Chapter 1

Introducing the Chow ring

Keynote Questions

As we indicated in the introduction, we will preface each chapter of this book with a series of “keynote questions:” examples of the sort of concrete problems that can be solved using the ideas and techniques introduced in that chapter. In general, the answers to these questions will be found in the same chapter. In the present case, we will not develop our roster of examples sufficiently to answer the keynote questions below until the second chapter; we include them here so that the reader can have some idea of “what the subject is good for” in advance.

(1) Let F_0, F_1 and $F_2 \in \Bbbk[X, Y, Z]$ be three general homogeneous cubic polynomials in three variables. Up to scalars, how many linear combinations $t_0F_0 + t_1F_1 + t_2F_2$ factor as a product of a linear and a quadratic polynomial? (Answer on page 65.)

(2) Let F_0, F_1, F_2 and $F_3 \in \Bbbk[X, Y, Z]$ be four general homogeneous cubic polynomials in three variables. How many linear combinations $t_0F_0 + t_1F_1 + t_2F_2 + t_3F_3$ factor as a product of three linear polynomials? (Answer on page 65.)

(3) If A, B, C are general homogeneous quadratic polynomials in three variables, for how many triples $t = (t_0, t_1, t_2)$ do we have

$$(A(t), B(t), C(t)) = (t_0, t_1, t_2)?$$

(Answer on page 55.)

(4) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $L \subset \mathbb{P}^3$ a general line. How many planes containing L are tangent to S? (Answer on page 50.)

(5) Let $L \subset \mathbb{P}^3$ be a line, and let S and $T \subset \mathbb{P}^3$ be surfaces of degrees s and t containing L. Suppose that the intersection $S \cap T$ is the union of L and a smooth curve C. What are the degree and genus of C? (Answer on page 71.)

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1.1 The goal of intersection theory

Though intersection theory has many and surprising applications, in its most basic form it gives information about the intersection of two subvarieties of a given variety. An early incarnation, and in some sense the model for all of intersection theory, is the theorem of Bézout: If plane curves $A, B \subset \mathbb{P}^2$ intersect transversely, then they intersect in $(\deg A)(\deg B)$ points (see Figure 1.3 on page 18).

If A is a line, this is a special case of Gauss' fundamental theorem of algebra: A polynomial $f(x)$ in one complex variable has $\deg f$ roots, if the roots are counted with multiplicity. Late in the 19th century it was understood how to attribute multiplicities to the intersections of any two plane curves without common components (we shall describe this in Section 1.3.7 below), so Bézout's theorem could be extended: The intersection of two plane curves without common components consists a collection of points with multiplicities adding up to $(\deg A)(\deg B)$.

In modern geometry we need to understand intersections of subvarieties in much greater generality. In this book we will mostly consider intersections of arbitrary subvarieties in a smooth ambient variety X. The goal of this chapter is to introduce a ring $A(X)$, called the *Chow ring* of X, and to associate to every subscheme $A \subset X$ a class $[A]$ in $A(X)$ generalizing the degree of a curve in $\mathbb{P}^2$. In Section 1.3.7 we will explain a far-reaching extension of Bézout's theorem:

Theorem 1.1 (Bézout's theorem for dimensionally transverse intersections)**.** *If $A, B \subset X$ are subvarieties of a smooth variety X and* $\operatorname{codim}(A \cap B) = \operatorname{codim} A + \operatorname{codim} B$, *then we can associate to each irreducible component C_i of $A \cap B$ a positive integer $m_{C_i}(A, B)$ in such a way that*

$$[A][B] = \sum m_{C_i}(A, B) \cdot [C_i].$$

The integer $m_{C_i}(A, B)$ is called the *intersection multiplicity of A and B along C_i*; giving a correct definition in this generality occupied algebraic geometers for most of the first half of the 20th century.

Though Theorem 1.1 is restricted to the case where the subvarieties A, B meet only in codimension $\operatorname{codim} A + \operatorname{codim} B$ (the case of *dimensionally proper intersection*), there is a very useful extension to the case where the codimensions of the components of the intersection are arbitrary; this will be discussed in Chapter 13.

Many important applications involve subvarieties defined as zero loci of sections of a vector bundle $\mathcal{E}$ on a variety X, and this idea has potent generalizations. It turns out that there is a way of defining classes $c_i(\mathcal{E}) \in A(X)$, called the *Chern classes* of $\mathcal{E}$, and the theory of Chern classes is a pillar of intersection theory. The third and final section of this chapter takes up a special case of the general theory that is of particular importance and relatively easy to describe: the first Chern class of a line bundle. This allows us to introduce the *canonical class*, a distinguished element of the Chow ring of any smooth variety, and show how to calculate it in simple cases. The general theory of Chern classes will be taken up in Chapter 5.

1.2 The Chow ring

We now turn to the definition and basic properties of the Chow ring. Then we introduce excision and Mayer–Vietoris theorems that allow us to calculate the Chow rings of many varieties. Most importantly we describe the functoriality of the Chow ring: the existence, under suitable circumstances, of pushforward and pullback maps.

Chow groups form a sort of homology theory for quasi-projective varieties; that is, they are abelian groups associated to a geometric object that are described as a group of cycles modulo an equivalence relation. In the case of a smooth variety, the intersection product makes the Chow groups into a graded ring, the Chow ring. This is analogous to the ring structure on the homology of smooth compact manifolds that can be imported, using Poincaré duality, from the natural ring structure on cohomology.

Throughout this book we will work over an algebraically closed ground field $\Bbbk$ of characteristic 0. Virtually everything we do could be formulated over arbitrary fields (though not every statement remains true in characteristic p), and occasionally we comment on how one would do this.

1.2.1 Cycles

Let X be any algebraic variety (or, more generally, scheme). The *group of cycles* on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties (reduced irreducible subschemes) of X. The group $Z(X)$ is graded by dimension: we write $Z_k(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension k (these are called k*-cycles*), so that $Z(X) = \bigoplus_k Z_k(X)$. A cycle $Z = \sum n_i Y_i$, where the Y_i are subvarieties, is *effective* if the coefficients n_i are all nonnegative. A *divisor* (sometimes called a *Weil divisor*) is an $(n-1)$-cycle on a pure n-dimensional scheme. It follows from the definition that $Z(X) = Z(X_{\text{red}})$; that is, $Z(X)$ is insensitive to whatever nonreduced structure X may have.

To any closed subscheme $Y \subset X$ we associate an effective cycle $\langle Y \rangle$: If $Y \subset X$ is a subscheme, and $Y_1, \dots, Y_s$ are the irreducible components of the reduced scheme Y_{red}, then, because our schemes are Noetherian, each local ring $\mathcal{O}_{Y,Y_i}$ has a finite composition series. Writing l_i for its length, which is well-defined by the Jordan–Hölder theorem (Theorem 0.3), we define the cycle $\langle Y \rangle$ to be the formal combination $\sum l_i Y_i$. (The coefficient l_i is called the *multiplicity* of the scheme Y along the irreducible component Y_i, and written $\operatorname{mult}_{Y_i}(Y)$; we will discuss this notion, and its relation to the notion of intersection multiplicity, in Section 1.3.8.)

In this sense cycles may be viewed as coarse approximations to subschemes.

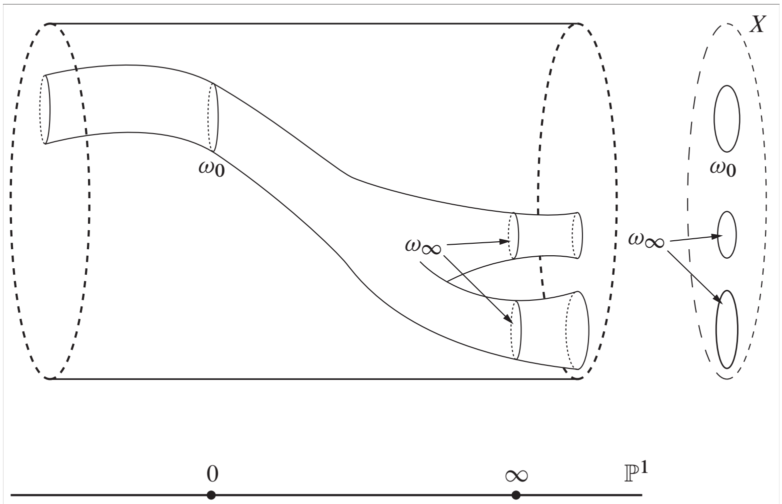

Figure 1.1 Rational equivalence between two cycles ω_0 and ω_∞ on X.

1.2.2 Rational equivalence and the Chow group

The *Chow group* of X is the group of cycles of X modulo *rational equivalence*. Informally, two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them — that is, a cycle on $\mathbb{P}^1 \times X$ whose restrictions to two fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are A_0 and A_1. Here is the formal definition:

Definition 1.2. Let $\operatorname{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (\{t_0\} \times X)\rangle - \langle \Phi \cap (\{t_1\} \times X)\rangle,$$

where $t_0, t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. We say that two cycles are *rationally equivalent* if their difference is in $\operatorname{Rat}(X)$, and we say that two subschemes are rationally equivalent if their associated cycles are rationally equivalent — see Figures 1.1 and 1.2.

Definition 1.3. The *Chow group* of X is the quotient

$$A(X) = Z(X)/\operatorname{Rat}(X),$$

the *group of rational equivalence classes of cycles on* X. If $Y \in Z(X)$ is a cycle, we write $[Y] \in A(X)$ for its equivalence class; if $Y \subset X$ is a subscheme, we abuse notation slightly and denote simply by $[Y]$ the class of the cycle $\langle Y \rangle$ associated to Y.

