Appendix C

Topology of algebraic varieties

Throughout this appendix we work with projective varieties $X \subset \mathbb{P}^N_{\mathbb{C}}$ — that is, with complex projective varieties. We can also view such a variety as a complex analytic, or *holomorphic*, subvariety of $\mathbb{P}^N_{\mathbb{C}}$ — that is, a subset locally defined by the vanishing of analytic equations — or, if X is smooth, as a complex submanifold of $\mathbb{P}^N_{\mathbb{C}}$. The topology induced from the standard topology on $\mathbb{P}^N_{\mathbb{C}}$, referred to as the *classical*, or sometimes *analytic*, topology, is much finer than the Zariski topology with which we have dealt in this text. Using it, we can consider geometric invariants of X such as the singular homology and cohomology groups $H_*(X, \mathbb{Z})$ and $H^*(X, \mathbb{Z})$.

In this appendix, we explain a little of what is known about such invariants. Throughout, when we speak of topological properties of X, we refer to the classical, or analytic, topology.

C.1 GAGA theorems

One might think that there would be many more holomorphic subvarieties of $\mathbb{P}^N$ than algebraic subvarieties, or that in passing from a smooth projective variety X over $\mathbb{C}$ to its underlying complex manifold we would be losing information, since regular functions are holomorphic but not conversely. But this is not the case:

Theorem C.1 (Chow)**.** *Every holomorphic subvariety of* $\mathbb{P}^N_{\mathbb{C}}$ *is algebraic.*

See for example Griffiths and Harris [1994, Section I.3] for a proof. Many further results in this direction were proven in Serre [1955/1956]. These are collectively known as the GAGA theorems, after the name of Serre's paper ("Géométrie algébrique et géométrie analytique").

https://doi.org/10.1017/CBO9781139062046.019 Published online by Cambridge University Press

It follows immediately from Chow's theorem that if X and Y are projective varieties over $\mathbb{C}$ then *any holomorphic map* $f : X \to Y$ *is algebraic* (Proof: Apply Theorem C.1 to the graph $\Gamma_f \subset X \times Y$). Not quite so immediate are the facts that any holomorphic vector bundle on a projective variety is algebraic and that if $\mathcal{E}$ is any such vector bundle on X then any global holomorphic section of $\mathcal{E}$ is algebraic. More generally, the Čech cohomology groups of $\mathcal{E}$ will be the same, whether computed for the sheaf of holomorphic sections of $\mathcal{E}$ in the analytic topology or the sheaf of regular sections in the Zariski topology. Thus, for example, the tangent space to the Picard group of X, which can be identified with the first Zariski cohomology $H^1(\mathcal{O}_X)$, may also be identified with the cohomology $H^1(\mathcal{O}_{X,\mathrm{an}})$ in the analytic topology.

In sum, as it applies to projective varieties, we should think of the classical topology and the analytic tools it brings as a new approach to the study of the same projective algebraic varieties and phenomena. This is the point of view taken, for example, in Griffiths and Harris [1994], and we will give references to that book.

Once one goes beyond projective varieties, there are many complex manifolds that are not algebraic: while every one-dimensional compact complex manifold is an algebraic curve, there are many natural examples of compact complex surfaces that are not algebraic.

C.2 Fundamental classes and Hodge theory

C.2.1 Fundamental classes

Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth projective variety. Since $\mathbb{P}^N_{\mathbb{C}}$ is compact as a topological space and X is a closed subspace, X is compact. Because $\mathbb{C}$ has an orientation preserved by holomorphic functions, X is automatically orientable. In particular, if X has dimension n then it is a compact oriented real $2n$-dimensional manifold, and has a fundamental class $[X] \in H_{2n}(X, \mathbb{Z})$. (In this appendix, when we talk about the homology or cohomology groups of X or of a subvariety of X we mean the singular homology or cohomology.) By Poincaré duality, capping with this class induces an isomorphism

$$H^{2n-k}(X, \mathbb{Z}) \xrightarrow{\cap [X]} H_k(X, \mathbb{Z}).$$

More generally, let $Y \subset \mathbb{P}^r$ be any k-dimensional variety. By a theorem of Łojasiewicz [1964] (see Hironaka [1975]), Y admits a finite triangulation in which the singular locus is a subcomplex. Since the singularities of Y occur in real codimension ≥ 2, the sum of the simplices of dimension $2k$ in the triangulation is a cycle, called the *fundamental cycle* of Y; the class of this cycle is called the *fundamental class* of Y and is again denoted by $[Y] \in H_{2k}(Y, \mathbb{Z})$. If $\nu : \widetilde{Y} \to Y$ is a desingularization of Y, then the fundamental class of Y is equal to the pushforward $[Y] = \nu_*[\widetilde{Y}]$.

https://doi.org/10.1017/CBO9781139062046.019 Published online by Cambridge University Press

If X is an n-dimension projective variety (possibly singular), we can use this idea to define a homomorphism

$$Z_k(X) \to H_{2k}(X, \mathbb{Z})$$

from the group of k-cycles on X to its homology group, defined by associating to any k-dimensional subvariety $i : Y \hookrightarrow X$ the pushforward $i_*[Y] \in H_{2k}(X, \mathbb{Z})$ of the fundamental class $[Y]$ of Y. We say that two k-cycles A and A' are *homologically equivalent* if the pushforwards of their fundamental classes are equal. This is a coarser equivalence relation than rational equivalence; that is, this map factors through a map

$$A_k(X) \to H_{2k}(X, \mathbb{Z}).$$

If we suppose that X is smooth, then composing with Poincaré duality we get a homomorphism

$$\eta : A^{n-k}(X) = A_k(X) \to H^{2n-2k}(X, \mathbb{Z}),$$

which we call the *fundamental class map*; the image of the class of a subvariety $Y \subset X$ will be denoted by η_Y. (If one wants to avoid invoking the triangulability of complex varieties, the map η can be defined in de Rham cohomology by arguing that if $Y \subset X$ is any k-dimensional subvariety then integration over Y gives a well-defined linear functional on closed modulo exact $2k$-forms on X; see Griffiths and Harris [1994, p. 61].) For example, if $X = \mathbb{P}^n$ then η is an isomorphism — $A(X) = \mathbb{Z}[\zeta]/(\zeta^{n+1}) = H^*(X, \mathbb{Z})$ — so the information in the fundamental class $[Y]$ is the dimension and degree of Y.

A crucial fact is that intersection products in $A(X)$ corresponds to cup products in $H^*(X, \mathbb{Z})$:

Theorem C.2. *The map η is a ring homomorphism; that is, it takes the intersection product in $A(X)$ to the cup product on $H^*(X, \mathbb{Z})$:*

$$\eta([A][B]) = \eta[A] \cup \eta[B].$$

For a proof, see Griffiths and Harris [1994, Section 0.4].

