Chapter 13

Excess intersections and the Chow ring of a blow-up

Keynote Questions

(a) Suppose that S_1, S_2 and $S_3 \subset \mathbb{P}^3$ are three surfaces of degrees s_1, s_2 and s_3 whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme $\Gamma \subset \mathbb{P}^3$. What is the degree of Γ? More generally, what happens when we replace L with a smooth curve of genus g and degree d? (Answer on page 450.)

(b) Suppose that S_1 and $S_2 \subset \mathbb{P}^4$ are two surfaces whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme $\Gamma \subset \mathbb{P}^4$. In terms of the geometry of S_1 and S_2, can we say what the degree of Γ is? Can we say what the degree of Γ is in terms of the degrees of S_1 and S_2 alone? As in the preceding question, what happens when we replace L with a smooth curve of genus g and degree d? (Answers on page 459.)

(c) Let $\Lambda \cong \mathbb{P}^{n-c} \subset \mathbb{P}^n$ be a codimension-c linear subspace, and let $Q_1, \dots, Q_n \subset \mathbb{P}^n$ be general quadric hypersurfaces containing Λ. If we write the intersection $\bigcap Q_i$ as the union

$$\bigcap_{i=1}^{n} Q_i = \Lambda \cup \Gamma,$$

what is the degree of Γ? (Answer on page 460.)

(d) Is it the case that every smooth curve $C \subset \mathbb{P}^3$ is the scheme-theoretic intersection of three surfaces? (Answer on page 452.)

(e) Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g. If $S, T \subset \mathbb{P}^3$ are smooth surfaces of degrees s and t containing C, at how many points of C are S and T tangent? (Answer on page 476.)

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Let X be a smooth projective variety and $S_1, \ldots, S_k \subset X$ subvarieties. Basic intersection theory, for example in the form of Bézout's theorem for proper intersections (Theorem 1.1), tells us that the sum of the classes of the irreducible components of the intersection, with appropriate multiplicities, is $\prod[S_i]$, but *only* under the hypothesis that the intersection has the expected dimension. The first three keynote questions of this chapter are examples of what are called *excess intersection* problems: situations in which we wish to describe something about improper intersections, where the intersection has components of dimension greater than expected.

A remarkable discovery of Fulton and MacPherson is that this is possible in surprising generality: Given a collection of cycles $S_i \subset X$, subject to mild hypotheses there is a canonical way of assigning a class $\gamma_{C_\alpha} \in A(C_\alpha)$ of the right dimension to each connected component C_α of the intersection $\bigcap S_i$ so that the sum of the pushforwards of these classes in $A(X)$ is equal to $\prod[S_i]$. Moreover, the γ_{C_α} are determined by local geometry. The result is sometimes called the *excess intersection formula*.

The first part of this chapter will be devoted to an exposition of this formula. We begin with some elementary examples, including Keynote Question (a), worked out by hand, which should make it at least plausible that such a formula should exist. We then present the general statement, Theorem 13.3. (An excellent account, with proofs, from which some of the material in this chapter is taken, is Fulton and MacPherson [1978]; see also Fulton [1984, Section 6.3].) We also give a heuristic argument which may help to explain the form that the excess intersection formula takes. As an application of the excess intersection formula, we answer Keynote Questions (b) and (c), and explain how the excess intersection formula applies to the problem of finding the number of conics tangent to five given conics, as suggested in Chapter 8.

The rest of the chapter is devoted to several related topics: the technique of specialization to the normal cone, the "key formula" for intersections in a subvariety, and a description of the Chow ring of the blow-up of a smooth variety along a smooth subvariety.

One note: We do not have the tools to give a proof of either the excess intersection formula or the Grothendieck Riemann–Roch formula, the subjects of this chapter and the next. Nonetheless, given their beauty and their importance in modern intersection theory, we wanted to give an exposition of both topics. Thus in the final two chapters of this book, we will not attempt to prove all the assertions made, focusing rather on what they say and heuristically why they might be true.

One aspect of this is that while up to now we have introduced Chern classes only on smooth varieties, here we want to invoke the fact that they may be defined much more generally for locally free sheaves on any scheme, and that they continue to satisfy Theorem 5.3. In particular, we can use part (d) of that theorem to define the product of an arbitrary Chow class with the Chern class of a bundle; in other words, a Chern class $c_i(\mathcal{E})$ of a vector bundle on an arbitrary scheme X defines an operation $A_k(X) \to A_{k-i}(X)$. This is the point of view taken by Fulton; the reader can find proofs of these assertions in Fulton [1984, Chapter 3].

13.1 First examples

13.1.1 The intersection of a divisor and a subvariety

Suppose that X is a smooth projective variety, $D \subset X$ is a Cartier divisor and $\iota : C \hookrightarrow X$ the inclusion of a subvariety C in X. If C intersects D generically transversely, then the intersection class $[C][D]$ of D and C is $[C \cap D]$; more generally, since D is Cohen–Macaulay, this holds as long as C is not contained in D. But what if C *is* contained in D? In this case $D \cap C = C$, so we would like to find a class γ_C on C that pushes forward to the class $[D][C]$ on X.

Since D is an effective divisor, it is the zero locus of a global section σ of the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ on X, and is equivalent to $(\sigma')_0 - (\sigma')_\infty$, the divisor of zeros minus poles, of any rational section σ' of $\mathcal{L}$. We can find a rational section σ' of $\mathcal{L}$ such that $(\sigma')_0$ and $(\sigma')_\infty$ are both generically transverse to C. (Reason: If $\mathcal{L}'$ is a very ample bundle then $\mathcal{L} \otimes \mathcal{L}'^n$ is very ample for large n, and thus both $\mathcal{L} \otimes \mathcal{L}'^n$ and $\mathcal{L}'^n$ have sections without poles that vanish on divisors generically transverse to C. If we call these sections τ and τ', then τ/τ' is a rational section of $\mathcal{L}$ with the desired property.) If we take γ_C to be the class of the cycle $\langle(\sigma')_0 \cap C\rangle - \langle(\sigma')_\infty \cap C\rangle$, then since intersection products are well-defined on rational equivalence classes we have

$$\iota_*(\gamma_C) = [(\sigma')_0 \cap C] - [(\sigma')_\infty \cap C] = [D][C] \in A(X),$$

as required.

We may think of the class γ_C as the class of a rational section of the line bundle $\mathcal{L}|_C$, or equivalently as the first Chern class of $\mathcal{L}|_C$. Anticipating what is to come, we note that

$$\mathcal{N}_{D/X} := \mathrm{Hom}_X(\mathcal{I}_{D/X}, \mathcal{O}_D) = \mathrm{Hom}_X(\mathcal{L}^{-1}, \mathcal{O}_D) = \mathcal{L}|_D.$$

Thus we have proven:

Proposition 13.1. *Suppose that X is a smooth projective variety. Let $\iota : C \hookrightarrow X$ be the inclusion of a subvariety of codimension k in a smooth variety X, and let $D \subset X$ be an effective Cartier divisor containing C. We have*

$$[C][D] = \iota_*\gamma_C \in A^{k+1}(X),$$

where

$$\gamma_C = c_1(\mathcal{N}_{D/X}|_C).$$

Under suitable hypotheses we can give a geometric interpretation of Proposition 13.1 that shows why the normal bundle of D, restricted to C, is relevant. For simplicity we will take C to be a smooth curve, and suppose that there is a rational deformation of the divisor $D \subset X$, that is, a one-parameter family of divisors $\mathcal{D} \subset \Delta \times X$, parametrized by $\mathbb{A}^1$, such that the special fiber D_0 is D. (This will be the case in particular whenever

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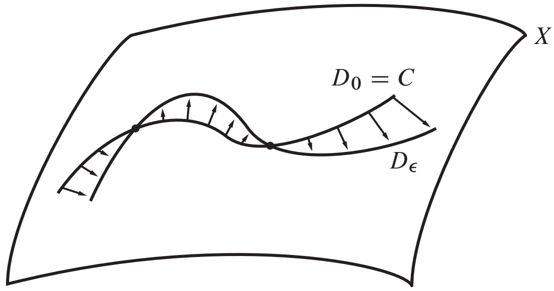

Figure 13.1 The limits of the points of intersection of C with the deformed divisor D_ϵ are zeros of the corresponding section of the normal bundle.

$h^0(\mathcal{O}_X(D)) > 1$.) Suppose moreover that a general member D_t of the deformation is transverse to C; that is,

$$D_t \cap C = \{p_1(t), \dots, p_k(t)\} \quad \text{for } t \neq 0,$$

with $k = \deg([D_t] \cdot [C]) = \deg([D] \cdot [C])$. Finally, suppose that the deformation is nontrivial to first order along C; that is, its restriction to the scheme $\operatorname{Spec} \Bbbk[t]/(t)^2 \subset \Delta$ supported at 0 corresponds to a section σ of the normal bundle $\mathcal{N}_{D/X}$ that is not identically 0 along C.

In this situation, we claim that the limit as $t \to 0$ of the divisor $D_t \cap C$ is the zero locus of the section σ, and thus represents the class $c_1(\mathcal{N}_{D/X}|_C)$. Heuristically, away from zeros of σ the deformation moves D away from itself (and hence away from C), while the points where σ is zero are stationary to first order.

To see this, take local analytic coordinates $(z_1, \dots, z_n)$ on X near a point of C such that, locally,

$$D = (z_n = 0) \quad \text{and} \quad C = (z_2 = \cdots = z_n = 0).$$

Locally, z_1 is a coordinate on C, and the normal bundle of D is trivial. We can write the family $\mathcal{D} \subset \Delta \times X$ as the zero locus of the function

$$z_n + t f_1(z_1, \dots, z_{n-1}) + t^2 f_2(z_1, \dots, z_{n-1}) + \cdots,$$

with $f_1(z_1, \dots, z_{n-1})$ representing the corresponding section of the normal bundle. Restricting to $\Delta \times C$, this becomes

$$t f_1|_C + t^2 f_2|_C + \cdots = t(f_1|_C + t f_2|_C + \cdots),$$

from which we see that a zero of $f_1|_C$ of order m will be a limit of exactly m points of intersection of C with D_t as $t \to 0$.

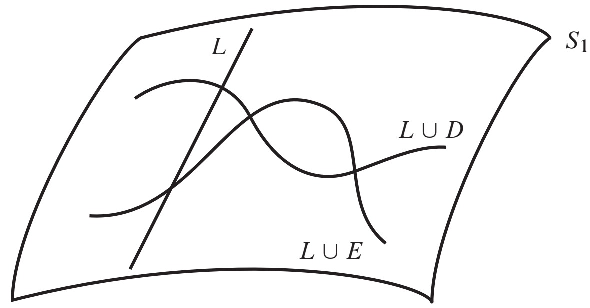

Figure 13.2 S_2 and S_3 intersect S_1 in $L \cup D$ and $L \cup E$.

13.1.2 Three surfaces in $\mathbb{P}^3$ containing a curve

Next we consider Keynote Question (a): Given three surfaces S_1, S_2 and $S_3 \subset \mathbb{P}^3$ of degrees s_i whose intersection consists of the disjoint union of a reduced line L and a zero-dimensional scheme Γ, we wish to find the degree of Γ. We will approach the question naively, solving it the way it might have been solved in the 19th century. We will then explain the modification necessary if L is replaced with an arbitrary smooth curve. This leads us to a formula exhibiting the main features of the general case.

By way of preparation, we would like to be able to assume that S_1 is smooth without changing the intersection $S_1 \cap S_2 \cap S_3$. We can do this if we reorder the surfaces so that the degree s_1 of S_1 is maximal among the s_i. With this choice we can replace S_1 by the zero locus of a general linear combination $F_1 + AF_2 + BF_3$, where F_i is the defining equation of S_i and A and B are general polynomials of degrees $s_1 - s_2$ and $s_1 - s_3$. The surface S_1 is then the general element of a linear system in $\mathbb{P}^3$ whose base locus is smooth of dimension less than $3/2$, and smoothness follows from Proposition 5.6.

Because S_1 is smooth, we can simplify the problem of computing $\deg \Gamma$ by working in the intersection ring $A(S_1)$. By hypothesis, the intersection of the two surfaces S_1 and S_2 may be written as

$$S_1 \cap S_2 = L + D \in Z_1(S_1)$$

for some divisor D on S_1. Similarly, write

$$S_1 \cap S_3 = L + E \in Z_1(S_1)$$

for some divisor E on S_1. (Note that D, E or even both may be 0.) If $p \in D \cap L$, then p is a singular point of $S_1 \cap S_2$. If also $p \in S_3$, then, since S_3 is a Cartier divisor, p is a singular point of $S_1 \cap S_2 \cap S_3$ as well, contradicting our hypothesis that the intersection is smooth along L. Thus $S_1 \cap S_2 \cap S_3$ is the disjoint union of L and $D \cap E$, so $\Gamma = D \cap E$. In particular the degree of Γ is the intersection number of the classes of D and E in $A(S_1)$.

