Chapter 0

Introduction

Es gibt nach des Verf. Erfarhrung kein besseres Mittel, Geometrie zu lernen, als das Studium des Schubertschen *Kalküls der abzählenden Geometrie*.

(There is, in the author's experience, no better means of learning geometry than the study of Schubert's *Calculus of Enumerative Geometry*.)

–B. L. van der Waerden (in a Zentralblatt review of *An Introduction to Enumerative Geometry* by Hendrik de Vries).

Why you want to read this book

Algebraic geometry is one of the central subjects of mathematics. All but the most analytic of number theorists speak our language, as do mathematical physicists, complex analysts, homotopy theorists, symplectic geometers, representation theorists. . . . How else could you get between such apparently disparate fields as topology and number theory in one hop, except via algebraic geometry?

And intersection theory is at the heart of algebraic geometry. From the very beginnings of the subject, the fact that the number of solutions to a system of polynomial equations is, in many circumstances, constant as we vary the coefficients of those polynomials has fascinated algebraic geometers. The distant extensions of this idea still drive the field forward.

At the outset of the 19th century, it was to extend this "preservation of number" that algebraic geometers made two important choices: to work over the complex numbers rather than the real numbers, and to work in projective space rather than affine space. (With these choices the two points of intersection of a line and an ellipse have somewhere to go as the ellipse moves away from the real points of the line, and the same for the point of intersection of two lines as the lines become parallel.) Over the course of the century, geometers refined the art of counting solutions to geometric problems — introducing the central notion of a *parameter space*, proposing the notions of an equivalence relation

https://doi.org/10.1017/CBO9781139062046.002 Published online by Cambridge University Press

on cycles and a product on the equivalence classes and using these in many subtle calculations. These constructions were fundamental to the developing study of algebraic curves and surfaces.

In a different field, it was the search for a mathematically precise way of describing intersections that underlay Poincaré's study of what became algebraic topology. We owe Poincaré duality and a great deal more in algebraic topology directly to this search. The difficulties Poincaré encountered in working with continuous spaces (now called manifolds) led him to develop the idea of a simplicial complex as well.

Despite a lack of precise foundations, 19th century enumerative geometry rose to impressive heights: for example, Schubert, whose *Kalkül der abzählenden Geometrie* (originally published in 1879, and reprinted 100 years later in [1979]) represents the summit of intersection theory in the late 19th century, calculated the number of twisted cubics tangent to 12 quadrics — and got the right answer (5,819,539,783,680). Imagine landing a jumbo jet blindfolded!

At the outset of the 20th century, Hilbert made finding rigorous foundations for Schubert calculus one of his celebrated problems, and the quest to put intersection theory on a sound footing drove much of algebraic geometry for the following century; the search for a definition of multiplicity fueled the subject of commutative algebra in work of van der Waerden, Zariski, Samuel, Weil and Serre. This progress culminated, towards the end of the century, in the work of Fulton and MacPherson and then in Fulton's landmark book *Intersection theory* [1984], which both greatly extended the range of intersection theory and for the first time put the subject on a precise and rigorous foundation.

The development of intersection theory is far from finished. Today the focus includes virtual fundamental cycles, quantum intersection rings, Gromov–Witten theory and the extension of intersection theory from schemes to stacks. In a different direction, there are computer systems that can do many of the computations in this book and many more; see for example the package *Schubert2* in *Macaulay2* (Grayson and Stillman [2015]) and the library *Schubert* in SINGULAR (Decker et al. [2015]).

A central part of a central subject of mathematics — of course you would want to read this book!

Why we wrote this book

Given the centrality of the subject, it is not surprising how much of algebraic geometry one encounters in learning enumerative geometry. And that is how this book came to be written, and why: Like van der Waerden, we found that intersection theory makes for a great "second course" in algebraic geometry, weaving together threads from all over the subject. Moreover, the new ideas encountered in this setting are not merely more abstract definitions for the student to memorize, but tools that help answer concrete questions.

