= c_4(\mathcal{E})(4\sigma_1 + 2\zeta) - c_4(\mathcal{E}) c_1(\mathcal{E}), \end{aligned}$$

where we have used the computation $c_1(\mathcal{T}_{\mathbb{G}(1,3)}) = 4\sigma_1$ from Proposition 5.18. We can compute $c_1(\mathcal{E})$ by the splitting principle or by calling

```
chern_1 (ZG**symmetricPower_d dual S)
```

and we get $c_1(\mathcal{E}) = 6\sigma_1 + 4\zeta$. Using the fact that ζ^2 restricts to zero on the preimage of a line in $\mathbb{P}^{19}$, this gives

$$\begin{aligned} 2 - 2g(C) &= \deg(27\sigma_{2,2} + 42\sigma_{2,1}\zeta)(4\sigma_1 + 2\zeta - 6\sigma_1 - 4\zeta) \\ &= \deg(-138\sigma_{2,2}\zeta) = -138, \end{aligned}$$

whence $g = 70$. Another view of this computation is suggested in Exercise 6.54.

Finally, consider a general pencil of quartic surfaces. By Exercise 6.64, no element of the pencil will contain more than one line. It is likewise true that no line will lie on more than one element of the pencil. (If a line lay on more than one element of the pencil, it would be a component of the base locus — but, since the pencil is general, the base locus is smooth and connected.) Thus, the number of quartic surfaces that contain a line in a general pencil of quartic surfaces is the number of lines that lie on some quartic surface in the pencil, that is, the degree of $\Phi(3,4,1) \cap \mathbb{P}^1$. Writing σ again for the section of $\pi_2^* \operatorname{Sym}^4 \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ defined above, this is

$$\deg\big(\zeta^{33} c_5(\pi_2^* \operatorname{Sym}^4 \mathcal{S}^* \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1))\big).$$

By the computation of $[\Phi(3,4,1)]$, this is 320. □

The coefficients of higher powers of ζ in the class of $\Phi(3,3,1)$ computed above have to do with the geometry of larger linear systems of cubics: For example, we will see how to answer questions about lines on a net of cubics in Exercise 6.50.

6.6.1 Lines on the quartic surfaces in a pencil

Here is a slightly different approach to Keynote Question (b). Given that the set of quartic surfaces that contain some line is a hypersurface Γ in the projective space $\mathbb{P}^{34}$ of quartic surfaces, we are asking for the degree of Γ.

To find that number, we look again at the bundle $\mathcal{E}$ on the Grassmannian $\mathbb{G}(1, 3)$, whose fiber over a point $L \in \mathbb{G}(1, 3)$ is the vector space

$$\mathcal{E}_L = H^0(\mathcal{O}_L(4)),$$

that is, the fourth symmetric power $\mathrm{Sym}^4 \mathcal{S}^*$ of the dual of the universal subbundle on $\mathbb{G}(1, 3)$. As before, the polynomials f and g generating the pencil define sections σ_f and σ_g of the bundle $\mathcal{E}$. The locus of lines $L \subset \mathbb{P}^3$ that lie on some element of the pencil is the locus where the values of the sections σ_f and σ_g are dependent, so the degree of this locus is the degree of the fourth Chern class $c_4(\mathcal{E}) \in A^4(\mathbb{G}(1, 3)) \cong \mathbb{Z}$. As before, this can be computed either with the splitting principle or with *Schubert2*, and one finds again the number 320.

We will see another way of calculating the genus of the curve Φ in the following chapter (after we have determined the number of singular cubic surfaces in a general pencil), by expressing Φ as a 27-sheeted cover of $\mathbb{P}^1$ and using Hurwitz's theorem.

6.7 Lines on a cubic with a double point

Identifying the Fano scheme $F_1(X)$ as the Hilbert scheme of lines on X has allowed us to give a necessary and sufficient condition for its smoothness, and to show that it is indeed smooth in certain cases. But there are aspects of its geometry that we cannot get at in this way, such as the multiplicity of $F_1(X)$ at a point L where it is not smooth. We might want to know, for example, if we can find a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $2n - 3$ whose Fano scheme of lines includes a point of multiplicity exactly 2, as in Harris [1979]; or, we might ask, if X has an ordinary double point, how does this affect the number of lines it will contain? To answer such questions we must go back to the local equations of $F_1(X)$ introduced (in more generality) at the beginning of this chapter.

