Chapter 5

Chern classes

Keynote Questions

(a) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. How many lines $L \subset \mathbb{P}^3$ are contained in S? (Answer on page 253.)

(b) Let F and G be general homogeneous polynomials of degree 4 in four variables, and consider the corresponding family $\{S_t = V(t_0F + t_1G) \subset \mathbb{P}^3\}_{t \in \mathbb{P}^1}$ of quartic surfaces in $\mathbb{P}^3$. How many members S_t of the family contain a line? (Answer on page 233.)

(c) Let F and G be general homogeneous polynomials of degree d in three variables, and let $\{C_t = V(t_0F + t_1G) \subset \mathbb{P}^2\}_{t \in \mathbb{P}^1}$ be the corresponding family of plane curves of degree d. How many of the curves C_t will be singular? (Answer on page 268.)

In this chapter we will introduce the machinery for answering these questions; the answers themselves will be found in Chapters 6 and 7.

5.1 Introduction: Chern classes and the lines on a cubic surface

Cartier divisors — defined through the vanishing loci of sections of line bundles — are of enormous importance in algebraic geometry. More generally, it turns out that many interesting varieties of higher codimension may be described as the loci where sections of vector bundles vanish, or where collections of sections become dependent; this reduces some difficult problems about varieties to easier, linear problems.

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

Chern classes provide a systematic way of treating the classes of such loci, and are a central topic in intersection theory. They will play a major role in the rest of this book. We begin with an example of how they are used, and then proceed to a systematic discussion.

To illustrate, we explain the Chern class approach to a famous classical result:

Theorem 5.1. *Each smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 distinct lines.*

Sketch: Given a smooth cubic surface $X \subset \mathbb{P}^3$ determined by the vanishing of a cubic form F in four variables, we wish to determine the degree of the locus in $\mathbb{G}(1,3)$ of lines contained in X.

We *linearize* the problem using the observation that, if we fix a particular line L in $\mathbb{P}^3$, then the condition that L lie on X can be expressed as four linear conditions on the coefficients of F. To see this, note that the restriction map from the 20-dimensional vector space of cubic forms on $\mathbb{P}^3$ to the four-dimension vector space $V_L = H^0(\mathcal{O}_L(3))$ of cubic forms on a line $L \cong \mathbb{P}^1 \subset \mathbb{P}^3$ is a linear surjection, and the condition for the inclusion $L \subset X$ is that F maps to 0 in V_L.

As the line L varies over $\mathbb{G}(1,3)$, the four-dimensional spaces V_L of cubic forms on the varying lines L fit together to form a vector bundle $\mathcal{V}$ of rank 4 on $\mathbb{G}(1,3)$. A cubic form F on $\mathbb{P}^3$, through its restriction to each V_L, defines an algebraic global section σ_F of this vector bundle. Thus the locus of lines contained in the cubic surface X is the zero locus of the section σ_F. Assuming for the moment that this zero locus is zero-dimensional, we call its class in $A(\mathbb{G}(1,3))$ the *fourth Chern class* of $\mathcal{V}$, denoted $c_4(\mathcal{V})$.

At this point all we have done is to give our ignorance a fancy name. But there are powerful tools for computing Chern classes of vector bundles, especially when those bundles can be built up from simpler bundles by linear-algebraic constructions. In the present situation, the spaces $H^0(\mathcal{O}_L(1))$ fit together to form the dual $\mathcal{S}^*$ of the tautological subbundle of rank 2 on $\mathbb{G}(1,3)$, and the bundle $\mathcal{V}$ is the symmetric cube $\operatorname{Sym}^3 \mathcal{S}^*$ of $\mathcal{S}^*$, which allows us to express the Chern classes of $\mathcal{V}$ in terms of those of $\mathcal{S}^*$, as in Example 5.16. At the same time, it is not hard to calculate the Chern classes $c_i(\mathcal{S}^*)$ directly; we do this in Section 5.6.2. Putting these things together, we will show in Chapter 6 that

$$\deg c_4(\mathcal{V}) = 27.$$

Of course, to prove Theorem 5.1 one still has to show that the number of lines on any smooth cubic surface is finite, and that the zeros of σ_F all occur with multiplicity 1; this will also be carried out in Chapter 6. $\square$

There are proofs of Theorem 5.1 that do not involve vector bundles and Chern classes. For example, one can show that any smooth cubic surface X can be realized as the blow-up of $\mathbb{P}^2$ in six suitably general points, and using this one can analyze the geometry of X in detail (see for example Manin [1986] or Reid [1988]). But the Chern class approach applies equally to results where no such analysis is available.

For example, we will see in Chapter 6 how to use the Chern class method to show that a general quintic threefold in $\mathbb{P}^4$ contains exactly 2875 lines (a computation that played an important role in the discovery of mirror symmetry; see for example Morrison [1993]), and that a general hypersurface of degree 37 in $\mathbb{P}^{20}$ contains exactly

$$47984924096538345636727806051910707603936407618172699855 15$$

lines, a fact of no larger significance whatsoever.

5.2 Characterizing Chern classes

Let $\mathcal{E}$ be a vector bundle on a variety X of dimension n. We will introduce Chern classes $c_i(\mathcal{E}) \in A_{n-i}(X)$, extending the definition of $c_1(\mathcal{L})$ for a line bundle $\mathcal{L}$ in Section 1.4. As with our treatment of the intersection product, we will give an appealingly intuitive characterization rather than a proof of existence.

Recall that we defined the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on a variety X to be

$$c_1(\mathcal{L}) = [\mathrm{Div}(\tau)] \in A_{n-1}(X)$$

for any rational section τ of $\mathcal{L}$. We define $c_i(\mathcal{L}) = 0$ for all $i \geq 2$. In this section we will characterize Chern classes $c_i(\mathcal{E})$ for any vector bundle $\mathcal{E}$ and any integer $i \geq 0$.

We first sketch the situation in the case of a bundle $\mathcal{E}$ generated by its global sections (this circumstance is in fact the case in most of our applications, and in particular in the example of the 27 lines given above). Let $r = \operatorname{rank} \mathcal{E}$.

In the case $r = 1$ already treated, the class $c_1(\mathcal{L})$ may be regarded as a measure of nontriviality: if $c_1(\mathcal{L}) = 0$, then $\mathcal{L}$ has a nowhere-vanishing section, whence $\mathcal{L} \cong \mathcal{O}_X$. We extend this idea of measuring nontriviality using the idea of the "degeneracy locus" of a collection of sections — roughly, this is the locus where the sections become linearly dependent in the fibers of $\mathcal{E}$. To make the meaning precise, we use multilinear algebra.

The bundle $\mathcal{E}$ is trivial if and only if it has r everywhere-independent global sections $\tau_0, \dots, \tau_{r-1}$; in this case, any set of r general sections will do. Thus a first measure of nontriviality is the locus where r general sections $\tau_0, \dots, \tau_{r-1}$ are dependent. If we write $\tau : \mathcal{O}_X^r \to \mathcal{E}$ for the map sending the i-th basis vector to τ_i, then this is the locus where τ fails to be a surjection, or, equivalently, where the determinant of τ is zero. We can interpret this as the vanishing of a special section of an exterior power of $\mathcal{E}$: It is the zero scheme of the section

$$\tau_0 \wedge \cdots \wedge \tau_{r-1} \in \bigwedge^r \mathcal{E}.$$

Since $\operatorname{rank} \mathcal{E} = r$, the bundle $\bigwedge^r \mathcal{E}$ has rank 1 and the class of the zero locus is by definition $c_1(\bigwedge^r \mathcal{E})$; this is a class in $A_{\dim X - 1}(X)$ depending only on the isomorphism class of $\mathcal{E}$. We call it the *first Chern class of* $\mathcal{E}$, written $c_1(\mathcal{E})$.

More generally, we can consider for any i the scheme where $r - i$ general sections of $\mathcal{E}$ fail to be independent, defined by the vanishing of

$$\tau_0 \wedge \cdots \wedge \tau_{r-i} \in \bigwedge^{r-i+1} \mathcal{E}.$$

This is called the *degeneracy locus* of the sections $\tau_0, \ldots, \tau_{r-i}$. Since these degeneracy loci are central to our understanding of (and applications of) Chern classes, we should first say what we expect them to look like.

To see how this should go, consider first the "degeneracy locus of one section." A section τ of $\mathcal{E}$ is locally given by r functions $f_1, \ldots, f_r$, so that by the principal ideal theorem the codimension of each component of $V(\tau)$ is at most r. Moreover, if $\mathcal{E}$ is generated by global sections and τ is a general section, then the function f_{i+1} will not vanish identically on any component of the locus where $f_1, \ldots, f_i$ vanish, and it follows that every component of $V(\tau)$ has codimension exactly r. Under our standing assumption of characteristic 0, a version of Bertini's theorem tells us that $V(\tau)$ is reduced as well. (This may fail in characteristic p, for example in the case of a line bundle whose complete linear system defines an inseparable morphism.) It turns out that this is typical.

Lemma 5.2. *Suppose that $\mathcal{E}$ is a vector bundle of rank r on a variety X, and let i be an integer with $1 \leq i \leq r$. Let $\tau_0, \ldots, \tau_{r-i}$ be global sections of $\mathcal{E}$, and let $D = V(\tau_0 \wedge \cdots \wedge \tau_{r-i})$ be the degeneracy locus where they are dependent.*

(a) No component of D has codimension $> i$.

(b) If the τ_i are general elements of a vector space $W \subset H^0(\mathcal{E})$ of global sections generating $\mathcal{E}$, then D is generically reduced and has codimension i in X.

