Algebraic Topology Homework 0

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EXERCISE 1. Construct an explicit deformation retraction of the torus with one point delted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Proof: We think of a torus as a product of the interval $I = [-1, 1]$ with iteself with parallel edges identified in matching orientation, i.e.

$$\mathbb{T}^2 = I^2/\sim$$

where $(-1, x) \sim (1, x)$ and $(x, -1) \sim (x, 1)$. The wedge of the longitudinal and meridian circles upon which we wish to deformation retract are precisely the boundary of I^2 under the quotient map: $S^1 \wedge S^1 = \pi(\partial I^2)$. What this means is that we can deformation retract a $I^2 - \{\text{pt}\}$ to its boundary and compose with the projection map $\pi : I^2 \longrightarrow \mathbb{T}^2$ in order to construct the desired deformation retrat of the punctured torus. This is much easier to visualize and, more importantly, easier to explicitly write down.

Without loss of generality, suppose pt $= (0, 0)$. Indeed, if it were any other point in the interior of I^2, we could simply apply a homeomorphism. Let $X = I^2 \setminus \{(0, 0)\}$, so that $\mathbb{T}^2 = X/\sim$.

For future Isaac: To construct the homotopy of X to its boundary, a good first attempt is to imagine the ray emenating from $(0, 0)$ and passing through some other point $(a, b) \in I^2$. This intersects ∂I^2 in exactly one place. The homotopy we'd like to write down linearly interpolates (a, b) to this unique intersection point with ∂I^2 in one unit time, so that all interior points reach the boundary at $t = 1$. However, this homotopy is rather a pain to write down, so we add a few steps to reduce the total work.

Consider the circle $S^1 = \{|(a, b) \in \mathbb{R}^2 \ | \ \|(a, b)\| = 1\} \subseteq I^2$. The map $f : X \longrightarrow S^1$ defined $x \mapsto \frac{x}{\|x\|}$ is a retract onto S^1. In particular, $f|_{\partial I^2}$ is a bijective map from ∂I^2 to S^1, and thus has inverse $g : S^1 \longrightarrow \partial I^2$. This is the map we intuitively described above restricted to the circle. We note that the composition $g \circ f$ is the map which first takes a point $x \in X$ to the point on S^1 correponding to x's "direction" and then sends the result to its corresponding point on ∂I^2.

We can now write down the homotopy $F : X \times I \longrightarrow X$:

$$F(x, t) = (1 - t)x + tg(f(x)).$$

This map is continuous, as it is the restriction, composition, product and sum of continuous functions on $\mathbb{R}^3 \setminus \{(x, y, z) \mid x = y = 0\}$. Furthermore, it is a deformation retraction of X onto ∂I^2, as it fixes ∂I^2 at every time step. The composition $\pi \circ F$ yields the desired deformation retraction on $\mathbb{T}^2$. $\square$

EXERCISE 2.

(a) Show that the composition of homotopy equivalences $X \longrightarrow Y$ and $Y \longrightarrow Z$ is a homotopy equivalence $X \longrightarrow Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \longrightarrow Y$ is an equivalence relation.

Proof:

(a) We first prove the following lemma.

Lemma 0.1. Let $f : X \to Y$ and $g : X \to Y$ be functions such that $f \simeq g$. Let $F : X \times [0,1] \to Y$ where $F(x,0) = f(x)$ and $F(x,1) = g(x)$ be the homotopy connecting f and g. Furthermore, let $f' : Y \to Z$ and $g' : Y \to Z$ be functions such that $f' \simeq g'$ connected by the $G : Y \times [0,1] \to Z$ where $G(x,0) = f'(x)$ and $G(x,1) = g'(x)$. We show that

$$f'f \simeq g'g$$

Proof. We want to find a homotopy $H : X \times [0,1] \to Z$ connecting $f'f$ and $g'g$. Let H be defined as $H(x,t) = G(F(x,t),t)$. H is continuous since G and F are continuous. Furthermore,

