2 Schemes

Exercise 2.1. *Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.*

Solution. Let $\phi : A \to A_f$ be the obvious ring homomorphism. This evidently defines a scheme morphism $\operatorname{Spec} A_f \to \operatorname{Spec} A$.

The ideal generated by the image $\phi\mathfrak{p}$ of a prime ideal $\mathfrak{p}$ of A is the set $A_f\mathfrak{p} = \{\frac{a}{f^n} : n \in \mathbb{N}, a \in \mathfrak{p}\}$ which is prime in A_f, and its inverse image $\phi^{-1}(A_f\mathfrak{p})$ is $\mathfrak{p}$, unless $f \in \mathfrak{p}$ in which case $A_f\phi\mathfrak{p} = A_f$. So the morphism $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is surjective onto the underlying space of $D(f)$.

Let $\mathfrak{p}, \mathfrak{q}$ be two points of $\operatorname{Spec} A_f$ that get sent to the same image in $\operatorname{Spec} A$. This means that all of their elements of the form $\frac{a}{1}$ are the same. Now $\frac{a}{f^n} \in \mathfrak{p}$ if and only iff $f^n \frac{a}{f^n} = a \in \mathfrak{p}$ if and only if $a \in \mathfrak{q}$ if and only if $\frac{1}{f^n} a = \frac{a}{f^n} \in \mathfrak{q}$ and so $\mathfrak{p} = \mathfrak{q}$. Hence, the morphism is injective on the underlying space of $\operatorname{Spec} A_f$.

This bijection of sets is continuous automatically since if comes from a ring homomorphism. To see that it is a homeomorphism we need to show that it is open. Let $\mathfrak{a} \subset A_f$ be an ideal and $\mathfrak{b} = (f) \cap \phi^{-1}\mathfrak{a} \subset A$. A prime ideal $\mathfrak{p} \in \operatorname{Spec} A_f$ is in the open complement of $V(\mathfrak{a})$ if and only if $\mathfrak{p} \not\supset \mathfrak{a}$ if and only if $\phi^{-1}\mathfrak{p} \not\supset \mathfrak{b}$ and conversely, $\mathfrak{q} \in \operatorname{Spec} A$ is in the open complement of $V(\mathfrak{b})$ if and only if $\mathfrak{q} \not\supset \mathfrak{b}$ if and only if $A_f\mathfrak{q} \not\supset \mathfrak{a}$ and so we have a homeomorphism.

It remains to show that the morphism of structure sheaves $\mathcal{O}_{\operatorname{Spec} A|_{D(f)}} \to F_*\mathcal{O}_{\operatorname{Spec} A_f}$ (where F is the scheme morphism) is an isomorphism. It is enough to check this on the stalks. Let $\mathfrak{p} \in D(f)$. The stalk of $\mathcal{O}_{\operatorname{Spec} A}$ is $A_\mathfrak{p}$ and the stalk of $F_*\mathcal{O}_{\operatorname{Spec} A_f}$ is $(A_f)_\mathfrak{p}$ where we confuse $\mathfrak{p}$ with its preimage in $\operatorname{Spec} A_f$. Since $f \neq \mathfrak{p}$ the morphism $A_\mathfrak{p} \to (A_f)_\mathfrak{p}$ induced by ϕ is clearly an isomorphism.

Exercise 2.2. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme.*

Solution. Let $\operatorname{Spec} A_i$ be an affine open cover for X. The intersection of each $\operatorname{Spec} A_i$ with U is an oen subset of $\operatorname{Spec} A_i$ which is therefore covered by basic open affines $D(f_{ij})$. Hence, we obtain an open affine cover $\operatorname{Spec}(A_i)_{f_ij}$ for U.

Exercise 2.3. Reduced Schemes.

a *Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.*

b *Let $(X, \mathcal{O}_X)$ be a scheme. Show that $X_{red} \overset{red}{=} (X, (\mathcal{O}_X)_{red})$ is a scheme. Show that there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism on the underlying topological spaces.*

c *Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced Show that there is a unique morphism $g : X \to Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \to Y$.*

The resoning behind the glueing here needs to be checked and/or made more explicit.

Solution. a Suppose that $(X, \mathcal{O}_X)$ is reduced. So $\mathcal{O}_X(U)$ has no nilpotent elements for each U. Let $P \in X$ be a point and consider a representative (U, s) of an element of the stalk. If this element is nilpotent, then there is some subneighbourhood $V \ni P$ of P on which s^n vanishes, but $\mathcal{O}_X(V)$ has no nilpotents, so s vanishes on V an therefore $(V, s|_V) = (U, s)$ is zero. Hence, the stalk has no nilpotents.

Suppose conversely, that each stalk has no nilpotents, and suppose that $s \in \mathcal{O}_X(U)$ is nilpotent, say $s^n = 0$. Then the germ of s^n is zero at each point in U. Since the stalks have no nilpotents, this means that the stalk of s vanishes at each point of U. But this means that $s = 0$ since a sheaf is a separated presheaf. So $\mathcal{O}_X(U)$ has no nilpotents.

b Suppose that $X = \operatorname{Spec} A$ is affine and denote by A_{red} the quotient $A/\mathfrak{N}$ where $\mathfrak{N} = \mathfrak{N}(A)$ is the nilradical of A. Since every prime ideal of A contains $\mathfrak{N}$, as topological spaces, sp $\operatorname{Spec} A = sp$ $\operatorname{Spec} A_{red}$. Now for a basic open affine $D(f)$ we have $\mathcal{O}_{\operatorname{Spec}(A_{red})}(D(f)) \cong (A/\mathfrak{N})_f \cong A_f/(\mathfrak{N}(A_f))$. That is, on a basic open affine U we have $\mathcal{O}_{\operatorname{Spec}(A_{red})}|_U \cong \mathcal{O}_{(\operatorname{Spec} A)_{red}}|_U$. Since the basic opens cover X this shows that $\operatorname{Spec}(A_{red}) \cong (X, (\mathcal{O}_X)_{red})$.

Now for a general scheme X, a cover of X with open affines $\operatorname{Spec} A_i$ gives a coer $\operatorname{Spec}(A_i)_{red}$ for $(X, (\mathcal{O}_X)_{red})$. Hence, the latter is a scheme.

