7 Projective Morphisms

Exercise 7.1. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.*

Solution. The morphism $\mathscr{L} \to \mathscr{M}$ of sheaves is surjective (resp. isomorphic) if and only if it is surjective (resp. isomorphic) on stalks (Exercise I.1.2). Furthermore, $\mathscr{L}$ and $\mathscr{M}$ being invertible means that they are locally free of rank one. So we are reduced to the question, given a local ring $(A, \mathfrak{m})$ and a surjective morphism $\phi : A \to A$ of A-modules, show that ϕ is an isomorphism. Since $\phi(a) = \phi(a \cdot 1) = a\phi(1)$ the morphism ϕ is determined by $b \mapsto b\phi(1)$. Since ϕ is surjective, there is some element $c \in A$ that gets mapped to 1, so $c\phi(1) = 1$ and therefore $\phi(1)$ is invertible. Then we can define $\psi : A \to A$ by $a \mapsto ac$ and this gives an inverse to ϕ. So ϕ is an isomorphism.

Exercise 7.2. *Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphisms $\phi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m-n-1$.*

Solution. Now since the s_i and t_i generate the same subspace of $\Gamma(X, \mathscr{L})$ each s_i can be written (possibly non-uniquely) as a k-linear combination $s_i = \sum a_{ij} t_j$ of the t_j. We choose the a_{ij} so that the corresponding $(n+1) \times (m+1)$ matrix has linearly independent rows. [1] The coefficients a_{ij} determine $n+1$ global sections $u_i = \sum a_{ij} x_j$ of $\mathcal{O}(1)$ on $\mathbb{P}^m$ and we have $\phi^* u_i = \phi^* \sum a_{ij} x_j = \sum a_{ij} \phi^* x_j = \sum a_{ij} t_j = s_i$. So the morphism $\rho : \mathbb{P}^m - L \to \mathbb{P}^n$ determined by the u_i satisfies $\rho \circ \phi = \psi$ by the uniqueness in Theorem II.7.1. It remains to see that ρ is a linear projection, which Hartshorne fails to define. We define it to be a morphism $\mathbb{P}^m - L \to \mathbb{P}^n$ defined by $n+1$ linearly independent global sections of $\mathcal{O}(1)$ where L is the closed subvariety determined by the global sections considered as homogeneous elements of degree 1 of the homogeneous coordinate ring. The since the global sections are linearly independent and of degree 1, L will be a linear subspace of $\mathbb{P}^n$ of projective dimension $m - n - 1$. We don't need the automorphism because we have probably defined linear projection in a more general way tha Hartshorne has in mind.

Exercise 7.3. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism. Then:*

[1] Let $\dim V = r + 1$ and notice that we suppose $n \leq m$. Now notice that we can find a subset of $\{s_i\}$ that are linearly independent (inductively, choose s_{i_j} not in the span of $s_{i_1}, s_{i_2}, \ldots, s_{i_{j-1}}$) and similarly for the t_i. Without loss of generality, we can assume that these linearly independent subsets are $\{s_0, \ldots, s_r\}$ and $\{t_0, \ldots, t_r\}$. Now for each $i = 0, \ldots, r$ we can express s_i uniquely as a linear combination of the t_j for $j = 0, \ldots, r$, thus obtaining an $(r+1) \times (r+1)$ matrix that is invertible. For each $i > r$, express s_i as $s_i = t_i + \sum_{j=0}^{r} a_{ij} t_j$. Then the full $(n+1) \times (m+1)$ matrix $[a_{ij}]$ consists of an upper left square $(r+1) \times (r+1)$ square which is invertible, and for each $i > r$ we have a nonzero entry in the ith column and zeros in the jth columns for $j > i$. So the rows are linearly independent.

a *either* $\phi(\mathbb{P}^n) = pt$ *or* $m \geq n$ *and* $\dim \phi(\mathbb{P}^n) = n$*;*

b *in the second case,* ϕ *can be obtained as the composition of (1) a d-uple embedding* $\mathbb{P}^n \to \mathbb{P}^N$ *for a uniquely determined* $d \geq 1$*, (2) a linear projection* $\mathbb{P}^N - L \to \mathbb{P}^m$*, and (3) an automorphism of* $\mathbb{P}^m$*. Also,* ϕ *has finite fibres.*

Solution. a A morphism from $\mathbb{P}^n$ to $\mathbb{P}^m$ is equivalant to giving a line bundle $\mathscr{L}$ on $\mathbb{P}^n$ and $m+1$ global sections $s_0, \dots, s_m$ that generate $\mathscr{L}$ at every point of $\mathbb{P}^n$. Consider the subsets $Z_i = \{P \in \mathbb{P}^n | (s_j)_P \notin \mathfrak{m}_P \mathscr{L}_P, j = 0, 1, \dots, i\}$. These are closed subset of $\mathbb{P}^n$ and $Z_i \supseteq Z_{i+1}$. Since the s_i generate $\mathscr{L}$ at every point $Z_m = \varnothing$. Now $\mathbb{P}^n$ has dimension n so either $Z_i = \varnothing$ for every i, or $m \geq n$. In the first case, the global sections are all of degree zero in the homogeneous coordinate ring of $\mathbb{P}^n$ so $d = 0$ and its image in $\mathbb{P}^m$ is a point. In the second case, we show that $\dim \phi(\mathbb{P}^n) = n$ by induction on m.

