4 Separated and Proper Morphisms

Exercise 4.1. *Show that a finite morphism is proper.*

Solution. Let $f : X \to Y$ be the finite morphism. Finite implies finite type so we only need to show that f is universally closed and separated.

f is separated. We want to show that $X \to X \times_Y X$ is a homeomorphism onto a closed subset of $X \times_Y X$. It is enough to show this locally so take an open affine cover $\{V_i = \operatorname{Spec} B_i\}$ of Y. Since f is finite, the preimages of the V_i are also affine, say $U_i = \operatorname{Spec} A_i$. Now $U_i \times_{V_i} U_i$ are open affine subsets of $X \times_Y X$ which cover the image of the diagonal and so it is enough to show that each $\Delta^{-1} U_i \times_{V_i} U_i \to U_i \times_{V_i} U_i$ is a closed immersion. Now the preimages are $\Delta^{-1} U_i \times_{V_i} U_i = U_i$ so we want to show that the scheme morphism induced by $A_i \otimes_{B_i} A_i \to A_i$ is a closed immersion. Since this ring homomorphism is surjective, the result follows from Exercise II.2.18(c).

f is universally closed. The proof of Exercise II.3.13(d) goes through to show that finite morphisms are stable under base change (in fact, the proof becomes easier). Secondly, we know that finite morphisms are closed (Exercise II.3.5) and therefore finite morphisms are universally closed.

Exercise 4.2. *Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced , or (b) Y is nonseparated.*

Solution. Let U be the dense open subset of X on which f and g agree. Consider the pullback square(s):

$$\begin{tikzcd}
U \arrow[r, equal] \arrow[d] & U \arrow[d] \\
Z \arrow[r, "\Delta'"] \arrow[d] & X \arrow[d, "f,g"] \\
Y \arrow[r, "\Delta"] & Y \times_S Y
\end{tikzcd}$$

Since Y is separated, the lower horizontal morphism is a closed immersion. Closed immersions are stable under base extension (Exercise II.3.11) and so $Z \to X$ is also a closed immersion. Now since f and g agree on U, the image of U in $Y \times_S Y$ is contained in the diagonal and so the pullback is, again U (at least topologically. But this means that $U \to X$ factors through Z, whose image is a closed subset of X. Since U is dense, this means that *sp* Z = *sp* X. Since $Z \to X$ is a closed immersion, the morphism of sheaves $\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. Consider an open affine $V = \operatorname{Spec} A$ of X. Restricted to V, the morphism $Z \cap V \to V$ continues to be a closed immersion and so $Z \cap V$ is an affine scheme, homeomorphic to V, determined by an ideal $I \subseteq A$. Since $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a homeomorphism, I is contained in the nilradical. But A is reduced and so $I = 0$. Hence, $Z \cap V = Z$ and therefore $Z = X$.

a Consider the case where $X = Y = \operatorname{Spec} k[x,y]/(x^2, xy)$, the affine line with nilpotents at the origin, and consider the two morphisms $f, g : X \to Y$, one the identity and the other defined by $x \mapsto 0$, i.e. killing the nilpotents at the origin. These agree on the complement of the origin which is a dense open subset but the sheaf morphism disagrees at the origin.

b Consider the affine line with two origins, and let f and g be the two open inclusions of the regular affine line. They agree on the complement of the origin but send the origin two different places.

Exercise 4.3. *Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give a example to show that this fails if X is not separated.*

Solution. Consider the pullback square

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

Since X is separated over S the diagonal is a closed immersion. Closed immersions are stable under change of base (Exercise II.3.11(a)) and so $U \cap V \to U \times_S V$ is a closed immersion. But $U \times_S V$ is affine since all of U, V, S are. So $U \cap V \to U \times_S V$ is a closed immersion into an affine scheme and so $U \cap V$ itself is affine (Exercise II.3.11(b)).

For an example when X is not separated consider the affine plane with two origins X and the two copies U, V of the usually affine plane inside it as open affines. The intersection of U and V is $\mathbb{A}^2 - \{0\}$ which is not affine.