The fundamental class map underlies much of the relevance of topology to geometry. The simplest invariant of a subvariety $A \subset X$ of a variety in general is its fundamental class in $H^*(X, Z)$, just as the simplest invariants of a subvariety of projective space are its dimension and degree.

This picture of the classes of subvarieties has a missing piece, encapsulated in one of the major open problems in the field: Which cohomology classes of a smooth variety are represented by linear combinations of classes of subvarieties? That is, what is the image of η? The *Hodge conjecture* (Section C.2.4) is an attempt to answer this question. Before explaining the statement, we provide some background.

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C.2.2 The Hodge decomposition

Hodge noticed that if one tensors the real cotangent bundle of a complex manifold with the complex numbers it splits in a natural way. We can explain this phenomenon as follows:

Let $p \in X \subset \mathbb{P}^N_{\mathbb{C}}$ be a point on a smooth, n-dimensional projective algebraic variety. Viewing X as a real $\mathcal{C}^\infty$ manifold of dimension $2n$, we consider the $\mathcal{C}^\infty$ cotangent bundle $\mathcal{T}^* X$. The fiber $T^*_p X$ has a natural complex structure coming from the fact that the differentials of real functions on a small open set are spanned by the differentials of the real and imaginary parts of complex analytic functions: for any such function $\varphi : X \to \mathbb{C}$ we have $i \cdot d(\mathrm{Re}(\varphi)) := d(\mathrm{Re}(i \cdot \varphi)) \ = -d(\mathrm{Im}(\varphi))$.

Now, any n-dimensional complex vector space V can be regarded as a real vector space of dimension $2n$. Though there is no natural splitting into real and imaginary parts (that is, no splitting that is invariant under complex-linear transformations), the *complexification* $\mathbb{C} \otimes_{\mathbb{R}} V$ *does* split naturally as $\mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C}^n \oplus \mathbb{C}^n$, where the complex structure is defined by multiplication in the first factor and the two summands are the $+1$ and -1 eigenspaces under the real linear transformation that is multiplication by $i \otimes i$.

In particular, the complexification of the real cotangent space of a complex analytic manifold X at a point p is naturally a direct sum of the complex cotangent space (the complex vector space of differentials of complex analytic functions at p) and a space that may be identified as the space of differentials of anti-holomorphic functions. If $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$ are complex analytic coordinates on an open set $U \subset X$ then the splitting may be written as

$$\mathbb{C} \otimes_{\mathbb{R}} T^*_p X \cong \mathbb{C}\langle dz_1, \dots, dz_n\rangle \oplus \mathbb{C}\langle d\overline{z}_1, \dots, d\overline{z}_n\rangle,$$

where $dz_\alpha = dx_\alpha + i\, dy_\alpha$ and $d\overline{z}_\alpha = dx_\alpha - i\, dy_\alpha$. A direct computation shows that $T^*_p X$, with the complex structure defined above, is mapped isomorphically to the fiber of the complex cotangent bundle under the composition of the inclusion into $\mathbb{C} \otimes T^*_p X$ and the projection modulo the space spanned by the differentials of the anti-holomorphic functions. Putting this together, we get:

Theorem C.3 (Hodge)**.** *If X is a complex manifold, then the complexified cotangent bundle $\mathcal{T}(X) \otimes \mathbb{C}$ splits as a direct sum*

$$\mathbb{C} \otimes \mathcal{T}^* X = \mathcal{T}^{*\prime} X \oplus \mathcal{T}^{*\prime\prime} X, \tag{C.1}$$

where, with respect to any analytic local coordinates $\{z_\alpha\}$ at any point p, the fibers of $\mathcal{T}^{*\prime} X$ and $\mathcal{T}^{*\prime\prime} X$ are given by

$$T^{*\prime}_p X = \mathbb{C}\langle dz_1, \dots, dz_n\rangle, \quad T^{*\prime\prime}_p X = \mathbb{C}\langle d\overline{z}_1, \dots, d\overline{z}_n\rangle.$$

Moreover, $\mathcal{T}^{*\prime} X$ is the underlying holomorphic vector bundle of Ω_X, the algebraic cotangent bundle of X.

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The decomposition (C.1) immediately gives rise to a direct sum decomposition of the exterior powers of $T_p^* X \otimes \mathbb{C}$:

$$\bigwedge^k(\mathbb{C} \otimes T_p^* X) = \bigoplus_{p+q=k} \bigwedge^p T_p^{*\prime} X \otimes \bigwedge^q T_p^{*\prime\prime} X.$$

From this we get a decomposition of the sheaf $\mathcal{A}^k(X)$ of complex-valued $\mathcal{C}^\infty$ differential forms of degree k on X:

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}, \tag{C.2}$$

where $\mathcal{A}^{p,q}$ is the sheaf of complex-valued $\mathcal{C}^\infty$ k-forms whose value at every point $p \in X$ lies in the summand $\bigwedge^p T_p^{*\prime} X \otimes \bigwedge^q T_p^{*\prime\prime} X$. A section of $\mathcal{A}^{p,q}$ is called a *form of type* (p,q), or a (p,q)-form. Note that while the bundle $\bigwedge^k \mathcal{T}^{*\prime} X$ has the structure of a holomorphic vector bundle, the sheaf $\mathcal{A}^{k,0}$ consists of all $\mathcal{C}^\infty$ sections of $\bigwedge^k \mathcal{T}^{*\prime} X$.

Hodge proved that when X is a smooth projective variety, and more generally when X is a Kähler variety, the decomposition (C.2) descends to the level of the de Rham cohomology of X: Any closed differential form φ on X can be written naturally as a sum

$$\varphi = \varphi^{(k,0)} + \varphi^{(k-1,1)} + \cdots + \varphi^{(0,k)},$$

where $\varphi^{(p,q)}$ is closed of type (p,q) and the de Rham cohomology class of $\varphi^{(p,q)}$ depends only on the class of φ.[1] Since the de Rham cohomology may be identified with the complexified singular cohomology, we get a decomposition of that space:

Theorem C.4 (Hodge decomposition)**.** *If X is a smooth projective variety, then the singular cohomology of X with complex coefficients decomposes as*

$$H^k(X,\mathbb{C}) = H^k_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the subspace of de Rham cohomology classes represented by forms of type (p,q). Complex conjugation interchanges $H^{p,q}(X)$ and $H^{q,p}(X)$.