We will describe the lines on a cubic surface with one ordinary double point. Other examples can be found in Exercise 6.55, where we will consider the case of cubic surfaces with more than one double point, and in Exercise 6.62, where we will show that it is possible to find a smooth quintic hypersurface $X \subset \mathbb{P}^4$ whose Fano scheme contains an isolated double point.

To this end, we will adapt the notation of Section 6.1.1 to the case of cubic surfaces. We work in an open neighborhood $U \subset \mathbb{G}(1, 3)$ of the line

$$L : x_2 = x_3 = 0,$$

where U consists of the lines not meeting the line $x_0 = x_1 = 0$. Any line in U can be written uniquely as the row space of a matrix of the form

$$A = \begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}$$

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(so $U \cong \mathbb{A}^4$, with coordinates a_2, a_3, b_2, b_3). Such a line has the parametrization

$$\mathbb{P}^1 \ni (s_0, s_1) \to (s_0, s_1)A = (s_0, s_1, a_2 s_0 + b_2 s_1, a_3 s_0 + b_3 s_1) \in \mathbb{P}^3.$$

Now let $X \subset \mathbb{P}^3$ be a cubic surface containing L, and suppose that the point $p = (1, 0, 0, 0) \in L$ is an ordinary double point of X; that is, the tangent cone to X at p is the cone over a smooth conic curve. We assume that X has no other singularities along L.

We may also suppose that the tangent cone to X at p is given by the equation $x_1 x_3 + x_2^2 = 0$. With these choices, the defining equation $g(x)$ of X can be written in the form

$$g(x) = x_0 x_1 x_3 + x_0 x_2^2 + \alpha x_1^2 x_2 + \beta x_1^2 x_3 + \gamma x_1 x_2^2 + \delta x_1 x_2 x_3 + \epsilon x_1 x_3^2 + k,$$

where $k \in (x_2, x_3)^3$. The condition that X be smooth along L except at p says that $\alpha \neq 0$; otherwise the coefficients $\alpha, \ldots, \epsilon$ are arbitrary.

As we saw in Section 6.4.3, the normal bundle $\mathcal{N}_{L/X}$ can be computed from the short exact sequence

$$0 \longrightarrow \mathcal{N}_{L/X} \longrightarrow \mathcal{O}_L^2(1) \xrightarrow{(g_2\ g_3)} \mathcal{O}_L(3),$$

where the g_3, g_3 are the coefficients of x_2, x_3 in the part of g that is not contained in (x_2, x_3); that is, $g_2 = \alpha x_1^2$ and $g_3 = x_0 x_1 + \beta x_1^2$.

Since the polynomial ring in s, t has unique factorization, the syzygies between these two forms are generated by the linear syzygy $\alpha x_1 g_3 - (x_0 + \beta x_1) g_2 = 0$, so $\mathcal{N}_{L/X} \cong \mathcal{O}_L$, and the tangent space to the Fano scheme is given by $T_{[L]} F_1(X) = H^0(\mathcal{N}_{L/X})$, which is one-dimensional. In particular, the Fano scheme is *not* "smooth of the expected dimension" at $[L]$.

We can now write down the local equations of $F_1(X)$ near L: If we substitute the four coordinates from the parametrization of a line in U into g, we get

$$g(s, t, a_2 s + b_2 t, a_3 s + b_3 t) = c_0 s^3 + c_1 s^2 t + c_2 s t^2 + c_3 t^3,$$

where the c_i are the polynomials in the $a_{i,j}$ that define the intersection of the Fano scheme with U. Writing this out, we find that, modulo terms of higher degree, the c_i are

$$\begin{aligned}
c_0 &= a_2^2, \\
c_1 &= a_3 + 2a_2 b_2 + \gamma a_2^2 + \delta a_2 a_3 + \epsilon a_3^2, \\
c_2 &= b_3 + \alpha a_2 + \beta a_3 + b_2^2 + 2\gamma a_2 b_2 + \delta(a_2 b_3 + a_3 b_2) + 2\epsilon a_3 b_3, \\
c_3 &= \alpha b_2 + \beta b_3 + \gamma b_2^2 + \delta b_2 b_3 + \epsilon b_3^2.
\end{aligned}$$

