Chapter 4

Grassmannians in general

Keynote Questions

(a) If $V_1, \dots, V_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four? (Answer on page 150.)

(b) Let $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ be a twisted cubic curve contained in the Grassmannian $\mathbb{G}(1,3) \subset \mathbb{P}^5$ of lines in $\mathbb{P}^3$, and let

$$S = \bigcup_{[\Lambda] \in C} \Lambda \subset \mathbb{P}^3$$

be the surface swept out by the lines corresponding to points of C. What is the degree of S? How can we describe the geometry of S? (Answer on page 145.)

(c) If $Q_1, Q_2 \subset \mathbb{P}^4$ are general quadric hypersurfaces and $S = Q_1 \cap Q_2$ their surface of intersection, how many lines does S contain? More generally, if Q_1 and Q_2 are general quadric hypersurfaces in $\mathbb{P}^{2n}$ and $X = Q_1 \cap Q_2$, how many $(n-1)$-planes does X contain? (Answer on page 157.)

(d) What is the degree of the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$, embedded in projective space via the Plücker embedding? (Answer on page 150.)

In this chapter, we will extend the ideas developed in Chapter 3 by introducing Schubert cycles and classes on $G(k,n)$, the Grassmannian of k-dimensional subspaces in an n-dimensional vector space V, and analyzing their intersections, a subject that goes by the name of the *Schubert calculus*. Of course we may also consider $G(k,n)$ in its projective guise as $\mathbb{G}(k-1,n-1)$, the Grassmannian of projective $(k-1)$-planes in $\mathbb{P}^{n-1}$, and in places where projective geometry is more natural (such as Sections 4.2.3 and 4.4) we will switch to the projective notation.

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4.1 Schubert cells and Schubert cycles

Let $G = G(k, V)$ be the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V. Generalizing the example of $\mathbb{G}(1,3) = G(2,4)$, the center of our study will be a collection of subvarieties of $G(k,n)$ called *Schubert varieties* or *Schubert cycles*, defined in terms of a chosen *complete flag* $\mathcal{V}$ in V, that is, a nested sequence of subspaces

$$0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

with $\dim V_i = i$.

The Schubert cycles are indexed, in a way that will be motivated below, by sequences $a = (a_1, \ldots, a_k)$ of integers with

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0.$$

(Such sequences are often described by *Young diagrams* — see Section 4.5.) For such a sequence a, we define the *Schubert cycle* $\Sigma_a(\mathcal{V}) \subset G$ to be the closed subset

$$\Sigma_a(\mathcal{V}) = \{\Lambda \in G \mid \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \text{ for all } i\}.$$

Theorem 1.7 shows that the class $[\Sigma_a(\mathcal{V})] \in A(G)$ does not depend on the choice of flag, since any two flags differ by an element of GL_n. In general, when dealing with a property independent of the choice of $\mathcal{V}$, we will shorten the name to Σ_a, and we define

$$\sigma_a := [\Sigma_a] \in A(G);$$

these, naturally, are called *Schubert classes*. We shall see (in Corollary 4.7) that $A(G)$ is a free abelian group and that the classes σ_a form a basis.

To simplify notation, we generally suppress trailing zeros in the indices, writing $\Sigma_{a_1,\ldots,a_s}$ in place of $\Sigma_{a_1,\ldots,a_s,0,\ldots,0}$. Also, we use the shorthand Σ_{p^r} to denote $\Sigma_{p,\ldots,p}$, with r indices equal to p.

To elucidate the rather awkward-looking definition of $\Sigma_a(\mathcal{V})$, suppose that $\Lambda \subset V$ is a k-plane. If Λ is general, then $V_i \cap \Lambda = 0$ for $i \leq n-k$, while $\dim V_{n-k+i} \cap \Lambda = i$ for $i > n - k$. Thus we may describe Σ_a as the set of Λ such that $\dim V_j \cap \Lambda \geq i$ occurs for a value of j that is a_i steps sooner than expected.

Equivalently, we may consider the sequence of subspaces of Λ

$$0 \subset (V_1 \cap \Lambda) \subset (V_2 \cap \Lambda) \subset \cdots \subset (V_{n-1} \cap \Lambda) \subset (V_n \cap \Lambda) = \Lambda. \tag{4.1}$$

Each subspace in this sequence is either equal to the one before it or of dimension one greater, and the latter phenomenon occurs exactly k times. The Schubert cycle $\Sigma_a(\mathcal{V})$ is the locus of planes Λ for which "the i-th jump in the sequence (4.1) occurs at least a_i steps early."

Here are two common special cases to bear in mind:

- The cycle of k-subspaces Λ meeting a given space of dimension l nontrivially is the Schubert cycle

$$\Sigma_{n-k+1-l}(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_l \neq 0\}.$$

In particular, the Schubert cycle of k-dimensional subspaces meeting a given $(n-k)$-dimensional subspace nontrivially is

$$\Sigma_1(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_{n-k} \neq 0\}.$$

This is a hyperplane section of G in the Plücker embedding. (But not every hyperplane section of G is of this form. This follows by a dimension count: the family of $(n-k)$-planes — the Grassmannian $G(n-k,n)$ — has dimension $k(n-k)$, whereas the space of linear forms in the Plücker coordinates has dimension $\binom{n}{k} - 1$.)

- The sub-Grassmannian of k-subspaces contained in a given l-subspace is the Schubert cycle

$$\Sigma_{(n-l)^k}(\mathcal{V}) = \{\Lambda \mid \Lambda \subset V_l\}.$$

Similarly, the sub-Grassmannian of planes containing a given r-plane is the Schubert cycle

$$\Sigma_{(n-k)^r}(\mathcal{V}) = \{\Lambda \mid V_r \subset \Lambda\}.$$

The cycles Σ_i, defined for $0 \leq i \leq n-k$, and the cycles Σ_{1^i}, defined for $0 \leq i \leq k$, are called *special* Schubert cycles. As we shall see in Section 5.8, their classes are intimately connected with the tautological sub and quotient bundles on G, and each of the corresponding sequences of classes forms a minimal generating set for the algebra $A(G)$.

Our indexing of the Schubert cycles is by no means the only one in use, but it has several good properties:

- It reflects the partial order of the Schubert cycles defined with respect to a given flag $\mathcal{V}$ by inclusion: if we order the indices termwise, that is, $(a_1, \ldots, a_k) \leq (a'_1, \ldots, a'_k)$ if and only if $a_i \leq a'_i$ for $1 \leq i \leq k$ (writing $a < a'$ when $a \leq a'$ and $a \neq a'$; that is, $a_i < a'_i$ for some i), then

$$\Sigma_a \subset \Sigma_b \iff a \geq b.$$

This follows immediately from the definition.

- It makes the codimension of a Schubert cycle apparent: By Theorem 4.1 below,

$$\operatorname{codim}(\Sigma_a \subset G) = \sum a_i,$$

so that $|a| := \sum a_i$ is the degree of σ_a in $A(G)$.

- It is preserved under pullback via the natural inclusions

$$i : G(k,n) \hookrightarrow G(k+1,n+1)$$

(whose image is the set of $(k+1)$-subspaces containing V_1) and

$$j : G(k,n) \hookrightarrow G(k,n+1)$$

(whose image is the set of k-subspaces contained in V_n); that is,

$$i^*(\sigma_a) = \sigma_a \quad \text{and} \quad j^*(\sigma_a) = \sigma_a.$$

Here we adopt the convention that when $a_1 > n-k$, or when $a_{k+1} > 0$, we take $\sigma_a = 0$ as a class in $A(G(k,n))$. (This convention is consistent with the restriction to sub-Grassmannians. For example, $\Sigma_{n-k+1} \subset G(k,n+1)$ is the subset of the k-planes containing a fixed general one-dimensional subspace, and thus the intersection of Σ_{n-k+1} with the $G(k,n)$ of subspaces contained in a fixed codimension-1 subspace is empty, so that $j^*\sigma_{n-k+1} = 0 \in A(G(k,n))$.) It follows that if we establish a formula

$$\sigma_a\sigma_b = \sum \gamma_{a,b;c}\,\sigma_c$$

in the Chow ring of $G(k,n)$, the same formula holds true in all $G(k',n')$ with $k' \le k$ and $n'-k' \le n-k$. Whenever it happens that i^* or j^* is an isomorphism on $A^{|a|+|b|}$, the formula will also hold in $A(G(k,n+1))$ or $A(G(k+1,n+1))$, respectively. Conditions for this are given in Exercise 4.32.

There is a natural isomorphism $G(k,V) \cong G(n-k,V^*)$ obtained by associating to a k-dimensional subspace $\Lambda \subset V$ the $(n-k)$-dimensional subspace $\Lambda^\perp \subset V^*$ consisting of all those linear functionals on V that annihilate Λ. This duality carries each Schubert cycle to another Schubert cycle. For example, one checks immediately that $\Sigma_i(W)$, which is the set of k-planes Λ meeting a fixed $(n-k+1-i)$-plane W nontrivially, is carried into the Schubert cycle Σ_{1^i} of $(n-k)$-planes Λ' such that $\dim(\Lambda' \cap W^\perp) \ge i$, that is, such that $\Lambda' + W^\perp \subsetneq V$. See Section 4.5 for the general case.

4.1.1 Schubert classes and Chern classes

Schubert classes provide fundamental invariants of vector bundles. Recall from Theorem 3.4 that, if $\mathcal{E}$ is a vector bundle of rank r generated by a space $W \cong \mathbb{k}^n$ of global sections on a variety X, then there is a map $X \to G(n-r,W)$ sending a point $x \in X$ to the subspace in W consisting of the sections vanishing at x. The pullbacks of the Schubert classes σ_a give a fundamental set of invariants of $\mathcal{E}$ called the *Chern classes* of $\mathcal{E}$ — see Section 5.6.2. We will see that every Schubert class is a polynomial in the special Schubert classes (see Section 4.7).

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4.1.2 The affine stratification by Schubert cells

As in the case of $\mathbb{G}(1,3) = G(2,4)$, the Grassmannian $G(k,n)$ has an affine stratification. To see this, set

$$\Sigma_a^\circ = \Sigma_a \setminus \left(\bigcup_{b>a} \Sigma_b \right).$$

The Σ_a° are called *Schubert cells*.

Theorem 4.1. *The locally closed subset $\Sigma_a^\circ \subset G$ is isomorphic to the affine space $\mathbb{A}^{k(n-k)-|a|}$; in particular Σ_a° is smooth and irreducible, and the Schubert variety Σ_a is irreducible and of codimension $|a|$ in $G(k,n)$. The tangent space to Σ_a° at a point $[\Lambda]$ is the subspace of $T_{[\Lambda]}G = \mathrm{Hom}(\Lambda, V/\Lambda)$ consisting of those elements φ that send*

$$V_{n-k+i-a_i} \cap \Lambda \ \subset \ \Lambda$$

into

$$\frac{V_{n-k+i-a_i} + \Lambda}{\Lambda} \ \subset \ V/\Lambda$$

for $i = 1, \dots, k$.

Proof: Choose a basis $(e_1, \dots, e_n)$ for V so that

$$V_i = \langle e_1, \dots, e_i \rangle.$$

Suppose $[\Lambda] \in \Sigma_a$, and consider the sequence (4.1) of subspaces of Λ. By definition, the first nonzero subspace in the sequence will be $V_{n-k+1-a_1} \cap \Lambda$, the first of dimension 2 will be $V_{n-k+2-a_2} \cap \Lambda$, and so on. Thus we may choose a basis $(v_1, \dots, v_k)$ for Λ with $v_1 \in V_{n-k+1-a_1}$, $v_2 \in V_{n-k+2-a_2}$, and so on. In terms of this basis, and the basis $(e_1, \dots, e_n)$ for V, the matrix representative of Λ has the form

$$\begin{pmatrix}
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & 0
\end{pmatrix}.$$

(This particular matrix corresponds to the case $k = 4$, $n = 9$ and $a = (3,2,2,1)$.) If Λ were general in G, and we chose a basis for Λ in this way, the corresponding matrix would look like

$$\begin{pmatrix}
* & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 \\
* & * & * & * & * & * & * & * & *
\end{pmatrix}.$$

Thus the Schubert index a_i is the number of “extra zeros” in row i.

