Lecture 2

$(K, |\cdot|)$ non-arch. valued field.

For $x \in K$ and $r \in \mathbb{R}_{>0}$, define

$$B(x,r) = \{ y \in K \mid |x-y| < r \}$$

$$\overline{B}(x,r) = \{ y \in K \mid |x-y| \leq r \}$$

Lemma 1.8: Let $(K, |\cdot|)$ be non-archimedean.

(i) If $z \in B(x,r)$, then $B(z,r) = B(x,r)$ ↗ Open balls don't have centres

(ii) If $z \in \overline{B}(x,r)$, then $\overline{B}(z,r) = \overline{B}(x,r)$

(iii) $B(x,r)$ is closed

(iv) $\overline{B}(x,r)$ is open

Proof: (i) Let $y \in B(x,r)$

$$|x-y| < r \Rightarrow |z-y| = |(z-x)+(x-y)| \leq \max(|z-x|, |x-y|) < r$$

Thus $B(x,r) \subseteq B(z,r)$

Reverse inclusion follows by symmetry.

(ii) same as i)

(iii) Let $y \notin B(x,r)$. If $z \in B(x,r) \cap B(y,r)$,

then $B(x,r) = B(z,r) = B(y,r) \Rightarrow y \in B(x,r)$ #

$\Rightarrow B(x,r) \cap B(y,r) = \emptyset$.

(iv) If $z \in \overline{B}(x,r)$, then $B(z,r) \subseteq \overline{B}(z,r) \underset{\text{(ii)}}{=} \overline{B}(x,r)$ □

§2 Valuation rings

Defn 2.1: Let K be a field. A valuation on K is a function $v: K^\times \to \mathbb{R}$ s.t.

(i) $v(xy) = v(x) + v(y)$

(ii) $v(x+y) \geq \min(v(x), v(y))$.

Fix $0 < \alpha < 1$. If v a valuation on K,

then $|x| = \begin{cases} \alpha^{v(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$

determines a non-arch. abs. val.

Conversely, a non-arch. abs. val. determines

a valuation $v(x) = \log_\alpha |x|$

Remark: • We ignore trivial valuation $v(x) = 0$ $\forall x \in K^\times$

• Say v_1, v_2 are equivalent if $\exists c \in \mathbb{R}_{>0}$ s.t.

$$v_1(x) = c\, v_2(x) \quad \forall x \in K^\times$$

Eg. • $K = \mathbb{Q}$, $v_p(x) = -\log_p |x|_p$ is the p-adic valuation.

• k field. $K = k(t) = \mathrm{Frac}(k[t])$ rational function field.

$v\left(t^n \frac{f(t)}{g(t)}\right) = n$, $f, g \in k[t]$, $f(0), g(0) \neq 0$.
$\rightsquigarrow$ t-adic valuation.

• $K = k((t)) = \mathrm{Frac}(k[[t]]) = \left\{ \sum_{i=n}^{\infty} a_i t^i \mid a_i \in k, n \in \mathbb{Z} \right\}$

the field of formal Laurent series over k.

$v\left(\sum_i a_i t^i\right) = \min\{ i \mid a_i \neq 0 \}$

is the t-adic valuation on K

Defn 2.2: Let $(K, |\cdot|)$ be a non-arch. valued field.

The valuation ring of K is defined to be

$$O_K = \{ x \in K \mid |x| \leq 1 \} \quad (= \bar{B}(0,1))$$

$$\left(= \{ x \in K^\times \mid v(x) \geq 0 \} \cup \{0\} \right)$$

Proposition 2.3: (i) O_K is an open subring of K.

(ii) The subsets $\{ x \in K \mid |x| \leq r \}$ and

$$\{ x \in K \mid |x| < r \}$$

for $r \leq 1$ are open ideals in O_K.

(iii) $O_K^\times = \{ x \in K \mid |x| = 1 \}$

Proof: $|0| = 0$

$|1| = 1 \quad (|1| = |1 \cdot 1| = |1|^2 \Rightarrow |1| = 1)$

4 $\quad \rightsquigarrow 0, 1 \in O_K$

$|-1| = |1| \Rightarrow |-x| = |x|$

Thus $x \in O_K \Rightarrow -x \in O_K$

If $x, y \in O_K$, then $|x+y| \leq \max(|x|, |y|) \leq 1$

$$\Rightarrow x + y \in O_K$$

If $x, y \in O_K$, then $|xy| = |x||y| \leq 1 \Rightarrow xy \in O_K$.