Proof: (a) This is Macaulay's "generalized unmixedness theorem." He proved it for the case of polynomial rings, and the general case was proved by Eagon and Northcott — see for example Eisenbud [1995, Exercise 10.9].

(b) Let W be an m-dimensional vector space of global sections of $\mathcal{E}$ that generate $\mathcal{E}$, and let $\varphi : X \to G(m-r, W)$ be the associated morphism sending $p \in X$ to the kernel of the evaluation map $W \to \mathcal{E}_p$. If $U \subset W$ is a subspace of dimension $r - i + 1$ spanned by $\tau_0, \ldots, \tau_{r-i}$, then the locus $V(\tau_0 \wedge \cdots \wedge \tau_{r-i}) \subset X$ is the preimage $\varphi^{-1}(\Sigma)$ of the Schubert cycle

$$\Sigma_i(U) = \{\Lambda \in G(m-r, W) \mid \Lambda \cap U \neq 0\}$$

of $(m-r)$-planes in W meeting U nontrivially. By Kleiman's theorem (Theorem 1.7), if $U \subset W$ is general this locus is generically reduced of codimension i. □

We can now characterize the Chern classes $c_i(\mathcal{E}) \in A^i(X)$ for vector bundles $\mathcal{E}$ on smooth varieties X and integers $i \geq 0$:

Theorem 5.3. *There is a unique way of assigning to each vector bundle $\mathcal{E}$ on a smooth quasi-projective variety X a class $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X)$ in such a way that:*

(a) (Line bundles) If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$, where $c_1(\mathcal{L}) \in A^1(X)$ is the class of the divisor of zeros minus the divisor of poles of any rational section of $\mathcal{L}$.

(b) (Bundles with enough sections) If $\tau_0, \dots, \tau_{r-i}$ are global sections of $\mathcal{E}$, and the degeneracy locus D where they are dependent has codimension i, then $c_i(\mathcal{E}) = [D] \in A^i(X)$.

(c) (Whitney's formula) If

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is a short exact sequence of vector bundles on X then

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}) \in A(X).$$

(d) (Functoriality) If $\varphi : Y \to X$ is a morphism of smooth varieties, then

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E})).$$

Although we will not prove Theorem 5.3 completely, we will explain some parts of the proof in Section 5.9 below. We will see below that these properties make many Chern class computations easy. Here are two tastes:

Corollary 5.4 (Sums of line bundles). *If $\mathcal{E}$ is the direct sum of line bundles $\mathcal{L}_i$, or more generally has a filtration whose quotients are line bundles $\mathcal{L}_i$, then*

$$c(\mathcal{E}) = \prod c(\mathcal{L}_i) = \prod (1 + c_1(\mathcal{L}_i));$$

that is, $c_i(\mathcal{E})$ is the result of applying the i-th elementary symmetric function to the classes $c_1(\mathcal{L}_i)$.

Proof: This follows from a repeated application of Whitney's formula. □

Corollary 5.5. *If $\mathcal{E}$ is a vector bundle on X of rank $>$ dim X, and $\mathcal{E}$ is generated by its global sections, then $\mathcal{E}$ has a nowhere-vanishing global section.*

Proof: By part (b), the degeneracy locus (vanishing locus) of one generic section has codimension $> \dim X$. □

The strong Bertini theorem

Corollary 5.5 has an interesting geometric consequence in the following strengthening of Bertini's theorem:

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

Proposition 5.6 (Strong Bertini)**.** *Let X be a smooth, n-dimensional quasi-projective variety, and $\mathcal{D}$ the linear system of divisors on X corresponding to the subspace $W \subset H^0(\mathcal{L})$ of sections of a line bundle $\mathcal{L}$. If the base locus of $\mathcal{D}$ — that is, the scheme-theoretic intersection*

$$Z = \bigcap_{D \in \mathcal{D}} D$$

— is a smooth k-dimensional subscheme of X, and $k < n/2$, then the general member of the linear system $\mathcal{D}$ is smooth everywhere.

The inequality $k < n/2$ is sharp. For example, take $X = \mathbb{P}^4$ and $\mathcal{D}$ the linear system of all hypersurfaces of degree $d \geq 2$ containing a fixed 2-plane Z. If $Y = V(F) \subset \mathbb{P}^4$ is any hypersurface of degree $d > 1$ containing Z, then the three partial derivatives of F corresponding to the coordinates on Z are identically zero and the two remaining partial derivatives of F must have a common zero somewhere along Z; thus Y is singular at some point of Z. For an extension of this example, see Exercise 5.45.

Proof of Proposition 5.6: To begin with, the classical Bertini theorem tells us that the general member D of the linear system $\mathcal{D}$ is smooth away from Z.

To see that it is also smooth along Z, suppose that D is the zero locus of a general section $\sigma \in W \subset H^0(\mathcal{L})$. Since σ vanishes on Z, it gives rise to a section $d\sigma$ of the tensor product $\mathcal{N}^*_{Z/X} \otimes \mathcal{L}$ of the conormal bundle $\mathcal{N}^*_{Z/X} = \mathcal{I}_{Z/X}/\mathcal{I}^2_{Z/X}$ with the line bundle $\mathcal{L}$; we can think of $d\sigma$ as the differential of σ along Z. The hypothesis that the sections $\sigma \in W$ generate the sheaf $\mathcal{I}_{Z/X} \otimes \mathcal{L}$, together with Lemma 5.2 and the fact that $\dim Z = k < n - k = \operatorname{rank}(\mathcal{N}^*_{Z/X})$, shows that $d\sigma$ is nowhere zero. $\square$

5.3 Constructing Chern classes

A construction of Chern classes for a bundle of rank r that is generated by global sections is implicit in Theorem 5.3 (b): $c_i(\mathcal{E})$ is the degeneracy locus of $r-i+1$ general global sections. An alternative way of stating the same thing is often useful. We have already proved this in Lemma 5.2, but it is worth stating it here explicitly:

Proposition 5.7. *Let $\mathcal{E}$ be a vector bundle of rank r on the smooth, quasi-projective variety X, and let $W \subset H^0(\mathcal{E})$ be an m-dimensional vector space of sections generating $\mathcal{E}$. If $\varphi : X \to G(m-r, W)$ denotes the associated morphism sending $p \in X$ to the kernel of the evaluation map $W \to \mathcal{E}_p$, then the i-th Chern class $c_i(\mathcal{E})$ is the pullback*

$$c_i(\mathcal{E}) = \varphi^*(\sigma_i)$$

of the Schubert class $\sigma_i \in A^i(G(m-r, W))$.

This allows us to construct Chern classes for globally generated bundles, and we will see in Section 5.9.1 how to prove basic facts about Chern classes, such as Whitney's formula, from this construction. To construct Chern classes for arbitrary bundles we use a different technique, the projectivization of a vector bundle. We will have much more to say about this construction in Chapter 9; for now we will simply state what is necessary to construct the Chern classes and to make use of a fundamental tool for computing with Chern classes, introduced in Section 5.4: the "splitting principle."

Definition 5.8. Let X be a scheme, and let $\mathcal{E}$ be a vector bundle of rank $r+1$ on X. By the *projectivization* of E we will mean the natural morphism

$$\pi_{\mathcal{E}} : \mathbb{P}\mathcal{E} := \operatorname{Proj}(\operatorname{Sym}\mathcal{E}^*) \to X.$$

By a *projective bundle* over X we mean a morphism $\pi : Y \to X$ that can be realized as $\pi_{\mathcal{E}}$ for some vector bundle $\mathcal{E}$ over X.

Thus the closed points of $\mathbb{P}\mathcal{E}$ correspond to pairs (x, ξ) with $x \in X$ and ξ a one-dimensional subspace $\xi \subset \mathcal{E}_x$ of the fiber $\mathcal{E}_x$ of $\mathcal{E}$. Ordinary projective space is of course the special case in which X is a point and $\mathcal{E}$ is a vector space.

The bundle $\pi : \mathbb{P}\mathcal{E} \to X$ comes equipped with a tautological line bundle

$$\mathcal{S}_{\mathcal{E}} := \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \subset \pi^*\mathcal{E},$$

constructed as the sheafification of the graded $\operatorname{Sym}\mathcal{E}^*$-module obtained by shifting the grading by -1, just as in the case of ordinary projective space.

Here is the result about projectivized vector bundles that serves to define the Chern classes in general:

Theorem 5.9. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth variety X, and let $\pi : \mathbb{P}\mathcal{E} \to X$ be the projectivized vector bundle. Let ζ be the first Chern class of the dual $\mathcal{S}_{\mathcal{E}}^*$ of the tautological bundle $\mathcal{S}_{\mathcal{E}}$ on $\mathbb{P}\mathcal{E}$.*

(a) The flat pullback map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is injective.
(b) The element $\zeta \in A(\mathbb{P}\mathcal{E})$ satisfies a unique monic polynomial $f(\zeta)$ of degree r with coefficients in $\pi^*(A(X))$.

Definition 5.10. Let $\mathcal{E}$ be a vector bundle of rank r on a smooth variety X. The Chern classes $c_i(\mathcal{E})$ are the unique elements of $A(X)$ such that

$$f(\zeta) = \zeta^r + \pi^* c_1(\mathcal{E})\zeta^{r-1} + \cdots + \pi^* c_r(\mathcal{E});$$

that is,

$$A(\mathbb{P}\mathcal{E}) = A(X)[\zeta]/(f(\zeta)).$$

In fact, this definition of Chern classes may be extended to singular varieties, as in Fulton [1984, Chapter 3], and this is a crucial element of the intersection theory of singular varieties: as we have seen (Example 2.22), it is simply not possible to define

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

products of arbitrary classes on singular varieties in general, but it is possible to define products with Chern classes of a vector bundle by restricting the vector bundle. For a proof of Theorem 5.9 in the smooth case, see Theorem 9.6; for the proof in general, see Fulton [1984, Chapter 3].

