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Equivariant intersection theory

(With an Appendix by Angelo Vistoli: The Chow ring of $\mathscr{M}_2$)

Dan Edidin[1,*], **William Graham**[2,**]

[1] Department of Mathematics, University of Missouri, Columbia MO 65211, USA
[2] Department of Mathematics, University of Georgia, Athens GA 30602, USA

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1 Introduction

The purpose of this paper is to develop an equivariant intersection theory for actions of linear algebraic groups on schemes and, more generally, algebraic spaces. The theory is based on our construction of equivariant Chow groups. These are algebraic analogues of equivariant cohomology groups which have all the functorial properties of ordinary Chow groups. In addition, they enjoy many of the properties of equivariant cohomology.

Previous work [Br, Gi, Vi1] (see also [Ny]) defined equivariant Chow groups using only invariant cycles on X. However, there are not enough invariant cycles on X to define equivariant Chow groups with nice properties, such as being a homotopy invariant, or having an intersection product when X is smooth (see Sect. 3.5). The definition of this paper is modeled after the Borel in equivariant cohomology. It is made possible by Totaro's Chow approximation of EG by open subsets of representations [To]. Consequently, an equivariant class is represented by an invariant cycle

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on $X \times V$, where V is a representation of G. By enlarging the definition of equivariant cycle, we obtain a rich theory which is closely related to other aspects of group actions on schemes and algebraic spaces.

After establishing the basic properties of equivariant Chow groups, this paper is mainly devoted to the relationship between equivariant Chow groups and Chow groups of quotient algebraic spaces and stacks. If G is a linear algebraic group acting on a space X, denote by $A_i^G(X)$ the i-th equivariant Chow group of X. If G acts properly on X then a quotient X/G exists in the category of algebraic spaces ([Ko], [K-M]); under some additional hypotheses (see [GIT]) if X is a scheme then X/G is a scheme. We prove that there is an isomorphism of $A^G_{i+\dim G}(X) \otimes \mathbf{Q}$ and $A_i(X/G) \otimes \mathbf{Q}$. If G acts with trivial stabilizers this holds without tensoring with $\mathbf{Q}$.

For an action which is not proper, there need not be a quotient in the category of algebraic spaces. However, there is always an Artin quotient stack $[X/G]$. We prove that the equivariant groups $A_*^G(X)$ depend only on the stack $[X/G]$ and not on its presentation as a quotient. If X is smooth, then $A_G^1(X)$ is isomorphic to Mumford's Picard group of the stack, and the ring $A_G^*(X)$ can naturally be identified as an integral Chow ring of $[X/G]$ (Sect. 3).

These results imply that equivariant Chow groups are a useful tool for computing Chow groups of quotient spaces and stacks. For example, Pandharipande [Pa1], [Pa2] has used equivariant methods to compute Chow rings of moduli spaces of maps of projective spaces as well as the Hilbert scheme of rational normal curves. In this paper, we compute the integral Chow rings of the stacks $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ of elliptic curves, and obtain a simple proof of Mumford's result [Mu] that $\mathrm{Pic}_{\mathrm{fun}}(\mathscr{M}_{1,1}) = \mathbf{Z}/12\mathbf{Z}$. In an appendix to this paper, Angelo Vistoli computes the Chow ring of $\mathscr{M}_2$, the moduli stack of smooth curves of genus 2.

Equivariant Chow groups are also useful in proving results about intersection theory on quotients. It is easy to show that if X is smooth then there is an intersection product on $A_*^G(X)$. The theorem on quotients therefore implies that there exists an intersection product on the rational Chow groups of a quotient of a smooth algebraic space by a proper action. The existence of such an intersection product was shown by Gillet and Vistoli, but only under the assumption that the stabilizers are reduced. This is automatic in characteristic 0, but typically fails in characteristic p. The equivariant approach does not require this assumption and therefore extends the work of Gillet and Vistoli to arbitrary characteristic. Furthermore, by avoiding the use of stacks, the proof becomes much simpler.

Finally, equivariant Chow groups define invariants of quotient stacks which exist in arbitrary degree, and associate to a smooth quotient stack an integral intersection ring which when tensored with $\mathbf{Q}$ agrees with rings defined by Gillet and Vistoli (Prop. 15). By analogy with quotient stacks, this suggests that there should be an integer intersection ring associated to an arbitrary smooth stack, which could be nonzero in degrees higher than the dimension of the stack.

We remark that besides the properties mentioned above, the equivariant Chow groups we define are compatible with other equivariant theories such as cohomology and K-theory. For instance, if X is smooth then there is a cycle map from equivariant Chow theory to equivariant cohomology (Sect. 2.8). In addition, there is a map from equivariant K-theory to equivariant Chow groups, which is an isomorphism after completion; and there is a localization theorem for torus actions, which can be used to give an intersection theoretic proof of residue formulas of Bott and Kalkman. These topics will be treated elsewhere.

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2 Definitions and basic properties

2.1 Conventions and notation

This paper is written in the language of algebraic spaces. It is possible to work entirely in the category of schemes, provided the mild technical hypotheses of Proposition 23 are satisfied. These hypotheses insure that if X is a scheme with a G-action, then the mixed spaces $X_G = (X \times U)/G$ are schemes. Here G acts freely on U (which is an open subset of a representation of G) and hence on $X \times U$. In the category of algebraic spaces, quotients of free actions always exist ([D-M] or [Ar]; see Proposition 22). By working with algebraic spaces we can therefore avoid the hypotheses of Proposition 23.

Another reason to work with algebraic spaces comes from a theorem of Kollár and Keel and Mori [Ko], [K-M], generalizing the result about free actions mentioned above. They prove that if G is a linear algebraic group acting properly on a separated algebraic space X, then a geometric quotient X/G exists as a separated algebraic space. Such quotients arise frequently in moduli problems. However, even if X is a scheme, the quotient X/G need not be a scheme. By developing the theory for algebraic spaces, we can apply equivariant methods to study such quotients.

For these reasons, the natural category for this theory is that of algebraic spaces. In Sect. 6.1 we explain why the intersection theory of [Fu] remains unchanged in this category.

Except in Sect. 6.2, all schemes and algebraic spaces are assumed to be quasi-separated and of finite type over a field k which can have arbitrary characteristic. A *smooth* space is assumed to be separated, implying that the diagonal $X \to X \times X$ is a regular embedding. For brevity, we will use the term variety to mean integral algebraic space (rather than integral *scheme*, as is usual). An algebraic group is always assumed to be linear. For simplicity of exposition, we will usually assume that our spaces are equidimensional.

Group actions If an algebraic group G acts on a scheme or algebraic space X then the action is said to be *closed* if the orbits of geometric points are closed in X. It is *proper* if the action map $G \times X \to X \times X$ is proper. If every point has an invariant neighborhood such that the action is proper in the neighborhood then we say the action is *locally proper*. It is *free* if the action map is a closed embedding.

If $G \times X \xrightarrow{j} X \times X$ is a group action, we will call $j^{-1}(\Delta_X) \to \Delta_X$ the stabilizer group scheme of X. Its fibers are the stabilizers of the points of X. A group action is said to have *finite stabilizer* if the map $j^{-1}(\Delta_X) \to \Delta_X$ is finite. Since a linear algebraic group is affine, the fibers of $j : G \times X \xrightarrow{j} X \times X$ are finite when G acts properly. Hence, a proper action has finite stabilizer. The converse need not be true [GIT, Example 0.4]. Note that a locally proper action has finite stabilizer as well.

Finally we say the action is *set-theoretically free* or *has trivial stabilizer* if the stabilizer of every point is trivial. Equivalently, this means $j^{-1}(\Delta_X) \to \Delta_X$ is an isomorphism. If the action is proper and has trivial stabilizer then it is free (Lemma 8).

A flat, surjective, equivariant map $X \xrightarrow{f} Y$ is called a principal bundle if G acts trivially on Y, and the map $X \times G \to X \times_Y X$ is an isomorphism. For smooth groups this condition is equivalent to local triviality in the étale topology, i.e., there is an étale cover $U \to Y$ such that $U \times_Y X \simeq U \times G$.

As noted above, if G acts set-theoretically freely on X then a geometric quotient X/G always exists in the category of algebraic spaces, and moreover, X is a principal G-bundle over X/G (Proposition 22).

2.2 Equivariant Chow groups

Let X be an n-dimensional algebraic space. We will denote the i-th equivariant Chow group of X by $A_i^G(X)$, and define it as follows.

Let G be a g-dimensional algebraic group. Choose an l-dimensional representation V of G such that V has an open set U on which G acts freely and whose complement has codimension more than $n - i$. Let $U \to U/G$ be the principal bundle quotient. Such a quotient automatically exists as an algebraic space; moreover, for any algebraic group, representations exist so that U/G is a scheme – see Lemma 9 of Sect. 6. The principal bundle $U \to U/G$ is Totaro's finite dimensional approximation of the classifying bundle $EG \to BG$ (see [To] and [E-G1]). The diagonal action on $X \times U$ is also free, so there is a quotient in the category of algebraic spaces $X \times U \to (X \times U)/G$ which is a principal G-bundle. We will usually denote this quotient by X_G. (See Proposition 23 for conditions that are sufficient to imply that X_G is a scheme.)

Definition-Proposition 1 *Set $A_i^G(X)$ (the i-th equivariant Chow group) to be $A_{i+l-g}(X_G)$, where A_* is the usual Chow group. This group is independent of the representation as long as $V - U$ has sufficiently high codimension.*

Proof. As in [To], we will use Bogomolov's double fibration argument. Let V_1 be another representation of dimension k such that there is an open subset U_1 with a principal bundle quotient $U_1 \to U_1/G$ and whose complement has codimension at least $n - i$. Let G act diagonally on $V \oplus V_1$. Then $V \oplus V_1$ contains an open set W which has a principal bundle quotient W/G and contains both $U \oplus V_1$ and $V \oplus U_1$. Thus, $A_{i+k+l-g}(X \times^G W) = A_{i+k+l-g}(X \times^G (U \oplus V_1))$ since $(X \times^G W) - (X \times^G (U \oplus V_1)$ has dimension smaller than $i + k + l - g$. On the other hand, the projection $V \oplus V_1 \to V$ makes $X \times^G (U \oplus V_1)$ a vector bundle over $X \times^G U$ with fiber V_1 and structure group G. Thus, $A_{i+k+l-g}(X \times^G (U \oplus V_1)) = A_{i+l-g}(X \times^G U)$. Likewise, $A_{i+k+l-g}(X \times^G W) = A_{i+k-g}(X \times^G U_1)$, as desired. □

Remark In the sequel, the notation $U \subset V$ will refer to an open set in a representation on which the action is free, and X_G will mean a mixed quotient $X \times^G U$ for any representation V of G. If we write $A_{i+l-g}(X_G)$ then $V - U$ is assumed to have codimension more than $n - i$ in V. (As above $n = \dim X$, $l = \dim V$ and $g = \dim G$.)

Equivariant cycles If $Y \subset X$ is an m-dimensional G-invariant subvariety (recall that variety means integral algebraic space), then it has a G-equivariant fundamental class $[Y]_G \in A_m^G(X)$. More generally, if V is an l-dimensional representation and $S \subset X \times V$ is an $m + l$-dimensional subvariety, then S has a G-equivariant fundamental class $[S]_G \in A_m^G(X)$. Thus, unlike ordinary Chow groups, $A_i^G(X)$ can be non-zero for any $i \leq n$, including negative i.

Proposition 1 *If $\alpha \in A_m^G(X)$, then there exists a representation V such that $\alpha = \sum a_i[S_i]_G$, where S_i are $m + l$ invariant subvarieties of $X \times V$, where l is the dimension of V.*

Proof. Cycles of dimension $m + l - g$ on X_G correspond exactly to invariant cycles of dimension $m + l$ on $X \times U$. Since $V - U$ has high codimension, invariant $m + l$ cycles on $X \times U$ extend uniquely to invariant $m + l$ cycles on $X \times V$. □

The representation V is not unique. For example, $[X]_G$ and $[X \times V]_G$ define the same equivariant class.

The projection $X \times U \to U$ induces a flat map $X_G \to U/G$ with fiber X. Restriction to a fiber gives a map $i^* : A_*^G(X) \to A_*(X)$ from equivariant Chow groups to ordinary Chow groups. The map is independent of the choice of fiber because any two points of U/G are rationally equivalent. For any G-invariant subvariety $Y \subset X$, $i^*([Y]_G) = [Y]$.

Before reading further, the reader may want to skip to Sect. 3 for examples.

2.3 Functorial properties

In this section all maps $f : X \to Y$ are assumed to be G-equivariant.

Let **P** be one of the following properties of morphisms of schemes or algebraic spaces: proper, flat, smooth, regular embedding or l.c.i.

Proposition 2 *If $f : X \to Y$ has property* **P**, *then the induced map $f_G : X_G \to Y_G$ also has property* **P**.

Proof. If $X \to Y$ has property **P**, then, by base change, so does the map $X \times U \to Y \times U$. The morphism $Y \times U \to Y_G$ is flat and surjective (hence faithfully flat), and $X \times U \simeq X_G \times_{Y_G} Y \times U$. Thus by descent [SGA I, Sect. 8.4–5], the morphism $X_G \to Y_G$ also has property **P**. $\square$

Proposition 3 *Equivariant Chow groups have the same functoriality as ordinary Chow groups for equivariant morphisms with property* **P**.

Proof. If $f : X \to Y$ has property **P**, then so does $f_G : X_G \to Y_G$. Define push-forward f_* or pullback f^* on equivariant Chow groups as the pullback or push-forward on the ordinary Chow groups of X_G and Y_G. The double fibration argument shows that this is independent of the choice of representation. $\square$

2.4 Chern classes

Let X be a algebraic space with a G-action, and let E be an equivariant vector bundle. Consider the quotient $E \times U \to E_G$.

Lemma 1 *$E_G \to X_G$ is a vector bundle.*

Proof. The bundle $E_G \to X_G$ is an affine bundle which is locally trivial in the étale topology since it becomes locally trivial after the smooth base change $X \times U \to X_G$. Also, the transition functions are linear since they are linear when pulled back to $X \times U$. Hence, $E_G \to X_G$ is a vector bundle over X_G. $\square$

Definition 1 *Define equivariant Chern classes $c_j^G(E) : A_i^G(X) \to A_{i-j}^G(X)$ by $c_j^G(E) \cap \alpha = c_j(E_G) \cap \alpha \in A_{i-j+l-g}(X_G)$.*

By the double fibration argument, the definition does not depend on the choice of representation.

