Lecture 20

1 **§ Statements of local class field theory**

K non-arch. local field.

Definition 17.1: An extension L/K is abelian if it is Galois and $\mathrm{Gal}(L/K)$ is an abelian group

Facts: Let L_1/K, L_2/K abelian

(i) L_1L_2/K abelian

(ii) If $L_1 \cap L_2 = K$, there is canonical iso.

$$\mathrm{Gal}(L_1L_2/K) \Rightarrow \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)$$

Fact (i) $\Rightarrow$ $\exists$ maximal abelian extension K^{ab} of K.

Eg. Let K^{ur} denote the max. unram. ext. of K inside K^{sep}.

$$K^{ur} = \bigcup_{m=1}^{\infty} K(\zeta_{q^m-1}) \quad (|k|=q), \quad k_{K^{ur}} \cong \overline{\mathbb{F}}_q.$$

$$\mathrm{Gal}(K^{ur}/K) \cong \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$$

so K^{ur} is abelian and hence $K^{ur} \subseteq K^{ab}$.

$\exists$ exact sequence

$$0 \longrightarrow I_{K^{ab}/K} \longrightarrow W(K^{ab}/K) \longrightarrow \mathbb{Z} \longrightarrow 0$$

For L/K unram, let $\mathrm{Fr}_{L/K} \in \mathrm{Gal}(L/K)$ corresponding

2 to $\mathrm{Fr}_{k_L/k} \in \mathrm{Gal}(k_L/k)$.

Theorem 17.2: (1) (Local Artin reciprocity) There exists a unique topological isomorphism (iso. of groups + homeo.)

$$\mathrm{Art}_K : K^\times \longrightarrow W(K^{ab}/K)$$

satisfying the following properties

(i) $\mathrm{Art}_K(\pi)|_{K^{ur}} = \mathrm{Fr}_{K^{ur}/K}$ for any uniformizer $\pi \in K$

(ii) For each finite subextension L/K in K^{ab}/K

$$\mathrm{Art}_K(N_{L/K}(L^\times))|_L = \{1\}.$$

Art_K is the Artin reciprocity map.

(2) Let L/K finite abelian. Then Art_K induces an iso.

$$\frac{K^\times}{N_{L/K}(L^\times)} \cong \frac{W(K^{ab}/K)}{W(K^{ab}/L)} \cong \mathrm{Gal}(L/K).$$

Remark (i) $\mathrm{Fr}_{K^{ur}/K}$ lifts $x \mapsto x^q$ in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ - the "arithmetic Frobenius." $\exists$ another normalization of Art_K with $\mathrm{Art}_K(\pi)|_{K^{ur}} = (\mathrm{Fr}_{K^{ur}/K})^{-1}$ ← geometric Frobenius

(ii) Special case of Local Langlands

(iii) Used to characterise global Artin map of global class field theory.

<u>Properties of Artin map</u>

• (Existence theorem) For $H \subseteq K^\times$ open finite index subgroup, $\exists L/K$ finite abelian s.t. $N_{L/K}(L^\times) = H$.

In particular, Art_K induces an (inclusion reversing) isomorphism of posets

$$\left\{\begin{matrix}\text{Open finite index}\\ \text{subgroups of } K^\times\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite abelian}\\ \text{extensions } L/K\end{matrix}\right\}$$

$$H \longmapsto (K^{ab})^{\mathrm{Art}_K(H)}$$

$$N_{L/K}(L^\times) \longleftarrow\!\shortmid \; L/K$$

- (Norm functoriality) Let L/K finite separable ext.