We have already seen that if $m < n$ then the image of ϕ is a point. Consider $n \leq m$ and ϕ is surjective then $\dim \phi(\mathbb{P}^n) = \dim \mathbb{P}^m = m$ and so $m = n$. If ϕ is not surjective then there is a point P not in the image, and so we can compose $\mathbb{P}^n \to \mathbb{P}^m - P$ with projection from the point $\mathbb{P}^m - P \to \mathbb{P}^{m-1}$. By the inductive hypothesis on $\phi' : \mathbb{P}^n \to \mathbb{P}^{m-1}$ either $\dim \phi'(\mathbb{P}^n) = n$ in which case $\dim \phi(\mathbb{P}^n) \geq n$ and is therefore n, or $\phi'(\mathbb{P}^n)$ is a point. If $\phi'(\mathbb{P}^n)$ is a point then $\phi(\mathbb{P}^n)$ is contained in the preimage of this point under the projection. But this preimage is isomorphic to $\mathbb{A}^1$. So we have a morphism $\mathbb{P}^n \to \mathbb{A}^1$. Since $\mathbb{P}^n$ is proper and connected, its image is proper (Exericse II.4.4) and connected, and the only proper connected subschemes of $\mathbb{A}^1$ singleton points. Hence, the image of $\mathbb{P}^n$ is a point.

Exercise 7.4. a *Use (7.6) to show that if* X *is a scheme of finite type over a noetherian ring* A*, and if* X *admits an ample invertible sheaf, then* X *is separated.*

b *Let* X *be the affine line over a field* k *with the origin doubled. Calculate* $\operatorname{Pic} X$*, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on* X*.*

Solution. a If X admits an ample invertible sheaf $\mathscr{L}$ then Theorem II.7.6 tells use that $\mathscr{L}^n$ is very ample for some $n > 0$ and so X admits an imbedding in projective space. So there is a morphism $X \to \mathbb{P}^n$ for some n that factors as an open imbedding followed by a closed imbedding. Projective space is separated and so the structural morphism $\mathbb{P}^n \to \operatorname{Spec} A$ is separated. But then $X \to \operatorname{Spec} A$ is a composition of an open immersion, a closed immersion, and $\mathbb{P}^n \to \operatorname{Spec} A$, all of which are separated. Hence $X \to \operatorname{Spec} A$ is separated.

b An invertible sheaf $\mathscr{L}$ on X restricts to invertible sheaves on U_0, U_1, the two copies of the affine line that we have constructed X out of. Using Proposition II.6.2 and Corollary II.6.16 we see that $\operatorname{Pic} U_i = 0$ so every invertible sheaf is isomorphic to the structure sheaf. So $\mathscr{L}$ is determined by the isomorphism $\mathcal{O}_{U_0}|_{U_1 \cap U_0} \tilde{\to} \mathscr{L}|_{U_1 \cap U_0} \tilde{\to} \mathcal{O}_{U_1}|_{U_1 \cap U_0}$. Using II.6.2 and II.6.16 again we see that $\operatorname{Pic} U_0 \cap U_1 = 0$ so $\mathscr{L}|_{U_0 \cap U_1} \cong \mathcal{O}_{U_0 \cap U_1}$ and therefore the isomorphism is an automorphism of $k[x, x^{-1}]$ as a module over itself. Automorphisms of this form are determined by a unit in the ring, and the units of $k[x, x^{-1}]$ are the polynomials of the form ax^n for $a \in k^*$ and $n \in \mathbb{Z}$. So every element of $\operatorname{Pic} X$ is determined by a polynomial of the form ax^n. Following our construction, it can be seen that the corresponding Cartier divisor is $\{(U_0, 1), (U_1, ax^n)\}$. In this form it can be seen that ax^n and by^m define the same invertible sheaf if and only if $n = m$, so $\operatorname{Pic} X \cong \mathbb{Z}$. Denote by $\mathscr{L}_n$ the invertible sheaf corresponding to $n \in \mathbb{Z}$.

  Given a Cartier divisor $\{(U_0, 1), (U_1, x^n)\}$, the corresponding invertible sheaf $\mathscr{L}_n$ is the subsheaf of $\mathscr{K}$ generated locally on U_0 by 1 and on U_1 by x^{-n}. A global section of $\mathscr{L}_n$ is a section on U_0 and a section on U_1 that agree on the intersection. That is, an element of $k[x]$ and an element of $x^{-n}k[x]$ that agree when restricted to $U_0 \cap U_1$. So the element of $x^{-n}k[x]$ must have homogeneous components of nonnegative degree, and so if $n > 0$ the local ring at the origin of U_1 cannot be generated by a global section. So each of the invertible sheaves $\mathscr{L}_n$ for $n > 0$ aren't generated by global sections.

  Now suppose that $\mathscr{L}$ is an ample invertible sheaf, say $\mathscr{L} = \mathscr{L}_n$. Then by Theorem II.7.6 $\mathscr{L}^m = \mathscr{L}_{nm}$ is very ample over $\operatorname{Spec} k$ for some $m > 0$. This means there is a morphism to some projective space $\phi : X \to \mathbb{P}^n_k$ such that $\mathscr{L}_{nm} \cong \phi^* \mathcal{O}(1)$. But since $\mathbb{P}^n$ is separated, the two origins get sent to the same point, and so the morphism factors through $X \xrightarrow{f} \mathbb{A}^1 \xrightarrow{g} \mathbb{P}^n$. Since $\operatorname{Pic} \mathbb{A}^1 = 0$ we have $g^* \mathcal{O}(1) \cong \mathcal{O}_{\mathbb{A}^1}$ and so $\phi^* \mathcal{O}(1) = f^* g^* \mathcal{O}(1) = f^* \mathcal{O}_{\mathbb{A}^1} = \mathcal{O}_X$. So $n = 0$. Now consider the coherent sheaf $\mathscr{L}_n$ for some $n > 0$. If $\mathcal{O}_X$ really were ample then there would be some i_0 such that for $i > i_0$ the sheaf $\mathscr{L}_n \otimes \mathcal{O}_X^{\otimes i}$ was generated by its global sections. But we have seen that this is not the case. So not even $\mathcal{O}_X$ is ample, and therefore there are no ample invertible sheaves on X.