This is reflected in the organization of the contents. A good example of this is Chapter 6 (“Lines on hypersurfaces”). The stated goal of the chapter is to describe the class, in the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$, of the scheme $F_1(X) \subset \mathbb{G}(1,n)$ of lines lying on a given hypersurface $X \subset \mathbb{P}^n$, as an application of the new technique of Chern classes. But this raises a question: how can we characterize the scheme structure on $F_1(X)$, and what can we say about the geometry of this scheme? In short, this is an ideal time to introduce the notion of a *Hilbert scheme*, which gives a general framework for these questions; in the present setting, we can explicitly write down the equations defining $F_1(X)$, and prove theorems about its local geometry. In the end, a large part of the chapter is devoted to this discussion, which is as it should be: A reader may or may not have any use for the knowledge that a general quintic hypersurface $X \subset \mathbb{P}^4$ contains exactly 2875 lines, but a functional understanding of Hilbert schemes is a fundamental tool in algebraic geometry.

What’s with the title?

The number in the title of this book is a reference to the solution of a classic problem in enumerative geometry: the determination, by Chasles, of the number of smooth conic plane curves tangent to five given general conics. The problem is emblematic of the dual nature of the subject. On the one hand, the number itself is of little significance: life would not be materially different if there were more or fewer. But the fact that the problem is well-posed — that there is a Zariski open subset of the space of 5-tuples $(C_1, \dots, C_5)$ of conics for which the number of conics tangent to all five is constant, and that we can in fact determine that number — is at the heart of algebraic geometry. And the insights developed in the pursuit of a rigorous derivation of the number — the recognition of the need for, and the introduction of, a new parameter space for plane conics, and the understanding of why intersection products are well-defined for this space — are landmarks in the development of algebraic geometry.

The rest of the title is from “1066 & All That” by W. C. Sellar and R. J. Yeatman, a parody of English history textbooks; in many ways the number 3264 of conics tangent to five general conics is as emblematic of enumerative geometry as the date 1066 of the Battle of Hastings is of English history.

What is in this book

We are dealing here with a fundamental and almost paradoxical difficulty. Stated briefly, it is that learning is sequential but knowledge is not. A branch of mathematics [...] consists of an intricate network of interrelated facts, each of which contributes to the understanding of those around it. When confronted with this network for the first time, we are forced to follow a particular path, which involves a somewhat arbitrary ordering of the facts.

–Robert Osserman.

Where to begin? To start with the technical underpinnings of a subject risks losing the reader before the point of all the preliminary work is made clear, but to defer the logical foundations carries its own dangers — as the unproved assertions mount up, the reader may well feel adrift.

Intersection theory poses a particular challenge in this regard, since the development of its foundations is so demanding. It is possible, however, to state fairly simply and precisely the main foundational results of the subject, at least in the limited context of intersections on smooth projective varieties. The reader who is willing to take these results on faith for a little while, and accept this restriction, can then be shown what the subject is good for, in the form of examples and applications. This is the path we have chosen in this book, as we will now describe.

Overture

The first two chapters may be thought of as an overture to the subject, introducing the central themes that will play out in the remainder of the book. In the first chapter, we introduce rational equivalence, the Chow ring, the pullback and pushforward maps — the “dogma” of the subject. (In regard to the existence of an intersection product and pullback maps, we do not give proofs; instead, we refer the reader to Fulton [1984].) We follow this in the second chapter with a range of simple examples to give the reader a sense of the themes to come: the computation of Chow rings of affine and projective spaces, their products and (some) blow-ups. To illustrate how intersection theory is used in algebraic geometry, we examine loci of various types of singular cubic plane curves, thought of as subvarieties of the projective space $\mathbb{P}^9$ parametrizing plane cubics. Finally, we briefly discuss intersection products of curves on surfaces, an important early example of the subject.

Grassmannians

The intersection rings of the Grassmannians are archetypal examples of intersection theory. Chapters 3 and 4 are devoted to them and their underlying geometry. Here we introduce *Schubert cycles*, whose classes form a basis for the Chow ring, and use them to solve a number of geometric problems, illustrating again how intersection theory is used to solve enumerative problems.

Chern classes

We then come to a watershed in the subject. Chapter 5 takes up in earnest a notion at the center of modern intersection theory, and indeed of modern algebraic geometry: Chern classes. As with the development of intersection theory, we focus on the classical characterization of Chern classes as degeneracy loci of collections of sections. This interpretation provides useful intuition and is basic to many applications of the theory.