It follows from the principal ideal theorem (Theorem 0.1) that the Chow group is graded by dimension:

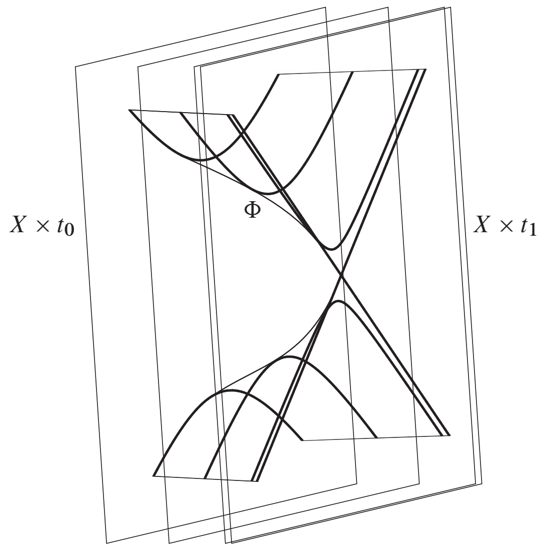

Figure 1.2 Rational equivalence between a hyperbola and the union of two lines in $\mathbb{P}^2$.

Proposition 1.4. *If X is a scheme then the Chow group of X is graded by dimension; that is,*

$$A(X) = \bigoplus A_k(X),$$

with $A_k(X)$ the group of rational equivalence classes of k-cycles.

Proof: If $\Phi \subset \mathbb{P}^1 \times X$ is an irreducible variety not contained in a fiber over X then, in an appropriate affine open set $\Phi \cap (\mathbb{A}^1 \times X) \subset \Phi$, the scheme $\Phi \cap (\{t_0\} \times X)$ is defined by the vanishing of the single nonzerodivisor $t - t_0$. It follows that the components of this intersection are all of codimension exactly 1 in Φ, and similarly for $\Phi \cap (\{t_1\} \times X)$. Thus all the varieties involved in the rational equivalence defined by Φ have the same dimension. □

When X is equidimensional we may define the *codimension* of a subvariety $Y \subset X$ as $\dim X - \dim Y$, and it follows that we may also grade the Chow group by codimension. When X is also smooth, we will write $A^c(X)$ for the group $A_{\dim X - c}$, and think of it as the group of codimension-c cycles, modulo rational equivalence. (It would occasionally be convenient to adopt the same notation when X is singular, but this would conflict with established convention — see the discussion in Section 2.5 below.)

1.2.3 Transversality and the Chow ring

We said at the outset that much of what we hope to do in intersection theory is modeled on the classical Bézout theorem: that if plane curves $A, B \subset \mathbb{P}^2$ of degrees d and e intersect transversely then they intersect in de points. Two things about this

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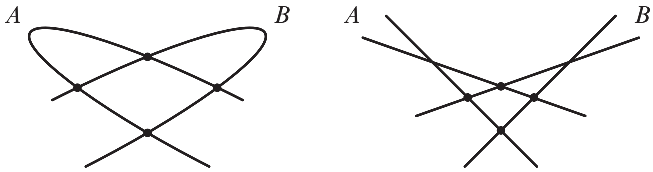

Figure 1.3 Two conics meet in four points.

result are striking. First, the cardinality of the intersection does not depend on the choice of curves, beyond knowing their degrees and that they meet transversely. Given this invariance, the theorem follows from the obvious fact that a union of d general lines meets a union of e general lines in de points (Figure 1.3).

Second, the answer, de, is a product, suggesting that some sort of ring structure is present. A great deal of the development of algebraic geometry over the past 200 years is bound up in the attempt to understand, generalize and apply these ideas, leading to precise notions of the sense in which intersection of subvarieties resembles multiplication. What makes the Chow groups useful is that, under good circumstances, the rational equivalence class of the intersection of two subvarieties A, B depends only on the rational equivalence classes of A and B, and this gives a product structure on the Chow groups of a smooth variety.

To make this statement precise we need some definitions. We say that subvarieties A, B of a variety X intersect *transversely* at a point p if A, B and X are all smooth at p and the tangent spaces to A and B at p together span the tangent space to X; that is,

$$T_p A + T_p B = T_p X,$$

or equivalently

$$\operatorname{codim}(T_p A \cap T_p B) = \operatorname{codim} T_p A + \operatorname{codim} T_p B.$$

We will say that subvarieties $A, B \subset X$ are *generically transverse*, or that they intersect *generically transversely*, if they meet transversely at a general point of each component C of $A \cap B$. The terminology is justified by the fact that the set of points of $A \cap B$ at which A and B are transverse is open. We extend the terminology to cycles by saying that two cycles $A = \sum n_i A_i$ and $B = \sum m_j B_j$ are generically transverse if each A_i is generically transverse to each B_j.

More generally, we will say subvarieties $A_i \subset X$ intersect transversely at a smooth point $p \in X$ if p is a smooth point on each A_i and $\operatorname{codim}\left(\bigcap T_p A_i\right) = \sum \operatorname{codim} T_p A_i$, and we say that they intersect generically transversely if there is a dense set of points in the intersection at which they are transverse.

As an example, if A and B have complementary dimensions in X (that is, if $\dim A + \dim B = \dim X$), then A and B are generically transverse if and only if they are transverse everywhere; that is, their intersection consists of finitely many points and they intersect transversely at each of them. (In this case we will accordingly drop the modifier "generically.") If $\operatorname{codim} A + \operatorname{codim} B > \dim X$, then A and B are generically transverse if and only if they are disjoint.

Theorem–Definition 1.5. *If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the condition:*

(∗) *If two subvarieties A, B of X are generically transverse, then*

$$[A][B] = [A \cap B].$$

This structure makes

$$A(X) = \bigoplus_{c=0}^{\dim X} A^c(X)$$

into an associative, commutative ring, graded by codimension, called the Chow ring *of X.*

Fulton [1984] gave a direct construction of the product of cycles on any smooth variety over any field, and proved that the products of rationally equivalent cycles are rationally equivalent. In a setting where the first half of the moving lemma (Theorem 1.6 below) holds, such as a smooth, quasi-projective variety over an algebraically closed field, this product is characterized by the condition (∗) of Theorem–Definition 1.5.

Even if X is smooth and A, B are subvarieties such that every component of $A \cap B$ has the expected codimension $\operatorname{codim} A + \operatorname{codim} B$, we cannot define $[A][B] \in A(X)$ to be $[A \cap B]$, because the class $[A \cap B]$ depends on more than the rational equivalence classes of A and B. This problem can be solved by assigning intersection multiplicities to the components; see Section 1.3.7.

1.2.4 The moving lemma

Historically, the proof of Theorem–Definition 1.5 was based on the *moving lemma*. This has two parts:

Theorem 1.6 (Moving lemma)**.** *Let X be a smooth quasi-projective variety.*

(a) For every $\alpha, \beta \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ with $[A] = \alpha$ and $[B] = \beta$.
(b) The class $[A \cap B]$ is independent of the choice of such cycles A and B.

A proof of the first part is given in Appendix A; this is sufficient to establish the uniqueness of a ring structure on $A(X)$ satisfying the condition (∗) of Theorem–Definition 1.5.

The second part, which historically was used to prove the existence portion of Theorem–Definition 1.5, is more problematic; as far as we know, no complete proof existed prior to the publication of Fulton [1984].

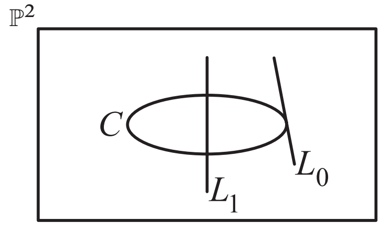

Figure 1.4 The cycle L_0 can be "moved" to the rationally equivalent cycle L_1, which is transverse to the given subvariety C.

The first half of the moving lemma is useful in shaping our understanding of intersection products and occasionally as a tool in the proof of assertions about them, and we will refer to it when relevant.

On a singular variety the moving lemma may fail: For example, if $X \subset \mathbb{P}^3$ is a quadric cone then any two cycles representing the class of a line of X meet at the origin, a singular point of X, and thus cannot be generically transverse (see Exercise 1.36). Further, the hypothesis of smoothness in Theorem 1.5 cannot be avoided: We will also see in Section 2.5 examples of varieties X where no intersection product satisfying the basic condition $(*)$ of Theorem 1.5 can be defined. The news is not uniformly negative: Intersection products *can* be defined on singular varieties if we impose some restrictions on the classes involved, as we will see in Proposition 1.31.

Kleiman's transversality theorem

There is one circumstance in which the first half of the moving lemma is relatively easy: when a sufficiently large group of automorphisms acts on X, we can use automorphisms to move cycles to make them transverse. Here is a special case of a result of Kleiman:

Theorem 1.7 (Kleiman's theorem in characteristic 0)**.** *Suppose that an algebraic group G acts transitively on a variety X over an algebraically closed field of characteristic 0, and that $A \subset X$ is a subvariety.*

(a) If $B \subset X$ is another subvariety, then there is an open dense set of $g \in G$ such that gA is generically transverse to B.
(b) More generally, if $\varphi : Y \to X$ is a morphism of varieties, then for general $g \in G$ the preimage $\varphi^{-1}(gA)$ is generically reduced and of the same codimension as A.
(c) If G is affine, then $[gA] = [A] \in A(X)$ for any $g \in G$.

Proof: (a) This is the special case $Y = B$ of (b).

(b) Let the dimensions of X, A, Y and G be n, a, b and m respectively. If $x \in X$, then the map $G \to X : g \mapsto gx$ is surjective and its fibers are the cosets of the stabilizer of x in G. Since all these fibers have the same dimension, this dimension must be $m - n$. Set

$$\Gamma = \{(x, y, g) \in A \times Y \times G \mid gx = \varphi(y)\}.$$

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Because G acts transitively on X, the projection $\pi : \Gamma \to A \times Y$ is surjective. Its fibers are the cosets of stabilizers of points in X, and hence have dimension $m - n$. It follows that Γ has dimension

$$\dim \Gamma = a + b + m - n.$$

On the other hand, the fiber over g of the projection $\Gamma \to G$ is isomorphic to $\varphi^{-1}(gA)$. Thus either this intersection is empty for general g, or else it has dimension $a + b - n$, as required.