Exercise 7.5. *Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.*

a *If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sectrions, then $\mathscr{L} \otimes \mathscr{M}$ is ample.*

b *If $\mathscr{L}$ is ample, and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.*

Solution. a Note that if $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of $\mathcal{O}_X$-modules that are generated by global sections $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_m\}$ then the tensor product, is generated by the global sections $\{f_i \otimes g_j\}$. Now consider some coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$. Since $\mathscr{L}$ is ample, there is some n_0 such that for all $n > n_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections. By the remark we just made, this implies that $(\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^d \cong \mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L})^n$ is generated by global sections. Hence, $\mathscr{L} \otimes \mathscr{M}$ is ample.

b $\mathscr{M}$ is coherent and so for sufficiently large i the sheaf $\mathscr{L}^i \otimes \mathscr{M}$ is generated by global sections. By the remark we made in part (a) this means that $(\mathscr{L}^i \otimes \mathscr{M})^d$ is generated by global sections for all positive d. Take $n = i+1$. For another arbitrary coherent sheaf $\mathscr{F}$, there is some d_0 such that for all $d > d_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^d$ is generated by global sections. It follows that $(\mathscr{F} \otimes \mathscr{L}^d) \otimes (\mathscr{L}^i \otimes \mathscr{M})^d \cong \mathscr{F} \otimes (\mathscr{L}^n \otimes \mathscr{M})^d$ is generated by global sections for all $d > d_0$. Hence, $\mathscr{L}^n \otimes \mathscr{M}$ is ample for sufficiently large n.

c $\mathcal{O}_X$ is a coherent sheaf so there is some d_0 such that for all $d > d_0$ $\mathcal{O}_X \otimes \mathscr{M}^d$ is generated by global sections. For an arbitrary coherent sheaf $\mathscr{F}$, there is some e_0 such that for all $e > e_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^e$ is generated by global sections. Choose n_0 bigger than e_0 and d_0. Then for all $n > n_0$ we have $\mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L})^n \cong (\mathscr{F} \otimes \mathscr{L}^n) \otimes (\mathcal{O}_X \otimes \mathscr{M}^n)$ is generated by global sections. So $\mathscr{L} \otimes \mathscr{M}$ is ample.

d

e From Theorem II.7.6 we see that there is some $n > 0$ for which $\mathscr{L}^n$ is very ample. Using $\mathscr{F} = \mathscr{L}$ in the definition of ample shows that there is some d_0 for which $\mathscr{L}^d$ is generated by global sections for all $d > d_0$. Then by the previous part $\mathscr{L}^d \otimes \mathscr{L}^n = \mathscr{L}^{d+n}$ is very ample for all $d + n > d_0 + n$.

Exercise 7.6. The Riemann-Roch Problem.

a Show that if D is very ample, and if $X \hookrightarrow \mathbb{P}^n$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of X.

b If D corresponds to a torsion element of $\operatorname{Pic} X$, of order r, then $\dim |nD| = 0$ if $r|n$ and $\dim |nD| = -1$ otherwise. In this case the function is periodic of period r.

Solution. a Recall that the Hilbert polynomial is the numerical polynomial associated to the Hilbert function $\phi : n \mapsto \dim_k S_n$ where S is the homogeneous coordinate ring of X. Via the embedding we can assocoiate $\mathscr{L}$ with $S(1)^{\sim}$ and then using Exercise II.5.9(b) we see that $S_n \to \Gamma(X, S(n)^{\sim}) = \Gamma(X, \mathscr{L}^n)$ is an isomorphism for all large enough n. So for all dn large enough have $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1 = \dim S_n - 1 = \phi(n) - 1$. For n large enough, by definition $\phi(n) = P_X(n)$ and so for n large enough we get $\dim |nD| = P_X(n) - 1$.

b If D is a torsion element of degree r then rD is trivial and so its corresponding line bundle is the structure sheaf, whose vector space of global sections has dimension one. So $\dim |rD| = \dim \Gamma(X, \mathcal{O}_X) - 1 = 1 - 1 = 0$. Similarly, if $n = rk$ for some integer k, then $\dim |rkD| = \dim \Gamma(X, \mathcal{O}_X^k) - 1 = \dim \Gamma(X, \mathcal{O}_X) - 1 = 1 - 1 = 0$.