Examining these polynomials, we see that c_1, c_2 and c_3 have independent differentials at the origin $a_2 = a_3 = b_2 = b_3 = 0$; thus, in a neighborhood of the origin the zero locus of these three is a smooth curve. Moreover, the tangent line to this curve is not contained in the plane $a_2 = 0$, so $c_0 = a_2^2$ vanishes to order exactly 2 on this curve.

Thus the component of $F_1(X)$ supported at L is zero-dimensional, and is isomorphic to $\operatorname{Spec} \Bbbk[\epsilon]/(\epsilon^2)$. In particular, it has multiplicity 2.

Having come this far, we can answer the question: If $X \subset \mathbb{P}^3$ is a cubic surface with one ordinary double point p, and X is otherwise smooth, how many lines will X contain? We have seen that the lines $L \subset X$ passing through p count with multiplicity 2, and those not passing through p with multiplicity 1. Since we know that the total count, with multiplicity, is 27, the only question is: How many distinct lines on X pass through p?

To answer this, take $p = (1, 0, 0, 0)$ as above and expand the defining equation $g(x)$ of X around p. Since p is a double point of X, we can write

$$g(x_0, x_1, x_2, x_3) = x_0 A(x_1, x_2, x_3) + B(x_1, x_2, x_3),$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. The lines on X through p then correspond to the common zeros of A and B. Moreover, if we write a line L through p as the span $L = \overline{p, q}$ with $q = (0, x_1, x_2, x_3)$, then, by Exercise 6.61, the condition that X be smooth along $L \setminus \{p\}$ is exactly the condition that the zero loci of A and B intersect transversely at (x_1, x_2, x_3). Thus there will be exactly six lines on X through p. Summarizing:

Proposition 6.38. *Let $X \subset \mathbb{P}^3$ be a cubic surface with an ordinary double point p. If X is smooth away from p, it contains exactly 21 lines: 6 through p and 15 not passing through p.*

(Compare this with the discussion starting on page 640 of Griffiths and Harris [1994].)

In Exercises 6.55–6.58, we will take up the case of cubics with more than one singularity, arriving ultimately at the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points*.

We have used the local equations of the Fano scheme only to describe the locus of lines on a single hypersurface. A similar approach gives some information about the lines on a linear system of hypersurfaces. As a sample application, we will see in Exercises 6.65 and 6.66 how to describe the singular locus of and tangent spaces to the locus $\Sigma \subset \mathbb{P}^{34}$ of quartic surfaces in $\mathbb{P}^3$ containing a line.

6.8 The Debarre–de Jong Conjecture

By Theorem 6.34, general hypersurfaces $X \subset \mathbb{P}^n$ of degree d all have Fano schemes $F_1(X)$ of the "expected" dimension $\varphi = 2n - d - 3$. On the other hand, it is easy to find smooth hypersurfaces of degree > 3 whose Fano schemes have dimension $> \varphi$; any smooth surface of degree > 3 in $\mathbb{P}^3$ that contains a line is such an example.

However, Corollary 6.33 shows that *every* smooth hypersurface of degree ≤ 3 has Fano scheme of dimension φ (that is, the open set of hypersurfaces for which $F_1(X)$ has the expected dimension contains the open set of smooth hypersurfaces when the degree is ≤ 3). Further, it was shown by Harris et al. [1998] that when $d \ll n$ every smooth hypersurface of degree d in $\mathbb{P}^n$ has a Fano scheme of lines of the correct dimension — in fact, all the $F_k(X)$ have the expected dimension when both d and k are much smaller than n. But the lower bound on n given there is very large, and examples are few. In general, we have no idea what to conjecture for the true bound required!