The homomorphism $X \to X_{red}$ is induced on an affine cover $\operatorname{Spec} A_i \subset X$ by the ring morphisms $A_i \to A_i/\mathfrak{N}(A_i)$. We have already seen that it is a homeomorphism on the underlying topological spaces.

c Let $V_i = \operatorname{Spec} B_i$ be an open affine cover for Y, and let $U_{ij} = \operatorname{Spec} A_{ij}$ be an open affine cover of $f^{-1}V_i$. As in the previous part $V_i^{red} = \operatorname{Spec} B_i^{red}$ is an open affine cover for Y_{red} and the morphism $Y_{red} \to Y$ is induced by the ring homomorphisms $B_i \to B_i^{red}$. Now since each A_{ij} is reduced, $\mathfrak{N}(B_i)$ is in the kernel of each of the ring homomorphisms $B_i \to A_{ij}$ and so these factor uniquely as $B_i \to B_i^{red} \to A_{ij}$. So the morphisms $U_{ij} \to V_i$ factor uniquely as $U_{ij} \to V_i^{red} \to V_i$. The same is true of each intersection of te U_{ij}'s and so this gives rise to a unique factorization $f^{-1}V_i \to V_i^{red} \to V_i$. These patch to give a unique factorization $X \to Y_{red} \to Y$.

Exercise 2.4. *Let A be a ring and let $(X, \mathcal{O})X$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$ we have an associated map on sheaves $f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map*

$$\alpha : \hom_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \hom_{\mathfrak{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

Show that α is bijective.

Solution. We will show that $\operatorname{Spec}(-) : \mathfrak{Ring} \to \mathfrak{Sch}$ is a right adjoint to $\Gamma(-, \mathcal{O}_-) : \mathfrak{Sch} \to \mathfrak{Ring}$. One way to show this is to provide two natural transformations

$$\eta : id_{Sch} \to \operatorname{Spec} \circ \Gamma \qquad \varepsilon : id_{Rings} \to \Gamma \circ \operatorname{Spec}$$

such that for all $A \in \mathfrak{Rings}$ and $X \in \mathfrak{Sch}$ we have

$$\Gamma\eta_X \circ \varepsilon_{\Gamma X} \cong id_{\Gamma X} \qquad \text{and} \qquad \operatorname{Spec} \varepsilon_A \circ \eta_{\operatorname{Spec} A} \cong id_{\operatorname{Spec} A}$$

The obvious choice for ε is the isomorphism of Proposition 2.2 (c). For a scheme X we define the natural transformation η as follows. Let $U_i = \operatorname{Spec} A_i$ be an affine cover of the scheme. Each restriction $\Gamma X \to A_i$ gives a morphism $\operatorname{Spec} A_i \to \operatorname{Spec} \Gamma X$ and since the restriction $\Gamma X \to \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_{ij})$ is the same as $\Gamma X \to \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_{ij})$ these morphisms glue to give a morphism $X \to \operatorname{Spec} \Gamma X$.

So now since ε is an isomorphism, we just have to show that for any scheme X and A a ring, $\Gamma\eta_X$ is an isomorphism, and $\eta_{\operatorname{Spec} A}$ is an isomorphism. For a scheme consider the morphism $\eta : X \to \operatorname{Spec} \Gamma X$ just defined. It comes with a sheaf morphism $\mathcal{O}_{\operatorname{Spec} \Gamma X} \to \eta_* \mathcal{O}_X$ whose global sections we want to be an isomorphism. This is indeed the case. Now for a ring A, we have a sheaf morphism $\operatorname{Spec} A \to \operatorname{Spec} \Gamma \operatorname{Spec} A$ and since $\Gamma \operatorname{Spec} A \cong A$ this too is an isomorphism. Hence, the functors are adjoints and so α is a bijection.

We can more explicitely describe the bijection now as sending $\phi : A \to \Gamma(X, \mathcal{O}_X)$ to the composition

$$X \overset{\eta_X}{\to} \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \overset{\operatorname{Spec} \phi}{\to} \operatorname{Spec} A$$

and sending $(f, f^\#) : X \to \operatorname{Spec} A$ to

$$A \overset{\varepsilon}{\cong} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \overset{f^\#(\operatorname{Spec} A)}{\to} \Gamma(X, \mathcal{O}_X)$$

Exercise 2.5. *Describe* $\operatorname{Spec} \mathbb{Z}$ *and show that it is a final object for the category of schemes.*

Solution. Description. There is one closed point (p) for every prime number p and one generic point (0). As the ideals of $\mathbb{Z}$ are (n), the closed subsets are finite sets of primes (the prime divisors of n) and the open sets are their complements, together with the empty set. As a consequence, every closed subset is of the form $D(n)$ for some integer n and so the structure sheaf takes an open set $D(n)$ to $\mathbb{Z}$ localized at the prime divisors of n that is, $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \{\frac{a}{b} : p \not| b \ \forall\ p|n\}$. The value of the structure sheaf on the whole space is $\mathbb{Z}$ (since it is affine).

Final object. We have seen by the adjunction between Spec and Γ that the morphisms from a scheme X to an affine scheme $\operatorname{Spec} A$ are in one to one correspondence with the morphisms $A \to \Gamma(X, \mathcal{O}_X)$. Since we consider only identity preserving ring homomorphisms, there is a unique one of these if $A = \mathbb{Z}$.

Exercise 2.6. *Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes.*

Solution. The zero ring as no points as there are no proper prime ideals. The structure sheaf takes the usual value on the empty open set. As the point set is empty, there is a unique morphism of topological spaces from *sp* $\operatorname{Spec} 0$ to any topological space X. If X is a scheme, then the structure sheaf pulls back to the structure sheaf of $\operatorname{Spec} 0$, which has a unique morphism to itself - the identity morphism. Hence, $\operatorname{Spec} 0$ is initial.

Exercise 2.7. *Let X be a scheme and K a field. Show that to give a morphism of* $\operatorname{Spec} K$ *to* X *it is equivalent to give a point* $x \in X$ *and an inclusion map* $k(x) \to K$.