Exercise 4.4. *Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S.*

Solution. Since $Z \to S$ is proper and $Y \to S$ separated it follows from Corollary II.4.8e that $Z \to Y$ is proper. Proper morphisms are closed and so $f(Z)$ is closed.

$f(Z) \to S$ is finite type. This follows from it being a closed subscheme of a scheme Y of finite type over S (Exercise II.3.13(a) and (c)).

$f(Z) \to S$ is separated. This follows from the change of base square and the fact that closed immersions are preserved under change of base.

$$\begin{tikzcd} f(Z) \arrow[r] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\ f(Z) \times_S f(Z) \arrow[r] & Y \times_S Y \end{tikzcd}$$

$f(Z) \to S$ is universally closed. Let $T \to S$ be some other morphism and consider the following diagram

$$\begin{tikzcd} T \times_S Z \arrow[r] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ T \times_S f(Z) \arrow[r] \arrow[d, "s'"] & f(Z) \arrow[d, "s"] \\ T \arrow[r] & S \end{tikzcd}$$

Our first task will be to show that $T \times_S Z \to T \times_S f(Z)$ is surjective. Suppose $x \in T \times_S f(Z)$ is a point with residue field $k(x)$. Following it horizontally we obtain a point $x' \in f(Z)$ with residue field $k(x') \subset k(x)$ and this lifts to a point $x'' \in Z$ with residue field $k(x'') \supset k(x')$. Let k be a field containing both $k(x)$ and $k(x'')$. The inclusions $k(x''), k(x) \subset k$ give morphisms $\operatorname{Spec} k \to T \times_S f(Z)$ and $\operatorname{Spec} k \to Z$ which agree on $f(Z)$ and therefore lift to a morphism $\operatorname{Spec} k \to T \times_S, Z$ giving a point in the preimage of x. So $T \times_S Z \to T \times_S f(Z)$ is sujective.

Now suppose that $W \subseteq T \times_S f(Z)$ is a closd subset of $T \times_S f(Z)$. Its vertical preimage $(f')^{-1}W$ is a closed subset of $T \times_S Z$ and since $Z \to S$ is universally closed the image $s' \circ f'((f')^{-1}(W))$ in T is closed. As f' is surjective, $f'((f')^{-1}(W)) = W$ and so $s' \circ f'((f')^{-1}(W)) = s'(W)$. Hence, $T \times_S f(Z)$ is closed in T.

Exercise 4.5. *Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k has center x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.*

a *If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.*

b *If X is proper over k, then every valuation of K/k has a unique center on X.*

c *Prove the converses of (a) and (b).*

d *If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$.*

Solution. a Let R be the valuation ring of a valuation on K. Having center on some point $x \in X$ is equivalent to an inclusion $\mathcal{O}_{x,X} \subseteq R \subseteq K$ (such that $\mathfrak{m}_R \cap \mathcal{O}_{x,X} = \mathfrak{m}_x$) which is equivalent to a diagonal morphism in the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur] & \operatorname{Spec} k \end{tikzcd}$$

But by the valuative criterion for separability this diagonal morphism (if it exists) is unique. Therefore, the center, if it exists, is unique.

b Same argument as the previous part.

c

d Suppose that there is some $a \in \Gamma(X, \mathcal{O}_X)$ such that $a \notin k$. Consider the image $a \in K$. Since k is algebraically closed, a is transcendental over k and so $k[a^{-1}]$ is a polynomial ring. Consider the localization $k[a^{-1}]_{(a^{-1})}$. This is a local ring contained in K and therefore there is a valuation ring $R \subset K$ that dominates it. Since $\mathfrak{m}_R \cap k[a^{-1}]_{(a^{-1})} = (a^{-1})$ we see that $a^{-1} \in \mathfrak{m}_R$.

Now since X is proper, there exists a unique dashed morphism in the diagram on the left.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd} \qquad \begin{tikzcd} K & \Gamma(X, \mathcal{O}_X) \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Taking global sections gives the diagram on the right which implies that $a \in R$ and so $v_R(a) \geq 0$. But $a^{-1} \in \mathfrak{m}_R$ and so $v_R(a^{-1}) > 0$ This gives a contradiction since $0 = v_R(1) = v_R(\frac{a}{a}) = v_R(a) + v_R(\frac{1}{a}) > 0$.