Since X is a variety it is smooth at a general point. Since G acts transitively, all points of X look alike, so X is smooth. Since any algebraic group in characteristic 0 is smooth (see for example Lecture 25 of Mumford [1966]), the fibers of the projection to $A \times Y$ are also smooth, so Γ itself is smooth over $A_{\mathrm{sm}} \times Y_{\mathrm{sm}}$. Since field extensions in characteristic 0 are separable, the projection $(\Gamma \setminus \Gamma_{\mathrm{sing}}) \to G$ is smooth over a nonempty open set of G, where Γ_{sing} is the singular locus of Γ. That is, the general fiber of the projection of Γ to G is smooth outside Γ_{sing}. If the projection of Γ_{sing} to G is not dominant, then $\varphi^{-1}(gA)$ is smooth for general g.

To complete the proof of generic transversality, we may assume that the projection $\Gamma_{\mathrm{sing}} \to G$ is dominant. Since G is smooth, the principal ideal theorem shows that every component of every fiber of $\Gamma \to G$ has codimension $\leq \dim G$, and thus every component of the general fiber has codimension exactly $\dim G$ in Γ. Since $\Gamma_{\mathrm{sing}} \to G$ is dominant, its general fiber has dimension $\dim \Gamma_{\mathrm{sing}} - \dim G < \dim \Gamma - \dim G$, so no component of a general fiber can be contained in Γ_{sing}. Thus $\varphi^{-1}(gA)$ is generically reduced for general $g \in G$.

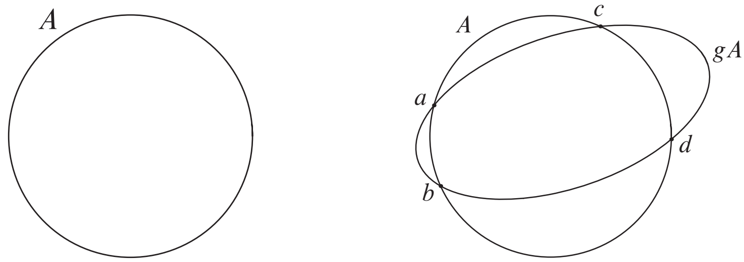
(c) We will prove this part only for the case where G is a product of copies of GL_n, as this is the only case we will use. For the general result, see Theorem 18.2 of Borel [1991].

In this case G is an open set in a product M of vector spaces of matrices. Let L be the line joining 1 to g in M. The subvariety

$$Z = \{(g, x) \in (G \cap L) \times X \mid g^{-1}x \in A\}$$

gives a rational equivalence between A and gA. □

The conclusion fails in positive characteristic, even for Grassmannians; examples can be found in Kleiman [1974] and Roberts [1972b]. However, Kleiman showed that the conclusion holds in general under the stronger hypothesis that G acts transitively on nonzero tangent vectors to X (each tangent space to the Grassmannian is naturally identified with a space of homomorphisms — see Section 3.2.4 — and the automorphisms preserve the ranks of these homomorphisms, so they do not act transitively on tangent vectors).



$$[A]^2 = [A]\,[gA] = [a + b + c + d]$$

Figure 1.5 The cycle A meets a general translate of itself generically transversely.

1.3 Some techniques for computing the Chow ring

1.3.1 The fundamental class

If X is a scheme, then the *fundamental class* of X is $[X] \in A(X)$. It is always nonzero. We can immediately prove this and a little more, and these first results suffice to compute the Chow ring of a zero-dimensional scheme:

Proposition 1.8. *Let X be a scheme.*

(a) $A(X) = A(X_{\text{red}})$.
(b) If X is irreducible of dimension k, then $A_k(X) \cong \mathbb{Z}$ and is generated by the fundamental class *of X. More generally, if the irreducible components of X are $X_1, \ldots, X_m$, then the classes $[X_i]$ generate a free abelian subgroup of rank m in $A(X)$.*

Proof: (a) Since both cycles and rational equivalences are generated by varieties we have $Z(X) = Z(X_{\text{red}})$ and $\text{Rat}(X) = \text{Rat}(X_{\text{red}})$.

(b) By definition the $[X_i]$ are among the generators of $A(X)$. Further, $\text{Rat}(X)$ is generated by varieties in $\mathbb{P}^1 \times X$, each of which is contained in some $\mathbb{P}^1 \times X_i$. □

Example 1.9 (Zero-dimensional schemes)**.** From Proposition 1.8 it follows that the Chow group of a zero-dimensional scheme is the free abelian group on the components.

1.3.2 Rational equivalence via divisors

The next simplest case is that of curves, and it is not hard to see that the Chow group of 0-cycles on a curve is the divisor class group.

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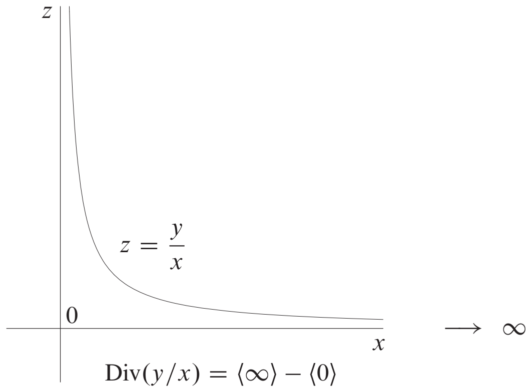

Figure 1.6 Graph of the rational function $z = y/x$ on the open set $y = 1$ in $\mathbb{P}^1$, showing that $[V(y)] - [V(x)] = 0$ in $A(\mathbb{P}^1)$.

More generally, for any variety X we can express the group $\mathrm{Rat}(X)$ of cycles rationally equivalent to 0 in terms of divisor classes: First, suppose that X is an affine variety. If $f \in \mathcal{O}_X$ is a function on X other than 0, then by Krull's principal ideal theorem (Theorem 0.1) the irreducible components of the subscheme defined by f are all of codimension 1, so the cycle defined by this subscheme is a divisor; we call it the *divisor of* f, denoted $\mathrm{Div}(f)$. If Y is any irreducible codimension-1 subscheme of X, we write $\mathrm{ord}_Y(f)$ for the order of vanishing of f along Y, so we have

$$\mathrm{Div}(f) = \sum_{Y \subset X \text{ irreducible}} \mathrm{ord}_Y(f)\langle Y\rangle.$$

If f, g are functions on X and $\alpha = f/g$, then we define the divisor $\mathrm{Div}(\alpha) = \mathrm{Div}(f/g)$ to be $\mathrm{Div}(f) - \mathrm{Div}(g)$; see Figure 1.6. This is well-defined because $\mathrm{ord}_Y(ab) = \mathrm{ord}_Y(a) + \mathrm{ord}_Y(b)$ for any functions defined on an open set. We denote by $\mathrm{Div}_0(\alpha)$ and $\mathrm{Div}_\infty(\alpha)$ the positive and negative parts of $\mathrm{Div}(\alpha)$ — in other words, the divisor of zeros of α and the divisor of poles of α, respectively.

We extend the definition of the divisor associated to a rational function to varieties X that are not affine as follows. The field of rational functions on X is the same as the field of rational functions on any open affine subset U of X, so if α is a rational function on X then we get a divisor $\mathrm{Div}(\alpha|_U)$ on each open subset $U \subset X$ by restricting α. These agree on overlaps, and thus define a divisor $\mathrm{Div}(\alpha)$ on X itself. We will see that the association $\alpha \mapsto \mathrm{Div}(\alpha)$ is a homomorphism from the multiplicative group of nonzero rational functions to the additive group of divisors on X.

Proposition 1.10. *If X is any scheme, then the group* $\mathrm{Rat}(X) \subset Z(X)$ *is generated by all divisors of rational functions on all subvarieties of X. In particular, if X is irreducible of dimension n, then $A_{n-1}(X)$ is equal to the divisor class group of X.*

See Fulton [1984, Proposition 1.6] for the proof.

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Example 1.11. It follows from Proposition 1.10 that two 0-cycles on a curve C (by which we mean here a one-dimensional variety) are rationally equivalent if and only if they differ by the divisor of a rational function. In particular, the cycles associated to two points on C are rationally equivalent if and only if C is birational to $\mathbb{P}^1$, the isomorphism being given by a rational function that defines the rational equivalence.

Example 1.12. If X is an affine variety whose coordinate ring R does not have unique factorization, then there may not be a "best" way of choosing an expression of a rational function α on X as a fraction, and $\mathrm{Div}_0(\alpha)$ need not be the same thing as $\mathrm{Div}(f)$ for any one representation $\alpha = f/g$ of α. For example, on the cone $Q = V(XZ - Y^2) \subset \mathbb{A}^3$, the rational function $\alpha = X/Y$ has divisor $L - M$, where L is the line $X = Y = 0$ and M the line $Y = Z = 0$; but, as the reader can check, α cannot be written in any neighborhood of the vertex $(0, 0, 0)$ of Q as a ratio $\alpha = f/g$ with $\mathrm{Div}(f) = L$ and $\mathrm{Div}(g) = M$.

1.3.3 Affine space

Affine spaces are basic building blocks for many rational varieties, such as projective spaces and Grassmannians, and it is easy to compute their Chow groups directly:

Proposition 1.13. $A(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$.

Proof: Let $Y \subset \mathbb{A}^n$ be a proper subvariety, and choose coordinates $z = z_1, \dots, z_n$ on $\mathbb{A}^n$ so that the origin does not lie in Y. We let

$$W^\circ = \{(t, tz) \subset (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n \mid z \in Y\} = V(\{f(z/t) \mid f(z) \text{ vanishes on } Y\}).$$

The fiber of W° over a point $t \in \mathbb{A}^1 \setminus \{0\}$ is tY, that is, the image of Y under the automorphism of $\mathbb{A}^n$ given by multiplication by t. Let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of W° in $\mathbb{P}^1 \times \mathbb{A}^n$. Note that W°, being the image of $(\mathbb{A}^1 \setminus 0) \times Y$, is irreducible, and hence so is W.