$\exists$ commutative diagram:

$$\begin{array}{ccc} L^\times & \xrightarrow{\mathrm{Art}_L} & W(L^{ab}/L) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ K^\times & \xrightarrow{\mathrm{Art}_K} & W(K^{ab}/K) \end{array}$$

Proposition 17.4: L/K finite abelian. Then

$$e_{L/K} = (\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)).$$

Proof: For $x \in L^\times$, we have

$$v_K(N_{L/K}(x)) = f_{L/K}\, v_L(x)$$

$\Rightarrow$ have surjection

$$\frac{K^\times}{N_{L/K}(L^\times)} \xrightarrow{v_K} \frac{\mathbb{Z}}{f_{L/K}\mathbb{Z}}.$$

with kernel

$$\frac{\mathcal{O}_K^\times N_{L/K}(L^\times)}{N_{L/K}(L^\times)} \cong \frac{\mathcal{O}_K^\times}{\mathcal{O}_K^\times \cap N_{L/K}(L^\times)}$$

$$\cong \frac{\mathcal{O}_K^\times}{N_{L/K}(\mathcal{O}_L^\times)}$$

Theorem 17.2(ii) $\Rightarrow$ $n = (K^\times : N_{L/K}(L^\times))$

$$= f_{L/K}(\mathcal{O}_K^\times : N_{L/K}(\mathcal{O}_L^\times)) \qquad \square$$

Corollary 17.5: L/K finite abelian. Then L/K is unramified iff $N_{L/K}(\mathcal{O}_L^\times) = \mathcal{O}_K^\times$. $\square$

§ Construction of $\mathrm{Art}_{\mathbb{Q}_p}$

Recall: $\mathbb{Q}_p^{ur} = \bigcup_{m=1}^{\infty} \mathbb{Q}_p(\zeta_{p^m-1}) = \bigcup_{p \nmid m} \mathbb{Q}_p(\zeta_m)$.

- $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ totally ramified deg $p^{n-1}(p-1)$, with

$$\theta_n : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) \xrightarrow{\sim} (\mathbb{Z}/p^n\mathbb{Z})^\times$$

For $n \geq m \geq 1$, there is a diagram

$$\begin{array}{ccc} \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p) & \twoheadrightarrow & \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p) \\ \downarrow{\scriptstyle \wr}\,\theta_n & & \downarrow{\scriptstyle \wr}\,\theta_m \\ (\mathbb{Z}/p^n\mathbb{Z})^\times & \underset{\text{canonical projection}}{\twoheadrightarrow} & (\mathbb{Z}/p^m\mathbb{Z})^\times \end{array}$$

Set $\mathbb{Q}_p(\zeta_{p^\infty}) = \bigcup_{n=1}^{\infty} \mathbb{Q}_p(\zeta_{p^n})$. Then $\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$ is Galois and we have

$$\theta : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \xrightarrow{\sim} \varprojlim_{n \geq 1} (\mathbb{Z}/p^n\mathbb{Z})^\times \simeq \mathbb{Z}_p^\times$$

We have $\underset{\text{tot. ram.}}{\mathbb{Q}_p(\zeta_{p^\infty})} \cap \underset{\text{unram}}{\mathbb{Q}_p^{ur}} = \mathbb{Q}_p$.

$\rightsquigarrow$ iso. $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times$.

Theorem 17.6: (Local Kronecker-Weber)

$$\mathbb{Q}_p^{ab} = \mathbb{Q}_p^{ur}\mathbb{Q}_p(\zeta_{p^\infty})$$

Pf: sketch later

Proof: sketch later. $\square$

Construct $\mathrm{Art}_{\mathbb{Q}_p}$ as follows

We have $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$.

$$p^n u \mapsto (n, u)$$

Then $\mathrm{Art}_{\mathbb{Q}_p}(p^n u) = \left(\left(\mathrm{Fr}_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}\right)^n, \theta^{-1}(u) \right)$

$$\widehat{\mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)} \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$$

Image lies in $W(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$

Remark: Definition of $\mathrm{Art}_{\mathbb{Q}_p}$ involves choice of a totally ramified $\mathbb{Q}_p(\zeta_{p^\infty})$ and the choice of uniformizer p, which determines the isomorphism

$$\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$$

The choices are related.

They "cancel out" and $\mathrm{Art}_{\mathbb{Q}_p}$ is canonical