5.4 The splitting principle

For more complicated examples, we will use Whitney's formula in conjunction with a result called the *splitting principle*, which may be stated as:

Theorem 5.11 (Splitting principle)**.** *Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

This remarkable result is really a corollary of the construction of projectivized vector bundles, applied via the next result:

Lemma 5.12 (Splitting construction)**.** *Let X be any smooth variety and $\mathcal{E}$ a vector bundle of rank r on X. There exists a smooth variety Y and a morphism $\varphi : Y \to X$ with the following two properties:*

(a) The pullback map $\varphi^* : A(X) \to A(Y)$ is injective.
(b) The pullback bundle $\varphi^*\mathcal{E}$ on Y admits a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \varphi^*\mathcal{E}$$

by vector subbundles $\mathcal{E}_i \subset \varphi^*\mathcal{E}$ with successive quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ locally free of rank 1.

Proof: We may construct $\varphi : Y \to X$ by iterating the projectivized vector bundle construction: First, on $Y_1 := \mathbb{P}\mathcal{E}$ we have a tautological subbundle $\mathcal{S}_1 \subset \pi_{\mathcal{E}}^*(\mathcal{E})$. Writing $\mathcal{Q}_1$ for the quotient, we next construct $Y_2 := \mathbb{P}\mathcal{Q}_1$. On Y_2 we have exact sequences

$$0 \longrightarrow \pi_{\mathcal{Q}_1}^*(\mathcal{S}_1) \subset \pi_{\mathcal{Q}_1}^*\pi_{\mathcal{E}}^*\mathcal{E} \longrightarrow \pi_{\mathcal{Q}_1}^*\mathcal{Q}_1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{S}_2 \longrightarrow \pi_{\mathcal{Q}_1}^*\mathcal{Q}_1 \longrightarrow \mathcal{Q}_2 \longrightarrow 0.$$

Continuing this way for $r-1$ steps we get a space $Y := Y_r$ such that the pullback of $\mathcal{E}$ to Y admits a filtration whose successive quotients are line bundles.

Finally, by Theorem 5.9, there is a class $\zeta \in A^1(\mathbb{P}\mathcal{E})$ in the Chow ring of any projective bundle $\pi : \mathbb{P}\mathcal{E} \to X$ that restricts to the hyperplane class on each fiber. By the push-pull formula, if $\mathcal{E}$ has rank r then for any class $\alpha \in A(X)$ we have

$$\pi_*(\zeta^{r-1}\pi^*\alpha) = \alpha,$$

from which we see that the pullback map $\pi^* : A(X) \to A(\mathbb{P}\mathcal{E})$ is injective. □

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

We will study projective bundles much more extensively in Chapter 9; in particular, we give a more fleshed out version of this argument in the proof of Lemma 9.7.

Proof of Theorem 5.11: With notation as in the theorem, we can use Whitney's formula (part (c) of Theorem 5.3) and our a priori definition of the Chern class of a line bundle to describe the Chern class of the pullback:

$$c(\varphi^*\mathcal{E}) = \prod_{i=1}^{r} c(\mathcal{E}_i/\mathcal{E}_{i-1});$$

by the first part of the lemma, this determines the Chern classes of $\mathcal{E}$. $\square$

5.5 Using Whitney's formula with the splitting principle

We will now illustrate the use of Whitney's formula with the splitting principle.

A first consequence is that the Chern classes of a bundle vanish above the rank, something we saw already in the case of bundles with enough sections.

Example 5.13 (Vanishing)**.** If $\mathcal{E}$ is a vector bundle of rank r, then $c_i(\mathcal{E}) = 0$ for $i > r$. Reason: If $\mathcal{E}$ split as $\bigoplus_{i=1}^{r} \mathcal{L}_i$ for line bundles $\mathcal{L}_i$ then, since $c(\mathcal{L}_i) = 1 + c_1(\mathcal{L}_i)$, Whitney's formula would imply that

$$c(\mathcal{E}) = \prod_{i=1}^{r}(1 + c_1(\mathcal{L}_i)),$$

which has no terms of degree $> r$.

Example 5.14 (Duals)**.** If $\mathcal{E} = \bigoplus \mathcal{L}_i$, then

$$c(\mathcal{E}^*) = \prod(1 + c_1(\mathcal{L}_i^*)) = \prod(1 - c_1(\mathcal{L}_i)),$$

since $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$ when $\mathcal{L}$ is a line bundle. Given this, Whitney's formula gives us the basic identity

$$c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E}).$$

By the splitting principle, this identity holds for any bundle.

Example 5.15 (Determinant of a bundle)**.** By the *determinant* $\det \mathcal{E}$ of a bundle $\mathcal{E}$ we mean the line bundle that is the highest exterior power $\det \mathcal{E} := \bigwedge^{\operatorname{rank} \mathcal{E}} \mathcal{E}$. We have already observed that if $\mathcal{E}$ is globally generated, then $c_1(\det \mathcal{E}) = c_1(\mathcal{E})$; this was one of our motivating examples. The splitting principle and Whitney's formula allow us to deduce this for arbitrary bundles: If we assume that $\mathcal{E} = \bigoplus \mathcal{L}_i$, then $\det \mathcal{E} = \bigotimes \mathcal{L}_i$ and hence

$$c_1(\det \mathcal{E}) = \sum c_1(\mathcal{L}_i) = c_1(\mathcal{E});$$

the splitting principle tells us this identity holds in general.

Example 5.16 (Symmetric squares)**.** Suppose that $\mathcal{E}$ is a bundle of rank 2. If $\mathcal{E}$ splits as a direct sum $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ of line bundles $\mathcal{L}$ and $\mathcal{M}$ with Chern classes $c_1(\mathcal{L}) = \alpha$ and $c_1(\mathcal{M}) = \beta$ then, by Whitney's formula, $c(\mathcal{E}) = (1+\alpha)(1+\beta)$, whence

$$c_1(\mathcal{E}) = \alpha + \beta \quad \text{and} \quad c_2(\mathcal{E}) = \alpha\beta.$$

Further, we would have

$$\operatorname{Sym}^2 \mathcal{E} = \mathcal{L}^{\otimes 2} \oplus (\mathcal{L} \otimes \mathcal{M}) \oplus \mathcal{M}^{\otimes 2},$$

from which we would deduce

$$\begin{aligned} c(\operatorname{Sym}^2 \mathcal{E}) &= (1+2\alpha)(1+\alpha+\beta)(1+2\beta) \\ &= 1 + 2(\alpha+\beta) + (2\alpha^2 + 8\alpha\beta + 2\beta^2) + 4\alpha\beta(\alpha+\beta). \end{aligned}$$

This expression may be rewritten in a way that involves only the Chern classes of $\mathcal{E}$: As the reader may immediately check, it is equal to

$$1 + 2c_1(\mathcal{E}) + \big(2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E})\big) + 4c_1(\mathcal{E})c_2(\mathcal{E}).$$

By the splitting principle, this is a valid expression for $c(\operatorname{Sym}^2 \mathcal{E})$ whether or not $\mathcal{E}$ actually splits.

We could use the same method to give formulas for the Chern classes of any symmetric or exterior power — or of any multilinear functor — applied to vector bundles whose Chern classes we know.

Together, the splitting principle and Whitney's formula give a powerful tool for calculating Chern classes, as we will see over and over in the remainder of this text; see Exercises 5.30–5.35 for more examples.

5.5.1 Tensor products with line bundles

As an application of the splitting principle, we will derive the relation between the Chern classes of a vector bundle $\mathcal{E}$ of rank r on a variety X and the Chern classes of the tensor product of $\mathcal{E}$ with a line bundle $\mathcal{L}$.

To do this, we start by assuming that $\mathcal{E}$ splits as a direct sum of line bundles

$$\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{M}_i;$$

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

let $\alpha_i = c_1(\mathcal{M}_i) \in A^1(X)$ be the first Chern class of $\mathcal{M}_i$, so that

$$c(\mathcal{E}) = \prod_{i=1}^{r}(1 + \alpha_i).$$

In other words, the elementary symmetric polynomials in the α_i are the Chern classes of $\mathcal{E}$:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \cdots + \alpha_r &= c_1(\mathcal{E}),\\
\sum_{1 \le i < j \le r} \alpha_i \alpha_j &= c_2(\mathcal{E}),\\
&\vdots\\
\alpha_1 \alpha_2 \dots \alpha_r &= c_r(\mathcal{E}).
\end{aligned}$$

Now let $\beta = c_1(\mathcal{L})$ be the first Chern class of $\mathcal{L}$. Since

$$\mathcal{E} \otimes \mathcal{L} = \bigoplus_{i=1}^{r} \mathcal{M}_i \otimes \mathcal{L},$$

we have, by Whitney's formula,

$$c(\mathcal{E} \otimes \mathcal{L}) = \prod_{i=1}^{r}(1 + \alpha_i + \beta). \tag{5.1}$$

