Problems from Hartshorne Chapter 2.2

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EXERCISE 1. Let A be an abelian group and defined the *constant presheaf* associated to A on the topological space X to be the preshseaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.

Proof: Let $\mathcal{C}$ be the constant sheaf on X, i.e. the sheaf defined as follows: for any open $U \subseteq X$, $\mathcal{C}(U)$ is the group of all continuous maps of U into A (where A is endowed with the discrete topology). Let $\mathcal{G}$ be any other sheaf on X.

Define $\theta : \mathcal{F} \longrightarrow \mathcal{C}$ as follows. For an open set U, let $\theta(U) : \mathcal{F}(U) = A \longrightarrow \mathcal{C}(U)A$ send a point $a \in A$ to the constant map $(x \mapsto a) \in \mathcal{C}(U)$.

Now suppose we have some morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$. We would like to define $\beta : \mathcal{C} \longrightarrow \mathcal{G}$ such that $\beta \circ \theta = \alpha$.

Fix an open subset $U \subseteq X$ and a section $f : U \longrightarrow A$ of $\mathcal{C}(U)$. Notice that $\{f^{-1}(a)\}_{a \in A}$ is an open cover of U and $f|_{f^{-1}(a)} = (x \mapsto a) = \theta(U)(a)$ for all $a \in A$. Consider the collection $\{\alpha(U)(a)\}_{a \in A}$ of sections in $\mathcal{G}(U)$. These satisfy the gluing compatibility condition, namely

$$\alpha(U)(a)|_{f^{-1}(a) \cap f^{-1}(b)} = \alpha(U)(b)|_{f^{-1}(a) \cap f^{-1}(b)}$$

and hence there is some element $g_f \in \mathcal{G}(U)$ such that $g_f|_{f^{-1}(a)} = \alpha(U)(a)|_{f^{-1}(a)}$ for all $a \in A$. We simply define $\beta(U)(f) = g_f$ to obtain a map $\beta(U) : \mathcal{C}(U) \longrightarrow \mathcal{G}(U)$. This satisfies the restriction requirements and hence β is a map of schemes. Furthermore, if $f = \theta(U)(a)$ for some $a \in A$, then f is the constant map $x \mapsto a$ and hence $f^{-1}(a) = U$, so $\beta(f) = \alpha(U)(a)$. This shows that $\alpha = \beta \circ \theta$, meaning $\mathcal{C}$ satisfies the universal property of the sheaf associated to $\mathcal{F}$. $\square$

EXERCISE 2.

(a) For any morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence ... $\longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \longrightarrow$ ... of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof:

(a) Recall that for any $V \subseteq X$ containing a point P we have the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\pi"] & \mathcal{G}(V) \arrow[d, "\pi"] \\
\mathcal{F}_P \arrow[r, "\varphi_P"] & \mathcal{G}_P
\end{tikzcd}$$

Start with an element $(t, V) \in \ker(\varphi_P)$. Then t is a section of $\mathcal{F}(V)$ by definition and by commutativity of the diagram we have that $\pi(\varphi(V)(t)) = (\varphi(V)(t), V) = 0$ in $\mathcal{G}_P$. This means that there is some open neighborhood $W \subset V$ of P such that $\varphi(U)(t)|_W = 0$ by the equivalence relation on $\mathcal{G}_P$, and since $\varphi(U)(t)|_W = \varphi(W)(t)$ we have that $\varphi(W)(t|_W) = 0$. Hence $t|_W = 0$ and so $t \in \ker \varphi(W)$. Hence $(t|_W, W) \in (\ker \varphi)_P$, and because $(t|_W, W)$ and (t, V) represent the same element in $\ker(\varphi_P)$, this shows the inclusion $\ker(\varphi_P) \subseteq (\ker \varphi)_P$.

For the other inclusion, take an element $(t, V) \in (\ker \varphi)_P$. This means that $t \in (\ker \varphi)(V) = \ker(\varphi(V))$ and hence $\varphi(V)(t) = 0$ in $\mathcal{G}(V)$. Composing with π gives $\pi(\varphi(V)(t)) = (\varphi(V)(t), V) = 0$ in $\mathcal{G}_P$. By commutativity, $\pi((t, V)) = (t, V) \in \mathcal{F}_P$ maps to 0 under φ_P, so $(t, V) \in \ker(\varphi_P)$. This gives us the other inclusion.