Solution. Since $\operatorname{Spec} K$ has a unique point, unique nonempty open set, and global sections K, given a point x we obtain immediately a continuous morphism of topological spaces $i : \operatorname{Spec} K \to X$. To define the sheaf morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_{\operatorname{Spec} K}$ note that $i_*\mathcal{O}_{\operatorname{Spec} K}$ is the skyscraper sheaf with ring of sections K so for every open set $U \ni x$ we need to give a ring homomorphism $\mathcal{O}_X(U) \to K$ in a way natural in U. We define these morphisms as the composition

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \to K$$

they are natural in U by definition of $\mathcal{O}_{X,x}$.

Conversly, given a scheme morphism $(i, i^{\#}) : \operatorname{Spec} K \to X$ we obtain a point $x = i((0))$, the image of the unique point of $\operatorname{Spec} K$. For the inclusion map, consider an affine open $\operatorname{Spec} A$ containing x. In $\operatorname{Spec} A$, the point x is a prime ideal $\mathfrak{p}$ and so if $\phi : A \to K$ is the corresponding ring homomorphism, $\mathfrak{p}$ is the kernel of ϕ and so we get an induced inclusion $k(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to K$.

Exercise 2.8. *Let X be a scheme over a field k. Show that to give a k-morphism of* $\operatorname{Spec} k[\epsilon]/\epsilon^2$ *to* X *is equivalent to giving a point* $x \in X$, *rational over* k *and an element of* $\hom_{k-Vec}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$.

Solution. Let $T = \operatorname{Spec} k[\epsilon]/\epsilon^2$.

Supose that we have a morphism of schemes $(\tau, \tau^{\#}) : T \to X$. We get a point x by taking the image of the unique point in T. To see that it is k-rational note that we have an inclusion of fields $k(x) \to k$ induced by τ but since the morphism τ is a k-morphism, this has to be compatible with the structural morphism to k. So we have inclusions $k \subset k(x) \subset k$ and therefore $k(x) = k$. Now consider $\tau^{\#}_x : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$, the stalk of $\tau^{\#}$. Taking an open affine $\operatorname{Spec} A$ containing x we can write this as $A_{\mathfrak{p}} \to k[\epsilon]/\epsilon^2$ where $\mathfrak{p}$ is the prime ideal correpsonding to x. This morphism is induced a ring homomorphism $\phi : A \to k[\epsilon]/\epsilon^2$ whose scheme morphism $T \to \operatorname{Spec} A$ sends (ϵ), the only point of T, to $\mathfrak{p}$. So $\phi^{-1}((\epsilon)) = \mathfrak{p}$ and therefore every element in $(\mathfrak{p}A_{\mathfrak{p}})^2 = \mathfrak{m}_x^2$ gets sent to (ϵ^2) which is zero. Hence, the composition

$$\mathfrak{m}_x \subset \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2 \to k$$

passes to a k-homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$.

Now suppose that we are given a point $x \in X$, rational over k, and an element $\phi \in \hom_{k-Vec}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. The morphism $\tau : T \to X$ of topological spaces is easily defined by sending the unique point of T to x. To define a mophism of sheaves $\tau^{\#} : \mathcal{O}_X \to \tau_*\mathcal{O}_T$ we need to give a morphism $\mathcal{O}_X(U) \to k[\epsilon]/\epsilon^2$ for every open subset $U \ni x$ containing x. We will give a morphism $\mathcal{O}_{X,x} \to k[e]/\epsilon^2$ and then define $\mathcal{O}_X(U)$ as the composition

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k[e]/\epsilon^2$$

Let α denote the morphism $\mathcal{O}_{X,x} \to k(x) = k$. Then we claim that

$$\begin{aligned} \mathcal{O}_{X,x} &\to k[e]/\epsilon^2 \\ f &\mapsto \alpha(f) + \phi(f - \alpha(f))\epsilon \end{aligned}$$

is a ring homomorphism. Assuming it is well-defined, it is immediate that it is k-linear, and so we just need to notice that $f - \alpha(f)$ really is in $\mathfrak{m}$ the maximal ideal [1] and that for $f, g \in \mathcal{O}_{X,x}$ the relation

$$\phi(fg - \alpha(fg)) = \phi(f - \alpha(f))\alpha(g) + \alpha(f)\phi(g - \alpha(g))$$

holds. [2]

Exercise 2.9. *If X is a scheme show that every (non)empty irreducible closed subset has a unique generic point.*

Solution. Claim 1: If η is a generic point of Z then η is in $Z \cap U$ for all open sets U that have nontrivial intersection with Z. Suppose that $\eta \neq Z \cap U$. Then η is in its complement $Z^c \cup U^c$. We know that $\eta \in Z$ and so $\eta \neq Z^c$ and therefore $\eta \in U^c$. Since U^c is closed and contains η, it must contain Z, the closure of η. Hence, $Z \cap U = \varnothing$.

Using the claim we have just proven, we can reduced to the affine case by chosing an open affine, say $\operatorname{Spec} A$, that has nontrivial intersection with Z.

Claim 2: If a closed subset $V(I) \subseteq \operatorname{Spec} A$ is irreducible then $\sqrt{I}$ is prime. Let $fg \in \sqrt{I}$ and consider the closed subsets $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$ where $I_1 = (f) + \sqrt{I}$ and $I_2 = (g) + \sqrt{I}$. If $h \in I_1 \cap I_2$ then we can write $h = af + i = bg + j$ for some $a, b \in A$ and $i, j \in \sqrt{I}$. Then $h^2 = abfg + ij + ibg + afj$ and so all these terms are in $\sqrt{I}$, so is h^2 and therefore so is h. So $I_1 + I_2 = \sqrt{I}$, but since $V(I)$ is irreducible this means either $I_1 = \sqrt{I}$ or $I_2 = \sqrt{I}$. Hence, either f or g are in $\sqrt{I}$ and so $\sqrt{I}$ is prime.

It is now straightforward to see that $\sqrt{I}$ is the unique generic point of $V(I)$.