Hodge also showed that the spaces $H^{p,q}(X)$, despite their apparently transcendental character, could be computed algebraically:

Theorem C.5.

$$H^{p,q}(X) = H^q(\Omega_X^p),$$

the q-sheaf cohomology space of the p-th exterior power of the sheaf of differential forms on X; in particular, $H^{k,0}(X)$ is the space of global holomorphic k-forms on X.

[1] More precisely, the Hodge theorem asserts that with respect to any given Hermitian metric on X every de Rham cohomology class is uniquely represented by a harmonic form; the Kähler condition on the metric ensures that the Laplacian commutes with the decomposition of a form by type.

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Proof sketch: This result follows from computing the homology of the *Dolbeault complex* which resolves Ω_X^p:

$$0 \to \Omega_X \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots ;$$

since the sheaves $\mathcal{A}^{p,i}$ are flasque, we can compute the Zariski cohomology of Ω_X^p as the homology of the complex

$$H^0\mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} H^0\mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \cdots$$

and this homology, at the q-th step of the resolution, is $H^{p,q}(X)$. □

The data we have just described, consisting of a lattice (that is, finitely generated free $\mathbb{Z}$-module) $\Lambda = H^k(X,\mathbb{Z})/\text{tors}$, together with a decomposition of its complexification

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \quad \text{with } H^{p,q} = \overline{H^{q,p}},$$

where $\overline{H^{q,p}}$ denotes the complex conjugate of $H^{q,p}$, is called a *Hodge structure* of weight k. When $\dim X = k$, so that $\Lambda = H^k(X, Z)$ is the middle-dimensional cohomology, the cup product is a unimodular inner product on the lattice Λ, and, since the wedge product of a (p,q)-form with a (p',q') form is a $(p+p',q+q')$-form, the subspaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal, that is, $H^{p,q} \cup H^{p',q'} = 0$ unless $(p',q') = (q,p)$. Hodge structures with these additional properties are said to be *polarized*.

C.2.3 The Hodge diamond

The dimensions $h^{p,q} = h^{p,q}(X)$ of the Hodge groups $H^{p,q} = H^{p,q}(X)$ of a smooth projective variety are often represented in a diamond, called the *Hodge diamond*, with $h^{n,n}$ at the top. For example, the Hodge diamond of a smooth quartic surface $X \subset \mathbb{P}^3$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

meaning, for example, that $h^{1,1}(X) = 20$. (Given the Lefschetz hyperplane theorem of Section C.4 below, the tools of Section 5.7.3 enable us to compute these numbers.)

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The Hodge diamond is left-right symmetric through complex conjugation since $H^{p,q}(X) = \overline{H^{q,p}(X)}$, but it is also top-bottom symmetric: the cup product is defined on de Rham cohomology by multiplication of differential forms, and thus $H^{p,q} \cup H^{p',q'} = 0$ if either $p + p' > n$ or $q + q' > n$. Given this, Poincaré duality shows that $H^{n-p,n-q}(X)$ is dual to $H^{p,q}(X)$, and in particular they have the same dimension.

An immediate consequence of the symmetry is that the odd Betti numbers of a smooth projective variety are even; for example, $b_1(X) = h^{1,0} + h^{0,1} = 2h^{1,0}$. This tells us, for example, that the compact complex manifold X given as the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the group of automorphisms generated by $\varphi : (z, w) \mapsto (2z, 2w)$, which is homeomorphic to $S^1 \times S^3$, cannot be a complex algebraic variety.

Holomorphic forms can be pulled back under birational transformations, and it follows that the Hodge numbers $h^{k,0}(X) = H^0(\Omega_X^k)$ are birational invariants. However, the other Hodge numbers $h^{p,q}(X)$ are not in general birational invariants.

The Hodge numbers are however deformation invariant, in the sense that they are constant in flat families X_λ of smooth projective varieties. Indeed, the Hodge numbers are upper-semicontinuous because they are the dimensions of the cohomology groups of coherent sheaves (Corollary B.12). On the other hand, their sum

$$\sum_{p+q=k} h^{p,q}(X_\lambda) = \dim H^k(X_\lambda)$$

is a topological invariant, and any two fibers in a flat family of smooth varieties are homeomorphic.

C.2.4 The Hodge conjecture

We now return to the question of which cohomology classes on a smooth projective variety X can be represented as linear combinations of the fundamental classes of algebraic varieties; that is, what is the image of $\eta : A(X) \to H^*(X)$?

Since the map on cohomology induced by a holomorphic map (such as the inclusion $Z \hookrightarrow X$, or the composition of the inclusion with a resolution $\tilde{Z} \to Z$ of the singularities of Z) respects Hodge type, the fundamental class of a codimension-k subvariety $Z \subset X$ is of type (k, k). Thus the image of $\mathbb{C} \otimes_{\mathbb{Z}} \eta$ lies in $\bigoplus_k H^{k,k}(X) \subset H^*(X, \mathbb{C})$.

The Hodge conjecture asserts that modulo torsion the converse is true:

Conjecture C.6 (Hodge)**.** *Every class in $H^{2k}(X, \mathbb{Q})$ whose image in $H^{2k}(X, \mathbb{C})$ lies in $H^{k,k}(X)$ is a rational linear combination of fundamental classes of algebraic subvarieties of X.*

It might seem more natural to make this conjecture without tensoring with $\mathbb{Q}$; but this statement, sometimes known as the "integral Hodge conjecture," is known to be false; see Atiyah and Hirzebruch [1961] and, for a recent survey, Totaro [2013].

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An important special case of the Hodge conjecture is the following result of Lefschetz, which is the integral Hodge conjecture in the codimension-1 case:

Theorem C.7 (Lefschetz $(1,1)$-theorem)**.** *Let X be a smooth projective variety. If $\alpha \in H^2(X,\mathbb{Z})$ is any class whose image in $H^2(X,\mathbb{C})$ lies in $H^{1,1}(X)$, then α is the fundamental class of a divisor on X.*

In particular, *any* torsion class in $H^2(X,\mathbb{Z})$ is algebraic — that is, it is the class of a divisor on X.

Proof: We will give an outline of the proof; the explicit calculations may be found in Griffiths and Harris [1994, p. 163]. As above, let $\mathcal{O}_{X,\mathrm{an}}$ be the sheaf of holomorphic functions, defined as a sheaf in the classical topology. Let $\mathcal{O}^*_{X,\mathrm{an}}$ denote the sheaf of nowhere-zero holomorphic functions; this is a sheaf of abelian groups with respect to multiplication (in particular, it is *not* a coherent sheaf). There is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X,\mathrm{an}} \xrightarrow{\exp} \mathcal{O}^*_{X,\mathrm{an}} \longrightarrow 0,$$

where $\mathbb{Z}$ denotes the sheaf of locally constant integer-valued functions and the exponential map exp sends a function f to e^f. (Note that this sequence is not exact in the Zariski topology, where a point $p \in X$ may not have any simply connected open neighborhoods.) The cohomology groups $H^i(\mathbb{Z})$ (in the classical topology) are naturally isomorphic to the singular homology groups $H^i(X,\mathbb{Z})$.