Now let's consider $\operatorname{im}(\varphi)$.

□

EXERCISE 3.

(a) Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U and there are elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ and an open set U such that $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is not surjective.

EXERCISE 14. Let $\mathcal{F}$ be a sheaf on X, and let $s \in \mathcal{F}(U)$ be a section over an open set U. The *support* of s, denote $\operatorname{Supp} s$ is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk of $\mathcal{F}_P$. Show that $\operatorname{Supp} s$ is a closed subset of U. We define the *support* of $\mathcal{F}$ $\operatorname{Supp} \mathcal{F}$, to be $\{P \in X \mid \mathcal{F}_P \neq 0\}$. It need not be a closed subset.

Proof: Consider the set $V = \{P \in U \mid s_P = 0\}$. For each $P \in V$ there then exists some W_P containing P and open in V such that $s_P = (s|_{W_P}, W_P) = 0$, i.e. so that $s|_{W_P} = 0$. We then have that $V = \bigcup_{P \in V} W_P$, and hence V is open. Because $\operatorname{Supp} s$ is the complement of V it is closed.

An example of a sheaf whose support is not a closed set in U is $j_!\mathbb{Z}$. Here $j : U \longrightarrow X$ is the inclusion and $j_! : \operatorname{Sh}(U, \mathbb{Z}) \longrightarrow \operatorname{Sh}(X, \mathbb{Z})$ is the functor where $j_!\mathcal{F}$ is the sheaf associated to the presheaf

$$V \mapsto \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}.$$

The sheaf $j_!\mathcal{F}$ has the property that $(j_!\mathcal{F})_x = \mathcal{F}_x$ if $x \in U$ and is 0 otherwise. Hence, the support of $j_!\mathbb{Z}$ is simply U, which is open, not necessarily closed. □

EXERCISE 15. Sheaf $\mathcal{H}om$. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X . For any open set $U \subseteq X$ show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaes has a natural structure of an abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}_U, \mathcal{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$, "sheaf hom" for short, and is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof: We first show that $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}_U)$ is an abelian group. This is easy; we simply define $(f+g)(U) = f(U) + g(U) \in \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{F}(U), \mathcal{G}(U))$. The zero morphism $0 : \mathcal{F} \longrightarrow \mathcal{G}$ defined $0(U)(s) = 0$ is the identity and the inverse of a map $f : \mathcal{F} \longrightarrow \mathcal{G}$ is the morphism $-f : \mathcal{F} \longrightarrow \mathcal{G}$ defined on sections by $(-f)(U)(s) = -f(U)(s)$. This addition is compatible with restrictions.

Note that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is indeed a presheaf – it associates an abelian group to every $U \subseteq X$ and for every inclusion $V \subseteq U$ we get a restriction $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}(\mathcal{F}|_V, \mathcal{G}_V)$ given by restriction a morphism $f : \mathcal{F}|_U \longrightarrow \mathcal{G}|_U$ to $\mathcal{F}|_V \longrightarrow \mathcal{G}|_V$ (here we are technically using the fact that $(\mathcal{F}|_U)|_V \cong \mathcal{F}|_V$). We therefore need only show the two locality conditions hold for $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Identitiy Axiom: Suppose f is a section of $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, i.e. that it is a map $f : \mathcal{F}|_U \longrightarrow \mathcal{G}|_U$, such that $f|_{V_i} = 0$ on some open cover $\{V_i\}$ of U. Take some other open set $W \subseteq U$ and let $W_i = W \cap V_i$. Take some section $s \in \mathcal{F}(W)$. For each i, the diagram

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "f(W)"] \arrow[d, "\rho"] & \mathcal{G}(W) \arrow[d, "\rho"] \\ \mathcal{F}(W_i) \arrow[r, "f(W_i)"] & \mathcal{G}(W_i) \end{tikzcd}$$

commutes and $f|_{W_i} = f(W_i)$ by definition, so we get that $f(W_i)(s|_{W_i}) = 0$ for each i. The commutativity of the diagram paired with the fact that $\mathcal{G}$ is a sheaf gives us that $f(W)(s) = 0$, since the $\mathcal{G}$ section $f(W)(s)$ restricts to zero on W_i for each i. Because s was chosen to be an arbitrary section $f(W)$ must be zero and because W was chosen to be an arbitrary open subset of U the morphism $f : \mathcal{F}|_U \longrightarrow \mathcal{G}|_U$ must be zero. This proves the first sheaf axiom.