For the case $r \not| n$ we will first show that $\dim \Gamma(X, \mathscr{L}) = 0$. Consider a global section $s \in \Gamma(X, \mathscr{L})$ and let $Z_i = \{P \in X | s_P^{\otimes i} \in \mathfrak{m}_P \mathscr{L}_P^{\otimes i}\}$. If we take an open affine subset U on which we have an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_U$ then s gives a section $t \in \mathcal{O}_U(U)$ and the set $Z_i \cap U$ is $\{P \in X | t_P^i \in \mathfrak{m}_P\}$ and so we see that from this that $Z_i = Z_1$ for all $i \geq 1$. Furthermore, since $\mathscr{L}^r = \mathcal{O}_X$, we see that $Z_{ri} = \varnothing$ or X since the only global sections of $\mathcal{O}_X$ are constants. Hence, $Z_1 = \varnothing$ or X. If $Z_1 = \varnothing$ then recalling the construction of D from $\mathscr{L}$ we see that $D = 0$ and so $r = 1$, and so $r|n$ for all n and we have the previous case. If $Z_1 = X$ then our original global section s was zero and so there are no nonzero global sections of $\mathscr{L}$.

Now for any $i = 1, \ldots, r-1$, the sheaf $\mathscr{L}^i$ is again a torsion sheaf of rank dividing r and so we see that $\mathscr{L}^i$ has no global sections for each of these i. Now $\Gamma(X, \mathscr{L}^n) = \Gamma(X, \mathscr{L}^{kr+i}) = \Gamma(X, \mathscr{L}^i)$ for some $i = 1, \ldots, r-1$ and so for any n that is not a multiple of r, there are no nonzero global sections of $\mathscr{L}^n$. Hence $\dim |nD| = \Gamma(X, \mathscr{L}^n) - 1 = 0 - 1 = -1$.

Exercise 7.7. Some Rational Surfaces. *Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.*

a *The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface.*

b *Show that the subsystem defined by $x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), (x_0 - x_1)x_2$ gives a closed immersion of X into $\mathbb{P}^4$.*

c *Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing though a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\widetilde{X}$, then this map extends to give a closed immersion of $\widetilde{X}$ in $\mathbb{P}^4$. Show that $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\widetilde{X}$ which do not meet.*

Solution. a Recall that the Veronese surface is the 2-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. That is, the embedding that corresponds the the ring homomorphism

$$(y_0, y_1, y_2, y_3, y_4, y_5) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2)$$

from $k[y_i]$ to $k[x_i]$. Consider the morphism $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ corresponding to the linear system $|D|$. In the proof of Theorem II.7.1 the morphism ϕ is defined via $\mathbb{P}^2_{s_i} \to D_+(y_i)$ where s_i is the $(i+1)$th basis vector of $|D|$.

Take $s_0 = x_0^2$. Then the morphism is $\operatorname{Spec}[\frac{x_1}{x_0}, \frac{x_2}{x_0}] \to \operatorname{Spec}[\frac{y_1}{y_0}, \dots, \frac{y_5}{y_0}]$ and is defined via the ring homomorphism

$$\left(\frac{y_1}{y_0}, \dots, \frac{y_5}{y_0}\right) \mapsto \left(\frac{x_2^2}{x_1^2}, \frac{x_0x_1}{x_1^2}, \frac{x_0x_2}{x_1^2}, \frac{x_1x_2}{x_1^2}\right)$$

Clearly, this agrees with the Veronese embedding described above. Now we can do the same thing for the other s_i or evoke Exercise II.4.2 to see that the two morphisms agree.

b We use the cirteria from Proposition 7.3. Since $D_+(x_i^2) = D_+(x_i)$, the five open sets corresponding to the chosen global sections cover $\mathbb{P}^2$, hence (1) is satisfied. Now we want to show that for every closed point $P \in \mathbb{P}^2$, the global sections whose germ are in $\mathfrak{m}_P\mathscr{L}_P$ generate $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$ as a $k(P)$-vector space.

Each closed point appears in one of the open affines $D_+(x_0), D_+(x_1), D_+(x_2)$. The system is symmetric under $x_1 \leftrightarrow x_2$ so show (2) is satisfied for all closed points in $D_+(x_1)$ will imply it for $D_+(x_2)$ and then that will leave the one remaining closed point $(1, 0, 0)$ that is not in $D_+(x_1) \cup D_+(x_2)$.

We start with $P = (1, 0, 0)$ which is the origin in $\mathbb{A}^2 \cong D_+(x_0)$. Choose coordinates $u = \frac{x_1}{x_0}, v = \frac{x_2}{x_0}$. We use the isomorphism $\mathcal{O}(2)|_{D_+(x_0)} \cong \mathcal{O}_X|_{D_+(x_0)}$ so for sections of $\mathcal{O}(2)$ we have $x_0^2 = 1$, $x_0^2\frac{x_1}{x_0} = u$, and $x_0^2\frac{x_2}{x_0} = v$. Then our global sections are $1, u^2, v^2, u(1-v), (1-u)v$. What we want to show is that $\mathfrak{m}_P/\mathfrak{m}_P^2 = (u, v)\mathcal{O}_P/(u, v)^2\mathcal{O}_P$ is generated by the linear combinations of the given global sections. Note that the images of u and v in this vector space are basis vectors. We need only the global sections $u(1-v), (1-u)v$ for in $(u, v)\mathcal{O}_P/(u, v)^2\mathcal{O}_P$ we have $uv = 0$ since $uv \in (u, v)^2$. So everything is fine.

While we are in $D_+(x_0)$ we do the point $(u+1, v+1)$ as well; we will see why later. We have the global section $u(1-v)+2 = u+1-(u+1)(v+1)$ which is $u+1$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $v(1-u)+2 = v+1-(u+1)(v+1)$ which is $v+1$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$. So our global sections generate the vector space $(u+1, v+1)\mathcal{O}_P/(u+1, v+1)^2\mathcal{O}_P$.

