Notes from the VaNTAGe Seminar

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1 2022-02-01: Computing isomorphism classes of abelian varieties over finite fields

We have a nice classification of abelian varieties $A/\mathbb{C}$ of dimension g. For such a variety, $A(\mathbb{C})$ is a torus: $T := \mathbb{C}^g/\Lambda$ where $\Lambda \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}$. In addition, T admits a non-degenerate *Riemann form* which in turn yields a polarization. The functor $A \mapsto A(\mathbb{C})$ therefore yields an equivalence of categories:

$$\{\text{abelian varieties}/\mathbb{C}\} \leftrightarrow \begin{array}{c} \mathbb{C}^g/\lambda \text{ with } \Lambda \simeq \mathbb{Z}^{2g} admitting \\ aRiemannform \end{array}.$$

In characteristic $p > 0$, such an equivalence *cannot* exit. There are supersingular elliptic curves with quaternionic endomorphism algebras. Nevertheless, over finite fields, we can obtain analogous results if we restrict ourselves to certain subcategories of abelian varieties.

Recall that an abelian variety $A/\mathbb{F}_q$, with q a power of a prime, comes equipped with a Frobenius endomorphism that induces an action

$$\mathrm{Frob}_A : T_\ell A \longrightarrow T_\ell A$$

for any $\ell \neq p$ where $T_\ell(A) = \operatorname{inj\,lim} A[\ell^n] \simeq \mathbb{Z}_\ell^{2g}$. This is a $\mathbb{F}_q$-linear map of vector spaces. The characteristic polynomial of this operator is denoted $h_A(x) := \mathrm{char}(\mathrm{Frob}_A)$, and it is a q-Weil polynomial and is isogeny invariant.

By Honda-Tate theory (that thing from the end of local fields) the association

$$\text{isogeny class of } A \;\leftrightarrow\; h_A(x)$$

is injective and allows us to enumerate all abelian varieties over $\mathbb{F}_q$ up to isogeny. We also have that $h_A(x)$ is squarefree if and only if $\mathrm{End}(A)$ is commutative.

Deligne came up with an equivalence of categories for *ordinary* abelian varieties over $\mathbb{F}_q$ analogous to that seen in abelian categories over $\mathbb{C}$.

We say that $A/\mathbb{F}_q$ is *ordinary* if half of the p-adic roots of h_A are units.

Theorem 1.1 (Deligne '69'). *Let $q = p^r$ with p prime. There is an equivalence of categories between the category of ordinary abelian varieties over $\mathbb{F}_q$ and the category of pairs (T,F) with $T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}$ and $T \xrightarrow{F} T$ which satisfy the following:*

- *$F \otimes \mathbb{Q}$ is semisimple*
- *the roots of $\mathrm{char}_{F\otimes\mathbb{Q}}(x)$ have absolute value $\sqrt{q}$ and half of them are p-adic units*
- *There exists $V : T \longrightarrow V$ such that $FV = VF = q$.*

This is equivalence is given by the functor $A \mapsto (T(A), F(A))$.

Let's consider the case that h_A is square free.

- Fix an *ordinary squarefree* q-Weil polynomial h :
- Put $K := \mathbb{Q}[x]/(h) = \mathbb{Q}[F]$, an étale algebra = product of number fields.
- Put $V = q/F$. Deligne's equivalence induces:

Theorem 1.2. $\{$*abelian varieties over* bF_q *in* $\mathcal{C}_h\}/\sim$ ...

Didn't get anything else :/