Gluing Axiom: Suppose now that we have morphisms $f_i : \mathcal{F}|_{V_i} \longrightarrow \mathcal{G}|_{V_i}$ on some open cover $\{V_i\}$ of an open set $W \subseteq U$ such that $f_i(V_i \cap V_j) = f_j(V_i \cap V_j)$. We can define a morphism $f : \mathcal{F}|_W \longrightarrow \mathcal{G}|_W$ which restricts to f_i on V_i as follows.

Fix an arbitrary section $s \in \mathcal{F}(W)$, restrict it to V_i and map it to $\mathcal{G}|_{V_i}$. This is $f_i(V_i)(s|_{V_i})$. The restriction of this $\mathcal{G}(V_i)$ section to $V_i \cap V_j$ is $f_i(V_i)(s|_{V_i})|_{V_j} = f_i(V_i)(s|_{V_i \cap V_j})$ by the commutativity requirement satisfied by $f_i(V_i)$ and furthermore $f_i(V_i)(s|_{V_i \cap V_j}) = f_j(V_j)(s|_{V_i \cap V_j}) = f_j(V_j)(s|_{V_j})|_{V_i}$ since f_i and f_j agree on overlaps. Hence $\{f_i(V_i)(s|_{V_i})\}_i$ form a collection of sections in $\mathcal{G}(V_i)$ which agree on overlaps, so there is some unique $x \in \mathcal{G}(W)$ which restricts to $f_i(V_i)(s|_{V_i})$ on V_i. Now define $f(W)(s) = x$. This is the only thing we could possibly do, since x is the unique element which satisfies $x|_{V_i} = f(V_i)(s|_{V_i})$ for all i. One can see that f is compatible with restrictions by definition (we *defínted* it by lifting restrictions on a cover) and that $f(W')$ is a homomorphism of abelian groups by tracing a sum $s + t$ of sections in $\mathcal{F}(W')$ through the same restriction diagrams and lifting to $\mathcal{G}(W')$. $\square$

EXERCISE 17. Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$ in the closure of P, and 0 elsewhere. Hence the name "skyscraper sheaf". Show that this sheaf could also be described as $i_*(A)$ where A denotes the constant sheaf A on the closed subspace $\{P\}^-$ and $i\{P\}^- \longrightarrow X$ is the inclusion.

Proof: Suppose $Q \in \{P\}^-$ so that every open set V containing Q also contains P. Then $i_P(A)(V) = A$ for every such set by definition, and the restriction map $i_P(A)(V) \longrightarrow i_P(A)(V')$ for $Q \in V' \subseteq V$ is the identity. Hence the stalk at $i_P(A)(V)$ is indeed A. If Q is not in the closure of $\{P\}$ then there is some open set V containing Q which avoids P. Hence $i_P(A)(V) = 0$ and the stalk at Q must necessarily be zero.

Suppose now that $i_*(A)$ is the pushforward of the constant sheaf on $\{P\}^-$ via the inclusion $i : \{P\}^- \longrightarrow X$. Any open subset of $\{P\}^-$ is given by the intersection of $\{P\}^-$ with $V \subseteq X$ open. If this intersection contains a point Q, then V necessarily contains P as well, since Q is in the closure of $\{P\}$. This means every nonempty open subset of $\{P\}^-$ contains P, and in particular, any two open subsets meet. This implies that $\{P\}^-$ is connected and thus the constant sheaf A on $\{P\}^-$ is simply the constant presheaf. The pushforward i_*A is then

$$i_*A(V) = A(i^{-1}(V)) = \begin{cases} A & i^{-1}(V) \text{ nonempty } \iff P \in V \\ 0 & i^{-1}(V) = \iff P \notin V \end{cases}.$$

This is exactly the skyscraper sheaf $\square$