Applications, I: Using the tools

We illustrate the use of Chern classes by taking up two classical problems: Chapter 6 deals with the question of how many lines lie on a hypersurface (for example, the fact that there are exactly 27 lines on each smooth cubic surface and 2875 lines on a general quintic threefold), and Chapter 7 looks at the singular hypersurfaces in a one-dimensional family (for example, the fact that a general pencil of plane curves of degree d has $3(d-1)^2$ singular elements). Using the basic technique of *linearization*, these problems can be translated into problems of computing Chern classes. These and the next few chapters are organized around geometric problems involving constructions of useful vector bundles and the calculation of their Chern classes.

Parameter spaces

Chapter 8 concerns an area in which intersection theory has had a profound influence on modern algebraic geometry: *parameter spaces* and their compactifications. This is illustrated with the five conic problem; there is also a discussion of the modern example of Kontsevich spaces, and an application of these.

Applications, II: Further developments

The remainder of the book introduces a series of increasingly advanced topics. Chapters 9, 10 and 11 deal with a situation ubiquitous in the subject, the intersection theory of projective bundles, and its applications to subjects such as projective duality and the enumerative geometry of contact conditions.

Chern classes are defined in terms of the loci where collections of sections of a vector bundle become dependent. These can be interpreted as loci where maps from trivial vector bundles drop rank. The Porteous formula, proved and applied in Chapter 12, generalizes this, expressing the classes of the loci where a map between two general vector bundles has a given rank or less in terms of the Chern classes of the two bundles involved.

Advanced topics

Next, we come to some of the developments of the modern theory of intersections. In Chapter 13, we introduce the notion of "excess" intersections and the *excess intersection formula*, one of the subjects that was particularly mysterious in the 19th century but elucidated by Fulton and MacPherson. This theory makes it possible to describe the intersection class of two cycles, even if the dimension of their intersection is "too large." Central to this development is the idea of *specialization to the normal cone*, a construction fundamental to the work of Fulton and MacPherson; we use this to prove

https://doi.org/10.1017/CBO9781139062046.002 Published online by Cambridge University Press

the famous “key formula” comparing intersections of cycles in a subvariety $Z \subset X$ to the intersections of those cycles in X, and use this in turn to give a description of the Chow ring of a blow-up.

Chapter 14 contains an account of Riemann–Roch formulas, leading up to a description of Grothendieck’s version. The chapter concludes with a number of examples and applications showing how Grothendieck’s formula can be used.

Appendices

The moving lemma

The literature contains a number of papers proving various parts of the moving lemma (see below for a statement). We give a careful proof of the first half of the lemma in Appendix A.

Cohomology and base change

Many results in this book will be proved by constructing an appropriate vector bundle and computing its Chern classes. The theorem on cohomology and base change (Theorem B.5) is a key tool in these constructions: We use it to show that, under appropriate hypotheses, the direct image of a sheaf is a vector bundle. We present a complete discussion of this important result in Appendix B.

Topology of algebraic varieties

When we treat algebraic varieties over an arbitrary field we use the Zariski topology, where an open set is defined as the locus where a polynomial function takes nonzero values. But if the ground field is the complex numbers, we can also use the “classical” topology: With this topology, a smooth projective variety over $\mathbb{C}$ is a compact, complex manifold, and tools like singular homology can help us study its geometry. Appendix C explains some of what is known in this direction, and also compares some of the possible substitutes for the Chow ring.

The Brill–Noether theorem

Appendix D explains an application of enumerative geometry to a problem that is central in the study of algebraic curves and their moduli spaces: the existence of special linear series on curves. We give the Kempf/Kleiman–Laksov proof of this theorem, which draws upon many of the ideas and techniques of the book, plus a new one: the use of topological cohomology in the context of intersection theory. This is also a wonderful illustration of the way in which enumerative geometry can be the essential ingredient in the proof of a purely qualitative result.

Relation of this book to *Intersection theory*

Fulton's book *Intersection theory* [1984] is a great work. It sets up for the first time a rigorous framework for intersection theory, and does so in a generality significantly extending and refining what was known before and laying out an enormous number of applications. It stands as an encyclopedic reference for the subject.

By contrast, the present volume is intended as a textbook in algebraic geometry, a second course, in which the classical side of intersection theory is a starting point for exploring many topics in geometry. We describe the intersection product at the outset, but do not attempt to give a rigorous proof of its existence, focusing instead on basic examples. We use concrete problems to motivate the introduction of new tools from all over algebraic geometry. Our book is not a substitute for Fulton's; it has a different aim. We do hope that it will provide the reader with intuition and motivation that will make reading Fulton's book easier.