Now, we can express the product on the right as a polynomial in β and the elementary symmetric polynomials in the α_i: For example, we have

$$c_1(\mathcal{E} \otimes \mathcal{L}) = \sum_{i=1}^{r}(\alpha_i + \beta) = c_1(\mathcal{E}) + r c_1(\mathcal{L}),$$

and likewise

$$\begin{aligned}
c_2(\mathcal{E} \otimes \mathcal{L}) &= \sum_{1 \le i < j \le r} (\alpha_i + \beta)(\alpha_j + \beta)\\
&= \sum_{1 \le i < j \le r} \alpha_i \alpha_j + (r-1)\beta \sum_{i=1}^{r} \alpha_i + \binom{r}{2}\beta^2\\
&= c_2(\mathcal{E}) + (r-1)c_1(\mathcal{E})c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2,
\end{aligned}$$

and so on. In general, we have:

Proposition 5.17. *If $\mathcal{E}$ is a vector bundle of rank r and $\mathcal{L}$ is a line bundle, then*

$$\begin{aligned}
c_k(\mathcal{E} \otimes \mathcal{L}) &= \sum_{l=0}^{k} \binom{r-l}{k-l} c_1(\mathcal{L})^{k-l} c_l(\mathcal{E})\\
&= \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E}).
\end{aligned}$$

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

Proof: This is just a matter of collecting the terms of degree l in the α_i and degree $k-l$ in β in the expression (5.1): we write

$$\begin{aligned}\prod_{i=1}^{r}(1+\alpha_i+\beta) &= \sum_{1\le i_1<\cdots<i_l\le r}(1+\beta)^{r-l}\alpha_{i_1}\cdots\alpha_{i_l}\\ &= \sum_l c_l(\mathcal{E})(1+\beta)^{r-l},\end{aligned}$$

and the proposition follows. □

5.5.2 Tensor product of two bundles

Whitney's formula and the splitting principle yield a formula for the Chern class of the tensor product of two bundles of any rank. But, as we will see in Exercises 5.35–5.36, the formula in general is quite complicated. Special cases, however, are amenable to explicit calculation; for example, we can handle the case of the first Chern class $c_1(\mathcal{E}\otimes\mathcal{F})$:

Proposition 5.18. *If $\mathcal{E},\mathcal{F}$ are vector bundles of ranks e and f respectively, then*

$$c_1(\mathcal{E}\otimes\mathcal{F}) = f\cdot c_1(\mathcal{E}) + e\cdot c_1(\mathcal{F}).$$

Proof: Suppose $\mathcal{E}=\bigoplus\mathcal{L}_i$ and $\mathcal{F}=\bigoplus\mathcal{M}_i$ are direct sums of line bundles, so that we can write

$$c(\mathcal{E}) = \prod_{i=1}^{e}(1+\alpha_i) \quad\text{and}\quad c(\mathcal{F}) = \prod_{j=1}^{f}(1+\beta_j)$$

with $c_1(\mathcal{L}_i)=\alpha_i$ and $c_1(\mathcal{M}_j)=\beta_j$; note that $c_1(\mathcal{E})=\alpha_1+\cdots+\alpha_e$ and $c_1(\mathcal{F})=\beta_1+\cdots+\beta_f$. We have then

$$\mathcal{E}\otimes\mathcal{F} = \bigoplus_{i,j=1,1}^{e,f}\mathcal{L}_i\otimes\mathcal{M}_j,$$

and correspondingly

$$c(\mathcal{E}\otimes\mathcal{F}) = \prod_{i,j=1,1}^{e,f}(1+\alpha_i+\beta_j).$$

In particular, this gives

$$\begin{aligned}c_1(\mathcal{E}\otimes\mathcal{F}) &= \sum_{i,j=1,1}^{e,f}(\alpha_i+\beta_j)\\ &= f\sum_{i=1}^{e}\alpha_i + e\sum_{j=1}^{f}\beta_j\\ &= fc_1(\mathcal{E})+ec_1(\mathcal{F}).\end{aligned}$$

□

There is one other case in which we can give a closed-form expression for a Chern class of a general tensor product: We will see, in Chapter 12, a formula for the top Chern class $c_{ef}(\mathcal{E} \otimes \mathcal{F})$ of a tensor product of bundles of ranks e and f.

There is also a different approach that allows us to express the characteristic classes of a general tensor product more comprehensibly: The *Chern character* $\mathrm{Ch}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ is a certain formal power series in the Chern classes of $\mathcal{E}$, with rational coefficients, that satisfies the attractive formulas

$$\mathrm{Ch}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{Ch}(\mathcal{E}) + \mathrm{Ch}(\mathcal{F}),$$
$$\mathrm{Ch}(\mathcal{E} \otimes \mathcal{F}) = \mathrm{Ch}(\mathcal{E}) \cdot \mathrm{Ch}(\mathcal{F}).$$

See Section 14.2.1 for more information.

5.6 Tautological bundles

We have seen how the splitting principle, in conjunction with Whitney's formula, allows us to express the Chern classes of bundles in terms of simpler ones. To apply this, of course, we need to have a roster of basic bundles whose Chern classes we know; in this section we will calculate the Chern classes of some of these.

5.6.1 Projective spaces

We start with the most basic of all bundles: the bundle $\mathcal{O}_{\mathbb{P}^r}(1)$ on projective space $\mathbb{P}^r$. We have

$$c_1(\mathcal{O}_{\mathbb{P}^r}(1)) = \zeta \in A^1(\mathbb{P}^r),$$

where ζ is the hyperplane class; similarly,

$$c_1(\mathcal{O}_{\mathbb{P}^r}(n)) = n \cdot \zeta \in A^1(\mathbb{P}^r)$$

for any $n \in \mathbb{Z}$.

This in turn allows us to compute the Chern class of the universal quotient bundle $\mathcal{Q}$ on $\mathbb{P}^r$: If $\mathbb{P}^r = \mathbb{P}V$, from the exact sequence

$$0 \longrightarrow \mathcal{S} = \mathcal{O}_{\mathbb{P}^r}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

we have

$$c(\mathcal{Q}) = \frac{1}{c(\mathcal{O}_{\mathbb{P}^r}(-1))} = \frac{1}{1-\zeta} = 1 + \zeta + \zeta^2 + \cdots + \zeta^r.$$

Note that we could also arrive at this directly from the description of Chern classes as degeneracy loci of sections: An element $v \in V$ gives rise to a global section σ of the bundle $\mathcal{Q}$; given k elements $v_1, \ldots, v_k \in V$, the corresponding sections $\sigma_1, \ldots, \sigma_k$

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

of $\mathcal{Q}$ will be linearly dependent at a point $x \in \mathbb{P}^r$ exactly when x lies in the $\mathbb{P}^{k-1}$ corresponding to the subspace $W = \langle v_1, \dots, v_k \rangle \subset V$ spanned by the v_i. Thus

$$c_{r-k+1}(\mathcal{Q}) = [\mathbb{P}^{k-1}] = \zeta^{r-k+1} \in A^{r-k+1}(\mathbb{P}^r).$$

5.6.2 Grassmannians

Let us consider next the case of the Grassmannian $G = G(k, n)$ of k-planes in an n-dimensional vector space V, and its universal sub and quotient bundles $\mathcal{S}$ and $\mathcal{Q}$.

We will start with $\mathcal{Q}$, since this bundle is globally generated, so that we can determine its Chern classes directly as degeneracy loci. Specifically, elements $v \in V$ give rise to sections σ of $\mathcal{Q}$ simply by taking their images in each quotient of V; that is, for a k-plane $\Lambda \subset V$, we set

$$\sigma(\Lambda) = \overline{v} \in V/\Lambda.$$

Now, given a collection $v_1, \dots, v_m \in V$, the corresponding sections will fail to be independent at a point $\Lambda \in G$ exactly when the corresponding $\overline{v}_i \in V/\Lambda$ are dependent, which is to say when Λ intersects the span $W = \langle v_1, \dots, v_m \rangle \subset V$ in a nonzero subspace — that is, when

$$\mathbb{P}\Lambda \cap \mathbb{P}W \neq \varnothing.$$

We may recognize this locus as the Schubert cycle $\Sigma_{n-k-m+1}(W)$, from which we conclude that the Chern class of $\mathcal{Q}$ is the sum

$$c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{n-k}.$$

Unlike $\mathcal{Q}$, the universal subbundle $\mathcal{S}$ does not have nonzero global sections, so we cannot use the characterization of Chern classes as degeneracy loci. But the dual bundle $\mathcal{S}^*$ does: If $l \in V^*$ is a linear form, we can define a section τ of $\mathcal{S}^*$ by restricting l to each k-plane $\Lambda \subset V$ in turn; in other words, we set

$$\tau(\Lambda) = l|_{\Lambda}.$$

Now, if we have m independent linear forms $l_1, \dots, l_m \in V^*$, the corresponding sections of $\mathcal{S}^*$ will fail to be independent at the point $\Lambda \in G$ — that is, some linear combination of the l_i will vanish identically on Λ — exactly when Λ fails to intersect the common zero locus U of the l_i properly, that is, when

$$\dim(\mathbb{P}\Lambda \cap \mathbb{P}U) \geq k - m.$$

Again, this locus is a Schubert cycle in G, specifically the cycle $\Sigma_{1,1,\dots,1}(U)$, and we conclude that

$$c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1} + \cdots + \sigma_{1,1,\dots,1};$$

from this we can deduce in turn that

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{1,1} + \cdots + (-1)^k \sigma_{1,1,\dots,1}.$$

Note that this description of $c(\mathcal{S})$ can also be deduced from our knowledge of $c(\mathcal{Q})$ and Corollary 4.10.

5.7 Chern classes of varieties

The most important vector bundles on a smooth variety X are its tangent bundle $\mathcal{T}_X$ and its dual, the cotangent bundle Ω_X. Their Chern classes are so important in geometry that the Chern class of the tangent bundle is usually just called the *Chern class of* X.