$$H(x,0) = G(F(x,0),0) = f'(f(x)) = f'f$$

and

$$H(x,1) = G(F(x,1),1) = g'(g(x)) = g'g$$

□

We now begin the proof. Let X and Y be homotopy equivalent and let Y and Z be homotopy equivalent. There must then exist functions $f : X \to Y$, $g : Y \to X$, $f' : Y \to Z$, $g' : Z \to Y$ such that

$$gf \simeq \mathrm{id}_X \qquad fg \simeq \mathrm{id}_Y \qquad g'f' \simeq \mathrm{id}_Y \qquad f'g' \simeq \mathrm{id}_Z$$

where $\mathrm{id}_X, \mathrm{id}_Y$ and id_Z denote the identity functions on X, Y and Z respectively. Note here that f is the homotopy equivalence of X and Y and that f' is the homotopy equivalence of Y and Z.

From the result of part (b), which is independent of this result, we know that $f' \simeq f'$, and thus by the above lemma we have that

$$fg \simeq \mathrm{id}_Y \Rightarrow f'fg \simeq f'\,\mathrm{id}_Y = f'$$

By the same logic, we have that

$$f'fgg' \simeq f'g'$$

and by the transitivity of the homotopy relation (also proven in part b) we conclude that since $f'g' \simeq \mathrm{id}_Z$,

$$f'fgg' \simeq \mathrm{id}_Z$$

Furthermore, since $g'f' \simeq \mathrm{id}_Y$,

$$gg'f' \simeq g\,\mathrm{id}_Y = g$$

and

$$gg'f'f \simeq gf.$$

Finally, by the transitivity of homotopic relations, since $gf \simeq \mathrm{id}_X$ we conclude that

$$gg'f'f \simeq \mathrm{id}_X \, .$$

Since there exist functions $f'f : X \to Z$ and $gg' : Z \to X$ such that $gg'f'f \simeq \mathrm{id}_X$ and $f'fgg' \simeq \mathrm{id}_Z$, we conclude that X and Z are homotopy equivalent by the composition $f'f$ of homotopy equivalences.

For the second part of the question, we say that $X \sim Y$ if X and Y are homotopy equivalent. We show that this relation is an equivalence relation.

Notice we have already proven that this relation has the transitive property, since

$$X \sim Y \text{ and } Y \sim Z \Rightarrow X \sim Z$$

It remains only to show that it also holds for the reflexive and symmetric properties.

Reflexive: Let X be a topological space. The identity id_X is a homotopy equivalence between X and X since $\mathrm{id}_X \circ \mathrm{id}_X \simeq \mathrm{id}_X$. Thus, $X \sim X$.

Symmetric: Let X and Y be homotopy equivalent topological spaces. There must then exist functions $f : X \to Y$ and $g : Y \to X$ such that

$$fg \simeq \mathrm{id}_Y \quad \text{and} \quad gf \simeq \mathrm{id}_X$$

This makes f a homotopic equivalence from X to Y. However, these are the same conditions required for g to be a homotopic equivalence from Y and X. Thus, $X \sim Y \Leftrightarrow Y \sim X$

Since the relation is reflexive, symmetric, and transitive, we conclude that it is an equivalence relation.

(b) We show that $\simeq$ is an equivalence relation.

Reflexive: Let $f : X \to Y$. Define $f_t : X \to Y$ where $t \in [0,1]$ such that $f_t(x) = f(x)$ for all $t \in [0,1]$. Since $f_0(x) = f(x)$ and $f_1(x) = f(x)$, we have $f \simeq f$ by the homotopy f_t

Symmetric: Let $f, g : X \to Y$, and assume that $f \simeq g$. Then there is some function $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. The reverse homotopy $G : X \times I \to Y, x \mapsto F(x, 1-t)$ therefore gives $G(x,0) = g(x)$ and $G(x,1) = f(x)$. Thus, $f \simeq g \Rightarrow g \simeq f$.

We can prove the converse by replacing f and g. Thus, homotopic equivalence is symmetric.