Following [GIT], we denote by $\mathrm{Pic}^G(X)$ the group of isomorphism classes of G-linearized locally free sheaves on X.

Theorem 1 *Let X be a locally factorial variety of dimension n. Then the map* $\mathrm{Pic}^G(X) \to A^G_{n-1}(X)$ *defined by* $L \mapsto (c_1(L) \cap [X]_G)$ *is an isomorphism.*

Proof. We know that the map $\mathrm{Pic}(X_G) \xrightarrow{\cap c_1(L_G)} A_{n-g+l-g}(X_G) = A^G_{n-1}(X)$ is an isomorphism. Since $X \times U \to X_G$ is a principal bundle, $\mathrm{Pic}(X_G) = \mathrm{Pic}^G(X \times U)$. The theorem now follows from the following lemma.

Lemma 2 *Let X be a locally factorial variety with a G-action.*

(a) *Let $U \overset{j}{\hookrightarrow} X$ be an invariant subvariety such that $X - U$ has codimension more than* 1. *Then the restriction map* $j^* : \mathrm{Pic}^G(X) \to \mathrm{Pic}^G(U)$ *is an isomorphism.*

(b) *Let V be a representation and let $\pi : X \times V \to X$ be the projection. Then* $\pi^* : \mathrm{Pic}^G(X) \to \mathrm{Pic}^G(X \times V)$ *is an isomorphism.*

Proof. We first prove (a).

Injectivity: Suppose $L \in \mathrm{Pic}^G(X)$ and j^*L is trivial. Since $\mathrm{Pic}(X) \cong \mathrm{Pic}(X - Y)$, this implies that as a bundle L must be trivial. A linearization of the trivial bundle on X is just a homomorphism $G \to \Gamma(X, \mathcal{O}^*_X)$. Since X is a variety and $X - U$ has codimension more than one, $\Gamma(X, \mathcal{O}^*_X) = \Gamma(U, \mathcal{O}^*_U)$. Thus a linearization of the trivial bundle is trivial on X if and only if it is trivial on U, proving injectivity.

Surjectivity: A linearization of L is a homomorphism of G into the group of automorphisms of L over X. To show that j^* is surjective, we must show that if $L \in \mathrm{Pic}(X)$ is linearizable on U then it is linearizable on X. But any isomorphism $\alpha : L|_U \to g^*L|_U$ extends to an isomorphism over X. (To see this, pick an isomorphism $\beta : L \to g^*L$; we know one exists because $\mathrm{Pic}(X) \cong \mathrm{Pic}(U)$ and L and g^*L are isomorphic on $X - Y$. Then $\alpha = \beta \cdot f$, where $f \in \Gamma(U, \mathcal{O}^*(U))$, but $\Gamma(X, \mathcal{O}^*_X) = \Gamma(U, \mathcal{O}^*_U)$, so α extends to X.) Hence L is linearizable on X.

The proof of (b) is similar. The key point is that if X is a variety and V is a vector space, then $\Gamma(X \times V, \mathcal{O}^*_{X \times V}) = \Gamma(X, \mathcal{O}^*_X)$, because if R is an integral domain, then the units in $R[t_1, \ldots, t_n]$ are the just the units of R. □

2.5 Exterior products

If X and Y have G-actions then there are exterior products $A^G_i(X) \otimes A^G_j(X) \to A^G_{i+j}(X \times Y)$. By Proposition 1 any $\alpha \in A^G_*(X)$ can be written as $\alpha = \sum a_i[S_i]_G$ where the S_i's are G-invariant subvarieties of $X \times V$ for some representation V.

Let V, W be representations of G of dimensions l and k respectively. Let $S \subset X \times V$, $T \subset Y \times W$, be G-invariant subvarieties of dimensions $i + l$ and $j + k$ respectively. Let $s : X \times V \times Y \times W \to X \times Y \times (V \oplus W)$ be the isomorphism $(x, v, y, w) \mapsto (x, y, v \oplus w)$.

Definition-Proposition 2 (*Exterior products*) *The assignment* $[S]_G \times [T]_G \mapsto [s(S \times T)]_G$ *induces a well defined exterior product map of equivariant Chow groups* $A_i^G(X) \otimes A_j^G(Y) \to A_{i+j}^G(X \times Y)$.

Proof. The proof follows from [Fu, Proposition 1.10] and the double fibration argument used above. $\square$

Given the above propositions, equivariant Chow groups have all the formal properties of ordinary Chow groups ([Fu, Chaps. 1–6]). In particular, if X is smooth, there is an intersection product on the equivariant Chow groups $A_*^G(X)$ which makes $\oplus A_*^G(X)$ into a graded ring.

2.6 Operational Chow groups

In this section we define equivariant operational Chow groups $A_G^i(X)$, and compare them with the operational Chow groups of X_G.

Define equivariant operational Chow groups $A_G^i(X)$ as operations $c(Y \to X) : A_*^G(Y) \to A_{*-i}^G(Y)$ for every G-map $Y \to X$. As for ordinary operational Chow groups ([Fu, Chapter 17]), these operations should be compatible with the operations on equivariant Chow groups defined above (pull-back for l.c.i. morphisms, proper push-forward, etc.). From this definition it is clear that for any X, $A_G^*(X)$ has a ring structure. The ring $A_G^*(X)$ is graded, and $A_G^i(X)$ can be non-zero for any $i \geq 0$.

Note that by construction, the equivariant Chern classes defined above are elements of the equivariant operational Chow ring.

Proposition 4 *If* X *is smooth of dimension* n, *then* $A_G^i(X) \simeq A_{n-i}^G(X)$.

Corollary 1 (*of Theorem* 1) *If* X *is a smooth variety with a* G*-action, then the map* $\mathrm{Pic}^G(X) \to A_G^1(X)$ *defined by* $L \mapsto c_1(L)$ *is an isomorphism.* $\square$

Proof of Proposition 4. Define a map $A_G^i(X) \to A_{n-i}^G(X)$ by the formula $c \mapsto c \cap [X]_G$. Define a map $A_{n-i}^G(X) \to A_G^i(X)$, $\alpha \mapsto c_\alpha$ as follows. If $Y \xrightarrow{f} X$ is a G-map, then since X is smooth, the graph $\gamma_f : Y \to Y \times X$ is a G-map which is a regular embedding. If $\beta \in A_*^G(Y)$ set $c_\alpha \cap \beta = \gamma_f^*(\beta \times \alpha)$.

Claim (cf. [Fu, Proposition 17.3.1]): $\beta \times (c \cap [X]_G) = c \cap (\beta \times [X]_G)$.

Given the claim, the formal arguments of [Fu, Proposition 17.4.2] show that the two maps are inverses.

Proof of Claim. By Proposition 1 and the linearity of equivariant operations, we may assume there is a representation V so that $\beta = [S]_G$ for a G-invariant subvariety $S \subset Y \times V$. Since S is G-invariant, the projection $p : S \times X \to X$ is equivariant. Thus,

$$[S]_G \times (c \cap [X]_G) = p^*(c \cap [X]_G) = c \cap p^*([X]_G)$$
$$= c \cap ([S \times X]_G) = c \cap ([S]_G \times [X]_G) \ . \qquad \square$$

Let V be a representation such that $V - U$ has codimension more than k, and set $X_G = X \times^G U$. In the remainder of the subsection we will discuss the relation between $A^k_G(X)$ and $A^k(X_G)$ (ordinary operational Chow groups).

Recall [Fu, Definition 18.3] that an envelope $\pi : \tilde{X} \to X$ is a proper map such that for any subvariety $W \subset X$ there is a subvariety $\tilde{W}$ mapping birationally to W via π. In the case of group actions, we will say that $\pi : \tilde{X} \to X$ is an *equivariant* envelope, if π is G-equivariant, and if we can take $\tilde{W}$ to be G-invariant for G-invariant W. If there is an open set $X^0 \subset X$ over which π is an isomorphism, then we say $\pi : \tilde{X} \to X$ is a *birational* envelope.

Lemma 3 *If $\pi : \tilde{X} \to X$ is an equivariant* (*birational*) *envelope, then $p : \tilde{X}_G \to X_G$ is a* (*birational*) *envelope* ($\tilde{X}_G$ *and* X_G *are constructed with respect to a fixed representation* V). *Furthermore, if X^0 is the open set over which π is an isomorphism* (*necessarily G-invariant*), *then p is an isomorphism over $X^0_G = X^0 \times^G U$.*

Proof. Fulton [Fu, Lemma 18.3] proves that the base extension of an envelope is an envelope. Thus $\tilde{X} \times U \xrightarrow{\pi \times id} X \times U$ is an envelope. Since the projection $X \times U \to X$ is equivariant, this envelope is also equivariant. If $W \subset X_G$ is a subvariety, let W' be its inverse image (via the quotient map) in $X \times U$. Let $\tilde{W}'$ be an invariant subvariety of $\tilde{X} \times U$ mapping birationally to W'. Since G acts freely on $\tilde{X} \times U$ it acts freely on $\tilde{W}'$, and $\tilde{W} = \tilde{W}'/G$ is a subvariety of $\tilde{X}_G$ mapping birationally to W. This shows that $\tilde{X}_G \to X_G$ is an envelope; it is clear that the induced map $\tilde{X}_G \to \tilde{X}$ is an isomorphism over X^G_0. $\square$

Suppose $\tilde{X} \xrightarrow{\pi} X$ is an equivariant envelope which is an isomorphism over X^0. Let $\{S_i\}$ be the irreducible components of $S = X - X^0$, and let $E_i = \pi^{-1}(S_i)$. Then $\{S_{iG}\}$ are the irreducible components of $X_G - X^0_G$ and $E_{iG} = \pi^{-1}(S_{iG})$.

Theorem 2 *If X has an equivariant smooth envelope $\pi : \tilde{X} \to X$ such that there is an open $X^0 \subset X$ over which π is an isomorphism, and $V - U$ has codimension more than k, then $A^k_G(X) = A^k(X_G)$.*

Proof. If $\pi : \tilde{X} \to X$ is an equivariant non-singular envelope, then $p : \tilde{X}_G \to X_G$ is also an envelope and $\tilde{X}_G$ is non-singular. Thus, by [Ki, Lemma 1.2] $p^* : A^*(X_G) \to A^*(\tilde{X}_G)$ is injective. The image of p^* is described inductively in [Ki, Theorem 3.1]. A class $\tilde{c} \in A^*(\tilde{X}_G)$ equals p^*c if and only if for each E_{iG} , $\tilde{c}_{|E_{iG}} = p^*c_i$ where $c_i \in A^*(E_i)$. This description follows from formal properties of operational Chow groups, and the exact sequence [Ki, Theorem 2.3]

$$A^*(X_G) \xrightarrow{p} A^*(\tilde{X}_G) \xrightarrow{p_1^* - p_2^*} A^*(\tilde{X}_G \times_{X_G} \tilde{X}_G)$$

where p_1 and p_2 are the two projections from $\tilde{X}_G \times_{X_G} \tilde{X}_G$.

By Proposition 4 above, we know that $A^k_G(\tilde{X}) = A^k(\tilde{X}_G)$. We will show that $A^k_G(X)$ and $A^k(X_G)$ have the same image in $A^k(\tilde{X}_G)$. By Noetherian induction we may assume that $A^k_G(S_i) = A^k((S_i)_G)$. To prove the theorem, it suffices to show that there is also an exact sequence of equivariant operational Chow groups

$$0 \to A^*_G(X) \xrightarrow{\pi^*} A^*_G(\tilde{X}) \xrightarrow{p_1^* - p_2^*} A^*_G(\tilde{X} \times_X \tilde{X})$$

This can be checked by working with the action of $A^*_G(X)$ on a fixed Chow group $A_i(X_G)$ and arguing as in Kimura's paper. □

Corollary 2 *If X is separated and has an equivariant resolution of singularities* (*in particular if the characteristic is* 0), *and $V - U$ has codimension more than k, then $A^k_G(X) = A^k(X_G)$.*

Proof (cf. [Ki, Remark 3.2]). We must show the existence of an equivariant envelope $\pi : \tilde{X} \to X$. By equivariant resolution of singularities, there is a resolution $\pi_1 : \tilde{X}_1 \to X$ such that π_1 is an isomorphism outside some invariant subscheme $S \subset X$. By Noetherian induction, we may assume that we have constructed an equivariant envelope $\tilde{S} \to S$. Now set $\tilde{X} = \tilde{X}_1 \cup \tilde{S}$. □

2.7 Equivariant higher Chow groups

In this section assume that X is quasi-projective (a quasi-projective algebraic space is a scheme [Kn, p.140]). Bloch ([Bl]) defined higher Chow groups $CH^i(X,m)$ as $H_m(Z^i(X,\cdot))$ where $Z^i(X,\cdot)$ is a complex whose k-th term is the group of cycles of codimension i in $X \times \Delta^k$ which intersect the faces properly. Since we prefer to think in terms of dimension rather than codimension we will define $A_p(X,m)$ as $H_m(Z_p(X,\cdot))$, where $Z_p(X,k)$ is the group of cycles of dimension $p+k$ in $X \times \Delta^k$ intersecting the faces properly. When X is equidimensional of dimension n, then $A_p(X,m) = CH^{n-p}(X,m)$.

If $Y \subset X$ is closed, there is a localization long exact sequence. The advantage of indexing by dimension rather than codimension is that the sequence exists without assuming that Y is equidimensional.

Lemma 4 *Let X be equidimensional, and let $Y \subset X$ be closed*; *then there is a long exact sequence of higher Chow groups*

$$\begin{aligned} \cdots \to A_p(Y,k) \to A_p(X,k) \to A_p(X-Y,k) \to \\ \cdots \to A_p(Y) \to A_p(X) \to A_p(X-Y) \to 0 \end{aligned}$$

(*there is no requirement that Y be equidimensional*).

Proof. This is a simple consequence of the localization theorem of [Bl]. We must show that the complex $Z_p(X - Y, \cdot)$ is quasi-isomorphic to the complex $\frac{Z_p(X,\cdot)}{Z_p(Y,\cdot)}$. By induction on the number of components, (when Y is irreducible this is Bloch's Theorem) it suffices to verify the quasi-isomorphism when Y is the union of two irreducible components Y_1 and Y_2.