Now consider the closed points in $D_+(x_1)$. Choose coordinates $u = \frac{x_0}{x_1}, v = \frac{x_2}{x_1}$. We use the isomorphism $\mathcal{O}(2)|_{D_+(x_1)} \cong \mathcal{O}_X|_{D_+(x_1)}$ so for sections of $\mathcal{O}(2)$ we have $x_1^2 = 1$, $x_1^2\frac{x_0}{x_1} = u$, and $x_1^2\frac{x_2}{x_1} = v$. Then our global sections are $u^2, 1, v^2, (u-v), -uv$. What we want to show is that $\mathfrak{m}_P/\mathfrak{m}_P^2 = (u-a, v-b)\mathcal{O}_P/(u-a, v-b)^2\mathcal{O}_P$ is generated by the linear combinations of the given global sections. If $a + b \neq 0$ then consider

$$\begin{aligned} uv - ab &= (u-a)(v+b) - b(u-a) + a(v-b) \\ (u-v) + (b-a) &= (u-a) + (v-b) \end{aligned}$$

written on the left as a linear combination of our global sections, and on the right as elements of $\mathfrak{m}_P = (u-a, v-b)\mathcal{O}_P$. We have

$$uv - ab = b(u-a) + a(v-b) + \Big((v-b)(u-a)\Big)$$

$$(u-v) + (b-a) = (u-a) - (v-b)$$

and so modulo $\mathfrak{m}_P^2$ these generate $\mathfrak{m}_P$ as long as $a+b \neq 0$ (recall that the images of $u-a$ and $v-b$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ are basis vectors).

So we have seen that (2) holds for all points not in the hypersurface $V(x_0 + x_2)$. Actually, everything that we did for $D_+(x_1)$ holds for $D_+(x_2)$ as well, with 1 switched with 2 so we actually see that (2) holds for all points not in $V(x_0+x_2)$ or $V(x_0+x_1)$. That is, all points except $(1,-1,-1)$. But we saw that it holds for $(1,-1,-1)$ earlier in $D_+(x_0)$. Hence, (2) holds for all closed points, and $X \to \mathbb{P}^4$ is a closed immersion.

c If we use coordinates $y_0, \dots, y_4$ for $\mathbb{P}^4$ and x_0, x_1, x_2 for $\mathbb{P}^2$ and take our point to be $(0,0,1) = \langle x_0, x_1 \rangle$, it can be seen by looking at the basic opens $D_+(x_0), D_+(x_1)$ and $D_+(x_2) - P$ that the linear system $\mathfrak{d}$ with basis vectors $x_0^2, x_1^2, x_0x_1, x_1x_2, x_0x_2$ maps U homeomorphically onto an open subset of the closed subvariety $V = V(y_2y_3 - y_0y_4, y_1y_3 - y_2y_4)$. Since the image of $\widetilde{X}$ is a closed subset and $U \cong \pi^{-1}$ is dense in $\widetilde{X}$, the closure V of the image of U must be the image of $\widetilde{X}$. Now picking a global section y_0 of $\mathcal{O}(1)$ it can be seen to correspond to the divisor $V(y_0,y_1,y_2) + V(y_0,y_2,y_3) + V(y_0,y_3,y_4)$ and so has degree 3.

The image of the line $ax_0 + bx_1 = 0$ (minus P) of $U \subset \mathbb{P}^2$ in V has as its closure the line $V(ay_0 + by_2, ay_1 + by_2, ay_4 + by_3)$ and it follows from some linear algebra that if the ratio $a:b$ is different to $a':b'$ then the two corresponding lines in $V \subset \mathbb{P}^4$ do not share a point.

Exercise 7.8. *Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 corresondence between sections of π and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.*

Solution. By Proposition 7.12 to give a morphism $X \to \mathbb{P}(\mathscr{E})$ over X (that is, a section) it is equivalent to give an invertible sheaf $\mathscr{L}$ on X and a surjective map of sheaves $\mathscr{E} \to \mathscr{L}$. So we are done.

Exercise 7.9. *Let X be a regular noetherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X.*

a *Show that* $\operatorname{Pic} \mathbb{P}(\mathscr{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

b *If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$ (over X) if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.*

Solution. a There is a natural morphism $\alpha : \operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$ defined by $(\mathscr{L}, n) \mapsto (\pi^* \mathscr{L}) \otimes \mathcal{O}(n)$. We claim that this gives the desired isomorphism. Let r be the rank of $\mathscr{E}$. Pick a point $\iota : x \hookrightarrow X$ and an open affine neighbourhood U of it on which $\mathscr{E}$ is free and let $k(x)$ be the residue field. On U we have $\pi^{-1} U = \mathbb{P}^{r-1}_U$ and so we obtain an embedding $\mathbb{P}^{r-1}_{k(x)} \to \mathbb{P}^{r-1}_U \to \mathbb{P}(\mathscr{E})$. Clearly, $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(n)|_U \cong \mathcal{O}_U(n)$ and we know that $\operatorname{Pic} \mathbb{P}^{r-1}_{k(x)} = \mathbb{Z}$ so we have obtain a left inverse to $\mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$. So it remains to show that α is surjective, and that $\operatorname{Pic} X \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$ is injective.