Existence of the intersection product

The *moving lemma* was for most of a century the foundation on which intersection theory was supposed to rest. It has two parts:

(a) Given classes $\alpha, \beta \in A(X)$ in the Chow group of a smooth, projective variety X, we can find representative cycles A and B intersecting generically transversely.
(b) The class of the intersection of these cycles is independent of the choice of A and B.

Using these assertions it is easy to define the intersection product on the Chow groups of a smooth variety: $\alpha\beta$ is defined to be the class of $A \cap B$, where A and B are cycles representing the classes α and β and intersecting generically transversely, and this is how intersection products were defined. The problem is that, while the first part can be and was proved rigorously, as far as we know there was prior to the publication of Fulton's book in 1984 no complete proof of the second part. Of course, part (b) is an immediate consequence of the existence of a well-defined intersection product (Fulton [1984, Section 8.3]), and so we refer the reader to Fulton's book for this key existence result.

Nonetheless, we feel that part (a) of the moving lemma is useful in shaping one's intuition about intersection products. Moreover, given the existence statement, part (a) of the moving lemma allows simpler and more intuitive proofs of a number of the basic assertions of the theory, and we will use it in that way. We therefore give a proof of part (a) in Appendix A, following Severi's ideas.

Keynote problems

To highlight the sort of problems we will learn to solve, and to motivate the material we present, we will begin each chapter with some *keynote questions*.

Exercises

One of the wonderful things about the subject of enumerative geometry is the abundance of illuminating examples that are accessible to explicit computation. We have included many of these as exercises. We have been greatly aided by Francesco Cavazzani; in particular, he has prepared solutions, which appear on a web site associated to this book.

Prerequisites, notation and conventions

What you need to know before starting

When it comes to prerequisites, there are two distinct questions: what you should know to start reading this book; and what you should be prepared to learn along the way.

Of these, the second is by far the more important. In the course of developing and applying intersection theory, we introduce many key techniques of algebraic geometry, such as deformation theory, specialization methods, characteristic classes, Hilbert schemes, commutative and homological algebra and topological methods. That is not to say that you need to know these things going in. Just the opposite, in fact: Reading this book is an occasion to learn them.

So what do you need before starting?

(a) An undergraduate course in classical algebraic geometry or its equivalent, comprising the elementary theory of affine and projective varieties. *An invitation to algebraic geometry* (Smith et al. [2000]) contains almost everything required. Other books that cover this material include *Undergraduate algebraic geometry* (Reid [1988]), *Introduction to algebraic geometry* (Hassett [2007]), *Elementary algebraic geometry* (Hulek [2012]) and, at a somewhat more advanced level, *Algebraic geometry, I: Complex projective varieties* (Mumford [1976]), *Basic algebraic geometry, I* (Shafarevich [1994]) and *Algebraic geometry: a first course* (Harris [1995]). The last three include much more than we will use here.

(b) An acquaintance with the language of schemes. This would be amply covered by the first three chapters of *The geometry of schemes* (Eisenbud and Harris [2000]).

(c) An acquaintance with coherent sheaves and their cohomology. For this, *Faisceax algébriques cohérents* (Serre [1955]) remains an excellent source (it is written in the language of varieties, but applies nearly word-for-word to projective schemes over a field, the context in which this book is written).

In particular, *Algebraic geometry* (Hartshorne [1977]) contains much more than you need to know to get started.

Language

Throughout this book, a *scheme* X will be a separated scheme of finite type over an algebraically closed field $\Bbbk$ of characteristic 0. (We will occasionally point out the ways in which the characteristic p situation differs from that of characteristic 0, and how we might modify our statements and proofs in that setting.) In practice, all the schemes considered will be quasi-projective. We use the term *integral* to mean reduced and irreducible; by a *variety* we will mean an integral scheme. (The terms "curve" and "surface," however, refer to one-dimensional and two-dimensional schemes; in particular, they are not presumed to be integral.) A subvariety $Y \subset X$ will be presumed closed unless otherwise specified. If X is a variety we write $\Bbbk(X)$ for the field of rational functions on X. A *sheaf* on X will be a coherent sheaf unless otherwise noted.

By a *point* we mean a closed point. Recall that a *locally closed* subscheme U of a scheme X is a scheme that is an open subset of a closed subscheme of X. We use the term "subscheme" (without any modifier) to mean a closed subscheme, and similarly for "subvariety."

