# Toric Geometry:
# Theorems and Definitions

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## Contents

**1 Dictionary** **2**

**2 What makes a toric variety?** **3**

2.1 Tori . . . 3

2.2 Toric Varieties . . . 3

2.3 Cones and Fans . . . 3

**3 Properties of Affine Toric Varieties** **5**

**4 Smoothness of Affine Toric Varieties** **6**

# 1 Dictionary

Toric geometry is concerned with the construction of varieties and schemes given by specifying semigroups and fans and other combinatorial objects. It is therefore useful to fix certain symbols.

- $N$: We define $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ and note that $N \cong \mathbb{Z}^n$.
- $M$ : We define $M$ to be the dual lattice of $N$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$.
- $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ : We define $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}^n$.

# 2 What makes a toric variety?

## 2.1 Tori

## 2.2 Toric Varieties

## 2.3 Cones and Fans

Throughout this section, let $T \cong (\mathbb{C}^*)^n$ and $N = \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$. Note that $N$ is the collection of 1-parameter subgroups of $T$, or the set of cocharacters if you prefer that terminology. In addition, every variety is an integral separated scheme of finite type over $\operatorname{Spec}\mathbb{C}$ unless otherwise specified.

**Definition 2.1.** A *rational polyhedral cone* $\sigma$ in $N$ is a set $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ given by the positive span of some finite subset of $N_{\mathbb{R}}$, i.e. a set

$$\sigma = \mathrm{cone}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} c_i v_i \;\middle|\; c_i \in \mathbb{R}_{\geq 0} \right\}.$$

By rescaling the cone basis set, we may assume $v_i \in N$ for each $1 \leq i \leq k$, and from now on will do so.

**Definition 2.2.** Let $\sigma = \mathrm{cone}\{v_1, \dots, v_k\}$ be a rational polyhedral cone. The *span* of $\sigma$ is the smallest vector subspace $V$ containing $\sigma$. We have that

$$V = \sigma + (-\sigma) = \{v_1, \dots, v_k\} = \{\sigma\}.$$

The *dimension* of $\sigma$ is the dimension of the span of $\sigma$. We say that $\sigma$ is *full-dimensional* if $\dim \sigma = \dim N_{\mathbb{R}} = n$.

**Definition 2.3.** A rational polyhedral cone is said to be *strictly convex* if it doesn't contain a line, i.e. if it doesn't contain a one dimensional affine subspace of $N_{\mathbb{R}}$.

Unless otherwise specified, by "cone" we mean "strictly convex rational polyhedral cone".

**Definition 2.4.** Given a cone $\sigma \subseteq N_{\mathbb{R}}$, the *dual cone* $\sigma\vee \subseteq M_{\mathbb{R}}$ is defined

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0, \ \forall v \in \sigma\}.$$

The pairing $\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \longrightarrow \mathbb{R}$ is simply the evaluation map $\langle m, u \rangle = m(u)$.

We further define the double dual $(\sigma^{\vee})^{\vee}$ by

$$(\sigma^{\vee})^{\vee} = \left\{ v \in N_{\mathbb{R}} \;\middle|\; \langle m, v \rangle \geq 0, \ \forall m \in \sigma^{\vee} \right\}$$

The following are fundamental facts regarding $\sigma$ and $\sigma^{\vee}$.

**Proposition 2.5.** Let $\sigma$ be a cone in $N$ and $\sigma^{\vee}$ be its dual.

(a) $\sigma^{\vee}$ is a rational polyhedral cone in $M$ (not necessarily strictly convex)

(b) $(\sigma^{\vee})^{\vee} = \sigma$

(c) $\sigma$ is full-dimensional if and only if $\sigma^{\vee}$ is strictly convex

**Definition 2.6.** A *fan* $\Sigma$ in $N$ is a collection of cones in $N$ such that

(i) if $\sigma \in \Sigma$ then every face of $\sigma$ belongs to $\Sigma$

(ii) if $\sigma_1, \sigma_2 \in \Sigma$ then $\sigma_1 \cap \sigma_2$ is a face of both $\sigma_1$ and $\sigma_2$.

We wish to construct varieties from cones and fans. Starting with a cone $\sigma$ in $N$, we will associate to it an affine variety $X_\sigma = \operatorname{Spec} R_\sigma$. Given a fan $\Sigma$, we will construct a variety $X_\Sigma$ by gluing together $X_{\sigma_1}$ and $X_{\sigma_2}$ along $X_{\sigma_1 \cap \sigma_2}$. We will first build affine toric varieties, and for that, we'll need affine semigroups.