Denote by $H \in A^1(S_1)$ the hyperplane class on S_1; since $S_1 \cap S_2 \sim s_2 H$ and $S_1 \cap S_3 \sim s_3 H$, we have

$$D \sim s_2 H - L \quad \text{and} \quad E \sim s_3 H - L$$

in $A^1(S_1)$. Thus, in $A(S_1)$ we have

$$\begin{aligned}[D][E] &= [s_2 H - L][s_3 H - L] \\ &= s_2 s_3 [H]^2 - (s_2 + s_3)[H][L] + [L]^2 \\ &= s_2 s_3 [H]^2 - (s_2 + s_3)[H][L] + \iota_* c_1 \mathcal{N}_{L/S_1},\end{aligned}$$

where ι denotes the inclusion of L in S_1, and $[L]^2 = \iota_* c_1 \mathcal{N}_{L/S_1}$ by the reasoning of Section 13.1.1.

To obtain a numerical result, we note that $\deg([H]^2) = \deg S_1 = s_1$, while $\deg([H][L]) = \deg L = 1$. We can compute $\deg(c_1(\mathcal{N}_{L/S_1})) = \deg([L]^2)$ as in Section 2.4 by using the adjunction formula twice. First, $K_L = [L]|_L + [K_{S_1}]|_L$ and $K_{S_1} = \mathcal{O}_{S_1}(s_1 - 4)$. Thus

$$\deg([L][K_{S_1}]) = s_1 - 4,$$

and since $\deg K_L = 2g(L) - 2 = -2$ we get

$$\deg([L]^2) = 2 - s_1.$$

This gives

$$\deg(\Gamma) = \deg([D][E]) = \prod s_i - \sum s_i + 2,$$

which is the answer to our first keynote question. Note that in this formula the number $\prod s_i = \deg[S_1][S_2][S_3]$ is the degree that the intersection would have if there were no curve component, so we can think of the remaining terms $\sum s_i - 2$ as representing “the contribution γ_L of the line to the intersection.”

For example, if $s_1 = s_2 = s_3 = 1$ we get $\deg \Gamma = 0$, corresponding to the fact that three planes meet in a linear space. More generally, if $s_1 = 1$ then we get $\deg \Gamma = (s_2 - 1)(s_3 - 1)$, corresponding to the fact that the residual curves D and E are plane curves of degrees $s_2 - 1$ and $s_3 - 1$, respectively.

We can make a similar computation if we replace the line L by any smooth curve C. If D has degree d and genus g then the adjunction formula, applied in the same way, gives

$$\deg(c_1(\mathcal{N}_{C/S_1})) = \deg([C]^2) = 2g - 2 - d(s_1 - 4).$$

It follows that in this more general case

$$\deg \Gamma = \deg([D][E]) = \prod s_i - d\left(\sum s_i\right) + 4d + 2g - 2.$$

We can interpret this formula as saying that the class $\prod[S_i]$ can be decomposed into the actual number of isolated points of intersection (with multiplicities) and a class, supported on the positive-dimensional component of $\bigcap S_i$, expressed in terms of various normal bundles. First, note that

$$\deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) = ds_i.$$

Next consider the natural exact sequence of ideal sheaves

$$0 \longrightarrow \mathcal{I}_{S_1/\mathbb{P}^3} \longrightarrow \mathcal{I}_{C/\mathbb{P}^3} \longrightarrow \mathcal{I}_{C/S_1} \longrightarrow 0.$$

Since S_1 is a Cartier divisor in $\mathbb{P}^3$ and C is a Cartier divisor on S_1, the left-hand term is a line bundle on $\mathbb{P}^3$ and the right-hand term is the line bundle on S_1. Thus the sequence restricts to an exact sequence of vector bundles on C and, applying the functor $\mathcal{H}om_{\mathcal{O}_S}(-, \mathcal{O}_C)$, we derive the exact sequence of restricted normal bundles

$$0 \longrightarrow \mathcal{N}_{C/S_1} \longrightarrow \mathcal{N}_{C/\mathbb{P}^3} \longrightarrow \mathcal{N}_{S_1/\mathbb{P}^3}|_C \longrightarrow 0.$$

In particular,

$$\begin{aligned}\deg c_1(\mathcal{N}_{C/\mathbb{P}^3}) &= \deg c_1(\mathcal{N}_{C/S_1}) + \deg c_1(\mathcal{N}_{S_1/\mathbb{P}^3}|_C)\\ &= 2g - 2 - d(s_1 - 4) + ds_1\\ &= 4d + 2g - 2.\end{aligned}$$

Putting this together, we have proven an excess intersection formula for our case:

Proposition 13.2. *Let S_1, S_2 and $S_3 \subset \mathbb{P}^3$ be surfaces of degrees s_1, s_2 and s_3 whose intersection consists of the disjoint union of a smooth curve C of degree d and genus g and a zero-dimensional scheme Γ. We have*

$$\begin{aligned}\deg\Big(\prod[S_i]\Big) &= \deg(\Gamma) + d\Big(\sum s_i\Big) - (4d + 2g - 2)\\ &= \deg(\Gamma) + \sum \deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) - \deg c_1(\mathcal{N}_{C/\mathbb{P}^3}).\end{aligned}$$

Another way to view this result is to imagine that we have one-parameter families $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \subset \Delta \times \mathbb{P}^3$ specializing to S_1, S_2 and S_3 such that the fibers $(S_1)_t$, $(S_2)_t$ and $(S_3)_t$ intersect transversely in $s_1s_2s_3$ points $p_1(t), \dots, p_{s_1s_2s_3}(t)$ for $t \neq 0$. Proposition 13.2 tells us that the number of points $p_i(t)$ that tend toward C as $\lambda \to 0$ is

$$\sum \deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) - \deg c_1(\mathcal{N}_{C/\mathbb{P}^3});$$

in other words, this is "the contribution of C to the total degree $s_1s_2s_3$ of the intersection." We will see in Section 13.3.1 a heuristic way to interpret this expression.

We can apply the formula of Proposition 13.2 to answer Keynote Question (d): If $C \subset \mathbb{P}^3$ is a smooth curve of degree d and genus g, can C necessarily be expressed as the (scheme-theoretic) intersection of three surfaces? By the formula of Proposition 13.2, the degrees s_i of the three surfaces would have to satisfy the equality

$$\prod s_i - d\Big(\sum s_i\Big) + 4d + 2g - 2 = 0.$$

For example, if $C \subset \mathbb{P}^3$ is an elliptic quintic curve, then C lies on no planes or quadrics (this follows from the genus formula for smooth curves on a plane and quadric), but when all s_i are ≥ 3 the quantity $\prod s_i - 5\big(\sum s_i\big) + 20$ is positive. (In Exercise 13.16 the reader is asked to answer the corresponding question for quintic curves $C \subset \mathbb{P}^3$ of genera 0 and 2.) This question was first answered by Peskine and Szpiro [1974] in a different way; the method given here is from Fulton [1984, Example 9.1.2]. Exercises 13.17 and 13.18 give further examples of such applications.

We will now explain how to restate Proposition 13.2 in forms that match the general expression of the excess intersection formula given in Theorem 13.3. The expression $\sum_i \deg c_1(\mathcal{N}_{S_i/\mathbb{P}^3}|_C) - \deg c_1(\mathcal{N}_{C/\mathbb{P}^3})$ that appears in Proposition 13.2 may be thought of as the degree of the component in $A_0(C)$ of the ratio of Chern classes

$$\frac{\prod c(\mathcal{N}_{S_i/\mathbb{P}^3}|_C)}{c(\mathcal{N}_{C/\mathbb{P}^3})}.$$

This expression also works for the components of $\bigcap S_i$ that are of the correct dimension — that is, of dimension 0. For if $p \in \bigcap S_i$ is an isolated zero-dimensional (possibly nonreduced) component, then the multiplicity of the reduced point p_{red} in the intersection is equal to the degree of p, as per the discussion in Section 1.3.8, and the normal bundle $\mathcal{N}_{p/\mathbb{P}^3}$ is isomorphic to the sum of the normal bundles of the S_i restricted to p, so that

$$\frac{\prod c(\mathcal{N}_{S_i/\mathbb{P}^3}|_p)}{c(\mathcal{N}_{p/\mathbb{P}^3})} = 1,$$

representing the fundamental class $[p] \in A_0(p)$. If for each connected component C_α of the intersection we define the class

$$\gamma_{C_\alpha} = \left\{ \frac{\prod_{i=1}^{3} c(\mathcal{N}_{S_i/\mathbb{P}^3}|_{C_\alpha})}{c(\mathcal{N}_{C_\alpha/\mathbb{P}^3})} \right\}_0 \in A_0(C_\alpha),$$

then we have shown

$$\prod_{i=1}^{3} [S_i] = \sum \iota_{C_\alpha *}(\gamma_{C_\alpha}),$$

where the sum is taken over all connected components C_α of $\bigcap S_i$.

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This expression for γ_{C_α} is symmetric in the S_i but, as we shall soon see, there is sometimes an advantage in using the following nonsymmetric form. We single out S_1, and note that since S_1 is smooth, C is actually a Cartier divisor on S_1. Thus it makes sense to speak of the normal bundle of C in S_1, and we have an exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{S_1/\mathbb{P}^3} \longrightarrow \mathcal{N}_{C_\alpha/\mathbb{P}^3} \longrightarrow \mathcal{N}_{C_\alpha/S_1} \longrightarrow 0.$$

Using Whitney's formula (Theorem 5.3), we get

$$c(\mathcal{N}_{C_\alpha/\mathbb{P}^3}) = c(\mathcal{N}_{S_1/\mathbb{P}^3})c(\mathcal{N}_{C_\alpha/S_1}).$$

Substituting the right side for the left in the formula for γ_{C_α}, we get

$$\gamma_{C_\alpha} = \left\{ \frac{\prod_{i=2}^{3} c(\mathcal{N}_{S_i/\mathbb{P}^3}|_{C_\alpha})}{c(\mathcal{N}_{C_\alpha/S_1})} \right\}_0.$$

Bearing in mind that the Segre class of a bundle is the inverse of the Chern class, we get the expressions

$$\begin{aligned} \gamma_{C_\alpha} &= \left\{ s(\mathcal{N}_{C_\alpha/\mathbb{P}^3}) \prod_{i=1}^{3} c(\mathcal{N}_{S_i/\mathbb{P}^3}|_{C_a}) \right\}_0 \\ &= \left\{ s(\mathcal{N}_{C_\alpha/S_1}) \prod_{i=2}^{3} c(\mathcal{N}_{S_i/\mathbb{P}^3}|_{C_\alpha}) \right\}_0. \end{aligned}$$

13.2 Segre classes of subvarieties

One might wonder why we have bothered to replace the inverse Chern class in the formula above with the Segre class. The reason is that Segre classes can be defined in a much more general context; indeed, this is the approach of Fulton [1984], where the Segre classes are used to define the Chern classes. What we need for the definition of Segre classes is really just a projective morphism $\pi : E \to C$ and a distinguished Cartier divisor class $\zeta \in A(E)$. Since we can compute the intersection of a Cartier class with any subvariety, we can define a codimension-i class on E as the i-th self-intersection ζ^i, and push $\sum \zeta^i$ forward to C. In the case where E is the projectivization of a bundle $\mathcal{N}$ on C and $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{N}}(1))$, the result is by definition the Segre class $s(\mathcal{N})$. More generally we say that $\pi : E \to C$ is a *cone* if $E = \operatorname{Proj} \mathcal{S}$, where $\mathcal{S}$ is a graded, locally finitely generated sheaf of algebras over $\mathcal{O}_C$ with $\mathcal{S}_0 = \mathcal{O}_C$, and π is the morphism corresponding to the inclusion. Taking ζ to be the line bundle on E associated to the sheaf of $\mathcal{S}$-modules $\mathcal{S}(1)$, we define the Segre class of the cone E to be

$$s(E) = \pi_* \left(\sum_k \zeta^k \right) \in A(C).$$

For our purposes it suffices to take the case where X is a variety, $C \subset X$ is a proper subscheme, and $\mathcal{S} = \bigoplus_n (\mathcal{I}^n_{C/X}/\mathcal{I}^{n+1}_{C/X})$, that is, the case where $\pi : E \to C$ is the exceptional divisor of the blow-up $B = \mathrm{Bl}_C X$ of X along C and ζ is the negative of the class of the exceptional divisor $E \subset B$, restricted to E. Following the pattern above we define the *Segre class of* C *in* X to be

$$s(C, X) := \pi_*\left(\sum_{k \geq 0} c_1(\mathcal{O}_E(1))^k\right),$$

where the intersections are taken in $A(E)$.

