Foundations of Data Science and Machine Learning – *Homework 2*

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EXERCISE 1.

EXERCISE 2.

(a) Fix $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ both with an ℓ_2-norm of 1. Suppose that $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^r$ is a linear map satisfying

$$(1-\varepsilon)\|\mathbf{x}+\mathbf{y}\|_2^2 \le \|\Phi(\mathbf{x}+\mathbf{y})\|_2^2 \le (1+\varepsilon)\|\mathbf{x}+\mathbf{y}\|_2^2 \tag{1}$$

and

$$(1-\varepsilon)\|\mathbf{x}-\mathbf{y}\|_2^2 \le \|\Phi(\mathbf{x}-\mathbf{y})\|_2^2 \le (1-\varepsilon)\|\mathbf{x}+\mathbf{y}\|_2^2. \tag{2}$$

Use the identity $4\langle \mathbf{u}, \mathbf{v}\rangle = \|\mathbf{u}+\mathbf{v}\|_2^2 - \|u - \mathbf{v}\|_2^2$ to show that

$$|\langle x, y\rangle - \langle \Phi(x), \Phi(y)\rangle \le \varepsilon\|\mathbf{x}\|_2\|y\|_2.$$

(b) If $\chi = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is an arbitrary fixed set in $\mathbb{R}^d$ and $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^r$ is a random matrix with independent mean-zero variance $1/r$ Gaussian entries, how should the embedding dimension r scale in terms of n, d, ε so that with probability at least 0.9 it holds

$$|\langle \mathbf{x}_j, \mathbf{x}_k\rangle - \langle \Phi(\mathbf{x}_j), \Phi(\mathbf{x}_k)\rangle| \le \varepsilon\|\mathbf{x}\|_2\|\mathbf{y}\|_2.$$

Proof: (a) Rearrange both equation (1) and (2) as follows:

$$-\varepsilon\|\mathbf{x}+\mathbf{y}\|_2^2 \le \|\Phi(\mathbf{x}+\mathbf{y})\|_2^2 - \|\mathbf{x}+\mathbf{y}\|_2^2 \le \varepsilon\|\mathbf{x}+\mathbf{y}\|_2^2 \tag{3}$$

$$-\varepsilon\|\mathbf{x}-\mathbf{y}\|_2^2 \le \|\Phi(\mathbf{x}-\mathbf{y})\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2 \le \varepsilon\|\mathbf{x}-\mathbf{y}\|_2^2. \tag{4}$$

Since $\|\mathbf{x}\|_2, \|\mathbf{y}\|_2 \le 1$, by the triangle inequality $\|\mathbf{x}-\mathbf{y}\|_2^2, \|\mathbf{x}+\mathbf{y}\|_2^2 \le 1$. Using this fact and adding equation (4) to the negative of (3) yields

$$-4\varepsilon \le \|\mathbf{x}+\mathbf{y}\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2 - (\|\Phi(x+y)\|_2^2 - \|\Phi(x-y)\|_2^2) \le 4\varepsilon.$$

Linearity of Φ together with the given identity then implies

$$-4\varepsilon \le 4\langle \mathbf{x}, \mathbf{y}\rangle - 4\langle \Phi(\mathbf{x}), \Phi(\mathbf{y})\rangle \le 4\varepsilon,$$

and so we have

$$\big|\langle \mathbf{x}, \mathbf{y}\rangle - \langle \Phi(\mathbf{x}), \Phi(\mathbf{y})\rangle\big| \le \varepsilon = \varepsilon\|\mathbf{x}\|_2\|\mathbf{y}\|_2$$

as desired.

(b)

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