Transitivity: Let $f, g, h : X \to Y$ be functions such that $f \simeq g$ and $g \simeq h$. Let $F, G : X \times I \to Y$ be the homotopies of f, g and g, h respectively. We define $H : X \times I \to Y$ as follows:

$$(x,t) \mapsto \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Notice that $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x, 2-1) = h(x)$. Furthermore, since $H(x, \frac{1}{2}) = F(x,1) = g(x) = G(x,0) = G(x, 1-1)$, by the pasting lemma H is continuous since it is continuous over $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. H is therefore a homotopy between f and h, and we conclude that the relation of homotopy is transitive.

Since the relation of homotopy is reflexive, symmetric, and transitive, we conclude that it is an equivalence relation.

$\square$

EXERCISE 3. Show that a retract of a contractible space is contractible.

Proof:

Lemma 0.2. *Let $f, g : X \to Y$ and $f \simeq g$. If $h : W \to X$ and $p : Y \to Z$ then $f \circ h \simeq g \circ h$ and $p \circ f \simeq p \circ g$.*

Proof. If $f \simeq g$ then there is a homotopy $F : X \times I \to Y$ such that $F_0 = f$ and $F_1 = g$. By composing each component function of F with h and p, i.e. $F_t \circ h$ and $p \circ F_t$, we obtain homotopies connecting $f \circ h$ and $g \circ h$, and $d \circ f$ and $d \circ g$ respectively. $\square$

Let X be a contractible space and let $r : X \to X$ be a retract of X to $A \subset X$. Since X is contractible, there exists some $x_0 \in X$ such that $\mathrm{id}_X \simeq c_{x_0}$ where $c_{x_0} : X \to X$ is the constant map $x \mapsto x_0$ for all $x \in X$.

Let $i_A : A \to X$ be the inclusion map of A. Notice that $r \circ \mathrm{id}_X \circ i : A \to A$ is exactly the identity map on A, and that by the lemma,

$$\mathrm{id}_X \simeq c_{x_0} \implies r \circ \mathrm{id}_X \circ i \simeq r \circ c_{x_0} \implies r \circ \mathrm{id}_X \circ i \simeq r \circ c_{x_0} \circ i = r \circ c_{x_0}$$

Thus,

$$\mathrm{id}_A = r \circ \mathrm{id}_X \circ i \simeq r \circ c_{x_0} = c_{r(x_0)}$$

where $c_{r(x_0)}$ denotes the constant function obtained by $r \circ c_{x_0}$. We conclude that A is contractible. $\square$

EXERCISE 4. Show that S^∞ is contractible.

EXERCISE 5.

(a) Show that the mapping cylinder of every map $f : S^1 \longrightarrow S^1$ is a CW complex.

(b) Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

EXERCISE 6. Show that a CW complex is contractible if is the union of two contractible subcomplexes whose intersection is also contractible.

Proof: We start with a lemma.

Lemma 0.3. Let A and B be contractible. Then $A \vee B$ is contractible.

Proof. Let A and B contract to $a \in A$ and $b \in B$ respectively. We show that $A \vee B \simeq \{a\} \vee \{b\} = \{x_0\}$, where $\{x_0\}$ is the result of the wedge sum between $\{a\}$ and $\{b\}$.

Since $A \simeq \{a\}$ and $B \simeq \{b\}$ we have the homotopy equivalences f, f', g, and g' where

$$A \underset{f'}{\overset{f}{}} \{a\} \qquad B \underset{g'}{\overset{g}{}} \{b\}.$$