For example, in the case where X is smooth and E is equidimensional (in particular, when C is locally a complete intersection), since the blow-up has the same dimension as X the dimension of E is $\dim X - 1$, so the relative dimension of E over C is $\operatorname{codim} C - 1$; the codimension-$k$ component $s_k(C, X)$ of $s(C, X)$ is thus

$$s_k(C, X) := \pi_*(\zeta^{\operatorname{codim} C - 1 + k}).$$

Indeed, if C is locally a complete intersection in X then

$$\mathcal{I}^n_{C/X}/\mathcal{I}^{n+1}_{C/X} \cong \mathrm{Sym}^n(\mathcal{I}_{C/X}/\mathcal{I}^2_{C/X}) = \mathrm{Sym}^n(\mathcal{N}^*_{C/X}),$$

so $\mathrm{Proj}\big(\bigoplus_n \mathcal{I}^n_{C/X}/\mathcal{I}^{n+1}_{C/X}\big) = \mathrm{Proj}(\mathcal{N})$ and thus

$$s(C, X) = s(\mathcal{N}_{C/X}) = c(\mathcal{N}_{C/X})^{-1},$$

as before.

13.3 The excess intersection formula

Putting together these ideas, and recalling that on any variety C we can take the product of an arbitrary class in $A(C)$ with the Chern class of a bundle on C, we can express a very general form of Bézout's theorem. It suffices to treat the case where we intersect just two subvarieties:

Theorem 13.3 (Excess intersection formula)**.** *If $S \subset X$ is a subvariety of a smooth variety X and T is a locally complete intersection subvariety of X, then*

$$[S][T] := \sum_C (\iota_C)_*(\gamma_C),$$

where:

- *The sum is taken over the connected components C of $S \cap T$.*
- *$\iota_C : C \to X$ denotes the inclusion morphism.*
- *$\gamma_C = \{s(C, S)c(\mathcal{N}_{T/X}|_C)\}_d \in A_d(C)$, where $d = \dim X - \operatorname{codim} S - \operatorname{codim} T$ is the "expected dimension" of the intersection.*

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If the subvariety S is locally a complete intersection as well, then we have a symmetric form

$$\gamma_C = \{s(C, X)c(\mathcal{N}_{S/X}|_C)c(\mathcal{N}_{T/X}|_C)\}_d.$$

Two notes on this statement. First, it should be emphasized that each connected component C of $S \cap T$ is to be taken with the scheme structure inherited from $S \cap T$; this is important, because it may affect the classes $s(C, S)$ and $s(C, X)$. Secondly, it might appear that when $S \subset X$ is a locally complete intersection we have an exact sequence of normal bundles

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/X} \longrightarrow \mathcal{N}_{S/X}|_C \longrightarrow 0,$$

from which it would follow by Whitney that

$$s(\mathcal{N}_{C/S}) = \frac{1}{c(\mathcal{N}_{C/S})} = \frac{c(\mathcal{N}_{S/X}|_C)}{c(\mathcal{N}_{C/X})} = s(\mathcal{N}_{C/X})c(\mathcal{N}_{S/X}|_C),$$

accounting for the difference in the two expressions above for γ_C. Sadly, the equality $s(C, S) = s(C, X)c(\mathcal{N}_{S/X})$ is not true in general, even when both S and T are locally complete intersections, as for example in the case $X = \mathbb{P}^3$, C and T are both equal to a line $L \subset \mathbb{P}^3$ and S is the union of two planes containing L. So the equality of the two expressions in this case is even subtler than it looks.

By induction, we could extend the formula to a formula for the intersection of S with an arbitrary number of locally complete intersection subvarieties $S_i \subset X$. In the case where the subvarieties S_i to be intersected are all hypersurfaces, there is a form of the excess intersection formula due to Wolfgang Vogel that is sometimes more easily adapted to computation; see Vogel [1984] and, for a comparison with Theorem 13.3, van Gastel [1990]. We will give a brief description of Vogel's approach in Section 13.3.6 below.

Theorem 13.3 is of great theoretical importance in at least three circumstances:

Intersection products on nonsmooth varieties: The formula can serve as a definition of the intersection product of locally complete intersection subvarieties of an *arbitrary* variety X; there is no smoothness used in defining the terms on the right-hand side of the formula. This is, in fact, the way that Fulton [1984] defined intersections in this case.

Intersection products on smooth varieties: Defining intersections with locally complete intersection subvarieties actually suffices to define all intersection products on a smooth variety! Indeed, suppose $T_1, T_2 \subset Y$ are arbitrary subvarieties. Since Y is smooth the diagonal $Y \cong \Delta \subset Y \times Y$ is a locally complete intersection subvariety, so we can use the asymmetric form of the formula on $X = Y \times Y$ to define classes on the connected components of $\Delta \cap (T_1 \times T_2) \cong T_1 \cap T_2$. Pushing these classes forward to $\Delta \cong X$ and adding, we get the intersection class of T_1 and T_2 in X. Again, this is the path followed in Fulton [1984].

General pullbacks: In Theorem 1.23, we characterized the pullback φ^* of equivalence classes of cycles along a projective morphism $\varphi : X \to Y$ of smooth varieties by taking the class of a cycle A on X to the class of $\varphi^{-1}(A)$, in the case where A is generically transverse to φ, in the sense that A is a linear combination of subvarieties of Y whose preimages in X are generically reduced of the same codimension; the moving lemma shows that we can find a cycle A with this property in any rational equivalence class, so that specifying the pullback class in this case determines φ^*. Theorem 13.3 gives a different way of defining $\varphi^*([A])$, without smoothness hypotheses and without the moving lemma, in the cases where $A \subset X$ is a closed locally complete intersection subscheme of X or the graph of φ is locally a complete intersection in $X \times Y$ (a situation that holds whenever both X and Y are smooth): In these cases one can apply Theorem 13.3 to compute the product of the class $[X \times A]$ and the class of the graph Γ_φ of φ as a class on the intersection $(X \times A) \cap \Gamma_\varphi \cong \varphi^{-1}(A)$ inside $A(X \times Y)$, and push the result forward along the projection to X. (Note that, as in the case of intersections, the pullback class $\varphi^*[A]$ is expressed as the pushforward of a class on the preimage $\varphi^{-1}(A)$; this is occasionally very useful, as in the proof of Proposition 13.12.) One must, of course, prove that this agrees with $[\varphi^{-1}(A)] \in A(X)$ in the case where A is generically transverse to φ. Once more, this is the route taken in Fulton [1984].

For the proof of Theorem 13.3 and the treatment of its consequences as above, we refer to Fulton [1984, Section 6.3], where all this is worked out. We will give only the proof of a special case, Theorem 13.7; however, this proof contains one of the major new ideas, from Fulton and MacPherson [1978], that went into the general theorem.

In the following section, we will give a heuristic argument that may help to explain the form of Theorem 13.3. Following that, we work out some examples, including the second and third keynote questions.

13.3.1 Heuristic argument for the excess intersection formula

The expression given in Theorem 13.3 for the class γ_C may seem to come out of nowhere. The following calculation may help explain where it is coming from; though it is not a suitable framework for a proof — it involves far too many extra hypotheses, and we have omitted the multiplicity calculations that would be necessary, even subject to those hypotheses, to make it into a proof — it will hopefully at least make the form of Theorem 13.3 more plausible.

To begin with, we make the following assumptions:

- X will be a smooth, projective variety of dimension n.
- S and $T \subset X$ will be smooth subvarieties of codimensions k and l respectively.
- The intersection $C = S \cap T$ will be smooth, with connected components C_α of codimension $k + l - m_\alpha$.

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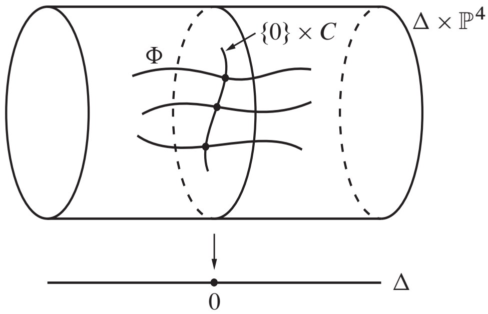

Figure 13.3 Limits of points of intersection of the deformed varieties S_λ and T_λ are singular points of $\mathcal{S} \cap \mathcal{T} = \Phi \cup (\{0\} \times C)$.

Again, we want to assign to each C_α a cycle class $\gamma_\alpha \in A^{m_\alpha}(C_\alpha)$ of dimension $k+l-n$, which represents the contribution of C_α to the total intersection $S \cap T$, that is, such that if we denote by $i_\alpha : C_\alpha \to X$ the inclusion, then

$$\sum_\alpha (i_\alpha)_*(\gamma_\alpha) = [S] \cdot [T] \in A^{k+l}(X).$$

We will do this by imagining that we can deform S and T to cycles S_λ and T_λ on X intersecting transversely, and asking for the limiting position of the intersection $S_\lambda \cap T_\lambda$ as $\lambda \to 0$. We therefore make a crucial (and very frequently counterfactual) fourth hypothesis:

- There exist families $\mathcal{S}, \mathcal{T} \subset \Delta \times X$, flat over a smooth rational curve Δ, with fibers $S_0 = S$ and $T_0 = T$, such that $S_\lambda \cap T_\lambda$ is transverse for $\lambda \neq 0$.

Given all this, we can express the intersection $\mathcal{S} \cap \mathcal{T}$ of the $(k+1)$- and $(l+1)$-folds $\mathcal{S}$ and $\mathcal{T}$ in the $(n+1)$-fold $\Delta \times X$ as a union

$$\mathcal{S} \cap \mathcal{T} = \Phi \cup D,$$

where Φ is flat over Δ, consisting of the components of the intersection $S_\lambda \cap T_\lambda$ for $\lambda \neq 0$ and their limits — that is, the closure in $\Delta \times X$ of the intersection of $\mathcal{S} \cap \mathcal{T}$ with $(\Delta \setminus \{0\}) \times X$ — and $D = \{0\} \times C$, as in Figure 13.3 above.

Now, we know that the cycle $\Xi = \Phi \cap (\{0\} \times X)$ has dimension $k+l-n$, and has class $[S] \cdot [T] \in A^{k+l}(X)$ as a cycle on X. Moreover, Ξ consists of the sum of its intersections $\Xi_\alpha = \Phi \cap D_\alpha$ with the connected components $D_\alpha = \{0\} \times C_\alpha$ of $S \cap T$. To find the class $[S] \cdot [T]$, accordingly, we have to figure out the class of the intersection $\Phi \cap D_\alpha$ for each connected component.

Finally, one way to characterize the points of $\Phi \cap D_\alpha$ is to observe that *they are the points $p \in D_\alpha$ where the tangent spaces $T_p\mathcal{S}$ and $T_p\mathcal{T} \subset T_p(\Delta \times X)$ fail to intersect in $T_p D_\alpha$*. Now, if we had

$$T_p D_\alpha = T_p \mathcal{S} \cap T_p \mathcal{T}$$

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for all $p \in D_\alpha$, we would have a direct sum decomposition of bundles

$$\mathcal{N}_{D_\alpha/\Delta\times X} = \mathcal{N}_{\mathcal{S}/\Delta\times X}|_{D_\alpha} \oplus \mathcal{N}_{\mathcal{T}/\Delta\times X}|_{D_\alpha}.$$

In general, we see that we have a map

$$\mathcal{N}_{D_\alpha/\Delta\times X} \to \mathcal{N}_{\mathcal{S}/\Delta\times X}|_{D_\alpha} \oplus \mathcal{N}_{\mathcal{T}/\Delta\times X}|_{D_\alpha}$$

between vector bundles on D_α of ranks $k + l - m_\alpha + 1$ and $k + l$, and the locus where this map fails to be injective (as a map of vector bundles) is exactly the cycle Ξ_α.

We have

$$c(\mathcal{N}_{D_\alpha/\Delta\times X}) = c(\mathcal{N}_{C_\alpha/X})$$

and

$$\mathcal{N}_{\mathcal{S}/\Delta\times X}|_{D_\alpha} = \mathcal{N}_{S/X}|_{C_\alpha},$$

and likewise

$$\mathcal{N}_{\mathcal{T}/\Delta\times X}|_{D} = \mathcal{N}_{T/X}|_{C_\alpha}.$$

Now we can apply Porteous to deduce that the class of $\Xi = \Phi \cap D_\alpha$ is

$$\gamma_\alpha = [\Xi_\alpha] = \left\{ \frac{c(\mathcal{N}_{S/X}|_{C_\alpha}) \cdot c(\mathcal{N}_{T/X}|_{C_\alpha})}{c(\mathcal{N}_{C_\alpha/X})} \right\}^{m_\alpha} \in A^{m_\alpha}(D_\alpha),$$

from which we arrive at the statement of Theorem 13.3.

13.3.2 Connected components versus irreducible components

One further remark is in order before we get to the examples. One might ask if it is possible to refine the excess intersection formula to associate a class γ_C to each *irreducible* component of the intersection, as opposed to each connected component, in such a way that when we push forward each γ_C into $A(X)$ and sum the results, we get the class $[S][T]$. The following example shows that this is not in fact possible.