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Now suppose that $\Lambda \in \Sigma_a^\circ$; that is, $\Lambda \in \Sigma_a$ but not in any of the smaller varieties $\Sigma_{a'}$ for $a' > a$. In this case $v_i \notin V_{n-k+i-a_i-1}$, so, for each i, the coefficient of $e_{n-k+i-a_i}$ in the expression of v_i as a linear combination of the e_α is nonzero, and this condition characterizes elements of Σ_a° among elements of Σ_a. (It follows in particular that Σ_a is the closure of Σ_a°.) Given that the coefficient of $e_{n-k+i-a_i}$ in v_i is nonzero, we can multiply v_i by a scalar to make the coefficient 1, obtaining a basis for Λ represented by the rows of a matrix of the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & 1 & 0 \end{pmatrix},$$

where the 1 in the i-th row appears in the $(n-k+i-a_i)$-th column for $i = 1, \dots, k$.

Finally, we can subtract a linear combination of $v_1, \dots, v_{i-1}$ from v_i to kill the coefficients of $e_{n-k+j-a_j}$ in the expression of v_i as a linear combination of the e_α for $j < i$, to arrive at a basis of Λ given by the row vectors of the matrix

$$A = \begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 1 & 0 \end{pmatrix}.$$

Setting $b = \{n-k+1-a_1, \dots, n-a_k\}$, we may describe this by saying that the b-th submatrix A_b of A (that is, the submatrix involving columns from b) is the identity matrix. We claim that Λ has a unique basis of this form. Indeed, any other basis of Λ has a matrix obtained from this one by left multiplication by a unique invertible $k \times k$ matrix g, and thus has submatrix $A_b = g$.

It follows that Σ_a° is contained in the open subset $U \subset G$ consisting of planes Λ complementary to the span of the $n-k$ basis vectors whose indices are not in b. By the same argument, any element of $U = \Sigma_0^\circ$ has a unique basis given by the rows of a $k \times n$ matrix with submatrix $A_b = I$, that is, of the form

$$A = \begin{pmatrix} * & * & 1 & * & 0 & 0 & * & 0 & * \\ * & * & 0 & * & 1 & 0 & * & 0 & * \\ * & * & 0 & * & 0 & 1 & * & 0 & * \\ * & * & 0 & * & 0 & 0 & * & 1 & * \end{pmatrix}.$$

Thus Σ_a° is a coordinate subspace of $U \cong \mathbb{A}^{k(n-k)}$ defined by the vanishing of $|a|$ coordinates, and it follows that Σ_a° is smooth and irreducible, and of codimension $|a|$, as claimed. Since Σ_a is the closure of Σ_a°, it is also irreducible and of codimension $|a|$ in $G(k,n)$ (but it may be singular; for example, one sees from the Plücker relation (Example 3.1) that $\Sigma_1 \subset G(2,4)$ is the cone in $\mathbb{P}^4$ over a smooth quadric in $\mathbb{P}^3$).

The statement about tangent spaces follows from the explicit coordinate description of Σ_a above. We identify the open set $U \cong \mathbb{A}^{k(n-k)}$ with the set of $k \times n$ matrices having an identity matrix in positions from b. Since the tangent space to an affine space may be identified with the corresponding vector space, the tangent space $\mathrm{Hom}(\Lambda, V/\Lambda)$ to $G(k,n)$ at Λ is given by the set of matrices in the positions from b' complementary to those in b, or more properly by the transposes of these matrices. Given such a tangent vector, we may complete it uniquely to a $k \times n$ matrix with submatrix $A_b = I$, and this (or rather its transpose) corresponds to the lifting $\Lambda \to V = V/\Lambda \oplus \Lambda$ inducing the identity map $\Lambda \to \Lambda$. Thus the set of tangent directions at Λ to the affine subspace Σ_a° is identified with the set of matrices in that subspace, and this corresponds precisely to the set of maps in $\mathrm{Hom}(\Lambda, V/\Lambda)$ whose lifting as above sends $V_{n-k+i-a_i}$ into $V_{n-k+i-a_i} + \Lambda$, as claimed. □

From Proposition 1.17 we see that $A(G)$ is at least generated as an abelian group by the classes σ_a, and the existence of the degree homomorphism $\deg : A^{k(n-k)} \to \mathbb{Z}$ that counts points shows that $A^{k(n-k)}(G)$ is actually free on the class of a point, which is the generator $\sigma_{(n-k)^k}$. In Corollary 4.7 we will prove that all the $A^i(G)$ are free, by intersection theory and results on transversality.

The description of the tangent spaces in Theorem 4.1 can be used to prove this transversality. Here is an example:

Corollary 4.2. *Let $G = G(k,n)$. Then*

$$(\sigma_{n-k})^k = (\sigma_{1^k})^{n-k} = \sigma_{(n-k)^k} \in A^{k(n-k)}(G);$$

that is, $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{n-k}$ are both equal to the class of a point in the Chow ring of G.

Proof: We know that $A_0(G)$ is generated by the class $\sigma_{(n-k)^k}$ of a point, so it suffices to show that both $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{(n-k)}$ are of degree 1.

We regard G as the variety of k-dimensional subspaces Λ of the n-dimensional vector space V. If $H \subset V$ is a codimension-1 subspace, then

$$\Sigma_{1^k}(H) = \{\Lambda \subset V \mid \Lambda \subset H\},$$

and the tangent space to $\Sigma_{1^k}(H)$ at the point corresponding to Λ is

$$T_{[\Lambda]}(\Sigma_{1^k}(H)) = \{\varphi \in \mathrm{Hom}(\Lambda, V/\Lambda)\} \mid \varphi(\Lambda) \subset H\}.$$

If $H_1, \ldots, H_{n-k}$ are general codimension-1 subspaces, then there is a unique k-plane Λ in $\bigcap_{i=1}^k \Sigma_{1^k}(H_i)$, namely, the intersection $\Lambda = \bigcap_{i=1}^k H_i$. Further, the tangent spaces intersect only in the zero homomorphism, so the intersection is transverse. This proves that $(\sigma_{1^k})^{(n-k)}$ is the class of a point.

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To prove the corresponding statement for $(\sigma_{n-k})^k$, we can make an analogous argument, or we can simply use duality: the isomorphism $G(k,n) \cong G(n-k,n)$ introduced above carries σ_{1^k} to σ_k, as we have already remarked, and preserves the degree of 0-cycles. □

4.1.3 Equations of the Schubert cycles

It is a remarkable fact that under the Plücker embedding $G = G(k,n) \hookrightarrow \mathbb{P}^N$ every Schubert cycle $\Sigma_a(\mathcal{V}) \subset G$ defined relative to the standard flag $V_i = \langle e_1, \dots, e_i \rangle$ is the intersection of G with a coordinate subspace of $\mathbb{P}^N$, that is, a subspace defined by the vanishing of an easily described subset of the Plücker coordinates. This is true even at the level of homogeneous ideals:

Theorem 4.3. *Let $\Sigma_a \subset G(k,n) \subset \mathbb{P}^N$ be a Schubert cycle, and let b be the strictly increasing k-tuple $b = (n-k+1-a_1, \dots, n-k+2-a_2, \dots)$. The homogeneous ideal of the Σ_a in $\mathbb{P}^N$ is generated by the homogeneous ideal of the Grassmannian (the Plücker relations, page 94) together with those Plücker coordinates $p_{b'}$ such that $b' \not\leq b$ in the termwise partial order.*

The equations of the Σ_a were studied in Hodge [1943], and this work led to the notions of a straightening law (Doubilet et al. [1974]) and Hodge algebra (De Concini et al. [1982]). A proof of Theorem 4.3 in terms of Hodge algebras may be found in the latter publication, along with a proof that the homogeneous coordinate ring of Σ_a is Cohen–Macaulay. The ideas have also been extended to homogeneous varieties for other reductive groups by Lakshmibai, Musili, Seshadri and their coauthors (see for example Seshadri [2007]). Avoiding this theory, we will prove Theorem 4.3 only in the easy case $G(2,4) = \mathbb{G}(1,3) \subset \mathbb{P}^5$. In Exercise 4.17 we invite the reader to give the easier proof of the set-theoretic version of Theorem 4.3.

Proof of Theorem 4.3 for $G(2,4)$: In $G(2,4)$, the Schubert cycle $\Sigma_{a,b}$ consists of those two-dimensional subspaces that meet V_{3-a} nontrivially and are contained in V_{4-b}. We must show that the homogeneous ideal of

$$\Sigma_{a,b} \subset G(2,4) \subset \mathbb{P}^5$$

is generated by the Plücker relation $g := p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3}$ together with the Plücker coordinates

$$\{p_{i,j} \mid (i,j) \not\leq (3-a, 4-b)\}$$

(note that the condition $(i,j) \not\leq (3-a, 4-b)$ means $i > 3-a$ or $j > 4-b$). Specifically:

- Σ_1 is the hyperplane section $p_{3,4} = 0 \subset G$; that is, it is the cone over the nonsingular quadric $\overline{g} = -p_{1,3}p_{2,4} + p_{14}p_{2,3}$ in $\mathbb{P}^3$.

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- Σ_2 is the plane $p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{1,1}$ is the plane $p_{1,4} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{2,1}$ is the line $p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.
- $\Sigma_{2,2}$ is the point $p_{1,3} = p_{1,4} = p_{2,3} = p_{2,4} = p_{3,4} = 0$.

A subspace $L \in \Sigma_{a,b}$ has a basis whose first vector is in V_{3-a}, and therefore has its last $a+1$ coordinates equal to 0, and whose second vector is in V_{4-b}, and thus has its last b coordinates equal to 0. If B is the matrix whose rows are the coordinates of these two vectors, then $p_{i,j}$ is (up to sign) the determinant of the submatrix of B involving the columns i and j. It follows that if $i > 3-a$ or $j > 4-b$, then $p_{i,j}(L) = 0$, so the given subsets of Plücker coordinates do vanish on the Schubert cycles as claimed.

To show that the ideals of the Schubert cycles are generated by the relation g and the given subsets, observe that each of the subsets is the ideal of the irreducible subvariety described above, and these have the same dimensions as the Schubert cycles. For example, we know that $\dim \Sigma_{1,1} = 2$, and the ideal

$$(g, p_{1,4}, p_{2,4}, p_{3,4}) = (p_{1,4}, p_{2,4}, p_{3,4}) \subset \Bbbk[p_{1,2}, \dots, p_{3,4}]$$

is the entire ideal of a plane. $\square$

4.2 Intersection products

4.2.1 Transverse flags

Throughout, we let $G = G(k, V)$ be the Grassmannian of k-dimensional linear subspaces of an n-dimensional vector space V. We start with one useful definition. As we said, Kleiman's theorem assures us (in characteristic 0) that, for a general pair of flags $\mathcal{V}$ and $\mathcal{W}$ on V, the Schubert cycles $\Sigma_a(\mathcal{V}), \Sigma_b(\mathcal{W}) \subset G$ intersect generically transversely. In this case, we can actually say explicitly what "general" means:

Definition 4.4. We say that a pair of flags $\mathcal{V}$ and $\mathcal{W}$ on V are *transverse* if any of the following equivalent conditions hold:

(a) $V_i \cap W_{n-i} = 0$ for all i.
(b) $\dim(V_i \cap W_j) = \min(0, i+j-n)$ for all i, j.
(c) There exists a basis $e_1, \dots, e_n$ for V in terms of which

$$V_i = \langle e_1, \dots, e_i \rangle \quad \text{and} \quad W_j = \langle e_{n+1-j}, \dots, e_n \rangle.$$

Note that any two transverse pairs can be carried into one another by a linear automorphism of V. Moreover, transverse pairs form a dense open subset in the space of all pairs of flags, so any statement proved for a general pair of flags (such as the generic transversality of the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W}) \subset G$) holds for any transverse pair, and vice versa.