Thus O_K is a ring. Since $O_K = \bar{B}(0,1)$ it is open.

(ii) similar to i).

(iii) Note that $|x||x^{-1}| = |x x^{-1}| = 1$

Thus $|x| = 1 \Leftrightarrow |x^{-1}| = 1 \Leftrightarrow x, x^{-1} \in O_K$

$\Leftrightarrow x \in O_K^\times$ $\square$

Notation: $\mathfrak{m} := \{ x \in O_K \mid |x| < 1 \}$ a max ideal ...

Notation: $m := \{x \in O_K \mid |x| < 1\}$ a max. ideal of O_K

- $k := O_K/m$ is the residue field.

Corollary 2.4: O_K is a local ring with unique max. ideal m.

(ring w/ unique max. ideal)

Proof: m' max ideal. $m' \neq m \Rightarrow \exists x \in m' \setminus m$

$\overset{2.3\,(iii)}{\Rightarrow} x$ a unit $\Rightarrow m' = R$ #.

Eg. $K = \mathbb{Q}$ with $|\cdot|_p$. $O_K = \mathbb{Z}_{(p)}$, $m = p\mathbb{Z}_{(p)}$.

$k = \mathbb{F}_p$

5

Defn 2.5: Let $v : K^\times \to \mathbb{R}$ be a valuation.

If $v(K^\times) \cong \mathbb{Z}$, we say v is a discrete valuation. K is said to be a discretely valued field. An element $\pi \in O_K$ is a uniformizer if $v(\pi) > 0$ and $v(\pi)$ generates $v(K^\times)$.

Eg. $K = \mathbb{Q}$ p-adic valuation
$K = k(t)$ t-adic valuation

$\Big\}$ d.v. fields

Remark: If v is a discrete valuation, can replace with equiv. one s.t. $v(K^\times) = \mathbb{Z}$. Call such v normalized valuations (then $v(\pi) = 1$, $\Rightarrow \pi$ unif.)

Lemma 2.6: Let v be a valuation on K.

TFAE:

(i) v is discrete

$\subseteq$ ✓

$\supseteq v(x^{-1}r) \geq 0$

(ii) O_K is a PID

$\Rightarrow r \in x O$

(ii) $\mathcal{O}_K$ is a PID $\Rightarrow I = x\mathcal{O}_K$

(iii) $\mathcal{O}_K$ is Noetherian.

(iv) m is principal.

Proof: (i)$\Rightarrow$(ii) Let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. Let $x \in I$ s.t. $v(x) = \min\{v(a) \mid a \in I\}$ which exists since v is discrete.

Then $\overset{\text{claim}}{x\mathcal{O}_K} = \{a \in \mathcal{O}_K \mid v(a) \geq v(x)\}$ is equal to I.

$\subseteq$ ✓ (I is an ideal)

$\supseteq$ $v(x^{-1}r) \geq 0 \Rightarrow r \in x\mathcal{O}_K$

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(iv) Write $m = \mathcal{O}_K x_1 + \dots + \mathcal{O}_K x_n$.

Wlog. $v(x_1) \leq v(x_2) \leq \dots \leq v(x_n)$.

Then $m = \mathcal{O}_K x_1$.

(iv) $\Rightarrow$ (i) Let $m = \mathcal{O}_K \pi$ for some $\pi \in \mathcal{O}_K$ and let $c = v(\pi)$. Then if $v(x) > 0$, $x \in m$ and hence $v(x) \geq c$. Thus $v(K^\times) \cap (0, c) = \emptyset$.
Since $v(K^\times)$ is a subgroup of $(\mathbb{R}, +)$, we have $v(K^\times) = c\mathbb{Z}$. □

Lemma 2.7: Let v be a discrete valuation on K and $\pi \in \mathcal{O}_K$ a uniformizer. $\forall x \in K^\times$, $\exists n \in \mathbb{Z}$ and

$u \in \mathcal{O}_K^\times$ s.t. $x = \pi^n u$. In particular $K = \mathcal{O}_K[\frac{1}{x}]$

for any $x \in \mathfrak{m}$ and hence $K = \mathrm{Frac}(\mathcal{O}_K)$

7 Proof: For $x \in K^\times$, let n s.t. $v(x) = v(\pi^n) = n\, v(\pi)$,

then $v(x\pi^{-n}) = 0 \Rightarrow u = x\pi^{-n} \in \mathcal{O}_K^\times$. $\square$