

# D-Modules, Unit $F$ -Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

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# 1 Questions

**Question 1.1.** What is the point of D-modules? Why do people care about them? What sort of questions do they answer? What insights do they provide?

**Question 1.2.** Given a scheme  $X$  over a field  $k$ , we define the sheaf  $\mathcal{E}nd_k(\mathcal{O}_X) = \mathcal{H}om_k(\mathcal{O}_X, \mathcal{O}_X)$  to be the sheaf of  $k$ -linear endomorphisms on  $\mathcal{O}_X$  as in [Gie75, Definition 0.3]. What I think this means is this: we have a structure morphism  $X \rightarrow \text{Spec } k$  and we can therefore think of  $\mathcal{O}_X$  as a  $\mathcal{O}_{\text{Spec } k}$ -module via the map  $f^\# : \mathcal{O}_{\text{Spec } k} \rightarrow f_* \mathcal{O}_X$ . However, this is rather stupid, since this means  $\mathcal{E}nd_k(\mathcal{O}_X)$  is a  $\mathcal{O}_{\text{Spec } k}$ -module, which should mean it takes open sets from  $\text{Spec } k$ . In the book [HTT08, Example A.4.2.] however, this sheaf takes open sets from  $X$ . What's going on here?

**Question 1.3.** How can one "naturally" make  $A_m$  a subalgebra of  $A_n$  when  $m \leq n$ ? It seems like there are  $n$  many subalgebras of  $A_n$  isomorphic to  $A_{n-1}$ , for example.

## 2 Some Non-Commutative Algebra

D-modules requires non-commutative algebra. Necessary facts are found here.

### 2.1 Filtered rings and modules

This subsection follows Ginzburg's notes quite closely, see [BIBTEX SETUP, GINZBURG D-MODULES Page 3].

**Definition 2.1 (Filtered Ring).** Let  $A$  be an associative ring with unit. We call  $A$  a *filtered ring* if an increasing filtration  $\dots \subset A_i \subset A_{i+1} \subset \dots$  by additive subgroups is given such that

- (i)  $A_i A_j \subset A_{i+j}$
- (ii)  $1 \in A_0$ ,
- (iii)  $\bigcup A_i = A$ , i.e. the filtration is *exhausting*.

Typically, either (a)  $\mathbb{N}$  or (b)  $\mathbb{Z}$  is chosen for the index set. In the former case  $A$  is said to be *positively filtered*. Note that (a) can be viewed as a special case of (b) by setting  $A_{-1} = 0$ . In the latter case we will consider the topology induced by the filtration by taking  $\{A_i\}_{i \in \mathbb{Z}}$  to be the base of open sets, and we then impose two additional conditions:

- (iv)  $\bigcap A_i = \{0\}$ , i.e. the topology defined by  $\{A_i\}$  is *separating*
- 1.  $A$  is complete with respect to this topology.

Finally, we denote by  $\text{gr } A$  the associated graded ring  $\bigoplus A_i / A_{i-1}$ .

### 3 Differential Operators and D-Modules

**Definition 3.1** (Quasi-coherent #1). Fix  $X$  a scheme over  $k$ ,  $\mathcal{O}_X$  the structure sheaf,  $\mathcal{F}$  a sheaf of  $\mathcal{O}_X$ -modules. We call  $\mathcal{F}$  a *quasi-coherent* sheaf of  $\mathcal{O}_X$ -modules (or simply an  $\mathcal{O}_X$ -modules) if it satisfies the condition

$$\text{If } U \subseteq X \text{ an open affine, } f \in \mathcal{O}_X(U), \text{ and } U_f = \{u \in U \mid f(u) \neq 0\},$$

then  $\mathcal{F}(U_f) = \mathcal{F}(U)_f = \mathcal{O}_X(U_f) \otimes_{\mathcal{O}_X(U)} \mathcal{F}$ .

**Definition 3.2** (Quasi-coherent #2). Let  $(X, \mathcal{O}_X)$  be a scheme. A sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  is quasi-coherent if  $X$  can be covered by affine opens  $U_i = \text{Spec } A_i$  such that for each  $i$  there exists an  $A_i$  module  $M_i$  with  $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ . We say  $\mathcal{F}_i$  is coherent if each  $M_i$  can be taken to be finitely generated.