Here is a lemma that will prove useful in intersecting Schubert cycles, though we will not use its full strength until the proof of Pieri's formula (Proposition 4.9):

Lemma 4.5. *Let $\Sigma_a(\mathcal{V}), \Sigma_b(\mathcal{W}) \subset G$ be Schubert cycles defined relative to transverse flags $\mathcal{V}$ and $\mathcal{W}$ on V. If $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ is a general point of their intersection, then:*

(a) Λ does not lie in any strictly smaller Schubert cycle $\Sigma_{a'}(\mathcal{V}) \subsetneq \Sigma_a(\mathcal{V})$.
(b) The flags induced by $\mathcal{V}$ and $\mathcal{W}$ on Λ (that is, consisting of intersections with Λ with flag elements V_α and W_β) are transverse.

Note that, by the first part, the flags $\Lambda^{\mathcal{V}}$ and $\Lambda^{\mathcal{W}}$ on Λ induced by $\mathcal{V}$ and $\mathcal{W}$ are, explicitly,

$$\Lambda_i^{\mathcal{V}} = \Lambda \cap V_{n-k+i-a_i} \quad \text{and} \quad \Lambda_i^{\mathcal{W}} = \Lambda \cap W_{n-k+i-b_i}, \qquad i = 1, \dots, k.$$

Proof of Lemma 4.5: The first part of the statement is immediate for dimension reasons: the flags $\mathcal{V}$ and $\mathcal{W}$ being transverse, the intersection $\Sigma_{a'}(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ will have dimension strictly less than $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$.

As for the second part, we have to show that the subspaces $\Lambda_i^{\mathcal{V}}$ and $\Lambda_{k-i}^{\mathcal{W}}$ are complementary, that is, that

$$\Lambda \cap V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}} \;=\; 0.$$

We do this by a dimension count: consider the incidence correspondence

$$\Phi = \left\{(\Lambda, [v]) \in (\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})) \times \mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}}) \mid v \in \Lambda\right\}.$$

We will show that $\dim \Phi < \dim(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W}))$, and thus the projection $\Phi \to \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ cannot be dominant, proving the lemma. Note that by the first part of the lemma we can replace Φ by the preimage of the complement U of

$$\mathbb{P}(V_{n-k+i-a_i-1} \cap W_{n-i-b_{k-i}}) \quad \text{and} \quad \mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}-1})$$

in $\mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}})$.

Since the flags $\mathcal{V}$ and $\mathcal{W}$ are transverse, we have

$$\dim \mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}}) = n - k - a_i - b_{k-i} - 1.$$

(If $a_i + b_{k-i} \geq n - k$, then the intersection $V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}}$ is 0 and Φ is correspondingly empty, so we are done in that case.) Next, suppose that $[v] \in U \subset \mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}})$. To describe the fiber of Φ over $[v]$, we consider the quotient space $V' = V/\langle v\rangle$, and the flags $\mathcal{V}'$ and $\mathcal{W}'$ on V' comprised of images of subspaces V_i and W_i under the projection $V \to V'$; that is,

$$V_j' = \begin{cases} (V_j + \langle v\rangle)/\langle v\rangle & \text{if } j < n-k+i-a_i, \\ V_{j+1}/\langle v\rangle & \text{if } j+1 \geq n-k+i-a_i, \end{cases}$$

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and similarly

$$W'_j = \begin{cases} (W_j + \langle v\rangle)/\langle v\rangle & \text{if } j < n-i-b_{k-i}, \\ W_{j+1}/\langle v\rangle & \text{if } j+1 \geq n-i-b_{k-i}. \end{cases}$$

Now we just observe that, if $(\Lambda, [v]) \in \Phi$, then the plane $\Lambda' = \Lambda/\langle v\rangle \subset V'$ belongs to the Schubert cycles

$$\Sigma_{a_1,\dots,\hat{a}_i,\dots,a_k}(\mathcal{V}') \quad \text{and} \quad \Sigma_{b_1,\dots,\widehat{b_{k-i}},\dots,b_k}(\mathcal{W}') \subset G(k-1, V').$$

Thus the fibers of Φ over $\mathbb{P}(V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}})$ have dimension

$$(k-1)(n-k) - \sum_{j\neq i} a_j - \sum_{j\neq k-i} b_j = (k-1)(n-k) - (|a| - a_i) - (|b| - b_{k-i}),$$

and altogether we have

$$\begin{aligned} \dim \Phi &= (n-k-a_i-b_{k-i}-1) + \big((k-1)(n-k) - (|a|-a_i) - (|b|-b_{k-i})\big) \\ &= k(n-k) - |a| - |b| - 1 \\ &< \dim(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})), \end{aligned}$$

as desired. □

4.2.2 Intersections in complementary dimension

As in the case of the Grassmannian $\mathbb{G}(1,3)$, we start our description of the Chow ring of G by evaluating intersections of Schubert cycles in complementary codimension. Here as before we use the fact that Schubert cycles $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ defined in terms of general flags $\mathcal{V}, \mathcal{W}$ always intersect generically transversely; this follows from Kleiman's theorem, or, in arbitrary characteristic, from Theorem 4.1.

Proposition 4.6. *If $\mathcal{V}$ and $\mathcal{W}$ are transverse flags in V and $\Sigma_a(\mathcal{V})$, $\Sigma_b(\mathcal{W})$ are Schubert cycles with $|a| + |b| = k(n-k)$, then $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ intersect transversely in a unique point if $a_i + b_{k+1-i} = n-k$ for each $i = 1,\dots,k$, and are disjoint otherwise. Thus*

$$\deg \sigma_a\sigma_b = \begin{cases} 1 & \textit{if } a_i + b_{k-i+1} = n-k \textit{ for all } i, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: As observed, since the two flags $\mathcal{V}$ and $\mathcal{W}$ are transverse, the Schubert cycles will meet generically transversely, and hence (since the intersection is zero-dimensional) transversely. Thus

$$\begin{aligned} \deg \sigma_a\sigma_b &= \#(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})) \\ &= \#\left\{ \Lambda \;\middle|\; \begin{matrix} \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i, \\ \dim(W_{n-k+i-b_i} \cap \Lambda) \geq i, \end{matrix} \quad \text{for all } i \right\}. \end{aligned}$$

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To evaluate the cardinality of this set, consider the conditions in pairs; that is, for each i, consider the i-th condition associated to the Schubert cycle $\Sigma_a(\mathcal{V})$:

$$\dim(V_{n-k+i-a_i} \cap \Lambda) \geq i$$

in combination with the $(k - i + 1)$-st condition associated to $\Sigma_b(\mathcal{W})$:

$$\dim(W_{n-i+1-b_{k-i+1}} \cap \Lambda) \geq k - i + 1.$$

If these conditions are both satisfied, then the subspaces

$$V_{n-k+i-a_i} \cap \Lambda \quad \text{and} \quad W_{n-i+1-b_{k-i+1}} \cap \Lambda,$$

having greater than complementary dimension in Λ, must have nonzero intersection; in particular, we must have

$$V_{n-k+i-a_i} \cap W_{n-i+1-b_{k-i+1}} \neq 0,$$

and, since the flags $\mathcal{V}$ and $\mathcal{W}$ are general, this in turn says we must have

$$n - k + i - a_i + n - i + 1 - b_{k-i+1} \geq n + 1,$$

or, in other words,

$$a_i + b_{k-i+1} \leq n - k.$$

If equality holds in this last inequality, the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k-i+1}}$ will meet in a one-dimensional vector space Γ_i, necessarily contained in Λ. (In the notation of Definition 4.4, $\Gamma_i = \langle e_{n-k+i-a_i} \rangle$.)

We have thus seen that $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ will be disjoint unless $a_i + b_{k-i+1} \leq n - k$ for all i. But from the equality

$$|a| + |b| = \sum_{i=1}^{k} (a_i + b_{k-i+1}) = k(n - k),$$

we see that if $a_i + b_{k-i+1} \leq n - k$ for all i, then we must have $a_i + b_{k-i+1} = n - k$ for all i. Moreover, in this case any Λ in the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ must contain each of the k subspaces Γ_i, so there is a unique such Λ, equal to the span of these one-dimensional spaces, as required. □

Corollary 4.7. *The Schubert classes form a free basis for $A(G)$, and the intersection forms $A^m(G) \times A^{\dim G - m}(G) \to \mathbb{Z}$ have the Schubert classes as dual bases.*

In view of the explicit duality between $A^m(G)$ and $A^{k(n-k)-m}(G)$ given by Proposition 4.6, it makes sense to introduce one more bit of notation: for any Schubert index $a = (a_1, \ldots, a_k)$, we will define the *dual index* to be the Schubert index $a^* = (n - k - a_k, \ldots, n - k - a_1)$. In these terms, Proposition 4.6 says that $\deg(\sigma_a \sigma_b) = 1$ if $b = a^*$ and is 0 otherwise.

Corollary 4.7 suggests a general approach to determining the coefficients in the expression of the class of a cycle as a linear combination of Schubert classes: If $\Gamma \subset G$ is any cycle of pure codimension m, we can write

$$[\Gamma] = \sum_{|a|=m} \gamma_a \sigma_a.$$

To find the coefficient γ_a, we intersect both sides with the Schubert cycle $\Sigma_{a^*}(\mathcal{V}) = \Sigma_{n-k-a_k,\ldots,n-k-a_1}(\mathcal{V})$ for a general flag $\mathcal{V}$; we then have

$$\gamma_a = \deg([\Gamma] \cdot \sigma_{a^*}) = \#(\Gamma \cap \Sigma_{a^*}(\mathcal{V})).$$

We have used exactly this approach — called the method of *undetermined coefficients* — in calculating classes of various cycles in $\mathbb{G}(1,3)$ in the preceding chapter; Proposition 4.6 and Corollary 4.7 say that it is more generally applicable in any Grassmannian. Explicitly, we have:

Corollary 4.8. *If $\alpha \in A^m(G)$ is any class, then*

$$\alpha = \sum_{|a|=m} \deg(\alpha\sigma_{a^*}) \cdot \sigma_a.$$

In particular, if σ_a and $\sigma_b \in A(G)$ are any Schubert classes on $G = G(k,n)$, then the product $\sigma_a\sigma_b$ is equal to

$$\sum_{|c|=|a|+|b|} \gamma_{a,b;c}\sigma_c,$$

where

$$\gamma_{a,b;c} = \deg(\sigma_a\sigma_b\sigma_{c^*}).$$

Since for general flags $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ the Schubert cycles $\Sigma_a(\mathcal{U})$, $\Sigma_b(\mathcal{V})$ and $\Sigma_{c^*}(\mathcal{W})$ are generically transverse by Kleiman's theorem, the coefficients $\gamma_{a,b;c} = \deg(\sigma_a\sigma_b\sigma_{c^*})$ are nonnegative integers. They are called *Littlewood–Richardson coefficients*, and they appear in many combinatorial and representation-theoretic contexts. If we adopt the convention that $\sigma_a = 0 \in A^{|a|}(G(k,n))$ if a fails to satisfy the conditions

$$n-k \geq a_1 \geq \cdots \geq a_k \geq 0 \quad \text{and} \quad a_l = 0 \text{ for all } l > k,$$

then the Littlewood–Richardson coefficients $\gamma_{a,b;c}$ depend only on the indices a, b and c, and not on k and n.

