

Information for Summer DRP 2022

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1 Starting out in Commutative Algebra

This is some sort of resource intended for those encountering commutative algebra for the first time.

1.1 Suggested Textbooks

- *Abstract Algebra* by Dummit and Foote. This will cover all of your bases and more. [\[DF04\]](#).
- *Algebra* by Michael Artin. Not an amazing textbook by any means, but brief and introductory. [\[Art91\]](#).
- *Introduction to Commutative Algebra* by Atiyah and Macdonald. Brief, yet somehow quite comprehensive and somewhat exhaustive. Wonderful selection of exercises. A sliver of God captured in print. Some printings have aged poorly and may melt your eyes if you stare at them for too long. [\[AM16\]](#)
- *Commutative Algebra with a View Towards Algebraic Geometry* by David Eisenbud. Another fantastic book, perhaps more advanced than others on this list. [\[Eis95\]](#).

1.2 List of Fundamental Topics

The following list of fundamental commutative algebra topics/concepts is meant to serve as a suggestion to a beginner encountering commutative algebra for the first time. It is intended merely as a suggestion. The user may find it particularly useful to look up definitions, examples, and facts mentioned in this list online as a supplement to whatever textbook they happen to be reading. Alternatively, one could simply read chapters 1-3 of Atiyah Macdonald, googling things as necessary and completing as many exercises as time permits.

As with any subject, its a good idea to make multiple passes through this material. We suggest spending some amount of time on each topic, moving on and then coming back periodically to complete exercises and discover new examples dealing with past topics throughout your studies.

(1) First ring theory definitions (1-2 sittings)

- (a) Definition of a ring
- (b) \mathbb{Z} as a ring, $\mathbb{Z}/n\mathbb{Z}$ as a ring
- (c) Polynomial rings $K[x_1, \dots, x_n]$
- (d) Examples of rings with and without commutativity, e.g. matrices and the ring of endomorphisms of a vector space $\text{End}_K(V)$ are both examples of noncommutative rings
- (e) Zero divisors, nilpotent elements, irreducible elements, units
- (f) Classes of rings, in particular reduced rings, integral domains, unique factorization rings, and fields.

Suggested Exercises:

- Exercises from sections 1 and 2 of [Art91, Chapter 11]
- (1) and (2) in Chapter 1 of [AM16, Chapter 1]

(2) Ideals and Homomorphisms (2-3 sittings)

- Definition of ideal
- Definition of ring homomorphism
- Quotient rings
- First isomorphism theorem for rings
- Every ideal is the kernel of some ring homomorphism
- Product rings
- The characteristic of a ring (important!)
- The characteristic of a field

Suggested Exercises:

- Exercises from sections 3 and 4 of [Art91, Chapter 1]
- Exercises 3-6 in [AM16, Chapter 1]

(3) Prime ideals and maximal ideals (3-4 sittings)

- Definition of prime ideals and maximal ideals
- Every prime ideal \mathfrak{p} of a ring A is the kernel of some ring homomorphism $\varphi : A \rightarrow R$, where R is an integral domain
- Every maximal ideal \mathfrak{m} of a ring A is the kernel of some surjective ring homomorphism $\varphi : A \rightarrow K$, where K is a field
- The proof of the following fact: “every proper ideal is contained in a maximal ideal” (the proof I have in mind makes use of Zorn’s lemma)
- The proof of the following (opposite) fact: “the set of primes in a commutative ring A , ordered with respect to inclusion, contains at least one minimal element.” That is, in every commutative ring A , there is a prime ideal \mathfrak{p} which does not properly contain any other prime ideal.

Suggested Exercises:

- Exercises 7-15 of [AM16, Chapter 1] (some of these are rather difficult at this stage)
- Exercises from Chapter 7.4 in [DF04]

(4) Modules (3-6 sittings)

- Basic definitions
- \mathbb{Z} -modules are exactly Abelian groups
- $K[x]$ -modules where K is a field [DF04, Chapter 10.1]
- Submodules, Quotient Modules

- (e) Direct sums of modules
- (f) Tensor Products of Modules* (somewhat difficult)

Suggested Exercises:

- Chapter 2 of [AM16]
- Chapter 10 of [DF04]

(5) Localization (3-6 sittings)

- (a) Definition of localization
- (b) Correspondence of ideals
- (c) Definition of a local ring
- (d) Localization at a prime ideal, i.e. $A_{\mathfrak{p}}$ for a prime $\mathfrak{p} \subseteq A$
- (e) Localization at an element, i.e. A_f for an element $f \in A$.
- (f) The field of fractions of an integral domain

Suggested Exercises:

- Chapter 3 of [AM16]
- Chapter 15.4 of [DF04]

(6) Miscellaneous Topics (many sittings)

- (a) Noetherian rings (read several characterizations of these)
- (b) Finitely generated modules over Noetherian rings (what are some characterizations of these?)
- (c) Height of an ideal in a Noetherian ring and Krull dimension
- (d) Artinian Rings
- (e) Integral extensions of rings
- (f) Hilbert's Nullstellensatz

2 Project Suggestions

These are ideas for directed reading projects suitable for students with some background in abstract algebra.