The fiber of W over the point $t = 1$ is just Y. On the other hand, since the origin in $\mathbb{A}^n$ does not lie in Y there is some polynomial $g(z)$ that vanishes on Y and has a nonzero constant term c. The function $G(t, z) = g(z/t)$ on $(\mathbb{A}^1 \setminus 0) \times \mathbb{A}^n$ then extends to a regular function on $(\mathbb{P}^1 \setminus 0) \times \mathbb{A}^n$ with constant value c on the fiber $\infty \times \mathbb{A}^n$. Thus the fiber of W over $t = \infty \in \mathbb{P}^1$ is empty, establishing the equivalence $Y \sim 0$ (see Figure 1.7). □

See Section 3.5.2 for a more systematic treatment of this idea. If you are curious about the fiber of W over $t = 0$, see Exercise 1.34.

1.3.4 Mayer–Vietoris and excision

We will use the next proposition in conjunction with Proposition 1.13 to find generators for the Chow groups of projective spaces and Grassmannians.

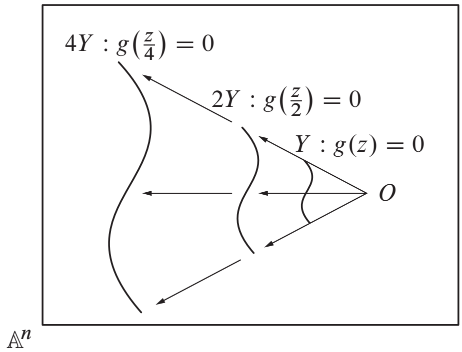

Figure 1.7 Scalar multiplication gives a rational equivalence between an affine variety not containing the origin and the empty set.

Proposition 1.10 makes it obvious that, if $Y \subset X$ is a closed subscheme, then the identification of the cycles on $\mathbb{P}^1 \times Y$ as cycles on $\mathbb{P}^1 \times X$ induces a map $\mathrm{Rat}(Y) \to \mathrm{Rat}(X)$, and thus a map $A(Y) \to A(X)$ (this is a special case of "proper pushforward;" see Section 1.3.6). Further, the intersection of a subvariety of X with the open set $U = X \setminus Y$ is a subvariety of U (possibly empty), so there is a restriction homomorphism $Z(X) \to Z(U)$. The rational equivalences restrict too, so we get a homomorphism of Chow groups $A(X) \to A(U)$ (this is a special case of "flat pullback;" see Section 1.3.6.)

Proposition 1.14. *Let X be a scheme.*

(a) *(Mayer–Vietoris) If X_1, X_2 are closed subschemes of X, then there is a right exact sequence*

$$A(X_1 \cap X_2) \longrightarrow A(X_1) \oplus A(X_2) \longrightarrow A(X_1 \cup X_2) \longrightarrow 0.$$

(b) *(Excision) If $Y \subset X$ is a closed subscheme and $U = X \setminus Y$ is its complement, then the inclusion and restriction maps of cycles give a right exact sequence*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0.$$

If X is smooth, then the map $A(X) \to A(U)$ is a ring homomorphism.

Before starting the proof, we note that we can restate the definition of the Chow group by saying that there is a right exact sequence

$$Z(\mathbb{P}^1 \times X) \longrightarrow Z(X) \longrightarrow A(X) \longrightarrow 0,$$

where the left-hand map takes the any subvariety $\Phi \subset \mathbb{P}^1 \times X$ to 0 if Φ is contained in a fiber, and otherwise to

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle.$$

Proof of Proposition 1.14: (b) There is a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & Z(Y \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial_Y"'] & Z(X \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial_X"'] & Z(U \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial_U"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[r] \arrow[d] & 0 \\
 & A(Y) \arrow[r] \arrow[d] & A(X) \arrow[r] \arrow[d] & A(U) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where the map $Z(Y) \to Z(X)$ takes the class $[A] \in Z(Y)$, where A is a subvariety of Y, to $[A]$ itself, considered as a class in X, and similarly for $Z(Y \times \mathbb{P}^1) \to Z(X \times \mathbb{P}^1)$. The map $Z(X) \to Z(U)$ takes each free generator $[A]$ to the generator $[A \cap U]$, and similarly for $Z(X \times \mathbb{P}^1) \to Z(U \times \mathbb{P}^1)$. The two middle rows and all three columns are evidently exact. A diagram chase shows that the map $A(X) \to A(U)$ is surjective, and the bottom row of the diagram above is right exact, yielding part (b).

(a) Let $Y = X_1 \cap X_2$. We may assume $X = X_1 \cup X_2$. We may argue exactly as in part (b) from the diagram

$$
\begin{tikzcd}
0 \arrow[r] & Z(Y \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial"'] & Z(X_1 \times \mathbb{P}^1) \oplus Z(X_2 \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial \oplus \partial"'] & Z(X \times \mathbb{P}^1) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X_1) \oplus Z(X_2) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & 0 \\
 & A(Y) \arrow[r] & A(X_1) \oplus A(X_2) \arrow[r] & A(X) \arrow[r] & 0
\end{tikzcd}
$$

where, for example, the map $Z(Y) \to Z(X_1) \oplus Z(X_2)$ takes a generator $[A] \in Z(Y)$ to $([A], -[A]) \in Z(X_1) \oplus Z(X_2)$ and the map $Z(X_1) \oplus Z(X_2) \to Z(X)$ is addition. $\square$

The map $A(Y) \to A(X)$ of part (b) sends the class $[Z] \in A(Y)$ of a subvariety Z of Y to the class $[Z] \in A(X)$ of the same subvariety, now viewed as a subvariety of X. As we will see in Section 1.3.6, this is a special case of the pushforward map $f_* : A(Y) \to A(X)$ associated to any proper map $f : Y \to X$. The map $A(X) \to A(U)$, sending the class $[Z] \in A(X)$ of a subvariety of X to the class $[Z \cap U] \in A(U)$ of its intersection with U, is a special case of a pullback map, also described in Section 1.3.6.

Corollary 1.15. *If $U \subset \mathbb{A}^n$ is a nonempty open set, then $A(U) = A_n(U) = \mathbb{Z} \cdot [U]$.*

1.3.5 Affine stratifications

In general we will work with very partial knowledge of the Chow groups of a variety, but when X admits an *affine stratification* — a special kind of decomposition into a

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union of affine spaces — we can know them completely. This will help us compute the Chow groups of projective space, Grassmannians, and many other interesting rational varieties.

We say that a scheme X is *stratified* by a finite collection of irreducible, locally closed subschemes U_i if X is a disjoint union of the U_i and, in addition, the closure of any U_i is a union of U_j — in other words, if $\overline{U_i}$ meets U_j, then $\overline{U_i}$ contains U_j. The sets U_i are called the *strata* of the stratification, while the closures $Y_i := \overline{U_i}$ are called the *closed strata*. (If we want to emphasize the distinction, we will sometimes refer to the strata U_i as the *open strata* of the stratification, even though they are not open in X.) The stratification can be recovered from the closed strata Y_i: we have

$$U_i = Y_i \setminus \bigcup_{Y_j \subsetneq Y_i} Y_j.$$

Definition 1.16. We say that a stratification of X with strata U_i is:

- *affine* if each open stratum is isomorphic to some $\mathbb{A}^k$; and
- *quasi-affine* if each U_i is isomorphic to an open subset of some $\mathbb{A}^k$.

For example, a complete flag of subspaces $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ gives an affine stratification of projective space; the closed strata are just the $\mathbb{P}^i$ and the open strata are affine spaces $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$.

Proposition 1.17. *If a scheme X has a quasi-affine stratification, then $A(X)$ is generated by the classes of the closed strata.*

Proof of Proposition 1.17: We will induct on the number of strata U_i. If this number is 1 then the assertion is Corollary 1.15.

Let U_0 be a minimal stratum. Since the closure of U_0 is a union of strata, U_0 must already be closed. It follows that $U := X \setminus U_0$ is stratified by the strata other than U_0. By induction, $A(U)$ is generated by the classes of the closures of these strata, and, by Corollary 1.15, $A(U_0)$ is generated by $[U_0]$. By excision (part (b) of Proposition 1.14) the sequence

$$\mathbb{Z} \cdot [U_0] = A(U_0) \longrightarrow A(X) \longrightarrow A(X \setminus U_0) \longrightarrow 0$$

is right exact. Since the classes in $A(U)$ of the closed strata in U come from the classes of (the same) closed strata in X, it follows that $A(X)$ is generated by the classes of the closed strata. $\square$

In general, the classes of the strata in a quasi-affine stratification of a scheme X may be zero in $A(X)$; for example, the affine line, with $A(\mathbb{A}^1) = \mathbb{Z}$, also has a quasi-affine stratification consisting of a single point and its complement, and we have already seen that the class of a point is 0. But in the case of an affine stratification, the classes are not only nonzero, they are independent:

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Theorem 1.18 (Totaro [2014]). *The classes of the strata in an affine stratification of a scheme X form a basis of $A(X)$.*

We will often use results that are consequences of this deep theorem, although in our cases much more elementary proofs are available, as we shall see.

1.3.6 Functoriality

A key to working with Chow groups is to understand how they behave with respect to morphisms between varieties. To know what to expect, think of the analogous situation with homology and cohomology. A smooth complex projective variety of (complex) dimension n is a compact oriented $2n$-manifold, so $H_{2m}(X, \mathbb{Z})$ can be identified canonically with $H^{2n-2m}(X, \mathbb{Z})$ (singular homology and cohomology). If we think of $A(X)$ as being analogous to $H_*(X, \mathbb{Z})$, then we should expect $A_m(X)$ to be a covariant functor from smooth projective varieties to groups, via some sort of pushforward maps preserving dimension. If we think of $A(X)$ as analogous to $H^*(X, \mathbb{Z})$, then we should expect $A(X)$ to be a contravariant functor from smooth projective varieties to rings, via some sort of pullback maps preserving codimension. Both these expectations are realized.