Corollary 4.8 shows that knowing the Littlewood–Richardson coefficients suffices to determine the products of all Schubert classes. In the case of the Grassmannians $G(2,n)$ they are either 0 or 1 (see Section 4.3), but a Littlewood–Richardson coefficient $\gamma_{a,b;c} > 1$ appears already in $G(3,6)$ (Exercise 4.35).

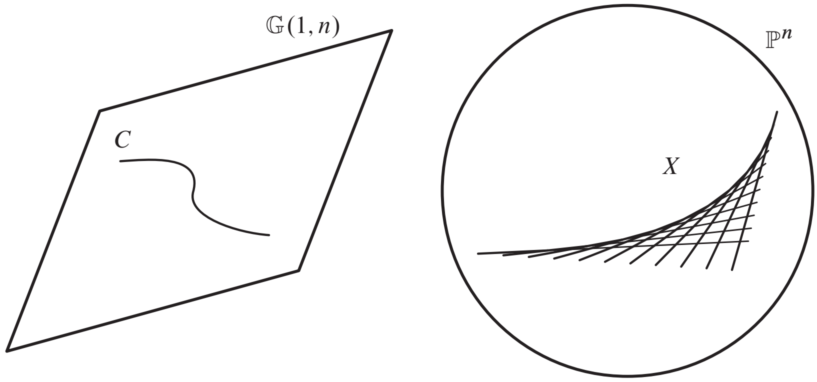

Figure 4.1 The surface $X \subset \mathbb{P}^n$ swept out by a one-parameter family $C \subset \mathbb{G}(1,n)$ of lines.

There exist beautiful algorithms for calculating the $\gamma_{a,b;c}$. We will give one in Section 4.7, and much more effective methods are given for example in Coşkun [2009] and Vakil [2006a]. But even simple questions such as, "when is $\gamma_{a,b;c} \neq 0$?" and "when is $\gamma_{a,b;c} > 1$?" do not seem to admit simple answers in general.

We will return to Schubert calculus shortly, but we take a moment here to use what we have already learned to answer Keynote Question (b).

4.2.3 Varieties swept out by linear spaces

Let $C \subset \mathbb{G}(k,n)$ be an irreducible curve, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of C; that is,

$$X = \bigcup_{[\Lambda]\in C} \Lambda \subset \mathbb{P}^n$$

(See Figure 4.1). We would like to relate the geometry of X to that of C; in particular, Keynote Question (b) asks us to find the degree of X when $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ is a twisted cubic curve.

To begin with, observe that X is indeed a closed subvariety of $\mathbb{P}^n$: If

$$\Phi = \{(\Lambda, p) \in \mathbb{G}(k,n) \times \mathbb{P}^n \mid p \in \Lambda\}$$

is the universal k-plane over $\mathbb{G}(k,n)$, as described in Section 3.2.3, and $\alpha : \Phi \to \mathbb{G}(k,n)$ and $\beta : \Phi \to \mathbb{P}^n$ are the projections, then we can write

$$X = \beta(\alpha^{-1}(C)).$$

Now, suppose that a general point $x \in X$ lies on a unique k-plane $\Lambda \in C$ — that is, the map $\beta : \alpha^{-1}(C) \to X \subset \mathbb{P}^n$ is birational, so that in particular $\dim(X) = k+1$. The degree of X is the number of points of intersection of X with a general $(n-k-1)$-plane

$\Gamma \subset \mathbb{P}^n$; since each of these points is a general point of X, and so lies on a unique k-plane Λ, the number is the number of k-planes Λ that meet Γ. In other words, we have

$$\begin{aligned}
\deg(X) &= \#(X \cap \Gamma) \\
&= \#(C \cap \Sigma_1(\Gamma)) \\
&= \deg([C] \cdot \sigma_1) \qquad \text{(by Kleiman's theorem)} \\
&= \deg(C),
\end{aligned}$$

where by the degree of C we mean the degree under the Plücker embedding of $\mathbb{G}(k,n)$.

These ideas allow us to answer Keynote Question (b): The surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a twisted cubic $C \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$, times the degree of the map β defined above, is equal to 3. Thus the surface X itself has degree 3 or 1. In the latter case, the curve C would be contained in a Schubert cycle $\Sigma_{1,1}$, and as we have seen in the description on page 138, this Schubert cycle is contained in the 2-plane in $\mathbb{P}^5$ defined by the vanishing of three Plücker coordinates. Since a twisted cubic is not contained in a 2-plane, this shows that the surface X has degree 3. More of the geometry of X is described in Exercises 4.23-4.25.

If $Z \subset \mathbb{G}(k,n)$ is a variety of any dimension m, we can form the variety $X \subset \mathbb{P}^n$ swept out by the planes of Z. Its degree — assuming it has the expected dimension $k + \dim Z$ and that a general point of X lies on only one plane $\Lambda \in Z$ — is expressible in terms of the Schubert coefficients of the class $[Z] \in A_m(\mathbb{G}(k,n))$, though it is not in general equal to the degree of Z. This is the content of Exercise 4.22; we will return to this question in Section 10.2, where we will see how to express the answer in terms of Chern and Segre classes.

4.2.4 Pieri's formula

One situation in which we can give a simple formula for the product of Schubert classes is when one of the classes has the special form $\sigma_b = \sigma_{b,0,\dots,0}$. Such classes are called *special Schubert classes*.

Proposition 4.9 (Pieri's formula)**.** *For any Schubert class $\sigma_a \in A(G)$ and any integer b,*

$$(\sigma_b \cdot \sigma_a) = \sum_{\substack{|c|=|a|+b \\ a_i \le c_i \le a_{i-1} \forall i}} \sigma_c$$

Proof:[1] By Corollary 4.8, Pieri's formula is equivalent to the assertion that, for any Schubert index c with $|c| = |a| + b$,

$$\deg(\sigma_a \sigma_b \sigma_{c^*}) = \begin{cases} 1 & \text{if } a_i \le c_i \le a_{i-1} \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

[1] This proof was shown to us by Izzet Coşkun

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To prove this, we will look at the corresponding Schubert cycles $\Sigma_a(\mathcal{V})$, $\Sigma_b(\mathcal{U})$ and $\Sigma_{c^*}(\mathcal{W})$, defined with respect to general flags $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{W}$; we will show that their intersection is empty if c_i violates the condition $a_i \leq c_i \leq a_{i-1}$ for any i, and consists of a single point if these inequalities are all satisfied. By Kleiman's theorem, the intersection multiplicity will be 1 in the latter case.

By definition,

$$\Sigma_a(\mathcal{V}) = \{\Lambda \mid \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \ \text{ for all } i\}$$

and

$$\Sigma_{c^*}(\mathcal{W}) = \{\Lambda \mid \dim(\Lambda \cap W_{i+c_{k+1-i}}) \geq i \ \text{ for all } i\}.$$

Set

$$A_i = V_{n-k+i-a_i} \cap W_{k+1-i+c_i},$$

so that either $A_i = 0$ or $\dim A_i = c_i - a_i + 1$. Combining the i-th condition in the first definition and the $(k+1-i)$-th condition in the second, we see that for any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ we have

$$\Lambda \cap A_i \neq 0.$$

If $c_i < a_i$ for some i then $A_i = 0$, so that $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) = \varnothing$, and $\deg \sigma_a \sigma_b \sigma_{c^*} = 0$, as required. Thus we may assume that $c_i \geq a_i$ for every i.

We claim that the A_i are linearly independent if and only if $c_i \leq a_{i-1}$ for all i. To see this, choose a basis e_i as in Section 4.2.2, so that $V_i = \langle e_1, \ldots, e_i \rangle$ and $W_j = \langle e_{n-j+1}, \ldots, e_n \rangle$. With this notation

$$A_i = \langle e_{n-k+i-c_i}, \ldots, e_{n-k+i-a_i} \rangle,$$

and the condition $c_i \leq a_{i-1}$ amounts to the condition that the two successive ranges of indices $n-k+i-1-c_{i-1}, \ldots, n-k+i-1-a_{i-1}$ and $n-k+i-c_i, \ldots, n-k+i-a_i$ do not overlap. In other words, if we let

$$A = \langle A_1, \ldots, A_k \rangle$$

be the span of the spaces A_i, then we have

$$\dim A \leq \sum c_i - a_i + 1 = k + b,$$

with equality holding if and only if $c_i \leq a_{i-1}$ for all i. Note that by Lemma 4.5 the plane Λ is spanned by its intersections with the A_i; that is, $\Lambda \subset A$.

Now we introduce the conditions associated with the special Schubert cycle $\Sigma_b(\mathcal{U})$. This is the set of k-planes that have nonzero intersection with a general linear subspace $U = U_{n-k+1-b} \subset V$ of dimension $n - k + 1 - b$. For there to be any $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfying this condition requires that $A \cap U \neq 0$, and hence, since U is general, that $\dim A \geq k + b$. Thus, if $c_i > a_{i-1}$ for any i, then we will have

$\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_b(\mathcal{U}) = \varnothing$. We can accordingly assume $c_i \le a_{i-1}$ for all i, and hence $\dim A = k + b$.

Finally, since $U \subset V$ is a general subspace of codimension $k+b-1$, it will meet A in a one-dimensional subspace. Choose v any nonzero vector in this intersection. Since $A = \bigoplus A_i$, we can write v uniquely as a sum

$$v = v_1 + \cdots + v_k \quad \text{with } v_i \in A_i.$$

Suppose now that $\Lambda \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ satisfies all the Schubert conditions above. Since $\Lambda \subset A$ and $\Lambda \cap U \neq 0$, Λ must contain the vector v, and, since Λ is spanned by its intersections with the A_i, it follows that Λ must contain the vectors v_i as well. Thus, we see that the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{U}) \cap \Sigma_{c^*}(\mathcal{W})$ will consist of the single point corresponding to the plane $\Lambda = \langle v_1, \dots, v_k \rangle$ spanned by the v_i, and we are done. □

As a corollary of the Pieri formula, we can prove a relation among the special Schubert classes that is an important special case of a theorem of Whitney used for computing Chern classes (Theorem 5.3):

Corollary 4.10. *In $A(G(k,n))$, we have*

$$(1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k})(1 - \sigma_1 + \sigma_{1,1} - \sigma_{1,1,1} + \cdots + (-1)^k \sigma_{1^k}) = 1.$$

Proof: We can use Pieri to calculate the individual products appearing in this expression. To start, Pieri tells us that

$$\sigma_l \sigma_{1^m} = \sigma_{l,1^m} + \sigma_{l+1,1^{m-1}}.$$

When we write out the terms of degree d in the product on the left, then, the sum telescopes: For $d > 0$,

$$\begin{aligned}\sum_{i=0}^{d} (-1)^i \sigma_{d-i}\sigma_{1^i} &= \sigma_d - (\sigma_d + \sigma_{d-1,1}) + (\sigma_{d-1,1} + \sigma_{d-2,1,1}) \\ &\qquad - \cdots + (-1)^{d-1}(\sigma_{2,1^{d-2}} + \sigma_{1^d}) + (-1)^d \sigma_{1^d} \\ &= 0.\end{aligned}$$