Exercise 4.6. *Let $f : X \to Y$ be a proper morphisms of affine varieties over k. Then f is a finite morphism.*

Solution. Since X and Y are affine varieties, by definition they are integral and so f comes from a ring homomorphism $B \to A$ where A and B are integral. Let $K = k(A)$. Then for valuation ring R of K that contains $\phi(B)$ we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

Since f is proper, the dashed arrow exists (uniquely, but we don't need this). From Theorem II.4.11A the integral closure of $\Phi(B)$ in K is the intersection of all valuation rings of K which contain $\phi(B)$. As the dashed morphism exists for any valuation ring K containing $\phi(B)$ so it follows that A is contained in the integral closure of $\phi(B)$ in K. Hence every element of A is integral over B, and this together with the hypothesis that f is of finite type implies that f is finite.

Exercise 4.7. Schemes over $\mathbb{R}$.

a Let X be a separated scheme of finite type over $\mathbb{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$ there is an open affine subset containing both of them. Show that there is a unique separated scheme X_0 of finite type over $\mathbb{R}$ such that $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$, and such that this isomorphism identifies the conjugation involution of X with the one on $X_0 \times_{\mathbb{R}} \mathbb{C}$.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbb{R}$, and X, σ will denote the corresponding scheme with oncolution over $\mathbb{C}$.

b *Show that X_0 is affine if and only if X is.*

c *If X_0, Y_0 are two such schemes over $\mathbb{R}$, then to give amorphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions.*

d *If $X \cong \mathbb{A}^1_{\mathbb{C}}$ then $X_0 \cong \mathbb{A}^1_{\mathbb{R}}$.*

e *If $X \cong \mathbb{P}^1_{\mathbb{C}}$ then either $X_0 \cong \mathbb{P}^1_{\mathbb{R}}$ or X_0 is isomorhic to the conic in $\mathbb{P}^2_{\mathbb{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^1 = 0$.*

Solution. a

b Since $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$ if X_0 is affine then certainly X is. Conversely, if $X = \operatorname{Spec} A$ is affine then as above, $X_0 = \operatorname{Spec}(A^\sigma)$.

c Certainly, given f_0 we get an f that commutes with the involution. Conversely, suppose that we are given f that commutes with σ. In the case where Y and X are affine $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ we get and induced morphism on σ invarants $A^\sigma \to B^\sigma$ and this gives us the morphism $X_0 \to Y_0$. If X and Y are not affine then take a cover of X by σ preserved open affines $\{U_i\}$ and for each i take a cover $\{V_{ij}\}$ of $f^{-1}U_i$ with each V_{ij} a σ preserved open affine of Y. Let $\pi : Y \to Y_0$ is the projection and recall that it is affine (part (b)). By the affine case, we get $\pi V_{ij} \to \pi U_i$ and by the way these are defined it can be seen that they glue together to give a morphism $Y_0 \to X_0$.

d See Case II of part (e).

e *Case I: σ has no closed fixed points.* Let $x \in X \cong \mathbb{P}^1_{\mathbb{C}}$ be a closed point and consider the space $U = X \backslash \{x, \sigma x\}$. Since σ has no fixed points and $PGL_{\mathbb{C}}(1)$ is transitive on pairs of distinct points we can find a $\mathbb{C}$-automorphism f that sends $(x, \sigma x)$ to $(0, \infty)$ and therefore assume that x and σx are 0 and ∞ and so $U \cong \operatorname{Spec} \mathbb{C}[t, t^{-1}]$. Note that the lift of σ is still $\mathbb{C}$-semilinear by the commutativity of the following diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & X \arrow[r, "\sigma"] \arrow[d] & X \arrow[r, "f^{-1}"] \arrow[d] & X \arrow[d] \\
\mathbb{C} \arrow[r, "id"] & \mathbb{C} \arrow[r, "\alpha"] & \mathbb{C} \arrow[r, "id"] & \mathbb{C}
\end{tikzcd}$$