A consequence of the finite-type hypothesis is that any subscheme Y of X has a *primary decomposition*: locally, we can write the ideal of Y as an irredundant intersection of primary ideals with distinct associated primes. We can correspondingly write Y globally as an irredundant union of closed subschemes Y_i whose supports are distinct subvarieties of X. In this expression, the subschemes Y_i whose supports are maximal — corresponding to the minimal primes in the primary decomposition — are uniquely determined by Y; they are called the *irreducible components* of Y. The remaining subschemes are called *embedded components*; they are not determined by Y, though their supports are.

If a family of objects is parametrized by a scheme B, we will say that a "general" member of the family has a given property P if the set $U(P) \subset B$ of members of the family with that property contains an open dense subset of B. When we say that a "very general" member has this property we will mean that $U(P)$ contains the complement of a countable union of proper subvarieties of B.

By the *projectivization* of a vector space V, denoted $\mathbb{P}V$, we will mean the scheme $\mathrm{Proj}(\mathrm{Sym}\, V^*)$ (where by $\mathrm{Sym}\, V$ we mean the symmetric algebra of V); this is the space whose closed points correspond to one-dimensional subspaces of V. This is opposite to the usage in, for example, Grothendieck and Hartshorne, where the points of $\mathbb{P}V$ correspond to one-dimensional quotients of V (that is, their $\mathbb{P}V$ is our $\mathbb{P}V^*$), but is in agreement with Fulton.

If X and $Y \subset \mathbb{P}^n$ are subvarieties of projective space, we define the *join* of X and Y, denoted $\overline{X, Y}$, to be the closure of the union of lines meeting X and Y at distinct points. If $X = \Gamma \subset \mathbb{P}^n$ is a linear space, this is just the cone over Y with vertex Γ; if X and Y are both linear subspaces, this is simply their span.

https://doi.org/10.1017/CBO9781139062046.002 Published online by Cambridge University Press

There is a one-to-one correspondence between vector bundles on a scheme X and locally free sheaves on X. We will use the terms interchangeably, generally preferring "line bundle" and "vector bundle" to "invertible sheaf" and "locally free sheaf." When we speak of the *fiber* of a vector bundle $\mathcal{F}$ on X at a point $p \in X$, we will mean the (finite-dimensional) vector space $\mathcal{F} \otimes \kappa(p)$, where $\kappa(p)$ is the residue field at p.

By a *linear system*, or *linear series*, on a scheme X, we will mean a pair $\mathcal{D} = (\mathcal{L}, V)$, where $\mathcal{L}$ is a line bundle on X and $V \subset H^0(\mathcal{L})$ a vector space of sections. Associating to a section $\sigma \in V \subset H^0(\mathcal{L})$ its zero locus $V(\sigma)$, we can also think of a linear system as a family $\{V(\sigma) \mid \sigma \in V\}$ of divisors $D \subset X$ parametrized by the projective space $\mathbb{P}V$; in this setting, we will sometimes abuse notation slightly and write $D \in \mathcal{D}$. By the *dimension* of the linear series we mean the dimension of the projective space $\mathbb{P}V$ parametrizing it, that is, $\dim V - 1$. Specifically, a one-dimensional linear system is called a *pencil*, a two-dimensional system is called a *net* and a three-dimensional linear system is called a *web*.

We write $\mathcal{O}_{X,Y}$ for the local ring of X along Y, and, more generally, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules we write $\mathcal{F}_Y$ for the corresponding $\mathcal{O}_{X,Y}$-module.

We can identify the Zariski tangent space to the affine space $\mathbb{A}^n$ with $\mathbb{A}^n$ itself. If $X \subset \mathbb{A}^n$ is a subscheme, by the *affine tangent space* to X at a point p we will mean the affine linear subspace $p + T_p X \subset \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ is a subscheme, by the *projective tangent space* to X at $p \in X$, denoted $\mathbb{T}_p X \subset \mathbb{P}^n$, we will mean the closure in $\mathbb{P}^n$ of the affine tangent space to $X \cap \mathbb{A}^n$ for any open subset $\mathbb{A}^n \subset \mathbb{P}^n$ containing p. Concretely, if X is the zero locus of polynomials F_α (that is, $X = V(I) \subset \mathbb{P}^n$ is the subscheme defined by the ideal $I = (\{F_\alpha\}) \subset \Bbbk[Z_0, \dots, Z_n]$), the projective tangent space is the common zero locus of the linear forms

$$L_\alpha(Z) = \frac{\partial F_\alpha}{\partial Z_0}(p) Z_0 + \cdots + \frac{\partial F_\alpha}{\partial Z_n}(p) Z_n.$$