Exercise 2.10. *Describe* $\operatorname{Spec} \mathbb{R}[x]$. *How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?*

Solution. $\operatorname{Spec} \mathbb{R}[x]$ has one point for every irreducible polynomial, together with the generic point (0). There is one closed point for every real number $(x - a)$ and one for every nonreal complex number $(x + \alpha)(x + \overline{\alpha})$ where $\alpha \in \mathbb{C}$. The residue field at the real numbers is $\mathbb{R}$ and at the “complex numbers” is $\mathbb{C}$. The closed sets are finite collections of points and the open sets their complements.

Exercise 2.11. *Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe* $\operatorname{Spec} k[x]$. *What are the residue field of its points? How many points are there with a given residue field?*

[1] This is a consequence of the composition $k \to \mathcal{O}_{X,x} \xrightarrow{\alpha} k(x) = k$ being the identity.

[2] This is more involved. By k-linearity of ϕ we just need to show that $fg - \alpha(fg)$ and $f\alpha(g) - \alpha(f)\alpha(g) + \alpha(f)g - \alpha(f)\alpha(g)$ get sent to the same place by ϕ. This will hapen if their difference is in $\mathfrak{m}_x^2$, and this can be seen by expanding $(f - \alpha(f)(g - \alpha(g)) \in \mathfrak{m}_x^2$.

Solution. Spec $k[x]$ has the generic point and one point for every (monic) irreducible polynomial. The residue field of a point corresponding to a polynomial of degree n is the finite field with p^n elements. To count how many irreducible polynomials there are of degree n, consider the field $\mathbb{F}_{p^n}$. Every irreducible polynomial $f(x)$ of degree n gives an element of $\mathbb{F}_{p^n}$ via the isomorphism $\mathbb{F}_p[x]/(f(x)) \to \mathbb{F}_{p^n}$ and every element α of $\mathbb{F}_{p^n}$ that is not contained in any subfields gives an irreducible polynomial of degree n by taking its minimal polynomial $\prod_{i=0}^{n-1}(x - \alpha^{p^i})$. These processes are inverses of each other and so we want to count the number of elements of $\mathbb{F}_{p^n}$ not contained in any subfields. This quantity is

$$\sum_{d|n} \mu(d) p^d$$

where

$$\mu(d) = \begin{cases} 0 & \text{if } d \text{ has repeated prime divisors} \\ (-1)^{(\#\text{ prime divisors}-1)} & \text{otherwise} \end{cases}$$

Exercise 2.12. Glueing Lemma. *Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an opent subset $U_{ij} \subseteq X_i$ and let if have the induced scheme structure. Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each $i, j, \phi_{ji} = \phi_{ij}^{-1}$, and (2) for each i, j, k $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_{ij} \cap U_{ik}$.*

The show that there is a scheme X, together with morphisms $\pi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.

Solution. First define a topological space X as the quotient of $\coprod X_i$ by the equivalence relation $x \sim y$ if $x = y$, or if there are i, j such that $x \in U_{ij} \subset X_i$, $y \in U_{ji} \subset X_j$, and $\phi_{ij}x = y$. This relation is reflexive by definition, symmetric since $\phi_{ji} = \phi_{ij}^{-1}$, and transitive since $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$, hence is really an equivalence relation. We take the quotient topology on $X = \coprod X_i / \sim$ (a set is open in X if and only if its preimage under $\coprod X_i \to X$ is open; in particular the image $\psi_i(X_i)$ of X_i is open for each i since its preimage is $X_i \coprod(\coprod_j U_{ji})$). Now for each i we have a sheaf $\psi_*\mathcal{O}_{X_i}$ on the image of X_i by pushing forward the structure sheaf of X_i, and on the intersections, we have the pushforward of the isomorphisms $\phi_{ij}^{\#}$, and these satisfy the required relation to use Exercise I.1.22 to glue the sheaves together obtaining a sheaf $\mathcal{O}_X$ together with isomorphisms $\psi_i : \mathcal{O}_X|_{\psi X_i} \xrightarrow{\sim} \psi_*\mathcal{O}_{X_i}$. So now we have a locally ringed space $(X, \mathcal{O}_X)$ and we immediately see that $\psi_i : X_i \to X$ is an isomorphism of locally ringed spaces onto an open locally ringed subspace of X. Hence, X is a scheme that satisfies (1). Tohat (2) is satisfied follows from our definition of the uderlying space of X as a quotient of $\coprod X_i$. To see (3) let x be a point in $\psi_i(U_{ij})$. Then the preimage of x in $\coprod X_i$ is certainly contained in X_j as well so $\psi(U_{ij}) \subseteq \psi_i(X_i) \cap \psi_j(X_j)$. Conversely, if $x \in \psi_i(X_i) \cap \psi_j(X_j)$ then there are $x_i \in X_i$ and $x_j \in X_j$ that are equivalent under $\sim$. Hence, $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $\psi_{ij}(x_i) = x_j$ so $x \in \psi_i U_{ij}$ and therefore $\psi(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$. (4) is fairly clear as well.

Exercise 2.13. *A topological space is* quasi-compact *if every open cover has a finite subcover.*

a *Show that a topological space is noetherian if and only if every open subset is quasi-compact.*

b *If X is an affine scheme show that $sp(X)$ is quasi-compact, but not in general noetherian.*

c *If A is a neotherian ring, show that $sp(\operatorname{Spec} A)$ is a noetherian toplogical space.*

d *Give an example to show that $sp(\operatorname{Spec} A)$ can be noetherian even when A is not.*

Solution. a Let X be noetherian, U an open subset and $\{U_i\}$ a cover of U. Define an increasing sequence of open subsets by $V_0 = \varnothing$ and $V_{i+1} = V_i \cup U_i$ where U_i is an element of the cover not contained in V_i. If we can always find such a U_i then we obtain a strictly increasing sequence of open subsets of X, which contradicts X being noetherian. Hence, there is some n for which $\cup_{i=1}^{n} U_i = U$ and therefore $\{U_i\}$ has a finite subcover.