By the GAGA theorems

$$H^1(\mathcal{O}^*_{X,\mathrm{an}}) = \mathrm{Pic}\, X,$$

and the coboundary map $\delta : H^1(\mathcal{O}^*_{X,\mathrm{an}}) \to H^2(X,\mathbb{Z})$ in the associated long exact sequence is the composition of the Chern class map $\mathrm{Pic}\, X \to A^1(X)$ with the fundamental class map $\eta : A^1(X) \to H^2(X,\mathbb{Z})$. A calculation shows that the maps

$$H^i(X,\mathbb{Z}) \to H^i(\mathcal{O}_{X,\mathrm{an}}) = H^{0,i}(X)$$

in the long exact sequence are the compositions of the maps $H^i(X,\mathbb{Z}) \to H^i(X,\mathbb{C})$ with the projections $H^i(X,\mathbb{C}) \to H^{0,i}(X)$. It follows that a class $\alpha \in H^2(X,\mathbb{Z})$ that maps to $H^{1,1}(X)$ is in the kernel of the induced map $H^i(X,\mathbb{Z}) \to H^i(\mathcal{O}_{X,\mathrm{an}})$, and hence in the image of δ. □

C.3 Comparison of rational equivalence with other cycle theories

In addition to the Chow ring $A(X)$ and the cohomology ring $H^*(X,\mathbb{Z})$ defined above, there are other cycle theories that we could have used in this text. In this section we will describe:

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- The ring A_{alg} of algebraic cycles modulo algebraic equivalence.
- The ring $H^*_{\mathrm{alg}}(X, \mathbb{Z})$ of algebraic cycles modulo homological equivalence.
- The ring $N^*(X)$ of algebraic cycles modulo numerical equivalence.

The reader will find more complete treatments in Fulton [1984, Chapter 19] and Fulton and MacPherson [1981]. A further way to treat algebraic cycles, which we will not discuss, is through K-theory (Fulton [1984, Section 20.5]). For interesting speculations about how this and other cycle theories might work even in the context of singular varieties, see Srinivas [2010].

For an example where the singular cohomology ring is useful (in a purely algebraic setting!), see Appendix D.

C.3.1 Algebraic equivalence

Rather than defining cycles to be equivalent only if there is a *rational* family of cycles interpolating between them — that is, a family parametrized by $\mathbb{P}^1$ — we can instead allow families parametrized by curves of any genus; that is, we say that two subvarieties $Z_1, Z_2 \subset X$ are *algebraically equivalent* if there is a reduced irreducible curve C and a subvariety $Y \subset C \times X$ such that the fibers of Y over two points $p, q \in C$ are Z_1 and Z_2. The resulting equivalence relation is called *algebraic equivalence*. The corresponding group $A_{\mathrm{alg}}(X)$ comes with a natural map

$$A(X) \to A_{\mathrm{alg}}(X).$$

All the items of the basic theory of Chow groups are true for the groups of cycles modulo algebraic equivalence; in particular, there is a decomposition by dimension $A_{\mathrm{alg}}(X) = \bigoplus A_{\mathrm{alg},d}(X)$, for a smooth variety X the groups $A^k_{\mathrm{alg}}(X) := A_{\mathrm{alg},\dim X - k}$ form a graded ring, and there are pushforward and pullback maps as in the case of the Chow rings.

By Bertini's theorem there is an irreducible curve through any two points of a projective variety X, and thus unlike in the case of the Chow groups we have $A_{\mathrm{alg},0}(X) \cong \mathbb{Z}$ via the degree map. The description of the group of cycles in codimension 1 is also simpler modulo algebraic equivalence than modulo rational equivalence: if X is smooth and projective over $\mathbb{C}$, then two divisors are algebraically equivalent if and only if they are homologically equivalent, and so $A^1_{\mathrm{alg}}(X) \hookrightarrow H^2(X, Z)$; in particular, $A^1_{\mathrm{alg}}(X)$ is finitely generated.

In other codimensions, however, the group of cycles modulo algebraic equivalence is as little understood as the Chow group. For example, if $X \subset \mathbb{P}^4$ is a smooth quintic threefold then we shall see in Section C.4.1 that two curves $C, C' \subset X$ are homologically equivalent if and only if they have the same degree. On the other hand, the 2875 lines on a general smooth quintic threefold X are linearly independent in $A^2_{\mathrm{alg}}(X)$ by Ceresa and Collino [1983], and it is not known whether $A^2_{\mathrm{alg}}(X)$ is even finitely generated.

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C.3.2 Algebraic cycles modulo homological equivalence

The group of algebraic cycles modulo homological equivalence is the image of the map $\eta : A(X) \to H^*(X, \mathbb{Z})$ defined in Section C.2.4, and is denoted by $H^*_{\mathrm{alg}}(X, \mathbb{Z})$. Since it is a subgroup of a finitely generated group, it is finitely generated (again, unlike the Chow groups). When X is smooth, η is a ring homomorphism, so this is a subring of $H^*(X, \mathbb{Z})$.

However, it still has quirks: For example, by the Noether–Lefschetz theorem (see for example Griffiths and Harris [1985]), if $U \subset \mathbb{P}^{34}$ is the space of smooth quartic surfaces, the function $r : U \to \mathbb{Z}$ associating to each surface S the rank of $H^2_{\mathrm{alg}}(S, \mathbb{Z})$ is nowhere continuous.

C.3.3 Numerical equivalence

We say that a class $\alpha \in A^k(X)$ is *numerically equivalent to 0* if the degree of the product $\alpha\beta$ is 0 for all classes $\beta \in A^{n-k}(X)$ of complementary dimension. The group of Chow classes modulo numerical equivalence is denoted by $N^*(X)$; when X is smooth, the intersection product gives this group the structure of a ring, and the quotient map $A(X) \to N^*(X)$ is a ring homomorphism.

Numerical equivalence has some of the advantages of cohomology and is easily available in all characteristics. In characteristic 0, any class homologically equivalent to 0 is evidently numerically equivalent to 0, so numerical equivalence represents a further coarsening of the notion of homological equivalence for varieties over $\mathbb{C}$.