Now σ induces an invertible semilinear $\mathbb{C}$-algebra homomorphism on $\mathbb{C}[t, t^{-1}]$. We will show that σ acts via $t \mapsto -t^{-1}$. The element t must get sent to something invertible and therefore gets sent to something of the form

at^k for some k an integer. [1] Since $\sigma^2 = id$ it follows that $k = \pm 1$. Furthermore, by considering σ on the function field $\mathbb{C}(t)$ it can be seen that $k = -1$ since otherwise the valution ring $\mathbb{C}[t]_{(t)} \subset \mathbb{C}(t)$ would be fixed, implying that σ has a fixed point. Now $t\sigma t = a$ is fixed by σ and σ acts by conjugation on constants, we see that $a \in \mathbb{R}$. If a is positive, the ideal $(t - \sqrt{a})$ is preserved contradicting the assumption of no fixed points, so $a \in \mathbb{R}_{\leq 0}$. Now we make a change of coordinates by replacing t with $\frac{1}{\sqrt{-a}}$. This amounts to choosing a slightly different element of $PGL(1)$ at the beginning when we were sending x and σx to 0 and ∞. With this new t our involution is $t \mapsto -t^{-1}$.

Now we rewrite $\mathbb{C}[t, t^{-1}]$ as $\frac{\mathbb{C}[\frac{X}{Z}, \frac{Y}{Z}]}{(1+\frac{XY}{Z^2})}$ via

$$\left\{ \begin{array}{c} \frac{X}{Z} = t^{-1} \\ \frac{Y}{Z} = -t \end{array} \right\}$$

so the involution acts by switching $\frac{X}{Z}$ and $\frac{Y}{Z}$ (and conjugation on scalars). Now consider the two subrings $\mathbb{C}[-t]$ and $\mathbb{C}[t^{-1}]$ of the function field $\mathbb{C}(t)$. We have isomorphisms

$$\frac{\mathbb{C}[\frac{Y}{X}, \frac{Z}{X}]}{(\frac{Y}{X} + (\frac{Z}{X})^2)} \cong \mathbb{C}[-t] \qquad t = \frac{Z}{X}$$

$$\frac{\mathbb{C}[\frac{X}{Y}, \frac{Z}{Y}]}{(\frac{X}{Y} + (\frac{Z}{Y})^2)} \cong \mathbb{C}[t^{-1}] \quad -t^{-1} = \frac{Z}{Y}$$

and σ acts by swapping these two rings (and conjugation on scalars). These three open affines patch together in a way compatible with σ to form an isomorphism

$$\text{Proj}\,\frac{\mathbb{C}[X, Y, Z]}{(XY + Z^2)} \cong \mathbb{P}^1_{\mathbb{C}}$$

where σ acts on the quadric by swapping X and Y, and conjugation on scalars. Making a last change of coordinates

$$U = \frac{1}{2}(X + Y) \qquad V = \frac{i}{2}(Y - X)$$

we finally get the isomorphism

$$\mathscr{Q} = \text{Proj}\,\frac{\mathbb{C}[X, Y, Z]}{(U^2 + V^2 + Z^2)} \cong \text{Proj}\,\frac{\mathbb{C}[X, Y, Z]}{(XY + Z^2)} \cong \mathbb{P}^1_{\mathbb{C}} = X$$

[1] *The group of units in* $\mathbb{C}[t, t^{-1}]$ *is* $\{at^k : a \neq 0, k \in \mathbb{Z}\}$. Suppose that $(\sum_{i=m'}^{m} a_i)(\sum_{i=n'}^{n} b_i) = 1$ the term of highest order in the product is $a_m b_n t^{n+m} = 1$ and so $n = 1 - m$. Similarly, the term of lowest order is $a_{m'} b_{n'} t^{n'+m'} = 1$ and so $n' = 1 - m'$. Now $n' \leq n = 1 - m \leq 1 - m' = n'$ and so $n = n'$. The same argument shows that $m = m'$. Hence, both elements of the product are of the form at^k for some k.

where σ acts on $\mathscr{Q}$ by conjugation of scalars alone. Hence

$$X_0 \cong \mathscr{Q}_0 = \operatorname{Proj} \frac{\mathbb{R}[X,Y,Z]}{(U^2+V^2+Z^2)}$$

