Lecture 17

Let K be a non-archimedean local field.
Write $U_K = \mathcal{O}_K^\times$, $\pi \in \mathcal{O}_K$ unif.

Definition 13.10: For $s \in \mathbb{Z}_{\geq 1}$ the s^{th} unit group $U_K^{(s)}$ is defined by
$$U_K^{(s)} = (1 + \pi^s \mathcal{O}_K, \times).$$
Set $U_K^{(0)} = U_K$. Then we have
$$\dots \subseteq U_K^{(s)} \subseteq U_K^{(s-1)} \subseteq \dots \subseteq U_K^{(0)} = U_K.$$

Proposition 13.11:

(i) $U_K^{(0)}/U_K^{(1)} \cong (k^\times, \times)$ $\quad (k := \mathcal{O}_K/\pi)$

(ii) $U_K^{(s)}/U_K^{(s+1)} \cong (k, +)$ $\quad s \geq 1$.

Proof: (i) Reduction mod π
$$\mathcal{O}_K^\times \to k^\times \quad \text{surjective}$$
with kernel $1 + \pi\mathcal{O}_K = U_K^{(1)}$.

(ii)
$$f : U_K^{(s)} \to k$$
$$1 + \pi^s x \mapsto x \bmod \pi$$
$$(1+\pi^s x)(1+\pi^s y) = 1 + \pi^s(x+y+\pi^s xy).$$
$x + y + \pi^s(xy) \equiv x + y \bmod \pi$, hence
$\therefore f$ group hom., surjective with $\ker(f) = U_K^{(s+1)}$ $\square$

Corollary 13.12: Let $[K, \mathbb{Q}_p] < \infty$. Infinite index subgroup of $\mathcal{O}_K^\times$ isomorphic to $(\mathcal{O}_K,+)$

Proof: $r > \frac{e}{p-1}$, $(\mathcal{O}_K,+) \cong U_K^{(r)}$

$U_K^{(r)} \subseteq U_K$ finite index by Prop 13.11. $\square$

Remark: Not true for K equal char. – exp. not defined.

Eg. $\mathbb{Z}_p$, $p>2$, $e=1$, can take $r=1$.

then $\mathbb{Z}_p^\times \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

$x \mapsto \left(x \bmod p, \frac{x}{[x \bmod]p}\right)$

$p=2$, take $r=2$.

$\mathbb{Z}_2^\times \xrightarrow{\sim} (\mathbb{Z}/4\mathbb{Z})^\times \times \underbrace{(1+4\mathbb{Z}_2)}_{} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2$

$x \mapsto \left(x \bmod 4, \frac{x}{\varepsilon(x)}\right)$

Get another proof that $\quad \varepsilon(x) = \begin{cases} +1 & x \equiv 1 \bmod 4 \\ -1 & x \equiv -1 \bmod 4 \end{cases}$

$$\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & p>2 \\ (\mathbb{Z}/2\mathbb{Z})^2 & p=2 \end{cases}$$

§ Higher ramification groups.

L/K finite Galois extension of non-arch local fields

$\pi_L \in \mathcal{O}_L$ unif.

Definition 14.1: v_L normalized valuation on L.

For $s \in \mathbb{R}_{\geq -1}$, the s^{th} ramification group is

$$G_s(L/K) = \{\sigma \in Gal(L/K) \mid v_L(\sigma(x)-x) \geq s+1 \ \forall x \in \mathcal{O}_L\}$$

Eg. $G_{-1}(L/K) = Gal(L/K)$

$G_0(L/K) = \{\sigma \in Gal(L/K) \mid \sigma(x) \equiv x \bmod \pi$

$G_0(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma(x) = x \bmod \pi_L \ \forall x \in \mathcal{O}_L\}$

$= \ker(\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(k_L/k))$.

$= I_{L/K}$.

Note: For $s \in \mathbb{Z}_{\geq 0}$

$$G_s(L/K) = \ker\left(\mathrm{Gal}(L/K) \to \mathrm{Aut}\left(\mathcal{O}_L/\pi_L^{s+1}\mathcal{O}_L\right)\right)$$

hence $G_s(L/K)$ normal in G.

We have

$$\ldots \subseteq G_s \subseteq G_{s-1} \subseteq \ldots \subseteq G_{-1} = \mathrm{Gal}(L/K)$$

· Remark: G_s only changes at integers. G_s, $s \in \mathbb{R}_{\geq -1}$ used to define upper numbering.

Theorem 14.2: (i)

For $s \geq 1$, $G_s = \{\sigma \in G_0 \mid v_L(\sigma(\pi_L) - \pi_L) \geq s+1\}$

(ii) $\bigcap_{n=0}^{\infty} G_n = \{1\}$

(iii) Let $s \in \mathbb{Z}_{\geq 0}$. $\exists$ injective group hom.

$$G_s/G_{s+1} \hookrightarrow U_L^{(s)}/U_L^{(s+1)}$$

induced by $\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L}$. This map is independent of the choice of π_L.