Proper pushforward

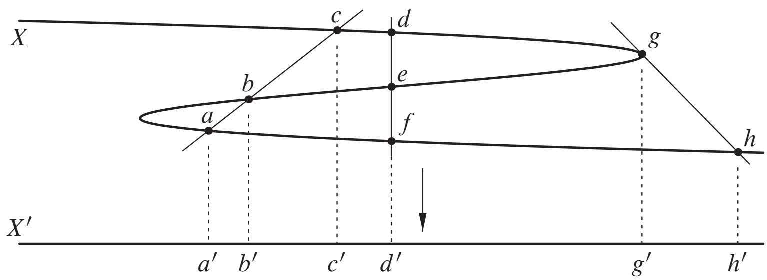
If $f : Y \to X$ is a proper map of schemes, then the image of a subvariety $A \subset Y$ is a subvariety $f(A) \subset X$. One might at first guess that the pushforward could be defined by sending the class of A to the class of $f(A)$, and this would not be far off the mark. But this would not preserve rational equivalence (an example is pictured in Figure 1.8). Rather, we must take multiplicities into account.

If $A \subset Y$ is a subvariety and $\dim A = \dim f(A)$, then $f|_A : A \to f(A)$ is *generically finite*, in the sense that the field of rational functions $\Bbbk(A)$ is a finite extension of the field $\Bbbk(f(A))$ (this follows because they are both finitely generated fields, of the same transcendence degree $\dim A$ over the ground field). Geometrically the condition can be expressed by saying that, for a general point $x \in f(A)$, the preimage $y := f|_A^{-1}(x)$ in A is a finite scheme. In this case the degree $n := [\Bbbk(A) : \Bbbk(f(A))]$ of the extension of rational function fields is equal to the degree of y over x for a dense open subset of $x \in f(A)$, and this common value n is called the *degree* of the covering of $f(A)$ by A. We must count $f(A)$ with multiplicity n in the pushforward cycle:

Definition 1.19 (Pushforward for cycles). Let $f : Y \to X$ be a proper map of schemes, and let $A \subset Y$ be a subvariety.

(a) If $f(A)$ has strictly lower dimension than A, then we set $f_*\langle A\rangle = 0$.
(b) If $\dim f(A) = \dim A$ and $f|_A$ has degree n, then we set $f_*\langle A\rangle = n \cdot \langle f(A)\rangle$.
(c) We extend f_* to all cycles on Y by linearity; that is, for any collection of subvarieties $A_i \subset Y$, we set $f_*\left(\sum m_i\langle A_i\rangle\right) = \sum m_i f_*\langle A_i\rangle$.

$$a + b + c \sim d + e + f \sim 2g + h$$



$$a' + b' + c' \sim 3d' \sim 2g' + h'$$

Figure 1.8 Pushforwards of equivalent cycles are equivalent.

With this definition, the pushforward of cycles preserves rational equivalence:

Theorem 1.20. *If $f : Y \to X$ is a proper map of schemes, then the map $f_* : Z(Y) \to Z(X)$ defined above induces a map of groups $f_* : A_k(Y) \to A_k(X)$ for each k.*

For a proof see Fulton [1984, Section 1.4].

It is often hard to prove that a given class in $A(X)$ is nonzero, but the fact that the pushforward map is well-defined gives us a start:

Proposition 1.21. *If X is proper over* $\operatorname{Spec} \Bbbk$, *then there is a unique map* $\deg : A(X) \to \mathbb{Z}$ *taking the class $[p]$ of each closed point $p \in X$ to 1 and vanishing on the class of any cycle of pure dimension* > 0.

As stated, Proposition 1.21 uses our standing hypothesis that the ground field is algebraically closed. Without this restriction we would have to count each (closed) point by the degree of its residue field extension over the ground field.

We will typically use this proposition together with the intersection product: If A is a k-dimensional subvariety of a smooth projective variety X and B is a k-codimensional subvariety of X such that $A \cap B$ is finite and nonempty, then the map

$$A_k(X) \to \mathbb{Z} : \ [Z] \mapsto \deg([Z][B])$$

sends $[A]$ to a nonzero integer. Thus no integer multiple $m[A]$ of the class A could be 0.

Pullback

We next turn to the pullback. Let $f : Y \to X$ be a morphism and $A \subset X$ a subvariety of codimension c. A good pullback map $f^* : Z(X) \to Z(Y)$ on cycles should preserve rational equivalence, and, in the nicest case, for example when $f^{-1}(A)$ is generically reduced of codimension c, it should be geometric, in the sense that

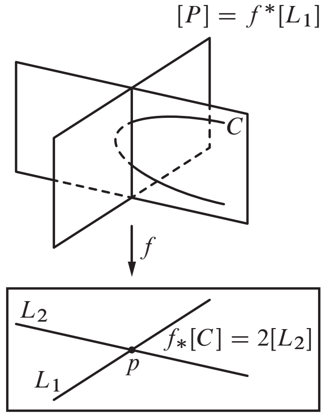


Figure 1.9 $2[p] = f_*([P][C]) = f_*([f^*L_1][C]) = [L_1]f_*[C] = [L_1][2L_2]$.

$f^*\langle A\rangle = \langle f^{-1}(A)\rangle$. This equality does not hold for all cycles, but does hold when A is a Cohen–Macaulay variety. (Recall that a scheme is said to be Cohen–Macaulay if all its local rings are Cohen–Macaulay. For a treatment of Cohen–Macaulay rings see Eisenbud [1995, Chapter 18].)

We start with a definition:

Definition 1.22. Let $f : Y \to X$ be a morphism of smooth varieties. We say a subvariety $A \subset X$ is *generically transverse* to f if the preimage $f^{-1}(A)$ is generically reduced and $\operatorname{codim}_Y(f^{-1}(A)) = \operatorname{codim}_X(A)$.

With that said, we have the following fundamental theorem:

Theorem 1.23. *Let $f : Y \to X$ be a map of smooth quasi-projective varieties.*

(a) There is a unique map of groups $f^* : A^c(X) \to A^c(Y)$ such that whenever $A \subset X$ is a subvariety generically transverse to f we have

$$f^*([A]) = [f^{-1}(A)].$$

This equality is also true without the hypothesis of generic transversality as long as $\operatorname{codim}_Y(f^{-1}(A)) = \operatorname{codim}_X(A)$ and A is Cohen–Macaulay. The map f^* is a ring homomorphism, and makes A into a contravariant functor from the category of smooth projective varieties to the category of graded rings.

(b) (Push-pull formula) The map $f_* : A(Y) \to A(X)$ is a map of graded modules over the graded ring $A(X)$. More explicitly, if $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$, then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

The last statement of this theorem is the result of applying appropriate multiplicities to the set-theoretic equality $f(f^{-1}(A) \cap B) = A \cap f(B)$; see Figure 1.9.

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One simple case of a projective morphism is the inclusion map from a closed subvariety $i : Y \subset X$. When X and Y are smooth, our definitions of intersections and pullbacks make it clear that, if A is any subvariety of X, then $[A][Y]$ is represented by the same cycle as $i^*([A])$ — except that these are considered as classes in different varieties. More precisely, we can write

$$[A][Y] = i_*(i^*[A]).$$

In this case the extra content of Theorem 1.23 is that this cycle is well-defined as a cycle on Y, not only as a cycle on X. Fulton [1984, Section 8.1] showed that it is even well-defined as a class on $X \cap Y$, and, more generally, he proved the existence of such a refined version of the pullback under a proper, locally complete intersection morphism (of which a map of smooth projective varieties is an example).

The uniqueness statement in Theorem 1.23 follows at once upon combining the moving lemma with the following:

Theorem 1.24. *If $f : Y \to X$ is a morphism of smooth quasi-projective varieties, then there is a finite collection of subvarieties $X_i \subset X$ such that if a subvariety $A \subset X$ is generically transverse to each X_i then A is generically transverse to f.*

(See Theorem A.6.) Note that this result depends on characteristic 0; it fails when f is not generically separable.

Pullback in the flat case

The flat case is simpler than the projective case for two reasons: first, the preimage of a subvariety of codimension k is always of codimension k; second, rational functions on the target pull back to rational functions on the source. We will use the flat case to analyze maps of affine space bundles.

Theorem 1.25. *Let $f : Y \to X$ be a flat map of schemes. The map $f^* : A(X) \to A(Y)$ defined on cycles by*

$$f^*(\langle A\rangle) := \langle f^{-1}(A)\rangle \quad \textit{for every subvariety } A \subset X$$

preserves rational equivalence, and thus induces a map of Chow groups preserving the grading by codimension.

When X and Y are smooth and f is flat, the two pullback maps agree, as one sees at once from the uniqueness statement in Theorem 1.23.

1.3.7 Dimensional transversality and multiplicities

When two subvarieties A, B of a smooth variety X meet generically transversely, then we have

$$[A][B] = [A \cap B] \in A(X). \tag{$*$}$$

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Does this formula hold more generally? Clearly it cannot hold unless the intersection $A \cap B$ has the expected dimension.

Theorem 1.26. *Let $A, B \subset X$ be subvarieties of a smooth variety X such that every irreducible component C of the intersection $A \cap B$ has codimension* $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$. *For each such component C there is a positive integer $m_C(A, B)$, called the* intersection multiplicity of A and B along C, *such that:*

(a) $[A][B] = \sum m_C(A, B)[C] \in A(X)$.

(b) $m_C(A, B) = 1$ *if and only if A and B intersect transversely at a general point of C.*

(c) In case A and B are Cohen–Macaulay at a general point of C, then $m_C(A, B)$ is the multiplicity of the component of the scheme $A \cap B$ supported on C. In particular, if A and B are everywhere Cohen–Macaulay, then

$$[A][B] = [A \cap B].$$

(d) $m_C(A, B)$ *depends only on the local structure of A and B at a general point of C.*

For further discussion of this result see Hartshorne [1977, Appendix A], and for a full treatment see Fulton [1984, Chapter 7]. In view of Theorem 1.26, we make a definition:

Definition 1.27. Two subschemes A and B of a variety X are *dimensionally transverse* if for every irreducible component C of $A \cap B$ we have $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$.

The reader should be aware that what we call "dimensionally transverse" is often called "proper" in the literature. We prefer "dimensionally transverse" since it suggests the meaning (and "proper" means so many different things!).