There is a conjecture, however, for the Fano schemes of lines. To motivate it, note that a general hypersurface $X \subset \mathbb{P}^{2m+1}$ containing an m-plane will be smooth (Exercise 6.68) and will contain a copy of the Grassmannian of lines in the m-plane, a variety of dimension $2m-2$. When $d > 2m+1$, this is larger than the expected dimension $\varphi = 2d - 3$ of $F_1(X)$. Another family of such examples is given in Exercise 6.67, but, just as in the examples above, that construction requires $d > n$.

Conjecture 6.39 (Debarre–de Jong)**.** *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d with $d \leq n$, then the Fano scheme $F_1(X)$ of lines on X has dimension $2n - 3 - d$.*

One striking aspect of the Debarre–de Jong conjecture is that the inequality $d \leq n$ for a smooth hypersurface $X \subset \mathbb{P}^n$ is exactly equivalent to the condition that the anti-canonical bundle ω_X^* is ample, though it is not clear what role this might play in a proof.

Conjecture 6.39 has been proven for $d \leq 5$ by de Jong and Debarre, and for $d \leq 8$ by Beheshti (see for example Beheshti [2006]). One might worry that proving the conjecture, even for small d, would involve high-dimensional geometry, but as we will now show, it would be enough to prove the conjecture for $n = d$.

Proposition 6.40. *If* $\dim F_1(X) = d - 3$ *for every smooth hypersurface of degree d in $\mathbb{P}^d$, and $d \leq n$, then* $\dim F_1(X) = 2n - d - 3$ *for every smooth hypersurface of degree d in $\mathbb{P}^n$.*

Proof: We have already treated the case of quadrics (Proposition 4.15), so we may assume that $3 \leq d \leq n$. Suppose that $X \subset \mathbb{P}^n$ is a smooth hypersurface and $L \subset X$ is a line. Let Λ be a general d-plane in $\mathbb{P}^n$ containing L, and let $Y = \Lambda \cap X$. By Lemma 6.41 below, Y is a smooth hypersurface of degree d in $\Lambda = \mathbb{P}^d$.

The Fano scheme $F_1(Y)$ is the intersection of $F_1(X)$ with the Schubert cycle $\Sigma_{m-n,m-n}(\Lambda) \subset \mathbb{G}(1,m)$; by the generalized principal ideal theorem,

$$d - 3 = \dim_L F_1(Y) \geq \dim_L F_1(X) - 2(n - d),$$

whence $\dim_L F_1(X) \leq 2n - d - 3$, as required. $\square$

We have used a special case of the following extension of Bertini's theorem:

Lemma 6.41. *Let $k < n < m$; let $X \subset \mathbb{P}^m$ be a smooth hypersurface and $L \cong \mathbb{P}^k \subset X$ a k-plane contained in X. If $\Lambda \cong \mathbb{P}^n \subset \mathbb{P}^m$ is a general n-plane containing L, then the intersection $Y = X \cap \Lambda$ is smooth if and only if $n - 1 \geq 2k$.*

Proof: If $n - 1 < 2k$ then Y must be singular, by Corollary 6.26.

For the converse, we may assume by an obvious induction that $n = m - 1$. Bertini's theorem implies that Y is smooth away from L. On the other hand, the locus of tangent hyperplanes $\mathbb{T}_p X$ to X at points $p \in L$ is a subvariety of dimension at most k in the dual projective space $\mathbb{P}^{m*}$, while the locus of hyperplanes containing L will be the $(m - k - 1)$-plane $L^\perp \subset \mathbb{P}^{m*}$. Thus, if $n - 1 = m - 2 \geq 2k$, so that $k < m - k - 1$, then not every hyperplane containing L is tangent to X at a point of L. It follows that, for general Λ, the intersection $Y = \Lambda \cap X$ is smooth. □

We can now prove Conjecture 6.39 for $d = 4$, and thereby give a negative answer for Keynote Question (d):

Theorem 6.42. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree 4, then the Fano scheme $F_1(X)$ has dimension $2n - 7$.*

Proof: Proposition 6.40 shows that it is enough to consider the case $n = 4$. Suppose $F \subset F_1(X)$ is an irreducible component with $\dim F \geq 2$, and let $L \in F$ be a general point. By Proposition 6.30, the normal bundle $\mathcal{N} = \mathcal{N}_{L/X}$ must be either $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$. Either way, all global sections of N take values in a line bundle contained in N. It follows that, for any point $p \in L$, the map $H^0(\mathcal{N}_{L/X}) \to (\mathcal{N}_{L/X})_p = T_p X / T_p L$ has rank at most 1. (Since $\dim H^0(N) \geq \dim T_L F \geq 2$, the normal bundle must in fact be $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-2)$, but we do not need this.)