Conjecturally, it is not much coarser: If the Hodge conjecture is true, then by Poincaré duality and the hard Lefschetz theorem (Theorem C.13) the pairing

$$H^{2k}_{\mathrm{alg}}(X, \mathbb{Q}) \times H^{2n-2k}_{\mathrm{alg}}(X, \mathbb{Q}) \to \mathbb{Q}$$

given by the intersection/cup product would be nondegenerate. Given this, it would follow that any class $\alpha \in H^*_{\mathrm{alg}}(X)$ numerically equivalent to 0 is torsion in $H^*(X, \mathbb{Z})$, or, in other words,

$$N^k(X) = H^{2k}_{\mathrm{alg}}(X)/\mathrm{tors}.$$

Since the Hodge conjecture is known in codimension 1, this statement is known for $k = 1$ and $n - 1$. This makes the notion of numerical equivalence particularly suitable for applications of intersection theory that deal largely with divisors and curves, such as the minimal model program; most papers in that area work with numerical equivalence.

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C.3.4 Comparing the theories

Summarizing the relationships of the theories we have defined, we have

$$\begin{tikzcd} A^*(X) \arrow[r, two heads] & A^*_{\mathrm{alg}}(X) \arrow[r, two heads] \arrow[dr] & H^*_{\mathrm{alg}}(X) \arrow[r, hook] \arrow[d] & H^*(X,\mathbb{Z}) \\ & & N^*(X) & \end{tikzcd}$$

with $N^*(X)$ conjecturally isomorphic to $H^*_{\mathrm{alg}}(X)/\mathrm{tors}$.

In fact, with small modifications to our arguments (and ignoring the necessary restriction to characteristic 0 when using cohomology), we could use any of the five cycle theories we have described for almost every intersection we compute in this book. The class of smooth projective varieties for which the theories coincide includes any variety with an affine stratification, and in particular all products of projective spaces and Grassmannians; it includes all homogeneous spaces for affine algebraic groups; it includes any projective bundle over a variety of this class, and all blow-ups of varieties in this class along subvarieties in this class. This class represents a tiny fraction of all varieties, but a large fraction of the set of varieties on which we can effectively carry out intersection theory.

C.4 The Lefschetz hyperplane theorem

The Lefschetz hyperplane theorem says that if $X \subset Y$ is a smooth, ample divisor on a smooth projective variety then in all but one dimension the cohomology of X is induced (in a sense we make precise below) from that of Y. The name comes from the use of the theorem to compare the topology of a projective variety with that of its general hyperplane section.

Theorem C.8 (Lefschetz hyperplane theorem)**.** *Let $X \subset Y$ be a smooth subvariety of codimension 1 on a smooth projective variety. If, as a divisor, X is ample, then the restriction map*

$$H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

is an isomorphism for $k < \dim(X)$ and injective for $k = \dim(X)$. Similarly, the map

$$H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z})$$

is an isomorphism for $k < \dim(X)$ and surjective for $k = \dim(X)$.

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Despite the apparent symmetry of these two assertions they are rather different: the morphism

$$H^{2(\dim X)-k} \cong H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}) \cong H^{2(\dim X+1)-k}$$

is induced by inclusion, whereas the homomorphism $H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ may be defined by the intersection of X with a generic translate of a cycle in $H^k(Y, \mathbb{Z})$. We will see the importance of this difference in the examples of Section C.4.1.

By Poincaré duality, the groups $H^k(X, \mathbb{Z})$ for $k > \dim(X)$ are isomorphic to the groups $H_k(X, \mathbb{Z})$ with $k < \dim(X)$, so this describes all the cohomology groups of X in terms of those of Y, except for one: the middle-dimensional cohomology of X.

See Milnor [1963] for an elegant proof of Theorem C.8 following an argument of Andreotti and Frankel [1959]. Here is a sketch: The hypothesis that X is ample in Y implies that the line bundle $\mathcal{L} = \mathcal{O}_Y(X)$ has a Hermitian metric $\|\cdot\|$ with positive curvature. If σ is a global section of $\mathcal{L}$ vanishing on X, then $\|\sigma\|$ is a Morse function on Y with minimum 0 along X. We may apply Morse theory and use the curvature statement to bound the index of the critical points of $\|\sigma\|$. It follows that Y has the homotopy type of a space obtained by attaching cells of dimension $\geq \dim(X)$ to X, from which the Lefschetz theorem follows.

Lefschetz' original proof [1950] applied only to very ample divisors $X \subset Y$, that is, divisors that could be realized as hyperplane sections of Y under some embedding $Y \subset \mathbb{P}^N$ in projective space (hence the name of the theorem). In this setting, Lefschetz took a general pencil of divisors $\{X_\lambda\}_{\lambda \in \mathbb{P}^1}$ including X, and so gave a map

$$\pi : \widetilde{Y} = \mathrm{Bl}_\Gamma(Y) \to \mathbb{P}^1$$

from the blow-up of Y along the base locus $\Gamma = \bigcap X_\lambda$ of this pencil to $\mathbb{P}^1$. The map π is almost a fiber bundle: the fibers are all homeomorphic to X except over the finite number of values $\lambda \in \mathbb{P}^1$ with X_λ singular. By studying the local geometry of the map around these points, Lefschetz was able to relate the cohomology of the variety $\widetilde{Y}$, and hence that of Y, to that of X. Lefschetz' analysis in the end yields more than just the statement above — for example, it tells us about the monodromy action on the cohomology of the elements of the pencil — and is worth reading, despite the difficulties of translation (both linguistic and, more challengingly, mathematical!).

C.4.1 Applications to hypersurfaces and complete intersections

We can apply Theorem C.8 to smooth hypersurfaces in $\mathbb{P}^N$, but also, inductively using Bertini's theorem, to smooth complete intersections in projective space. In the following discussion we will write ζ for the class of a hyperplane in projective space, as well as its restriction to a projective variety X.

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Corollary C.9. *If $X \subset \mathbb{P}^{n+c}$ is a smooth complete intersection of dimension n, then the map*

$$H_k(X,\mathbb{Z}) \to H_k(\mathbb{P}^{n+c},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

induced by the inclusion $X \subset \mathbb{P}^{n+1}$ is an isomorphism for $k < n$ and surjective for $k = n$, and the restriction map

$$H^k(\mathbb{P}^{n+c},\mathbb{Z}) \to H^k(X,\mathbb{Z})$$

is an isomorphism for $k < n$ and injective for $k = n$.