□

4.3 Grassmannians of lines

Let $G = G(2, V)$ be the Grassmannian of two-dimensional subspaces of an $(n+1)$-dimensional vector space V, or, equivalently, lines in the projective space $\mathbb{P}V \cong \mathbb{P}^n$. The Schubert cycles on G with respect to a flag $\mathcal{V}$ are of the form

$$\Sigma_{a_1,a_2}(\mathcal{V}) = \{\Lambda \mid \Lambda \cap V_{n-a_1} \neq 0 \text{ and } \Lambda \subset V_{n+1-a_2}\}.$$

In this case, Pieri's formula (Proposition 4.9) allows us to give a closed-form expression for the product of any two Schubert classes:

Proposition 4.11. *Assuming that* $a_1 - a_2 \geq b_1 - b_2$,

$$\begin{aligned}\sigma_{a_1,a_2}\sigma_{b_1,b_2} &= \sigma_{a_1+b_1,a_2+b_2} + \sigma_{a_1+b_1-1,a_2+b_2+1} + \cdots + \sigma_{a_1+b_2,b_1+a_2} \\ &= \sum_{\substack{|c|=|a|+|b| \\ a_1+b_1 \geq c_1 \geq a_1+b_2}} \sigma_{c_1,c_2}.\end{aligned}$$

Proof: We will start with the simplest cases, where the intersection of general Schubert cycles is again a Schubert cycle: If $b_1 = b_2 = b$, then the Schubert cycle $\Sigma_{b,b}(\mathcal{W})$ is equal to

$$\{\Lambda \mid \Lambda \subset W_{n-b}\},$$

so that for any a_1, a_2 we have

$$\begin{aligned}\Sigma_{a_1,a_2}(\mathcal{V}) \cap \Sigma_{b,b}(\mathcal{W}) &= \left\{\Lambda \;\middle|\; \begin{array}{r} \Lambda \cap V_{n-1-a_1} \neq 0, \\ \Lambda \subset V_{n-a_2}, \\ \Lambda \subset W_{n-b} \end{array}\right\} \\ &= \left\{\Lambda \;\middle|\; \begin{array}{r} \Lambda \cap (V_{n-1-a_1} \cap W_{n-b}) \neq 0, \\ \Lambda \subset (V_{n-a_2} \cap W_{n-b}) \end{array}\right\} \\ &= \Sigma_{a_1+b,a_2+b}(V_{n-1-a_1} \cap W_{n-b}, V_{n-a_2} \cap W_{n-b}).\end{aligned}$$

Thus by Kleiman's theorem we have

$$\sigma_{a_1,a_2}\sigma_{b,b} = \sigma_{a_1+b,a_2+b}. \tag{4.2}$$

Now, suppose we want to intersect an arbitrary pair of Schubert classes σ_{a_1,a_2} and σ_{b_1,b_2}. We can write

$$\begin{aligned}\sigma_{a_1,a_2}\sigma_{b_1,b_2} &= (\sigma_{a_1-a_2,0}\sigma_{a_2,a_2})(\sigma_{b_1-b_2,0}\sigma_{b_2,b_2}) \\ &= \sigma_{a_1-a_2,0}\,\sigma_{b_1-b_2,0}\,\sigma_{a_2+b_2,a_2+b_2},\end{aligned}$$

and if we can evaluate the product of the first two terms in the last expression, we can use (4.2) to finish the calculation.

But this is exactly what Pieri gives us: if $a \geq b$, Pieri says that

$$\sigma_{a,0}\sigma_{b,0} = \sigma_{a+b,0} + \sigma_{a+b-1,1} + \cdots + \sigma_{a,b},$$

and the general statement follows. □

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We can use this description of the Chow ring of $\mathbb{G}(1,n)$ (and a little combinatorics) to answer Keynote Question (d): What is the degree of the Grassmannian $\mathbb{G}(1,n) = G(2,n+1)$ under the Plücker embedding? We observe first that, since the hyperplane class on $\mathbb{P}(\bigwedge^2 \Bbbk^{n+1})$ pulls back to the class $\sigma_1 \in A^1(G(2,n+1))$, we have

$$\deg(G(2,n+1)) = \deg(\sigma_1^{2n-2}).$$

To evaluate this product, we make a directed graph with the Schubert classes σ_a in $G(2,n+1)$ as vertices and with the inclusions among the corresponding Schubert cycles $\Sigma_a(\mathcal{V})$ indicated by arrows (the graph shown is the case $n=5$):

$$\begin{tikzcd}
\sigma_0 & & & & \\
\sigma_1 \arrow[u] & & & & \\
\sigma_2 \arrow[u] & \sigma_{1,1} \arrow[ul] & & & \\
\sigma_3 \arrow[u] & \sigma_{2,1} \arrow[ul] \arrow[u] & & & \\
\sigma_4 \arrow[u] & \sigma_{3,1} \arrow[ul] \arrow[u] & \sigma_{2,2} \arrow[ul] & & \\
 & \sigma_{4,1} \arrow[ul] \arrow[u] & \sigma_{3,2} \arrow[ul] \arrow[u] & & \\
 & & \sigma_{4,2} \arrow[ul] \arrow[u] & \sigma_{3,3} \arrow[ul] & \\
 & & & \sigma_{4,3} \arrow[ul] \arrow[u] & \\
 & & & & \sigma_{4,4} \arrow[ul]
\end{tikzcd}$$

In terms of this graph, the rule expressed in Proposition 4.11 for multiplication by σ_1 is simple: The product of any Schubert class $\sigma_{a,b}$ with σ_1 is the sum of all immediate predecessors of $\sigma_{a,b}$ — that is, the Schubert classes in the row below $\sigma_{a,b}$ that are connected to $\sigma_{a,b}$ by an arrow. In particular, the degree $\deg((\sigma_1)^{2n-2})$ of the Grassmannian is the number of paths upward through this diagram starting with $\sigma_{n-1,n-1}$ and ending with $\sigma_{0,0}$. If we designate such a path by a sequence of $n-1$ "1"s and $n-1$ "2"s, corresponding to whether the first or second indices change (these are the vertical and diagonal arrows in the graph shown) reading from left to right, there are never more "2"s than "1"s. Equivalently, if we associate to a "1" a left parenthesis and to a "2" a right parenthesis, this is the number of ways in which $n-1$ pairs of parentheses can appear in a grammatically correct sentence. This is called the $(n-1)$-st *Catalan number*; a standard combinatorial argument (see, for example, Stanley [1999]) gives

$$c_{n-1} = \frac{(2n-2)!}{n!(n-1)!}.$$

In sum, we have:

Proposition 4.12. *The degree of the Grassmannian $G(2, n+1) \subset \mathbb{P}(\bigwedge^2 \Bbbk^{n+1})$ is*

$$\deg G(2, n+1) = \frac{(2n-2)!}{n!(n-1)!}.$$

This number also represents the answer to the enumerative problem of how many lines in $\mathbb{P}^n$ meet each of $2n-2$ general $(n-2)$-planes $V_1, \ldots, V_{2n-2} \subset \mathbb{P}^n$.

Pieri's formula (Proposition 4.9) gives us the means to answer the generalization of Keynote Question (d) to all Grassmannians: Since σ_1 is the class of the hyperplane section of the Grassmannian in its Plücker embedding, the degree of the Grassmannian in that embedding is the degree of $\sigma_1^{k(n-k)}$. This will be worked out (with the aid of the hook formula from combinatorics) in Exercise 4.38; the answer is that

$$\deg(G(k,n)) = (k(n-k))! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!}.$$

We can also use the description of $A(\mathbb{G}(1,n))$ given in Proposition 4.11 to answer Keynote Question (a): If $V_1, \ldots, V_4 \subset \mathbb{P}^{2n+1}$ are four general n-planes, how many lines $L \subset \mathbb{P}^{2n+1}$ meet all four? The answer is the cardinality of the intersection $\bigcap \Sigma_n(V_i) \subset \mathbb{G}(1, 2n+1)$; given transversality — a consequence of Kleiman's theorem in characteristic 0, and checked directly in arbitrary characteristic via the description of tangent spaces to Schubert cycles in Theorem 4.1 — this is the degree of the product $\sigma_n^4 \in A(\mathbb{G}(1, 2n+1))$. Applying Proposition 4.11, we have

$$\sigma_n^2 = \sigma_{2n} + \sigma_{2n-1,1} + \cdots + \sigma_{n+1,n-1} + \sigma_{n,n};$$

since each term squares to the class of a point and all pairwise products are zero, we have

$$\deg(\sigma_n^4) = n+1,$$

and this is the answer to our question.

We will see in Exercise 4.26 another way to arrive at this number, in a manner analogous to the alternative solution to the four-line problem given in Section 3.4.1; Exercise 4.27 gives a nice geometric consequence.

4.4 Dynamic specialization

In Section 3.5.1, we started to discuss the method of *specialization*, and used it to determine the products of some Schubert classes. We can compute intersection numbers in other cases only by using a stronger and more broadly applicable version of this technique, called *dynamic specialization*.

Recall that in Section 3.5.1 we described an alternative approach to establishing the relation $\sigma_1^2 = \sigma_{11} + \sigma_2$ in the Chow ring of the Grassmannian $\mathbb{G}(1,3)$. Instead of taking two general translates of the Schubert cycle $\Sigma_1(L) \subset \mathbb{G}(1,3)$ — whose intersection was necessarily generically transverse, but whose intersection class required additional work to calculate — we considered the intersection $\Sigma_1(L) \cap \Sigma_1(L')$, where $L, L' \subset \mathbb{P}^3$ were not general, but incident lines. The benefit here is that now the intersection is visibly a union of Schubert cycles: Specifically, if $p = L \cap L'$ is their point of intersection and $H = \overline{L, L'}$ their span, we have

$$\Sigma_1(L) \cap \Sigma_1(L') = \Sigma_2(p) \cup \Sigma_{1,1}(H).$$

The trade-off is that we cannot just invoke Kleiman to see that the intersection is indeed generically transverse; this can however be established directly by using the description of the tangent spaces to the two cycles given in Theorem 4.1.

Suppose now we are dealing with the Grassmannian $G = \mathbb{G}(1,4)$ of lines in $\mathbb{P}^4$ and we try to use an analogous method to determine the product $\sigma_2^2 \in A^4(G)$ — that is, the class of the locus of lines meeting each of two given lines. We would try to find a pair of lines $L, M \subset \mathbb{P}^4$ such that the two cycles

$$\Sigma_2(L) = \{\Lambda \mid \Lambda \cap L \neq \varnothing\} \quad \text{and} \quad \Sigma_2(M) = \{\Lambda \mid \Lambda \cap M \neq \varnothing\}$$

representing the class σ_2 are special enough that the class of the intersection is clear, but still sufficiently general that they intersect generically transversely.

However, there are no such pairs of lines. If the lines L and M are disjoint, they are effectively a general pair, and the intersection is not a union of Schubert cycles. On the other hand, if L meets M at a point p, then the locus of lines through p forms a three-dimensional component of the intersection $\Sigma_2(L) \cap \Sigma_2(M)$, so the intersection is not even dimensionally transverse.

We can nevertheless consider a family of lines $\{M_t\}$ in $\mathbb{P}^4$, parametrized by $t \in \mathbb{A}^1$, with M_t disjoint from L for $t \neq 0$ and with M_0 meeting L at a point p. This gives a family of intersection cycles $\Sigma_2(L) \cap \Sigma_2(M_t)$. To make this precise, we consider the subvariety

$$\Phi^\circ = \{(t, \Lambda) \in \mathbb{A}^1 \times G \mid t \neq 0 \text{ and } \Lambda \in \Sigma_2(L) \cap \Sigma_2(M_t)\}$$

and its closure $\Phi \subset \mathbb{A}^1 \times G$. Since M_t is disjoint from L for $t \neq 0$, the fiber $\Phi_t = \Sigma_2(L) \cap \Sigma_2(M_t)$ of Φ over $t \neq 0$ represents the class σ_2^2, and it follows that Φ_0 does as well. The key point is that when we look at the fiber Φ_0 *we are looking not at the intersection* $\Sigma_2(L) \cap \Sigma_2(M_0)$ *of the limiting cycles, but rather at the flat limit of the intersection cycles* $\Sigma_2(L) \cap \Sigma_2(M_t)$, which is necessarily of the expected dimension.