Let $Y \subset X$ be the subvariety swept out by the lines of $F \subset F_1(X)$. By Proposition 6.22, Y can have dimension at most 2. But by hypothesis, Y contains a two-dimensional family of lines. From Proposition 6.3 we conclude that Y is a 2-plane. Corollary 6.26 tells us this is impossible, and we are done. □

6.8.1 Further open problems

The Debarre–de Jong conjecture deals with the dimension of the family of lines on a hypersurface $X \subset \mathbb{P}^n$, but we can also ask further questions about the geometry of $F_1(X)$: for example, whether it is irreducible and/or reduced. Exercises 6.70–6.73, in which we show that the Fano scheme $F_1(X)$ of lines on the Fermat quartic hypersurface $X \subset \mathbb{P}^4$ is neither, shows that the Debarre–de Jong statement cannot be strengthened for all $d \leq n$. But — based on our knowledge of examples — it does seem to be the case that the smaller d is relative to n, the better behaved $F_1(X)$ is for an arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ of degree d. For example, the following questions are open:

(a) Is $F_1(X)$ is reduced and irreducible if $d \leq n - 1$ and $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d?

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(b) Can we bound the dimension of the singular locus of $F_1(X)$ in terms of d? (The arguments above show that for $d = 3$ the Fano scheme $F_1(X)$ is smooth, while for $d \geq n$ it may not be reduced. What about the range $4 \leq d \leq n - 1$?)

The analogous questions for $F_k(X)$ with $k > 1$ are completely open. We can ask, for example: Given d and k, what is the largest n such that there exists a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d with $\dim F_k(X) > (k+1)(n-k) - \binom{k+d}{d}$? Again, Harris et al. [1998] says that such n are bounded, but the bound given there is probably far too large.

Finally, we can ask: Why the Fano schemes instead of other Hilbert schemes? Why not look, for example, at rational curves of any degree e on a hypersurface? Here the field is wide open. Specifically, we have an "expected" dimension: Since a rational curve $C \subset \mathbb{P}^n$ is given parametrically as the image of a map $f : \mathbb{P}^1 \to \mathbb{P}^n$, which is specified by $n+1$ homogeneous polynomials of degree e on $\mathbb{P}^1$, and two such $(n+1)$-tuples have the same image if and only if they differ by a scalar or by an automorphism of $\mathbb{P}^1$, the space $\mathcal{H}$ of such curves has dimension $(n+1)(e+1)-4$. On the other hand, the condition for $X = V(F)$ to contain such a curve $C \cong \mathbb{P}^1$ is that $f^*F = 0 \in H^0(\mathcal{O}_{\mathbb{P}^1}(de))$, which may be regarded as $ed + 1$ conditions on X. If we expect these conditions to be independent then we would expect the fibers of the incidence correspondence

$$\Psi = \{(X, C) \in \mathbb{P}^N \times \mathcal{H} \mid C \subset X\}$$

over $\mathcal{H}$ to have dimension $N - (de + 1)$, and Ψ correspondingly to have dimension

$$(n+1)(e+1) - 4 + N - (de+1) = N + (n-d)e + n + e - 4.$$

This leads us to:

Conjecture 6.43. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree d, then X contains a rational curve of degree e if and only if*

$$\lambda(n, d, e) := (n-d)e + n + e - 4 \geq 0;$$

when this inequality is satisfied the family of such curves on X has dimension $\lambda(n, d, e)$.

