Graduate Research Plan

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Throughout this document, let R be a Noetherian commutative ring of positive prime characteristic p. Such a ring comes equipped with the Frobenius endomorphism, a ring morphism $F : R \to R$ defined $F(r) = r^p$. This delightful additional symmetry provides several powerful ways to classify singularities, the bad points of geometric curves, and when a singularity admits a description via F, we creatively refer to it as an F-singularity. The project I propose as part of my graduate research broadly concerns these F-singularities. I specifically wish to investigate the following question: if R is a strongly F-regular ring of prime characteristic $p > 0$, is the symbolic Rees algebra Noetherian at divisorial ideals? An affirmative answer to this conjecture, which I will now refer to as the ”Rees is Noether conjecture,” would have enormous ramifications in prime characteristic commutative algebra and algebraic geometry. Among other things, it would imply the ”weak implies strong” conjecture, one of the longest-standing problems in this area of study.

Background Tight closure theory, first defined by Hochster and Huneke [1], is one way in which we take advantage of the Frobenius endomorphism. For an ideal $I \subseteq R$, we define $I^{[p^e]} := F^e(I)R$, that is, we let $I^{[p^e]}$ denote the extension of I under the eth iterate of the Frobenius map. The tight closure of an ideal I is the ideal I^* which consists of elements $x \in R$ such that there is some element $c \in R$ which avoids all minimal primes and satisfies $cx^{p^e} \in I^{[p^e]}$ for all sufficiently large natural numbers e.

There exist some rings in which all ideals are equal to their tight closure, i.e. rings for which tight closure operation is trivial. We call such rings weakly F-regular rings. It is unknown whether localization commutes with tight closure in this restricted case, and thus rings for which each localization is again weakly F-regular are called F-regular rings. Finally, if we denote by $F_*^e R$ the R-module obtained by restricting the action of R on itself via the Frobenius endomorphism, we may define the notion of strong F-regularity. We say that a ring R is strongly F-regular if for each nonzero $r \in R$ there exists a natural number e and an R-linear map $\varphi : F_*^e R \to R$ such that $\varphi(F_*^e r) = 1$. As the nomenclature suggests, we know strong $\implies$ $F-$regular $\implies$ weak, so the ”weak implies strong” conjecture would tell us that these three separate notions of F-regularity are actually all equivalent.

Paired with the work of the minimal model program, the ”Rees is Noether conjecture” would solve this long standing problem. The Rees ring is a familiar construction in commutative algebra, and we form the *symbolic* Rees algebra in much the same way. Recall that the symbolic power of a prime ideal $\mathfrak{p}$ is defined as follows: $\mathfrak{p} = R \cap \mathfrak{p}^{(n)} R_{\mathfrak{p}}$. We may extend this definition to an arbitrary ideal I by instead taking $\mathfrak{p}^n R_{\mathfrak{p}}$ to be the intersection of the ideals $I^n R_{\mathfrak{p}}$ with $\mathfrak{p} \in \mathrm{Ass}(I)$. If I is a a *divisorial ideal*, that is, an ideal whose associated primes are all height 1, then we define the symbolic Rees algebra $\mathcal{R}(I)$ as follows:

$$\mathcal{R}(I) = R \oplus I \cdot t \oplus I^{(2)} \cdot t^2 \oplus \dots = \bigoplus_{i \geq 0} I^{(i)} \cdot t^i \subseteq R[t].$$

Given that this is a ring, an algebraist would immediately ask: is $\mathcal{R}(I)$ Noetherian? In general, the answer is no. However, if we additionally require that R is strongly F-regular, it is conjectured that $\mathcal{R}(I)$ is always Noetherian. This is exactly the ”Rees is Noether conjecture”. Several cases are already known; for instance, it is true for local 3-dimensional strongly F-regular rings essentially of finite type over fields of characteristic $p > 5$ [2]. Aberbach and Polstra used a weaker form of this fact to prove that a ring of the above form is F-regular exactly when it is strongly F-regular. By adapting Aberbach and Polstra's proof, the solution to the ”Rees is Noether conjecture” would prove the general case of the ”weak implies strong” conjecture.

