Part III Commutative Algebra

Solutions to Example Sheet I, 2021

1. (a) is by induction on n. The base case is $n = 2$. Note in any event $I_1I_2 \subseteq I_1 \cap I_2$ by the definition of ideal. If I_1, I_2 are coprime, then there exists $x \in I_1, y \in I_2$ such that $x + y = 1$. Then if $z \in I_1 \cap I_2$, $xz + zy = z$ lies in I_1I_2.

   Now assume the result is true for $I_1, \ldots, I_{n-1}$, and let $J = \prod_{i=1}^{n-1} I_i = \bigcap_{i=1}^{n-1} I_i$. Since $I_i + I_n = (1)$ we have for each $1 \le i \le n-1$ that $x_i + y_i = 1$ for some $x_i \in I_i$, $y_i \in I_n$, so

   $$\prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1}(1 - y_i) \equiv 1 \pmod{I_n}.$$

   Thus $I_n + J = (1)$, so the case $n = 2$ and the induction case allow us to conclude.

   (b): $\Rightarrow$: Wlog will show I_1, I_2 are corpime. By surjectivity, there exists $x \in A$ such that $\phi(x) = (1, 0, \ldots, 0)$, so $x \equiv 1 \pmod{I_1}$ and $x \equiv 0 \pmod{I_2}$. Thus

   $$1 = (1 - x) + x \in I_1 + I_2,$$

   showing I_1, I_2 are coprime.

   $\Leftarrow$: Wlog it is enough to show there exists an $x \in A$ with $\phi(x) = (1, 0, \ldots, 0)$. We have equations $u_i + v_i = 1$ with $u_i \in I_1, v_i \in I_i$ for each $i \ge 2$. Take $x = \prod_{i=2}^n v_i = \prod_{i=2}^n (1 - u_i) \equiv 1 \pmod{I_1}$. Also, $x \equiv 0 \pmod{I_i}$, $i \ge 1$, so $\phi(x)$ is as desired.

   (c) Immediate since $\ker \phi = \bigcap_{i=1}^n I_i$.

2. Tensoring

   $$0 \to I \to A \to A/I \to 0$$

   with M gives

   $$I \otimes_A M \to A \otimes_A M \to (A/I) \otimes_A M \to 0.$$

   Since there is an isomorphism $A \otimes_A M \to M$ given by $a \otimes m \mapsto am$, it is clear the image of $I \otimes_A M$ in M is IM, hence the result.

3. If m, n are coprime, then $m(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ as m is invertible in $\mathbb{Z}/n\mathbb{Z}$. Hence we get the result immediately from Q2.

4. (a) We may view k as $k[x, y]/(x, y)$, so that again by Q2 the tensor product is $k[u, v]/(x, y)k[u, v]$. Note $(x, y)k[u, v]$ is the ideal $(u, uv) = (u) \subseteq k[u, v]$, so that the tensor product is $k[u, v]/(u) \cong k[v]$.

   (b) I claim that the tensor product is isomorphic to $k[u, w]/(u^2 - w^2)$, which we may prove by demonstrating the universal property for tensor product of rings. In particular, we have a commutive diagram

   $$\begin{tikzcd} k[u,w]/(u^2-w^2) & k[v] \arrow[l, "\psi"'] \\ k[v] \arrow[u, "\varphi"] & k[x] \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

   with φ given by $v \mapsto u$ and ψ given by $v \mapsto w$. So suppose given a ring R with maps $\phi', \psi' : k[v] \to R$ such that $\varphi' \circ f = \psi' \circ g$. Then to obtain $h : k[u, w]/(u^2 - w^2) \to R$ with $h \circ \varphi = \varphi'$, $h \circ \psi = \psi'$, we must have $h(u) = \varphi'(v)$, $h(w) = \psi'(v)$. Thus h is uniquely determined if it exists. However, it exists because $\varphi'(v^2) = \psi' \circ f(x) = \psi' \circ g(x) = \psi'(v^2)$, so $h(u^2 - w^2) = 0$.

