III Local fields

<u>Definition 7.1</u>: Let $(K, |\cdot|)$ be a valued field. K is a <u>local field</u> if it is complete and locally compact.

E.g. $\mathbb{R}$ and $\mathbb{C}$ are local fields.

<u>Proposition 7.2</u>: Let $(K, |\cdot|)$ be a non-arch. complete valued field. TFAE:

(i) K is locally compact.

(ii) $\mathcal{O}_K$ is compact.

(iii) v is discrete and $k := \mathcal{O}_K/m$ is finite.

Proof: (i) $\Rightarrow$ (ii) Let $U \ni 0$ be a compact neighbourhood of 0. Then $\exists x \in \mathcal{O}_K$ s.t. $x\mathcal{O}_K \subseteq U$. Since $x\mathcal{O}_K$ is closed, $x\mathcal{O}_K$ is compact $\Rightarrow$ $\mathcal{O}_K$ is compact

($x\mathcal{O}_K \xrightarrow{x^{-1}} \mathcal{O}_K$ is a homeomorphism

(ii) $\Rightarrow$ (i) $\mathcal{O}_K$ compact $\Rightarrow$ $a + \mathcal{O}_K$ compact $\forall a \in K$
$\Rightarrow$ K locally compact.

(ii) $\Rightarrow$ (iii) Let $x \in m$, and $A_x \subseteq \mathcal{O}_K$ be a set of coset reps. for $\mathcal{O}_K/x\mathcal{O}_K$.

Then $\mathcal{O}_K = \bigcup_{a \in A_x} a + x\mathcal{O}_K$ a disjoint open

$y \in A_x$

cover.

$\Rightarrow A_x$ is finite by compactness of $\mathcal{O}_K$

$\Rightarrow \mathcal{O}_K / x\mathcal{O}_K$ is finite

$\Rightarrow \mathcal{O}_K / m$ is finite.

Suppose v is not discrete.

Let $x = x_1, x_2, x_2, \ldots$ s.t.

$v(x_1) > v(x_2) > v(x_3) > \ldots > 0$.

Then $x\mathcal{O}_K \subsetneq x_2\mathcal{O}_K \subsetneq x_3\mathcal{O}_K \subsetneq \ldots \subsetneq \mathcal{O}_K$.

But $\mathcal{O}_K / x\mathcal{O}_K$ is finite so can only have finitely many subgroups. ※

(iii) $\Rightarrow$ (ii) Since $\mathcal{O}_K$ is a metric space, it suffices to show $\mathcal{O}_K$ is sequentially compact. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{O}_K$ and fix $\pi \in \mathcal{O}_K$ a uniformizer in $\mathcal{O}_K$.

Since $\pi^i\mathcal{O}_K / \pi^{i+1}\mathcal{O}_K \cong k$, $\mathcal{O}_K / \pi^i\mathcal{O}_K$ is finite ($\mathcal{O}_K \supseteq \pi\mathcal{O}_K \supseteq \ldots \supseteq \pi^i\mathcal{O}_K$), $\forall i \geq 1$.

Since $\mathcal{O}_K / \pi\mathcal{O}_K$ is finite, $\exists a \in \mathcal{O}_K / \pi\mathcal{O}_K$ and a subsequence $(x_{1,n})_{n=1}^{\infty}$ s.t. $x_{1,n} \equiv a \bmod \pi$

We define $y_1 = x_{1,1}$.

Since $\mathcal{O}_K / \pi^2\mathcal{O}_K$ is finite, $\exists a_2 \in \mathcal{O}_K / \pi^2\mathcal{O}_K$

and a subsequence $(x_{2,n})^{\infty} \subseteq (x_{1,n})_{n=1}^{\infty}$ s.t.

and a subsequence $(x_{2n})_{n=1}^{\infty}$ of $(x_{1n})_{n=1}^{\infty}$ s.t.

$x_{2n} \equiv a_2 \mod \pi^2 \mathcal{O}_K$.

Define $y_2 = x_{22}$.

Continuing in this fashion, we obtain sequences $(x_{in})_{n=1}^{\infty}$ for $i = 1, 2, \dots$

s.t. (1) $(x_{i+1\,n})_{n=1}^{\infty}$ is a subsequence of $(x_{in})_{n=1}^{\infty}$

(2) For any i, $\exists a_i \in \mathcal{O}_K / \pi^i \mathcal{O}_K$ s.t.

$x_{in} \equiv a_i \mod \pi^i \quad \forall n$.

Then necessarily $a_i \equiv a_{i+1} \mod \pi^i \ \forall i$.

Now choose $y_i = x_{ii}$; this defines a subsequence $(x_n)_{n=1}^{\infty}$. Moreover $y_i \equiv a_i \equiv a_{i+1} \equiv y_{i+1} \mod \pi^i$

Thus y_i is Cauchy, hence converges by completeness. □

Eg. (i) $\mathbb{Q}_p$ is a local field.