Proof: We induct on the codimension c of the complete intersection X; the case $c = 1$ is the Lefschetz theorem itself. Now suppose that

$$X = V(f_1) \cap \cdots \cap V(f_c)$$

with $\deg f_1 \geq \cdots \geq \deg f_c$. To carry out the inductive step, it suffices to know that

$$X = V(f_1) \cap \cdots \cap V(f_{c-1})$$

is smooth. To achieve this, we may replace $f_1, \ldots, f_{c-1}$ by general forms $f_1', \ldots, f_{c-1}'$ of the same degrees in the ideal $(f_1, \ldots, f_c)$. By Bertini's theorem,

$$X' = V(f_1') \cap \cdots \cap V(f_{c-1}')$$

is smooth away from the base locus X. But since X is a complete intersection in X' any singular point on X' that lies on X would be a singular point on X as well. □

As an application of Corollary C.9, observe that if $Z \subset X$ is any subvariety of codimension $k < n/2$ in X then Corollary C.9 tells us that the class $[Z] \in H^{2k}(X,\mathbb{Z})$ of Z is the restriction to X of an integral cohomology class on $\mathbb{P}^{n+c}$; in other words,

$$[Z] = (\alpha \cdot \zeta^k)|_X$$

for some $\alpha \in \mathbb{Z}$.

In Corollary 6.26 we showed that a smooth hypersurface in projective space cannot contain a linear space of more than half its dimension. Using Corollary C.9, we can say much more:

Corollary C.10. *Let $X \subset \mathbb{P}^{n+c}$ be a smooth complete intersection of dimension n and degree d. If $Z \subset X$ is any subvariety with $\dim(Z) > n/2$, then the degree of Z is divisible by d. In particular, if $Z \subset \mathbb{P}^{n+1}$ is any nondegenerate subvariety of dimension $> n/2$ and prime degree, then Z is contained in no smooth hypersurface.*

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Proof: Let k be the codimension of Z in X. By Lefschetz, we have $[Z] = \alpha\zeta^k$ for some $\alpha \in \mathbb{Z}$, whence

$$\begin{aligned}\deg(Z) &= \deg([Z]\cdot\zeta^{n-k})\\ &= \deg(\alpha\zeta^n)\\ &= \alpha d.\end{aligned}$$

□

Thus, for example, a smooth nondegenerate surface of degree 3 in $\mathbb{P}^4$ (every such surface is a cubic scroll $S(1,2)$) lies on no smooth threefold of degree $d > 3$ in $\mathbb{P}^4$. (In fact, it lies on no smooth hypersurfaces at all; the cases $d = 2$ and 3 can be handled by direct examination.)

There is a substantial strengthening in the case of codimension-1 subvarieties:

Corollary C.11. *If X is a smooth complete intersection of dimension $n \geq 3$ in projective space, any subvariety $Z \subset X$ of codimension 1 in X is the intersection of X with a hypersurface; equivalently, the homogeneous coordinate ring of X is factorial, so every unmixed codimension-1 subscheme of X is the intersection of X with a hypersurface.*

Proof: Since $n = \dim X > 1$, we have

$$H^1(X,\mathbb{Z}) = 0.$$

From the identification of Pic X with $H^1(\mathcal{O}_X^*)$ and the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X,\mathrm{an}} \xrightarrow{\exp} \mathcal{O}_{X,\mathrm{an}}^* \longrightarrow 0$$

introduced in Section C.2.4, it follows that every line bundle on X is determined by its topological Chern class in $H^2(X,\mathbb{Z})$. Since $n > 2$, the map $H^2(\mathbb{P}^N,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is surjective, so every class of codimension 2 on X is the restriction of a class on projective space. Thus every line bundle on X has the form $\mathcal{O}_X(m)$ for some m.

Now suppose that X is a hypersurface of degree d in $\mathbb{P}^{n+1}$. From the sheaf sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(m-d) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0$$

and the fact that $H^1(\mathcal{O}_{\mathbb{P}^{n+1}}(m-d)) = 0$, we deduce that every global section of $\mathcal{O}_X(m)$ is the restriction to X of a global section of $\mathcal{O}_{\mathbb{P}^{n+1}}(m)$ — that is, a homogeneous polynomial of degree m. The same argument, applied inductively, yields the same statement for complete intersections.

Putting this together, we see that if $Z \subset X$ is a divisor then $\mathcal{O}_X(Z) = \mathcal{O}_X(m)$ for some m, and moreover the global section of $\mathcal{O}_X(Z)$ vanishing on Z is the restriction to X of a homogeneous polynomial of degree m. This proves the first statement of the corollary.

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For the factoriality, it suffices to show that every codimension-1 prime ideal P in the homogeneous coordinate ring S_X is principal. We know that the subvariety S defined by P is the complete intersection of X and a hypersurface $V(f)$, and it follows that $(f) = P \cap Q$, where Q contains a power of the irrelevant maximal ideal. But since $S_X/(f)$ is a complete intersection, it is unmixed (see for example Eisenbud [1995, Theorem 11.5]), and we deduce that $(f) = P$.

The statement for unmixed codimension-1 subschemes follows at once. □

Grothendieck eliminated topology and generalized this result substantially; see Call and Lyubeznik [1994] for the statement and a relatively simple proof.

C.4.2 Extensions and generalizations

There have been numerous extensions of the Lefschetz hyperplane theorem:

- One can replace the hypothesis "complete intersection of ample divisors" with the hypothesis "zero locus of a section of an ample vector bundle" (see for example Matsumura [2014]).
- More fundamentally, it became clear from Milnor's proof that over $\mathbb{C}$ the result should be strengthened to a result on homotopy in place of homology; see for example Barth [1975] for a development in this direction.
- Singularities on Y away from X seem to play a secondary role. For statements allowing such singularities, see for example Okonek [1987] and Hamm [1995].
- Another stream of activity has centered on subvarieties "of low codimension" in projective space that are *not* complete intersections. It turns out that these are very special (see for example the survey Hartshorne [1974]), so it is reasonable to hope that there might be analogous theorems for them. For example, we have:

Theorem C.12 (Larsen [1973]). *If $X \subset \mathbb{P}^{n+r}$ is a smooth codimension-r subvariety then the restriction map*

$$H^k(\mathbb{P}^{n+r}_{\mathbb{C}}, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for $k \le n - r$ and injective for $k = n - r + 1$.

Thus, for example, if $X \subset \mathbb{P}^5$ is a smooth threefold then $H^1(X, \mathbb{Z}) = 0$. See also the simplified proof in Barth [1975], which includes a homotopy-theoretic version.

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C.5 The hard Lefschetz theorem and Hodge–Riemann bilinear relations

In this section we briefly describe two further results on the topology of varieties that are frequently used (though not in this text). A reference for both is Griffiths and Harris [1994, Section 0.7]. The first is called the "Hard Lefschetz theorem" in English; the more colorful French name is "Théorème de Lefschetz vache."