5. In fact we may write $A[x] \cong \bigoplus_{i=0}^\infty A$ as A-modules, i.e., $A[x]$ is a free A-module. Thus we will first show that arbitrary direct sums commute with tensor product, i.e., let $\{M_i\}_{i \in I}$ be a collection of A-modules with I an arbitrary index set. Let $M = \oplus_{i \in I} M_i$, so there are natural maps $\varphi_i : M_i \to M$. For an A-module N, we define a map

   $$f : \bigoplus_{i \in I}(M_i \otimes_A N) \to M \otimes_A N$$

   by $f((\alpha_i)_{i \in I}) = \sum_{i \in I}(\varphi_i \otimes \mathrm{id})(\alpha_i)$. Here $\varphi_i \otimes \mathrm{id}$ is the obvious map $M_i \otimes_A N \to M \otimes_A N$ induced by φ_i. Conversely, we define a map

   $$g : M \otimes_A N \to \bigoplus_{i \in I}(M_i \otimes_A N)$$

using the universal property of tensor product by defining a bilinear map with domain $M \times N$ given by $((m_i)_{i\in I}, n) \mapsto (m_i \otimes n)_{i\in I}$. These maps are easily seen to be inverse to each other by calculating $f \circ g$ and $g \circ f$ on generators, giving the isomorphism. [Alternatively, show the direct sum of tensor products satisfies the same universal property as $M \otimes_A N$ does. It is worth noting that tensor product *does not* distribute over infinite products.]

Now in fact any direct sum $\bigoplus_{i\in I} A$ is a flat A-module: given an injective map $f : M_1 \to M_2$, the above discussion shows that after tensoring we get $\bigoplus_{i\in I} M_1 \to \bigoplus_{i\in I} M_2$ which is just f on each component, hence injective.

6. (a) $\Rightarrow$: Suppose $S^{-1}M = 0$. Let $m_1, \dots, m_n$ be a generating set of M. Then since $m_i/1 = 0$ in $S^{-1}M$, there exists $s_1, \dots, s_n \in S$ such that $s_i m_i = 0$ for each i. Let $s = \prod_{i=1}^n s_i \in S$. Then $sm_i = 0$ for all i, so $sM = 0$. $\Leftarrow$: Suppose there exists an $s \in S$ such that $sM = 0$. Then for any $m \in M$, $m/1 = 0$ in $S^{-1}M$ since $sm = 0$. Thus $sM = 0$.

   (b) We first define a ring homomorphism $(ST)^{-1}A \to U^{-1}(S^{-1}A)$ using the universal property of localization. Note that if $s{\cdot}t \in ST$, its image in $U^{-1}(S^{-1}A)$, $(t/1){\cdot}s$, is invertible, since both $t/1$ and s are invertible. Thus the canonical ring map $A \to U^{-1}(S^{-1}A)$ factors through a well-defined ring map $(ST)^{-1}A \to U^{-1}(S^{-1}A)$. Note this map is given by $a/(s \cdot t) \mapsto (a/s)/(t/1)$.

   This map is clearly surjective, as $(a/s)/(t/1)$ is the image of $a/s \cdot t$. For injectivity, suppose $a/(s \cdot t) = 0$. Then there exists $s' \in S, t' \in T$ such that $as't' = 0$. Thus we may write $(a/s)/(t/1) = (as'/ss')/(t/1)$, which is zero since $(as')(t'/1) = 0$ in $S^{-1}A$.

   (c) There is a map $\varphi : S^{-1}B \to T^{-1}B$ given by $b/s \mapsto b/f(s)$. One needs to check this is well-defined, as we didn't state a universal property for localization of A-modules. But note that if $b/s = b'/s'$, there exists $s'' \in S$ with $(bs' - b's)s'' = 0$. But this is precisely the statement that $(bf(s') - b'f(s))f(s'') = 0$, by the definition of the A-module structure on B. Thus $b/f(s) = b'/f(s')$. This map is a homomorphism of $S^{-1}A$-modules. Indeed, it easily is seen to preserve sums, and to see it preserves products, note $\varphi((a/s) \cdot (b/s')) = \varphi((ab)/(ss')) = ab/f(ss') = (a/f(s)) \cdot (b/f(s'))$. Further, φ is clearly surjective. For injectivity, if $\varphi(b/s) = 0$, then there exists an $s' \in S$ such that $bf(s') = 0$. But then $b/s = 0$ since $bs' = 0$.