By the Bloch's theorem, $Z_p(X - (Y_1 \cup Y_2), \cdot) \simeq \frac{Z_p(X-Y_1,\cdot)}{Z_p(Y_2-(Y_1\cap Y_2),\cdot)}$ and $Z_p(X - Y_1, \cdot) \simeq \frac{Z_p(X,\cdot)}{Z_p(Y_1)}$ (here $\simeq$ denotes quasi-isomorphism). By induction on dimension, we can assume that the lemma holds for schemes of smaller dimension, so $Z_p(Y_2 - (Y_1 \cap Y_2), \cdot) \simeq \frac{Z_p(Y_2,\cdot)}{Z_p(Y_1\cap Y_2)}$. Finally note that $\frac{Z_p(Y_2,\cdot)}{Z_p(Y_1\cap Y_2,\cdot)} \simeq \frac{Z_p(Y_1\cup Y_2,\cdot)}{Z_p(Y_1,\cdot)}$. Combining all our quasi-isomorphisms we have

$$Z_p(X - (Y_1 \cup Y_2), \cdot) \simeq \frac{\frac{Z_p(X,\cdot)}{Z_p(Y_1,\cdot)}}{\frac{Z_p(Y_1\cup Y_2),\cdot}{Z_p(Y_1,\cdot)}} \simeq \frac{Z_p(X,\cdot)}{Z_p(Y_1 \cup Y_2,\cdot)}$$

as desired. □

If X is quasi-projective with a linearized G-action, we can define equivariant higher Chow groups $A_i^G(X, m)$ as $A_{i+l-g}(X_G, m)$, where X_G is formed from an l-dimensional representation V such that $V - U$ has high codimension (note that X_G is quasi-projective, by [GIT, Prop. 7.1]). The homotopy property of higher Chow groups shows that $A_i^G(X, m)$ is well defined.

Warning. Since the homotopy property of higher Chow groups has only been proved for quasi-projective varieties, our definition of higher equivariant Chow groups is only valid for quasi-projective varieties with a linearized action. However, if G is connected and X is quasi-projective and normal, then by Sumihiro's equivariant completion [Su] and [GIT, Corollary 1.6], any action is linearizable.

One reason for constructing equivariant higher Chow groups is to obtain a localization exact sequence:

Proposition 5 *Let X be equidimensional and quasi-projective with a linearized G-action, and let $Y \subset X$ be an invariant subscheme. There is a long exact sequence of higher equivariant Chow groups*

$$\begin{gathered}\cdots \to A_p^G(Y, k) \to A_p^G(X, k) \to A_p^G(X - Y, k) \to \\ \cdots \to A_p^G(Y) \to A_p^G(X) \to A_p^G(X - Y) \to 0 \ .\end{gathered}$$ □

2.8 Cycle maps

If X is a complex algebraic variety with the action of a complex algebraic group, then we can define equivariant Borel-Moore homology $H_{BM,i}^G(X)$ as

$H_{BM,i+2l-2g}(X_G)$ for $X_G = X \times^G U$. As for Chow groups, the definition is independent of the representation, as long as $V - U$ has sufficiently high codimension, and we obtain a cycle map

$$\mathrm{cl} : A_i^G(X) \to H_{BM,2i}^G(X)$$

compatible with the usual operations on equivariant Chow groups (cf. [Fu, Chap. 19]).

Let $EG \to BG$ be the classifying bundle. The open subsets $U \subset V$ are topological approximations to EG. For, if ϕ is a map of the j-sphere S^j to U, we may view ϕ as a map $S^j \to V$. Extend ϕ to a map $B^{j+1} \to V$. We may assume that the extended map is smooth and transversal to $V - U$. If $j + 1 < 2i$, where i is the complex codimension of $V - U$, then transversality implies that the extended map does not intersect $V - U$. Thus we have extended ϕ to a map $B^{j+1} \to U$. Hence $\pi_j(U) = 0$ for $j < 2i - 1$.

Note that $H_{BM,i}^G(X)$ is not the same as $H_i(X \times^G EG)$, However, if X is smooth, then X_G is also smooth, and $H_{BM,i}(X_G)$ is dual to $H^{2n-i}(X_G) = H^{2n-i}(X \times^G EG) = H_G^{2n-i}(X)$, where n is the complex dimension of X. In this case we can interpret the cycle map as giving a map

$$\mathrm{cl} : A_G^i(X) \to H_G^{2i}(X) \ .$$

If X is compact, and the open sets $U \subset V$ can be chosen so that U/G is projective, then Borel-Moore homology of X_G coincides with ordinary homology, so $H_{BM*}^G(X)$ can be calculated with a compact model. In general, however, U/G is only quasi-projective. If G is finite, then U/G is never projective. If G is a torus, then U/G can be taken to be a product of projective spaces. If $G = GL_n$, then U/G can be taken to be a Grassmannian (see the example in Sect. 3.1)

If G is semisimple, then U/G cannot be chosen projective, for then the hyperplane class would be a non-torsion element in A_G^1, but by Proposition 6, $A_G^* \otimes \mathbf{Q} \cong S(\hat{T})^W \otimes \mathbf{Q}$, which has no elements of degree 1. Nevertheless for semisimple (or reductive) groups we have, by Proposition 6, a cycle map

$$\mathrm{cl} : A_*^G(X)_{\mathbf{Q}} \to H_{BM*}^G(X;\mathbf{Q}) \simeq H_{BM*}^T(X;\mathbf{Q})^W$$

and if X is compact then the last group can be calculated with a compact model.

3 Examples

In this section we calculate some examples of equivariant Chow groups, particularly for connected groups. The point of this is to show that computing *equivariant* Chow groups is no more difficult than computing ordinary Chow groups, and in the case of quotients, equivariant Chow theory gives a way of computing ordinary Chow groups. Moreover, since many of the varieties with computable Chow groups (such as G/P's, Schubert varieties, spherical varieties, etc.) come with group actions, it is natural to study their equivariant Chow groups.

3.1 Representations and subsets

For some groups there is a convenient choice of representations and subsets. In the simplest case, if $G = \mathbf{G}_m$ then we can take V to an l-dimensional representation with all weights -1, $U = V - \{0\}$, and $U/G = \mathbf{P}^{l-1}$. If $G = T$ is a (split) torus of rank n, then we can take $U = \oplus_1^n (V - \{0\})$ and $U/T = \Pi_1^n \mathbf{P}^{l-1}$. If $G = GL_n$, take V to be the vector space of $n \times p$ matrices ($p > n$), with GL_n acting by left multiplication, and let U be the subset of matrices of maximal rank. Then U/G is the Grassmannian $\mathrm{Gr}(n,p)$. Likewise, if $G = SL(n)$, then U/G fibers over $\mathrm{Gr}(n,p)$ as the complement of the 0-section in the line bundle $\det(S) \rightarrow \mathrm{Gr}(n,p)$, where S is the tautological rank n subbundle on $\mathrm{Gr}(n,p)$.

3.2 Equivariant Chow rings of points

The equivariant Chow ring of a point was introduced in [To]. If G is connected reductive, then $A_G^* \otimes \mathbf{Q}$ and (if G is special) A_G^* are computed in [E-G1]. The computation given there does not use a particular choice of representations and subsets. The result is that $A_G^* \otimes \mathbf{Q} \cong S(\hat{T})^W \otimes \mathbf{Q}$, where T is a maximal torus of G, $S(\hat{T})$ the symmetric algebra on the group of characters $\hat{T}$, and W the Weyl group. If G is special this result holds without tensoring with $\mathbf{Q}$.

Proposition 6 *Let G be a connected reductive group with split maximal torus T and Weyl group W. Then $A_*^G(X) \otimes \mathbf{Q} = A_*^T(X)^W \otimes \mathbf{Q}$. If G is special the isomorphism holds with integer coefficients.*

Proof. If G acts freely on U, then so does T. Thus for a sufficiently large representation V, $A_i^T(X) = A_{i+l-t}((X \times U)/T)$ and $A_i^G(X) = A_{i+l-g}((X \times U)/G)$ (here l is the dimension of V, t the dimension of T and g the dimension of G) (cf. [*Vi* 4]). On the other hand, $(X \times U)/T$ is a G/T bundle over $(X \times U)/G$. Thus $A_k((X \times U/T)) \otimes \mathbf{Q} = A_{k+g-t}((X \times U)/G)^W \otimes \mathbf{Q}$ and if G is special, then the equality holds integrally ([E-G1]) and the proposition follows. $\square$

For G equal to $\mathbf{G}_m$ or $GL(n)$ the choice of representations in Sect. 3.1 makes it easy to compute A_G^* directly, without appealing to the result of [E-G1]. If $l > i$, then $A^i_{\mathbf{G}_m} = A^i(\mathbf{P}^{l-1}) = \mathbf{Z} \cdot t^i$, where $t = c_1(\mathcal{O}(1))$. Thus, $A^*_{\mathbf{G}_m}(pt) = \mathbf{Z}[t]$. More generally, for a split torus of rank n, $A_T^*(pt) = \mathbf{Z}[t_1, \ldots, t_n]$. Likewise, for p sufficiently large, $A^*_{GL_n}(pt) = A^i(\mathrm{Gr}(n,p))$ is the free abelian group of homogeneous symmetric polynomials of degree i in n variables (polynomials in the Chern classes of the rank n tautological sub-bundle). Thus $A^*_{GL_n}(pt) = \mathbf{Z}[c_1, \dots, c_n]$ where c_i has degree i. Likewise $A^*_{SL_n}(pt) = \mathbf{Z}[c_2, \dots c_n]$.

There is a map $A^*_{GL_n} \to A^*_T$, where T is a maximal torus. This is a special case of a general construction: if G acts on X and $H \subset G$ is a subgroup, then there is a pull-back $A^G_*(X) \to A^H_*(X)$. This map is induced by pulling back along the flat map $X_H = X \times^H U \to X_G = X \times^G U$. We can identify the map $A^*_{GL_n} \to A^*_T$ concretely as the map $\mathbf{Z}[c_1, \dots c_n] \to \mathbf{Z}[t_1, \dots, t_n]$ given by $c_i \mapsto e_i(t_1, \dots, t_n)$ (here e_i denotes the i-th symmetric polynomial), so $A^*_{GL_n}(pt)$ can be identified with the sub-ring of symmetric polynomials in $\mathbf{Z}[t_1, \dots, t_n]$. This is a special case of the result of Proposition 6.

More elaborate computations are required to compute A^*_G for other reductive groups (if one does not tensor with $\mathbf{Q}$). The cases $G = O(n)$ and $G = SO(2n+1)$ have been worked out by Pandharipande [Pa2] and Totaro. There is a conjectural answer for $G = SO(2n)$, verified by Pandharipande for $n = 2$.

Equivariant Chern classes over a point An equivariant vector bundle over a point is a representation of G. If $T = \mathbf{G}_m$, equivariant line bundles correspond to the 1-dimensional representation L_a where T acts by weight a. If (as above) we approximate BT by $(V - \{0\})/T = \mathbf{P}(V)$, where T acts on V with all weights -1, then the tautological sub-bundle corresponds to the representation L_{-1}. Hence $c_T(L_a) = at$.

3.3 Equivariant Chow rings of $\mathbf{P}^n$

We calculate $A^*_T(\mathbf{P}^n)$, where $T = \mathbf{G}_m$ acts diagonally on $\mathbf{P}^n$ with weights $a_0, \dots, a_n$ (i.e., $g \cdot (x_0 : x_1 : \cdots : x_n) = (g^{a_0}x_0 : g^{a_1}x_1 : \cdots : g^{a_n}x_n)$). In this case, $X_T \to U/T$ is the $\mathbf{P}^n$ bundle

$$\mathbf{P}(\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_n)) \to \mathbf{P}^{l-1} \ .$$

Thus $A^*(X_T) = A^*(\mathbf{P}^{l-1})[h]/(p(h,t))$ where t is the generator for $A^1(\mathbf{P}^{l-1})$ and

$$p(h,t) = \sum_{i=0}^{n} h^i \, e_i(a_0 t, \dots, a_n t) \ .$$

Letting the dimension of the representation go to infinity we see that $A^*_T(\mathbf{P}^n) = \mathbf{Z}[t,h]/p(h,t)$. Note that $A^*_T(\mathbf{P}^n)$ is a module of rank $n+1$ over the T-equivariant Chow ring of a point.

Assume that the weights of the T-action are distinct. Then the fixed point set $(\mathbf{P}^n)^T$ consists of the points $p_0, \dots, p_n$, where $p_r \in \mathbf{P}^n$ is the point which is non-zero only in the r-th coordinate. The inclusion $i_r : p_r \hookrightarrow \mathbf{P}^n$ is a regular embedding. The equivariant normal bundle is the equivariant vector bundle over the point p_r corresponding to the representation $V_r = \oplus_{s\neq r} L_{a_s}$. The equivariant push-forward i_{r*} is readily calculated. For example, if $n = 1$ then i_{r*} takes α to $\alpha \cdot (h + a_s t)$ (where $s \neq r$). Hence the map $i_* : A^T_*((\mathbf{P}^1)^T) \rightarrow A^*_T(\mathbf{P}^1)$ becomes an isomorphism after inverting t (and tensoring with $\mathbf{Q}$ if a_0 and a_1 are not relatively prime). This is a special case of the localization theorem for torus actions [E-G2].

We remark that the calculation of $A^*_T(\mathbf{P}^n)$ can be viewed as a special case of the projective bundle theorem for equivariant Chow groups (which follows from the projective bundle theorem for ordinary Chow groups), since $\mathbf{P}^n$ is a projective bundle over a point, which is trivial but not equivariantly trivial.

3.4 Computing Chow rings of quotients

By Theorem 3 the rational Chow groups of the quotient of a variety by a group acting with finite stabilizers can be identified with the equivariant Chow groups of the original variety. If the original variety is smooth then the rational Chow groups of the quotient inherit a canonical ring structure (Theorem 4).

For example, let W be a representation of a split torus T and let $X \subset W$ be the open set on which T acts properly. Since representations of T split into a direct sum of invariant lines, it easy to show that $W - X$ is a finite union of invariant linear subspaces $L_1, \ldots, L_r$. When $T = \mathbf{G}_m$ and $X = W - \{0\}$ then the quotient is a twisted projective space.

Let $\mathscr{R} \subset \hat{T}$ be the set of weights of T on W. If $L \subset W$ is an invariant linear subspace, set $\chi_L = \Pi_{\chi\in\mathscr{R}} \chi^{d(L,\chi)}$, where $d(L,\chi)$ is the dimension of the χ-weight space of V/L.

Proposition 7 *There is a ring isomorphism* $A^*(X/T)_{\mathbf{Q}} \simeq S(\hat{T})/(\chi_{L_1}, \ldots \chi_{L_r})$.