Example 13.4. Let $L_1, \ldots, L_4 \subset \mathbb{P}^2$ be four concurrent (but distinct) lines, and consider two reduced effective cycles S and T on $\mathbb{P}^2$, where S is the sum $L_1 + L_2 + L_3$ and $T = L_1 + L_2 + L_4$, as in Figure 13.4. Since S and T are cubic curves, the product $[S][T] \in A(\mathbb{P}^2)$ is the class of (any) cycle consisting of nine points. On the other hand, the intersection of the underlying algebraic sets is $C := L_1 \cup L_2$, and the two lines L_1 and L_2 play completely symmetric roles. Since 9 is an odd number, there is no canonical way of dividing the intersection cycle into a cycle on L_1 and a cycle on L_2.

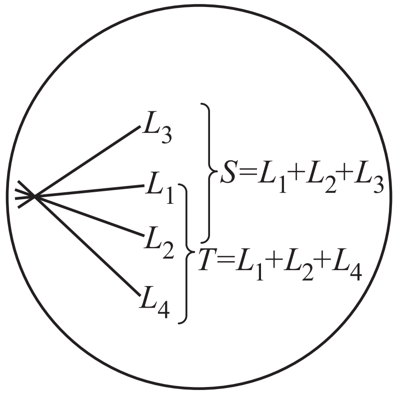

Figure 13.4 The product $[S][T]$ is a cycle of nine points on $L_1 \cup L_2$.

13.3.3 Two surfaces in $\mathbb{P}^4$ containing a curve

We can use Theorem 13.3 to answer the second of our keynote questions: Given smooth surfaces S and $T \subset \mathbb{P}^4$ of degrees s and t whose intersection consists of a smooth curve C of genus g and degree d and a collection Γ of reduced points, we ask, as before, what we can say about the degree of Γ.

Because S and T are assumed smooth, they are locally complete intersections. Moreover, since Γ is reduced each point $p \in \gamma$ satisfies $\gamma_p = [p]$. Thus the asymmetric form of the expression for γ_L in Theorem 13.3 gives

$$\begin{aligned} st &= \deg \Gamma + \deg \left\{ \frac{c(N_{T/\mathbb{P}^4}|_C)}{c(\mathcal{N}_{C/S})} \right\}_0 \\ &= \deg \Gamma + \deg c_1(N_{T/\mathbb{P}^4}|_C) - \deg c_1(N_{C/S}). \end{aligned}$$

Now $\deg c_1(\mathcal{N}_{C/S})$ is the degree of the class $[C]^2$, computed in $A(S)$, which we write $[C_S]^2$. On the other hand, the adjunction formula gives $\deg K_C = 2g - 2 = \deg K_T|_C + \deg[C_T]^2$, and

$$K_T = c_1(\mathcal{N}_{T/\mathbb{P}^4}) + K_{\mathbb{P}^4}|_T = c_1(\mathcal{N}_{T/\mathbb{P}^4}) + 5H,$$

where H is the hyperplane class in $A^1(T)$ (Hartshorne [1977, Proposition II.8.20]). Thus

$$\deg c_1(\mathcal{N}_{T/\mathbb{P}^4}|_C) = 2g - 2 + 5d - \deg[C_T]^2.$$

Putting this all together, we get

$$st = \deg \Gamma + 2g - 2 + 5d - \deg[C_S]^2 - \deg[C_T]^2,$$

which becomes $\deg \Gamma = st - 3 + \deg[L_S]^2 + \deg[L_T]^2$ when $C = L$, a line. For example, if $S, T \subset \mathbb{P}^4$ are 2-planes containing the line L, then Γ is empty and indeed the formula gives $\deg \Gamma = 0$. For other examples, see Exercise 13.19. In contrast to

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the answer to our first keynote question, this does not depend only on the degrees of S and T (that is, their classes in $A(\mathbb{P}^4)$), but on their geometry; the simplest example of this is described in Exercise 13.20 below.

13.3.4 Quadrics containing a linear space

As a second application of Theorem 13.3, we will answer Keynote Question (c). Let $Q_1, \dots, Q_n \subset \mathbb{P}^n$ be general quadric hypersurfaces containing a codimension-c plane $\Lambda \cong \mathbb{P}^{n-c} \subset \mathbb{P}^n$.

Proposition 13.5. *Let $\Lambda \subset \mathbb{P}^n$ be a plane of codimension c. If $Q_1, \dots, Q_n$ are quadric hypersurfaces that are general among those containing Λ, then*

$$\bigcap_{i=1}^{n} Q_i = \Lambda \cup \Gamma$$

as schemes, where Γ is a set of

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{c-1} = 2^n - \binom{n}{c} - \cdots - \binom{n}{n}$$

reduced points, disjoint from Λ.

At one extreme, when $c = 1$ each Q_i is the union of Λ with a generic hyperplane, and the intersection of these hyperplanes is a single point outside Λ. At the other extreme, when $c = n$, so that Λ is a point, the set Γ consists of all but one of the 2^n points in the complete intersection of the Q_i.

More interesting geometrically is the case where $c = 2$. Suppose Λ is the zero locus of the linear forms X_0 and X_1. Then each of the quadrics Q_i can be written as the zero locus of a linear combination of X_0 and X_1:

$$Q_i = V(F_i), \quad \text{where } F_i = X_0 L_i + X_1 M_i$$

for some linear forms L_i and M_i. Now consider the $2 \times (k+1)$ matrix

$$\Phi = \begin{pmatrix} X_0 & M_1 & M_2 & \cdots & M_k \\ X_1 & L_1 & L_2 & \cdots & L_k \end{pmatrix}.$$

Away from the locus $\Lambda = V(X_0, X_1)$ where X_0 and X_1 both vanish, the rank-1 locus of Φ is just the intersection of the quadrics Q_i; in other words,

$$\Gamma = \{X \in \mathbb{P}^n \mid \operatorname{rank}(\Phi(X)) \le 1\}.$$

The degree of Γ is then given by Porteous' formula; as we worked out in Section 12.4.1, this has degree $k + 1$. This is a special case of what is sometimes called the *Steiner construction*; see Griffiths and Harris [1994, Section 4.3] for more information.

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Proof of Proposition 13.5: We first claim that Λ is a reduced connected component of the intersection and Γ consists of a collection of reduced points. In particular, since Λ is a connected component of the intersection, we may apply Theorem 13.3. Since the ideal of Λ is generated by c linear forms l_i, the conormal bundle $\mathcal{N}^* := \mathcal{I}_\Lambda/\mathcal{I}_\Lambda^2$ of Λ is $\mathcal{O}_\Lambda^c(-1)$. The equation of a quadric hypersurface vanishing on Λ is a linear combination of the l_i with linear coefficients, and thus defines a general section of $\mathcal{N}(-1)$. Thus if we fix equations for the Q_i we get a general map

$$\mathcal{O}_\Lambda^n(-2) \to \mathcal{O}_\Lambda^c(-1).$$

Since a general $c \times n$ matrix has minors vanishing in codimension $n - c + 1$, this map is locally a surjection everywhere on Λ, which is to say that the Q_i cut out Λ scheme-theoretically. Since $\bigcap Q_i$ is smooth along Λ, no other component of $\bigcap Q_i$ meets Λ. In addition, since the linear series spanned by the Q_i has no base points outside of Λ, it follows that the "residual" scheme Γ consists of reduced points, as claimed.

The normal bundle of Q_i in $\mathbb{P}^n$ is $\mathcal{O}_{\mathbb{P}^n}(2)|_{Q_i} = \mathcal{O}_{Q_i}(2)$. Thus, writing $\zeta \in A^1(\Lambda)$ for the hyperplane class, we have

$$c(\mathcal{N}_{Q_i/\mathbb{P}^n}|_\Lambda) = 1 + 2\zeta.$$

From $\mathcal{N}_{\Lambda/\mathbb{P}^n} \cong \mathcal{O}_\Lambda(1)^{\oplus c}$, we get

$$c(\mathcal{N}_{\Lambda/\mathbb{P}^n}) = (1+\zeta)^c.$$

Thus, in the notation of Theorem 13.3, the contribution γ_Λ of the component Λ to the intersection is given by

$$\gamma_\Lambda = \left\{ \frac{(1+2\zeta)^n}{(1+\zeta)^c} \right\}_0 \in A_0(\Lambda) \cong \mathbb{Z}.$$

This is the coefficient of z^{n-c} in

$$\frac{(1+2z)^n}{(1+z)^c},$$

which is

$$x := \sum_{j=0}^{n-c} (-1)^j 2^{n-c-j} \binom{n}{n-c-j} \binom{c+j-1}{j},$$

and we must show that it is equal to $\sum_{i=c}^n \binom{n}{i}$, the number of subsets with $\geq c$ elements in a set S of cardinality n.

Now

$$2^{n-c-j} \binom{n}{n-c-j} \binom{c+j-1}{j} = \#\{A \subset B \subset S \mid \#B = n-c-j\} \binom{c+j-1}{j}.$$

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Thus we may regard the alternating sum x as counting subsets by inclusion-exclusion: Setting $k = n - c$, a given subset A of cardinality a is counted a total of

$$\sum (-1)^j \binom{n-a}{k-a-j} \binom{n-k+j-1}{j}$$

times, and we recognize the last expression as the coefficient of z^{k-a} in the power series expansion of

$$\frac{(1+z)^{n-a}}{(1+z)^{n-k}},$$

which is 1 since the quotient is $(1+z)^{k-a}$. $\square$

We can also verify the statement of Proposition 13.5 by specializing to the case where each quadric Q_i is a union of a general hyperplane H_i containing Λ and a second general hyperplane H_i'. Each point of $\bigcap Q_i$ is the point of intersection of a subset of d hyperplanes $\{H_i\}_{i \in D}$ and $n - d$ hyperplanes $\{H_i'\}_{i \notin D}$, and the point is outside Λ if and only if $d < c$. This establishes a bijection between the points of intersection outside Λ and subsets of $\{1, \dots, n\}$ of cardinality $< c$, yielding the result. Indeed, given that the number of points depends only on the rational equivalence class of the Q_i and the fact that Λ is a component of their intersection (the *principle of specialization*), this argument gives an alternative proof of Proposition 13.5.

13.3.5 The five conic problem

Theorem 13.3 gives another way to solve the five conic problem treated in Chapter 8: How many conics $C \subset \mathbb{P}^2$ are tangent to each of five given conics $C_1, \dots, C_5$. The problem was first solved by this method in Fulton and MacPherson [1978]; a short version appears in Fulton [1984, p. 158]. Here we simply sketch the necessary ideas.

As we saw in Chapter 8, the set of conics tangent to C_i is a sextic hypersurface $Z_i \subset \mathbb{P}^5$ in the space $\mathbb{P}^5$ parametrizing all plane conics. As we also saw there, the hypersurfaces $Z_1, \dots, Z_5 \subset \mathbb{P}^5$ do not intersect properly; rather, they all contain the Veronese surface $S \subset \mathbb{P}^5$ corresponding to double lines. Thus, if T denotes the component of the (scheme-theoretic) intersection $\bigcap Z_i$ supported on S, we have

$$\bigcap_{i=1}^{5} Z_i = T \cup \Gamma,$$

and the problem is to determine the cardinality of Γ. In Chapter 8 we did this by replacing $\mathbb{P}^5$ by the space of *complete conics*. Now we can apply Theorem 13.3 directly in $\mathbb{P}^5$. To do this, we need:

(a) *The Chern class of the restriction to S of the normal bundle of $Z_i \subset \mathbb{P}^5$*. This is the easiest part: Let $\zeta \in A^1(S)$ be the class of a line in $S \cong \mathbb{P}^2$, and let $\eta \in A^1(\mathbb{P}^5)$ be the hyperplane class on $\mathbb{P}^5$; note that the restriction of η to S is 2ζ. Since the Z_i are sextic hypersurfaces, $\mathcal{N}_{Z_i/\mathbb{P}^5} = \mathcal{O}_{Z_i}(6)$ and so

$$c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) = 1 + 12\zeta.$$

(b) *The multiplicity of Z_i along S*. By Riemann–Hurwitz, a general pencil of plane conics including a double line $2L$ will have four other elements tangent to C_i, so that $\operatorname{mult}_S(Z_i) = 2$. (See Exercise 13.31.)