Recall that if X is smooth and C is a component of $A \cap B$, then by Theorem 0.2 we have $\operatorname{codim} C \leq \operatorname{codim} A + \operatorname{codim} B$, so in this case the condition of dimensional transversality is that A and B intersect in the smallest possible dimension. (But note that $A \cap B$ may also be empty. In this case too, A and B are transverse.)

The Cohen–Macaulay hypothesis in part (c) is necessary: in Example 2.6 we will see a case where the intersection multiplicity is not given by the multiplicities of the components of the intersection scheme.

Given that we sometimes have $[A \cap B] \neq [A][B]$, it is natural to look for a correction term. This was found by Jean-Pierre Serre; we will describe it in Theorem 2.7, following Example 2.6.

Remarkably, it is often possible to describe the intersection product $[A][B]$ of the classes of subvarieties $A, B \subset X$ geometrically even when they are not dimensionally transverse. See Chapter 13.

Just as we say that cycles $A = \sum m_i A_i$ and $B = \sum n_j Bj$ are generically transverse if A_i and B_j are generically transverse for all i, j, we say that A and B are dimensionally transverse if A_i and B_j are dimensionally transverse for every i, j.

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The following explains the amount by which generic transversality is stronger than dimensional transversality.

Proposition 1.28. *Subschemes A and B of a variety X are generically transverse if and only if they are dimensionally transverse and each irreducible component of $A \cap B$ contains a point where X is smooth and $A \cap B$ is reduced.*

In particular, the proposition shows that, if X is smooth and A, B are dimensionally transverse subschemes that meet in a subvariety C, then A and B are generically transverse along C. The hypothesis that X is smooth cannot be dropped: For example, in the coordinate ring $\Bbbk[s^2, st, t^2]$ the ideal (s^2) defines a double line through the vertex that meets the reduced line defined by (st, t^2) in a reduced point.

Proof: If A and B are generically transverse, then each irreducible component C of $A \cap B$ contains a smooth point $p \in X$ such that A and B are smooth and transverse at p. It follows that C is smooth at p, and thus, in particular, C is reduced at p.

To prove the converse, let C be an irreducible component of $A \cap B$. Since the set of smooth points of X is open, and since by hypothesis C contains one, the smooth points of X that are contained in C form an open dense subset of C. Since $A \cap B$ is generically reduced, the open set where C is reduced is also dense, and it follows that the same is true for the smooth locus of C. Thus there is a point $p \in C$ that is smooth on both C and X. We must show that A and B are smooth at p.

The Zariski tangent space to C at p is the intersection of the Zariski tangent spaces $T_p A$ and $T_p B$ in $T_p X$. Since C and X are smooth at p,

$$\begin{aligned}\dim C = \dim T_p C &= \dim T_p A + \dim T_p B - \dim T_p X \\ &= \dim T_p A + \dim T_p B - \dim X.\end{aligned}$$

By hypothesis,

$$\dim C = \dim A + \dim B - \dim X.$$

Since $\dim T_p A \geq \dim A$ and $\dim T_p B \geq \dim B$, we must have $\dim T_p A = \dim A$ and $\dim T_p B = \dim B$, proving that A and B are smooth at p as well. Since the tangent spaces of A, B, X at p are equal to the corresponding Zariski tangent spaces, the equality

$$\dim T_p C = \dim T_p A + \dim T_p B - \dim T_p X$$

above completes the proof. □

1.3.8 The multiplicity of a scheme at a point

In connection with the discussion of intersection multiplicities above, we collect here the basic facts about the multiplicity of a scheme at a point; for details, see Eisenbud [1995, Chapter 12]. We will also indicate, at least in some cases, how intersection multiplicities are related to multiplicities of schemes.

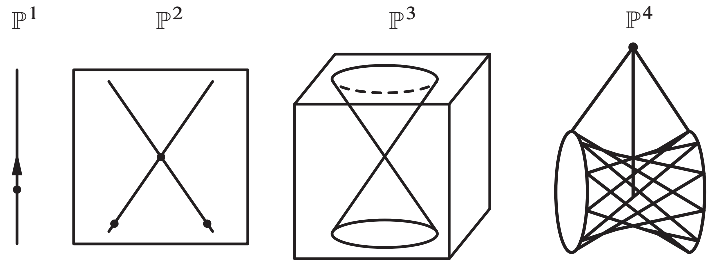

Figure 1.10 Ordinary double points of hypersurfaces of dimension 0, 1, 2 and 3.

Any discussion of the multiplicity of a scheme at a point begins with the case of a hypersurface in a smooth n-dimensional variety Z. In this case, we can be very explicit: If $p \in Z$ and $X \subset Z$ is a hypersurface given locally around p as the zero locus of a regular function f, we can choose local coordinates $z = (z_1, \dots, z_n)$ on Z in a neighborhood of p and expand f around p, writing

$$f(z) = f_0 + f_1(z) + f_2(z) + \cdots$$

with $f_k(z)$ homogeneous of degree k. The hypersurface X contains p if $f_0 = f(p) = 0$, and is then singular at p if $f_1 = 0$. In general, we say that X has *multiplicity* m at p if $f_0 = \cdots = f_{m-1} = 0$ and $f_m \neq 0$; we write $\operatorname{mult}_p(X)$ for the multiplicity of X at p. (If $m = 2$ we say that p is a *double point* of X; if $m = 3$ we say p is a *triple point*, and so on.) We define the *tangent cone* TC_pX of X at p to be the zero locus of f_m in the affine space $\mathbb{A}^n$ with coordinates $(z_1, \dots, z_n)$, and similarly we define the *projectivized tangent cone* $\mathbb{T}C_pX$ of X at p to be the scheme in $\mathbb{P}^{n-1}$ defined by f_m.

We can say this purely in terms of the local ring $\mathcal{O}_{Z,p}$, without the need to invoke local coordinates: If $\mathfrak{m} \subset \mathcal{O}_{Z,p}$ is the maximal ideal, the multiplicity of X at p is the largest m such that $f \in \mathfrak{m}^m$. We can then take f_m to be the image of f in the quotient $\mathfrak{m}^m/\mathfrak{m}^{m+1}$. Note that since

$$\mathfrak{m}^m/\mathfrak{m}^{m+1} = \operatorname{Sym}^m(\mathfrak{m}/\mathfrak{m}^2) = \operatorname{Sym}^m T_p^*Z,$$

the vector space of homogeneous polynomials of degree m on the Zariski tangent space T_pZ, we can view the projectivized tangent cone as a subscheme of $\mathbb{P}T_pZ$. (Note that the projectivized tangent cone may be nonreduced even though X itself is reduced at p, as in the case of a cusp, given locally as the zero locus of $y^2 - x^3$.) The multiplicity can also be characterized in these terms simply as the degree of the projectivized tangent cone.

For example, the simplest possible singularity of a hypersurface X, generalizing the case of a node of a plane curve, is called an *ordinary double point*. This is a point $p \in X$ such that the equation of X can be written in local coordinates with $p = 0$ as above with $f_0 = f_1 = 0$ and where f_2 is a *nondegenerate quadratic form* — that is, the projectivized tangent cone to X at p is a smooth quadric. Indeed, examples are the cones over smooth quadrics — see Figure 1.10. (Here it is important that the characteristic is not 2: A quadric in $\mathbb{P}^{n-1}$ is smooth if the generator f_2 of its ideal, together with the derivatives of f_2, is an irrelevant ideal; when the characteristic is not 2, Euler's formula $2f_2 = \sum z_i \partial f_2/\partial z_i$ shows that it is equivalent to assume that the partial derivatives of f_2 are linearly independent, and this is the property we will often use. In characteristic 2 — where a symmetric bilinear form is also skew-symmetric — *no* quadratic form in an odd number of variables has this property.)

How do we extend this definition to arbitrary schemes X? The answer is to start by defining the tangent cones. We can do this explicitly in terms of local coordinates $z = (z_1, \dots, z_n)$ on a smooth ambient variety Z containing X: We define the tangent cone to be the subscheme of $\mathbb{A}^n$ defined by the leading terms of *all* elements of the ideal $I \subset \mathcal{O}_{Z,p}$ of X at p, and the projectivized tangent cone to be the corresponding subscheme of $\mathbb{P}^{n-1}$.

As before, this can be said without recourse to local coordinates (or, for that matter, any ambient variety Z). To start, we filter the local ring $\mathcal{O}_{X,p}$ by powers of its maximal ideal $\mathfrak{m}$:

$$\mathcal{O}_{X,p} \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \cdots .$$

We then form the associated graded ring

$$A = \Bbbk \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots ,$$

and define the tangent cone and projectivized tangent cone to be Spec A and Proj A respectively. Note that since the ring A is generated in degree 1, we have a surjection

$$\mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{Sym}(T_p^* X) \to A,$$

so that these can be viewed naturally as subschemes of the Zariski tangent space $T_p X$ and its projectivization, respectively. As we will see shortly, one important feature of these constructions is that we always have

$$\dim TC_p X = \dim X \quad \text{and} \quad \dim \mathbb{T}C_p X = \dim X - 1,$$

even though the dimension of the Zariski tangent space may be larger.

We now define the multiplicity $\mathrm{mult}_p(X)$ of X at p to be the degree of the projectivized tangent cone $\mathbb{T}C_p X$, viewed as a subscheme of the projective space $\mathbb{P}T_p X$. In purely algebraic terms, we can express this directly in terms of the Hilbert polynomial of the graded ring A: If we set

$$h_A(m) = \dim_{\Bbbk} A_m,$$

then for $m \gg 0$ the function h_A will be equal to a polynomial $p_A(m)$ of degree $\dim X - 1$, called the *Hilbert polynomial* of A. The multiplicity $\mathrm{mult}_p(X)$ is then equal to $(\dim X - 1)!$ times the leading coefficient of the Hilbert polynomial $p_A(m)$.

It follows from the theory that the multiplicity $\mathrm{mult}_{Y_i}(Y)$ of a scheme Y along an irreducible component Y_i of Y, as introduced in Section 1.2.1 in connection with the definition of the cycle associated to a scheme, is equal to the multiplicity of Y at a general point of Y_i.