Proof. By Theorem 3, $A^*(X/T)_{\mathbf{Q}} = A^*_T(X)_{\mathbf{Q}}$. Since $W - X$ is a union of linear subspaces $L_1, \ldots L_r$, we have an exact sequence (ignoring the shifts in degrees)

$$\oplus A^*_T(L_i) \xrightarrow{i_*} A^*_T(W) \rightarrow A^*_T(U) \rightarrow 0 \ .$$

Identifying $A^*_T(W)$ with $A^*_T(pt) = S(\hat{T})$, we see that $A^T(U) = S(\hat{T})/\mathrm{im}(i_*)$. Since each invariant linear subspace is the intersection of invariant hypersurfaces, the image of $A^*_T(L_i)$ in $A^*_T(W) = S(\hat{T})$ is the ideal $\chi_{Li} S(\hat{T})$. $\square$

Remark The preceding proposition is a simpler presentation of a computation in [Vi2, Sect. 4]. In addition, we do not need to assume that the stabilizers are reduced, so there is no restriction on the characteristic.

In [E-S], Ellingsrud and Strømme considered representations V of G for which all G-semi-stable points are stable for a maximal torus of G, such that G acts freely on the set $V^s(G)$ of G-stable points. In this case they gave a presentation for $A^*(V^s/G)$. Using Theorem 3, it can be shown that their presentation is valid (with $\mathbf{Q}$ coefficients) even if G doesn't act freely on $V^s(G)$.

In a more complicated example, Pandharipande [Pa1] used equivariant Chow groups to compute the rational Chow ring of the moduli space, $M_{\mathbf{P}^r}(\mathbf{P}^k, d)$ of maps $\mathbf{P}^k \to \mathbf{P}^r$ of degree d. This moduli space is the quotient $U(k,r,d)/GL(k+1)$, where $U(k,r,d) \subset \oplus_0^r H^0(\mathbf{P}^k, \mathcal{O}_{\mathbf{P}^k}(d))$ is the open set parameterizing base-point free $r+1$-tuples of polynomials of degree d on $\mathbf{P}^k$. His result is that for any d, $A^*(M_{\mathbf{P}^r}(\mathbf{P}^k, d))_{\mathbf{Q}}$ is canonically isomorphic to the rational Chow ring of the Grassmannian $\mathrm{Gr}(\mathbf{P}^k, \mathbf{P}^r)$.

3.5 Intersecting equivariant cycles, an example

Let $X = k^3 - \{0\}$ and let T denote the 1-dimensional torus acting with weights 1, 2, 2. We let $A^i[X/T]$ denote the group of invariant cycles on X of codimension i, modulo the relation $\mathrm{div}(f) = 0$, where f is a T-invariant rational function on an invariant subvariety of X. We will show that there is no (reasonable) intersection product, with integer coefficients, on $A^*[X/T]$. We will also compare $A^*[X/T]$ to $A_T^*(X)$, which does have an integral intersection product.

Clearly $A^0[X/T] = \mathbf{Z} \cdot [X]$. An invariant codimension 1 subvariety is the zero set of a weighted homogeneous polynomial $f(x,y,z)$, where x has weight 1 and y and z have weights 2. If f has weight n then the cycle defined by f is equivalent to the cycle $n \cdot p$, where p is the class of the plane $x = 0$. Thus $A^1[X/T] = \mathbf{Z} \cdot p$. The invariant codimension 2 subvarieties are just the T-orbits. If we let l denote the class of the line $x = y = 0$, then we see that the orbit $T \cdot (a,b,c)$ is equivalent to l if $a = 0$, and to $2l$ otherwise. Thus $A^2[X/T] = \mathbf{Z} \cdot l$. Finally, $A^i[X/T] = 0$ for $i \geq 3$.

If Z_1 and Z_2 are the cycles defined by $x = 0$ and $y = 0$, then Z_1 and Z_2 intersect transversely in the line $x = y = 0$. Thus, in a "reasonable" intersection product we would want $2p^2 = [Z_1] \cdot [Z_2] = l$ or $p \cdot p = \frac{1}{2} l$. But $\frac{1}{2} l$ is not an integral class in $A^2[X/T]$, so such an intersection product does not exist.

Now consider the equivariant groups $A_T^*(X)$. We model BT by $\mathbf{P}^N$, where N is arbitrarily large; then the mixed space X_T corresponds to the complement of the 0-section in the vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. Thus $A_T^*(X) = \mathbf{Z}[t]/(4t^3)$.

Each invariant cycle on X defines an element of $A_T^*(X)$, so there is a natural map $A^*[X/T] \to A_T^*(X)$. This map takes p to t and l to $2t^2$. The equivariant theory includes the extra cycle, t^2, necessary to define an integral intersection product. We can view elements of $A_T^*(X)$ as cycles on $X \times V$, where V is a representation of T with all weights-1. The class t^2 is represented by the cycle $x = 0, \phi = 0$ where ϕ is any linear function on V.

Another example was suggested by the referee. Let X be the blow up of $\mathbf{P}^2$ at a point p. Let G be the subgroup of PGL(3) fixing p; then G acts on X, since it acts on $\mathbf{P}^2$ and on the tangent space at p. The exceptional divisor E is invariant, so it defines an equivariant cycle $[E]_G$. However, the product $[E]_G^2$ cannot be represented by an invariant cycle, since G acts on X without fixed points.

4 Intersection theory on quotients

One of the most important properties of equivariant Chow groups is that they compute the rational Chow groups of a quotient by a group acting with finite stabilizer. They can also be used to show that the rational Chow groups of a moduli space which is a quotient (by a group) of a smooth algebraic space have an intersection product – even when there are infinitesimal automorphisms.

4.1 Chow groups of quotients

Let G be a g-dimensional group acting on a algebraic space X. Following Vistoli, we define a *quotient* $X \xrightarrow{\pi} Y$ to be a map which has the following properties (cf. [GIT, Definition 0.6(i–iii)]): π commutes with the action of G, the geometric fibers of π are the orbits of the geometric points of X, and π is universally submersive, i.e., $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is, and this property is preserved by base change. (This is called a topological quotient in [Ko, Definition 2.7].) Unlike what Mumford calls a geometric quotient, we do not require that $\mathcal{O}_Y = \pi_*(\mathcal{O}_X)^G$. The advantage of this definition is that it is preserved under base change. In characteristic 0 there are no inseparable extensions, so for normal varieties our quotient is in fact a geometric quotient [GIT, Prop. 0.2]. For proper actions, a geometric quotient is unique [GIT, Ko]. If $L \subset K$ is an inseparable extension and G_K is a group defined over K, then both Spec L and Spec K are quotients of G_K by G_K. This example shows that quotients need not be unique in characteristic p.

Proposition 8 (a) *If G acts with trivial stabilizer on an algebraic space X and $X \rightarrow Y$ is the principal bundle quotient then $A^G_{i+g}(X) = A_i(Y)$.*

(b) *If in addition X is quasi-projective and G acts linearly, then $A^G_{i+g}(X, m) = A^G_i(X, m)$ for all $m \geq 0$.*

Proof. If the stabilizers are trivial, then $(V \times X)/G$ is a vector bundle over the quotient Y. Thus X_G is an open set in this bundle with arbitrarily high codimension, and the proposition follows from homotopy properties of (higher) Chow groups. □

Theorem 3 (a) *Let X be an algebraic space with a (locally) proper G-action and let $X \xrightarrow{\pi} Y$ be a quotient. Then*

$$A^G_{i+g}(X) \otimes \mathbf{Q} \simeq A_i(Y) \otimes \mathbf{Q} \ .$$

(b) *If in addition X is quasi-projective with a linearized G-action, and the quotient Y is quasi-projective, then*

$$A^G_{i+g}(X, m) \otimes \mathbf{Q} \simeq A^G_i(Y, m) \otimes \mathbf{Q}$$

Theorem 4 *With the same hypotheses as in Theorem* 3(*a*), *there is an isomorphism of operational Chow rings*

$$\pi^* : A^*(Y)_{\mathbf{Q}} \xrightarrow{\simeq} A^*_G(X)_{\mathbf{Q}} \ .$$

Moreover if X is smooth, then the map $A^*(Y)_{\mathbf{Q}} \xrightarrow{\cap[Y]} A_*(Y)_{\mathbf{Q}}$ is an isomorphism. In particular, if X is smooth, the rational Chow groups of the quotient space $Y = X/G$ have a ring structure, which is independent of the presentation of Y as a quotient of X by G.

Remarks (1) By [Ko] or [K-M], if G acts locally properly on a (locally separated) algebraic space, then a (locally separated) geometric quotient $X \xrightarrow{\pi} Y$ always exists in the category of algebraic spaces. Moreover, the result of [K-M] holds under the weaker hypothesis that G acts on X with finite stabilizer. However, the quotient Y need not be locally separated. We expect that Theorems 3 and 4 still hold in this case, but our proof does not go through.

(2) The hypotheses in Theorem 3(b) are purely technical. They are necessary because the localization theorem for higher Chow groups has only been proved for quasi-projective schemes. If the localization theorem were proved for algebraic spaces, Theorem 3(b) would hold in this case.

(3) Checking that an action is proper can be difficult. If G is reductive, then [GIT, Proposition 0.8 and Converse 1.13] give criteria for properness. In particular if X is contained in the set of stable points for some linearized action of G on X then the action is proper. If $X \rightarrow Y$ is affine and Y is quasi-projective the action is also proper. Not surprisingly, checking that an action is locally proper is slightly easier. In particular if G is reductive and G, acts with finite stabilizers, then the action is locally proper if X can be covered by invariant affine open sets.

(4) In practice, many interesting varieties arise as quotients of smooth varieties by connected algebraic groups which act with finite stabilizers. Examples include simplicial toric varieties and various moduli spaces such as curves, vector bundles, stable maps, etc. Theorem 4 provides a tool to compute Chow groups of such varieties (see Sect. 3 for some examples).

(5) As noted above, Theorem 4 shows that there exists an intersection product on the rational Chow groups of quotients of smooth varieties and algebraic spaces. There is a long history of work on this problem. In characteristic 0, Mumford [Mu] proved the existence of an intersection product on the rational Chow groups of $\bar{\mathscr{M}}_g$, the moduli space of stable curves. Gillet [Gi] and Vistoli [Vi1] constructed intersection products on quotients in arbitrary characteristic, provided that the stabilizers of geometric points are reduced. (In characteristic 0 this condition is automatic, but it can fail in positive characteristic.) In characteristic 0, Gillet ([Gi, Thm 9.3]) showed that his product on $\bar{\mathscr{M}}_g$ agreed with Mumford's, and in [Ed, Lemma 1.1] it was shown that Vistoli's product also agreed with Mumford's. If the stabilizers are reduced, we show that our product agrees with Gillet's and Vistoli's (Proposition 15). Hence, Gillet's product and Vistoli's agree for quotient stacks, answering a question in [Vi1]. Moreover, Theorem 4 does not require that the stabilizers be reduced and is therefore true in arbitrary characteristic, answering [Vi1, Conjecture 6.6] affirmatively for moduli spaces of quotient stacks.

(6) Equivariant intersection theory gives a nice way of working with cycles on a singular moduli space $\mathscr{M}$ which is a quotient X/G of a smooth variety by a group acting with finite stabilizer. Given a family $Y \xrightarrow{p} B$ of schemes parameterized by $\mathscr{M}$, there is a map $B \xrightarrow{f} \mathscr{M}$. For a subvariety $W \subset \mathscr{M}$ we wish to define a class $f^*([W]) \in A_*B$ corresponding to how the image of B intersects W. This can be done (after tensoring with $\mathbf{Q}$) using equivariant theory.

By Theorem 3, there is an isomorphism $A_*(\mathscr{M})_{\mathbf{Q}} = A_*^G(X)$ which takes $[W]$ to the equivariant class $w = \frac{e_W}{i_W}[f^{-1}W]_G$ (where e_W and i_W are defined in Sect. 4.2 below). Let $E \to B$ be the principal G-bundle $B \times_{[X/G]} X$ (the fiber product is a scheme, although the product is taken over the quotient stack $[X/G]$; typically, E is the structure bundle of a projective bundle $\mathbf{P}(p_*L)$ for a relatively very ample line bundle L on Y). Since X is smooth, there is an equivariant pull-back $f_G^* : A_*^G(X) \to A_*^G(E)$ of the induced map $E \xrightarrow{f_G} X$, so we can define a class $f_G^*[W] \in A_*^G(E)$. Identifying $A_*^G(E)$ with $A_*(B)$ we obtain our class $f^*[W]$. When $\mathscr{M} = \bar{\mathscr{M}}_g$ is the moduli space of stable curves of genus g, these methods can be used to rederive formulas of [Ed, Sect. 3] for intersections with various nodal loci.

4.2 Preliminaries

This section contains some results about quotients that will be used in proving Theorem 3. The reader may wish to read the proofs after the proof of Theorem 3.

Let G act locally properly on X with quotient $X \xrightarrow{\pi} Y$. The field extension $K(Y) \subset K(X)^G$ is purely inseparable by [Bo, p. 43], and thus finite because both $K(Y)$ and $K(X)^G$ are intermediate extensions of $k \subset K(X)$ and $K(X)$ is a finitely generated extension of k. Set $i_X = [K(X)^G : K(Y)]$.

Write e_X for the order of the stabilizer at a general point of X. This is the degree of the finite map $S(\mathrm{id}_X) \to X$ where $S(\mathrm{id}_X)$ is the stabilizer of the identity morphism as defined in [GIT, Definition 0.4]. Note that the map $S(\mathrm{id}_X) \to X$ can be totally ramified. This occurs exactly when the stabilizer of a general geometric point is non-reduced. Finally, set $\alpha_X = \frac{e_X}{i_X}$.

Lemma 5 *Let $K = K(Y)$ be the ground field, and suppose $\pi : X \to Y = \mathrm{Spec}\, K$ is a quotient of a variety X by a group G over K. Then π factors as $X \to \mathrm{Spec}\, K(X)^G \to \mathrm{Spec}\, K(Y)$.*

Proof. First, X is normal. To see this, let $Z \subset X$ be the set of non-normal points, a proper G-invariant subset of X. If L is an algebraically closed field containing K, write $X_L = X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$, $G_L = G \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L$. Now, Z_L is a proper G-invariant subset of X_L. Since X_L is a single G_L-orbit, Z_L is empty. The map $X_L \to X$ is surjective, by the going up theorem; hence Z is empty, so X is normal.

