Lecture 19

Last time: L/K Galois

$$\mathrm{Gal}(L/K) \cong \varprojlim_{\substack{K \subseteq F \subseteq L \\ F/K \text{ finite Galois}}} \mathrm{Gal}(F/K). \quad \text{Ex sheet 4}$$

Example: $K = \mathbb{F}_q$, $L = \overline{\mathbb{F}}_q$ alg. closure.

$$\{F/K \text{ finite Galois}\} \xleftrightarrow{1:1} \mathbb{N}_{\geq 1}$$

$$\mathbb{F}_{q^n} \longleftrightarrow n$$

$\mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^n}$ iff $m \mid n$

$\exists$ commutative diagram

$$\begin{tikzcd}
\mathrm{Fr}_q \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \arrow[r, two heads] \arrow[d, phantom, "\wr\wr"] & \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \ni \mathrm{Fr}_q \arrow[d, phantom, "\wr\wr"] \\
1 \in \mathbb{Z}/n\mathbb{Z} \arrow[r, two heads, "\text{proj}"] & \mathbb{Z}/m\mathbb{Z} \ni 1
\end{tikzcd}$$

$$\Rightarrow \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \varprojlim_{n \in (\mathbb{N}_{\geq 1}, |)} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}$$

$$\mathrm{Fr}_q \longleftrightarrow 1$$

profinite completion of $\mathbb{Z}$

Let $\langle \mathrm{Fr}_q \rangle \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ subgp generated by Fr_q.

The inclusion $\langle \mathrm{Fr}_q \rangle \subseteq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ corresponds to

$$\mathbb{Z} \subseteq \hat{\mathbb{Z}}$$

Ex Sheet 3: $\hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$.

16.3 $\Rightarrow$ Can endow $\mathrm{Gal}(L/K)$ with profinite topology

Note: L/K finite $\to$ discrete topology

Theorem 16.4: (Fundamental theorem of
2 Galois theory) Let L/K Galois. $\exists$ bijection

$$\{F/K \text{ subextensions of } L/K\} \xleftrightarrow{1-1} \{\text{Closed subgroups of } Gal(L/K)\}$$

$$F \longmapsto Gal(L/F)$$

$$L^H \longleftarrow H \leq Gal(L/K)$$

Moreover; • F/K finite iff $Gal(L/F)$ open

• F/K Galois iff $Gal(L/F)$ normal in $Gal(L/K)$

Proof: Ex sheet 4.

§ Weil Group

K local field. L/K separable algebraic ext.

Definition 16.5: (i) L/K is unramified if F/K is unramified for all F/K finite subextension

(ii) L/K is totally ramified if F/K is totally ramified $\forall$ F/K finite subext.

Proposition 16.6: Let L/K unramified. Then L/K is Galois and

$$Gal(L/K) \cong Gal(k_L/k).$$

3 Proof: Every finite subext. F/K is unram, hence Galois $\Rightarrow$ L/K is normal and separable, hence L/K Galois.

k_L/k Galois since k is finite.

Moreover $\exists$ commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[r, "\mathrm{res}"] \arrow[d, phantom, "16.3 \,\|\wr"] & \mathrm{Gal}(k_L/k) \arrow[d, "i"] \\
\varprojlim\limits_{\substack{F/K \text{ finite} \\ F \subseteq L}} \mathrm{Gal}(F/K) \arrow[r, two heads] & \varprojlim\limits_{\substack{F/K \text{ finite} \\ F \subseteq L}} \mathrm{Gal}(k_{F}/k)
\end{tikzcd}$$

The natural map $\left\{ \begin{smallmatrix} F/K \text{ finite} \\ F \subseteq L \end{smallmatrix} \right\} \to \left\{ \begin{smallmatrix} l/k \text{ finite} \\ l \subseteq k_L \end{smallmatrix} \right\}$

is a bijection, since F is unramified. Thus

$$\varprojlim_{\substack{F/K \text{ finite} \\ F \subseteq L}} \mathrm{Gal}(k_F/k) \overset{13.4}{\cong} \varprojlim_{\substack{l/k \text{ finite} \\ l \subseteq k_L}} \mathrm{Gal}(l/k) \overset{16.3}{\cong} \mathrm{Gal}(k_L/k)$$

$\Rightarrow i$ iso $\square$

Ex Sheet 3. L_1/K, L_2/K finite unram. $\Rightarrow$ L_1L_2/K unram. Thus for any L/K, $\exists$ max unram. subext. K_0/K

Let L/K Galois. There is surjection

$$\mathrm{res} : \mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(K_0/K) \cong \mathrm{Gal}(k_L/k)$$

and we write $I_{L/K}$ for the kernel of res (Inertia subgroup).