Suppose every open subset of X is quasi-compact. Suppose $U_1 \subset U_2 \subset \dots$ is an increasing sequence of open subsets of X. Then $\{U_i\}$ is a cover for $\cup U_i$. Since this has a finite subcover, there must be some n for which $U_n = U_{n+1}$, hence, the sequence stabilizes and X is noetherian.

b Let $\{U_i\}$ be an open cover for $sp(X)$. The complements of U_i are closed and therefore determined by ideals I_i in $A = \Gamma(\mathcal{O}_X, X)$. Since $\cup U_i = X$ the I_i generate the unit ideal and hence $1 = \sum_{j=1}^{n} f_j g_{i_j}$ for some f_j where $g_{i_j} \in I_{i_j}$. Then $\{I_{i_1}, \dots, I_{i_n}\}$ also generate the unit ideal and therefore we have a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$.

An example of a non noetherian affine scheme is $\operatorname{Spec} k[x_1, x_2, \dots]$ which has a decreasing chain of closed subsets $V(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, x_3) \supset \dots$.

c A decreasing sequence of close subsets $Z_1 \supset Z_2 \supset \dots$ corresponds to an increasing sequence $I_1 \subset I_2 \subset \dots$ of ideals of A. Since A is noetherian this stabilizes at some point and therefore, so does the sequence of closed subsets.

d If A is the ring of p-adic integers, then there is one prime idea so the space is noetherian, but there is an increasing chain of ideals $(0) \subset (p) \subset (p^2) \subset \dots$.

Exercise 2.14. a *Let S be a graded ring. Show that $\operatorname{Proj} S = \varnothing$ if and only if every element of S_+ is nilpotent.*

b *Let $\phi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T | \mathfrak{p} \not\supseteq \phi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$ and show that ϕ determines a natural morphism $f : U \to \operatorname{Proj} S$.*

c *The morphism f can be an isomorphism even when ϕ is not. For example, suppose that $\phi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.*

d *Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong \operatorname{Proj} S$.*

Solution. a Since every prime ideal contains every nilpotent, if every element of S_+ is nilpotent then $\mathfrak{p} \supset S_+$ for every homogeneous prime ideal $\mathfrak{p}$. Hence, $\operatorname{Proj} S$ is empty.

Conversely, suppose $\operatorname{Proj} S$ is empty and consider $s \in S_+$. Let $\mathfrak{p} \subset S$ be a prime ideal, and consider the homogeneous prime ideal $\mathfrak{q} = \sum_{d \geq 0} \mathfrak{p} \cap S_d \subset \mathfrak{p}$ (check that it is prime!). Since $\operatorname{Proj} S$ is empty, $D_+(s)$ is empty, so every homogeneous prime ideal contains s. Hence If $\mathfrak{p}$ contains s. Since every prime ideal contains s, it is nilpotent, so every element of S_+ is nilpotent.

b Let $\mathfrak{p} \in U$. Then $\phi(S_+) \not\subseteq \mathfrak{p}$ and so unless $S_+ = 0$, there is some $f \in S_+$ such that $\phi f \notin \mathfrak{p}$. If $\phi f_i \in \mathfrak{p}$ for every homogeneous component f_i of f then $\phi f \in \mathfrak{p}$, so there is some homogeneous component f_i of f such that $\phi f_i \notin \mathfrak{p}$. Hence, we have found a basic open $D_+(\phi f_i)$ that contains $\mathfrak{p}$. Moreover, $D_+(\phi f_i)$ is contained in U since every prime in $D_+(\phi f_i)$ doesn't contain ϕf_i and so doesn't contain $\phi(S_+)$. The basic opens of this kind cover U and therefore it is open since it is union of open sets.

For $\mathfrak{p} \in U$ define $f(\mathfrak{p}) = \phi^{-1}\mathfrak{p}$. Since $\mathfrak{p} \not\supseteq \phi(S_+)$ we have $\phi^{-1}\mathfrak{p} \not\supseteq S_+$ so the morphism is well defined. As in the affine case, this morphism preserves all ideals, and therefore closed subsets so it is a continuous morphism of topological spaces. As in the affine case, the morphism of sheaves $f^\#$ is induced by $S_{(\phi^{-1}\mathfrak{p})} \to T_{(\mathfrak{p})}$ for $\mathfrak{p} \in U$.

c *The open U is* $\operatorname{Proj} T$. Let $\mathfrak{p} \in \operatorname{Proj} T$, suppose that $\mathfrak{p} \supseteq \phi(S_+)$ and let $t \in T_e$ with $i > 0$. Since ϕ_d is an isomorphism for $d \geq d_0$, there is some $s \in S_{ed_0}$ such that $\phi_{ed_0} s = t^{d_0}$. Since $\mathfrak{p} \supseteq \phi(S_+)$ this means that $\phi_{ed_0} s = t^{d_0} \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $t \in \mathfrak{p}$. But $\mathfrak{p} \subseteq T_+$ contradicts the assumption that $\mathfrak{p} \in \operatorname{Proj} T$, so $\mathfrak{p} \not\subseteq \phi(S_+)$. Since $\mathfrak{p}$ was arbitrary, this shows that $U = \operatorname{Proj} T$.