Tangent cones and blow-ups

There is another characterization of the projectivized tangent cone that will be very useful to us in what follows.

We start by recalling some basic facts about blow-ups. Blowing-up is an operation that associates to any scheme Z and subscheme Y a morphism $\pi : B = \mathrm{Bl}_Y(Z) \to Z$. The general operation is described and characterized in Chapter 4 of Eisenbud and Harris [2000]; in the present circumstances, we will be concerned with the case where Y is a smooth point $p \in Z$.

The *exceptional divisor* $E \subset B$ is defined to be $\pi^{-1}(Y)$, the preimage of Y in B. If $X \subset Z$ is any subscheme, we define its *strict transform* $\widetilde{X} \subset B$ to be the closure in B of the preimage $\pi^{-1}(X \setminus Y \cap X)$ of X away from Y.

Suppose that X is embedded in a smooth ambient variety Z of dimension n, and consider the blow-up of Z at p. In this case the exceptional divisor E is isomorphic to $\mathbb{P}^{n-1}$. Unwinding the definitions, we can see that *the projectivized tangent cone* $\mathbb{T}C_pX$ *to* X *at* p *is the intersection of* $\widetilde{X}$ *with* $E \cong \mathbb{P}^{n-1}$. This gives us immediately that $\dim \mathbb{T}C_pX = \dim X - 1$.

Again, we can say this without having to choose an embedding of X in a smooth Z: Since blow-ups behave well with respect to pullbacks (see Proposition IV-21 of Eisenbud and Harris [2000]), we could simply say that $\mathbb{T}C_pX$ is the exceptional divisor in the blow-up $\mathrm{Bl}_p(X)$.

Multiplicities and intersection multiplicities

The notions of multiplicity (of a scheme at a point) and intersection multiplicities (of two subschemes meeting dimensionally transversely in a smooth ambient variety) are closely linked: If $p \in X \subset \mathbb{P}^n$ is a point on a subscheme of pure dimension k and $\Lambda \cong \mathbb{P}^{n-k} \subset \mathbb{P}^n$ is a general $(n-k)$-plane containing p, then the intersection multiplicity $m_p(X, \Lambda)$ is equal to $\mathrm{mult}_p(X)$.

This statement can be generalized substantially:

Proposition 1.29. *Let X and Y be two subschemes of complementary dimension intersecting dimensionally properly in a smooth variety Z, and $p \in X \cap Y$ any point of intersection. If the projectivized tangent cones $\mathbb{T}C_pX$ and $\mathbb{T}C_pY$ are disjoint in $\mathbb{P}T_pZ$, then*

$$m_p(X, Y) = \mathrm{mult}_p(X) \cdot \mathrm{mult}_p(Y).$$

This proposition is proved in Section 2.1.10. In general, there is only the inequality $m_p(X, Y) \geq \mathrm{mult}_p(X) \cdot \mathrm{mult}_p(Y)$; see Fulton [1984, Chapter 12].

1.4 The first Chern class of a line bundle

Many of the most interesting and useful classes in the Chow groups come from vector bundles via the theory of Chern classes. The simplest case is that of the first Chern class of a line bundle, which we will now describe. We will introduce the theory in more generality in Chapter 5.

If $\mathcal{L}$ is a line bundle on a variety X and σ is a rational section, then on an open affine set U of a covering of X we may write σ in the form f_U/g_U and define $\mathrm{Div}(\sigma)|_U = \mathrm{Div}(f) - \mathrm{Div}(g)$. This definition agrees where two affine open sets overlap, and thus defines a divisor on X, which is a *Cartier divisor* (see Hartshorne [1977, Section II.6]). Moreover, if τ is another rational section of $\mathcal{L}$ then $\alpha = \sigma/\tau$ is a well-defined rational function, so

$$\mathrm{Div}(\sigma) - \mathrm{Div}(\tau) = \mathrm{Div}(\alpha) \equiv 0 \bmod \mathrm{Rat}(X).$$

Thus for any line bundle $\mathcal{L}$ on a quasi-projective scheme X we may define the *first Chern class*

$$c_1(\mathcal{L}) \in A(X)$$

to be the rational equivalence class of the divisor σ for any nonzero rational section σ. (If we were working over an arbitrary scheme, we would have to insist that the numerator and denominator of our section were locally nonzerodivisors.) Note that there is no distinguished cycle in the equivalence class. As a first example, we see that $c_1(\mathcal{O}_{\mathbb{P}^n}(d))$ is the class of any hypersurface of degree d; in the notation of Section 2.1 it is $d\zeta$, where ζ is the class of a hyperplane.

Recall that the Picard group $\mathrm{Pic}(X)$ is by definition the group of isomorphism classes of line bundles $\mathcal{L}$ on X, with addition law $[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}']$.

Proposition 1.30. *If X is a variety of dimension n, then c_1 is a group homomorphism*

$$c_1 : \mathrm{Pic}(X) \to A_{n-1}(X).$$

If X is smooth, then c_1 is an isomorphism.

If $Y \subset X$ is a divisor in a smooth variety X, then the ideal sheaf of Y is a line bundle denoted $\mathcal{O}_X(-Y)$, and its inverse in the Picard group is denoted $\mathcal{O}_X(Y)$. The inverse of the map c_1 above takes $[Y]$ to $\mathcal{O}_X(Y)$.

Proof of Proposition 1.30: To see that c_1 is a group homomorphism, suppose that $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on X. If σ and σ' are rational sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively, then $\sigma \otimes \sigma'$ is a rational section of $\mathcal{L} \otimes \mathcal{L}'$ whose divisor is $\mathrm{Div}(\sigma) + \mathrm{Div}(\sigma')$.

Now assume that X is smooth and projective. Since the local rings of X are unique factorization domains, every codimension-1 subvariety is a Cartier divisor, so to any divisor we can associate a unique line bundle and a rational section. Forgetting the section, we get a line bundle, and thus a map from the group of divisors to $\mathrm{Pic}(X)$. By

Proposition 1.10, rationally equivalent divisors differ by the divisor of a rational function, and thus correspond to different rational sections of the same bundle. It follows that the map on divisors induces a map on $A_{n-1}(X)$, inverse to the map c_1. □

If X is singular, the map $c_1 : \mathrm{Pic}(X) \to A_{n-1}(X)$ is in general neither injective or surjective. For example, if X is an irreducible plane cubic with a node, then $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is not a monomorphism (Exercise 1.35). On the other hand, if $X \subset \mathbb{P}^3$ is a quadric cone with vertex p, then $A_1(X) = \mathbb{Z}$ and is generated by the class of a line, and the image of $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is $2\mathbb{Z}$ (Exercise 1.36).

Another case when the moving lemma is easy is when the class of the cycle to be moved has the form $c_1(\mathcal{L})$ for some line bundle $\mathcal{L}$. We also get a useful formula for the product of any class with $c_1(\mathcal{L})$:

Proposition 1.31. *Suppose that X is a smooth quasi-projective variety and $\mathcal{L}$ is a line bundle on X. If $Y_1, \dots, Y_n$ are any subvarieties of X, then there is a cycle in the class of $c_1(\mathcal{L})$ that is generically transverse to each Y_i. If X is smooth and $Y \subset X$ is any subvariety, then*

$$c_1(\mathcal{L}) \cdot [Y] = c_1(\mathcal{L}|_Y).$$

The class $c_1(\mathcal{L}|_Y)$ on the right-hand side of the formula is actually a class in $A(Y)$, so to be precise we should have written $i_*(c_1(\mathcal{L}|_Y))$, where $i : Y \hookrightarrow X$ is the inclusion and i_* the pushforward map, first encountered in Proposition 1.14 and defined in general in Section 1.3.6. This imprecision points to an important theoretical fact: Even on a singular variety (or scheme) X one can form the intersection product of any class with the first Chern class of a line bundle, defined (when the class is the class of a subscheme) via the prescription $c_1(\mathcal{L}) \cdot [Y] = c_1(\mathcal{L}|_Y)$ above.

This intersection is actually defined by the formula as a class on Y, not just a class on X. This is the beginning of the theory of "refined intersection products" defined in Fulton [1984]. When we define other Chern classes of vector bundles we shall see that the same construction works in that more general case.

We imposed the hypothesis of smoothness in Proposition 1.31 because we have only discussed products in this context. In fact, the formula could be used to define an action of a class of the form $c_1(\mathcal{L})$ on $A(X)$ much more generally. This is the point of view taken by Fulton.

Sketch of proof of Proposition 1.31: Since X is quasi-projective, there is an ample bundle $\mathcal{L}'$ on X. For a sufficiently large integer n both the line bundles $\mathcal{L}'^{\otimes n}$ and $\mathcal{L}'^{\otimes n} \otimes \mathcal{L}$ are very ample, so by Bertini's theorem there are sections $\sigma \in H^0(\mathcal{L}'^{\otimes n})$ and $\tau \in H^0(\mathcal{L}'^{\otimes n} \otimes \mathcal{L})$ whose zero loci $\mathrm{Div}(\sigma)$ and $\mathrm{Div}(\tau)$ are generically transverse to each Y_i. The class $c_1(\mathcal{L})$ is rationally equivalent to the cycle $\mathrm{Div}(\sigma) - \mathrm{Div}(\tau)$, proving the first assertion. Moreover, $c_1(\mathcal{L})[Y_i] = [\mathrm{Div}(\sigma) \cap Y_i] - [\mathrm{Div}(\tau) \cap Y_i]$ by Theorem–Definition 1.5. Since $\mathrm{Div}(\sigma) \cap Y_i = \mathrm{Div}(\sigma|_{Y_i})$, and similarly for τ, we are done. □

genus	$\deg(K_X)$	topology	curvature	$\dim \mathrm{Aut}(X)$	cover	points
0	< 0	$\chi > 0$	> 0	3	$\mathbb{CP}^1$	infinite
1	0	$\chi = 0$	0	1	$\mathbb{C}$	infinite
≥ 2	> 0	$\chi < 0$	< 0	finite	Δ	finite

Table 1.1 Behavior of curves for $\deg(K_X) < 0$, $\deg(K_X) = 0$ and $\deg(K_X) > 0$.