For example, if X is a smooth curve then its tangent bundle is a line bundle, so its Chern class has the form $1+c_1(\mathcal{T}_X)$. Here $c_1(\mathcal{T}_X) = -c_1(\Omega_X)$ is the anticanonical class, whose degree is $2-2g$, where g is the genus of X. In general, if X is a smooth complex projective manifold of dimension n then Theorem 5.21 below says that $\deg c_n(\mathcal{T}_X)$ is the topological Euler characteristic of X.

5.7.1 Tangent bundles of projective spaces

We start by calculating the Chern classes of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space. This is straightforward, given the Euler sequence of Section 3.2.4: We have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \longrightarrow 0$$

and hence

$$c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1},$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the hyperplane class.

We could also derive this from the identification $\mathcal{T} = \mathrm{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$, where $S = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $\mathcal{Q}$ are the universal sub and quotient bundles, by applying Proposition 5.17.

Note that this calculation implies the algebraic/projective version of the "hairy coconut" theorem: Since $c_n(\mathcal{T}_{\mathbb{P}^n}) = (n+1)\zeta^n \neq 0$, there does not exist a nowhere-zero vector field on $\mathbb{P}^n$.

5.7.2 Tangent bundles to hypersurfaces

We can combine the formula above for the Chern classes of the tangent bundle to projective space $\mathbb{P}^r$ and Whitney's formula to calculate the Chern classes of the tangent bundle to a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d.

To do this, we use the standard normal bundle sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^n}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^n} \longrightarrow 0$$

and the identification

$$\mathcal{N}_{X/\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(X)|_X = \mathcal{O}_X(d)$$

established in Section 1.4.2. Letting ζ_X denote the restriction to X of the hyperplane class on $\mathbb{P}^n$, we can write

$$\begin{aligned} c(\mathcal{T}_X) &= \frac{c(\mathcal{T}_{\mathbb{P}^n}|_X)}{\mathcal{N}_{X/\mathbb{P}^n}} \\ &= \frac{(1+\zeta_X)^{n+1}}{1+d\zeta_X} \\ &= \left(1+(n+1)\zeta_X+\binom{n+1}{2}\zeta_X^2+\cdots\right)(1-d\zeta_X+d^2\zeta_X^2+\cdots). \end{aligned}$$

We can generalize this calculation to complete intersections:

Example 5.19 (Chern classes of complete intersections)**.** Suppose that

$$X = Z_1 \cap \cdots \cap Z_k \subset \mathbb{P}^n$$

is the complete intersection of k hypersurfaces of degrees $d_1, \ldots, d_k$ defined by forms F_i of degrees d_i. The relations among the F_i are generated by the Koszul relations $F_j F_i - F_i F_j = 0$. This means that if we restrict to Y, where the F_i vanish, we get

$$\mathcal{I}_{Y/X}/\mathcal{I}_{Y/X}^2 = \mathcal{I}_{Y/X}|Y = \bigoplus_i \mathcal{O}_Y(-d_i),$$

so the normal bundle $\mathcal{N} = \mathcal{N}_{X/\mathbb{P}^n}$ of X in $\mathbb{P}^n$ is a direct sum $\mathcal{N} = \bigoplus \mathcal{O}_X(d_i)$. Applying Whitney's formula, we get

$$c(\mathcal{T}_X) = \frac{(1+\zeta_X)^{n+1}}{\prod(1+d_i\zeta_X)}.$$

5.7.3 The topological Euler characteristic

Recall that the *topological Euler characteristic* of a manifold M is by definition $\chi_{\text{top}}(M) := \sum(-1)^i \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$, where $H^i(M;\mathbb{Q})$ is the singular cohomology group. When M is a smooth projective variety over $\mathbb{C}$, it may be regarded as a manifold with respect to the classical, or analytic, topology, so $\chi_{\text{top}}(M)$ makes sense in this case.

Theorem 5.20 (Poincaré–Hopf theorem)**.** *If M is a smooth compact orientable manifold and σ is a vector field with isolated zeros, then*

$$\chi_{\mathrm{top}}(M) = \sum_{\{x \mid \sigma(x)=0\}} \mathrm{index}_x(\sigma).$$

A beautiful account of this classic result can be found in Milnor [1997]. Now suppose that M is a smooth complex projective variety. If the tangent bundle $\mathcal{T}_X$ is generated by global sections, then it has a section σ that vanishes at only finitely many points, and vanishes simply there. Since this section is represented locally by complex analytic functions, its index at each of its zeros will be 1, and we may replace the sum in the Poincaré–Hopf theorem by the number of its zeros — in other words, the degree of the top Chern class of $\mathcal{T}_X$. An elementary topological argument (see, for example, Chapter 3 of Griffiths and Harris [1994]) shows that this is true more generally:

Theorem 5.21. *If X is a smooth n-dimensional projective variety, then*

$$\chi_{\mathrm{top}}(X) = \deg c_n(\mathcal{T}_X).$$

Example 5.22 (Euler characteristic of $\mathbb{P}^n$)**.** Since $c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ is the class of a hyperplane, we deduce that

$$\chi_{\mathrm{top}}(\mathbb{P}^n) = \deg(c_n(\mathcal{T}_{\mathbb{P}^n})) = n+1.$$

Of course this is immediate from the fact that $H^{2i}(\mathbb{P}^n, \mathbb{Q}) = \mathbb{Q}$ for $i = 0, \dots, n$ while $H^{2i+1}(\mathbb{P}^n, \mathbb{Q}) = 0$ for all i.

Example 5.23 (Blow-up of a surface)**.** Sometimes one can use Theorem 5.21 to compute a Chern class. For example, the blow-up Y of a complex surface X at a point p can be described topologically as the union of $X \setminus D$ with a tubular neighborhood of the exceptional curve, which is a copy of $\mathbb{P}^1$. Thus

$$\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(X) - \chi_{\mathrm{top}}(p) + \chi_{\mathrm{top}}(\mathbb{P}^1) = \chi_{\mathrm{top}}(X) - 1 + 2 = \chi_{\mathrm{top}}(X) + 1,$$

and we deduce that $\deg c_2(\mathcal{T}_Y) = \deg c_2(\mathcal{T}_X) + 1$. (One can generalize this formula algebraically, and identify the class $c(\mathcal{T}_Y)$, by using the Chern classes of coherent sheaves that are not vector bundles; see for example Section 14.2.1, and, for the computation, Fulton [1984, Section 15.4].)

Example 5.24 (Euler characteristic of a hypersurface)**.** Now let X be a smooth hypersurface of degree d in $\mathbb{P}^n$. From the normal bundle sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^n}|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}^n} \longrightarrow 0$$

and the fact that $\mathcal{N}_{X/\mathbb{P}^n} \cong \mathcal{O}_X(d)$, we have

$$c(\mathcal{T}_X) = \frac{(1+\zeta_X)^{n+1}}{(1+d\zeta_X)} = ((1+\zeta_X)^{n+1})(1 - d\zeta_X + d^2(\zeta_X)^2 + \cdots).$$

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

Taking the component of degree $\dim X = n-1$, we get

$$c_{n-1}(\mathcal{T}_X) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^i \zeta_X^{n-1}.$$

Since the degree of ζ_X^{n-1} is the number of points of intersection of $n-1$ general hyperplanes on the $(n-1)$-dimensional variety X, we have $\zeta_X^{n-1} = d$. Thus, finally,

$$\chi_{\text{top}}(X) = \deg(c_{n-1}(\mathcal{T}_X)) = \sum_{i=0}^{n-1} (-1)^i \binom{n+1}{n-1-i} d^{i+1}.$$

We can get still more from this formula: The Lefschetz hyperplane theorem (see Section C.4) tells us that the integral cohomology groups of X are all equal to the corresponding cohomology groups of projective space, except for the middle one $H^{n-1}(X)$; that is, the Betti numbers $b_i = \dim_{\mathbb{Q}} H^i(M;\mathbb{Q})$ other than b_{n-1} are 1 in even dimensions and 0 in odd. (In fact, the analogous statement is true for any smooth complete intersection: All the cohomology groups except the middle are equal to those of projective space.) Thus the Euler characteristic determines the middle Betti number b_{n-1}. In Table 5.1, we give the results of this calculation in a few of the cases where it is most frequently used.

hypersurface	χ	b_{n-1}
quadric surface	4	2
cubic surface	9	7
quartic surface	24	22
quintic surface	55	53
quadric threefold	4	0
cubic threefold	−6	10
quartic threefold	−56	60
quintic threefold	−200	204
quadric fourfold	6	2
cubic fourfold	27	23

Table 5.1 Euler characteristics of favorite hypersurfaces.

It is interesting to compare this computation with what we already knew: A smooth quadric surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, from which we can see directly both the Euler characteristic and the second Betti number; a smooth cubic surface in $\mathbb{P}^3$ is the blow-up of $\mathbb{P}^2$ at six points, so the Euler characteristic is $3+6$; the quadric fourfold may also be viewed as the Plücker embedding of the Grassmannian $\mathbb{G}(1,3)$, whose cohomology has as basis its six Schubert cycles, and whose middle cohomology in particular has basis given by the two Schubert cycles $\sigma_{1,1}$ and σ_2.