Surjectivity. Let $\mathfrak{p} \in \operatorname{Proj} S$. Define $\mathfrak{q} = \sqrt{\langle \phi \mathfrak{p} \rangle}$ to be the radical of the *homogeneous* ideal generated by $\phi\mathfrak{p}$ the image of ϕ (note that radicals of homogeneous ideals are homogeneous). We will show that (i) $\phi^{-1}\mathfrak{q} = \mathfrak{p}$, and (ii) $\mathfrak{q}$ is prime. We start with (i). The inclusion $\phi^{-1}\mathfrak{q} \supseteq \mathfrak{p}$ is clear, so suppose we have $a \in \phi^{-1}\mathfrak{q}$. Then $\phi a^n \in \langle \phi \mathfrak{p} \rangle$ for some integer n. This means that $\phi a^n = \sum b_i \phi s_i$ for some $b_i \in T$ and $s_i \in \mathfrak{p}$. If we take a high enough m, then the every monomial in the b_i will be in $T_{\geq d_0}$, and since we have isomorphisms $T_d \cong S_d$ for $d \geq d_0$ this means that these monomials correspond to some $c_j \in S$. The element $(\sum b_i \phi s_i)^m$ is a polynomial in the ϕs_i whose coefficients are monomials of degree m in the b_i, and this corresponds in S to a polynomial in the s_i with coefficients in the c_j, which is in $\mathfrak{p}$, as all the s_i are. Hence, $\phi a^{nm} \in \phi\mathfrak{p}$ and so $a^{nm} \in \mathfrak{p}$ and therefore $a \in \mathfrak{p}$. So $\phi^{-1}\mathfrak{q} \subseteq \mathfrak{p}$ and combining this with the other inclusion shows that $\phi^{-1}\mathfrak{q} = \mathfrak{p}$. (ii) Suppose that $ab \in \mathfrak{q}$ for some $a, b \in T$. Then using the same reasoning as for (i) we see that $(ab)^{nm} \in \phi\mathfrak{p}$ for some n, m such that $(ab)^{nm} \in T_{\geq d_0}$. If necassary, take higher power so that $a^{nmk}, b^{nmk} \in T_{\geq d_0}$ as well. Using the isomorphism $T_{\geq d_0} \cong S_{\geq d_0}$ this means that a^{nmk}, b^{nmk} correspond to elements of S and we see that their product is in $\mathfrak{p}$. Hence, one of a^{nmk} or b^{nmk} are in $\mathfrak{p}$, say a^{nmk}. Then $a^{nmk} \in \phi\mathfrak{p}$ and so $a \in \mathfrak{q}$. So $\mathfrak{q}$ is prime.

Injectivity. Suppose that $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} T$ have the same image under $f : \operatorname{Proj} T \to \operatorname{Proj} S$. Then $\phi^{-1}\mathfrak{p} = \phi^{-1}\mathfrak{q}$. Consider $t \in \mathfrak{p}$. Since $t \in \mathfrak{p}$ we have $t^{d_0} \in \mathfrak{p}$ and since ϕ_d is an isomorphism for $d \geq d_0$ it follows that there is a unique $s \in S$ with $\phi s = t^{d_0}$. The element s is in $\phi^{-1}\mathfrak{p}$ and so since $\phi^{-1}\mathfrak{p} = \phi^{-1}\mathfrak{q}$ this implies that $s \in \phi^{-1}\mathfrak{q}$. So $\phi s = t^{d_0} \in \mathfrak{q}$. Now $\mathfrak{q}$ is prime and so $t \in \mathfrak{q}$. Hence $\mathfrak{p} \subseteq \mathfrak{q}$. By symmetry $\mathfrak{q} \subseteq \mathfrak{p}$ as well and therefore $\mathfrak{p} = \mathfrak{q}$.

Isomorphism of structure sheaves. Since $\operatorname{Proj} S$ is covered by open affines of the form $D_+(s)$ for some homogeneous element of S, it is enough to check the isomorphism on these. Note that $D_+(s) = D_+(s^i)$ so we can assume that the degree of s is $\geq d_0$. With this assumption it can be seen that $f^{-1}D_+(s) = D_+(t) \subseteq \operatorname{Proj} T$ where t is the element of T corresponding to s under the isomorphism $S_{\deg s} \to T_{\deg s}$ since a homogeneous prime ideal $\mathfrak{q} \subset T$ gets sent to $D_+(s)$ if and only if s is not in its preimage, if and only if t is not in $\mathfrak{q}$. So our task is to show that the morphism $S_{(s)} \to T_{(t)}$ is an isomorphism. If $\frac{f}{s^n}$ gets sent to zero then $0 = t^m\phi f = \phi(s^m)\phi f$ for some m (choose $m > 0$ so that we don't have to handle the case $\deg f = 0$ separately), and so $s^m f \in \ker\phi$. Taking a high enough power of $s^m f$ puts it in one of the S_d for which $S_d \to T_d$ is an isomorphism and so $s^m f = 0$ and therefore $\frac{f}{s^n} = 0$ so our morphism is injective. Now suppose that $\frac{f}{t^n} \in T_{(t)}$. This is equal in $T_{(t)}$ to $\frac{t^{d_0} f}{t^{n+d_0}}$ and now $t^{d_0} f$ has degree high enough to have a preimage in S. So our morphism is surjective.

Exercise 2.15. *a Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.*

b If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

c Now show that if V, W are any two varieties over k, then the natural map

$$\hom_{\mathfrak{Var}}(V, W) \to \hom_{\mathfrak{Sch}/k}(t(V), t(W))$$

is bijective.

Solution. a Every point of $t(V)$ is by definition, an irreducible closed subset of V. If P is not a closed point its corresponding irreducible closed subset Z is not a point, but a subvariety of V of dimension greater than zero. Then by Theorem 1.8A the transcendence degree of its residue field over k is greater than zero. Hence, a residue field of k implies P is closed. Conversely, a closed point comes gives a residue field of transcendence degree zero, and since k is algebraically closed this means that $k(P) = k$.

b The morphism of structure sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a morphism of residue field $k(f(P)) \to k(P)$. Since X and Y are schemes over k, these residue fields are both extensions of k. So if $k(P) = k$ then we have a tower $k \hookrightarrow k(f(P)) \hookrightarrow k$, and so $k(f(P)) \cong k$.