Case II: σ has at least one fixed point. Now suppose that σ fixes a closed point x. This means that σ restricts to a semilinear automorphism of the complement of the fixed point $\operatorname{Spec}\mathbb{C}[t] \subset \mathbb{P}^1_{\mathbb{C}}$. Since σ is invertible, t gets sent to something of the form $at+b$. There exists a change of coordinates $s = ct+d$ such that $\sigma s = s$ and so in these new coordinates we get a σ invariant isomorphism $X \cong \mathbb{P}^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 4.8. *Let $\mathscr{P}$ be a property of morphisms of schemes such that:*

a *a closed immersion has $\mathscr{P}$;*

b *a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;*

c *$\mathscr{P}$ is stable under base extension.*

Then show that

d *a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;*

e *if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;*

f *If $f : X \to Y$ has $\mathscr{P}$, then $f_{red} : X_{red} \to Y_{red}$ has $\mathscr{P}$.*

Solution. d Let $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ be the morphisms. The morphism $f \times f'$ is a composition of base changes of f and f' as follows:

$$\begin{tikzcd}
 & X \arrow[dd] \\
X \times X' \arrow[ur] \arrow[dd] & \\
 & Y \\
Y \times X' \arrow[ur] \arrow[dr] \arrow[dd] & \\
 & X' \arrow[dd] \\
Y \times Y' \arrow[dr] & \\
 & Y'
\end{tikzcd}$$

Therefore $f \times f'$ has property $\mathscr{P}$.

e Same argument as above but we should also note that since g is separated the diagonal morphism $Y \to Y \times_Z Y$ is a closed embedding and therefore satsifies $\mathscr{P}$.

$$\begin{tikzcd}
 & Y \arrow[dd] \\
X \arrow[ur] \arrow[dd] & \\
 & Y \times_Z Y \\
X \times_Z Y \arrow[ur] \arrow[dr] \arrow[dd] & \\
 & X \arrow[dd] \\
Y \arrow[dr] & \\
 & Z
\end{tikzcd}$$

f Consider the factorization

$$\begin{tikzcd}
X_{red} \arrow[dr, "\Gamma_{f_{red}}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "f_{red}"'] & & \\
 & Y_{red} \times_Y X_{red} \arrow[r] \arrow[d] & X_{red} \arrow[d] \\
 & Y_{red} \arrow[r] & Y
\end{tikzcd}$$

The morphism $X_{red} \to X \to Y$ is a composition of a closed immersion and a morphism with property scP and therefore it has property $\mathscr{P}$. Therefore the vertical morphism out of the fibre product is a base change of a morphism with property $\mathscr{P}$ and therefore, itself has property $\mathscr{P}$. To se that f_{red} has property $\mathscr{P}$ it therefore remains only to see that the graph $\Gamma_{f_{red}}$ has property $\mathscr{P}$ for then f_{red} will be a composition of morphisms with property $\mathscr{P}$. To see this, recall that the graph is following base change

$$\begin{tikzcd}
X_{red} \arrow[r] \arrow[d, "\Gamma"] & Y_{red} \arrow[d, "\Delta"] \\
X_{red} \times_Y Y_{red} \arrow[r] & Y_{red} \times_Y Y_{red}
\end{tikzcd}$$

But $Y_{red} \times_Y Y_{red} = Y_{red}$ and $\Delta = id_{Y_{red}}$ and so Δ is a closed immersion. Hence, Γ is a base change of a morphism with property $\mathscr{P}$.

Exercise 4.9. *Show that a composition of projective morphisms is projective. Conclude that projective morphisms have properties (a)-(f) of Exercise II.4.8 above.*

Solution. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two projective morphisms. This gives rise to a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[dr, "f"'] & \mathbb{P}^r \times Y \arrow[r, "id\times g'"] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s \times Z \arrow[d] \\
& Y \arrow[r, "g'"] \arrow[dr, "g"'] & \mathbb{P}^s \times Z \arrow[d] \\
& & Z
\end{tikzcd}$$

where f' and g' (and therefore $id \times g'$) are closed immersions. Now using the Segre embedding the projection $\mathbb{P}^r \times \mathbb{P}^s \times Z \to Z$ factors as

$$\mathbb{P}^r \times \mathbb{P}^s \times Z \to \mathbb{P}^{rs+r+s} \times Z \to Z$$

So since the Segre embedding is a closed immersion then we are done since we have found a closed immersion $X \to \mathbb{P}_Z^{rs+r+s}$ which factors $g \circ f$.