We proved the conjecture in this chapter for $e = 1$, but the general case is difficult (the case $n = 4$, $d = 5$ alone is the *Clemens conjecture*, which has been the object of much study in its own right). Recently, however, there has been substantial progress: see Beheshti and Mohan Kumar [2013] and Riedl and Yang [2014].

Note that the analog of the Debarre–de Jong conjecture in this setting — that the dimension estimate of Conjecture 6.43 holds for an arbitrary smooth $X \subset \mathbb{P}^n$ of degree $d \leq n$ — is false; one counterexample is given in Exercise 6.74. But it might hold when d satisfies a stronger inequality with respect to n, perhaps for $d \leq n/e$.

6.9 Exercises

Exercise 6.44. Show that the expected number of lines on a hypersurface of degree $2n-3$ in $\mathbb{P}^n$ (that is, the degree of $c_{2n-2}(\mathrm{Sym}^{2n-3}\mathcal{S}^*) \in A(\mathbb{G}(1,n))$) is always positive, and deduce that *every hypersurface of degree* $2n-3$ *in* $\mathbb{P}^n$ *must contain a line*. (This is just a special case of Corollary 6.32; the idea here is to do it without a tangent space calculation.)

Exercise 6.45. Let $X \subset \mathbb{P}^4$ be a general quartic threefold. By Theorem 6.42, X will contain a one-parameter family of lines. Find the class in $A(\mathbb{G}(1,4))$ of the Fano scheme $F_1(X)$, and the degree of the surface $Y \subset \mathbb{P}^4$ swept out by these lines.

Exercise 6.46. Find the class of the scheme $F_2(Q) \subset \mathbb{G}(2,5)$ of 2-planes on a quadric $Q \subset \mathbb{P}^5$. (Do the problem first, then compare your answer to the result in Proposition 4.15.)

Exercise 6.47. Find the expected number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$, that is, the degree of $c_{15}(\mathrm{Sym}^4\mathcal{S}^*) \in A(\mathbb{G}(2,7))$.

Exercise 6.48. We can also use the calculation carried out in this chapter to count lines on complete intersections $X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$, simply by finding the classes of the schemes $F_1(Z_i)$ of lines on the hypersurfaces Z_i and multiplying them in $A(\mathbb{G}(1,n))$. Do this to find the number of lines on the intersection $X = Y_1 \cap Y_2 \subset \mathbb{P}^5$ of two general cubic hypersurfaces in $\mathbb{P}^5$.

Exercise 6.49. Find the Chern class $c_3(\mathrm{Sym}^3\mathcal{S}^*) \in A^3(\mathbb{G}(1,3))$ as a multiple of the class $\sigma_{2,1}$. Why is this coefficient equal to the degree of the curve of lines on the cubic surfaces in a pencil? Note that this computation does not use the universal Fano scheme Φ.

Exercise 6.50. Let $\{X_t \subset \mathbb{P}^3\}_{t\in\mathbb{P}^2}$ be a general net of cubic surfaces in $\mathbb{P}^3$.

(a) Let $p \in \mathbb{P}^3$ be a general point. How many lines containing p lie on some member X_t of the net?
(b) Let $H \subset \mathbb{P}^3$ be a general plane. How many lines contained in H lie on some member X_t of the net?

Compare your answer to the second half of this question to the calculation in Chapter 2 of the degree of the locus of reducible plane cubics!

Exercise 6.51. Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 3$. Show that if $F_1(X)$ is positive-dimensional, then either X is a cone or X has a positive-dimensional singular locus.

Exercise 6.52. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold and

$$\{S_t = X \cap H_t\}_{t\in\mathbb{P}^1}$$

a general pencil of hyperplane sections of X. What is the degree of the surface swept out by the lines on the surfaces S_t, and what is the genus of the curve parametrizing them?

Exercise 6.53. Prove Theorem 6.13 using the methods of Section 6.7, that is, by writing the local equations of $F_k(X) \subset \mathbb{G}(k,n)$

Exercise 6.54. Let $\{S_t\}_{t\in\mathbb{P}^1}$ be a general pencil of cubic surfaces, and let Φ be the incidence correspondence

$$\Phi = \{(t,L) \in \mathbb{P}^1 \times \mathbb{G}(1,3) \mid L \subset S_t\}.$$

Using Propositions 6.38 and 7.4, show that the projection $\Phi \to \mathbb{P}^1$ has degree 27 and has six branch points over each of the 32 values of t for which S_t is singular, and deduce again the conclusion of Corollary 6.37 that the genus of Φ is 70.

