Title Goes Here

Isaac Martin

Last compiled September 16, 2022

---

§ *Chapter 1*

EXERCISE 1. Let $(S, \mathcal{S})$ be a measurable space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$. Show that the following sets are also in $\mathcal{S}$.

(1) the set C_1 of all $x \in S$ such that $x \in A_n$ for at least 5 different values of n.

(2) the set C_2 of all $x \in S$ such that $x \in A_n$ for exactly 5 different values of n.

(3) the set C_3 of all $x \in S$ such that $x \in A_n$ for all but finitely many n ("finitely many" includes none).

(4) the set C_4 of all $x \in S$ such that $x \in A_n$ for at most finitely many values of n.

Proof: (1)

$\square$

EXERCISE 2. (Atomic structure of algebras). A **partition** of a set S is a family $\mathcal{P}$ of non-empty subsets of S with the property that each $x \in S$ belongs to exactly one $A \in \mathcal{P}$.

(1) How many algebras are there on the set $S = \{1, 2, 3\}$?

(2) By constructing a bijection between the two families, show that the number of different algebras on a finite set S is equal to the number of different partitions of S. Note: the elements of the partition corresponding to an algebra are said to be its atoms.

(3) Does there exist an algebra with 754 elements?

EXERCISE 3. Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable if it is either monotone or convex.

Proof: Define the following collection of subsets of $\mathbb{R}$: $A = \{(-\infty, a] \mid a \in \mathbb{R}\}$. I claim that $\sigma(A) = \mathcal{B}(\mathbb{R})$, i.e. that A generates the Borel algebra on $\mathbb{R}$. To see this, it suffices to show that all open intervals are contained in $\sigma(A)$. Fix an open interval $(a, b) \subseteq \mathbb{R}$. Sigma algebras are closed under setminus, so $(-\infty, b] \setminus (-\infty, a] = (a, b]$. To obtain an open set, take a countable sequence of these half-open intervals and union them:

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right].$$

Hence $(a, b) \in \sigma(A)$, and since $A \subseteq \mathcal{B}(\mathbb{R})$, we have $\sigma(A) = \mathcal{B}(\mathbb{R})$. We now move on to the problem at hand.

Suppose first that f is monotone increasing, so that $x \leq y \implies f(x) \leq f(y)$. By a theorem from class, it suffices to prove f is measurable on a generating set for the Borel algebra. Consider the interval $(-\infty, a)$ and set $b = \inf f^{-1}(a)$. $\square$