Proof: Let $K_0 \subseteq L$ be max. unramified extension of K in L. Upon replacing K by K_0, WMA L/K totally ramified

(i) Theorem 15.8 $\Rightarrow \mathcal{O}_L = \mathcal{O}_K[\pi_L]$.

Suppose $v_L(\sigma(\pi_L) - \pi_L) \geq s+1$.

Let $x \in \mathcal{O}_L$, then $x = f(\pi_L)$, $f(X) = \mathcal{O}_K[X$

$$\sigma(x) - x = \sigma(f(\pi_L)) - f(\pi_L)$$

use $X^n - Y^n = (X-Y)(X^{n-1} + \dots + Y^{n-1})$

$$= f(\sigma(\pi_L)) - f(\pi_L)$$

$$= (\sigma(\pi_L) - \pi_L)\, g(\pi_L), \quad g(X) \in \mathcal{O}_K[$$

Thus $v_L(\sigma(x) - x) = v_L(\sigma(\pi_L) - \pi_L) + \underbrace{v_L(g(\pi_L))}_{\geq 0}$

$$\geq s+1.$$

(ii) Suppose $\sigma \in Gal(L/K)$, $\sigma \neq 1$.

Then $\sigma(\pi_L) \neq \pi_L$ because $L = K(\pi_L)$,

and hence $v_L(\sigma(\pi_L) - \pi_L) < \infty$.

Thus $\sigma \notin G_s$ for $s >> 0$.

(iii) Note: For $\sigma \in G_s$, $s \in \mathbb{Z}_{\geq 0}$,

$$\sigma(\pi_L) \in \pi_L + \pi_L^{s+1} \mathcal{O}_L$$

$$\Rightarrow \frac{\sigma(\pi_L)}{\pi_L} \in 1 + \pi_L^s \mathcal{O}_L$$

We claim $\varphi : G_s \longrightarrow U_L^{(s)} / U_L^{(s+1)}$

$$\sigma \longmapsto \frac{\sigma(\pi_L)}{\pi_L}$$

a group hom. with kernel G_{s+1}.

For $\sigma, \tau \in G_s$, let $\tau(\pi_L) = u\pi_L$, $u \in \mathcal{O}_L^\times$

Then $\underline{\sigma\tau(\pi_L)} = \underline{\sigma(\tau(\pi_L))} \cdot \underline{\tau(\pi_L)}$

$\pi_L \qquad \tau(\pi_L) \qquad \pi_L$

$$= \frac{\sigma(u)}{u} \cdot \frac{\sigma(\pi_L)}{\pi_L} \cdot \frac{\tau(\pi_L)}{\pi_L}$$

But $\sigma(u) \in u + \pi_L^{s+1}\theta_L$ since $\sigma \in G_s$, thus $\frac{\sigma(u)}{u} \in U_L^{(s+1)}$ and hence

$$\frac{\sigma\tau(\pi_L)}{\pi_L} \equiv \frac{\sigma(\pi_L)}{\pi_L} \cdot \frac{\tau(\pi_L)}{\pi_L} \mod U_L^{(s+1)}$$

$\Rightarrow \varphi$ is a group hom.

Moreover $\ker(\varphi) = \{\sigma \in G_s \mid \sigma(\pi_L) \equiv \pi_L \mod \pi_L^{s+2}\}$

$$= G_{s+1}.$$

If $\pi_L' = a\pi_L$ is another uniformiser, $a \in U_L$.

Then $\frac{\sigma(\pi_L')}{\pi_L'} = \frac{\sigma(a)}{a} \frac{\sigma(\pi_L)}{\pi_L} \equiv \frac{\sigma(\pi_L)}{\pi_L} \mod U_L^{(s+1)}$.

$\square$

Corollary 14.3: $Gal(L/K)$ is solvable.

Proof: By Proposition 13.10 + Theorem 14.2 + Theorem 13.4, for $s \in \mathbb{Z}_{\geq -1}$

$$G_s/G_{s+1} \cong \text{a subgroup of } \begin{cases} Gal(k_L/k) & \text{if } s = -1. \\ (k_L^\times, \times) & \text{if } s = 0 \\ (k_L, +) & \text{if } s \geq 1. \end{cases}$$

Thus G_s/G_{s+1} is solvable for $s \geq -1$.

Conclude using Theorem 14.2(ii). $\square$

Let char $k = 0$. Then $|G_0/G_1|$ is coprime

[illegible]
to p and $|G_1| = p^n$ some $n \geq 0$. Thus G_1 is the unique (since normal) Sylow-p subgroup of $G_0 = I_{L/K}$.

Definition 14.4: The group G_1 is called the wild inertia group and G_0/G_1 is the tame quotient.