Injectivity of α. Suppose that $\pi^* \mathscr{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. Then by Proposition II.7.11 we see that $\pi_*(\pi^* \mathscr{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$ and by the Projection Formula (Exercise II.5.1(d)) we have $\mathscr{L} \otimes \pi_* \mathcal{O}(n) \cong \mathcal{O}_X$. Again by Proposition II.7.11 we know that $\pi_* \mathcal{O}(n)$ is the degree n part of the symmetric algebra on $\mathscr{E}$ and since rank $\mathscr{E} \geq 2$ this implies that $n = 0$ and $\mathscr{L} \cong \mathcal{O}_X$. Hence α is injective.

Surjectivity of α. Let $\{U_i\}$ be an open cover of X for which $\mathscr{E}$ is locally trivial, and such that each U_i is integral and separated. We can find such a cover since every affine scheme is separated, and X is regular implies that the local rings are reduced. The subschemes $V_i \stackrel{def}{=} \mathbb{P}(\mathscr{E}|_{U_i}) \cong U_i \times \mathbb{P}^{r-1}$ form an open cover of $\mathbb{P}(\mathscr{E})$ and since X is regular, each U_i is regular, and in particular, regular in codimension one, and hence satisfies (*), so we can apply Exercise II.6.1 to find that $\operatorname{Pic} V_i \cong \operatorname{Pic} U_i \times \mathbb{Z}$.

Now if $\mathscr{L} \in \operatorname{Pic} \mathbb{P}(\mathscr{E})$ then for each i, by restricting we get an element $\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i \in \operatorname{Pic} V_i \cong \operatorname{Pic} U_i \times \mathbb{Z}$ together with transition isomorphisms

$$\alpha_{ij} : (\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i)|_{V_{ij}} \to (\mathcal{O}_j(n_j) \otimes \pi_j^* \mathscr{L}_j)|_{V_{ji}}$$

that satisfy the cocycle condition. These isomorphisms pushforward to give isomorphisms

$$\alpha_{ij} : \pi_*(\mathcal{O}_i(n_i)|_{V_{ij}}) \otimes \mathscr{L}_i \to \pi_*(\mathcal{O}_j(n_j)|_{V_{ji}}) \otimes \mathscr{L}_j$$

via the projection formula. A quick look at Proposition II.7.11 and considering ranks, we see that $n_i = n_j$. Furthermore, it can be seen from the definition of $\mathbb{P}(\mathscr{E})$ that $\mathcal{O}_j(n)|_{V_{ij}} = \mathcal{O}_{ij}(n)$ and so our isomorphism α_{ij} is $\mathcal{O}_{ij}(n) \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij}(n) \otimes \pi_j^* \mathscr{L}_j|_{V_{ij}}$. Tensoring this with $\mathcal{O}_{ij}(-n)$ we get isomorphisms $\mathcal{O}_{ij} \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij} \otimes \pi_j^* \mathscr{L}_j$ and the projection formula together with II.7.11 again then tells us that we have isomorphisms $\beta_{ij} : \mathscr{L}_i|_{U_{ij}} \cong \mathscr{L}_j|_{U_{ij}}$, and it can be shown that these satisfy the cocycle condition as a cosequence of the α_{ij} satisfying it. Hence, we can glue the $\mathscr{L}_i$ together to obtain a sheaf $\mathscr{M}$ on X such that $\pi^* \mathscr{M} \otimes \mathcal{O}(n)$ is isomorphic to $\mathscr{L}$ on each connected component of X (where n depends on the component).

b One direction follows immediately from Lemma II.7.9 but we choose to do it more explicitely, using Yoneda's Lemma.

Suppose we have $Z \xrightarrow{f} Y \xrightarrow{g} X$ for arbitrary schemes Y and Z and morphisms f, g. Proposition II.7.12 says that we have an isomorphism $\hom_X(Y, \mathbb{P}(\mathscr{E})) \xrightarrow{\sim} \{$ quotient invertible sheaves $g^*\mathscr{E} \to \mathscr{L}\}$ and that this is given by $(Y \xrightarrow{u} \mathbb{P}(\mathscr{E})) \mapsto (u^*\pi^*\mathscr{E} \to u^*\mathcal{O}(1))$. It is straightforward that the following square commutes

$$\begin{tikzcd} \hom_X(Y, \mathbb{P}(\mathscr{E})) \arrow[r] \arrow[d, "- \circ g"] & \{ \text{ quotient invertible sheaves of } g^*\mathscr{E}\} \arrow[d, "g^*"] \\ \hom_X(Z, \mathbb{P}(\mathscr{E})) \arrow[r] & \{ \text{ quotient invertible sheaves of } f^*\mathscr{E}\} \end{tikzcd}$$

since $(ab)^* \cong b^*a^*$ and so we actually have an isomorphism of functors between $\hom_X(-, \mathbb{P}(\mathscr{E}))$ and the functor $F_{\mathscr{E}}$ that sends a scheme $g : Y \to X$ over X to the set of quotient invertible sheaves of $g^*\mathscr{E}$.

Now Yoneda's Lemma says that if two representable functors are isomorphic then their representatives are isomorphic. If we have an isomorphism $\mathscr{E} \cong \mathscr{E}' \otimes \mathscr{L}$ we get an induced isomorphism $F_{\mathscr{E}} \cong F_{\mathscr{E}'}$ by sending a quotient invertible sheaf $g^*\mathscr{E} \to \mathscr{M}$ to $g^*(\mathscr{E}' \otimes \mathscr{L}) \to \mathscr{M}$ and then $g^*\mathscr{E}' \to \mathscr{M} \otimes (g^*\mathscr{L})^{-1}$. It can be checked that this is functorial using Exercise II.6.8(a), and so we obtain via Yondea, an isomorphism $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$.