**Definition 2.7.** A *semigroup* is a set $S$ together with an associative binary operation and an identity element. This is what some (most) people seem to call a monoid – it's a category with a single point. To be an *affine semigroup*, $S$ must additionally satisfy:

- $S$ is commutative. We will write the binary operation as $+$ and the identity element as 0 to reflect this. Note that this means a finite set $A \subseteq S$ therefore generates
$$\mathbb{N}A = \left\{ \sum_{m \in A} a_m m \,\middle|\, a_m \in \mathbb{N} \right\}.$$
- $S$ is finitely generated, i.e. there is a finite set $A \subseteq S$ such that $\mathbb{N}A = S$.
- The semigroup can be embedded in a lattice $M$.

We focus first on building a variety $X_\sigma$ from a cone $\sigma$ in $N$. Here is our construction/definition.

**Construction 2.8.** Given a cone $\sigma \subseteq N_\mathbb{R}$ and its dual cone $\sigma^\vee \subseteq M_\mathbb{R}$, we define
$$S_\sigma := \sigma \cap M \tag{1}$$
to be the semigroup associated to $\sigma$. We then consider the group algebra over $\mathbb{C}$ with basis $S_\sigma$:
$$\mathbb{C}[S_\sigma] = \left\{ \sum_{i=1}^{r} c_i \cdot z^{m_i} \,\middle|\, c_i \in \mathbb{C},\ m_i \in S_\sigma \subseteq M \right\}. \tag{2}$$
The addition on $\mathbb{C}[S_\sigma]$ is formal. The multiplication is defined $z^{m_i} \cdot z^{m_j} = z^{m_i + m_j}$ and is extended by distribution. E.g. we have that $\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[t_1, ..., t_n]$ and $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[t_1^{\pm}, ..., t_n^{\pm}]$.

Finally, we define
$$X_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma] \tag{3}$$
to be the affine toric variety associated to $\sigma$. Note that because we will eventually build toric varieties from fans whose affine pieces are given by pieces of the above form, we sometimes denote $X_\sigma$ by $U_\sigma$ instead.

It is still left to show that $X_\sigma$ constructed in this way is in fact a toric variety.

**Proposition 2.9.** (Cox-Little-Scheck) If $\sigma$, $S_\sigma$, and $X_\sigma$ are as in Construction (2.8) then $X_\sigma$ is an affine toric variety.

***Proof.*** See page 31 of Cox-Little Scheck. Fill it in later. □

One might ask, "Why do we define $S_\sigma$ as a subset of the dual lattice $M$ rather than the lattice $N$? Surely we could take $S_\sigma = \sigma \cap N$ and get an equally reasonable result."

**COME BACK TO THE ABOVE QUESTION. CONSIDER REVERSE CONSTRUCTION – GIVEN AFFINE TORIC VARIETY $T \subseteq X$ CONSTRUCT A SEMIGROUP (HOWEVER ONE DOESN'T ALWAYS GET A CONE)**

# 3 Properties of Affine Toric Varieties

**Definition 3.1.** An affine semigroup $S \subseteq M$ is said to be *saturated* if for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in S$ implies $m \in S$.

An affine semigroup $S$ is saturated if and only if $S = S_\sigma = \sigma^\vee \cap M$ for some strongly convex rational polyhedral cone $\sigma \subseteq N$. In terms of toric varieties, this means the following:

**Proposition 3.2.** Let $V$ be an affine toric variety with torus $T_N$. Then the following are equivalent:

(i) $V$ is normal (for us this means $V \cong \operatorname{Spec} R$ for some integrally closed domain $R$.)

(ii) $V \cong \operatorname{Spec}(\mathbb{C}[S])$, where $S \subseteq M$ is some saturated affine semigroup

(iii) $V \cong \operatorname{Spec} \mathbb{C}[S_\sigma] = X_\sigma$, where $S_\sigma = \sigma^\vee \cap M$ and $\sigma \subseteq N_\mathbb{R}$ is a strongly convex rational polyhedral cone.

*Proof:* $\square$

Notice that embedded in the equivalence $(b) \iff (c)$ from Theorem (3.2) is the fact that a semigroup is affine if and only if it is isomorphic to $S_\sigma$ for some strongly convex rational polyhedral cone $\sigma$.

# 4 Smoothness of Affine Toric Varieties

The main goal of this section is a classification of smooth affine toric varieties associated to cones $\sigma$. This is Theorem (4.4). Before we proceed, however, we prove several useful lemmas. Throughout this section $X_\sigma$ is an affine toric variety associated to a cone $\sigma \subseteq M_{\mathbb{R}}$.