The fiber Φ_0 is contained in the intersection $\Sigma_2(L) \cap \Sigma_2(M_0)$, but has smaller dimension. Thus a line Λ arising as the limit of lines Λ_t meeting both L and M_t must satisfy some additional condition beyond meeting both L and M_0, and to characterize Φ_0 we need to say what that condition is.

For $t \neq 0$, the lines L and M_t together span a hyperplane $H_t = \overline{L, M_t} \cong \mathbb{P}^3 \subset \mathbb{P}^4$. Let H_0 be the hyperplane that is the limit of the H_t as t goes to 0. If $\{\Lambda_t\}$ is a family of lines with Λ_t meeting both L and M_t for $t \neq 0$, then the limiting line Λ_0 must be contained in H_0.

Of course, if Λ_0 does not pass through the point $p = L \cap M_0$, then it must be contained in the 2-plane $P = \overline{L, M_0}$, so the new condition $\Lambda_0 \subset H_0$ is redundant. In sum, we conclude that the support of Φ_0 must be contained in the union of the two two-dimensional Schubert cycles

$$\Phi_0 \subset \{\Lambda \mid \Lambda \subset P\} \cup \{\Lambda \mid p_0 \in \Lambda \subset H_0\} = \Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0).$$

We will see in Exercise 4.28 that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$, and in Exercise 4.29 that Φ_0 is generically reduced. Thus, the cycle associated to the scheme Φ_0 is exactly the sum $\Sigma_{2,2}(P) + \Sigma_{3,1}(p_0, H_0)$, and we can deduce the formula

$$\sigma_2^2 = \sigma_{3,1} + \sigma_{2,2} \in A^4(\mathbb{G}(1,4)).$$

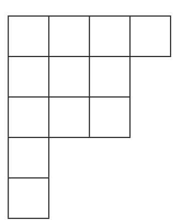
This is a good example of the method of dynamic specialization, in which we consider not a special pair of cycles representing given Chow classes and intersecting generically transversely, but a family of representative pairs specializing from pairs that do intersect transversely to a pair that may not. We then must describe the limit of the intersections (not the intersection of the limits). This technique is a starting point for the general algorithms of Coşkun [2009] and Vakil [2006a]. For another example of its application, see Griffiths and Harris [1980].

Often, as in the examples cited above, to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages; see Exercise 4.30 for an example. To see this idea carried out in a much broader context, see Fulton [1984, Chapter 11].

4.5 Young diagrams

For many purposes, it is convenient to represent the Schubert class $\sigma_{a_1,\ldots,a_k}$ by a *Young diagram*; that is, as a collection of left-justified rows of boxes with the i-th row of length a_i. For example, $\sigma_{4,3,3,1,1}$ would be represented by

$\sigma_{4,3,3,1,1} \quad \longleftrightarrow$



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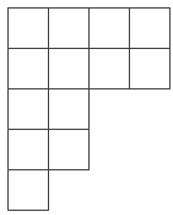
(Warning: there are many different conventions in use for interpreting the correspondence between Schubert classes and Young diagrams!) The condition that $n-k \geq a_1 \geq a_k \geq 0$ means that the Young diagram fits into a box with k rows and $n-k$ columns, and the rows of the diagram are weakly decreasing in length from top to bottom. As another example, the relation between a Schubert class σ_a and the dual Schubert class σ_{a^*}, described in Proposition 4.6, could be described by saying that the Young diagrams of $\sigma = \sigma_a$ and $\tau = \sigma_{a^*}$, after rotating the latter 180°, are complementary in the $k \times (n-k)$ box; if $\sigma = \sigma_{4,3,3,1,1} \in A(G(5,10))$, for example, then τ is as shown in the following:

σ	σ	σ	σ	τ
σ	σ	σ	τ	τ
σ	σ	σ	τ	τ
σ	τ	τ	τ	τ
σ	τ	τ	τ	τ

;

that is,

$\tau \quad \longleftrightarrow$  .

As a first application of this correspondence, we can count the Schubert classes as follows:

Corollary 4.13. $A(G(k,n)) \cong \mathbb{Z}^{\binom{n}{k}}$ *as abelian groups.*

Proof: The number of Schubert classes is the same as the number of Young diagrams that fit into a $k \times (n-k)$ box of squares B. To count these, we associate to each Young diagram Y in B its "right boundary" L: this is the path consisting of horizontal and vertical segments of unit length which starts from the upper-right corner of the $k \times (n-k)$ box and ends at the lower-left corner of the box, such that the squares in Y are those to the left of L. (For example, in the case of the Young diagram associated to $\sigma_{4,3,3,1,1} \subset G(5,10)$, illustrated above, we may describe L by the sequence $h, v, h, v, v, h, h, v, v$ where h and v denote horizontal and vertical segments, respectively, and we start from the upper-right corner.)

Of course the number of h terms in any such boundary must be $n-k$, the width of the box, and the number of v terms must be k, the height of the box. Thus the length of the boundary is n, and giving the boundary is equivalent to specifying which k steps will be vertical; that is, the number of Young diagrams in B is $\binom{n}{k}$, as required. □

The correspondence between Schubert classes and Young diagrams behaves well with respect to many basic operations on Grassmannians. For example, under the duality $G(k,n) \cong G(n-k,n)$, the Schubert cycle corresponding to the Young diagram Y is

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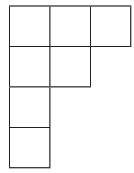
taken to the Schubert cycle corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-to-southeast. For example, if

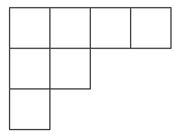
$$\sigma_{3,2,1,1} \in A(G(4,7)) \quad \longleftrightarrow$$



,

then the corresponding Schubert cycle in $G(3,7)$ is

$$\sigma_{4,2,1} \in A(G(3,7)) \quad \longleftrightarrow$$

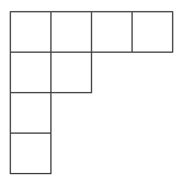


.

This is reasonably straightforward to verify, and is the subject of Exercise 4.31.

Pieri's formula can also be described in terms of Young diagrams: It says that for any Schubert class σ_b and any special Schubert class $\sigma_a = \sigma_{a,0,\dots,0}$, the Schubert classes appearing in the product $\sigma_a\sigma_b$ (all with coefficient 1) correspond to Young diagrams obtained from the Young diagram of σ_b *by adding a total of a boxes, with at most one box added to each column*, as long as the result is still a Young diagram: for example, if we want to multiply the Schubert class

$$\sigma_{4,2,1,1} \in A(G(4,8)) \quad \longleftrightarrow$$



by the Schubert class σ_1, we can add a box in either the first, second, third or fifth row, to obtain the expression

$$\sigma_1\sigma_{4,2,1,1} = \sigma_{5,2,1,1} + \sigma_{4,3,1,1} + \sigma_{4,2,2,1} + \sigma_{4,2,1,1,1}.$$

The combinatorics of Young diagrams is an extremely rich subject with many applications. For an introduction, see for example Fulton [1997].

4.5.1 Pieri's formula for the other special Schubert classes

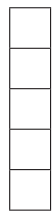
Let V be an n-dimensional vector space and $\mathcal{V} = V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$ a flag in V. As we observed in Section 4.1, for any integer a with $1 \le a \le n-k$ the isomorphism $G(k,V) \cong G(n-k,V^*)$ carries the special Schubert cycle

$$\Sigma_a = \{\Lambda \in G(k,V) \mid \Lambda \cap V_{n-k+1-a} \neq 0\}$$

defined relative to the flag $\mathcal{V}$ to the Schubert cycle

$$\Sigma_{1,\dots,1}(\mathcal{V}^\perp) = \{\Lambda \in G(n-k, V^*) \mid \dim(\Lambda \cap V_{n-k+a-1}) \geq a\}$$

defined relative to the flag $\mathcal{V}^\perp$ formed by the annihilators of the V_i. The Schubert classes $\sigma_{1,\dots,1}$ (often written σ_{1^a})

$\sigma_{1^5} \quad \longleftrightarrow$ 

are also referred to as *special Schubert classes*. The correspondence between Schubert classes and Young diagrams makes it easy to translate Pieri's formula into a formula for multiplication by σ_{1^a}: The Schubert classes appearing in the product $\sigma_{1^a}\sigma_b$ (all with coefficient 1) correspond to Young diagrams obtained from the Young diagram of σ_b *by adding a total of a boxes, with at most one box added to each row*, as long as the result is still a Young diagram:

Theorem 4.14 (Pieri's formula, part II). *For any Schubert class $\sigma_b = \sigma_{b_1,\dots,b_k} \in A(G(k,n))$ and any integer a with $1 \leq a \leq n-k$,*

$$\sigma_{1^a}\sigma_b = \sum_{\substack{|c|=a+|b| \\ b_i \leq c_i \leq b_i+1 \ \forall i}} \sigma_c$$

4.6 Linear spaces on quadrics

We can generalize the calculation in Section 3.6 of the class of the locus of lines on a quadric surface to a description of the class of the locus of planes of any dimension on a smooth quadric hypersurface of any dimension.

To begin with, since our field $\Bbbk$ has characteristic $\neq 2$, a nonsingular form $Q(x)$ of degree 2 on $\mathbb{P}V$ can be written in the form $Q(x) = q(x,x)$, where $q(x,y)$ is a nonsingular symmetric bilinear form $V \times V \to \Bbbk$. A linear subspace $\mathbb{P}W \subset \mathbb{P}V$ lies on the quadric $Q(x) = 0$ if and only if W is *isotropic* for q, that is, $q(W,W) = 0$. Thus, we want to find the class of the locus $\Phi \subset G = G(k,V)$ of isotropic k-planes for q.

To start, we want to find the dimension of Φ. There are a number of ways to do this; probably the most elementary is to count bases for isotropic subspaces. To find a basis for an isotropic subspace, we can start with any vector v_1 with $q(v_1,v_1) = 0$, then choose $v_2 \in \langle v_1\rangle^\perp \setminus \langle v_1\rangle$ with $q(v_2,v_2) = 0$, $v_3 \in \langle v_1,v_2\rangle^\perp \setminus \langle v_1,v_2\rangle$ with $q(v_3,v_3) = 0$, and so on. Since $\langle v_1,\dots,v_i\rangle \subset \langle v_1,\dots,v_i\rangle^\perp$, this necessarily terminates when $i \geq n/2$; in other words, *a nondegenerate quadratic form will have no isotropic subspaces of*

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dimension strictly greater than half the dimension of the ambient space. (We could also see this by observing that q defines an isomorphism of V with its dual V^* carrying any isotropic subspace $\Lambda \subset V$ into its annihilator $\Lambda^\perp \subset V^*$.)

In this process, the allowable choices for v_1 correspond to points on the quadric $Q(x) = 0$; those for v_2 correspond to the points on the quadric $Q|_{v_1^\perp}$, and so forth. In general, the v_i form a locally closed subset of V of dimension $n - i$. Thus the space of all bases for isotropic k-planes has dimension

$$(n-1) + \cdots + (n-k) = k(n-k) + \binom{k}{2}.$$

Since there is a k^2-dimensional family of bases for a given isotropic k-plane, the space of such planes has dimension

$$k(n-k) + \binom{k}{2} - k^2 = k(n-k) - \binom{k+1}{2},$$

or in other words the cycle Φ has codimension $\binom{k+1}{2}$ in $G(k, V)$ when $k \le n/2$, and is empty otherwise.