7. There is an obvious ring homomorphism $k[x, z] \to k[x, y, z]/(xy - z^2)$ defined by $x \mapsto x$, $z \mapsto z$, and we may then compose this map with the canonical map to $(k[x, y, z]/(xy - z^2))_x$. Since the image of $x \in k[x, z]$ is invertible, this gives a ring map
$$\varphi : k[x, z]_x \to (k[x, y, z]/(xy - z^2))_x$$
by the universal property. This map is surjective: note that in the ring on the right, we may write $y = z^2/x$. Thus any polynomial in x, y, z, x^{-1} can be written as a polynomial in x, z, x^{-1}, and hence is in the image of φ. For injectivity, note that $k[x, y, z]/(xy - z^2)$ is an integral domain as $xy - z^2$ is irreducible, hence $(xy - z^2)$ is prime. Thus if $f(x, z)/x^r \in k[x, z]_z$ has $\varphi(f(x, z)/x^r) = 0$, we in fact must have $f(x, z) = 0$ in the ring $k[x, y, z]/(xy-z^2)$, i.e., $f(x, z) \in (xy - z^2)$. Thus we may write $f = (xy - z^2)g(x, y, z)$ for some $g(x, y, z) \in k[x, y, z]$. Consider g as a polynomial in y, i.e., $g = g_0(x, z) + yg_1(x, z) + \cdots + y^n g_n(x, z)$, with $g_n \neq 0$. Then the highest degree term in y in the product is $xy^{n+1}g_n(x, z)$, which is non-zero as $g_n \neq 0$. But $f(x, z)$ contains no term with a y in it, a contradiction. Thus $f \notin (xy - z^2)$ and the map is injective.

8. If $m, m' \in T(M)$, then there exists $a, a' \in A \setminus \{0\}$ such that $am = 0 = a'm'$. But then $(aa')(m + m') = 0$, so $m + m' \in T(M)$. If $a'' \in A$, then $a(a''m) = 0$, so $a''m \in T(M)$. Thus $T(M)$ is a sub-module

   (a) Suppose $m \in M$ with image $\bar{m} \in M/T(M)$ lying in $T(M/T(M))$, i.e., there exists $a \in A \setminus \{0\}$ such that $a\bar{m} = 0$. Thus $am \in T(M)$, so there exists $a' \in A$ such that $a'am = 0$, but then $m \in T(M)$.

   (b) If $m \in T(M)$, there exists $a \in A \setminus \{0\}$ with $am = 0$. But then $0 = f(am) = af(m)$, so $f(m) \in T(N)$.

   (c) Injectivity on the left follows immdiately from injectivity of $M_1 \to M_2$. Let $f : M_1 \to M_2$, $g : M_2 \to M_3$ be the maps, and f_T, g_T the corresponding maps on torsion modules. Since $g \circ f = 0$, $g_T \circ f_T = 0$, and hence $\mathrm{im} f_T \subseteq \ker g_T$. Conversely, let $m_2 \in \ker g_T$. Then there exists $m_1 \in M_1$ with $f(m_1) = m_2$ by exactness of the original sequence. However, since f is injective and m_2 is torsion, we must have m_1 torsion. Hence $\ker g_T \subseteq \mathrm{im} f_T$ as desired.

   (d) Let $m/s \in T(S^{-1}M)$. Then there exists $a/s' \in S^{-1}A$ such that $(a/s')(m/s) = 0$, i.e., there exists $s'' \in S$ such that $as''m = 0$. So $m \in T(M)$, so $m/s \in S^{-1}T(M)$. Conversely, given any $m \in T(M)$, there exists an $a \in A \setminus \{0\}$ with $am = 0$. Thus $(a/1)(m/s) = a(m/s) = 0$, so whenever $m/s \in S^{-1}T(M)$, we have $m/s \in T(S^{-1}M)$, giving the desired equality.

9. Define a map $\varphi : F \to F$ by taking $e_i = (0, \dots, 1, \dots, 0)$ (the 1 in the i^{th} place) to x_i. Since $x_1, \dots, x_n$ generate F, this map is surjective, and we would like to show it is then necessarily injective. As injectivity and surjectivity are local properties, we may show this after localizing at a maximal ideal, and hence may assume that A is local. Now let $K = \ker \varphi$, giving an exact sequence
$$0 \to K \to F \to F \to 0.$$

We wish to show $K = 0$. Now tensor this exact sequence with $k = A/\mathfrak{m}$, where $\mathfrak{m}$ is the unique maximal ideal of A. Since k is a field, certainly $F \otimes_A k \to F \otimes_A k$ is an isomorphism. However, F is a flat A-module, hence by the hint, we have an exact sequence

$$0 \to K \otimes_A k \to F \otimes_A k \to F \otimes_A k \to 0,$$

so $K \otimes_A k = 0$. But $K \otimes_A k = K/\mathfrak{m}K$, and if we know that K is finitely generated, then by Nakayama's lemma, we see $K = 0$.