Now if $\mathrm{Spec}\, A \subset X$ is an open affine subset, then $K(Y) \subset A$, $K(X)^G$ is integral over $K(Y)$, and A is integrally closed in $K(X)$. We conclude that $K(X)^G \subset A$, which implies the result. $\square$

Remark. The fact that X_L is a single G_L-orbit is essential to the result. For example, suppose $K = \mathbf{F}_p(t)$, $A = K[u,v]/(u^p - tv^p)$, $X = \mathrm{Spec}\, A$, $G = \mathbf{G}_m$ acting by $g \cdot (u,v) = (gu, gv)$. The geometric points of X_L form two G_L-orbits, since $(X_L)_{\mathrm{red}} = \mathbf{A}^1_L$. The conclusion of the lemma fails since u/v is in $K(X)^G$ but not in A.

Proposition 9 *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \pi' \downarrow & & \pi \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of varieties with Y and Y' quotients and f and g finite and surjective. Then

$$\frac{[K(X') : K(X)]}{[K(Y') : K(Y)]} = \frac{\alpha_X}{\alpha_{X'}} = \frac{e_X}{e_{X'}} \cdot \frac{i_{X'}}{i_X}$$

Proof. Since $Y' \to Y$ is a finite, surjective map of varieties spec $K(Y)^*_Y Y' =$ spec $K(Y')$. To calculate degrees we may base change to spec $K(Y)$. Thus we replace Y' and Y by spec $K(Y')$ and spec $K(Y)$ and X and X' by their corresponding generic fibers. By Lemma 5 we have a commutative diagram of quotients

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ \mathrm{spec}(K(X')^G) & \to & \mathrm{spec}(K(X)^G) \\ \downarrow & & \downarrow \\ \mathrm{spec}(K(Y')) & \to & \mathrm{spec}(K(Y)) \end{array} .$$

Since $i_{X'} := [K(X')^G : K(Y')]$ and $i_X := [K(X)^G : K(Y)]$, it suffices to prove that

$$\frac{[K(X') : K(X)]}{[K(X')^G : K(X)^G]} = \frac{e_X}{e_{X'}} \quad .$$

By [Bo, Prop. 2.4] the extensions $K(X')^G \subset K(X')$ and $K(X)^G \subset K(X)$ are separable (transcendental). Thus, after finite separable base extensions, we may assume that there are sections $s' : \mathrm{spec}(K(X')^G) \to X'$ and $s : \mathrm{spec}(K(X)^G) \to X$.

The section s gives us a finite surjective map $G_K \to X$, where $G_K = G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K(X)^G$. The degree of this map is e_X because $G_K \to X$ is the pullback of the action morphism $G \times X \to X$ via the map $X \to X \times X$ given by $x \mapsto (x, s(\mathrm{Spec}\, K))$. Likewise, $e_{X'} = [K(G_{K'}) \colon K(X)]$, where $G_{K'} = G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K(X')^G$. Therefore,

$$\frac{e_X}{e_{X'}} = \frac{[K(G_K) : K(X)]}{[K(G_{K'}) : K(X')]} = \frac{[K(X') : K(X)]}{[K(X')^G : K(X)^G]} \quad ,$$

since $[K(G_{K'}) : K(G_K)] = [K(X')^G : K(X)^G]$. This completes the proof. $\square$

The following proposition is an analogue of [Vi1, Prop. 2.6] and [Se1, Thm 6.1]. Our proof is similar to Vistoli's.

Proposition 10 *Suppose that G acts locally properly on an algebraic space X. Let $X \to Y$ be a quotient. Then there is a commutative diagram of quotients, with X' a normal algebraic space*:

$$\begin{array}{ccc} X' & \rightarrow & X \\ \downarrow & & \downarrow \\ Y' & \rightarrow & Y \end{array}$$

where $X' \to Y'$ is a principal G-bundle (*in particular G acts with trivial stabilizer on X'*) *and the horizontal maps are finite and surjective.*

Remark It is clear from the proof that we can in fact make X' and Y' schemes. If X and Y are quasi-projective, then so are X' and Y'. If the action on X is proper, then the action on X' is free.

Proof. By [GIT, Lemma, p. 14], there is a finite map $Y' \to Y$, with Y' normal, so that the pull-back $X_1 \xrightarrow{\pi} Y'$ has a section in a neighborhood of every point. Cover Y' by a finite number of open sets $\{U_\alpha\}$ so that $X_1 \to Y'$ has local sections $U_\alpha \xrightarrow{s_\alpha} V_\alpha$ where $V_\alpha = \pi^{-1}(U_\alpha)$.

Define a G-map

$$\phi_\alpha : G \times U_\alpha \to V_\alpha$$

by the Cartesian diagram

$$\begin{array}{ccc} G \times U_\alpha & \xrightarrow{\phi_\alpha} & V_\alpha \\ \downarrow & & {\scriptstyle id \times s_\alpha \circ \pi} \downarrow \\ G \times V_\alpha & \to & V_\alpha \times V_\alpha \end{array} .$$

The action is locally proper so we can, by shrinking V_α, assume that ϕ_α is proper. Since it is also quasi-finite, it is finite.

To construct a principal bundle $X' \to Y'$ we must glue the $G \times U_\alpha$'s along their fiber product over X. To do this we will find isomorphisms $\phi_{\alpha\beta} : s_\alpha(U_{\alpha\beta}) \to s_\beta(U_{\alpha\beta})$ which satisfy the cocycle condition.

For each α, β, let $I_{\alpha\beta}$ be the space which parameterizes isomorphisms of s_α and s_β over $U_{\alpha\beta}$ (i.e. a section $U_{\alpha\beta} \to I_{\alpha\beta}$ corresponds to a global isomorphism $s_\alpha(U_{\alpha\beta}) \to s_\beta(U_{\alpha\beta})$). The space $I_{\alpha\beta}$ is finite over $U_{\alpha\beta}$ (but possibly totally ramified in characteristic p) since it is defined by the cartesian diagram

$$\begin{array}{ccc} I_{\alpha\beta} & \to & U_{\alpha\beta} \\ \downarrow & & {\scriptstyle 1 \times s_\beta} \downarrow \\ G \times U_{\alpha\beta} & \xrightarrow{1 \times \phi_\alpha} & U_{\alpha\beta} \times V_{\alpha\beta} \end{array}$$

(Note that $I_{\alpha\alpha}$ is the stabilizer of $s_\alpha(U_\alpha)$.)

Over $U_{\alpha\beta\gamma}$ there is a composition

$$I_{\alpha\beta} \times_{U_{\alpha\beta\gamma}} I_{\beta\gamma} \to I_{\alpha\gamma}$$

which gives multiplication morphisms which are surjective when $\gamma = \beta$.

After a suitable finite (but possibly inseparable) base change, we may assume that there is a section $U_{\alpha\beta} \to I_{\alpha\beta}$ for every irreducible component of $I_{\alpha\beta}$. (Note that $I_{\alpha\beta}$ need not be reduced.) Fix an open set U_α. For $\beta \neq \alpha$ choose a section $\phi_{\alpha\beta} : U_{\alpha\beta} \to I_{\alpha\beta}$. Since the $I_{\alpha\beta}$'s split completely and $I_{\alpha\alpha}$ is a group over $U_{\alpha\alpha}$ (in the sense of [GIT, Definition 0.1]), there are sections $\phi_{\beta\alpha} : U_{\alpha\beta} \to I_{\beta\alpha}$ so that $\phi_{\alpha\beta} \cdot \phi_{\beta\alpha}$ is the identity section of $U_{\alpha\alpha}$. For any β, γ we can define a section of $I_{\beta\gamma}$ over $U_{\alpha\beta\gamma}$ as the composition $\phi_{\beta\alpha} \cdot \phi_{\alpha\gamma}$. Because $I_{\beta\gamma}$ splits, the $\phi_{\beta\alpha}$'s extend to sections over $U_{\beta\gamma}$.

By construction, the $\phi_{\beta\gamma}$'s satisfy the cocycle condition. We can now define X' by gluing the $G \times U_\beta$'s along the $\phi_{\beta\gamma}$'s. □

4.3 Proof of Theorems 3 and 4

To simplify the notation we give the proofs assuming that G is connected (so the inverse image in X of a subvariety of Y is irreducible). All coefficients, including those of cycle groups, are assumed to be rational.

Proof of Theorem 3. Let Δ^m be the m-simplex of [Bl]. If G acts locally properly on X, then G acts locally properly on $X \times \Delta^m$ by acting trivially on the second factor. In this case, the boundary map of the higher Chow group complex preserves invariant cycles, so there is a sub-complex of invariant cycles $Z_*(X, \cdot)^G$. Set

$$A_*([X/G], m) = H_m(Z_*(X, \cdot)^G, \partial) \ .$$

This construction is well defined even when X is an arbitrary algebraic space.

Now if $X \to Y$ is a quotient, then so is $X \times \Delta^m \xrightarrow{\pi} Y \times \Delta^m$. Define a map $\pi^* : Z_k(X, m) \otimes \mathbf{Q} \to Z_{k+g}(X, m)^G \otimes \mathbf{Q}$ for all m as follows. If $F \subset Y \times \Delta^m$ is a $k + m$-dimensional subvariety intersecting the faces properly, then $H = (\pi^{-1}F)_{\mathrm{red}}$ is a G-invariant $(k + m + g)$-dimensional subvariety of $X \times \Delta^m$ which intersects the faces properly. Thus, $[H] \in Z_{k+g}(X, m)^G$. Set $\pi^*[F] = \alpha_H[H] \in Z^G_{k+g}(X, m)$, where $\alpha_H = e_H/i_H$ is defined as above. Since G-invariant subvarieties of $X \times \Delta^m$ correspond exactly to subvarieties of $Y \times \Delta^m$, π^* is an isomorphism of cycles for all m.

This pull-back has good functorial properties:

Proposition 11 (a) *If*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \pi' \downarrow & & \pi \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a commutative diagram of quotients with f and g proper, then $f_*\pi'^* = \pi^* g_*$ as maps $Z(Y', m) \to Z_*(X, m)^G$.

(b) *Suppose $T \subset X$ is a G-invariant subvariety. Let $S \subset Y$ be its image under the quotient map. Set $U = X - T$ and $V = U/G$, so there is a diagram of quotients*:

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ \pi \downarrow & & \pi \downarrow \\ V & \xrightarrow{j} & Y \ . \end{array}$$

Then $\pi^* j^* = j^* \pi^*$ as maps from $Z_k(Y, m)$ to $Z^G_{k+g}(U, m)$.

Proof. Part (a) follows immediately from Proposition 9. For (b), if $\alpha = [F]$ and $H = \pi^{-1}(F)_{\mathrm{red}}$, then $\pi^* j^* \alpha$ and $j^* \pi^* \alpha$ are both multiples of $[H \cap U]$. Since $e_{[H \cap U]} = e_{[H]}$, and $i_{[H \cap U]} = i_{[H]}$, the multiples are the same, proving (b). $\square$

Proposition 12 (a) *The map π^* commutes with the boundary operator defining higher Chow groups. In particular, there is an induced isomorphism of Chow groups*

$$A_k(Y,m) \simeq A_{k+g}([X/G],m)$$

(*note that the higher Chow groups $A_k(Y,m)$ are defined as groups even if Y is only an algebraic space*).

(b) *In the setting of Proposition* 11(*b*), *if X is quasi-projective with a linearized G-action, and the quotient Y is quasi-projective, then there is a commutative diagram of higher Chow groups*

$$\begin{array}{ccccccccc} \cdots \to & A_*([T/G],m) & \to & A_*([X/G],m) & \to & A_*([U/G],m) & \to \cdots \\ & \uparrow & & \uparrow & & \uparrow & \\ \cdots \to & A_*(S,m) & \to & A_*(Y,m) & \to & A_*(V,m) & \to \cdots \end{array}$$

where the vertical maps are isomorphisms. Hence the top row of this diagram is exact.

Proof. To prove (a), since π^* is an isomorphism on the level of cycles, once we show that π^* commutes with the boundary operator, it will follow that the induced map on Chow groups is an isomorphism. If

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \pi' \downarrow & & \pi \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a commutative diagram of quotients with f and g finite and surjective, then f_* and g_* are surjective as maps of cycles. By Proposition 11(a) it suffices to prove $\pi'^* : Z_*(Y') \to Z_*(X')^G$ commutes with ∂. By Proposition 10, there exists such a commutative diagram of quotients with $\pi' : X' \to Y'$ a principal bundle. Since π' is flat, π'^* commutes with ∂. This proves (a). Part (b) follows from (a), Proposition 11 and the localization theorem for higher Chow groups. □

Define a map $\alpha : Z_*([X/G],m) \to Z_*(X_G,m)$ by $[F] \mapsto [F]_G$. This map commutes with proper push-forward and flat pull-back. Arguing as in Proposition 12(a), we see that α commutes with the boundary operator defining higher Chow groups and hence induces a map on Chow groups (again denoted α).

The proof of Theorem 3 is completed by the following proposition.

Proposition 13 (a) *If X is an algebraic space with a* (*locally*) *proper G-action, then the map $\alpha : A_*([X/G]) \to A_*^G(X)$ is an isomorphism.*

(b) *If X is quasi-projective with a proper linearized G-action and a quasi-projective quotient $X \to Y$ exists, then $\alpha : A_*([X/G],m) \to A_*^G(X,m)$ is an isomorphism for $m \geqslant 0$.*

Proof of (*a*). Let Y be the quotient of X by G. We will prove that the composition $\alpha \circ \pi^* : A_*(Y) \to A_*^G(X)$ is an isomorphism. By Proposition 10 there is a commutative diagram of quotients with g and f finite and surjective

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \pi' \downarrow & & \pi \downarrow \\ Y' & \xrightarrow{f} & Y \end{array}$$

such that $X' \to Y'$ is a principal bundle.

Since $X' \to X$ and $Y' \to Y$ are finite and surjective, [Ki, Theorem 1.8] (which extends to the equivariant setting) says that there are exact sequences

$$\begin{array}{c} A_*^G(X' \times_X X') \xrightarrow{g_{1*}-g_{2*}} A_*^G(X') \xrightarrow{g_*} A_*^G(X) \to 0 \\ A_*^G(Y' \times_Y Y') \xrightarrow{f_{1*}-f_{2*}} A_*^G(Y') \xrightarrow{f_*} A_*^G(Y) \to 0 \end{array}$$

where g_i and f_i are the projections to X' and Y'. Set $X'' = X' \times_X X'$, and set $Y'' = X''/G$. The natural map $p : Y'' \to (Y' \times_Y Y')$ is finite and surjective, so the push-forward $A_*(Y'') \to A_*(Y' \times_Y Y')$ is a surjection. Hence the second sequence remains exact if we replace $Y' \times_Y Y'$ by Y'' (and f_i by $f_i \circ p$).