Having determined the dimension of Φ, we ask now for its class in $A(G(k, V))$. Following the method of undetermined coefficients, we write

$$[\Phi] = \sum_{|a| = \binom{k+1}{2}} \gamma_a \sigma_a,$$

with

$$\begin{aligned} \gamma_a &= \#(\Phi \cap \Sigma_{n-k-a_k,\ldots,n-k-a_1}(\mathcal{V})) \\ &= \#\{\Lambda \mid q|_\Lambda \equiv 0 \text{ and } \dim(\Lambda \cap V_{i+a_i}) \ge i \text{ for all } i\}. \end{aligned}$$

To evaluate γ_a, suppose that $\Lambda \subset V$ is a k-plane in this intersection. The subspace $V_{a_i+i} \subset V$ being general, the restriction $q|_{V_{a_i+i}}$ of q to it will again be nondegenerate. Since $q|_{V_{a_i+i}}$ has an isotropic i-plane, we must have $a_i + i \ge 2i$, or in other words

$$a_i \ge i \quad \text{for all } i.$$

But by hypothesis $\sum a_i = \binom{k+1}{2}$, so we must have equality in each of these inequalities. In other words, $\gamma_a = 0$ *for all* a *except the index* $a = (k, k-1, \ldots, 2, 1)$.

It remains to evaluate the coefficient

$$\gamma_{k,k-1,\ldots,2,1} = \#\{\Lambda \mid q(\Lambda, \Lambda) = 0 \text{ and } \dim(\Lambda \cap V_{2i}) \ge i \text{ for all } i\}, \tag{4.3}$$

where the equality holds by Kleiman's theorem. We claim that this number is 2^k.

We prove this inductively. To start, note that the restriction $q|_{V_2}$ of q to the two-dimensional space V_2 has two one-dimensional isotropic spaces, and Λ will necessarily contain exactly one of them: it cannot contain both, since Φ is disjoint from any Schubert cycle $\Sigma_b(\mathcal{V})$ with $|b| > k(n-k) - \binom{k+1}{2}$.

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We may thus suppose that Λ contains the isotropic subspace $W \subset V_2$, so that Λ is contained in $W^{\perp}$. Now, since $q(W, W) \equiv 0$, q induces a nondegenerate quadratic form q' on the $(n-2)$-dimensional quotient $W' = W^{\perp}/W$, and the quotient space

$$\Lambda' = \Lambda/W \subset W^{\perp}/W$$

is a $(k-1)$-dimensional isotropic subspace for q'. Moreover, since the spaces V_{2i} are general subspaces of V containing V_2, the subspaces

$$V'_{2i-2} = (V_{2i} \cap W^{\perp})/W \subset W^{\perp}/W$$

form a general flag in $W' = W^{\perp}/W$, and we have

$$\dim(\Lambda' \cap V'_{2i-2}) \geq i - 1 \quad \text{for all } i.$$

Inductively, there are 2^{k-1} isotropic $(k-1)$-planes $\Lambda' \subset W'$ satisfying these conditions, and so there are 2^k planes $\Lambda \subset W$ satisfying the conditions of (4.3). We have proven:

Proposition 4.15. *Let q be a nondegenerate quadratic form on the n-dimensional vector space V and $\Phi \subset G(k, V)$ the variety of isotropic k-planes for q. Assuming $k \leq n/2$, the class of the cycle Φ is*

$$[\Phi] = 2^k \sigma_{k,k-1,\ldots,2,1}.$$

As an immediate application of this result, we can answer Keynote Question (c). To begin with, we asked how many lines lie on the intersection of two quadrics in $\mathbb{P}^4$. To answer this, let $Q, Q' \subset \mathbb{P}^4$ be two general quadric hypersurfaces and $X = Q_1 \cap Q_2$. The set of lines on X is the intersection $\Phi \cap \Phi'$ of the cycles of lines lying on Q and Q'; by Kleiman's theorem these are transverse, and so we have

$$\#(\Phi \cap \Phi') = \deg(4\sigma_{2,1})^2 = 16.$$

More generally, if Q and $Q' \subset \mathbb{P}^{2n}$ are general quadrics, we ask how many $(n-1)$-planes are contained in their intersection; again, this is the intersection number

$$\#(\Phi \cap \Phi') = \deg(2^n \sigma_{n,n-1,\ldots,1})^2 = 4^n.$$

4.7 Giambelli's formula

Pieri's formula tells us how to intersect an arbitrary Schubert class with one of the special Schubert classes $\sigma_b = \sigma_{b,0,\ldots,0}$. Giambelli's formula is complementary, in that it tells us how to express an arbitrary Schubert class in terms of special ones; the two together give us (in principle) a way of calculating the product of two arbitrary Schubert classes.

We will state Giambelli's formula and indicate one method of proof; see Chapter 12 for some special cases and Fulton [1997] for a proof in general.

Proposition 4.16 (Giambelli's formula)**.**

$$\sigma_{a_1,a_2,\dots,a_k} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{vmatrix}.$$

Thus, for example, we have

$$\sigma_{2,1} = \begin{vmatrix} \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_2\sigma_1 - \sigma_3,$$

which we can then use together with Pieri to evaluate $\sigma_{2,1}^2$, for example. Giambelli's formula also reproduces some formulas we have derived already by other means: For example, when $a_1 = a_2 = 1$ it gives

$$\sigma_{1,1} = \begin{vmatrix} \sigma_1 & \sigma_2 \\ \sigma_0 & \sigma_1 \end{vmatrix} = \sigma_1^2 - \sigma_2,$$

or in other words $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$.

As the last two examples suggest, we could deduce Giambelli's formula from Pieri's formula. For example, in the 2×2 case, we can expand the determinant and apply Pieri's formula to obtain

$$\begin{aligned} \begin{vmatrix} \sigma_a & \sigma_{a+1} \\ \sigma_{b-1} & \sigma_b \end{vmatrix} &= \sigma_a\sigma_b - \sigma_{a+1}\sigma_{b-1} \\ &= (\sigma_{a,b} + \sigma_{a+1,b-1} + \cdots + \sigma_{a+b}) - (\sigma_{a+1,b-1} + \cdots + \sigma_{a+b}) \\ &= \sigma_{a,b}. \end{aligned}$$

More generally, we could prove Giambelli's formula inductively by expanding the determinant in Proposition 4.16 by cofactors along the right-hand column; Exercise 4.39 asks the reader to do this in the 3×3 case.

Giambelli's formula implies that the Chow ring $A(G)$ of a Grassmannian G is generated as a ring by the special Schubert classes, and we can ask about the polynomial relations among these classes. There is a surprisingly simple and elegant description of these relations, which we will derive in Section 5.8, from the fact that the special Schubert classes are exactly the Chern classes of the universal bundles on the Grassmannian.

Giambelli's formula and Pieri's formula together give an algorithm for calculating the product of any two Schubert classes: Use Giambelli to express either as a polynomial in the special Schubert classes, and then use Pieri to evaluate the product of this polynomial with the other. But this is a terrible idea for computation except in low-dimensional examples: Because Giambelli's formula is determinantal, the number of products involved increases rapidly with k and n. Nor is it easy to use Giambelli's

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formula to prove qualitative results about products of Schubert classes; it is not even clear from this approach that such a product is necessarily a nonnegative linear combination of Schubert classes. The algorithms of Coşkun and Vakil referred to earlier (Coşkun [2009] and Vakil [2006a]) are far better in these regards.

4.8 Generalizations

Much of the analysis we have given here of the Chow rings of Grassmannians applies more generally to any compact homogeneous space for a semisimple algebraic group. In this section, we will describe some of these spaces, and indicate how the analysis goes in some of the simplest non-Grassmannian cases.

4.8.1 Flag manifolds

Let V be a vector space of dimension n and $(k_1, \dots, k_m)$ any sequence of integers with $0 < k_1 < \cdots < k_m < n$. We define the *flag manifold* $F(k_1, \dots, k_m; V)$ to be the space of nested sequences of subspaces of V of dimensions $k_1, \dots, k_m$; that is,

$$F(k_1, \dots, k_m; V) = \Big\{(\Lambda_1, \dots, \Lambda_m) \in \prod G(k_i, V) \;\Big|\; \Lambda_1 \subset \cdots \subset \Lambda_m\Big\}.$$

As in the case of the Grassmannian, when only the dimension of V matters we also use the symbol $F(k_1, \dots, k_m; n)$; also as in the case of Grassmannians, we will sometimes use the projective notation

$$\mathbb{F}(k_1, \dots, k_m; \mathbb{P}V) = \Big\{(\Lambda_1, \dots, \Lambda_m) \in \prod \mathbb{G}(k_i, \mathbb{P}V) \;\Big|\; \Lambda_1 \subset \cdots \subset \Lambda_m\Big\}$$

for $0 \le k_1 < \cdots < k_m < \dim \mathbb{P}V$. We leave as an exercise the verification that the condition $\Lambda_1 \subset \cdots \subset \Lambda_m$ defines a closed subscheme of the product $\prod G(k_i, V)$. (This follows immediately from the case $m = 2$, which is the content of Exercise 3.21.) In particular, $F(k_1, \dots, k_m; V)$ is a projective variety.

At one extreme we have the case $m = n-1$, that is, $(k_1, \dots, k_m) = (1, 2, \dots, n-1)$; the variety

$$F(1, 2, \dots, n-1; V) \subset \prod_{k=1}^{n-1} G(k, V)$$

is called the *full flag manifold*, and maps to all the other flag manifolds $F(k_1, \dots, k_m; V)$ via projections to subproducts of Grassmannians. At the other, the cases with $m = 1$ are just the ordinary Grassmannians, and the cases with $m = 2$ are called *two-step flag manifolds*. We have already encountered some of these: the variety

$$\mathbb{F}(0, k; V) = \{(p, \Lambda) \in \mathbb{P}V \times \mathbb{G}(k, \mathbb{P}V) \mid p \in \Lambda\}$$

is often called the *universal k-plane in $\mathbb{P}V$*.

Many of the aspects of the geometry of Grassmannians we have explored in the last two chapters hold more generally for flag manifolds. In particular, a flag manifold $\mathbb{F}$ admits an affine stratification, the classes of whose closed strata (again called *Schubert classes*) freely generate the Chow ring $A(\mathbb{F})$ as a group. It is possible to describe the ring structure on $A(\mathbb{F})$ in terms of these generators (see Coşkun [2009]), but there is an alternative: as we will see in Chapter 9 it is possible (and easier in some settings) to determine the ring $A(\mathbb{F})$ by realizing the flag manifold as a series of projective bundles.

4.8.2 Lagrangian Grassmannians and beyond

There are important generalizations of flag manifolds that are homogeneous spaces for semisimple algebraic groups other than GL_n. For example:

(a) *Lagrangian Grassmannians*: If V is a vector space of dimension $2n$ with a nondegenerate skew-symmetric bilinear form $Q : V \times V \to \Bbbk$, the Lagrangian Grassmannians $LG(k, V)$ parametrize k-dimensional subspaces $\Lambda \subset V$ that are isotropic for Q; that is, such that $Q(\Lambda, \Lambda) = 0$. More generally, we have *Lagrangian flag manifolds*, parametrizing flags of such subspaces.

(b) *Orthogonal Grassmannians*: As in the previous case, we consider a vector space V with nondegenerate bilinear form Q, but now Q is symmetric. The orthogonal Grassmannian parametrizes isotropic subspaces, and likewise the orthogonal flag manifolds parametrize flags of isotropic subspaces. (In case $\dim V$ is even, we have to allow for the fact that the space of maximal isotropic planes has two connected components.)