5.7.4 First Chern class of the Grassmannian

In theory, we should be able to use the identification of the Chern classes $c(\mathcal{S})$ and $c(\mathcal{Q})$ to derive the Chern class of the tangent bundle $\mathcal{T}_G$, which by Theorem 3.5 is isomorphic to $\mathrm{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^* \otimes \mathcal{Q}$. In general, unfortunately, this knowledge remains theoretical: As we indicated in Section 5.5.2, the formula for the Chern class of the tensor product of two bundles of higher rank is complicated. But we can at least use Proposition 5.18 to give the first Chern class $c_1(\mathcal{T}_G)$; since $c_1(\mathcal{S}^*) = c_1(\mathcal{Q}) = \sigma_1$, we have:

Proposition 5.25. *The first Chern class of the tangent bundle of the Grassmannian $G = G(k, n)$ is*

$$c_1(\mathcal{T}_G) = n \cdot \sigma_1.$$

We see from this also that the canonical class K_G of G is $-n\sigma_1$. Note that this agrees with our prior calculations in the case $k = 1$ of projective space $\mathbb{P}^{n-1}$, and in the case $k = 2$ and $n = 4$, where the Grassmannian $G(2, 4)$ may be realized as a quadric hypersurface in $\mathbb{P}^5$ and we can apply the results of Section 5.7.2.

5.8 Generators and relations for $A(G(k, n))$

We have seen in Corollary 4.7 that the Chow ring of the Grassmannian is a free abelian group generated by the Schubert cycles. It follows moreover from Giambelli's formula (Proposition 4.16) that it is generated multiplicatively by just the *special Schubert cycles*, which are the Chern classes of the universal subbundle. We will now see that Whitney's formula and the fact that the Chern classes of a bundle vanish above the rank of the bundle provide a complete description of the relations among the special Schubert cycles, and that these form a complete intersection.

Theorem 5.26. *The Chow ring of the Grassmannian $G(k, n)$ has the form*

$$A(G(k, n)) = \mathbb{Z}[c_1, \dots, c_k]/I,$$

where $c_i \in A^i(G(k, n))$ is the i-th Chern class of the universal subbundle $\mathcal{S}$ and the ideal I is generated by the terms of total degree $n - k + 1, \dots, n$ in the power series expansion

$$\frac{1}{1 + c_1 + \cdots + c_k} = 1 - (c_1 + \cdots + c_k) + (c_1 + \cdots + c_k)^2 - \cdots \in \mathbb{Z}[\![c_1, \dots, c_k]\!].$$

Moreover, I is a complete intersection.

For example, the Chow ring of $G(3,7)$ is $\mathbb{Z}[c_1, c_2, c_3]/I$, where I is generated by the elements

$$c_1^5 + 4c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 + 2c_2c_3,$$
$$c_1^6 + 5c_1^4c_2 + 6c_1^2c_2^2 + c_2^3 + 4c_1^3c_3 + 6c_1c_2c_3 + c_3^2,$$
$$c_1^7 + 6c_1^5c_2 + 10c_1^3c_2^2 + 4c_1c_2^3 + 5c_1^4c_3 + 12c_1^2c_2c_3 + 3c_2^2c_3 + 3c_1c_3^2,$$

and these elements form a regular sequence.

The proof of Theorem 5.26 uses two results from commutative algebra, Proposition 5.27 and Lemma 5.28, which are variations on some frequently used results; readers may wish to familiarize themselves with them before reading the proof of Theorem 5.26. Recall that the *socle* of a finite-dimensional graded algebra T is the submodule of elements annihilated by all elements of positive degree. In particular, if d is the largest degree such that $T_d \neq 0$, then the socle of T contains T_d. For a somewhat different proof, and the generalization to flag bundles of arbitrary vector bundles, see Grayson et al. [2012].

Proof: Set $A = A(G(k,n))$ and write t_i for the degree-i part of the power series expansion of $1/(1 + c_1 + \cdots + c_k)$, so that $t_0 = 1$, $t_1 = -c_1$, $t_2 = c_1^2 - c_2, \dots$. Let $J = (t_{n-k+1}, \dots, t_n)$, and let $R = \mathbb{Z}[c_1, \dots, c_k]/J$.

Corollary 4.10, which is the special case of Whitney's formula (Theorem 5.3, part (c)) applied to the tautological sequence of vector bundles

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^n_{G(k,n)} \longrightarrow \mathcal{Q} \longrightarrow 0$$

on $G(k,n)$, shows that $c(\mathcal{Q}) = 1/c(\mathcal{S})$. Since $\mathcal{Q}$ has rank $n-k$, the classes $c_i(\mathcal{Q})$ vanish for all $i > n-k$, and it follows that there is a ring homomorphism

$$\varphi : R \to A, \quad t_i \mapsto c_i(\mathcal{Q}).$$

Under this homomorphism, the class c_i goes to the Schubert cycle $c_i(\mathcal{S}) = (-1)^i\sigma_{1^i}$ (where the subscript denotes a sequence of 1 repeated i times). Recall from Corollary 4.2 that $\sigma_{1^k}^{n-k}$ is the class of a point.

We will show that for any field F the sequence $t_{n-k+1}, \dots, t_n$ is a regular sequence in $R \otimes_{\mathbb{Z}} F$, and the induced map

$$R' := R \otimes_{\mathbb{Z}} F \xrightarrow{\varphi' := \varphi \otimes_{\mathbb{Z}} F} A' := A \otimes_{\mathbb{Z}} F$$

is an isomorphism. Since A is a finitely generated abelian group, the surjectivity of φ follows from this result using Nakayama's lemma and the two cases $F = \mathbb{Z}/(p)$ and $F = \mathbb{Q}$. On the other hand, by Corollary 4.7, A is a free abelian group so, as an abelian group, $\varphi(R)$ is free. Thus the kernel of φ is a summand of R, so the injectivity of φ follows from the injectivity, for every choice of F, of φ'. Using Lemma 5.28 inductively, this also follows that $t_{n-k+1}, \dots, t_n$ is a regular sequence, proving the theorem.

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

To show that $t_{n-k+1}, \ldots, t_n$ is a regular sequence in R' it suffices, since the t_i have positive degree, to show that

$$F[c_1, \ldots, c_k]/J$$

has Krull dimension zero. Since F was arbitrary it suffices, by the Nullstellensatz, to show that, if $f_i \in F$ are substituted for the c_i in such a way that $t_{n-k+1} = \cdots = t_n = 0$, then all the f_i are zero.

Indeed, after such a substitution we see that $1/(1 + f_1x + f_2x^2 + \cdots + f_kx^k) = p(x) + q(x)$, where $p(x)$ is a polynomial of degree $\leq n-k$ and $q(x)$ is a rational function vanishing to order at least $n+1$ at 0. We may rewrite this as

$$\frac{1 - p(x)(1 + f_1x + f_2x^2 + \cdots + f_kx^k)}{1 + f_1x + \cdots + f_kx^k} = q(x).$$

However, the denominator of the left-hand side is nonzero at the origin, and the numerator has degree at most n. Since $q(x)$ vanishes to order at least $n+1$ at the origin, both sides must be identically zero; that is $p(x) = 1, q(x) = 0$, and thus each $f_i = 0$, as required.

Combining this information with Proposition 5.27, we get:

- The dimension of R' (as a vector space over F) is $\binom{n}{k}$.
- The highest degree d such that $R'_d \neq 0$ is $k(n-k)$.
- Since a complete intersection is Gorenstein (Eisenbud [1995, Corollary 21.19]), every nonzero ideal of R' contains $R'_{k(n-k)}$.

We now return to the map φ'. By Corollary 4.13, the rank of $A(G(k, n))$ is also $\binom{n}{k}$; thus to show that φ' is an isomorphism, it suffices to show that its kernel is zero. We know that $(\sigma_{1^k})^{n-k}$ is in the image of φ', so $\operatorname{Ker}\varphi'$ does not contain $R_{k(n-k)}$. Since $R'_{k(n-k)}$ is the socle of R', the kernel of φ' must be zero. $\square$

We have used the following two results from commutative algebra:

Proposition 5.27. *Suppose that F is a field and that*

$$T = F[x_1, \ldots, x_k]/(g_1, \ldots, g_k)$$

is a zero-dimensional graded complete intersection with $\deg x_i = \delta_i > 0$ *and* $\deg g_i = \epsilon_i > 0$. *The Hilbert series of T is*

$$H_T(d) := \sum_{u=0}^{\infty} \dim_F T_u d^u = \frac{\prod_{i=1}^k (1 - d^{\epsilon_i})}{\prod_{i=1}^k (1 - d^{\delta_i})}.$$

The degree of the socle of T is $\sum_{i=0}^k \epsilon_i - \sum_{i=0}^k \delta_i$, and the dimension of T is

$$\dim_F T = \frac{\prod_{i=1}^k (\epsilon_i - 1)}{\prod_{i=1}^k (\delta_i - 1)}.$$

Proof: We begin with the Hilbert series. The polynomial ring $F[x_1, \ldots, x_k]$ is the tensor product of the one-variable polynomial rings $F[x_i]$, so

$$H_{F[x_1,\ldots,x_k]}(d) := \frac{1}{\prod_{i=1}^k (1 - d_i^\delta)}.$$

We can put in the relations one-by-one using the exact sequences

$$0 \longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_i)(-\epsilon_i) \xrightarrow{g_{i+1}} F[x_1, \ldots, x_k]/(g_1, \ldots, g_i) \longrightarrow F[x_1, \ldots, x_k]/(g_1, \ldots, g_{i+1}) \longrightarrow 0,$$

and using induction we see that

$$H_T(d) = H_{F[x_1,\ldots,x_k]/(g_1,\ldots,g_k)}(d) = \frac{\prod_{i=1}^k (1 - d^{\epsilon_i})}{\prod_{i=1}^k (1 - d^{\delta_i})}.$$

A priori this is a rational function of degree $s := \sum_{i=1}^k \epsilon_i - \sum_{i=1}^k \delta_i$. Since we know from the computation above that T is a finite-dimensional vector space over F, the Hilbert series must be a polynomial. Thus it is a polynomial of degree s, so the largest degree in which T is nonzero is s.