Suppose that we have an isomorphism $\alpha : \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ with inverse β. Since α_* and α^* are adjoints, we obtain for every quasi-coherent sheaf $\mathscr{F}$ on $\mathbb{P}(\mathscr{E})$ a morphism $\alpha^*\alpha_*\mathscr{F} \to \mathscr{F}$. If we choose an open affine subset $U = \operatorname{Spec} A$ of $\mathbb{P}(\mathscr{E})$, this morphism on U takes the form $(({}_BM) \otimes_B A)^\sim \to M$ where $\operatorname{Spec} B = \alpha(U)$, $A \cong B$ is induced by α and M is an A-module. This is an isomorphism and so $\alpha^*\alpha_*\mathscr{F} \to \mathscr{F}$ is an isomorphism. Now take $\mathscr{F} = \mathcal{O}(1)$. Then we have $\alpha^*\alpha_*\mathcal{O}(1) \cong \mathcal{O}(1)$ and so $\alpha_*\mathcal{O}(1) \cong \beta^*\mathcal{O}(1)$ since β is the inverse to α. We know that $\beta^*\mathcal{O}(1)$ is in the Picard group of $\mathbb{P}(\mathscr{E}')$ and so by part (a) it has the form $((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n)$ for some invertible sheaf $\mathscr{L}$ on X and some integer n. Pushing the isomorphism $\alpha_*\mathcal{O}(1) \cong ((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n)$ forward through π' and using the Projection formula (Exercise II.5.1d) and Proposition II.7.11 gives

$$\begin{aligned} \mathscr{E} \cong \pi_*\mathcal{O}(1) \cong (\pi')_*\alpha_*\mathcal{O}(1) \cong (\pi')_* \Bigg(((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n) \Bigg) \\ \cong \mathscr{L} \otimes (\pi')_*\mathbb{P}_{\mathbb{P}(\mathscr{E}')}(n) \cong \mathscr{L} \otimes \mathscr{S}^n(\mathscr{E}') \end{aligned}$$

Now since the rank of $\mathscr{E}'$ is $r \geq 2$, the rth degree of the symmetric algebra on $\mathscr{E}'$ has rank $\binom{n+r}{r}$ and so $n = 1$ and we have an isomorphism $\mathscr{E} \cong \mathscr{L} \otimes \mathscr{E}'$ for some line bundle $\mathscr{L}$.

Exercise 7.10. $\mathbb{P}^n$-Bundles Over a Scheme. *Let X be a noetherian scheme.*

a *By anaogy with Exercise II.5.18, define the notion of a projective bundle over X.*

b *If $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^n$-bundle over X.*

c *Assume that X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X. Can you weaken the hypothesis "X regular"?*

d *Conclude (in the case X regular) that there is a one-to-one correspondence between $\mathbb{P}^n$-bundlesover X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n+1$ under the equivlanece $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X.*

Solution. a A projective bundle of rank n over X is a scheme P and a morphism $f : P \to X$, together with additional data consisting of an open covering $\{U_j\}$ of X, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{P}^n_{U_i}$, such that for any i, j and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{P}^n_V = \operatorname{Proj} A[x_0, x_1, \ldots, x_n]$ is given by a linear automorphism θ of $A[x_0, x_1, \ldots, x_n]$.

b We take an affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{E}$ is free on U_i. So we have isomorphims $\mathscr{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus(n+1)}$. By definition of $\pi : \mathbb{P}(\mathscr{E}) \to X$ we have $\pi^{-1}U_i = \operatorname{Proj} \mathscr{S}(\mathscr{E})(U_i) \cong \operatorname{Proj} \mathscr{S}(\mathcal{O}_{U_i}^{\oplus(n+1)})(U_i) = \operatorname{Proj} A_i[x_0, \ldots, x_n] = \mathbb{P}^n_{U_i}$ where $\mathscr{S}(\mathscr{F})$ is the symmetric algebra associated to a locally free sheaf $\mathscr{F}$. So we have our automorphisms ψ_i. Now for any open affine subscheme $V = \operatorname{Spec} A$ of $U_i \cap U_j$, again from the definition of $\mathbb{P}(\mathscr{E})$ we have an isomorphism $\pi^{-1}V \cong \mathbb{P}^n_V$ and the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{P}^n_V$ is defined via the automorphism $\mathcal{O}_{U_i}^{n+1}|_V \cong \mathcal{O}_{U_j}^{n+1}|_V$ coming from the restriction morphisms $\mathscr{E}(U_i) \to \mathscr{E}(V) \leftarrow \mathscr{E}(U_j)$. Clearly this is of the desired form.

c

d Given a locally free sheaf of rank $n+1$ we obtain a projective bundle $\mathbb{P}(\mathscr{E})$ by part (b) of this question, so $\mathbb{P}(-) : \mathscr{L}oc_{n+1}(X) \to \mathscr{P}\mathscr{B}_n(X)$ is a map from locally free sheaves of rank $n+1$ to projective bundles of rank n. Conversely, given a projective bundle P, by part (c) we obtain a locally free sheaf $\mathscr{E} = \mathscr{E}_P$ of rank $n+1$ and an isomorphism $\mathbb{P}(\mathscr{E}) \cong P$, so we have a map $\mathscr{E}_- : \mathscr{P}\mathscr{B}_n(X) \to \mathscr{L}oc_{n+1}(X)$ which is a right inverse to $\mathbb{P}(-)$. The only thing left to see is that $\mathscr{E}_-$ is a left inverse to $\mathbb{P}(-)$ as well. So suppose that we have a localy free sheaf $\mathscr{F}$ or rank $n+1$ on X. Then we have seen that $\mathbb{P}(\mathscr{E}_{\mathbb{P}(\mathscr{F})}) \cong \mathbb{P}(\mathscr{F})$. But by Exercise II.7.9(b) this implies that $\mathscr{E}_{\mathbb{P}(\mathscr{F})} \cong \mathscr{F} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$. So we have the desired one-to-one correspondence after we note that $\mathbb{P}$ is still well defined on $\mathscr{L}oc_{n+1}(X)$ modulo the equivalence relation (again by Exercise II.7.9(b)).