2.1 *F*-Singularities

Geometry is about topological spaces (shapes) with maps (functions) to some other set, usually a field. In differential geometry, the shapes are called manifolds and the functions are differentiable functions to \mathbb{R} . In algebraic geometry, the “shapes” are called a varieties and the functions are either polynomials or rational functions mapping to a field K , which we call the *base field*. That is, a variety X is essentially just a topological space with functions $f : X \rightarrow K$ which locally “look like” polynomials. A “singularity” is a point on X which intuitively has a badly behaved tangent space, see the [cusp in this graph](#) for instance or the [node at the origin](#) of this graph.

More explicitly, we associate to every point $P \in X$ something called a *local ring* R_P . If R_P is what we call *regular*, then P is a regular point. If R_P is not regular, then P is a singular point, or a *singularity*. Singularities are a big area of study in algebraic geometry, and they fall into two cases depending on the *characteristic* $\text{char}(K)$ of the base field K .

- $\text{char}(K) = 0$. This case is more or less understood at this point, after much sweat and blood.
- $\text{char}(K) > 0$. This case has evaded complete classification, and it is this case with which we would be concerned. **Singularities in positive characteristic are collectively referred to as *F-Singularities***, for reasons that would become quickly clear.

Since a singularity $P \in X$ is really understood via the properties of its local ring R_P , one can forget about all of the algebraic geometry surrounding P and instead study the commutative algebra of R_P . With this framing, a “*F-Singularity*” is simply a local ring R of prime characteristic which is not regular.

A participant in this project would learn

- Lots of commutative algebra
- The basic theory of finite fields
- The properties of the Frobenius endomorphism $F : R \rightarrow R$
- Kunz’s theorem
- Different classes of *F-Singularities* (e.g. *F*-finite singularities, *F*-rational singularities, strongly *F*-regular singularities, etc.)

as well as additional topics depending on the interest of the student.

2.2 Galois Theory and Solvability by Radicals

In number theory, it is natural to consider fields which are nested inside one another. We say that a field K is a *subfield* of a field L if there is an injective homomorphism $K \hookrightarrow L$ from K to L . This is equivalent to saying that L contains a copy of the field K , and for this reason, we typically think of K as a subset of L and write $K \subseteq L$. For example, $\mathbb{Q} \subseteq \mathbb{R}$ and $\mathbb{R} \subseteq \mathbb{C}$.

There are several natural questions to ask in situations such as these. One is “what is the dimension of L when considered to be a vector space over K ?” In the case of $\mathbb{R} \subseteq \mathbb{C}$, the answer is “2”, since every complex number can be written $a + ib$ for real numbers a, b . Another question one might ask is “are there any intermediate extensions F such that $K \subseteq F \subseteq L$, and if so, how many are there?” In the case of the field extension $\mathbb{Q} \subseteq \mathbb{C}$, the answer “yes, a lot of them”. For instance, the field

$$\mathbb{Q}(i) = \{a + ib \mid a, b \in \mathbb{Q}\}$$

is a subfield of \mathbb{C} which contains \mathbb{Q} .

A slightly less obvious question is this: “what are the field automorphisms of L which fix K , and what does the collection of these automorphisms tell us?” The answers to this question end up being surprisingly deep, and in fact have spawned entire new subfields of math. Incidentally, they also provide answers to our two other questions above. If $K \subseteq L$ is a field extension, then an automorphism of $\sigma : L \rightarrow L$ is said to *fix* K

pointwise if $\sigma(x) = x$ for each $x \in K$. If every automorphism of L fixes the points in K and nothing more, then L/K is called a *Galois extension* and the group of automorphisms $\text{Aut}(L/K)$ is called the *Galois group* $\text{Gal}(L/K)$ of L/K . Galois theory was first used to prove the nonexistence of a quintic formula and has since become a vital part the algebraist/number theorist/algebraic geometer's toolkit. It notably played a role in Wiles's proof of Fermat's Last Theorem and represents (ha...) one half of the conjectured correspondence in the famed Langland's Program.

A student pursuing this project would first need to learn basic field theory, including finite field extensions and finite fields. They would then study finite Galois extensions and learn the fundamental theorem of Galois theory. The project could then progress in several different directions. The student could prove the quintic is not soluble by radicals, describe which polygons are constructible, or show that it is impossible to trisect an angle using only a compass and straightedge. Alternatively, if the student has had some exposure to topology, they could learn about infinite Galois extensions, delve into local fields, or even study aspects of the inverse Galois problem.

References

- [AM16] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. economy. Addison-Wesley Series in Mathematics. For the 1969 original see [MR0242802]. Westview Press, Boulder, CO, 2016, pp. ix+128. ISBN: 978-0-8133-5018-9; 0-201-00361-9; 0-201-40751-5.
- [Art91] Michael Artin. *Algebra*. Prentice Hall, Inc., Englewood Cliffs, NJ, 1991, pp. xviii+618. ISBN: 0-13-004763-5.
- [DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. Third. John Wiley & Sons, Inc., Hoboken, NJ, 2004, pp. xii+932. ISBN: 0-471-43334-9.
- [Eis95] David Eisenbud. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785. ISBN: 0-387-94268-8; 0-387-94269-6. DOI: [10.1007/978-1-4612-5350-1](https://doi.org/10.1007/978-1-4612-5350-1). URL: <https://doi.org/10.1007/978-1-4612-5350-1>.