Research Plan

It is no secret that the ”Rees is Noether conjecture” is exceptionally difficult, especially since the only techniques yet employed to approach it utilize the highly technical results of the minimal model program (MMP). Therefore, my plan has two main points:

Study special cases of the conjecture. My previous work on strongly F-regular rings and their divisor class groups has given me a good handle on the theory surrounding the study of F-singularities. By additionally assuming that the ring R is local, I can make use of several useful F-invariants, such as the Hilbert-Kunz multiplicity and F-signature. Although there have been efforts to globalize these invariants [3], many results are only known in the local case. Likewise, restrictions on the dimension of R have proven fruitful in the past. Work is being done to extend the hypotheses considered by Aberbach and Polstra to the $p = 2$ and $p = 3$ cases, however, more research is required to bypass the "essentially of finite type" assumption.

Learn more about the minimal model program. By gaining familiarity with the MMP, I will learn how I might employ its techniques to the "Rees is Noether conjecture". To do this, I would require mentors and collaborators well-established in the minimal model program whose interests also extend to the study of F-singularities. Due to this, I believe I could align my research well at the University of Michigan. There I would have access to Dr. Melvin Hochster and Dr. Karen Smith, pioneers of the study of F-singularities, as well as some of the most active contributors to the minimal model program such as Jakub Witaszek. Because of my established connections with Karl Schwede and Anurag Singh, frequent contributors of Dr. Hochster, Smith, and even Witaszek, I believe the preparations I am making during my final year will allow me to take full advantage of Michigan's expertise.

Intellectual Merit

As previously noted, an affirmative answer to "Rees is Noether" would imply the "weak implies strong" conjecture, and thus put to rest a crucial open problem. There are several other notable consequences to this as well. For instance, an integral domain R is said to be a *splinter* if for every module finite extension ring S of R, there is a surjective map $\varphi : S \to R$ which splits. The work of Chiecchio and collaborators [4] shows that if the symbolic Rees ring is finitely generated in the case that R is strongly F-regular (and again, R has prime characteristic $p > 0$), then the splinter condition and the strong F-regularity condition define the same class of rings. In this way, if the symbolic Rees algebra were indeed finitely generated for strongly F-regular rings, we would unite four classes of F-singularities: the three types of F-regular rings and splinters.

A solution to the "Rees is Noether conjecture" would also provide new ways to explicitly describe famously nebulous objects. For example, if it were true, for a strongly F-regular ring R with a finitely generated divisor class group one could obtain a $\mathbb{Q}$-factorization of the spectrum of R. This would provide new techniques for studying strongly F-regular rings as geometric objects.

Broader Impacts

The process of breaking down a mathematical result into its core constituents promotes its accessibility and helps highlight its role within the broader context of mathematics, so it is vital that new, cutting edge areas of research take the time to revisit their past results. One of the best ways to accomplish this is through the mentorship of undergraduates, as it requires making technical concepts accessible. My mentors understood this, and it is part of the reason I was introduced to F-signature in the first place. With their help, I was able to discover new, simpler proofs of famous results such as the existence of F-signature for finitely generated modules. As discussed in my personal statement, I, too, intend to spend a substantial amount of my time as a graduate student mentoring undergraduates. Not only will this provide up and coming students with valuable research experience, it will benefit the field by exercising the process of simplifying past results, thereby promoting the role of F-singularities within the broader field of mathematics.

References

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[2] Ian Aberbach and Thomas Polstra. Local cohomology bounds and test ideals, 2020.

[3] Alessandro De Stefani, Thomas Polstra, and Yongwei Yao. Globalizing f-invariants, 2016.

[4] Alberto Chiecchio, Florian Enescu, Lance Edward Miller, and Karl Schwede. Test ideals in rings with finitely generated anti-canonical algebras. *J. Inst. Math. Jussieu*, 17(4):979–980, 2018.