By a *one-parameter family* we will always mean a family $\mathcal{X} \to B$ with B smooth and one-dimensional (an open subset of a smooth curve, or spec of a DVR or power series ring in one variable), with marked point $0 \in B$. In this context, "with parameter t" means t is a local coordinate on the curve, or a generator of the maximal ideal of the DVR or power series ring.

Basic results on dimension and smoothness

There are a number of theorems in algebraic geometry that we will use repeatedly; we give the statements and references here. When X is a scheme, by the *dimension* of X we mean the Krull dimension, denoted $\dim X$. If X is an irreducible variety and $Y \subset X$ is a subvariety, then the *codimension* of Y in X, written $\operatorname{codim}_X Y$ (or simply $\operatorname{codim} Y$ when X is clear from context), is $\dim X - \dim Y$; more generally, if X is any scheme

https://doi.org/10.1017/CBO9781139062046.002 Published online by Cambridge University Press

and Y is a subvariety, then $\operatorname{codim}_X Y$ denotes the minimum of

$$\{\operatorname{codim}_{X'} Y \mid X' \text{ is a reduced irreducible component of } X\}.$$

More on dimension and codimension can be found in Eisenbud [1995].

We will often use the following basic result of commutative algebra:

Theorem 0.1 (Krull's principal ideal theorem)**.** *An ideal generated by n elements in a Noetherian ring has codimension $\leq n$.*

See Eisenbud [1995, Theorem 10.2] for a discussion and proof. We will also use the following important extension of the principal ideal theorem:

Theorem 0.2 (Generalized principal ideal theorem)**.** *If $f : Y \to X$ is a morphism of varieties and X is smooth, then, for any subvariety $A \subset X$,*

$$\operatorname{codim} f^{-1}A \leq \operatorname{codim} A.$$

In particular, if A, B are subvarieties of X, and C is an irreducible component of $A \cap B$, then $\operatorname{codim} C \leq \operatorname{codim} A + \operatorname{codim} B$.

The proof of this result can be reduced to the case of an intersection of two subvarieties, one of which is locally a complete intersection, by expressing the inverse image $f^{-1}A$ as an intersection with the graph $\Gamma_f \subset X \times Y$ of f. In this form it follows from Krull's theorem. The result holds in greater generality; see Serre [2000, Theorem V.3]. Smoothness is necessary for this (Example 2.22).

A module M is said to be of finite length if it has a finite maximal sequence of submodules. Such a sequence is called a *composition series*, and we will call the length of the sequence the *length* of the module. The following theorem shows this length is well-defined:

Theorem 0.3 (Jordan–Hölder theorem)**.** *A module M of finite length over a commutative local ring R has a maximal sequence of submodules $M \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$ Moreover, any two such maximal sequences are isomorphic; that is, they have the same length and composition factors (up to isomorphism).*

Theorem 0.4 (Chinese remainder theorem)**.** *A module of finite length over a commutative ring is the direct sum of its localizations at finitely many maximal ideals.*

For discussion and proof see Eisenbud [1995, Chapter 2], especially Theorem 2.13.

Theorem 0.5 (Bertini)**.** *If $\mathcal{D}$ is a linear system on a variety X in characteristic 0, the general member of $\mathcal{D}$ is smooth outside the base locus of $\mathcal{D}$ and the singular locus of X.*

Note that applying Bertini repeatedly, we see as well that if $D_1, \ldots, D_k$ are general members of the linear system $\mathcal{D}$ then the intersection $\bigcap D_i$ is smooth of dimension $\dim X - k$ away from the base locus of $\mathcal{D}$ and the singular locus of X.

This is the form in which we will usually apply Bertini. But there is another version that is equivalent in characteristic 0 but allows for an extension to positive characteristic:

Theorem 0.6 (Bertini)**.** *If $f : X \to \mathbb{P}^n$ is any generically separated morphism from a smooth, quasi-projective variety X to projective space, then the preimage $f^{-1}(H)$ of a general hyperplane $H \subset \mathbb{P}^n$ is smooth.*