The subgroup of GL_n that fixes a full flag in $\Bbbk^n$ is the group B of upper-triangular matrices. This is called a *Borel* subgroup of GL_n. Since GL_n acts transitively on flags, the set of all such flags is GL_n / B; it can be given the structure of an algebraic variety by taking the regular functions on the quotient to be B-invariant functions on GL_n. (With this structure, it is isomorphic to the flag manifold as we have defined it.) More generally, one could look at partial flags (for example, a single k-dimensional subspace); these are fixed by groups of block upper-triangular matrices, called *parabolic subgroups*. Thus for example the ordinary Grassmannian $G(k, n)$ has the form GL_n / P, where P is a parabolic subgroup.

It turns out that there is a natural way of defining Borel subgroups and parabolic subgroups in any semisimple group, and the Lagrangian and orthogonal Grassmannians may similarly be defined as quotients of the groups SO_n and Sp_n. The theory of general flag manifolds to which this leads is an extremely rich branch of mathematics. See for example Fulton and Harris [1991].

4.9 Exercises

Exercise 4.17. Use the description of the points of the Schubert cells given in Theorem 4.1 to show that Theorem 4.3 holds at least set-theoretically.

Exercise 4.18. Let $X \subset G(2,4)$ be an irreducible surface, and suppose that

$$[X] = \gamma_2\sigma_2 + \gamma_{1,1}\sigma_{1,1} \in A^2(G(2,4)).$$

Show that γ_2 and $\gamma_{1,1}$ are nonnegative, and that if $\gamma_2 = 0$ then $\gamma_{1,1} = 1$. (In general, it is not known what pairs $(\gamma_2, \gamma_{1,1})$ occur!)

Exercise 4.19. Let $S \subset \mathbb{P}^4$ be a surface of degree d, and $\Gamma_S \subset \mathbb{G}(1,4)$ the variety of lines meeting S.

(a) Find the class $\gamma_S = [\Gamma_S] \in A^1(\mathbb{G}(1,4))$.
(b) Use this to answer the question: if $S_1, \dots, S_6 \subset \mathbb{P}^4$ are general translates (under GL_5) of surfaces of degrees $d_1, \dots, d_6$, how many lines in $\mathbb{P}^4$ will meet all six?

Exercise 4.20. Let $C \subset \mathbb{P}^4$ be a curve of degree d, and $\Gamma_C \subset \mathbb{G}(1,4)$ the variety of lines meeting C.

(a) Find the class $\gamma_C = [\Gamma_C] \in A^2(\mathbb{G}(1,4))$.
(b) Use this to answer the question: if C_1, C_2 and $C_3 \subset \mathbb{P}^4$ are general translates of curves of degrees d_1, d_2 and d_3, how many lines in $\mathbb{P}^4$ will meet all three?

The following exercise is the first of a series regarding the variety $T_1(S)$ of lines tangent to a surface S in $\mathbb{P}^n$. More will follow in Exercises 7.30, 10.39 and 12.20.

Exercise 4.21. Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g, and $T_1(S) \subset \mathbb{G}(1,n)$ the variety of lines tangent to S. To find the class of the cycle $T_1(S)$, we need the intersection numbers $[T_1(S)] \cdot \sigma_3$ and $[T_1(S)] \cdot \sigma_{2,1}$. Find the latter.

Exercise 4.22. Let $Z \subset \mathbb{G}(k,n)$ be a variety of dimension m, and consider the variety $X \subset \mathbb{P}^n$ swept out by the linear spaces corresponding to points of Z; that is,

$$X = \bigcup_{[\Lambda] \in Z} \Lambda \subset \mathbb{P}^n.$$

For simplicity, assume that a general point $x \in X$ lies on a unique k-plane $\Lambda \in Z$.

(a) Show that X has dimension $k+m$ and degree equal to the intersection number $\deg(\sigma_m \cdot [Z])$.
(b) Show that this is not in general the degree of Z.

Exercises 4.23-4.25 deal with the geometry of the surface described in Keynote Question (b): the surface $X \subset \mathbb{P}^3$ swept out by the lines corresponding to a general twisted cubic $C \subset \mathbb{G}(1,3)$, whose degree we worked out in Section 4.2.3. To make life easier, we will assume that C is general, and in particular that it lies in a general 3-plane section of $\mathbb{G}(1,3)$. See also Section 9.1.1.

Exercise 4.23. To start, use the fact that the dual of $\mathbb{G}(1,3) \subset \mathbb{P}^5$ has degree 2 to show that a general twisted cubic $C \subset \mathbb{G}(1,3)$ lies on the Schubert cycles $\Sigma_1(L)$ and $\Sigma_1(M)$ for some pair of skew lines $L, M \subset \mathbb{P}^3$.

Exercise 4.24. Show that for skew lines $L, M \subset \mathbb{P}^3$, the intersection $\Sigma_1(L) \cap \Sigma_1(M)$ is isomorphic to $L \times M$ via the map sending a point $[\Lambda] \in \Sigma_1(L) \cap \Sigma_1(M)$ to the pair $(\Lambda \cap L, \Lambda \cap M) \in L \times M$, and that it is the intersection of $\mathbb{G}(1,3)$ with the intersection of the hyperplanes spanned by $\Sigma_1(L)$ and $\Sigma_1(M)$.

Exercise 4.25. Using the fact that $C \subset \Sigma_1(L) \cap \Sigma_1(M)$ has bidegree $(2,1)$ in $\Sigma_1(L) \cap \Sigma_1(M) \cong L \times M \cong \mathbb{P}^1 \times \mathbb{P}^1$ (possibly after switching factors), show that for some degree-2 map $\varphi : L \to M$ the family of lines corresponding to C may be realized as the locus

$$C = \{\overline{p, \varphi(p)} \mid p \in L\}.$$

Show correspondingly that the surface

$$X = \bigcup_{[\Lambda] \in C} \Lambda \subset \mathbb{P}^3$$

swept out by the lines of C is a cubic surface double along a line, and that it is the projection of a rational normal surface scroll $S(1,2) \subset \mathbb{P}^4$.

In Section 4.3 we calculated the number of lines meeting four general n-planes in $\mathbb{P}^{2n+1}$. In the following two exercises, we will see another way to do this (analogous to the alternative count of lines meeting four lines in $\mathbb{P}^3$ given in Section 3.4.1), and a nice geometric sidelight.

Exercise 4.26. Let $\Lambda_1, \dots, \Lambda_4 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be four general n-planes. Calculate the number of lines meeting all four by showing that the union of the lines meeting Λ_1, Λ_2 and Λ_3 is a Segre variety $S_{1,n} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and using the calculation in Section 2.1.5 of the degree of $S_{1,n}$.

Exercise 4.27. By the preceding exercise, we can associate to a general configuration $\Lambda_1, \dots, \Lambda_4$ of k-planes in $\mathbb{P}^{2k+1}$ an unordered set of $k+1$ cross-ratios. Show that two such configurations $\{\Lambda_i\}$ and $\{\Lambda'_i\}$ are projectively equivalent if and only if the corresponding sets of cross-ratios coincide.

The next two exercises deal with the example of dynamic specialization given in Section 4.4, and specifically with the family Φ of cycles described there.

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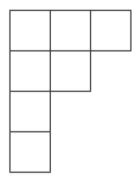
Exercise 4.28. Show that the support of Φ_0 is all of $\Sigma_{2,2}(P) \cup \Sigma_{3,1}(p_0, H_0)$.

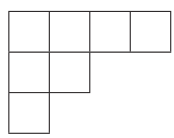
Exercise 4.29. Verify the last assertion made in the calculation of σ_2^2; that is, show that Φ_0 has multiplicity 1 along each component.
Hint: Argue that by applying a family of automorphisms of $\mathbb{P}^4$ we can assume that the plane H_t is constant, and use the calculation of the preceding chapter.

Exercise 4.30. A further wrinkle in the technique of dynamic specialization is that to carry out the calculation of an intersection of Schubert cycles we may have to specialize in stages. To see an example of this, use dynamic specialization to calculate the intersection σ_2^2 in the Grassmannian $\mathbb{G}(1, 5)$.
Hint: You have to let the two 2-planes specialize first to a pair intersecting in a point, then to a pair intersecting in a line.

Exercise 4.31. Suppose that the Schubert class $\sigma_a \in A(G(k, n))$ corresponds to the Young diagram Y in a $k \times (n - k)$ box B. Show that under the duality $G(k, n) \cong G(n - k, n)$ the class σ_a is taken to the Schubert class σ_b corresponding to the Young diagram Z that is the *transpose* of Y, that is, the diagram obtained by flipping Y around a 45° line running northwest-to-southeast. For example, if

$\sigma_{3,2,1,1} \in A(G(4, 7)) \quad \longleftrightarrow$  ,

then the corresponding Schubert class in $G(3, 7)$ is

$\sigma_{4,2,1} \in A(G(3, 7)) \quad \longleftrightarrow$  .

Exercise 4.32. Let $i : G(k, n) \to G(k+1, n+1)$ and $j : G(k, n) \to G(k, n+1)$ be the inclusions obtained by sending $\Lambda \subset \mathbb{k}^n$ to the span of Λ and e_{n+1} and to Λ respectively. Show that the map $i^* : A^d(G(k+1, n+1)) \to A^d(G(k, n))$ is a monomorphism if and only if $n - k \geq d$, and that $j^* : A^d(G(k, n+1)) \to A^d(G(k, n))$ is a monomorphism if and only if $k \geq d$. (Thus, for example, the formula

$$\sigma_1^2 = \sigma_2 + \sigma_{11},$$

which we established in $A(\mathbb{G}(1, 3))$, *holds true in every Grassmannian.*)

Exercise 4.33. Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, nondegenerate curve of degree d and genus g, and let $S_1(C) \subset \mathbb{G}(1, r)$ be the variety of chords to C, as defined in Section 3.4.3 above. Find the class $[S_1(C)] \in A_2(\mathbb{G}(1, r))$.

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Exercise 4.34. Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface, and let $T_k(Q) \subset \mathbb{G}(k,n)$ be the locus of planes $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap Q$ is singular. Show that

$$[T_k(Q)] = 2\sigma_1.$$

Exercise 4.35. Find the expression of $\sigma_{2,1}^2$ as a linear combination of Schubert classes in $A(G(3,6))$. This is the first example of a product of two Schubert classes where another Schubert class appears with coefficient > 1.

Exercise 4.36. Using Pieri's formula, determine all products of Schubert classes in the Chow ring of the Grassmannian $\mathbb{G}(2,5)$.

Exercise 4.37. Let Q, Q' and Q'' be three general quadrics in $\mathbb{P}^8$. How many 2-planes lie on all three? (Try first to do this without the tools introduced in Section 4.2.4.)

Exercise 4.38. Use Pieri to identify the degree of $\sigma_1^{k(n-k)}$ with the number of standard tableaux, that is, ways of filling in a $k \times (n-k)$ matrix with the integers $1, \dots, k(n-k)$ in such a way that every row and column is strictly increasing. Then use the "hook formula" (see, for example, Fulton [1997]) to show that this number is

$$(k(n-k))! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!}.$$

Exercise 4.39. Deduce Giambelli's formula in the 3×3 case (that is, the relation

$$\begin{vmatrix} \sigma_a & \sigma_{a+1} & \sigma_{a+2} \\ \sigma_{b-1} & \sigma_b & \sigma_{b+1} \\ \sigma_{c-2} & \sigma_{c-1} & \sigma_c \end{vmatrix} = \sigma_{a,b,c}$$

for any $a \geq b \geq c$) by assuming Giambelli in the 2×2 case, expanding the determinant by cofactors along the last column and applying Pieri.