**Remark 3.3.** If  $A$  is a ring and  $M$  an  $A$ -module, the sheaf associated to  $M$  is denoted by  $\tilde{M}$  and is formed as follows. For each  $\mathfrak{p} \in \text{Spec } A$ ,  $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$  is the localization with respect to  $\mathfrak{p}$ . Given an open set  $U \subseteq \text{Spec } A$ , define

$$\tilde{M}(U) = \left\{ s : U \rightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \mid s(\mathfrak{p}) \in M_{\mathfrak{p}}, \text{ and locally } s = \frac{m}{f}, m \in M, f \in A \right\}.$$

More verbosely, this last condition means that for each  $\mathfrak{p} \in U$  there is a neighborhood  $V \subseteq U$  of  $\mathfrak{p}$  such that for each  $\mathfrak{q} \in V$ ,  $f \notin \mathfrak{q}$  and  $s(\mathfrak{q}) = \frac{m}{f} \in M_{\mathfrak{q}}$ .

Alternatively, one may define

$$\tilde{M}(U_f) = M_f,$$

and then

$$\tilde{M}(U) = \varinjlim_{U_f \subseteq U} \tilde{M}(U_f).$$

Note that  $U_f$  is implied to be a distinguished open in one of the  $U_i$ , so really we need to take the limit above over all  $U_f$  in all  $U_i$  which intersect  $U$  nontrivially. This is a non-issue if  $U$  is affine.

**Lemma 3.4.** The following are equivalent conditions for  $\mathcal{F}$  a sheaf of  $\mathcal{O}_X$  modules:

- (a)  $\mathcal{F}$  is the direct limit of its coherent subschemes
- (b) For any Zariski open affine subset  $U \subseteq X$  and any  $f \in \mathcal{O}(U)$  one has  $\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$ .

A *quasi-coherent* sheaf is then one which satisfies these conditions.

**Lemma 3.5** (Noether Normalization Lemma). Let  $k$  be a field,  $A$  a finitely generated  $k$ -algebra. Then there exists algebraically independent elements  $y_1, \dots, y_d$  in  $A$  for some positive  $d$  such that  $A$  is finitely generated as a module over  $k[y_1, \dots, y_n]$ .

**Remark 3.6.** The Noether normalization lemma provides a way to define differential operators using a manifold-esque coordinate approach. I prefer the following coordinate-free approach provided by Gröthendieck, however.

**Definition 3.7** (Differential Operators). Let  $A$  be a commutative ring. For any pair of  $A$ -modules  $M, N$  we define the module  $\mathcal{D}iff_A^k(M, N)$  inductively as follows:

$$(i) \mathcal{D}iff_A^0(M, N) = \text{Hom}_A(M, N)$$

$$(ii) \mathcal{D}iff_A^{k+1}(M, N) = \left\{ \text{additive maps } u : M \rightarrow N \mid \forall a \in A, (au - ua) \in \mathcal{D}iff_A^k(M, N) \right\}$$

It follows from the definition that  $\mathcal{D}iff_A^k(M, N) \subset \mathcal{D}iff_A^{k+1}(M, N)$ . We define

$$\mathcal{D}iff_A(M, N) := \bigcup_k \mathcal{D}iff_A^k(M, N).$$

In the case that  $M = N = A$ , we write  $\mathcal{D}iff_A(M)$  and note that it is a filtered almost commutative ring.

The case in which we will be most interested is when  $M = N = A$ , i.e. when we consider  $A$  to be a ring over itself. Let's repeat the above construction for this case.

**Definition 3.8.** Let  $A$  be a commutative  $K$ -algebra for  $K$  a characteristic 0 field. Let  $D \in \text{End}(R)$ . We define the **order** of  $D$  inductively.

- $\text{ord}(D) = 0$  if  $[a, D] = -[D, a] = 0$  for all  $a \in A$ .
- $\text{ord}(D) = n \in \mathbb{Z}_{\geq 0}$  if  $\text{ord}(D) \neq k$  for all  $k < n$  and if  $\text{ord}([a, D]) = k_a$  for some  $k_a < n$  for each  $a \in A$ .

The set  $D^n(R)$  is defined to be the  $K$ -vector space of all operators of order  $\leq n$ .

**Definition 3.9.** A **derivation**  $D \in \text{End}(R)$  is an operator which satisfies the Leibniz rule

$$D(ab) = aD(b) + D(a)b$$

for every  $a, b \in A$ . The set of all derivations  $\text{Der}_K(A) \subseteq \text{End}_K(A)$  is a  $K$ -vector space and a left  $A$ -module under the action  $(a \cdot D)(b) = a(D(b))$ .

All derivations are order 1 operators. As one might hope, they're actually *all* order 1 operators.