(c) *The Chern classes of the normal bundle of $S \subset \mathbb{P}^5$*. In terms of the hyperplane classes $\eta \in A^1(\mathbb{P}^5)$ and $\zeta \in A^1(S)$, we have

$$c(\mathcal{T}_S) = (1+\zeta)^3 = 1 + 3\zeta + 3\zeta^2$$

and

$$c(\mathcal{T}_{\mathbb{P}^5}|_S) = (1+\eta)^6|_S = 1 + 12\zeta + 60\zeta^2.$$

Applying the Whitney formula to the sequence

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}_{\mathbb{P}^5}|_S \longrightarrow \mathcal{N}_{S/\mathbb{P}^5} \longrightarrow 0,$$

we conclude that

$$\begin{aligned} c(\mathcal{N}_{S/\mathbb{P}^5}) &= \frac{1 + 12\zeta + 60\zeta^2}{1 + 3\zeta + 3\zeta^2} \\ &= 1 + 9\zeta + 30\zeta^2, \end{aligned}$$

and inverting this we have

$$s(\mathcal{N}_{S/\mathbb{P}^5}) = 1 - 9\zeta + 51\zeta^2.$$

(d) *The scheme-theoretic intersection of the hypersurfaces Z_i*. This is easy to state: The component of $\bigcap Z_i$ supported on S is exactly the scheme $T = V(\mathcal{I}^2_{S/\mathbb{P}^5})$ defined by the square of the ideal $\mathcal{I}_{S/\mathbb{P}^5}$. We will not give a proof here; given part (b) above, the statement is equivalent to the statement that the proper transforms of the Z_i in the blow-up of $\mathbb{P}^5$ along S have no common intersection in the exceptional divisor, which is proved Griffiths and Harris [1994, Chapter 6]. Alternatively, via the isomorphism of the blow-up with the space of complete conics, it is tantamount to the statement, proved in Section 8.2.3, that every complete conic tangent to each of $C_1, \dots, C_5$ is smooth.

Given part (d), the blow-up of $\mathbb{P}^5$ along T is the same as the blow-up along S, but with the exceptional divisor doubled. Applying the definition of Section 13.2, the k-th graded piece of the Segre class $s(T, \mathbb{P}^5)$ is 2^{k+3} times the corresponding graded piece of $s(S, \mathbb{P}^5)$, so that

$$s(T, \mathbb{P}^5) = 8 - 144\zeta + 1632\zeta^2.$$

Thus the contribution of S to the degree of the intersection $\bigcap Z_i$ is

$$\begin{aligned}\deg\left(\prod c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)\cdot s(T,\mathbb{P}^5)\right) &= \deg((1+12\zeta)^5(8-144\zeta+1632\zeta^2))\\ &= 1632-60\cdot 144+1440\cdot 8\\ &= 4512,\end{aligned}$$

and the degree of Γ is correspondingly $7776-4512=3264$.

13.3.6 Intersections of hypersurfaces in general: Vogel's approach

We take a moment here to describe Vogel's approach to the problem of excess intersection, in the case where the varieties being intersected are all hypersurfaces. In fact, we have already seen this approach carried out in a special case: The method described here is exactly what we did in the case of the intersection of three surfaces in $\mathbb{P}^3$ considered in Section 13.1.2.

Briefly, let X be a smooth, n-dimensional projective variety and $D_1,\dots,D_k$ a collection of hypersurfaces in X. Assume the intersection $Y=\bigcap D_i$ has components of dimension strictly greater than $n-k$, as well as components of the expected dimension $n-k$. For simplicity, say

$$\bigcap_{i=1}^{k} D_i = \Phi\cup\Gamma,$$

with Φ and Γ smooth and disjoint, Φ of pure dimension $n-k+m$ and Γ of pure dimension $n-k$. Can we find the class of the sum Γ of the components of the expected dimension $n-k$?

In fact, we can, if we know something about the geometry of the divisors D_i and their intersection. What we want to do here is intersect the hypersurfaces D_i one at a time, and focus on the first step where the intersection fails to have the expected dimension; if we allow ourselves to change the order of the divisors D_i, we can assume that this occurs after we have intersected $k-m+1$ of the divisors, at which point Φ appears as a component of excess intersection. The point is, if we back up one step, we see that *the previous intersection* $D_1\cap\cdots\cap D_{k-m}$ *must have been reducible*: if $\Gamma\neq\varnothing$, then we must have

$$D_1\cap\cdots\cap D_{k-m}=\Phi\cup B_0.$$

Now back up one further step, and consider the intersection $S=D_1\cap\cdots\cap D_{k-m-1}$, which has dimension $n-k+m+1$. We have an equation of divisors on S

$$D_{k-m}\cap S = A+B_0,$$

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and similarly we can write

$$D_{k-m+\alpha} \cap S = A + B_\alpha$$

for each $\alpha = 0, \ldots, m$. We can then express the cycle Γ as a proper intersection of $m+1$ divisors on the variety S:

$$\Gamma = B_0 \cap \cdots \cap B_m,$$

and if we know enough about the geometry of Φ — specifically, if we can evaluate the products of powers $[\Phi]^l \in A^l(S)$ with products of the classes $[D_\alpha]|_S \in A^1(S)$ — we can then evaluate the class of this intersection as

$$[\Gamma] = \prod_{\alpha=0}^{m} ([D_{k-m+\alpha}]|_S - [\Phi]) \in A^{m+1}(S) \to A^{n-k}(X).$$

As we said, this approach was the one we used in Section 13.1.2; for some other examples, try Exercises 13.17 and 13.18.

13.4 Intersections in a subvariety

One frequently occurring situation in which excess intersection arises is the case of cycles A and B on a smooth variety X that happen to both lie on a proper subvariety $Z \subsetneq X$; in this case, the generalized principal ideal theorem (Theorem 0.2) says that their intersection has dimension at least $\dim A + \dim B - \dim Z > \dim A + \dim B - \dim X$. Thus as cycles on X their intersection cannot even be dimensionally transverse. Nevertheless we can relate their intersection class $[A][B] \in A(Z)$ in Z to the intersection of their classes on X. (To avoid confusion, we will denote the classes of A and B, viewed as cycles on X, by $\iota_*[A], \iota_*[B] \in A(X)$.)

Proposition 13.6 (Key formula). *Let $\iota : Z \to X$ be an inclusion of smooth projective varieties of codimension m, and let $\mathcal{N} = \mathcal{N}_{Z/X}$ be the normal bundle of Z in X. If $\alpha \in A^a(Z)$ and $\beta \in A^b(Z)$, then*

$$\iota_*\alpha \cdot i_*\beta = \iota_*(\alpha \cdot \beta \cdot c_m(\mathcal{N}_{Z/X})) \in A^{a+b+2m}(X).$$

Proposition 13.6 follows easily from the important special case where $B = Z$. Using the fact that $\alpha[Z] = \alpha \in A(Z)$, this takes the following form:

Theorem 13.7. *Let $\iota : Z \to X$ be an inclusion of smooth projective varieties of codimension m, and let $\mathcal{N}_{Z/X}$ be the normal bundle of Z in X. For any class $\alpha \in A(Z)$ we have*

$$\iota^*(\iota_*\alpha) = \alpha \cdot c_m(\mathcal{N}_{Z/X}) \in A^{a+m}(Z).$$

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Proof of Proposition 13.6 from Theorem 13.7: From $\iota^*(\iota_*\alpha) = \alpha\mathcal{N}_{Z/X}$, we use the push-pull formula to get

$$\iota_*(\alpha\mathcal{N}_{Z/X}\beta) = \iota_*(\iota^*(i_*\alpha)\beta) = (\iota_*\alpha)(\iota_*\beta). \qquad \square$$

Note that Theorem 13.7 and Proposition 13.6 may be deduced from the more general Theorem 13.3. The result is easier to visualize in the special case, however, and it will give us an occasion to describe a key technique, that of *specialization to the normal cone*.

One way to see the plausibility of Theorem 13.7 is to consider the special case where the normal bundle $\mathcal{N}_{Z/X}$ extends to a bundle $\mathcal{N}$ with enough sections on all of X (this happens, for example, when Z is a suitably positive divisor): In that case, $[Z] = c_m(\mathcal{N})$, so at least

$$\iota_*(\iota^*(\iota_*\alpha)) = \iota_*[Z]\alpha = c_m(\mathcal{N})\alpha.$$

The situation above can actually be realized topologically: Consider the complex case, and suppose that $\mathcal{N}_{Z/X}$ has enough sections. Topologically, the normal bundle looks like a tubular neighborhood of Z in X, and again $c_m(\mathcal{N}_{Z/X})$ is the class of the zero locus of a general section σ. Thinking of the image σ as a perturbation Z' of Z within the tubular neighborhood, the set where $\sigma = 0$ corresponds to the intersection $Z' \cap Z$ — that is, the self-intersection of Z as a subvariety of X.

Neither of these ideas suffice to prove Theorem 13.7, even in the complex analytic case: Rational equivalence is more subtle than homological equivalence, and the tubular neighborhood theorem that we used is false in the category of complex analytic or algebraic varieties. (For example, no analytic neighborhood of a conic curve $C \subset \mathbb{P}^2$ is biholomorphic to any neighborhood of the zero section in the normal bundle $\mathcal{N}_{C/\mathbb{P}^2}$; see Exercise 13.23.) However, the technique of "specialization to the normal cone" (also called "deformation to the normal cone"), introduced in Fulton and MacPherson [1978] (see also the references at the end of Fulton [1984, Chapter 5]), provides a flat degeneration from the neighborhood of Z in X to the neighborhood of Z in its normal bundle, which suffices. We will take up this technique in the next section, and then return to the proof of Theorem 13.7 in the following one.

Note that as a special case of Theorem 13.7, we see that if $\iota : Z \subset X$ is an inclusion of smooth projective varieties of codimension m then the square of the class $[Z] \in A^m(X)$ is the pushforward $\iota_*(c_m(\mathcal{N}_{Z/X}))$ of the top Chern class of the normal bundle. (We can use this, for example, to determine the self-intersection of a linear space $\Lambda \cong \mathbb{P}^m \subset X$ on a smooth hypersurface $X \subset \mathbb{P}^{2m+1}$, as suggested in Exercise 13.22.) More generally, associating to a class $\zeta \in A^m(X)$ the pushforwards $\iota_*(c_k(\mathcal{N}_{Z/X}))$ of the other Chern classes of the normal bundle of a smooth representative $\iota : Z \to X$ (that is, a smooth subvariety $Z \subset X$ with $[Z] = \zeta$) represents the analog, in the Chow setting, of the *Steenrod squares* in algebraic topology.

13.4.1 Specialization to the normal cone

Suppose again that X is a smooth projective variety of dimension n and $Z \subset X$ is a smooth subvariety of codimension m. Let

$$\mu : \mathcal{X} = \mathrm{Bl}_{\{0\}\times Z}(\mathbb{P}^1 \times X) \to \mathbb{P}^1 \times X$$

be the blow-up of $\mathbb{P}^1 \times X$ along the subvariety $\{0\} \times Z$, and write $E \subset \mathcal{X}$ for the exceptional divisor. As a variety, E is the projectivization of the normal bundle

$$\mathcal{N}_{\{0\}\times Z/\mathbb{P}^1\times X} \cong \mathcal{N}_{Z/X} \oplus \mathcal{O}_Z.$$

Thus E is the compactification of the total space of the normal bundle $\mathcal{N} = \mathcal{N}_{Z/X}$ described in Section 9.4.2.

We think of $\mathcal{X}$ as a family of projective varieties over $\mathbb{P}^1$ via the composition $\alpha = \pi_1 \circ \mu : \mathcal{X} \to \mathbb{P}^1 \times X \to \mathbb{P}^1$. Because $\mathcal{X}$ is a variety, the family is flat. The fibers X_t of $\mathcal{X}$ over $t \neq 0$ are all isomorphic to X, while the fiber X_0 of $\mathcal{X}$ over $t = 0$ consists of two irreducible components: the proper transform $\widetilde{X}$ of $\{0\} \times X$ in $\mathcal{X}$ (isomorphic to the blow-up $\mathrm{Bl}_Z(X)$) and the exceptional divisor $E \cong \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Z)$, with the two intersecting along the "hyperplane at infinity" $\mathbb{P}\mathcal{N} \subset \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_X)$ in E, which is the exceptional divisor in the first component $\widetilde{X} \cong \mathrm{Bl}_Z(X)$.

Now consider the subvariety $\mathbb{P}^1 \times Z \subset \mathbb{P}^1 \times X$, and let $\mathcal{Z}$ be its proper transform in $\mathcal{X}$. Since $\{0\} \times Z$ is a Cartier divisor in $\mathbb{P}^1 \times Z$, the morphism μ carries $\mathcal{Z}$ isomorphically to Z.

Write $|\mathcal{N}| \subset \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Z)$ for the open subset that is isomorphic to the total space of the normal bundle $\mathcal{N}$ of Z in X. Because $\mathbb{P}^1 \times Z$ intersects $\{0\} \times X$ transversely in $\{0\} \times Z$, the intersection $\mathcal{Z} \cap X_0$ is the zero section $\mathcal{N}_0 \subset |\mathcal{N}|$. In particular, $\mathcal{Z}$ does not meet the component $\widetilde{X}$ of the zero fiber $\mathcal{X}_0$.