Let $\mathrm{Fr}_{k_L/k} \in \mathrm{Gal}(k_L/k)$ be the Frobenius $x \mapsto x^{|k|}$ and let $\langle \mathrm{Fr}_{k_L/k} \rangle$ subgp generated by $\mathrm{Fr}_{k_L/k}$

Definition 16.7: Let L/K Galois. The Weil group $W(L/K) \subseteq \mathrm{Gal}(L/K)$ is $\mathrm{res}^{-1}(\langle \mathrm{Fr}_{k_L/k} \rangle)$

Remark: If k_L/k finite $W(L/K) = Gal(L/K)$

$W(L/K) \subsetneq Gal(L/K)$ if k_L/k infinite

$\exists$ commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_{L/K} \arrow[r] \arrow[d, equal] & W(L/K) \arrow[r] \arrow[d, hook] & \langle Fr_{k_L/k} \rangle \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I_{L/K} \arrow[r] & Gal(L/K) \arrow[r] & Gal(k_L/k) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Endow $W(L/K)$ with weakest topology s.t.

1) $W(L/K)$ is a topological group

2) $I_{L/K}$ is an open subgp of $W(L/K)$ with its subspace topology.

i.e. open sets are translates of open sets in $I_{L/K}$ by elements of $W(L/K)$.

WARNING: If k_L/k infinite, <u>NOT</u> the subspace topology on $W(L/K) \subseteq Gal(L/K)$.

Eg. $I_{L/K}$ not open in ~~subspace topology~~.

Proposition 16.8: L/K Galois

(i) $W(L/K)$ is dense in $Gal(L/K)$.

(ii) If F/K finite subext. of L/K. Then

$$W(L/F) = W(L/K) \cap Gal(L/F).$$

(iii) If F/K finite Galois subext. Then

$$W(L/K) \twoheadrightarrow Gal(F/K)$$

$$\frac{W(L/K)}{W(L/F)} \cong \mathrm{Gal}(F/K).$$

Prof: (i) $W(L/K)$ dense in $\mathrm{Gal}(L/K)$

$\Leftrightarrow$ $\forall F/K$ finite Galois subext,

$W(L/K)$ intersects every coset of $\mathrm{Gal}(L/F)$.

$\Leftrightarrow$ $\forall F/K$ finite Galois, $W(L/K) \twoheadrightarrow \mathrm{Gal}(F/K)$

$\exists$ diagram

$$\begin{tikzcd}
0 \arrow[r] & I_{L/K} \arrow[r] \arrow[d, "a"] & W(L/K) \arrow[r] \arrow[d, "b"] & \langle Fr_{k_L/k} \rangle \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & I_{F/K} \arrow[r] & \mathrm{Gal}(F/K) \arrow[r] & \mathrm{Gal}(k_F/k) \arrow[r] & 0
\end{tikzcd}$$

K_0/K max unram. ext. contained in L.

Then $K_0 \cap F$ = max. unram. ext. contained in F.

Then $\mathrm{Gal}(L/K_0) \twoheadrightarrow \mathrm{Gal}(F/K_0 \cap F)$ surj $\Rightarrow$ a surj.

$\mathrm{Gal}(k_F/k)$ is generated by $Fr_{k_F/k}$ $\Rightarrow$ c surj.

Diagram chase $\Rightarrow$ b surj.

(ii) F/K finite. $\exists$ diagram

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[r, two heads] & \mathrm{Gal}(k_L/k) \arrow[r, phantom, "\supseteq"] & \langle Fr_{k_L/k} \rangle \\
\mathrm{Gal}(L/F) \arrow[r, two heads] \arrow[u, hook] & \mathrm{Gal}(k_L/k_F) \arrow[r, phantom, "\supseteq"] \arrow[u, hook] & \langle Fr_{k_L/k_F} \rangle \arrow[u, hook]
\end{tikzcd}$$

For $\sigma \in \mathrm{Gal}(L/F)$

$\sigma \in W(L/F) \Leftrightarrow \sigma|_{k_L} \in \langle Fr_{k_L/k_F} \rangle$

We $\mathrm{Gal}(k_L/k_F) \cap \langle Fr_{k_L/k} \rangle = \langle Fr_{k_L/k_F} \rangle$ $\Leftrightarrow \sigma|_{k_L} \in \langle Fr_{k_L/k} \rangle$

$\Leftrightarrow \sigma \in W(L/K)$.

(iii) $$W(L/K)/W(L/F) \overset{(ii)}{=} \frac{W(L/K)}{W(L/K)\cap \mathrm{Gal}(L/F)}$$

$$= \frac{W(L/K)\,\mathrm{Gal}(L/F)}{\mathrm{Gal}(L/F)}$$

?

$$\overset{(i)}{=} \frac{\mathrm{Gal}(L/K)}{\mathrm{Gal}(L/F)} = \mathrm{Gal}(F/K).$$

$\square$