**Lemma 4.1.** Let $\sigma = \mathrm{cone}(v_1, ..., v_k) \subseteq N_{\mathbb{R}}$ be a cone. Suppose $\{v_1, ..., v_k\}$ forms some part of a $\mathbb{Z}$-basis for $N$. Then $X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma \leq n$.

***Proof.*** Choose a basis $e_1, ..., e_n$ for $N$ such that $v_i = e_i$ for $1 \leq i \leq k$ (that $\{v_1, ..., v_k\}$ is a $\mathbb{Z}$-basis for $N$ exactly makes this possible). This implies that $S_\sigma = \sigma^\vee \cap M$ is generated by

$$e_1^*, ..., e_k^*, \pm e_{k+1}^*, ..., \pm e_n^* \in M.$$

To see this, it helps to note the $e_i^*$ for $k+1 \leq i \leq n$ are exactly the basis vectors of $M$ which are zero on $\sigma$. This means

$$\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, ..., t_k, t_{k+1}^{\pm}, ..., t_n^{*}] = \mathbb{C}[t_1, ..., t_n] \otimes_{\mathbb{C}} \mathbb{C}[t_{k+1}^{\pm}, ..., t_n^{\pm}].$$

$\square$

**Lemma 4.2.** There exists a bijection correspondence

$$\begin{pmatrix} \text{closed points} \\ \text{of } X_\sigma \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{semigroup} \\ \text{homomorphisms} \\ S_\sigma \to \mathbb{C} \end{pmatrix}.$$

***Proof.*** We have the following one-to-one correspondences:

$$\left\{ \begin{matrix} \text{closed points} \\ \text{in X} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{maps of schemes} \\ \text{over } \mathbb{C} \\ \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[S_\sigma] \end{matrix} \right\} \leftrightarrow^{(*)} \left\{ \begin{matrix} \text{semigroup morphisms} \\ S_\sigma \to \mathbb{C} \end{matrix} \right\}.$$

Only $(*)$ is new. A morphism of schemes $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}[S_\sigma]$ over $\mathbb{C}$ yields a morphism of $\mathbb{C}$-algebras $\varphi$ : $\operatorname{Spec}\mathbb{C}[S_\sigma] \to \operatorname{Spec}\mathbb{C}$, which in turn determines an affine semigroup homomorphism $S_\sigma \to \mathbb{C}$ (with $\mathbb{C}$ considered to be an affine semigroup under multiplication) since $\varphi(z^{t+s}) = \varphi(z^t)\varphi(z^s)$. Likewise, any homomorphism of affine semigroups $\psi : S_\sigma \to \mathbb{C}$ can be extended to an algebra homomorphism $\varphi : \mathbb{C}[S_\sigma] \to \mathbb{C}$ by making a choice for the field automorphism $\varphi|_{\mathbb{C}}$. Normally, we would have two choices for $\varphi|_{\mathbb{C}}$, but in order for this to be compatible with the structure maps on $\operatorname{Spec}\mathbb{C}$ and $\operatorname{Spec}\mathbb{C}[S_\sigma]$, we have only *one* choice. Therefore $\varphi$ is uniquely determined by the image of $S_\sigma$ in $\mathbb{C}$, and since it is a $\mathbb{C}$-algebra homomorphism it corresponds to a unique map of schemes over $\mathbb{C}$. $\square$

**Definition 4.3.** Define $x_\sigma \in X_\sigma$ to be the point corresponding to the semigroup map

$$S_\sigma \xrightarrow{x_\sigma} \mathbb{C},\ m \mapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases},$$

where

$$\sigma^\perp = \{m \in M_{\mathbb{R}} \mid \langle u, m \rangle = 0,\ \forall u \in \sigma\}.$$

We now proceed to Theorem (4.4).

**Theorem 4.4.** *Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone and $X_\sigma$ be the associated affine toric variety. The following are equivalent:*

*(i)* *$X_\sigma$ is smooth*

*(ii)* *$\sigma$ is generated by a subset of a $\mathbb{Z}$-basis for $N$*

*(iii)* *$X_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where $k = \dim \sigma$.*

***4.4.*** Lemma (4.1) gives us $(ii) \implies (iii)$. The fibre product of smooth schemes with smooth structure maps is again smooth, so $(iii) \implies (i)$ is clear. It is only left to prove $(i) \implies (ii)$. $\square$