Let

$$\nu = \pi_2 \circ \mu : \mathcal{X} \to X$$

be the composition of the blow-up map $\mu : \mathcal{X} \to \mathbb{P}^1 \times X$ with the projection on the second factor. For $t \in \mathbb{P}^1$ other than 0, ν carries X_t isomorphically to X. As for the fiber X_0, on the component $\widetilde{X} \subset X_0$ the map ν is the blow-down map $\widetilde{X} = \mathrm{Bl}_Z X \to X$, while on the component E the map ν is the composition $i \circ \pi$ of the bundle map $\pi : E \cong \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Z) \to Z$ with the inclusion $i : Z \hookrightarrow X$. The situation is summarized in Figure 13.5.

We may describe the situation by saying that, in the family $\mathcal{X} \to \mathbb{P}^1$, as $t \in \mathbb{P}^1$ approaches 0 the neighborhood of Z in the fibers specializes from the neighborhood of Z in X to the neighborhood of Z in the total space of its normal bundle in X. More formally:

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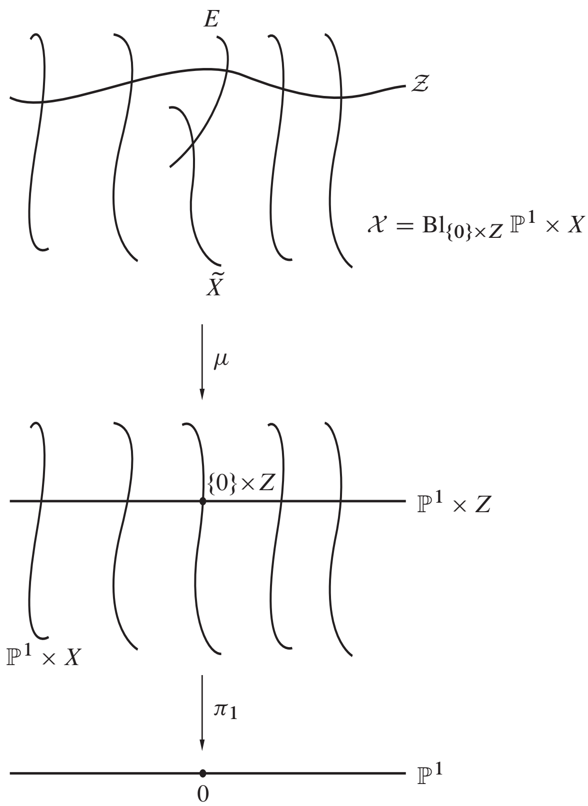

Figure 13.5 Specialization to the normal cone.

Theorem 13.8. *There is a flat family $\alpha : \mathcal{X} \to \mathbb{P}^1$ containing a subvariety $\mathcal{Z} \cong Z \times \mathbb{P}^1$ such that the restriction to $\mathbb{P}^1 \setminus \{0\}$ is isomorphic to*

$$(\mathbb{P}^1 \setminus \{0\}) \times X \hookleftarrow (\mathbb{P}^1 \setminus \{0\}) \times Z$$

and such that an open neighborhood of $Z \cong \alpha^{-1}(0) \cap \mathcal{Z}$ in $\alpha^{-1}(0)$ is isomorphic to the total space of the normal bundle of Z in X.

If we have a family $\{A_t\}_{t\in\mathbb{P}^1}$ of cycles in X that we would like to intersect with Z, then we can use this construction to transform the intersection of A_0 with Z into the intersection of the fiber of the proper transform of $\mathcal{A}$ in $\mathcal{X}$ with the zero section in the compactified normal bundle $E \cong \mathbb{P}(\mathcal{N} \oplus \mathcal{O}_Z)$. We can use our knowledge of the Chow rings of projective bundles to analyze this intersection.

13.4.2 Proof of the key formula

We now return to the proof of Theorem 13.7. Using Theorem 13.8 we will reduce the general case to the one treated in Proposition 9.16, where X is the total space of the compactification $\mathbb{P}(\mathcal{N}_{Z/X} \oplus \mathcal{O}_X)$ of the normal bundle of Z.

We will use the notation introduced in Section 13.4.1. The idea of the proof is that under the specialization to the normal cone the class $i^*i_*\alpha$ is deformed into the rationally equivalent class $j^*j_*\alpha$, where

$$j : Z \to \mathbb{P}(\mathcal{N}_{Z/X} \oplus \mathcal{O}_Z) = E$$

is the section sending Z to the zero section $\mathcal{N}_0 \subset |\mathcal{N}| \subset \overline{\mathcal{N}} := \mathbb{P}(\mathcal{N}_{Z/X} \oplus \mathcal{O}_Z)$. By Proposition 9.16, $j^*j_*\alpha = \alpha \cdot c_m(\mathcal{N})$.

Proof of Theorem 13.7: We may assume that α is the class of an irreducible subvariety $A \subset Z$. Let $\mathcal{A}$ and $\mathcal{Z}$ be the proper transforms of the subvarieties $A \times \mathbb{P}^1$ and $Z \times \mathbb{P}^1$. Since $Z \times \mathbb{P}^1$ meets $X \times \{0\}$ transversely, $\mathcal{Z} \cong Z \times \mathbb{P}^1$ via the projection, and this isomorphism also induces an isomorphism $\mathcal{A} \cong A \times \mathbb{P}^1$. To simplify notation, we will write $A_t \subset Z_t \subset X_t$ for the copies in the general fiber of $\mathcal{X}$, but we will write $A \subset Z \subset E$ instead of $A_0 \subset Z_0 \subset E$ for the fibers $A \subset Z$ contained in $E \subset X_0$.

By the moving lemma, we can find a cycle $\mathcal{C}$ on $\mathcal{X}$ linearly equivalent to $\mathcal{A}$ and generically transverse to $\mathcal{Z}$, to Z, to E, to X_t and to Z_t. The family $\mathcal{A}$ meets X_t and E generically transversely in A_t and A, respectively, so the equality $[\mathcal{C}] = [\mathcal{A}] \in A(\mathcal{X})$ restricts to equalities $i_*[A_t] = i_*[\mathcal{A} \cap X_t] = [\mathcal{C} \cap X_t] \in A(X_t)$ and $j_*[A] = [\mathcal{C} \cap E] \in A(E)$. Since $\mathcal{C}$ meets Z_t and Z generically transversely as well, we have $i^*i_*[A_t] = [\mathcal{C} \cap Z_t]$ and $j^*j_*[A] = [\mathcal{C} \cap E]$.

By generic transversality, neither Z_t nor Z can be contained in $\mathcal{C}$. It follows that after removing any components that do not dominate $\mathbb{P}^1$ the cycle $\mathcal{C}$ in $\mathcal{Z} \cong Z \times \mathbb{P}^1$ is a rational equivalence between $i^*i_*[A]$ and $j^*j_*[A]$. By Proposition 9.16, $j^*j_*[A] = [A] \cdot c_m(\mathcal{N})$, as required. □

13.5 Pullbacks to a subvariety

Theorem 13.7 has an important extension which we will explain here and use in Section 13.6 to compute the relations in the Chow ring of a blow-up. Suppose that $\pi : X' \to X$ is a morphism of smooth varieties and $Z \subset X$ is a subvariety. Set $Z' = \pi^{-1}(Z)$ and let $\pi' : Z' \to Z$ be the restriction of π. If $A \subset Z \subset X$, then the expected dimension of $\pi^{-1}(A)$ is $\dim A + \dim X' - \dim X$; however — at least when Z and Z' are smooth — the actual dimension of $\pi^{-1}(A) = \pi'^{-1}(A)$ is at least $\dim A + \dim Z' - \dim Z$. Since fiber dimension is only semicontinuous, it may well happen that $\dim Z' - \dim Z > \dim X' - \dim X$. When this occurs, A cannot be transverse to π, in the sense that $\pi^{-1}(A)$ is generically reduced of the right codimension, even when it is transverse to π'. As we shall see, this is exactly the situation when π is the blow-up of a smooth (or locally complete intersection) subvariety Z. Thus it is interesting to try to compute $\pi^*[A] \in A(X')$; more precisely, writing $i : Z \to X$ and $i' : Z' \to X'$ for the inclusions, we wish to compute $\pi^*(i_*[A])$ in terms of $\pi'^*([A])$ or

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perhaps $i'_*\pi'^*([A])$. The following picture may help keep track of the notation:

$$\begin{tikzcd} \pi^{-1}(A) \arrow[r, hook] & Z' \arrow[r, "i'"] \arrow[d, "\pi'"] & X' \arrow[d, "\pi"] \\ A \arrow[r, hook] & Z \arrow[r, "i"] & X \end{tikzcd}$$

Theorem 13.7 does exactly what we want in the special case where $\pi = i : Z \hookrightarrow X$ is an inclusion of smooth varieties: then $\pi' : Z' \to Z$ is the identity, $i' = i$, and we saw that

$$\begin{aligned} \pi^*(i_*[A]) &= i^*(i_*[A]) \\ &= [A]c_m\mathcal{N}_{Z/X} \\ &= i'_*(\pi'^*([A])\pi'^*(c_m\mathcal{N}_{Z/X})). \end{aligned}$$

The next result is a direct generalization. In order to state it, we have to assume that Z' is smooth; this is unnecessarily restrictive, but to state the theorem in its correct generality requires the formalism of Fulton [1984].

Theorem 13.9. *Suppose that $Z \subset X$ is a smooth subvariety of a smooth variety X, and that $\pi : X' \to X$ is a morphism from another smooth variety. Let $Z' = \pi^{-1}(Z)$ and assume that Z' is smooth, with connected components C_α of dimension c_α. Write $i : Z \to X$ and $i'_\alpha : C_\alpha \to X'$ for the inclusion maps, and likewise π_α for the restriction of π to C_α. For any class $\beta \in A_b(Z)$,*

$$\pi^*(i_*\beta) = \sum_\alpha (i'_\alpha)_*\{\pi_\alpha^*(\beta c(\mathcal{N}_{Z/X}))s(C_\alpha, X')\}_{b+\dim X'-\dim X}.$$

For the proof, see Fulton [1984, Chapter 6].

13.5.1 The degree of a generically finite morphism

Let $\varphi : X \to Y$ be a generically finite projective morphism to a smooth variety Y. If $q \in Y$ is a point such that $\varphi^{-1}(q)$ is finite, then the degree of φ is the number of points of $\varphi^{-1}(q)$, counted with appropriate multiplicity. By the moving lemma, any point $q \in Y$ is rationally equivalent to a cycle of points that is transverse to the map φ in the sense of Definition 1.22, so $\deg\varphi = \deg\varphi^*[q]$ for *any* point q. Using Theorem 13.9 we can express this in terms of the local geometry of X near $\varphi^{-1}(q)$:

Corollary 13.10. *Let $\varphi : X \to Y$ be a generically finite surjective map of smooth projective varieties. If $q \in Y$ is any point, then*

$$\deg(\varphi) = \deg\varphi^*[q] = \deg\{s(\varphi^{-1}(q), X)\}_0.$$

Proof: The normal bundle of q in Y is trivial, so the given formula follows from Theorem 13.9. □

As a trivial example, consider the degree-1 map $\varphi : X = \mathrm{Bl}_q Y \to Y$ that is the blow-up of an n-dimensional variety Y at a point q. Since the normal bundle to the exceptional divisor $E = \varphi^{-1}(q) \cong \mathbb{P}^{n-1}$ is $\mathcal{O}_E(-1)$, we have

$$s(\mathcal{N}_{E/X}) = \frac{1}{1-\zeta},$$

and the coefficient of ζ^{n-1} is indeed 1.

A nontrivial example where the principle of Corollary 13.10 is decisive is the beautiful calculation by Donagi and Smith [1980] of the degree of the "Prym map" in genus 6. Here φ is the map from the space R_6 of unramified covers of curves of genus 6 to the space A_5 of abelian varieties of dimension 5 (both of dimension 15) defined by the "Prym construction;" while it does not seem possible to enumerate the points of a general fiber, Donagi and Smith were able to calculate its degree by looking at a very special point (the Prym of a double cover of a smooth plane quintic) over which the fiber has three components: a point, a curve and a surface!

13.6 The Chow ring of a blow-up

We can now describe the Chow ring of a blow-up of a smooth projective variety along a smooth subvariety. After reviewing some basic facts about blow-ups, we give a set of generators and calculate their products. We illustrate the results in Section 13.6.3. In the last section we complete the story by describing the relations among the generators.

Throughout this section we will use the following notation. Let X be a smooth projective variety and $Z \subset X$ a smooth subvariety of codimension m. Write $i : Z \to X$ for the inclusion map. Let $\pi : W = \mathrm{Bl}_Z X \to X$ be the blow-up of X along Z. Let $E \subset W$ be the exceptional divisor and $j : E \hookrightarrow W$ the inclusion, so that we have the diagram

$$\begin{tikzcd} W = \mathrm{Bl}_Z X \arrow[d, "\pi"'] & E \arrow[l, hook', "j"'] \arrow[d, "\pi_E"] \\ X & Z \arrow[l, hook', "i"'] \end{tikzcd}$$

We write $\mathcal{N} = \mathcal{N}_{Z/X}$ for the normal bundle of Z in X.