Exercise 2.16. *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of x at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.*

a If $U = \operatorname{Spec} B$ is an open affine subscheme of X and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

b Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.

c Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0, f^n b$ is the restriction of an element of A.

d With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Solution. a A point x is in $U \cap X_f$ if and only if it is in U and the stalk f_x of f is not in the maximal ideal at x. Since U is affine we can take x to be a prime $\mathfrak{p} \in \operatorname{Spec} B$ and so the maximal ideal of the local ring is $\mathfrak{m} = \mathfrak{p}B_{\mathfrak{p}}$. The element $\overline{f}$ is in $\mathfrak{m}$ if and only if $\overline{f} \in \mathfrak{p}$ and so $U \cap X_f = D(\overline{f})$. Since a subset of a topological space is open if and only if it is open in every element of an open cover, we conclude that X_f is open in X.

b Let $U_i = \operatorname{Spec} A_i$ be an affine cover of X, finite since X is quasi-compact. The restriction of a to $U_i \cap X_f = \operatorname{Spec}(A_i)_f$ is zero for each i and so $f^{n_i} a = 0$ in A_i for some n_i. Choose an n bigger than all the n_i. Then $f^n a = 0$ in each $\operatorname{Spec} A_i$, and so since the $\operatorname{Spec} A_i$ cover X and $\mathcal{O}_X$ is a sheaf (in particular since it is a separated presheaf), this implies that $f^n a = 0$.

c Let $U_i = \operatorname{Spec} A_i$ (different from the previous part!!!). The restriction of b to each intersection $X_f \cap U_i$ can be written in the form $\frac{b_i}{f^{n_i}}$ for some $n_i \in \mathbb{N}$. Since there are finitely many affines, we can choose the expression so that all the n_is are the same, say n. In other words, we have found $b_i \in A_i$ such that $f^n b|_{U_i \cap X_f} = b_i$. Now consider $b_i - b_j$ on $U_i \cap U_j$. Since $U_i \cap U_j$ is quasi-compact and the restriction of $b_i - b_j$ to $U_i \cap U_j \cap X_f = (U_i \cap U_j)_f$ vanishes, we can apply the previous part to find m_{ij} such that $f^{m_{ij}}(b_i - b_j) = 0$ on $U_i \cap U_j$. Again, we choose m bigger than all the m_{ij} so that they are all the same. So the situation now is that we have sections $f^m b_i$ on each U_i that agree on the intersections. Hence, they lift to some global section $c \in \Gamma(X, \mathcal{O}_X)$. Now consider $c - f^{n+m}b$ on X_f. Its restriction to each $U_i \cap X_f$ is $f^m b_i - f^m b_i = 0$ and so $c = f^{n+m}b$ on X_f. Hence, $f^{n+m}b$ is the restriction of the global section c.

d Consider the morphism $A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$. If an element $\frac{a}{f^n}$ is in the kernel then $a|_{X_f} = 0$ and so by part (b) we have $f^m a = 0$ as global sections for some m. Hence, $\frac{a}{f^n}$ is zero and the morphism is injective. Now suppose we have a section b on X_f. By part (c) there is an m such that $f^m b$ is the restriction of some global section, say c. Hence, we have found $\frac{c}{f^m} \in A_f$ that gets sent to b so the morphism is surjective.

Exercise 2.17. A Criterion for Affineness.

a *Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.*

b *A scheme X is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A.*

Solution. a Take a cover of Y by open affines V_j and then cover each intersection $V_j \cap U_i$ by basic open affines of V_j. So we end up with a cover of Y composed of affines W_k, such that each one is an subset of some U_i. Since $f^{-1}(U_i) \to U_i$ is an isomorphism, its restriction to $f^{-1}(W_k) \to W_k$ will be for any $W_k \subset U$, so X is now covered by the same set of affines as Y. It can be checked that the gluing morphisms are the same and so Y and X are both isomorphic to the scheme obtained by gluing together the W_k and these isomorphisms are compatible with f.

b If A is affine we can take $f_1 = 1$.

Suppose then that we have elements $f_1, \ldots, f_r \in A$, that each $X_{f_i} = \operatorname{Spec} A_i$, and that the f_i generate A. We always have a morphism $f : X \to \operatorname{Spec} A$ and we wish to show that this is an isomorphism. Since the f_i generate A the basic opens $D(f_i) = \operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$. It is immediate that their preimages are X_{f_i} which we have already assumed are affine $X_{f_i} \cong \operatorname{Spec} A_i$. So our morphism f restricts to a morphism $\operatorname{Spec} A_i \to \operatorname{Spec} A_{f_i}$ which comes from a ring homomorphism $\phi_i : A_{f_i} \to A_i$. If we can show that the ϕ_i are isomorphisms then the result will follow from the previous part of this exercise.

Stated more clearly, we want to show that

$$\phi_i : \Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_X)$$

is an isomorphism for each i.

Injectivity. Let $\frac{a}{f_i^n} \in A_{f_i}$ and suppose that $\phi_i \frac{a}{f_i^n} = 0$. This means that it also vanishes in each of the intersections $X_{f_i} \cap X_{f_j} = \operatorname{Spec}(A_j)_{f_i}$ so for each j there is some n_j such that $f_i^{n_j} a = 0$ in A_j. Choose an m bigger than all the n_j. Now the restriction of $f_i^m a$ to each open set in a cover vanishes, therefore $f_i^m a = 0$. So $\frac{a}{f_i^n} = 0$ in A_{f_i}.

Surjectivity. Let $a \in A_i$. For each $j \neq i$ we have $\mathcal{O}_X(X_{f_i f_j}) \cong (A_j)_{f_i}$ so $a|_{X_{f_i f_j}}$ can be written as $\frac{b_j}{f_i^{n_j}}$ for some $b_j \in A_j$ and $n_j \in \mathbb{N}$. That is, we have elements $b_j \in A_j$ whose restriction to $X_{f_i f_j}$ is $f_i^{n_j} a$. Since there are finitely many, we can choose them so that all the n_i are the same, say n.

Now on the triple intersections $X_{f_i f_j f_k} = \operatorname{Spec}(A_j)_{f_i f_k} = \operatorname{Spec}(A_k)_{f_i f_j}$ we have $b_j - b_k = f_i^n a - f_i^n a = 0$ and so we can find $m_{jk} \in \mathbb{N}$ so that $f_i^{m_{jk}}(b_j - b_k) = 0$ on $X_{f_j f_k}$. Replacing each m_{jk} by m larger than all of them, the relation $f_i^m(b_j - b_k)$ still holds. So now we have a section $f_i^m b_j$ for each X_{f_j} $j \neq i$ together with a section $f_i^{n+m} a$ on X_{f_i} and these sections agree on all the intersections. This gives us a global section d whose restriction to X_{f_i} is $f_i^{n+m} a$ and so $\frac{d}{f_i^{n+m}}$ gets mapped to a by ϕ_i.