We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} A_*^G(X'') & \to & A_*^G(X') & \to & A_*^G(X) & \to 0 \\ \uparrow & & \uparrow & & \uparrow & \\ A_*(Y'') & \to & A_*(Y') & \to & A_*(Y) & \to 0 \end{array},$$

where the vertical maps are $\pi''^* \circ \alpha''$, $\pi'^* \circ \alpha'$, and $\pi \circ \alpha$, respectively. By Proposition 8, the first two maps are isomorphisms, so by the 5-lemma, the third is as well.

Proof of (*b*). We are going to use the localization exact sequences for higher equivariant Chow groups, and for the invariant Chow groups $A_*([X/G], m)$ (Proposition 12(b)). This is why we must assume that X is quasi-projective and that a quasi-projective quotient exists.

By Proposition 10, we can find a finite surjective map $g : X' \to X$, where G acts freely on X'. The map $\alpha' : A_*([X'/G], m) \to A_*^G(X', m)$ is an isomorphism, as noted in (a). By Noetherian induction and the localization long exact sequences it suffices to prove the result when X is replaced by the open subset over which g is flat, so assume this. Because g is also finite, $g_* g^*$ is multiplication by the degree of g. Hence (since we are using rational coefficients) the flat pull-back g^* makes $A_*^G(X, m)$ a summand in $A_*^G(X', m)$ and $A_*([X/G], m)$ a summand in $A_*([X'/G], m)$. Since $\alpha' \circ g^* = g^* \circ \alpha$, the summand $A_*^G([X/G], m) \subset A_*^G([X'/G], m)$ is isomorphic to the summand $A_*^G(X, m) \subset A_*^G([X/G], m)$. This proves the proposition, and with it Theorem 3. □

Proof of Theorem 4 The proof is similar to [Vi1, Proposition 6.1]. Let $\pi : X \to Y$ be the quotient map. We define a pull-back $\pi^* : A^*(Y) \to A^*_G(X)$ as follows: Suppose $c \in A^i(Y)$, $Z \to X$ is a G-equivariant morphism, and $z \in A^G_*(Z)$. For any representation, there are maps $Z_G \to X_G \to Y$. The class z is represented by a class $z_U \in A_{*+l-g}(Z_G)$ for some mixed space $Z^G_* U$. Define

$$\pi^* c \cap z = c \cap z_U \in A_{*+l-g-i}(Z_G) \simeq A^G_{*-i}(Z)$$

As usual, this definition is independent of the representation, so $\pi^* c \cap \alpha \in A^G_*(Z)$.

Let $\hat{\pi} : A_*(Y) \to A^G_{*+g}(X)$ denote the isomorphism of Theorem 3.

Lemma 6 *If* $c \in A^*(Y)$ *and* $y \in A_*(Y)$, *then*

$$\hat{\pi}(c \cap y) = \pi^* c \cap \hat{\pi} y \ .$$

Proof. If $X \to Y$ is a principal G-bundle, the lemma holds since the map $\hat{\pi} : A_*(Y) \to A^G_{*+g}(X) \simeq A_{*+g}(X_G)$ is just the pull-back induced by flat map $X_G \to Y$, and the operations in $A^*(Y)$ are compatible with flat pull-back. For an arbitrary quotient $X \to Y$, consider a commutative diagram as in Proposition 10, and pick $y' \in A_*(Y')$ with $f_*(y') = y$. Then $\hat{\pi}(c \cap y) = \hat{\pi} f_*(f^* c \cap y')$. By Proposition 11, this equals $g_* \hat{\pi}'(f^* c \cap y')$. Since $X' \to Y'$ is a principal bundle, this equals $g_*(\pi'^* f^* c \cap \hat{\pi}' y')$, which in turn equals

$$g_*(g^* \pi^* c \cap \hat{\pi}' y') = \pi^* c \cap g_* \hat{\pi}' y' = \pi^* c \cap \hat{\pi} f_* y' = \pi^* c \cap \hat{\pi} y \ .$$

This proves the lemma. □

We now show that π^* is an isomorphism. For injectivity, it suffices (by base change) to show that if $\pi^* c = 0$ then $c \cap y = 0$ for all $y \in A_*(Y)$; this holds since $\hat{\pi}(c \cap y) = 0$ by the lemma, and $\hat{\pi}$ is an isomorphism.

The proof of surjectivity is more difficult. Given $d \in A^*_G(X)$ define $c \in A^*(Y)$ as follows: If $Y' \to Y$ and $y' \in A_*(Y')$, set

$$c \cap y' = \hat{\pi}'^{-1}(d \cap \hat{\pi}' y) \ ,$$

where $\pi' : X' = X \times_Y Y' \to Y'$ is the pull-back quotient. We must show that

$$\pi^* c = d \ . \tag{1}$$

We begin with a preliminary construction. For any mixed space X_G we define a map $r : A^i_G(X) \to A^i(X_G)$. If $c \in A^*_g(X)$, $Z \to X_G$, and $z \in A_*(Z)$, define $r(c) \cap z$ as follows. Let $Z_U \to Z$ be the pull-back of the principal G-

bundle, $X \times U \to X_G$. Since $Z_U \to Z$ is a principal bundle, we identify $A_*(Z)$ with $A^G_{*+g}(Z_U)$. Let $z_U \in A^G_*(Z_U)$ correspond to $z \in A_*(Z)$, and define $r(c) \cap z$ to be the class corresponding to $c \cap z_U$.

This construction has the following property. If $X' \to X$ is equivariant, it induces a map $X'_G \to X_G$. Claim: If $x' \in A^G_*(X')$ corresponds to $x'_G \in A_*(X'_G)$, then $c \cap x'$ corresponds to $r(c) \cap x'_G$. The reason is that $(X'_G)_U \cong UxX' \xrightarrow{p_2} X'$, so via p_2^* we identify $A^G_*((X'_G)_U)$ with $A^G_*(X')$. Thus we have identifications

$$\begin{array}{ccccc} A^G_*(X') & \xrightarrow{\simeq} & A^G_*((X'_G)_U) & \xrightarrow{\simeq} & A_*(X'_G) \\ x' & \mapsto & p_2^*(x') & \mapsto & x'_G \ . \end{array}$$

Hence we have identifications

$$c \cap x' \mapsto c \cap p_2^*(x') \mapsto r(c) \cap x'_G \ ,$$

where the second is by definition, and the first because $c \cap p_2^*(x') = p_2^*(c \cap x')$. This proves the claim.

We can now prove (1). We must show that if $Z \to X_G$ and $z \in A_*(Z)$, then $\pi^* c \cap z = d \cap c$. We have $Z_G \to X_G \to Y$; by definition of π^*, if we represent z by $z_G \in A_*(Z_G)$ then $\pi^* c \cap z$ is represented by $c \cap z_G$. Consider the diagram

$$\begin{array}{ccc} Z_G \times_Y X = X' & \to & X \\ \downarrow \pi' & & \downarrow \pi \\ Z_G = Y' & \to & Y \ . \end{array}$$

We want to compare the class $c \cap z_U$ with the class on Z_G representing the equivariant class $d \cap z$. We will compare their pull-backs under the isomorphism $\hat{\pi}'$. By definition of $c, \hat{\pi}'(c \cap z_G) = d \cap \hat{\pi}'(z_G) \in A^G_*(Z_G \times_Y X)$. On the other hand, $d \cap z$ is represented by $r(d) \cap z_G$; so $\hat{\pi}'(r(d) \cap z_G) = (\pi'^* r(d)) \cap \hat{\pi}'(z_G)$. Thus, we need to show

$$d \cap \hat{\pi}'(z_G) = (\pi'^* r(d)) \cap \hat{\pi}'(z_G) \ .$$

This is proved as in the claim above; we omit details. This proves (1). We conclude that π^* is an isomorphism, as desired.

To complete the proof of the theorem, recall that $\hat{\pi}(c \cap y) = \pi^* c \cap \hat{\pi} y$. By Proposition 4 the map $A^*_G(X) \xrightarrow{\cap [X]_G} A^G_*(X)$ is an isomorphism (with **Z** coefficients). Since $\hat{\pi}([Y]) = \alpha_X [X]_G$, the map $c \mapsto c \cap [Y]$ is an isomorphism (with **Q** coefficients). □

5 Intersection theory on quotient stacks and their moduli

If G acts on an algebraic space X, a quotient $[X/G]$ exists in the category of stacks ([D-M, Example 4.8]; see below). This section relates equivariant Chow groups to Chow groups of quotient stacks.

We show that for proper actions, with rational coefficients, equivariant Chow groups coincide with the Chow groups defined by Gillet in terms of integral sub-stacks. Thus, in this case the intersection products of Gillet and Vistoli are the same. For an arbitrary action, the equivariant Chow groups are an invariant of the quotient stack. As an application, we calculate the Chow rings of the moduli stacks of smooth and stable elliptic curves.

5.1 Definition of quotient stacks

We recall the definition of quotient stack. For an introduction to stacks see [D-M], [Vi1]. Let G be a linear algebraic group acting on a scheme (or algebraic space). The quotient stack $[X/G]$ is the Artin stack associated to the groupoid $G \times X \;\rightarrow\; X \times X$. If B is a scheme, sections of $[X/G](B)$ are principal G-bundles $E \rightarrow B$ together with an equivariant map $E \rightarrow X$. Morphisms in $[X/G](B)$ are G-bundle isomorphisms which preserve the map to X. In particular, if G acts on X with trivial stabilizers then all morphisms are trivial, $[X/G]$ is a sheaf in the étale topology and the quotient is in fact an algebraic space (Proposition 22).

If G acts with finite reduced stabilizers then the diagonal $[X/G] \rightarrow [X/G] \times [X/G]$ is unramified and $[X/G]$ is a Deligne-Mumford stack. If G acts (locally) properly then the diagonal $[X/G] \rightarrow [X/G] \times [X/G]$ is (locally) proper and the stack is (locally) separated. In characteristic 0 any (locally) separated stack is Deligne-Mumford. However, in characteristic p this need not be true since the stabilizers of geometric points can be non-reduced.

5.2 Quotient stacks for proper actions

If G acts locally properly on X with reduced stabilizers then the quotient stack $[X/G]$ is locally separated and Deligne-Mumford. The rational Chow groups $A_*([X/G]) \otimes \mathbf{Q}$ were first defined by Gillet [Gi] and coincide with the groups $A_*([X/G]) \otimes \mathbf{Q}$ defined above. More generally, if G acts with finite stabilizers which are not reduced, then Gillet's definition can be extended and we can define the "naive" Chow groups $A_k([X/G])_{\mathbf{Q}}$ as the group generated by k-dimensional integral substacks modulo rational equivalences. In this context, Proposition 13 can be restated in the language of stacks as

Proposition 14 *Let G be a g-dimensional group which acts locally properly on an algebraic space X (so the quotient $[X/G]$ is a locally separated Artin stack). Then $A_i^G(X) \otimes \mathbf{Q} = A_{i-g}([X/G]) \otimes \mathbf{Q}$.* $\square$

Remark Although $A_*^G(X) \otimes \mathbf{Q} = A_*([X/G]) \otimes \mathbf{Q}$, the integral Chow groups may have non-zero torsion for all $i < \dim X$.

With the identification of $A_*^G(X) \otimes \mathbf{Q}$ and $A_*([X/G]) \otimes \mathbf{Q}$ there are three intersection products on the rational Chow groups of a smooth Deligne-Mumford quotient stack – the equivariant product, Vistoli's product defined using a Gysin pull-back for regular embeddings of stacks, and Gillet's product defined using the product in higher K-theory. The next proposition shows that they are identical.

Proposition 15 *If X is smooth and $[X/G]$ is a locally separated Deligne-Mumford stack (so G acts with finite, reduced stabilizers) then the intersection products on $A_*([X/G])_{\mathbf{Q}}$ defined by Vistoli and Gillet are the same as the equivariant product on $A_*^G(X)_{\mathbf{Q}}$.*

Proof. If V is an l-dimensional representation, then all three products agree on the smooth quotient space $(X \times U)/G$ [Vi1], [Gr]. Since the flat pull-back of stacks $f : A^*([X/G])_{\mathbf{Q}} \to A^*((X \times U)/G)_{\mathbf{Q}}$ commutes with all 3 products, and is an isomorphism up to arbitrarily high codimension, the proposition follows. $\square$

5.3 Integral Chow groups of quotient stacks

Suppose G acts arbitrarily on X. Consider the (possibly non-separated Artin) quotient stack $[X/G]$. The next proposition shows that the equivariant Chow groups do not depend on the presentation as a quotient, so they are an invariant of the stack.

Proposition 16 *Suppose that $[X/G] \simeq [Y/H]$ as quotient stacks. Then $A_{i+g}^G(X) \simeq A_{i+h}^H(Y)$, where* $\dim G = g$ and $\dim H = h$.

Proof. Let V_1 be an l-dimensional representation of G, and V_2 an M dimensional representation of H. Let $X_G = X \times^G U_1$ and $Y_H = X \times^H U_2$, where U_1 (resp. U_2) is an open set on which G (resp. H) acts freely. Since the diagonal of a quotient stack is representable, the fiber product $Z = X_G \times_{[X/G]} Y_H$ is an algebraic space. This space is a bundle over X_G and Y_H with fiber U_2 and U_1 respectively. Thus, $A_{i+l}(X_G) = A_{i+l+m}(Z) = A_{i+m}(Y_H)$ and the proposition follows. $\square$

As a consequence of Proposition 16 we can define the integral Chow groups of a quotient stack $\mathscr{F} = [X/G]$ by $A_i(\mathscr{F}) = A_{i-g}^G(X)$ where $g = \dim G$.

Proposition 17 *If $\mathscr{F}$ is smooth, then $\oplus A_*(\mathscr{F})$ has an integral ring structure.* $\square$

Following [Mu, Definition, p. 64] we define the Picard group $\mathrm{Pic}_{\mathrm{fun}}(\mathscr{F})$ of an algebraic stack $\mathscr{F}$ as follows. A element $\mathscr{L} \in \mathrm{Pic}_{\mathrm{fun}}(\mathscr{F})$ assigns to any map $S \xrightarrow{F} \mathscr{F}$ of a scheme S, an isomorphism class of a line bundle $L(F)$ on S. Moreover the assignment must satisfy the following compatibility conditions.