Recall from Hartshorne [1977] or Eisenbud and Harris [2000, Theorem IV-23] that if $\mathcal{I} = \mathcal{I}_Z \subset \mathcal{O}_X$ denotes the ideal sheaf of Z then the blow-up W is Proj of the *Rees algebra*

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots .$$

The preimage E of $Z = V(\mathcal{I}) \subset X$ is then

$$E = \mathrm{Proj}(\mathcal{A} \otimes \mathcal{O}_X/\mathcal{I}) = \mathrm{Proj}(\mathcal{O}_X/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots).$$

Since $Z \subset X$ is smooth it is locally a complete intersection, so the conormal bundle $\mathcal{N}^*_{Z/X} = \mathcal{I}/\mathcal{I}^2$ is locally free and

$$\mathcal{I}^k/\mathcal{I}^{k+1} = \operatorname{Sym}^k \mathcal{N}^*_{Z/X}$$

(see Eisenbud [1995, Exercise 17.14]). Thus we may make the identification

$$E = \operatorname{Proj}(\operatorname{Sym}\mathcal{N}^*_{Z/X}) = \mathbb{P}\mathcal{N}_{Z/X}.$$

We write $\zeta \in A^1(E)$ for the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{N}}(1)$.

13.6.1 The normal bundle of the exceptional divisor

Proposition 13.11. *The normal bundle of $E = \mathbb{P}\mathcal{N}_{Z/X}$ in W is*

$$\mathcal{N}_{E/W} = \mathcal{O}_{\mathbb{P}\mathcal{N}_{Z/X}}(-1),$$

the tautological subbundle on $\mathbb{P}\mathcal{N}_{Z/Y}$.

Proof: With notation as above, $\mathcal{I}_E/\mathcal{I}_E^2$ is the line bundle associated to the module

$$\mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots = \mathcal{O}_{\mathbb{P}\mathcal{N}_{Z/X}}(1),$$

so that

$$\mathcal{N}_{E/W} = \mathcal{H}om_W(\mathcal{I}_E/\mathcal{I}_E^2, \mathcal{O}_W) \cong \mathcal{O}_{\mathbb{P}\mathcal{N}_{Z/X}}(-1),$$

as stated. □

The proof of Proposition 13.11 just given works for any locally complete intersection subscheme of any scheme (and in complete generality, if we replace $\mathcal{N}_{Z/X}$ by the normal cone). In the case where Z and X are smooth, we can give a geometric proof and show that the isomorphism is induced by the differential $d\pi$ of the projection $\pi : W \to X$.

To this end, we first observe that, since the restriction of π to E is the projection from $\mathbb{P}\mathcal{N}$ to Z, the differential $d\pi$ induces a surjection from the tangent space $T_q E$ at a point q of E to $T_p Z$, where $p = \pi(q)$. Thus $d\pi$ induces a map

$$\overline{d\pi} : \mathcal{N}_{E/W} = \mathcal{T}_W/\mathcal{T}_E \to \pi^*(\mathcal{T}_X/\mathcal{T}_Z) = \pi^*\mathcal{N}_{Z/X}.$$

It now suffices to see that $\overline{d\pi}$ carries the one-dimensional vector space that is the fiber of $\mathcal{N}_{E/W}$ at a point q to the one-dimensional subspace of $\mathcal{N}_{Z/X,\pi(q)} = \pi^*(\mathcal{N}_{Z/X})_q$ that corresponds to the point q regarded as an element of $\mathbb{P}\mathcal{N}$. Indeed, if $C \subset X$ is the germ of a smooth curve passing through p with tangent space $L \subset T_p X$ corresponding to q, then the proper transform $\widetilde{C}$ of C passes through the point $q \in E$, and $d\pi$ carries the tangent space of $\widetilde{C}$ at q to L, as required.

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13.6.2 Generators of the Chow ring

We maintain the notation introduced at the beginning of Section 13.6. Using the identification $E \cong \mathbb{P}\mathcal{N}$, we let $\zeta \in A^1(E)$ be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}\mathcal{N}}(1)$. In these terms Proposition 13.11 implies that the first Chern class of the normal bundle of E in W is $-\zeta$.

Proposition 13.12. *The Chow ring $A(W)$ is generated by $\pi^*A(X)$ and $j_*A(E)$, that is, classes pulled back from X and classes supported on E. The rules for multiplication are*

$$\begin{aligned} \pi^*\alpha \cdot \pi^*\beta &= \pi^*(\alpha\beta) && \text{for } \alpha, \beta \in A(X), \\ \pi^*\alpha \cdot j_*\gamma &= j_*(\gamma \cdot \pi_E^* i^*\alpha) && \text{for } \alpha \in A(X),\ \gamma \in A(E), \\ j_*\gamma \cdot j_*\delta &= -j_*(\gamma \cdot \delta \cdot \zeta) && \text{for } \gamma, \delta \in A(E). \end{aligned}$$

Proof: The first formula is the statement that pullback $\pi^* : A(X) \to A(W)$ is a ring homomorphism. For the second, we note that $\pi^*\alpha \cdot j_*\gamma = j_*(j^*\pi^*\alpha \cdot \gamma)$ by the push-pull formula, while $j^*\pi^*\alpha = \pi_E^* i^*\alpha$ by functoriality (both part of Theorem 1.23). Since the normal bundle of E in W has first Chern class $-\zeta$, the third formula is a special case of Proposition 13.11.

To conclude we must show that $A(W)$ is generated by $\pi^*A(X)$ and $j_*A(E)$. Let $U = W \setminus E \cong X \setminus Z$. Let $k : U \to W$ be the inclusion. Suppose that $A \subset W$ is a subvariety. If $A \subset E$ we are done; else π maps A generically isomorphically onto $\pi(A)$, so $\pi_*[A] = [\pi(A)]$.

By the moving lemma there is a cycle B on X that is rationally equivalent to $\pi(A)$ and generically transverse to π (Definition 1.22), so that, by Theorem 1.23, $\pi^*[B] = [\pi^{-1}(B)]$. It is enough to show that $[A] - \pi^*[B]$ is in the image of j_*.

From the right exact sequence

$$A(E) \longrightarrow A(W) \xrightarrow{k^*} A(U) \longrightarrow 0$$

of Proposition 1.14, we see that it suffices to show that $k^*\pi^*([B]) = k^*[A]$. Since $[\pi(A)] = [B]$, we have

$$k^*([A]) = [A \cap U] = \pi_U^*([\pi(A)]) = \pi_U^*[B] = k^*\pi^*[B]$$

by functoriality, completing the argument. □

13.6.3 Example: the blow-up of $\mathbb{P}^3$ along a curve

The first nontrivial case, the blow-up of a smooth curve in $\mathbb{P}^3$, is already interesting. We first establish some notation:

Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, and let $\pi : W \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C. We write E, ζ, i and j as in the general discussion above, so that we have the diagram

$$\begin{tikzcd} W = \mathrm{Bl}_C \mathbb{P}^3 \arrow[d, "\pi"'] & E \cong \mathbb{P}\mathcal{N} \arrow[l, hook', "j"'] \arrow[d, "\pi_E"] \\ \mathbb{P}^3 & C \arrow[l, hook', "i"'] \end{tikzcd}$$

In addition, we write $h \in A^1(\mathbb{P}^3)$ for the class of a plane, $e = [E]$, and $\tilde{h}$ for the pullback $\pi^* h \in A^1(W)$. Finally, for $D \in Z^1(C)$ any divisor, we will denote by $F_D = \pi_E^* D \in Z^1(E)$ the corresponding linear combination of fibers of $E \to C$, and similarly for divisor classes.

Proposition 13.13. *If W is the blow-up of $\mathbb{P}^3$ along a smooth curve C, then, with notation as above,*

$A^0(W) = \mathbb{Z}$,	*generated by the class of W;*
$A^1(W) = \mathbb{Z}^2$,	*generated by $e, \tilde{h}$;*
$A^2(W)$	*is generated by $e^2 = -j_*(\zeta)$, F_D for $D \in A^1(C)$ and $\tilde{h}^2$;*
$A^3(W) = \mathbb{Z}$,	*generated by the class of a point.*

Other products among these classes are

$$\deg(e \cdot F_D) = \deg D, \qquad \tilde{h} \cdot F_D = 0, \qquad \deg(\tilde{h}^3) = 1,$$
$$\deg(\tilde{h}^2 e) = 0, \qquad \deg(\tilde{h} e^2) = -d, \qquad \deg(e^3) = -4d - 2g + 2.$$

Proof: As for any variety, $A^0(W) \cong \mathbb{Z}$, generated by the fundamental class $[W]$ of W. Since W is rational, $A^3(W) \cong \mathbb{Z}$ is generated by the class of a point.

Proposition 13.12 shows that $A^1(W)$ is generated by the classes $\tilde{h}$ and $e := [E]$. The map π_* sends $\tilde{h}$ to h and sends $[E]$ to 0, so we have an exact sequence

$$A^0(E) \xrightarrow{j_*} A^1(W) \xrightarrow{\pi_*} \mathbb{Z} \cdot h \cong Z \longrightarrow 0.$$

Also, by Proposition 13.12, $j^*(e) = j^* j_*[E] = -\zeta$. By Theorem 9.6, ζ generates a subgroup of $A^1(E)$ that is isomorphic to $\mathbb{Z}$, and thus the map

$$\mathbb{Z} = \mathbb{Z} \cdot [E] = A^0(E) \to A^1(W)$$

is a monomorphism. It follows that $A^1(W) \cong \mathbb{Z}^2$, freely generated by $\tilde{h}$ and e.

It will be convenient to introduce the notation $\tilde{l} = \pi^* l \in A^2(W)$ for the pullback of a line. Since $A^2(\mathbb{P}^3)$ is freely generated by l, Proposition 13.12 and Theorem 9.6 together show that $A^2(W)$ is generated by $\tilde{l}$, $j_*\zeta$ and $j_* F_D$ for $D \in A^1(C)$.

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If we represent h as the class of a general hyperplane H, then we can see by considering cycles that

$$\tilde{h}^2 = \pi^*[H]^2 = \pi^* l = \tilde{l} \quad \text{and} \quad \tilde{h} \cdot e = j_* F_{(C \cap H)},$$

while

$$e^2 = -j_* \zeta$$

by Proposition 13.12.

Similarly, we see by considering cycles that

$$\deg(\tilde{h} \cdot l) = 1 \quad \text{and} \quad \deg(\tilde{h} \cdot j_* f_D) = 0$$

for any divisor class D on C, while

$$\deg(\tilde{h} \cdot j_* \zeta) = d,$$

which follows from the push-pull formula and the equality $\pi_*(j_* \zeta) = [C] = dl \in A^2(\mathbb{P}^3)$. This determines the pairing between $A^1(W) \times A^2(W) \to A^3(W) = \mathbb{Z}$.

Likewise,

$$\begin{aligned} \deg(e \cdot l) &= 0, \\ \deg(e \cdot j_* f_D) &= -\deg D \qquad \text{for any divisor class } D \text{ on } C, \\ \deg(e \cdot j_* \zeta) &= -\deg c_1(\mathcal{N}_{C/\mathbb{P}^3}) = -4d - 2g + 2. \end{aligned}$$ □

Three surfaces in $\mathbb{P}^3$ revisited

As a first application of this description, we revisit the first keynote question of this chapter, or rather its generalization to Proposition 13.2. Again, let $S_1, S_2, S_3 \subset \mathbb{P}^3$ be surfaces of degrees s_1, s_2 and s_3 whose scheme-theoretic intersection consists of the disjoint union of a smooth curve C of degree d and genus g and a zero-dimensional scheme Γ.

We can get rid of the component C of the intersection by pulling back the problem to the blow-up W of $\mathbb{P}^3$ along C. If we let $\widetilde{S}_i$ be the proper transform of S_i, then the fact that $\bigcap S_i = C$ scheme-theoretically implies that the intersection $\bigcap \widetilde{S}_i$ does not meet E. Thus

$$\bigcap \widetilde{S}_i = \pi^{-1}(\Gamma);$$

in particular,

$$\deg(\Gamma) = \deg\Big(\prod [\widetilde{S}_i]\Big).$$

In terms of the generators of $A(W)$ given in Proposition 13.13,

$$[\widetilde{S}_i] = s_i \tilde{h} - e,$$

and so our answer is

$$\deg(\Gamma) = \deg \prod (s_i \tilde{h} - e).$$

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By Proposition 13.13, then

$$\deg(\Gamma) = \prod s_i - d\Big(\sum s_i\Big) + 4d + 2g - 2,$$

as before.