Theorem C.13 (Hard Lefschetz theorem)**.** *Let $X \subset \mathbb{P}^r$ be a smooth projective variety of dimension n. If $\zeta \in H^2(X, \mathbb{C})$ is the class of a hyperplane section of X, then the map*

$$\bigcup \zeta^k : H^{n-k}(X, \mathbb{C}) \to H^{n+k}(X, \mathbb{C})$$

is an isomorphism for all $k = 1, \ldots, n$.

We define the *primitive cohomology groups* of X for $m \leq n$ as

$$P^{n-k}(X) := \operatorname{Ker}\Big(\bigcup \zeta^{k+1} : H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})\Big),$$

and using Theorem C.13 we get the *Lefschetz decomposition*

$$H^m(X, \mathbb{C}) = \bigoplus \zeta^k \cdot P^{m-2k}(X)$$

for $m \leq n$.

Since the class ζ is of type $(1, 1)$, the Lefschetz decomposition is compatible with the Hodge decomposition: If for $k = p + q$ we set

$$P^{p,q}(X) = P^k \cap H^{p,q}(X),$$

then we have

$$P^k(X) = \bigoplus_{p+q=k} P^{p,q}(X).$$

In these terms, we can state the *Hodge–Riemann bilinear relations*. First, we define a Hermitian inner product

$$Q : H^{n-k}(X, \mathbb{C}) \times H^{n-k}(X, \mathbb{C}) \to \mathbb{C}$$

on the cohomology of X in middle degree or lower by

$$Q(\alpha, \beta) = (\alpha \cup \overline{\beta} \cup \zeta^k)[X].$$

Theorem C.14 (Hodge–Riemann bilinear relations)**.** *For $\alpha \in P^{p,q}(X)$,*

$$i^{p-q}(-1)^{\binom{n-k}{2}} Q(\alpha, \overline{\alpha}) > 0.$$

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When n is even we can use this to express the index of the cup product on $H^n(X, \mathbb{R})$ in terms of the dimensions of the primitive cohomology groups of X. For example, when X is a surface, the Lefschetz decomposition of $H^2(X, \mathbb{C})$ has the form

$$H^2(X, \mathbb{C}) = P^2(X) \oplus \mathbb{C}\langle\zeta\rangle,$$

and we conclude the signature of the cup product on $H^2(X, \mathbb{R})$ is $(2h^{2,0} + 1, h^{1,1} - 1)$. As a consequence:

Corollary C.15. *Let $S \subset \mathbb{P}^N$ be a smooth surface of degree d. If $\gamma = [C]$ is the class of a curve C of degree e on S, then*

$$\deg(\gamma^2) \leq \frac{e^2}{d}.$$

Proof: If γ is a multiple of the class ζ of a hyperplane section this is immediate (and we have equality above); if not, this is just the statement that the intersection pairing on the group $\mathbb{Z}\langle\zeta, \gamma\rangle \subset A^1(S)$ generated by the classes ζ and γ has signature $(1, 1)$ (and we have a strict inequality). □

Note that the statement of Corollary C.15 makes no reference to homology, and in fact it is true over any field; see Beauville [1996].

C.6 Chern classes in topology and differential geometry

If X is a smooth complex projective algebraic variety and $\mathcal{E}$ is an algebraic vector bundle on X, then applying the map $\eta : A(X) \to H^*(X, \mathbb{Z})$ to the Chern class of $\mathcal{E}$ we get a class in $H^*(X, \mathbb{Z})$, which we also denote by $c(\mathcal{E})$ and call the *topological Chern class* of $\mathcal{E}$.

It is possible to define $c(E) \in H^*(X, \mathbb{Z})$ much more generally, for any continuous complex vector bundle E on any reasonably nice topological space, say a simplicial complex. Here we think of E as a topological space, with a map $\pi : E \to X$ whose fibers are complex vector spaces $\mathbb{C}^n$, rather than as a locally free sheaf, and we denote by $E^\times$ the complement of the zero section in E.

The topological Chern classes are among the *characteristic classes* of vector bundles. Such classes were defined by Stiefel and Whitney (see for example Whitney [1941]) before they came into algebraic geometry. In this section we will sketch the original topological idea behind the construction.

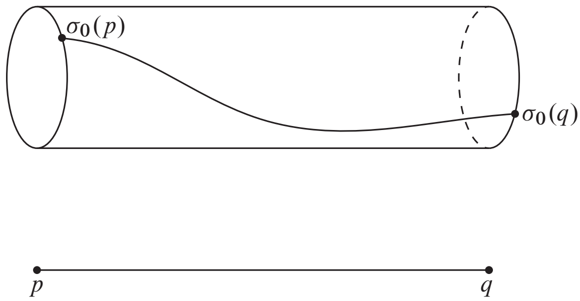

Figure C.1 Extending the section σ_0 over a 1-simplex.

C.6.1 Chern classes and obstructions

As we have seen, the top Chern class in algebraic geometry may be thought of as an obstruction to the existence of a nowhere-vanishing global section. In the category of manifolds (or more generally simplicial complexes) and continuous maps, we can make this precise, as follows:

Let E be a complex vector bundle on a topological space X that is the underlying space of a simplicial complex. We start by choosing an arbitrary section σ_0 of $E^\times$ over the 0-skeleton X_0 of X — that is, we select arbitrary nonzero vectors $\sigma_0(p) \in E_p^\times$ in the fibers of E over the vertices $p \in X$ of our complex.

Next we extend σ_0 to a section σ_1 of $E^\times$ defined over the 1-skeleton of X. We can always do this: Since a closed 1-cell $\Delta_1 \subset X$ is contractible, the restriction of E to Δ_1 is trivial, so that finding a nonzero section of E over Δ_1 with the assigned values $\sigma_0|_{\partial\Delta_1}$ on the boundary $\partial\Delta_1$ amounts to finding an arc $\gamma : [0, 1] \to \mathbb{C}^n \setminus \{0\}$ with given starting and ending points; since $\mathbb{C}^n \setminus \{0\}$ is connected, we can always do this (see Figure C.1).

We continue in this way, extending σ_1 to a section of $E^\times$ over successively larger skeleta of X: Given a section σ_{k-1} of $E^\times$ over the $(k-1)$-skeleton X_{k-1}, for each k-simplex $\Delta_k \subset X$ we trivialize the bundle on Δ_k, and view the problem of extending σ_{k-1} over Δ_k as that of giving a homotopy of $\sigma_{k-1}|_{\partial\Delta_k}$ with the constant map. We can certainly make the extension as long as $\pi_{k-1}(\mathbb{C}^n \setminus \{0\}) = 0$.