Exercise 4.10. Chow's Lemma. *Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.*

a *Reduce to the case X irreducible.*

b *Show that X can be covered by a finite number of open subsets $U_i, i = 1, \dots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.*

c *Let $U = \bigcap U_i$ and consider the map*

$$U \to X \times_S P_1 \times_s \cdots \times_S P_n$$

deduced from the give maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure. Let $g : X' \to X$ be the projection onto the first factor, and let $h : X' \to P = P_1 \times_S \cdots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

d *Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.*

Exercise 4.11. Valuative criteria using discrete valuation rings.

a *If $\mathcal{O}, \mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$.*

b *Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (resp. proper) if and only if the criterion of 4.3 (resp. 4.7) holds for all discrete valuation rings.*

Exercise 4.12. Examples of Valuation Rings. *Let k be an algebraically closed field.*

a *If K is a function field of dimension 1 over k, then every valuation ring of K/k (except for K itself) is discrete.*

b *If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.*

   (a) *If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1.*

   (b) *If $f : X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation rign of K/k with center at the closed point x_0 on X.*

   (c) *Let $x_0 \in X$ be a closed point. Let $f : X_1 \to X$ be the blowing up of x_0 and let $E_1 = f^{-1}x_0$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2 : X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}x_1$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{X_{i+1},x_{i+1}}$ dominates $\mathcal{O}_{X_i,x_i}$. Let $R_0 = \cup_{i=0}^{\infty}\mathcal{O}_{X_i,x_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k. Show that R is a valuation ring of K/k and that it has center x_0 on X. When is R a discrete valuation ring?*

Solution. a Let $R \subset K$ be a valuation ring of K. We will show that $\mathfrak{m}_R$ is principal, which will imply that R is discrete. Let $t \in \mathfrak{m}_R$. If $(t) = \mathfrak{m}_R$ then we are done. If not choose some $s \in \mathfrak{m}_R \backslash (t)$. Note that t is transcendental over k. To see this, suppose that it satisfies some polynomial $\sum_{i=0}^{n} a_i t^i = 0$ chosen so that $a_0 \neq 0$. Then $a_0 = t \sum a_i t^{i-1}$ and so $a_0 \in (t)$. But a_0 is a unit and so we get a contradiction, hence there is no such polynomial. Now since K has dimension 1 and t is transcendental, K is a finite algebraic extension of $k(t)$. The element $s \notin (t)$ and so it is algebraic over k. Hence, it satisfies some polynomial with coefficients in $k(t)$. Let $\sum_{i=0}^{n} a_i s^i = 0$ be such a polynomial, chosen so that $a_0 \neq 0$. Again, this implies that $a_0 = s \sum a_i s^{i-1}$. Write $a_0 = \frac{f(t)}{g(t)}$. Then we have $\frac{f(t)}{g(t)} = s \sum a_i s^{i-1}$ and so $f(t) = g(t)s \sum a_i s^{i-1}$ implying that $f(t) \in (s) \subseteq \mathfrak{m}_R$. Since $tin\mathfrak{m}_R$, the polynomial $f(t)$ can't have any constant term (otherwise this term would be in $\mathfrak{m}_R$ contradicting the fact that it is a proper ideal) and so $t \in (s)$ and hence $(s) \supset (t)$. If $(s) = \mathfrak{m}_R$ we are done. If not, repeat the process to obtain increasing chain of ideals $(t) \subset (s) \subset (s_1) \subset \ldots$ all contained in $\mathfrak{m}_R$. Since R is noetherian, this chain must terminate and we find so s_i such that $(s_i) = \mathfrak{m}_R$. Hence, $\mathfrak{m}_R$ is principal, and therefore by Theorem I.6.2A the valuation ring R is discrete.