Tangencies along a curve

We can also use Proposition 13.13 to answer Keynote Question (e): If $C \subset \mathbb{P}^3$ is a smooth curve of degree d and genus g, and $S, T \subset \mathbb{P}^3$ smooth surfaces of degrees s and t containing C, at how many points of C are S and T tangent?

We will count these points with the multiplicity defined by the intersection multiplicity of the sections of the normal bundle of C determined by S and T. Equivalently, these are the intersection multiplicities of the proper transforms $\widetilde{S}$ and $\widetilde{T}$ with the exceptional divisor of the blow-up W of $\mathbb{P}^3$ along C. With notation above, Proposition 13.13 yields

$$\begin{aligned}\deg([\widetilde{S}] \cdot [\widetilde{T}] \cdot [E]) &= \deg((s\tilde{h} - e)(t\tilde{h} - e)e)\\ &= -(s+t)\deg(\tilde{h}e^2) + \deg(e^3)\\ &= (s+t)d - 4d - 2g + 2.\end{aligned}$$

Thus, for example, two planes meeting along a line are nowhere tangent, but two quadrics Q_1, Q_2 containing a twisted cubic curve C will be tangent twice along C — as we can see directly, since the intersection $Q_1 \cap Q_2$ will consist of the union of C and a line meeting C twice.

One can check that the intersection multiplicity of the intersection of $\widetilde{S}$ and $\widetilde{T}$ and E at a point q with image $p \in C$ is 1 when $S \cap T$ is in a neighborhood of p the union of C with a curve D meeting C transversely in S — in other words, when $S \cap T$ has a node at p.

13.6.4 Relations on the Chow ring of a blow-up

There is one item missing in the description of the Chow ring of a blow-up provided by Proposition 13.12: a criterion for deciding whether a given class is zero. This is provided by the following:

Theorem 13.14. *Let $i : Z \to X$ be the inclusion of a smooth codimension-m subvariety in a smooth variety X, $\pi : W \to X$ the blow-up of X along Z and E the exceptional divisor, with inclusion $j : E \to W$. If $\mathcal{Q}$ is the universal quotient bundle on $E \cong \mathbb{P}\mathcal{N}_{Z/X}$, there is a split exact sequence of additive groups, preserving the grading by dimension:*

$$0 \longrightarrow A(Z) \xrightarrow{(i_*\ \ h)} A(X) \oplus A(E) \xrightarrow{(\pi^*\ \ j_*)} A(W) \longrightarrow 0,$$

where $h : A(Z) \to A(E)$ is defined by $h(\alpha) = -c_{m-1}(\mathcal{Q})\pi_E^*(\alpha)$.

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Proof: We adopt the notation from the beginning of Section 13.6.

Note that $\mathcal{Q} = \pi_E^*\mathcal{N}/\mathcal{O}_{\mathbb{P}\mathcal{N}}(-1)$, a bundle of rank $m-1$ on E. Theorem 13.9 shows that

$$\pi^* i_*(\alpha) = j_*(c_{m-1}(\mathcal{Q})\pi_E^*(\alpha));$$

that is, the two maps in the exact sequence compose to zero. As for the surjectivity of the right map, this was part of our preliminary description of $A(W)$ in Proposition 13.12.

Finally, to prove that the left map is a split monomorphism, it is enough to prove that h is a split monomorphism. We will show that $\pi_E \circ h = 1$ on $A(Z)$. To this end, we compute the Chern class of $\mathcal{Q}$ in terms of ζ and the Chern class of $\mathcal{N}$:

$$c(\mathcal{Q}) = \frac{c(\mathcal{N})}{1-\zeta}, \quad \text{so } c_{m-1}(\mathcal{Q}) = \zeta^{m-1} + c_1(\mathcal{N})\zeta^{m-2} + \cdots + c_{m-1}(\mathcal{N}),$$

where we are regarding the $c_i(\mathcal{N})$ as elements of $A(E)$ via the ring homomorphism π_E^*. Since $c_{m-1}(\mathcal{Q})$ is monic in ζ, Lemma 9.7 shows that

$$\pi_{E*}(h(\alpha)) = \alpha,$$

as required. $\square$

13.7 Exercises

Exercise 13.15. Show that the formula of Proposition 13.2 applies more generally if we replace $\mathbb{P}^3$ by an arbitrary smooth projective threefold X — that is, under the hypotheses of the proposition, we have

$$\deg(\Gamma) = \deg([S]\cdot[T]\cdot[U]) - \deg(\mathcal{N}_{S/X}|_L) - \deg(\mathcal{N}_{T/X}|_L) - \deg(\mathcal{N}_{U/X}|_L) + \deg(\mathcal{N}_{L/X}).$$

Exercise 13.16. (a) Show that a smooth quintic curve $C \subset \mathbb{P}^3$ of genus 2 is the scheme-theoretic intersection of three surfaces in $\mathbb{P}^3$.

(b) Show that a smooth rational quintic curve $C \subset \mathbb{P}^3$ is the scheme-theoretic intersection of three surfaces in $\mathbb{P}^3$ if and only if it lies on a quadric surface; conclude that some rational quintics are expressible as such intersections and some are not.

Exercise 13.17. Let $S, T, U, V \subset \mathbb{P}^4$ be smooth hypersurfaces of degrees d, e, f and g respectively, and suppose that

$$S \cap T \cap U \cap V = C \cup \Gamma,$$

with C a smooth curve of degree a and genus g and Γ a zero-dimensional scheme disjoint from C. What is the degree of Γ?

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Exercise 13.18. Let $S, T, U, V, W \subset \mathbb{P}^5$ be smooth hypersurfaces of degree d, and suppose that

$$S \cap T \cap U \cap V \cap W = \Lambda \cup \Gamma$$

with Λ a 2-plane and Γ a zero-dimensional scheme disjoint from Λ. What is the degree of Γ?

Exercise 13.19. Verify the answer to Keynote Question (b), given in Section 13.3.3, in the following cases:

(a) S is a smooth surface of degree d in a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing a line L, and T is a general 2-plane in $\mathbb{P}^4$ containing L.
(b) S and T are smooth quadric surfaces.

Exercise 13.20. Let $S = S(1,2) \subset \mathbb{P}^4$ be a cubic scroll, as in Section 9.1.1. Show directly that a general 2-plane $T \subset \mathbb{P}^4$ containing a line of the ruling of S meets S in one more point, but a general 2-plane containing the directrix of S (that is, the line of S transverse to the ruling) does not meet S anywhere else.

Exercise 13.21. Let $\mathbb{P}^8$ be the space of 3×3 matrices and $\mathbb{P}^5 \subset \mathbb{P}^8$ the subspace of symmetric matrices. Show that the Veronese surface in $\mathbb{P}^5$ is the intersection of $\mathbb{P}^5$ with the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, and verify the excess intersection formula in this case.

Exercise 13.22. Let $X \subset \mathbb{P}^5$ be a smooth hypersurface of degree d and $\Lambda \cong \mathbb{P}^2 \subset X$ a 2-plane contained in X. Use Theorem 13.7 to determine the degree of the self-intersection $[\Lambda]^2 \in A^4(X)$ of Λ in X. Check this in the cases $d = 1$ and 2.

Exercise 13.23. Let $X = 2C \subset \mathbb{P}^2$ be a double conic, that is, a subscheme defined by the square of a quadratic polynomial whose zero locus is a smooth conic curve $C \subset \mathbb{P}^2$. Show that the dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X(1)$, and applying Riemann–Roch deduce that X is not hyperelliptic; that is, it does not admit a degree-2 map to $\mathbb{P}^1$. Conclude that as asserted in Section 13.4 no analytic neighborhood of C in $\mathbb{P}^2$ is biholomorphic to an analytic neighborhood of the zero section in the normal bundle $\mathcal{N}_{C/\mathbb{P}^2}$. (See Bayer and Eisenbud [1995] for more about this.)

In Exercises 13.24–13.26 we adopt the notation of Section 13.6.3; in particular, we let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g, $W = \mathrm{Bl}_C \mathbb{P}^3$ the blow-up of $\mathbb{P}^3$ along C and $E \subset W$ the exceptional divisor.

Exercise 13.24. Let $q \in \mathbb{P}^3$ be any point not on the tangential surface of C, and let $\Gamma \subset E \subset W$ be the curve of intersections with E of the proper transforms of lines $\overline{p, q}$ for $p \in C$. Find the class of Γ in $A(W)$.

Exercise 13.25. Let $B \subset \mathbb{P}^3$ be another curve, of degree m, and suppose that B meets C in the points of a divisor D on C. Show that the class of the proper transform $\widetilde{B} \subset W$ of B is

$$[\widetilde{B}] = ml - F_D.$$

Exercise 13.26. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree e containing C, $\widetilde{S} \subset W$ its proper transform and $\Sigma_S = \widetilde{S} \cap E$ the curve on E consisting of normal vectors to C contained in the tangent space to S. Find the class $[\Sigma_S] \in A(W)$ of Σ_S in the blow-up

(a) by applying Proposition 9.13; and
(b) by multiplying the class $[\widetilde{S}]$ by the class $[E]$ in $A(W)$.

Exercise 13.27. In Section 9.3.2, we observed that the blow-up $X = \mathrm{Bl}_{\mathbb{P}^k} \mathbb{P}^n$ of $\mathbb{P}^n$ along a k-plane was a $\mathbb{P}^{k+1}$-bundle over $\mathbb{P}^{n-k-1}$, and used this to describe the Chow ring of X. We now have another description of the Chow ring of X. Compare the two, and in particular:

(a) Express the generators of $A(X)$ given in this chapter in terms of the generators given in Section 9.3.2.
(b) Verify the relations among the generators given here.

Exercise 13.28. Redo Exercise 13.17 by the method of Section 13.6.3, that is, by blowing up the positive-dimensional component of the intersection.

Exercise 13.29. Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex p and $\sigma : X = \mathrm{Bl}_p Q \to Q$ its blow-up at p. Let $\pi = \pi_q : Q \to \mathbb{P}^2$ be the projection from a general point $q \in \mathbb{P}^3$ and $f = \pi \circ \sigma : X \to \mathbb{P}^2$ the composition of the blow-up map with the projection. Find the degree of f by looking at the fiber over $\pi(p)$.

Exercise 13.30. Prove that if $Z \subset A$ is a locally complete intersection subscheme inside the projective variety A then $s(Z, A) = s(\mathcal{N}_{Z/A})$.

Exercise 13.31. Let $\mathbb{P}^N$ be the space of plane curves of degree d and $X \subset \mathbb{P}^N$ the locus of d-fold lines dL. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree m, and let $\Sigma \subset \mathbb{P}^N$ be the locus of curves tangent to C (that is, intersecting C in fewer than dm distinct points).

(a) Let $\mathcal{D} \subset \mathbb{P}^N$ be a general line. Show that every curve $D \in \mathcal{D}$ is either transverse to C or meets C in exactly $dm - 1$ points, and use Riemann–Hurwitz to conclude that

$$\deg(\Sigma) = 2md + m(m-3).$$

(b) Now suppose that $\mathcal{D} \subset \mathbb{P}^N$ is a general line meeting the locus X of d-fold lines. Show that $\mathcal{D}$ meets Σ in $2md + m(m-3) - m(d-1)$ other points, and conclude that

$$\mathrm{mult}_X(\Sigma) = m(d-1).$$

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Exercise 13.32. Recalling the definition of a circle from Section 2.3.1, we say that a *sphere* $Q \subset \mathbb{P}^3$ is a quadric containing the "circle at infinity" $W = X^2 + Y^2 + Z^2 = 0$. Let $Q_1, \dots, Q_4 \subset \mathbb{P}^3$ be four general spheres, and let $S_i \subset \mathbb{G}(1,3)$ be the locus of lines tangent to Q_i. Using Theorem 13.3 applied to the intersection $\bigcap S_i$, find the number of lines tangent to all four.

Exercise 13.33. Let $S \subset \mathbb{P}^5$ be the Veronese surface. Find the Chow ring $A(X)$, where $X = \mathrm{Bl}_S(\mathbb{P}^5)$ is the blow-up of $\mathbb{P}^5$ along S.

Exercise 13.34. Use the result of the preceding exercise to re-derive the number of conics tangent to five conics, as suggested in Section 8.1

Exercise 13.35. In Section 3.5.5 we saw how to determine the number of common chords of two general twisted cubics $C, C' \subset \mathbb{P}^3$ by specializing C and C' to curves of types $(1,2)$ and $(2,1)$ on a smooth quadric surface $Q \subset \mathbb{P}^3$. As noted there, if we specialized both curves to curves of type $(1,2)$ on Q, there would be a positive-dimensional family of common chords. Use Theorem 13.3 to analyze this case, and to show that the intersection number $\deg([\Psi_2(C)][\Psi_2(C')])$ of the cycles $\Psi_2(C), \Psi_2(C') \subset \mathbb{G}(1,3)$ is 10.