We first encounter difficulty when $k = 2n$: Since $\pi_{2n-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{Z}$, we may not be able to extend σ_{2n-1} over X_{2n}. As a measure of the obstruction, we define a simplicial $2n$-cochain $\alpha \in C^{2n}(X, \mathbb{Z})$ by

$$\alpha(\Delta_{2n}) = [\sigma_{2n-1}|_{\partial\Delta_k}] \in \pi_{2n-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{Z}.$$

One can show that the cochain α is a cocycle and that while α depends on the choices made its cohomology class does not.

In this context we define the *topological Chern class $c_n(E)$ of E* to be

$$c_n(E) = [\alpha] \in H^{2n}(X, \mathbb{Z}).$$

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The nonvanishing of $c_n(E)$ is an *obstruction* to finding a nowhere-zero section of E in the sense that the nonvanishing of $[\alpha]$ implies that we cannot have a nowhere-zero section of E, but the converse is not true: the vanishing of $[\alpha]$ says only that we can find such a section over the $2n$-skeleton X_{2n}.

The other topological Chern classes are defined similarly. To measure the obstruction to finding k everywhere-independent sections of E, we introduce the bundle of k-frames in E: the space

$$F_k(E) = \{(p, v_1, \dots, v_k) \mid p \in X \text{ and } v_1, \dots, v_k \in E_p \text{ are independent}\}.$$

The fibers of $F_k(E)$ over X are *frame manifolds*

$$F_k = \{v_1, \dots, v_k \in \mathbb{C}^n \mid v_1 \wedge \cdots \wedge v_k \neq 0\},$$

and a simple calculation with the long exact sequence in homotopy shows that we have

$$\pi_i(F_k) = \begin{cases} 0 & \text{if } i < 2n - 2k + 1, \\ \mathbb{Z} & \text{if } i = 2n - 2k + 1. \end{cases}$$

Thus we get a class in $H^{2n-2k+2}(X, \mathbb{Z})$, which is an obstruction to finding a section of $F_k(E)$. This class agrees with the Chern class $c_{n-k+1}(E)$ in the cases where the latter is defined.

One can show that when E is an algebraic vector bundle with enough sections this definition of the Chern classes agrees with their characterization as classes of degeneracy loci. For example, if σ is a global holomorphic section of E vanishing on a generically reduced subscheme $A \subset X$ of codimension n, we can choose the simplicial structure on X to be transverse to the zero locus of σ, meaning that $V(\sigma)$ is disjoint from the $(2n-1)$-skeleton of X and intersects each $2n$-simplex of X transversely in a finite number of points. If we choose the section σ_{2n-1} of $E^\times$ over the $(2n-1)$-skeleton on X to be the restriction of σ to X_{2n-1}, then the obstruction cocycle α will associate to each $2n$-simplex Δ_{2n} the number of its points of intersection with $V(\sigma)$, counted with appropriate sign. Unwinding the definitions, the cohomology class of this cocycle is exactly the Poincaré dual of the fundamental class of $V(\sigma)$.

C.6.2 Chern classes and curvature

Under slightly stronger hypotheses, Chern [1946] characterized the classes $c_k(E)$ in terms of curvature. Suppose that X is a differentiable manifold and E a differentiable complex vector bundle on X. Choose a Riemannian metric on the manifold X and a Hermitian metric on the vector bundle E — that is, a Hermitian inner product on each fiber of the bundle, varying differentiably with the point, so that in terms of a local trivialization of E the entries of the Hermitian matrix giving the inner product on E_p are $\mathcal{C}^\infty$ functions of p. The metric on the bundle E defines a notion of *parallel transport*:

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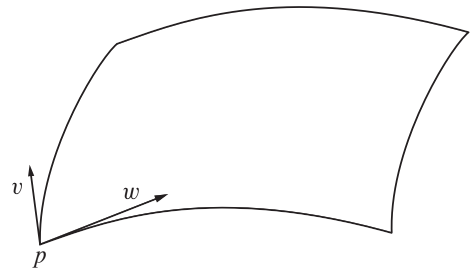

Figure C.2 A small geodesic quadrilateral with vertex at p.

given an arc γ in X, the metric determines a canonical way to identify the fibers of E over the points of the arc.

Now, suppose that $v, w \in T_p X$ are tangent vectors at a point $p \in X$. We can form a small geodesic square with a vertex at p and sides at p that are geodesics of length ϵ in the directions v and w; see Figure C.2.

Parallel transport around the perimeter of this square yields an automorphism $\varphi(v, w, \epsilon)$ of the fiber E_p; since $\varphi(v, w, \epsilon)$ goes to the identity as $\epsilon \to 0$, we arrive at an endomorphism $A(v, w)$ of E_p

$$A(v, w) = \lim_{\epsilon \to 0} \frac{\varphi(v, w, \epsilon) - \mathrm{Id}}{\epsilon}.$$

The endomorphism $A(v, w)$ is bilinear and skew-symmetric in v and w, so that we can think of A as an element of $\mathcal{H}om(\bigwedge^2 T_p X, \mathrm{End}(E_p))$; as p varies this gives us a global section Θ of the bundle

$$\mathcal{H}om(\textstyle\bigwedge^2 T(X), \mathrm{End}(E)) = \Omega_X^2 \otimes \mathrm{End}(E).$$

The section Θ is called the *curvature form* of the metric.

In terms of a local trivialization of E near p, we may think of Θ as an $n \times n$ matrix of differentiable 2-forms, called the *curvature matrix* of the metric. If we change the trivialization of E, the matrix Θ is replaced by its conjugate under the change-of-basis matrix. The coefficients of the characteristic polynomial of this matrix are thus well-defined global forms ω_{2k} on X for $k = 0, 1, \dots, n$.

Chern showed that:

(a) The forms ω_{2k} are closed.
(b) The de Rham cohomology classes $[\omega_{2k}] \in H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$ are independent of the choice of metric.
(c) The class $[\omega_{2k}]$ is the image of the class $c_k(E) \in H^{2k}(X, \mathbb{Z})$ under the natural map $H^{2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{C}) = H^{2k}_{\mathrm{dR}}(X, \mathbb{C})$.

See Chern [1946] for a proof.

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The analogs of many of the theorems proved in Chapter 5 and elsewhere for the algebraic Chern classes $c(E) \in A(X)$ hold true in the topological setting as well: The Whitney product formula and the splitting principle, and their consequences, such as the formula for the Chern class of a tensor product with a line bundle (Proposition 5.17), hold more generally for topological vector bundles, and indeed can be proved along similar lines. In particular, the analog of the formula of Theorem 9.6 for the Chow ring of a projective bundle holds true for the cohomology ring of a projective bundle as well (see for example Bott and Tu [1982]), a fact we will use in Appendix D.

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