To see that K is finitely generated, we proceed as follows. Since the original map φ is surjective, we may find $f_1, \ldots, f_n \in F$ with $\varphi(f_i) = e_i$, where $e_1, \ldots, e_n$ is the standard basis as above. Define a map $\psi : F \to F$ by $\psi(e_i) = f_i$. Then $\varphi \circ \psi = id_F$, we see in fact we may write $F \cong \operatorname{im}(\psi \circ \varphi) \oplus \ker \varphi$ via the map

$$m \mapsto \big((\psi \circ \varphi)(m), m - (\psi \circ \varphi)(m)\big).$$

Indeed, note $\varphi(m - (\psi \circ \varphi)(m)) = \varphi(m) - \varphi(m) = 0$ so $m - (\psi \circ \varphi)(m) \in \ker \varphi$. The above map is clearly injective as m can be recovered from the image of m. It is surjective: given (m_1, m_2) in the target module, we have $m_1 = \psi \circ \varphi(m)$ for some $m \in F$, and $m_2 - (m - \psi \circ \varphi(m)) \in \ker \varphi$, so the image of $m + (m_2 - (m - \psi \circ \varphi(m)))$ is (m_1, m_2). [We have proved that $\ker \varphi$ is a direct summand of F, and in fact this argument works for any surjective map $\varphi : M \to F$ for any module M.]

So we see that $\ker \varphi$ is a quotient of F, hence finitely generated.

10. (a) We need (1) $\emptyset$ and $\operatorname{Spec} A$ are closed sets, but this is true as $\emptyset = V(A)$ and $\operatorname{Spec} A = V(0)$. (2) Closed sets are closed under finite union, which is true as $\bigcup_{i=1}^n V(I_i) = V(I_1 \cap \cdots \cap I_n)$. Indeed, if $\mathfrak{p} \supseteq I_i$ for some i, then $\mathfrak{p} \supseteq I_1 \cap \cdots \cap I_n$, while if $\mathfrak{p} \supseteq I_1 \cap \cdots \cap I_n$, $\mathfrak{p} \supseteq I_i$ for some i one of the easy exercises on the first day handout. (3) Closed sets are closed under arbitrary intersection, as $\bigcap_{i \in I} V(I_i) = V(\sum_{i \in I} I_i)$. Indeed, if $\mathfrak{p} \supseteq I_i$ for each $i \in I$, then $\mathfrak{p} \supseteq \sum_{i \in I} I_i$, and conversely.

    (b) Let $U = \operatorname{Spec} A \setminus V(I)$ be an open set. We need to show that given any $\mathfrak{p} \in U$, there is an $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. Note $D(f) \subset U$ if and only if $f \notin \mathfrak{q} \Rightarrow I \not\subseteq \mathfrak{q}$. Thus take $f \in I$ with $f \notin \mathfrak{p}$. Then $D(f) \subset U$ and $\mathfrak{p} \in D(f)$.

    (c) If $a \cdot a' \in \varphi^{-1}(\mathfrak{p})$, then $\varphi(aa') \in \mathfrak{p}$ so either $\varphi(a)$ or $\varphi(a')$ lie in $\mathfrak{p}$. But then either a or a' lies in $\varphi^{-1}(\mathfrak{p})$. Thus $\varphi^{-1}(\mathfrak{p})$ is prime.

    To show continuity, it is enough to show the inverse image of a closed set is closed. But for $I \subseteq A$,

$$\begin{aligned} (\varphi^*)^{-1}(V(I)) &= \{\mathfrak{p} \in \operatorname{Spec} B \,|\, I \subseteq \varphi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} B \,|\, \varphi(I) \subseteq \mathfrak{p}\} \\ &= V(I^e). \end{aligned}$$