The dimension of T is the value of $H_T(d)$ at $d = 1$. The product $(1-d)^k$ obviously divides both the numerator and the denominator of the expression for the Hilbert series above. After dividing, we get

$$H_T(d) = \frac{\prod_{i=1}^k \sum_{j=0}^{\epsilon_i - 1} d^j}{\prod_{i=1}^k \sum_{j=0}^{\delta_i - 1} d^j}.$$

Setting $d = 1$ in this expression gives us the desired result. □

The other result from commutative algebra that we used is a version of the fact that regular sequences in a local ring can be permuted (Eisenbud [1995, Corollary 17.2]). The same result holds in the local case when every element of the regular sequence has positive degree, but the case we need is slightly different, since one element of the regular sequence is an integer. The result may also be viewed as a variation on the local criterion of flatness (Eisenbud [1995, Section 6.4]).

Lemma 5.28. *Suppose that R is a finitely generated graded algebra over $\mathbb{Z}$, with algebra generators in positive degrees, and that $f \in R$ is a homogeneous element. If R is free as a $\mathbb{Z}$-module and $f \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ is a monomorphism for every prime p, then f is a monomorphism and $R/(f)$ is free as a $\mathbb{Z}$-module as well.*

Proof: Since R is free, so is every submodule; in particular fR is free, and the kernel K of multiplication by f is a free summand of R. It follows that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \subset R \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$. Since this ideal is obviously contained in the kernel of multiplication by f

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

on $R \otimes \mathbb{Z}/(p)$, we see that $K \otimes_{\mathbb{Z}} \mathbb{Z}/(p) = 0$. Since K is free, this implies that $K = 0$ as well; that is, f is a nonzerodivisor on R, and the diagram

$$\begin{tikzcd}
0 \arrow[r] & R(-1) \arrow[r, "f"] \arrow[d, "p"] & R \arrow[r] \arrow[d, "p"] & R/fR \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & R(-1) \arrow[r, "f"] & R \arrow[r] & R/fR \arrow[r] & 0
\end{tikzcd}$$

has exact rows. A diagram chase (the *snake lemma*) shows that p is a nonzerodivisor on R/fR. Since p was an arbitrary prime, R/fR is a torsion-free abelian group. Since R is finitely generated and f is homogeneous, R/fR is a direct sum of finitely generated abelian groups, and torsion-freeness implies freeness. □

5.9 Steps in the proofs of Theorem 5.3

Though the locus D in item (b) of Theorem 5.3 depends very much on the sections τ_i chosen, Theorem 5.3 asserts that the class $[D]$ does not, so long as it has the "expected" codimension. This point is worth understanding directly: We start with the case $k = r$ of the top Chern class. If τ and τ' are two sections of $\mathcal{E}$ whose zero loci are of codimension r, then we can interpolate between $V(\tau)$ and $V(\tau')$ with the family

$$\Phi = \{([s,t], p) \in \mathbb{P}^1 \times X \mid s\tau(p) + t\tau'(p) = 0\}.$$

This gives a rational equivalence between $V(\tau)$ and $V(\tau')$: Since Φ has codimension at most r everywhere, components of Φ intersecting the fibers over 0 or $\infty \in \mathbb{P}^1$ must dominate $\mathbb{P}^1$, and taking the union of these components we get a rational equivalence between the class of the zero locus of τ and that of τ'.

The same argument works in the general case: If both $\tau_0, \dots, \tau_{r-i}$ and $\tau'_0, \dots, \tau'_{r-i}$ are collections of sections with degeneracy loci of codimension i, we set

$$\Phi = \left\{([s,t], p) \in \mathbb{P}^1 \times X \mid p \in V(s\tau_0 + t\tau'_0 \wedge \cdots \wedge s\tau_{r-i} + t\tau'_{r-i})\right\}.$$

Using Lemma 5.2, one can show that the components of Φ dominating $\mathbb{P}^1$ give a rational equivalence between $V(\tau_0 \wedge \cdots \wedge \tau_{r-i})$ and $V(\tau'_0 \wedge \cdots \wedge \tau'_{r-i})$.

5.9.1 Whitney's formula for globally generated bundles

Though we will not prove the existence of Chern classes satisfying the properties of Theorem 5.3, it is instructive to see how Whitney's formula (property (c) in Theorem 5.3) follows in the case of a globally generated bundle from facts about the Grassmannian.

Suppose that $\mathcal{E}$ and $\mathcal{F}$ are globally generated bundles on a variety X. Denote the ranks of $\mathcal{E}$ and $\mathcal{F}$ by e and f respectively. We will show that

$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}) \in A(X),$$

or equivalently

$$c_i(\mathcal{E} \oplus \mathcal{F}) = \sum_{i=j+k} c_j(\mathcal{E})c_k(\mathcal{F})$$

for $i \geq 0$.

In the extreme cases $i = 1$ and $i = e + f$ we can see this at once: In the first of these cases, Whitney's formula says that

$$c_1(\mathcal{E} \oplus \mathcal{F}) = c_1(\mathcal{E}) + c_1(\mathcal{F}).$$

If $\sigma_1, \dots, \sigma_e \in H^0(\mathcal{E})$ and $\tau_1, \dots, \tau_f \in H^0(\mathcal{F})$ are general sections, then the degeneracy locus of the $e + f$ sections

$$(\sigma_1, 0), \dots, (\sigma_e, 0), (0, \tau_1), \dots, (0, \tau_f) \in H^0(\mathcal{E} \oplus \mathcal{F})$$

is the sum, as divisors, of the degeneracy loci $V(\sigma_1 \wedge \cdots \wedge \sigma_e)$ and $V(\tau_1 \wedge \cdots \wedge \tau_f)$. Here we are using the identification

$$\bigwedge^{e+f}(\mathcal{E} \oplus \mathcal{F}) = \bigwedge^e \mathcal{E} \otimes \bigwedge^f \mathcal{F}.$$

In the second case, Whitney's formula says that

$$c_{e+f}(\mathcal{E} \oplus \mathcal{F}) = c_e(\mathcal{E})c_f(\mathcal{F}).$$

To see this, let σ and τ be general sections of $\mathcal{E}$ and $\mathcal{F}$ respectively. The zero locus $V((\sigma, \tau))$ of the section $(\sigma, \tau) \in H^0(\mathcal{E} \oplus \mathcal{F})$ is then the intersection of the zero loci $V(\sigma)$ and $V(\tau)$; by Lemma 5.2 applied to $\mathcal{F}|_{V(\sigma)}$, it will have the expected codimension $e + f$ and the equality above follows.

For the general case we adopt the alternative characterization of Chern classes of Proposition 5.7: If $V \subset H^0(\mathcal{E})$ is an n-dimensional subspace generating $\mathcal{E}$, we have a map $\varphi_V : X \to G(n - e, V)$ sending p to the subspace $V_p \subset V$ of sections vanishing at p; the k-th Chern class of $\mathcal{E}$ is then the pullback $\varphi_V^* \sigma_k$ of the Schubert class $\sigma_k \in A^k(G(n - e, V))$.

Let $V \subset H^0(\mathcal{E})$ and $W \subset H^0(\mathcal{F})$ be generating subspaces, of dimensions n and m; let φ_V and φ_W be the corresponding maps. The subspace $V \oplus W \subset H^0(\mathcal{E} \oplus \mathcal{F})$ is again generating, and gives a map

$$\varphi_{V \oplus W} : X \to G(n + m - e - f, V \oplus W).$$

Let

$$\varphi_V \times \varphi_W : X \to G(n - e, V) \times G(m - f, W)$$

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

be the product map. We have

$$\varphi_{V\oplus W} = \eta \circ (\varphi_V \times \varphi_W),$$

where $\eta : G(n-e, V) \times G(m-f, W) \to G(n+m-e-f, V \oplus W)$ is the map sending a pair of subspaces of V and W to their direct sum.

Lemma 5.29. *Let V and W be vector spaces of dimensions n and m. For any s and t, let*

$$\eta : G(s, V) \times G(t, W) \to G(s+t, V \oplus W)$$

be the map sending a pair (Λ, Γ) to $\Lambda \oplus \Gamma$. If α and β are the projection maps on $G(s, V) \times G(t, W)$, then, for any k,

$$\eta^*(\sigma_k) = \sum_{i+j=k} \alpha^*\sigma_i \cdot \beta^*\sigma_j.$$

Given Lemma 5.29, Whitney's formula (in our special case) follows: with φ_V, φ_W and $\varphi_{V\oplus W}$ as above, we have

$$c_k(\mathcal{E} \oplus \mathcal{F}) = \varphi^*_{V\oplus W}(\sigma_k) = \sum_{i+j=k} \varphi^*_V(\sigma_i)\varphi^*_V(\sigma_j) = \sum_{i+j=k} c_i(\mathcal{E})c_j(\mathcal{F}).$$

Note that Lemma 5.29 is a direct (and substantial) generalization of the calculation in Section 2.1.4 of the class of the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$. Specifically, if $V = W$, $m = n = r+1$ and $s = t = 1$, then the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$ is the preimage under the map $\eta : \mathbb{P}V \times \mathbb{P}V \to G(2, V \oplus V)$ of the Schubert cycle $\Sigma_n(V)$ of 2-planes intersecting the diagonal $V \subset V \oplus V$. Thus Lemma 5.29 in this case yields the formula of Section 2.1.4.

