Problems from Hartshorne Chapter 2.2

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EXERCISE 2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \longrightarrow K$.

Proof: Suppose first that we have a map $f : \operatorname{Spec} K \longrightarrow X$. Topologically, this is determined solely by choosing an image $x \in f(P)$ for the sole point $P \in \operatorname{Spec} K$. Sheaf theoretically, this consists of a map $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_K$ (by $\mathcal{O}_K$ we mean $\mathcal{O}_{\operatorname{Spec} K}$). This induces a local ring map on the stalk at P: $f^\sharp_P : \mathcal{O}_{X,x} \longrightarrow (f_*\mathcal{O}_K)_P = K$, meaning that the maximal ideal $\mathfrak{m}_x$ in $\mathcal{O}_{X,x}$ is sent to the maximal ideal $(0) \subseteq K$, meaning that $\mathfrak{m}_x = \ker f^\sharp_P$. This in turn implies that $f^\sharp_P$ factors through the quotient $\pi : \mathcal{O}_{X,x} \mapsto k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ and hence induces a map $k(x) \longrightarrow K$. This map is necessarily an inclusion since every ring homomorphism of fields is injective.

Now suppose we have an injection $p : k(x) \hookrightarrow K$. We can then define a map $f^\sharp_x : \mathcal{O}_{X,x} \longrightarrow K$ by $f^\sharp = p \circ \pi$, where $\pi : \mathcal{O}_{X,x} \longrightarrow k(x)$ is the quotient map. This is precisely a map on between the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{K,P}$. If we define $f : \operatorname{Spec} K \longrightarrow X$ by $P \mapsto x$ and $f^\sharp(U) : \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_K(U) = K$ by $f^\sharp(U) = f^\sharp_x \circ \iota$ where $\iota : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$ is the natural localization map, then $(f, f^\sharp)$ is a map of schemes. Note that for any open set $U \subseteq X$ not containing x the map $f^\sharp : \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_K(U)$ is necessarily the zero map, since $f_*\mathcal{O}_K(U) = \mathcal{O}_K(f^{-1}(U)) = \mathcal{O}_K(\varnothing) = 0$. $\square$

EXERCISE 2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof: The ring $k[x]$ is a PID since k is a field, so the prime ideals are all principally generated by irreducible polynomials $f \in k[x]$. $\square$

EXERCISE 2.18.

(a) Let A be a ring, $X = \operatorname{Spec} A$ $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \longrightarrow B$ be a ring homomorphism and let $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sehaves $f^\sharp : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_X$ is injective. Show furthermore in that case f is *dominant*, i.e. $f(\operatorname{Spec} B)$ is dense in X.

Proof: (a) Recall that the nilradical of any ring is equal to the intersection of all its prime ideals. Therefore

$$f \text{ is nilpotent } \iff f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \iff V(f) = \operatorname{Spec} A \iff D(f) = \varnothing.$$

(b) Note first that if $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_*\mathcal{O}_{\operatorname{Spec} B}$ is injective then it is injective on global sections and hence $\varphi = f^\sharp(\operatorname{Spec} A) : A \longrightarrow B$ is injective. Suppose instead that $f^\sharp$ is not injective, so that there is some $U \subseteq \operatorname{Spec} A$ such that $f^\sharp(U) : \mathcal{O}_{\operatorname{Spec} A}(U) \longrightarrow \mathcal{O}_{\operatorname{Spec}(B)}(f^{-1}(U))$ which is not an injective ring homomorphism. By taking $f \in A$ such that $D(f) \subseteq U$ (which exists since the sets $D(f)$ are basic opens) we can assume that $U = D(f)$. In this case, the map $f^\sharp(D(f))$ is the map $\varphi_f : A_f \longrightarrow B_{\varphi(f)}$. If this is not injective, then there is some $n \in \mathbb{N}$ such that $\varphi(f^n)\varphi(a) = 0 \implies \varphi(f^n \cdot a) = 0$ such that $f^n a \neq 0$, and hence φ is not injective.

Suppose now that $\varphi : A \longrightarrow B$ is injective. The map f is dominant if and only if $f(\operatorname{Spec} B)$ has nontrivial intersection with every (nonempty) basic open $D(f)$. Fix then a nonempty $D(f)$, which by part (a) means f is not nilpotent. Localizing at f yields a map $\varphi_f : A_f \longrightarrow B_{\varphi(f)}$. Pulling back a maximal ideal $\mathfrak{m} \in \operatorname{Spec} B_{\varphi(f)}$ by φ_f yields a prime ideal in $\mathfrak{p}$ in A_f, and then by the correspondence between $\operatorname{Spec} A_f$ and primes in $\operatorname{Spec} A$ which do not contain f, we get that $f(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m}) = \mathfrak{p} \in D(f)$. Hence the image of f is dense in $\operatorname{Spec} A$.

$\square$