(i) Let $S_1 \xrightarrow{F_1} \mathscr{F}$, $S_2 \xrightarrow{F_2} \mathscr{F}$ and $S_2 \xrightarrow{F_2} \mathscr{F}$ be maps of schemes to $\mathscr{F}$. If there is a map $S_1 \xrightarrow{f} S_2$ such that $F_1 = F_2 \circ f$ then there is an isomorphism $\phi(f) : L(F_1) \simeq f^*(L(F_2))$.

(ii) If $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$ are maps of schemes such that $F_2 = F_3 \circ g$ and $F_1 = F_2 \circ f$ then there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} L(F_1) & \overset{\phi(F_1)}{\rightarrow} & f^*(L(F_2)) \\ \phi(g \circ f)\Big\downarrow & & f^*\phi(g)\Big\downarrow \\ (g \circ f)^*(L(F_3)) & = & f^*(g^*(L(F_3))). \end{array}$$

The product $\mathscr{L} \otimes \mathscr{M}$ assigns to $S \xrightarrow{f} \mathscr{F}$ the line bundle $L_f \otimes M_f$.

Proposition 18 *Let X be a smooth variety with a G-action. Then $A^1_G(X) = \mathrm{Pic}_{\mathrm{fun}}([X/G])$.*

Proof. By Theorem 1, $A^1_G(X) = \mathrm{Pic}^G(X)$. The latter group is naturally isomorphic to $\mathrm{Pic}_{\mathrm{fun}}([X/G])$. $\square$

More generally, if $\mathscr{F}$ is any stack, we can define the integral operational Chow ring $A^*(\mathscr{F})$ as follows. An element $c \in A^k(\mathscr{F})$ defines an operation $A_*B \xrightarrow{c_f} A_{*-k}B$ for any map of a scheme $B \xrightarrow{f} \mathscr{F}$. The operations should be compatible with proper push-forward, flat pull-back and intersection products for maps of schemes to $\mathscr{F}$ (cf. [Fu, Definition 17.1] and [Vi1, Definition 5.1]). (This definition differs slightly from Vistoli's because we use integer coefficients and only consider compatibility with maps of schemes.)

Proposition 19 *Let $\mathscr{F} \simeq [X/G]$ be a smooth quotient stack. Then $A^*(\mathscr{F}) = A^*_G(X)$.*

Proof. Giving a map $B \xrightarrow{f} [X/G]$ is equivalent to giving a principal G-bundle $E \to B$ together with an equivariant map $E \to X$. An element of $A^*_G(X)$ defines an operation on $A^G_*(E) = A_*(B)$, hence an operational class in $A^*(\mathscr{F})$. Conversely an operational class $c \in A^k(\mathscr{F})$ defines an operation on $A_*(X_G)$ corresponding to the map $X_G \to \mathscr{F}$ associated to the principal bundle $X \times U \to X_G$. Set $d = c \cap [X_G] \in A^G_{\dim X - k}(X)$. Since X is smooth, the latter group is isomorphic to $A^k_G(X)$. $\square$

Remark. Proposition 16 suggests that there should be a notion of Chow groups of an arbitrary algebraic stack which can be nonzero in arbitrarily high degree. This situation would be analogous to the cohomology of quasi-coherent sheaves on the étale (or flat) site (cf. [D-M, p. 101]).

5.4 The Chow ring of the moduli stack of elliptic curves

In this section we will work over a field of characteristic not equal to 2 or 3, and compute the Chow ring of the moduli stacks $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ of elliptic curves. A. Vistoli has independently obtained the results of this section, also using equivariant intersection theory. He has also calculated the Chow ring of $\mathscr{M}_2$, the moduli stack of elliptic curves. This calculation will appear as an appendix to this article [Vi3].

Construction of the moduli stack The stacks $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ are defined as follows. A section of $\mathscr{M}_{1,1}$ (resp. $\bar{\mathscr{M}}_{1,1}$) over S is the data $(C \xrightarrow{\pi} S, \sigma)$ where $C \to S$ is a smooth (resp. possibly nodal) curve of genus 1 and $\sigma : S \to C$ is a smooth section.

Our construction of $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ is similar to [Mu, Sect. 4]. Let $\mathbf{P}(V) = \mathbf{P}^9$ be the projective space of homogeneous degree 3 forms in variables x, y, z. Let $X \simeq \mathbf{A}^3 \subset \mathbf{P}(V)$ be the affine subspace parameterizing forms proportional to

$$y^2z - (x^3 + e_1x^2z + e_2xz^2 + e_3z^3) \ ,$$

with e_1, e_2, e_3 arbitrary elements of the field k. Let $G = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix} \middle| a^3 = c^2d \neq 0 \right\}$. The image of G in PGL(3) consists of projective transformations which stabilize X.

Since $a \neq 0$ for all $g \in G$, we can normalize and identify G with the subgroup $\left\{ \begin{pmatrix} 1 & 0 & B \\ 0 & A & 0 \\ 0 & 0 & A^{-2} \end{pmatrix} \middle| A \neq 0 \right\}$ of $GL(3)$. An element $g = \begin{pmatrix} 1 & 0 & B \\ 0 & A & 0 \\ 0 & 0 & A^{-2} \end{pmatrix}$ acts on (e_1, e_2, e_3) by

$$\begin{aligned}(e_1, e_2, e_3) \mapsto (&A^{-2}e_1 + 3B,\ A^{-4}e_2 + 2A^{-2}Be_1 + 3B^2, \\ &A^{-6}e_3 + A^{-4}Be_2 + A^{-2}B^2e_1 + B^3) \ .\end{aligned}$$

Let $U \subset X$ be the open set where the polynomial $(x^3 + e_1x^2 + e_2x + e_3)$ has distinct roots over the algebraic closure of the field of definition.

Likewise, let $W \subset X$ be the open set where $x^3 + e_1x^2 + e_2x$ has at least 2 distinct roots.

Proposition 20 (a) $\mathscr{M}_{1,1} \simeq [U/G]$. (b) $\bar{\mathscr{M}}_{1,1} \simeq [W/G]$.

Proof of (*a*). Let $Y^\bullet$ be the functor such that a section of $Y^\cdot$ over S' is a triple $(X' \xrightarrow{f'} S', \sigma', \phi')$, where (X', σ') is an elliptic curve with section σ' and ϕ' is an isomorphism of the flag $\mathscr{O}_{S'} \subset \mathscr{O}_{S'}^{\oplus 2} \subset \mathscr{O}_{S'}^{\oplus 3}$ with the flag $f'_*(\mathscr{O}_{X'}(\sigma)) \subset f'_*(\mathscr{O}_{X'}(\sigma)^{\otimes 2}) \subset f'_*(\mathscr{O}_{X'}(\sigma)^{\otimes 3})$. This functor is represented by the scheme Y parameterizing cubic forms in 3 variables with a flex at $(1 : 0 : 0)$, since there is a universal triple $(\mathscr{X} \xrightarrow{F} Y, \sigma_Y, \Phi)$ on Y. There is an obvious action of the group $B \subset GL(3)$ of upper triangular matrices on Y. Claim: The quotient stack $[Y/B]$ is $\mathscr{M}_{1,1}$. Proof: Given a family $X \xrightarrow{f} S$ of elliptic curves, there is a canonical principal B-bundle $S' \to S$ associated to the flag of vector bundles $f_*(\mathscr{O}_X(\sigma)) \subset f_*(\mathscr{O}_X(\sigma)^{\otimes 2}) \subset f_*(\mathscr{O}_X(\sigma)^{\otimes 3})$. By construction, the pull-back of this flag to S' is equipped with an isomorphism with the standard flag in $\mathscr{O}_{S'}^{\oplus 3}$. Thus from a family of elliptic curves over S, we obtain a principal B-bundle $S' \to S$ together with an equivariant map $S' \to Y$, i.e., a section of $[Y/B]$ over S. The converse is similar.

Thus, it suffices to prove that $[U/G] \simeq [Y/B]$. Let Y' be the quotient of Y by the natural $\mathbf{G}_m$-action. Then $[Y/B] = [Y'/B']$, where B' is the group of upper triangular matrices in PGL(3). Identify Y' with the locally closed subscheme of $\mathbf{P}^9$ corresponding to cubics with a flex at $(1 : 0 : 0)$. Identifying $G \subset B'$, the inclusion $U \subset Y'$ is G-equivariant for the corresponding actions of G and B'. Thus we obtain smooth (representable) morphisms of quotient stacks

$$[U/G] \to [Y'/G] \to [Y'/B'] \simeq \mathscr{M}_{1,1} \quad .$$

Two curves in U are isomorphic iff they are in the same G-orbit, so the map $[U/G] \to \mathscr{M}_{1,1}$ is quasi-finite. Moreover, every elliptic curve can be embedded in $\mathbf{P}^2$ as a double cover of $\mathbf{P}^1$ branched at ∞ and 3 finite points. Thus the map is surjective. Finally, a direct check shows that the stabilizer of the G-action on a point of U is exactly the automorphism group of the corresponding elliptic curve. Thus the geometric fibers are single points. Hence, the map is étale, surjective of degree 1, and thus an isomorphism of stacks.

The proof of (b) is essentially identical. □

We now compute the Chow ring of this stack.

Proposition 21 (a) $A^*(\mathscr{M}_{1,1}) = \mathbf{Z}[t]/12t$. (b) $A^*(\bar{\mathscr{M}}_{1,1}) = \mathbf{Z}[t]/24t^2$.

Proof. (a) By definition, $A^*(\mathscr{M}_{1,1}) = A^*_G(U)$. The equivariant Chow ring is not hard to calculate. Let $T \subset G$ be the diagonal maximal torus. Then G is a unipotent extension of T, so $A^*_T(U) = A^*_G(U)$. Now T acts diagonally on $X = \{(e_1, e_2, e_3)\}$ with weights $(-2, -4, -6)$. Let $S = X - U$; then S is the discriminant locus in $\mathbf{A}^3$, which can be identified as the image of the big diagonal (i.e., the image under the S_3 quotient map $A^3 \to X \simeq A^3$ which maps $(a,b,c) \mapsto (a+b+c, ab+bc+ac, abc)$) and has equation $f(e_1,e_2,e_3) = 4e_2^3 + 27e_3^2 - 18e_1e_2e_3 - e_1^2e_2^2 + 4e_1^3e_3$. The form f is homogeneous of weighted degree -12 with respect to the T-action on X. Since $S \subset X$ is a divisor, $A_T^*(U) = A_T^*(X)/([S]_T)$ where $([S]_T)$ denotes the T-equivariant fundamental class of S. Since f has weight 12, $[S]_T = 12t \in A_T^*(X) = Z[t]$. Therefore, $A_T^*(U) = \mathbf{Z}[t]/12t$.

(b) The complement of W in X is the image of the small diagonal $a=b=c$ under the degree 6 map $\mathbf{A}^3 \to \mathbf{A}^3$. The small diagonal has T-equivariant fundamental class $4t^2$, the T fundamental class of $X - W$ is $24t^2$. Hence $A^*(\bar{\mathscr{M}}_{1,1}) = \mathbf{Z}[t]/24t^2$ as claimed. □

Remark From our computation we see $A^1(\mathscr{M}_{1,1}) = \mathrm{Pic}_{\mathrm{fun}}(\mathscr{M}_{1,1}) = \mathbf{Z}/12$, a fact which was originally proved by Mumford [Mu]. With an appropriate sign convention, the class $t \in Z[t]/12t$ is just $c_1(L)$ where L is the generator of $\mathrm{Pic}_{\mathrm{fun}}(\mathscr{M}_{1,1})$ which assigns to a family of elliptic curves $X \xrightarrow{\pi} S$ the line bundle $\pi_*(\omega_{X/S})$ (the Hodge bundle). Thus the monomial at^n corresponds to the class which assigns to a family $X \to S$ of elliptic curves, the class $c_1(L^{\otimes a})^n \cap [S] \in A_*(S)$.

Angelo Vistoli observed that this can be seen directly as follows: The unipotent radical of G acts freely on U (or W) and the quotient is the space of forms $y^2z = x^3 + \alpha xz^2 + \beta z^3$ with no double (or triple) roots. The torus action is given by $t \cdot \alpha = t^{-4}\alpha$ and $t \cdot \beta = t^{-6}\beta$, and the space is the total space of the $\mathbf{G}_m$ bundle over $\mathscr{M}_{1,1}$ (or $\bar{\mathscr{M}}_{1,1}$) corresponding to the Hodge bundle. Thus, the Chow ring of $\mathscr{M}_{1,1}$ (resp. $\bar{\mathscr{M}}_{1,1}$) is generated by the first Chern class of the Hodge bundle.

6 Some technical facts

6.1 Intersection theory on algebraic spaces

Unfortunately, while most results about schemes generalize to algebraic spaces, most references deal exclusively with schemes. In particular, this is the case for [Fu], the basic reference for the intersection theory used in this paper. The purpose of this section is to indicate very briefly how this theory generalizes to algebraic spaces.

We recall from [Kn] the definition of algebraic spaces, and basic facts about them. If X is a scheme, the functor $X^\bullet = \mathrm{Hom}(\cdot, X)$ from $(\mathrm{Schemes})^{\mathrm{opp}}$ to (Sets) is a sheaf in either the Zariski or étale topologies. With this as motivation, an algebraic space is defined to be a functor $A : (\mathrm{Schemes})^{\mathrm{opp}} \to (\mathrm{Sets})$ such that:

(1) A is a sheaf in the étale topology.

(2) (Local representability) There is a scheme U and a sheaf map $U^{\cdot} \to A$ such that for any scheme V with a map $V^\bullet \to A$, the fiber product

(of sheaves) $U^\bullet \times_A V^\bullet$ is represented by a scheme, and the map $U^\bullet \times_A V^\bullet \to V^\bullet$ is induced by an étale surjective map of schemes.

Knutson also imposes a technical hypothesis of quasi-separatedness, which states that the map $U^\bullet \times_A U^\bullet \to U^\bullet \times U^\bullet$ is quasi-compact.

A morphism of algebraic spaces is a natural transformation of functors.

The map $X \mapsto X^{\cdot}$ is a fully faithful embedding of (Schemes) into (Algebraic spaces). We identify X with $X^{\cdot}$ and henceforth use the same notation X for both of these. The scheme X is called a representable étale covering of A (or an étale atlas for A); it can be chosen to be a disjoint union of affine schemes. Thus, just as a scheme has a Zariski covering by affine schemes, an algebraic space has an étale covering by affine schemes.