11. *i)* $\Rightarrow$ *ii)*. If $\operatorname{Spec} A$ is disconnected, then we can write $\operatorname{Spec} A = V(I_1) \cup V(I_2)$ with $V(I_1) \cap V(I_2) = \emptyset$. Now $\emptyset = V(I_1) \cap V(I_2) = V(I_1 + I_2)$, so $I_1 + I_2 = A$ as $I_1 + I_2$ is thus not contained in any prime ideal. Also, $\operatorname{Spec} A = V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$, so $I_1 \cap I_2$ is contained in every prime ideal of A, i.e., $I_1 \cap I_2 \subseteq \sqrt{0}$. In particular, there exists $a \in I_1$, $b \in I_2$ such that $a + b = 1$, so that $V((a)) \cap V((b)) = \emptyset$. In any event $ab \in I_1 \cap I_2 \subseteq \sqrt{0}$. Thus there exists $n > 0$ such that $(ab)^n = 0$. Now $V((a^n)) = V(a)$, $V((b^n)) = V(b)$, so $V((a^n)) \cap V((b^n)) = \emptyset$. So there exists $e_1 \in (a^n)$, $e_2 \in (b^n)$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Thus $e_1 = e_1(e_1 + e_2) = e_1^2 + e_1 e_2 = e_1^2$, and similarly $e_2 = e_2^2$. This gives ii).

    ii) $\Rightarrow$ *iii)* Let $A_1 = A/(e_2)$, $A_2 = A/(e_1)$. Then there is an obvious ring homomorphism $A \to A_1 \times A_2$. This is an isomorphism by the Chinese Remainder Theorem. Indeed, the ideals $(e_1), (e_2)$ are coprime because $e_1 + e_2 = 1$, and because they are coprime, $(e_1) \cap (e_2) = (e_1 e_2) = (0)$.

    iii) $\Rightarrow$ *i)* Let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then $V((e_1)) \cup V((e_2)) = V(0) = \operatorname{Spec} A$ and $V((e_1)) \cap V((e_2)) = V(A) = \emptyset$, showing $\operatorname{Spec} A$ is disconnected.

12. *More about the spectrum.*

    (a) There is a one-to-one correspondence between primes of $S^{-1}A$ and primes of A disjoint from S, given by contraction and extension of prime ideals, so in particular the induced map φ^* is an inclusion. To show it is a homeomorphism we need to show that the closed sets of $\operatorname{Spec} S^{-1}A$ are precisely those of the form $(\varphi^*)^{-1}(V(I)) = V(I^e)$. But we showed in lecture that every ideal of $S^{-1}A$ is an extended ideal, so this is true.

    In case $S = \{1, f, \ldots, \}$, then $f^n \in \mathfrak{p} \Leftrightarrow f \in \mathfrak{p}$ for $\mathfrak{p} \subseteq A$ prime, so the set of prime ideals of $S^{-1}A$ is in one-to-one correspondence with the set of prime ideals of A not containing f, i.e., $D(f)$.

(b) Write $S^{-1}\varphi : S^{-1}A \to S^{-1}B$ for the induced map. Unwinding the definitions, to show the induced map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} S^{-1}A$ agrees with the restriction of φ^* to $S^{-1}Y$, it suffices to observe the following. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & S^{-1}A \arrow[d] \\ B \arrow[r] & S^{-1}B \end{tikzcd}$$

Now given $\mathfrak{p}$ a prime ideal of $S^{-1}B$, we may pull it back in two ways around the square. Of course, it doesn't matter which way you go because the square is commutative, hence the statement.

To see that $(\varphi^*)^{-1}(S^{-1}X) = S^{-1}Y$, let $\mathfrak{p} \in Y$ such that $\varphi^*(\mathfrak{p}) \in S^{-1}X$, i.e., $\varphi^{-1}(\mathfrak{p})$ is disjoint from S. But then $\mathfrak{p}$ is disjoint from $\varphi(S)$, so $\mathfrak{p} \in S^{-1}Y$. This shows that $(\varphi^*)^{-1}(S^{-1}X) \subseteq S^{-1}Y$, while the opposite inclusion is obvious.

(c) We note that the set of primes of A/I is in one-to-one correspondence with the primes of A containing I, so indeed $\operatorname{Spec} A/I$ can be identified with $V(I)$. Further, this identification is induced by the quotient map $A \to A/I$. Hence we may use the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[d] \\ B \arrow[r] & B/J \end{tikzcd}$$

as before to conclude.

(d) This is just combining parts (b) and (c), observing also that under the various natural identifications, the unique prime of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is identified with $\mathfrak{p} \in X$.