Proof: As in the earlier calculation of the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$, we will use the method of undetermined coefficients. Note that the product $G(s, V) \times G(t, W)$ can be stratified by products of Schubert cells; thus, by Proposition 1.17 the products $\alpha^*\sigma_a \cdot \beta^*\sigma_b$ span $A(G(s, V) \times G(t, W))$. (In particular, we have $A_0(G(s, V) \times G(t, W)) = \mathbb{Z}$.) Moreover, intersection products in complementary dimensions between classes of this type again have a simple form: We have

$$\deg\big((\alpha^*\sigma_a\beta^*\sigma_b)(\alpha^*\sigma_c\beta^*\sigma_d)\big) = \begin{cases} 1 & \text{if } a_i + c_{s-i+1} = n-s \text{ for all } i \text{ and} \\ & \qquad b_j + d_{m-j+1} = m-t \text{ for all } j, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we see that $A(G(s, V) \times G(t, W))$ is freely generated by the classes $\alpha^*\sigma_a\beta^*\sigma_b$, and that the intersection pairing in complementary dimensions is nondegenerate. Thus, to prove the equality of Lemma 5.29 it will be enough to show that both sides have the same product with any class $\alpha^*\sigma_a \cdot \beta^*\sigma_b$. Specifically, we need to show that for products $\alpha^*\sigma_a \cdot \beta^*\sigma_b$ of dimension k (that is, with $|a| + |b| = s(n-s) + t(m-t) - k$)

we have

$$\deg(\eta^*\sigma_k \cdot \alpha^*\sigma_a \cdot \beta^*\sigma_b) = \begin{cases} 1 & \text{if } a = (n-s,\dots,n-s,n-s-i) \text{ and} \\ & \quad b = (m-t,\dots,m-t,m-t-j) \\ & \qquad \text{for some } i+j=k, \\ 0 & \text{otherwise.} \end{cases}$$

We start with the "otherwise" half. Note that, by the dimension condition $|a|+|b| = s(n-s)+t(m-t)-k$, the condition $a = (n-s,\dots,n-s,n-s-i)$ and $b = (m-t,\dots,m-t,m-t-j)$ for some $i+j=k$ is equivalent to saying that the sum of the last two indices a_s and b_t is $a_s+b_t = n-s+m-t-k$; in all other cases it will be strictly greater.

Start by choosing general flags $V_1 \subset \cdots \subset V_n = V$, $W_1 \subset \cdots \subset W_m = W$ and $U_1 \subset \cdots \subset U_{n+m} = V \oplus W$. Then

$$\Sigma_a(\mathcal{V}) \subset \{\Lambda \subset V \mid \Lambda \subset V_{n-a_s}\}$$

and

$$\Sigma_b(\mathcal{W}) \subset \{\Gamma \subset W \mid \Lambda \subset W_{m-a_t}\},$$

so

$$\eta(\alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b) \subset \{\Omega \subset V \oplus W \mid \Omega \subset V_{n-a_s} \oplus W_{m-a_t}\}.$$

But

$$\Sigma_k(\mathcal{U}) = \{\Omega \subset V \oplus W \mid \Omega \cap U_{n-s+m-t-k+1} \neq 0\},$$

and, if $a_s + b_t > n-s+m-t-k$, then $(V_{n-a_s} \oplus W_{m-a_t}) \cap U_{n-s+m-t-k+1} = 0$; thus

$$\eta^{-1}\Sigma_k \cap \alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b = \varnothing$$

and the product of the corresponding classes is zero.

Similarly, in case $a = (n-s,\dots,n-s,n-s-i)$ and $b = (m-t,\dots,m-t,m-t-j)$ for some $i+j=k$, the intersection $U = (V_{n-a_s} \oplus W_{m-a_t}) \cap U_{n-s+m-t-k+1}$ will be one-dimensional. Since

$$\Sigma_a(\mathcal{V}) = \left\{\Lambda \subset V \;\middle|\; \begin{array}{r} V_{s-1} \subset \Lambda \text{ and} \\ \Lambda \subset V_{n-a_s} \end{array}\right\}$$

and

$$\Sigma_b(\mathcal{W}) = \left\{\Gamma \subset W \;\middle|\; \begin{array}{r} W_{t-1} \subset \Lambda \text{ and} \\ \Lambda \subset W_{m-a_t} \end{array}\right\},$$

we see that the intersection $\eta^{-1}\Sigma_k \cap \alpha^{-1}\Sigma_a \cap \beta^{-1}\Sigma_b$ will consist of the single point (Λ,Γ), where $\Lambda \subset V$ is the span of V_{s-1} and the projection $\pi_1(U)$ and likewise $\Gamma \subset W$ is the span of W_{t-1} and the image $\pi_2(U)$. That the intersection is transverse follows from Kleiman's theorem in characteristic 0, and from direct examination of the tangent spaces in general. □

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press

5.10 Exercises

Many of the following exercises give applications of Whitney's formula and the splitting principle. We will be assuming the basic facts that if

$$\mathcal{E} = \bigoplus_{i=1}^{e} \mathcal{L}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^{f} \mathcal{M}_i$$

are direct sums of line bundles, then

$$\begin{aligned} \operatorname{Sym}^k \mathcal{E} &= \bigoplus_{1 \le i_1 \le \cdots \le i_k \le r} \mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_k}, \\ \textstyle\bigwedge^k \mathcal{E} &= \bigoplus_{1 \le i_1 < \cdots < i_k \le r} \mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_k}, \\ \mathcal{E} \otimes \mathcal{F} &= \bigoplus_{i,j=1,1}^{e,f} \mathcal{L}_i \otimes \mathcal{M}_j. \end{aligned}$$

Exercise 5.30. Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\bigwedge^2 \mathcal{E}$ in terms of those of $\mathcal{E}$ by invoking the splitting principle and Whitney's formula

Exercise 5.31. Verify your answer to the preceding exercise by observing that the wedge product map

$$\mathcal{E} \otimes \textstyle\bigwedge^2 \mathcal{E} \to \bigwedge^3 \mathcal{E} = \det(\mathcal{E})$$

yields an identification $\bigwedge^2 \mathcal{E} = \mathcal{E}^* \otimes \det(\mathcal{E})$ and applying the formula for a tensor product with a line bundle.

Exercise 5.32. Let $\mathcal{E}$ be a vector bundle of rank 4. Express the Chern classes of $\bigwedge^2 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 5.33. Let $\mathcal{E}$ be a vector bundle of rank 3. Express the Chern classes of $\operatorname{Sym}^2 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 5.34. Let $\mathcal{E}$ be a vector bundle of rank 2. Express the Chern classes of $\operatorname{Sym}^3 \mathcal{E}$ in terms of those of $\mathcal{E}$.

Exercise 5.35. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of rank 2. Express the Chern classes of the tensor product $\mathcal{E} \otimes \mathcal{F}$ in terms of those of $\mathcal{E}$ and $\mathcal{F}$.

Exercise 5.36. Just to get a sense of how rapidly this gets complicated: Do the preceding exercise for a pair of vector bundles $\mathcal{E}$ and $\mathcal{F}$ of ranks 2 and 3.

Exercise 5.37. Apply Exercise 5.35 to find all the Chern classes of the tangent bundle $\mathcal{T}_G$ of the Grassmannian $G = G(2, 4)$.

Exercise 5.38. Find all the Chern classes of the tangent bundle $\mathcal{T}_Q$ of a quadric hypersurface $Q \subset \mathbb{P}^5$. Check that your answer agrees with your answer to the last exercise!

Exercise 5.39. Calculate the Chern classes of the tangent bundle of a product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces

Exercise 5.40. Find the Euler characteristic of a smooth hypersurface of bidegree (a, b) in $\mathbb{P}^m \times \mathbb{P}^n$.

Exercise 5.41. Using Whitney's formula, show that for $n \geq 2$ the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective space is not a direct sum of line bundles.

Exercise 5.42. Find the Betti numbers of the smooth intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, and of the intersection of three quadrics in $\mathbb{P}^5$. (Both of these are examples of *K3 surfaces*, which are diffeomorphic to a smooth quartic surface in $\mathbb{P}^3$.)

Exercise 5.43. Find the Betti numbers of the smooth intersection of two quadrics in $\mathbb{P}^5$. This is the famous *quadric line complex*, about which you can read more in Griffiths and Harris [1994, Chapter 6].

Exercise 5.44. Show that the cohomology groups of a smooth quadric threefold $Q \subset \mathbb{P}^4$ are isomorphic to those of $\mathbb{P}^3$ ($\mathbb{Z}$ in even dimensions, 0 in odd), but its cohomology ring is different (the square of the generator of $H^2(Q, \mathbb{Z})$ is twice the generator of $H^4(Q, \mathbb{Z})$). (This is a useful example of the fact that two compact, oriented manifolds can have the same cohomology groups but different cohomology rings, if you are ever teaching a course in algebraic topology.)

Exercise 5.45. Let $S \subset \mathbb{P}^4$ be a smooth complete intersection of hypersurfaces of degrees d and e, and let $Y \subset \mathbb{P}^4$ be any hypersurface of degree f containing S. Show that if f is not equal to either d or e, then Y is necessarily singular.
Hint: Assume Y is smooth, and apply Whitney's formula to the sequence

$$0 \longrightarrow \mathcal{N}_{S/Y} \longrightarrow \mathcal{N}_{S/\mathbb{P}^4} \longrightarrow \mathcal{N}_{Y/\mathbb{P}^4}|_S \longrightarrow 0$$

to arrive at a contradiction.

https://doi.org/10.1017/CBO9781139062046.007 Published online by Cambridge University Press