Exercise 6.55. Extending the results of Section 6.7, suppose that X is a general cubic surface having two ordinary double points $p, q \in X$. Describe the scheme structure of $F_1(X)$ at the point corresponding to the line $L = \overline{p,q}$, and in particular determine the multiplicity of $F_1(X)$ at L.

Exercise 6.56. Let $X \subset \mathbb{P}^3$ be a cubic surface and $p, q \in X$ isolated singular points of X; let $L = \overline{p,q}$. Show that L is an isolated point of $F_1(X)$ and that the multiplicity $\operatorname{mult}_L F_1(X)$ is ≥ 4.

Exercise 6.57. Let $X \subset \mathbb{P}^3$ be a cubic surface and $p_1, \dots, p_\delta$ isolated singular points of X. Show that no three of the points p_i are collinear.

Exercise 6.58. Use the result of the preceding two exercises to deduce the statement that *a cubic surface $X \subset \mathbb{P}^3$ can have at most four isolated singular points*.

Exercise 6.59. Using the methods of Section 6.7, show that there exists a pair (X, Λ) with $X \subset \mathbb{P}^7$ a quartic hypersurface and $\Lambda \subset X$ a 2-plane such that Λ is an isolated, reduced point of $F_2(X)$.

Exercise 6.60. Using the result of Exercise 6.59, show that the number of 2-planes on a general quartic hypersurface $X \subset \mathbb{P}^7$ is the number calculated in Exercise 6.47 (that is, the Fano scheme $F_2(X)$ is reduced for general X).

Exercise 6.61. To complete the proof of Proposition 6.38, let $X \subset \mathbb{P}^3$ be a cubic surface with one ordinary double point $p = (1,0,0,0)$, given as the zero locus of the cubic

$$F(Z_0, Z_1, Z_2, Z_3) = Z_0 A(Z_1, Z_2, Z_3) + B(Z_1, Z_2, Z_3),$$

where A is homogeneous of degree 2 and B homogeneous of degree 3. If we write a line $L \subset X$ through p as the span $L = \overline{p,q}$, with $q = (0, Z_1, Z_2, Z_3)$, show that X is smooth along $L \setminus \{p\}$ if and only if the zero loci of A and B intersect transversely at (Z_1, Z_2, Z_3).

Exercise 6.62. Show that there exists a smooth quintic threefold $X \subset \mathbb{P}^4$ whose scheme $F_1(X)$ of lines contains an isolated point of multiplicity 2.

Exercise 6.63. Let Φ be the incidence correspondence of triples consisting of a hypersurface $X \subset \mathbb{P}^n$ of degree $d = 2n - 3$, a line $L \subset X$ and a singular point p of X lying on L; that is,

$$\Phi = \{(X, L, p) \in \mathbb{P}^N \times \mathbb{G}(1, n) \times \mathbb{P}^n \mid p \in L \subset X \text{ and } p \in X_{\text{sing}}\}.$$

Show that Φ is irreducible.

Exercise 6.64. Let $\mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$.

(a) Show that the closure of the locus of quartics containing a pair of skew lines has dimension 32.
(b) Show that the closure of the locus of quartics containing a pair of incident lines also has dimension 32.
(c) Deduce that if $\{X_t = V(t_0 F + t_1 G)\}$ is a general pencil of quartics, then no member X_t of the pencil will contain more than one line.

Exercise 6.65. Suppose that F and G are two quartic polynomials on $\mathbb{P}^3$, and that $\{X_t = V(t_0 F + t_1 G)\}$ is the pencil of quartics they generate; let σ_F and σ_G be the sections of the bundle $\operatorname{Sym}^4 \mathcal{S}^*$ on $\mathbb{G}(1, 3)$ corresponding to F and G. Let X_t be a member of the pencil containing a line $L \subset \mathbb{P}^3$.