Define the functions $h : A \vee B \to \{x_0\}$ and $h' : \{x_0\} \to A \vee B$, where A and B are identified by $f'(x) \sim g'(x)$. Define

$$h(x) = x_0 \qquad h'(x) = f'(x) = g'(x)$$

We will show that these are homotopy equivalences and inverse homotopy equivalences respectively. Notice that

$$h \circ h' = \mathrm{id}_{\{x_0\}} = \simeq \mathrm{id}_{\{x_0\}}$$

and since $\mathrm{id}_{A \vee B} |A = \mathrm{id}_A$ and $\mathrm{id}_{A \vee B} |B = \mathrm{id}_B$, given that

$$h' \circ h|A = f' \circ f \simeq \mathrm{id}_A = \mathrm{id}_{A \vee B} |A \qquad h' \circ h|B = g' \circ g \simeq \mathrm{id}_B = \mathrm{id}_{A \vee B} |B$$

we conclude that $h' \circ h \simeq \mathrm{id}_{A \vee B}$. Thus $A \vee B$ is contractible to x_0. $\square$

We now proceed to the problem. Let X be a CW-Complex and let A and B be subcomplexes. We show that if $X = A \cup B$ and if A, B, and $A \cap B$ are all contractible then X is contractible.

First, notice that if A and B are subcomplexes of X, then $A \cap B$ is also a subcomplex of X. A subcomplex is, by definition, a closed collection of cells of a CW-complex. Since A and B are closed, so also is $A \cap B$. It remains to show that $A \cap B$ is a collection of n-cells. Let C^n_α denote the αth n-cell contained in A and D^n_β denote the βth n-cell contained in B. Suppose that $x \in D^n_{\beta_i}$ and $x \in C^n_{\alpha_j}$. Since a CW-complex is defined to be the disjoint union of cells, if $x \in D^n_{\beta_i}$ and $x \in C^n_{\alpha_j}$ then we can conclude that $C^n_{\alpha_j} = D^n_\beta$. This must be true of every two cells $D^n_{\beta_i}$ and $C^n_{\alpha_j}$, thus, if $x \in C^n_{\alpha_j}$ either $x \notin D^n_{\beta_i}$ or $C^n_{\alpha_j} = D^n_\beta$. Since A and B are both comprised solely of various cells, if $x \in A \cap B$ then it is contained in a cell shared by both A and B. We therefore know that if A and B are both subcomplexes, so also is $A \cap B$.

Since $A \cap B$ is a subcomplex of X, $(X, A \cap B)$ is a CW pair and by Prop (0.16) we know that X has the homotopy extension property. Since $A \cap B$ is contractible, there exists a nullhomotopy $\psi : A \cap B \times I \to A \cap B$ and some homotopy $\Psi : X \times I \to X$ such that $\Psi|(A \cap B) = \psi$. Notice then that $\Psi_1(A \cap B) = \{x_0\}$ for some $x_0 \in A \cap B$.

Now, since $A \cap B$ is a subcomplex of A and a subcomplex of B, the sets A and $A/(A \cap B)$ are homotopic, as are B and $B/(A \cap B)$. By Prop (0.17) we know that $A/(A \cap B) \simeq A \simeq \{a_0\}$ and $B/(A \cap B) \simeq B \simeq \{b_0\}$. Here we consider a_0 and b_0 to be the points to which A and B contract respectively. Thus, by the lemma, the wedge sum $A/(A \cap B) \vee B/(A \cap B)$ identified by $a_0 \sim b_0$ must also be contractible.

$A/(A \cap B)$ is the set A with all of the intersection $A \cap B$ identified to a point. Likewise, $B/(A \cap B)$ is the set B with all of the intersection $A \cap B$ identified to a point. The wedge sum of these two sets is therefore equal (up to a bijection) to the set$\Psi_1(X)$, obtained by contracting $A \cap B$ to a point. To be precise, we choose to identify $A \cap B$ with $\{x_0\}$. Thus, we have that

$$X \underset{\Psi}{\simeq} A/(A \cap B) \vee B/(A \cap B) \simeq \{x_0\}$$

we conclude that X is contractible. $\square$

EXERCISE 7. Use Corollary 0.20 to show that if (X, A) has the homotopy extension property, then $X \times I$ deformation retracts to $X \times \{0\} \cup A \times I$. Deduce from this that Proposition 0.18 holds more generally for any pair (X_1, A) satisfying the homotopy extension property.