There are several ways to think of algebraic spaces in relation to schemes. One way is to think of a (normal) algebraic space as a quotient of a scheme by a finite group (see [Ko]). Another is to think of an algebraic space as a quotient of a scheme by an étale equivalence relation. More precisely, in the setting of the above definition, let R denote the scheme $U \times_A U$; then A is a categorical quotient of $R \to U \times U$ [Kn, II.1.3]. Finally, any algebraic space has an open dense subset which is isomorphic to a scheme [Kn, II.6.7].

A key fact of algebraic spaces that we use is:

Proposition 22 *Let X be an algebraic space with a set-theoretically free action of* (smooth group) *G. Then a quotient $X \to Y$ exists in the category of algebraic spaces. Moreover this quotient is a principal bundle.*

Sketch of proof. Consider the functor $Y = [X/G]$ whose sections over B are principal G-bundles $E \to B$ together with an equivariant map $E \to X$. Since principal bundles can be constructed locally in the étale topology, and G acts without stabilizers, one can check that Y is a sheaf in the étale topology (cf. [D-M, Example 4.8]).

To show that Y is an algebraic space, we must construct an étale atlas for Y. This follows from [D-M, Theorem 4.21]. We give a proof below. Since X is an algebraic space and we are working locally in the étale topology we may assume X is a scheme. Consider the surjective map of étale sheaves $X \to Y$. It suffices to show that every closed point $x \in X$ is contained in a locally closed subscheme $Z \subset X$ such that $G \times Z \to X$ and hence $Z \to Y$ is étale.

Let $Gx \simeq G$ be the G orbit of X. Then Gx is the fiber of $X \to Y$ containing x. Let Z be a locally closed subscheme of X defined by lifts to $\mathcal{O}_X$ of the local equations for $x \in Gx$ so that $\dim Z = \dim X - \dim G$ and the scheme theoretic intersection $Z \cap Gx$ is $x \in Gx$. Since G is smooth, the point $x \in G$ is cut out by a regular sequence of length g where $g = \dim G$. Hence, by construction $Z \subset X$ is defined by a regular sequence in a neighborhood of x.

Since G acts without stabilizers, the action map $G \times X \xrightarrow{\Psi} X$ is smooth, since $G \times X \xrightarrow{\mathrm{id} \times \Psi} G \times X$ is an isomorphism. Now $G \times Z \to G \times X$ is a regular embedding in a neighborhood of $(1, x)$. Thus by [EGA4, Theorem 17.12], the map $G \times Z \to X$ is smooth of relative dimension 0, i.e. étale, in a neighborhood of $(1, x)$.

Finally, $X \to Y$ is a principal bundle, since $Y = [X/G]$ and (tautologically) we have $X \times_{[X/G]} X = X \times G$. □

Using representable étale coverings, one can extend basic definitions about schemes to algebraic spaces. Much of this is done in [Kn], where more complete definitions and details can be found. Here are some examples. Any sheaf on the category of schemes (e.g. $\mathcal{O}$) extends uniquely to a sheaf on the category of algebraic spaces: if U is an affine étale covering of the space A, and R is as above, then $\mathcal{O}(A) = \mathrm{Ker}(\mathcal{O}(U) \to \mathcal{O}(R))$. Likewise, a property P of schemes is called stable if given an étale covering $\{X_i \to X\}$, X has P if and only if X_i has P. Any stable property of schemes extends to a property of algebraic spaces by defining it in terms of representable étale coverings. Thus, one can speak of algebraic spaces which are normal, smooth, reduced, n-dimensional, etc. Similarly, if P is a stable property of maps of schemes such that P either (a) is local on the domain, or (b) satisfies effective descent, then P extends to a property of maps of algebraic spaces. For example, one can speak of maps of algebraic spaces which are (a) faithfully flat, flat, étale, universally open, etc., or (b) open immersions, closed immersions, affine or quasi-affine morphisms, etc.

Likewise, again using representable étale coverings, one can extend facts and constructions about schemes, e.g. Proj, fiber products, divisors, etc., to algebraic spaces; again much of this is done in [Kn].

The definition of Chow groups of schemes given in [Fu] generalizes immediately to algebraic spaces. (A similar definition was given for stacks in [Gi].) If X is an algebraic space, define the group of k-cycles $Z_k(X)$ to be the free abelian group generated by integral subspaces of dimension k. To define rational equivalence, first note that if X is an integral algebraic space, then the group of rational functions on X is defined. (Indeed, by the above remarks, X has an open dense subspace X^0 which is a variety, and the rational functions on X are the same as those on X^0.) If Y is an integral subspace of X of codimension 1 and f is a rational function on X, then the order of vanishing of f along Y, denoted $\mathrm{ord}_Y(f)$, can be defined by taking an étale map $\phi : U \to X$, where U is a variety and $\phi(U)$ has nonempty intersection with Y, and setting $\mathrm{ord}_Y(f) = \mathrm{ord}_{\phi^{-1}(Y)}(\phi^* f)$, where the right hand side is the definition for schemes in [Fu]. If W is a $k+1$-dimensional integral subspace of X and f is a rational function on W, define $\mathrm{div}(f) \in Z_k(X)$ to be $\sum \mathrm{ord}_Y(f)[Y]$ where the sum is over all codimension 1 integral subspaces of Y. Then, exactly as in [Fu], define $\mathrm{Rat}_k(X)$ to be the subgroup of $Z_k(X)$ generated by all $\mathrm{div}(f)$, for f and W as above, and define the Chow groups $A_k(X) = Z_k(X)/\mathrm{Rat}_k(X)$.

The arguments of [Fu, Chapters 1–6] can be carried over almost unchanged to show that Chow groups of algebraic spaces have the same functorial properties as Chow groups of schemes.

As an illustration, we will discuss the construction of Gysin homomorphisms for regular embeddings, which is the central construction of the first six chapters of [Fu]. If $X \to Y$ is any closed embedding of algebraic spaces then we can define the cone C_XY as for schemes, since the Spec construction for sheaves (in the étale topology) of $\mathcal{O}_Y$ algebras defines an algebraic space over Y. If $X \to Y$ is a regular embedding of codimension d, then $C_XY = N_XY$ is a vector bundle of rank d. We can then define a specialization homomorphism $Z_k(Y) \to Z_k(C_XY)$ as in [Fu, Sect. 5.2]. The deformation to the normal bundle construction of [Fu, Sect. 5.1] goes through unchanged, since the blow-up $Y \times \mathbf{P}^1$ along the subspace $X \times \infty$ is defined in the category of algebraic spaces. (The existence of blow-ups is a consequence of the Proj construction for graded algebras over algebraic spaces.) Thus as in Fulton, the specialization map passes to rational equivalence.

In particular if $X \to Y$ is a regular embedding of codimension d, then the construction of [Fu, Chap. 6] goes through, and we obtain a (refined) Gysin homomorphism. If X is a (separated) smooth algebraic space, then the diagonal map is a regular embedding. Therefore, the integral Chow groups of X have an intersection product.

Note also that algebraic spaces have an operational Chow ring with the same formal properties as that of [Fu, Chap. 17]. This follows from the fact that the ordinary Chow groups of algebraic spaces have the same functorial properties as Chow groups of schemes. In particular, if X is a smooth algebraic space of dimension n, then $A^i(X) = A_{n-i}(X)$, with the map defined as in [Fu].

Remark For algebraic stacks which have automorphisms, the diagonal is not a regular embedding in the sense we have defined. If the stack is Deligne-Mumford, then the diagonal is a local embedding (i.e. unramified). For such morphisms, Vistoli constructed a Gysin pull-back with $\mathbf{Q}$ coefficients on the Chow groups of integral substacks. To obtain a good intersection theory with $\mathbf{Z}$ coefficients on arbitrary algebraic stacks, a different definition of Chow groups is required. For quotient stacks, equivariant Chow groups give a good definition. This point is discussed in Sect. 5.

6.2 Actions of group schemes over a Dedekind domain

Let R be a Dedekind domain, and set $S = \mathrm{Spec}(R)$. Intersection theory remains valid for schemes and thus algebraic spaces defined over S [Fu, Sect. 20.2]. Thus, the equivariant theory will work for actions of smooth affine group schemes over S, provided we can find finite-dimensional representations of G/S where G acts generically freely. The following lemma shows that this can always be done if the fibers of G/S are connected.

Lemma 7 *Let G/S be a smooth affine group scheme defined over $S = \text{Spec } R$, where R is a Dedekind domain. Then there exists a finitely generated projective S-module E, such that G/S acts freely on an open set $U \subset E$ whose complement has arbitrarily high codimension.*

Proof. By [Se2, Lemma 1] the coordinate ring $R(G)$ is a projective R-module with a G action. The group G embeds into a finitely generated $R(G)$ submodule F. Since R is a Dedekind domain F is also projective. By [Se1, Proposition 3] F is contained in an invariant finitely generated submodule E (which is also projective). Since G acts freely on itself it acts freely on an open of E/S. Replacing E by $E \times_S \cdots \times_S E$ we obtain a representation on which G acts freely on an open set $U \subset E$, such that $E - U$ has arbitrarily high codimension. □

Thus if X/S is an algebraic space over S, we can construct a mixed space $X_G = X \times_G U$, where U is as in the lemma. We then define the i-th equivariant Chow group as $A_{i+l-g}(X_G)$ where $l = \dim(U/S)$ and $g = \dim\,(G/S)$. Since most of the results of intersection theory hold for algebraic spaces over a Dedekind domain, most of the results on equivariant Chow groups also hold, including the following:

(1) The functorial properties with respect to proper, flat and l.c.i maps hold.

(2) If X/S is smooth, there is an intersection product on $A^G_*(X)$ for X/S smooth.

(3) If G acts freely on X with quotient $X \to Y$ then $A^G_{*+g}(X) = A_*(Y)$.

(4) If G/S acts (locally) properly on X/S, then the theorem of [Ko], [K-M] implies that a quotient $X \to Y$ exists as an algebraic space over S. The results of Sect. 4 (for non-higher Chow groups) generalize, and $A^G_{*+g}(X)_{\mathbf{Q}} = A_*(Y)_{\mathbf{Q}}$.

Remark. Facts (3) and (4) imply that any moduli space over Spec $\mathbf{Z}$ which is the quotient of a smooth algebraic space by a proper action has a rational Chow ring.

6.3 Some facts about group actions and quotients

Here we collect some useful results about actions of algebraic groups.

Lemma 8 *Suppose that G acts properly on an algebraic space X. If the stabilizers are trivial, then the action is free.*

Proof. We must show that the action map $G \times X \to X \times X$ is a closed embedding. The properness of the action implies that this map is proper and quasi-finite, hence finite. Since the stabilizers are trivial, the map is unramified so it is an embedding in a neighborhood of every point of $G \times X$. Finally, the map is injective on geometric points, hence an embedding. □

Lemma 9 ([E-G1]) *Let G be an algebraic group. For any $i > 0$, there is a representation V of G and an open set $U \subset V$ such that $V - U$ has codimension more than i and such that a principal bundle quotient $U \to U/G$ exists in the category of schemes.*

Proof. Embed G into $GL(n)$ for some n. Assume that V is a representation of $GL(n)$ and $U \subset V$ is an open set such that a principal bundle quotient $U \to U/GL(n)$ exists. Since $GL(n)$ is special, this principal bundle is locally trivial in the Zariski topology. Thus U is locally isomorphic to $W \times GL(n)$ for some open $W \subset U/GL(n)$. A quotient U/G can be constructed by patching the quotients $W \times GL(n) \to W \times (GL(n)/G)$. (It is well-known that a quotient $GL(n)/G$ exists [Bo].)

We have thus reduced to the case $G = GL(n)$. Let V be the vector space of $n \times p$ matrices with $p > i + n$, and let $U \subset V$ be the open set of matrices of maximal rank. Then $V - U$ has codimension $p - n + 1$ and the quotient U/G is the Grassmannian $\mathrm{Gr}(n, p)$. $\square$

The following proposition gives conditions under which the mixed space X_G is a scheme. Recall that a group is special if every principal bundle is locally trivial in the Zariski topology. The groups $GL(n)$, $SL(n)$, $Sp(2n)$, as well as solvable groups, are special; $\mathrm{PGL}(n)$ and $SO(n)$, as well as finite groups, are not [Sem-Chev].

Proposition 23 *Let G be an algebraic group, let U be a scheme on which G acts freely, and suppose that a principal bundle quotient $U \to U/G$ exists. Let X be a scheme with a G-action. Assume that one of the following conditions holds:*

(1) *X is (quasi)-projective with a linearized G-action, or*

(2) *G is connected and X is equivariantly embedded as a closed subscheme of a normal variety, or*

(3) *G is special.*

Then a principal bundle quotient $X \times U \to X \times^G U$ exists in the category of schemes.

Proof. If X is quasi-projective with a linearized action, then there is an equivariant line bundle on $X \times U$ which is relatively ample for the projection $X \times U \to U$. By [GIT, Prop. 7.1] a principal bundle quotient $X \times^G U$ exists.

Now suppose that X is normal and G is connected. By Sumihiro's theorem [Su], X can be covered by invariant quasi-projective open sets which have a linearized G-action. Thus, by [GIT, Prop. 7.1] we can construct a quotient $X_G = X \times^G U$ by patching the quotients of the quasi-projective open sets in the cover.

If X equivariantly embeds in a normal variety Y, then by the above paragraph a principal bundle quotient $Y \times U \to Y \times^G U$ exists. Since G is affine, the quotient map is affine, and $Y \times U$ can be covered by affine invariant open sets. Since $X \times U$ is an invariant closed subscheme of $Y \times U$, $X \times U$ can also be covered by invariant affines. A quotient $X \times^G U$ can then be constructed by patching the quotients of the invariant affines.

Finally, if G is special, then $U \to U/G$ is a locally trivial bundle in the Zariski topology. Thus $U = \bigcup\{U_\alpha\}$ where $\phi_\alpha : U_\alpha \simeq G \times W_\alpha$ for some open $W_\alpha \subset U/G$. Then $\psi_\alpha : X \times U_\alpha \to X \times W_\alpha$ is a quotient, where ψ_α is defined by the formula $(x, w, g) \mapsto (g^{-1}x, w)$. (Here we assume that G acts on the left on both factors of $X \times U_\alpha$). $\square$

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