(a) Find the condition on F and G for L to be a reduced point of $V(\sigma_F \wedge \sigma_G) \subset \mathbb{G}(1, 3)$.
(b) Show that this is equivalent to the condition that the point $(t, L) \in \mathbb{P}^1 \times \mathbb{G}(1, 3)$ is a simple zero of the map $\pi_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \to \pi_2^* \operatorname{Sym}^4 \mathcal{S}^*$ introduced in the proof of Theorem 6.36.

Exercise 6.66. Let $\Sigma \subset \mathbb{P}^{34}$ be the space of quartic surfaces in $\mathbb{P}^3$ containing a line. Interpret the condition of the preceding problem in terms of the geometry of the pencil $\mathcal{D}$ around the line L, and use this to answer two questions:

(a) What is the singular locus of Σ?
(b) What is the tangent hyperplane $\mathbb{T}_X \Sigma$ at a smooth point corresponding to a smooth quartic surface X containing a single line?

The following two exercises give constructions of smooth hypersurfaces containing families of lines of more than the expected dimension.

Exercise 6.67. Let $Z \subset \mathbb{P}^{n-2}$ be any smooth hypersurface. Show that the cone $\overline{p, Z} \subset \mathbb{P}^{n-1}$ over Z in $\mathbb{P}^{n-1}$ is the hyperplane section of a smooth hypersurface $X \subset \mathbb{P}^n$, and hence that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Exercise 6.68. Take $n = 2m + 1$ odd, and let $\Lambda \subset \mathbb{P}^n$ be an m-plane. Show that there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ of any given degree d containing Λ, and deduce once more that for $d > n$ there exist smooth hypersurfaces $X \subset \mathbb{P}^n$ whose Fano scheme $F_1(X)$ of lines has dimension strictly greater than $2n - 3 - d$.

Note that the construction of Exercise 6.68 cannot be modified to provide counterexamples to the Debarre–de Jong conjecture, since by Corollary 6.26 there do not exist smooth hypersurfaces $X \subset \mathbb{P}^n$ containing linear spaces of dimension strictly greater than $(n-1)/2$. The following exercise shows that the construction of Exercise 6.67 is similarly extremal. It requires the use of the second fundamental form (see Section 7.4.3).

Exercise 6.69. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 2$. Show that X can have at most finitely many hyperplane sections that are cones.

To see some of the kinds of odd behavior the variety of lines on a smooth hypersurface can exhibit, short of having the wrong dimension, the following series of exercises will look at the Fermat quartic $X \subset \mathbb{P}^4$, that is, the zero locus

$$X = V(Z_0^4 + Z_1^4 + Z_2^4 + Z_3^4 + Z_4^4).$$

The conclusion is that $F_1(X)$ has 40 irreducible components, each of which is everywhere nonreduced! We start with a useful more general fact:

Exercise 6.70. Let $S = \overline{p, C} \subset \mathbb{P}^3$ be the cone with vertex p over a plane curve C of degree $d \geq 2$, and $L \subset S$ any line. Show that the tangent space $T_L F_1(S)$ has dimension at least 2, and hence that $F_1(S)$ is everywhere nonreduced.

Exercise 6.71. Show that X has 40 conical hyperplane sections Y_i, each a cone over a quartic Fermat curve in $\mathbb{P}^2$.

Exercise 6.72. Show that the reduced locus $F_1(Y_i)_{\text{red}}$ has class $4\sigma_{3,2}$.

Exercise 6.73. Using your answer to Exercise 6.45, conclude that

$$F_1(X) = \bigcup_{i=1}^{40} F_1(Y_i);$$

in other words, $F_1(X)$ is the union of 40 double curves.

Exercise 6.74. Show that:

(a) There exist smooth quintic hypersurfaces $X \subset \mathbb{P}^5$ containing a 2-plane $\mathbb{P}^2 \subset \mathbb{P}^5$.
(b) For such a hypersurface X, the family of conic curves on X has dimension strictly greater than the number $\lambda(5, 5, 2)$ of Conjecture 6.43.