(ii) $\mathbb{F}_p((t))$ is a local field.

More on inverse limits.

Let $(A_n)_{n=1}$ a sequence of sets/groups/rings and $\varphi_n : A_{n+1} \to A_n$ homomorphisms.

<u>Definition 7.3:</u> Assume A_n is finite. The <u>profinite topology</u> on $A := \varprojlim_n A_n$ is the weakest topology on A s.t. $A \to A_n$ is continuous $\forall n$, where A_n are equipped

with the discrete topology.

Fact: $A = \varprojlim_n A_n$ with profinite topology, is compact, totally disconnected and Hausdorff.

Proposition 7.4: Let K be a local field. Under the isomorphisms

$$\mathcal{O}_K \cong \varprojlim \mathcal{O}_K/\pi^n \mathcal{O}_K$$

($\pi \in \mathcal{O}_K$ a uniformizer), the topology on $\mathcal{O}_K$ coincides with the profinite topology.

Proof: One checks that the sets

$$\mathcal{B} := \{ a + \pi^n \mathcal{O}_K \mid n \in \mathbb{N}_{\geq 1},\ a \in A_{\pi^n} \}$$

A_{π^n} is a set of coset reps for $\mathcal{O}_K/\pi^n\mathcal{O}_K$

is a basis of open sets in both topologies.

For $|\cdot|$: clear.

For profinite topology: $\mathcal{O}_K \to \mathcal{O}_K/\pi^n\mathcal{O}_K$ is continuous iff $a + \pi^n \mathcal{O}_K$ open $\forall a \in A_{\pi^n}$.

Thus $\mathcal{B}$ is basis for profinite topology. $\square$

Lemma 7.5: Let K be a non-arch. local field and L/K a finite extension. Then L is a local field.

Proof: Theorem 6.1 $\Rightarrow$ L complete and discretely valued.

Suffices to show $k_L := \mathcal{O}_L/m_L$ is finite.

Let $\alpha_1, \dots, \alpha_n$ be basis for L as a K-v.s.

$\| \, \|_{\sup}$ (sup norm) equiv. to $|\cdot|_L$ implies there exists $r > 0$ s.t.

$$\mathcal{O}_L \subseteq \{ x \in L : \|x\|_{\sup} \leq r \}$$

Take $a \in K$ s.t. $|a| \geq r$, then

$$\mathcal{O}_L \subseteq \bigoplus_{i=1}^{n} a \alpha_i \mathcal{O}_K$$

$\Rightarrow \mathcal{O}_L$ is fin. gen. as a module over $\mathcal{O}_K$.

$\Rightarrow k_L$ is fin. gen. over k. $\square$

<u>Definition 7.7</u>: A discretely valued field $(K, |\cdot|)$ has equal characteristic if $\mathrm{char}(K) = \mathrm{char}(k)$. Otherwise it has mixed characteristic.

Eg. $\mathbb{Q}_p$ has mixed char.

Note: If K is a non-arch. local field, K has mixed char. (resp. equal char.) iff $\mathrm{char} K = 0$ (resp. $\mathrm{char} K > 0$).

<u>Theorem 7.8</u>: Let K be a non. arch. local field of equal characteristic $p > 0$. Then

$$K \cong \mathbb{F}_{p^n}((t)) \quad \text{some } n \geq 1.$$

Proof: K complete discretely valued, $\mathrm{char}\, K > 0$.

Moreover $k \cong \mathbb{F}_{p^n}$ is finite, hence perfect.

By Theorem 5.7, $K \cong \mathbb{F}_{p^n}((t))$. $\square$

<u>Lemma 7.9</u>: An abs. value $|\cdot|$ on a field is non-archimedean iff $|n|$ is bounded $\forall n \in \mathbb{Z}$.

Proof: "$\Rightarrow$" Since $|-1| = 1$, $|-n| = |n|$, thus suffices to show $|n|$ bounded for $n \geq 1$.

Then $|n| = |1+1+\dots+1| \leq 1$.

"$\Leftarrow$" Suppose $|n| \leq B$ $\forall n \in \mathbb{Z}$.

Let $x, y \in K$ with $|x| \leq |y|$. Then we have

$$|x+y|^m = \left|\sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i}\right|$$

$$\leq \sum_{i=0}^{m} \left|\binom{m}{i} x^i y^{m-i}\right|$$

$$\leq |y|^m (m+1) B$$

& Taking m^{th} roots gives

$$|x+y| \leq |y| \big[\underbrace{(m+1)B\big]^{1/m}}_{\to 1 \text{ as } m \to \infty}$$

$$|x+y| \leq |y| = \max(|x|, |y|)$$ $\square$