1.4.1 The canonical class

Perhaps the most fundamental example of the first Chern class of a line bundle is the *canonical class*, which we will define here; in the following section, we will describe the *adjunction formula*, which gives us a way to calculate it in many cases.

Let X be a smooth n-dimensional variety. By the *canonical bundle* ω_X of X we mean the top exterior power $\bigwedge^n \Omega_X$ of the cotangent bundle Ω_X of X; this is the line bundle whose sections are regular n-forms. By the *canonical class* we mean the first Chern class $c_1(\omega_X) \in A^1(X)$ of this line bundle. Perhaps reflecting the German language history of the subject, this class is commonly denoted by K_X.

The canonical class is probably the single most important indicator of the behavior of X, geometrically, topologically and arithmetically. For example, the only topological invariant of a smooth projective curve X over the complex field $\mathbb{C}$ is its genus $g = g(X)$, and we have

$$\deg(K_X) = 2g - 2.$$

Virtually every aspect of the geometry over $\mathbb{C}$ and the arithmetic over $\mathbb{Q}$ of X are fundamentally different depending on whether $\deg K_X$ is negative, zero or positive, corresponding to $g = 0, 1$ or $g \geq 2$, as can be seen in Table 1.1. (Here the topology is represented by the topological Euler characteristic, the differential geometry by the curvature of a metric with constant curvature, the complex analysis by the isomorphism class as a complex manifold of the universal cover and the arithmetic by the number of rational points over a suitably large finite extension of $\mathbb{Q}$.)

Example 1.32 (Projective space)**.** We can easily determine the canonical class of a projective space. To do this, we have only to write down a rational n-form ω on $\mathbb{P}^n$ and determine its divisors of zeros and poles. For example, if $X_0, \ldots, X_n$ are homogeneous coordinates on $\mathbb{P}^n$ and

$$x_i = \frac{X_i}{X_0}, \quad i = 1, \ldots, n,$$

are affine coordinates on the open set $U \cong \mathbb{A}^n \subset \mathbb{P}^n$ given by $X_0 \neq 0$, we may take ω to be the rational n-form given in U by

$$\omega = dx_1 \wedge \cdots \wedge dx_n.$$

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The form ω is regular and nonzero in U, so we have only to determine its order of zero or pole along the hyperplane $H = V(X_0)$ at infinity. To this end, let $U' \subset \mathbb{P}^n$ be the open set $X_n \neq 0$, and take affine coordinates $y_0, \dots, y_{n-1}$ on U' with $y_i = X_i/X_n$. We have

$$x_i = \begin{cases} y_i/y_0 & \text{for } i = 1, \dots, n-1, \\ 1/y_0 & \text{for } i = n, \end{cases}$$

so that

$$dx_i = \begin{cases} (1/y_0)dy_i - (y_i/y_0^2)dy_0 & \text{for } i = 1, \dots, n-1, \\ -(1/y_0^2)dy_0 & \text{for } i = n. \end{cases}$$

Taking wedge products, we see that

$$\omega = dx_1 \wedge \cdots \wedge dx_n = \frac{(-1)^n}{y_0^{n+1}} dy_0 \wedge \cdots \wedge dy_{n-1},$$

whence

$$\mathrm{Div}(\omega) = -(n+1)H,$$

so

$$K_{\mathbb{P}^n} = -(n+1)\zeta,$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the class of a hyperplane.

1.4.2 The adjunction formula

Let X again be a smooth variety of dimension n, and suppose that $Y \subset X$ is a smooth $(n-1)$-dimensional subvariety. There is a natural way to relate the canonical class of Y to that of X: If we compare the tangent bundle $\mathcal{T}_Y$ of Y with the restriction $\mathcal{T}_X|_Y$ to Y of the tangent bundle $\mathcal{T}_X$ of X, we get an exact sequence

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0,$$

where the right-hand term $\mathcal{N}_{Y/X}$ is called the *normal bundle* of Y in X. Taking exterior powers, this gives an equality of line bundles

$$(\textstyle\bigwedge^n \mathcal{T}_X)|_Y \cong \bigwedge^{n-1} \mathcal{T}_Y \otimes \mathcal{N}_{Y/X},$$

so that

$$\textstyle\bigwedge^{n-1} \mathcal{T}_Y \cong (\bigwedge^n \mathcal{T}_X)|_Y \otimes \mathcal{N}_{Y/X}^*,$$

and dualizing we have

$$\omega_Y \cong \omega_X|_Y \otimes \mathcal{N}_{Y/X}.$$

Moreover, we can compute $\mathcal{N}_{Y/X}$ in another way. There is an exact sequence

$$0 \longrightarrow \mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 \xrightarrow{\delta} \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0,$$

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where the map δ sends the germ of a function to the germ of its differential (see, for example, Eisenbud [1995, Proposition 16.3]). This identifies $\mathcal{N}^*_{Y/X}$ with the locally free sheaf $\mathcal{I}_{Y/X}/\mathcal{I}^2_{Y/X}$. When Y is a Cartier divisor in X, the case of primary interest for us, the ideal sheaf $I_{Y/X}$ of Y in X is the line bundle $\mathcal{O}_X(-Y)$, and the sheaf $I_{Y/X}/I^2_{Y/X} = \mathcal{O}_Y \otimes I_{Y/X}$ is its restriction to Y, denoted $\mathcal{O}_Y(-Y)$; thus

$$\mathcal{N}_{Y/X} \cong \mathcal{O}_X(Y)|_Y.$$

Combining this with the previous expression, we have what is commonly called the *adjunction formula*:

Proposition 1.33 (Adjunction formula)**.** *If $Y \subset X$ is a smooth $(n-1)$-dimensional subvariety of a smooth n-dimensional variety, then*

$$\omega_Y = \omega_X|_Y \otimes \mathcal{O}_X(Y)|_Y,$$

which we usually write as $\omega_X(Y)|_Y$. In particular, if Y is a smooth curve in a smooth complete surface X, then the degree of K_Y is given by an intersection product:

$$\deg K_Y = \deg\big((K_X + [Y])[Y]\big).$$

1.4.3 Canonical classes of hypersurfaces and complete intersections

We can combine the adjunction formula with the calculation in Example 1.32 to calculate the canonical classes of hypersurfaces, and more generally of complete intersections, in projective space. To start, let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. We have

$$\omega_X = \omega_{\mathbb{P}^n}(X)|_X = \mathcal{O}_X(d-n-1).$$

Thus

$$K_X = (d-n-1)\zeta,$$

where $\zeta = c_1(\mathcal{O}_X(1)) \in A^1(X)$ is the class of a hyperplane section of X.

More generally, suppose

$$X = Z_1 \cap \cdots \cap Z_k$$

is a smooth complete intersection of hypersurfaces $Z_1, \ldots, Z_k$ of degrees $d_1, \ldots, d_k$. Applying adjunction repeatedly to the partial intersections $Z_1 \cap \cdots \cap Z_i$, we see that

$$\omega_X = \mathcal{O}_X\Big(-n-1+\sum d_i\Big)$$

and so

$$K_X = \Big(-n-1+\sum d_i\Big)\zeta.$$

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This argument is not complete, because even though X is assumed smooth the partial intersections $Z_1 \cap \cdots \cap Z_i$ may not be. One way to complete it is to extend the definition of the canonical bundle to possibly singular complete intersections — the adjunction formula is true in this greater generality. Alternatively, if we order the hypersurfaces $Z_i = V(F_i)$ so that $d_1 \geq \cdots \geq d_k$ and replace F_i by a linear combination

$$F_i' = F_i + \sum_{j=i+1}^{k} G_j F_j,$$

with G_j general of degree $d_i - d_j$, the hypersurfaces $Z_i' = V(F_i')$ will have intersection X, and by Bertini's theorem the partial intersections will be smooth.

1.5 Exercises

Exercise 1.34. Let $Y \subset \mathbb{A}^n$ be a subvariety not containing the origin, and let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of the locus

$$W^\circ = \{(t, z) \mid z \in t \cdot Y\},$$

as in the proof of Proposition 1.13. Show that the fiber of W over $t = 0$ is the cone with vertex the origin $0 \in \mathbb{A}^n$ over the intersection $\overline{Y} \cap H_\infty$, where $\overline{Y} \subset \mathbb{P}^n$ is the closure of Y in $\mathbb{P}^n$ and $H_\infty = \mathbb{P}^n \setminus \mathbb{A}^n$ is the hyperplane at infinity.

Exercise 1.35. Show that if X is an irreducible plane cubic with a node, then $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is not a monomorphism, as follows: Show that there is no biregular map from X to $\mathbb{P}^1$. Use this to show that if $p \neq q \in X$ are smooth points, then the line bundles $\mathcal{O}_X(p)$ and $\mathcal{O}_X(q)$ are nonisomorphic. Show, however, that the zero loci of their unique sections, the points p and q, are rationally equivalent.

Exercise 1.36. Show that if $X \subset \mathbb{P}^3$ is a quadric cone with vertex p then $A_1(X) = \mathbb{Z}$ and is generated by the class of a line, and show that the image of $c_1 : \mathrm{Pic}(X) \to A_1(X)$ is $2\mathbb{Z}$ by showing that the image consists of the subgroup of classes of curves lying on X that have even degree as curves in $\mathbb{P}^3$. In particular, the class of a line on X is not in the image.
Hint: Do this by showing that no curve $C \subset X$ of odd degree can be a Cartier divisor on X: If such a curve meets the general line of the ruling of X at δ points away from p and has multiplicity m at p, then intersecting C with a general plane through p we see that $\deg(C) = 2\delta + m$; it follows that m is odd, and hence that C cannot be Cartier at p. Thus, the class $[M]$ of a line of the ruling cannot be $c_1(\mathcal{L})$ for any line bundle $\mathcal{L}$.