**Lemma 3.10.**  $D^1(A) = \text{Der}_K(R) + R$ . (See proof in *A Primer on D-modules* page 21.)

We can now define the ring of differential operators on  $A$ .

**Definition 3.11.** Let  $A$  be a  $K$ -algebra with  $K$  a characteristic zero field. The set of all finite order operators on  $A$  forms a noncommutative ring with pointwise addition and composition as multiplication. We denote this ring by  $D(A)$  and we call it the **ring of differential operators**:

$$D(A) = \bigcup_{n \in \mathbb{Z}_{\geq 0}} D^n(A) \subseteq \text{End}_K(A).$$

It's obvious that the addition of two finite order operators yields a differential operator with order equal to the maximum order of the two summands, it's *not* obvious that the composition of two finite order operators yields a finite order operator. We therefore require the following proposition for this definition to work:

**Proposition 3.12.** If  $D \in D^n(A)$  and  $D' \in D^m(A)$  then  $D \circ D', D' \circ D \in D^{n+m}(A)$ .

### 3.1 First examples of Differential Operators

**Example 3.13.** Let  $K$  be a field of characteristic zero and recall that  $K[x_1, \dots, x_n]$  is an infinite dimensional  $K$  vector space. We define  $\hat{x}_i, \partial_i \in \text{End}_K(K[x_1, \dots, x_n])$  by  $\hat{x}_i f \mapsto x_i \cdot f$  and  $\partial_i f \mapsto \frac{\partial f}{\partial x_i}$ . One can then check that  $[\partial_j, \hat{x}_j] = \partial_j \hat{x}_j - \hat{x}_j \partial_j = \delta_{ij} \text{id}$  where  $\text{id}$  is the identity operator on  $K[x_1, \dots, x_n]$  and  $\delta_{ij}$  is the Kronecker delta. In other words,

$$[\partial_i, \hat{x}_i](f) = f \quad \text{and} \quad [\partial_i, \hat{x}_j](f) = 0$$

when  $i \neq j$ . This is quite easy to check for an arbitrary polynomial but is nonetheless quite magical:

$$\begin{aligned} \partial_x (x \cdot (3x^2 + 2y)) &= 9x^2 + 2y, \\ x \cdot (\partial_x (3x^2 + 2y)) &= 6x^2, \\ (\partial_x \cdot \hat{x} - \hat{x} \cdot \partial_x)(3x^2 + 2y) &= 3x^2 + 2y, \end{aligned}$$

but

$$\partial_x (y \cdot (3x^2 + 2y)) - y \cdot (\partial_x (3x^2 + 2y)) = 6xy - 6xy = 0.$$

**Definition 3.14.** The  $n$ th Weyl algebra of  $K$  is the  $2n$ -dimensional  $K$ -subalgebra of  $\text{End}_K(K[x_1, \dots, x_n])$  generated by  $\hat{x}_1, \dots, \hat{x}_n, \partial_1, \dots, \partial_n$ , and is denoted by  $A_n(K)$  or  $A_n$  when the field is known. We let  $A_0(K) = K$ .

Note also that for  $m \leq n$ , we can make  $A_m$  a subalgebra of  $A_n$  in a "natural way".

### 3.2 D-Modules

**Definition 3.15.** A  $\mathcal{D}$ -module is a sheaf over the sheaf  $\mathcal{D}_X$  of regular differential operators over a variety (scheme, manifold, analytic complex manifold) which is quasi-coherent as an  $\mathcal{O}_X$ -module.

## 4 Bernstein-Sato Polynomial

**Theorem 4.1** (Björk, Kashiwara). *Let  $X$  be a smooth variety over the complex numbers and let  $f$  be a non-invertible regular function on  $X$  (i.e. a locally rational function whose numerator is non-invertible). There exists a polynomial  $b(s) \in \mathbb{C}[s]$  and a polynomial  $P(s) \in \mathcal{D}_X[s]$  whose coefficients are differential operators on  $X$ , such that the relation*

$$P(s)f^{s+1} = b(s) \cdot f^s$$

*holds formally in the  $\mathcal{D}_X$ -module  $\mathcal{O}_X[\frac{1}{f}, s] \cdot f^s$ . Here,  $f^{s+1}$  stands for  $f \cdot f^s$ .*

## References

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- [HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*. Vol. 236. Progress in Mathematics. Translated from the 1995 Japanese edition by Takeuchi. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. xii+407. ISBN: 978-0-8176-4363-8. DOI: 10.1007/978-0-8176-4523-6. URL: <https://doi.org/10.1007/978-0-8176-4523-6>.