Exercise 2.18. a *Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.*

b *Let $\phi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that ϕ is injective if and only if the map of sheaves $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant.*

c *With the same notation, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective.*

d *Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective, then ϕ is surjective.*

Lemma 1. *Let $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$ be a scheme morphims of affine schemes with corresponding ring homomorphism $\phi : A \to B$. Then for a point $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $(f_* \mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{p}}$ is $S^{-1}B = B \otimes_A A_{\mathfrak{p}}$ where $S = \phi(A \backslash \mathfrak{p})$.*

Proof. Since we can shrink every open subset U containing $\mathfrak{p}$ to one of the form $D(a)$ with $a \in A$, we can compute the stalk by taking the colimit over these. Notice that the preimage of $D(a)$ is $D(\phi a) \subseteq \operatorname{Spec} B$. [3] So $(f_* \mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{p}}$ is then

[3] If a prime $\mathfrak{q} \in \operatorname{Spec} B$ is in the preimage of $D(a)$ then $\phi^{-1}\mathfrak{q} \in D(a)$ and so $a \notin \phi^{-1}\mathfrak{q}$ and therefore $\phi a \notin \mathfrak{q}$. Conversely, if a prime $\mathfrak{q}$ is in $D(\phi a)$ then $\phi a \notin \mathfrak{q}$ and so $a \notin \phi^{-1}\mathfrak{q}$ so $\phi^{-1}\mathfrak{q} \in D(a)$.

the colimit of $\mathcal{O}_{\operatorname{Spec} B}$ evaluated at opens $D(a)$ with $a \notin \mathfrak{p}$, that is, the colimit of $B_{\phi a}$ for $a \notin \mathfrak{p}$. This is $S^{-1}B$. To see that it is the same as the tensor product, use the universal property of tensor products. □

Solution. a If f is nilpotent then $f^n = 0$ for some $n \in \mathbb{N}$ and so $f^n \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Hence, $f \in \mathfrak{p}$ for every prime ideal and therefore, $\mathfrak{p} \notin D(f)$ for every prime ideal $\mathfrak{p}$.

b If the map of sheaves is injective then in particular, taking global sections, we see that $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, f_*\mathcal{O}_Y)$ is injective. That is, $A \to B$ is injective. Conversely, suppose $A \to B$ is injective, pick a prime $\mathfrak{p} \in \operatorname{Spec} A$, and consider the stalk $A_\mathfrak{p} \to S^{-1}B$ of the morphism $f^\#$ at $\mathfrak{p}$ where $S = A \backslash \mathfrak{p}$ (see Lemma 1). That this is injective follows immediately from $A \to B$ being injective.

To see that it is dominant consider the complement of the closure of the image. That is, the biggest open set that doesn't intersect the image. This is covered by open affines of the form $D(f)$ where $f \in \phi^{-1}\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} B$. For such an f, we have $\phi f \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} B$ and so ϕf is in the nilradical, so ϕf is nilpotent. Since ϕ is injective, this means, f is nilpotent, so $D(f)$ is empty. So the closure of the image is the entire space.

c We immediately have a bijection between primes of A containing I and primes of $A/I \cong B$ where I is the kernel of ϕ. We already know the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is continuous so we just need to see that it is open to find that it is a homeomorphism. Note that for $f + I \in A/I$ the preimage of $D(f) \subset \operatorname{Spec} A$ is $D(f + I) \subset \operatorname{Spec}(A/I)$, so basic opens of $\operatorname{Spec}(A/I)$ are open in the image (with the induced topology). Since arbitrary unions of open sets are open, and the basic opens are a base for the topology, the image of every open set is open. The stalk $A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ of the sheaf morphism at $\mathfrak{p} \in \operatorname{Spec} A$ is clearly surjective.

d If $f^\#$ is surjective then it is surjective on each stalk. So for an element $b \in B$, for each point $\mathfrak{p}_i \in \operatorname{Spec} A$ there is an open neighbourhood which we can take to be a basic open $D(f_i)$ of $\operatorname{Spec} A$ such that the germ of b is the image of some $\frac{a_i}{f_i^{n_i}} \in A_{f_i}$. That is, $f_i^{m_i}(a_i - f_i^{n_i} b) = 0$ in B. Since all affine schemes are quasi-compact, we can find a finite set of the $D(f_i)$ that cover $\operatorname{Spec} A$, which means we can assume all the n_i and m_i are the same, say n and m. Since $D(f_i)$ is a cover, the f_i generated A and therefore so do the f_i^{n+m}, so we can write $1 = sum g_i f_i^{n+m}$ for some $g_i \in A$. We now have

$$b = \sum g_i f_i^{n+m} b = \sum g_i f_i^m a_i \in \operatorname{im} \phi$$

So ϕ is surjective.

Exercise 2.19. *Let A be a ring. Show that the following conditions are equivalent:*

a Spec A *is disconnected;*

b there exist nonzero elements $e_1, e_2 \in A$ *such that* $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$.

c A *is isomorphic to a direct product* $A_1 \times A_2$ *of two nonzero rings.*

Solution. ($1 \Rightarrow 3$) If Spec A is disconnected then it is the udisjoint union of two open sets, say as $\operatorname{Spec} A = U \coprod V$. In particular, this means that U and V are also both closed sets, and therefor correspond to ideals, say I and J. That is, $U = \operatorname{Spec} A/I$ and $V = \operatorname{Spec} A/J$. It follows that $\operatorname{Spec} A = \operatorname{Spec}(A/I) \times (A/J)$ and therefore $A = A_1 \times A_2$ where $A_1 = A/I$ and $A_2 = A/J$.

($3 \Rightarrow 2$) Choose $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

($2 \Rightarrow 1$) Since $e_1 e_2 = 0$, for every prime, either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$. The closed sets $V((e_1)), V((e_2))$ cover Spec A. Now if a prime $\mathfrak{p}$ is in both these closed sets then $e_1, e_2 \in \mathfrak{p}$ and therefore $1 = e_1 + e_2 \in \mathfrak{p}$ and so $\mathfrak{p} = A$. So the closed sets $V((e_1)), V((e_2))$ are disjoint. Since we have a cover of Spec A by disjoint closed sets, Spec A is disconnected.