Exercise 7.11. *a If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $d \geq 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$.*

b *If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I} \cdot \mathscr{J}$ give isomorphic blowings-up.*

c *If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1}U \to U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.*

Solution. a By definition, the blowing up of $\mathscr{I}$ is $\mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n)$, and the blowing up of $\mathscr{I}^d$ is $\mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^{nd})$. Locally–that is on an affine subscheme U of X, these blowing ups are $\operatorname{Proj} \bigoplus_{n\geq 0} \mathscr{I}(U)^n$ and $\operatorname{Proj} \bigoplus_{n\geq 0} \mathscr{I}(U)^{dn}$. By Exercise II.5.13 we know that these are isomorphic, and so if we can show that the isomorphism from Exercise II.5.13 is natural we are done, since these local isomorphisms will then agree on the pairwise intersections $U_i \cap U_j$ of two open affine subschemes. That is, we want to show that for a morphism of graded rings $T \to S$, the square commutes

$$\begin{tikzcd} \operatorname{Proj} S \arrow[r] \arrow[d] & \operatorname{Proj} T \arrow[d] \\ \operatorname{Proj} S^{(d)} \arrow[r] & \operatorname{Proj} T^{(d)} \end{tikzcd}$$

But in the proof of Exercise II.5.13, the horizontal morphisms come from inclusions $S^{(d)} \to S$ and $T^{(d)} \to T$ and so this square commutes. So we are done.

b This follows from Lemma II.7.9 or we can use Yoneda's Lemma as follows.

Proposition II.7.14 says that $\widetilde{X}$ represents the functor that sends Z to the set of morphisms $f : Z \to X$ such that $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z. Now since $\mathscr{J}$ is invertible, $f^*\mathscr{J}$ is invertible, and so if we can show that $f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z \cong (f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes f^*\mathscr{J}$, then $f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z$ will be invertible if and only if $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$ is invertible and so the two functors represented by the blowings-up of $\mathscr{I}$ and $\mathscr{I} \cdot \mathscr{J}$ will be isomorphic, implying that the blowing-ups themselves are isomorphic.

The sheaf $f^{-1}\mathscr{J} \cdot \mathcal{O}_Z$ is the image of $f^*\mathscr{J} \to \mathcal{O}_Z$, so we have natural maps

$$\begin{aligned}(f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes f^*\mathscr{J} \to (f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes (f^{-1}\mathscr{J} \cdot \mathcal{O}_Z) \to & f^{-1}\mathscr{I} \cdot f^{-1}\mathscr{J} \cdot \mathcal{O}_Z \\ &= f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z\end{aligned}$$

Since $\mathscr{J}$ is invertible, it is locally isomorphic to $\mathcal{O}_X$, and so $f^*\mathscr{J}$ is locally isomorphic to $\mathcal{O}_Z$. Let U be an open subset of U on which we have an isomorphism $f^*\mathscr{J}|_U \cong \mathcal{O}_Z|_U$. Then $f^*\mathscr{J}|_U$ is of the form $(\mathcal{O}_Z|_U)s$ for some section $s \in \mathcal{O}_Z(U)$ (that is, s generates $f^*\mathscr{J}|_U$ as a free $\mathcal{O}_Z|_U$ module). Assuming that $U = Z$ so that we can stop writing $|_U$ everywhere, our morphisms above become

$$(f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes s \xrightarrow{\sim} f^{-1}\mathscr{I} \cdot \mathcal{O}_Z \; s = f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z$$

Since we only need to check isomorphisms of sheaves locally, we are done.

Exercise 7.12. *Let X be a noetherian scheme and let Y, Z be two closed subschemes, neither one containing the other. Let $\widetilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathscr{I}_Y + \mathscr{I}_Z$). Show that the strict transform $\widetilde{Y}$ and $\widetilde{Z}$ of Y and Z in $\widetilde{X}$ do not meet.*

Solution. Suppose that they do meet at some point $P \in \widetilde{X}$. The image of this point πP in X is contained in some open affine scheme $U = \operatorname{Spec} A$ and the preimage of this open is $\pi^{-1}U = \operatorname{Proj} \bigoplus_{d \geq 0} (I_Y + I_Z)^d$ where $I_Y = \mathscr{I}_Y(U), I_Z = \mathscr{I}_Z(U)$. The intersections of Y and Z with U are $Y \cap U = \operatorname{Spec}(A/I_Y)$, and $Z \cap U = \operatorname{Spec}(A/I_Z)$ and the preimage of these opens of Y and Z are $\pi^{-1}(U \cap Y) = \operatorname{Proj} \bigoplus_{d \geq 0} ((I_Y + I_Z)(A/I_Y))^d \subset \widetilde{Y}$ and similarly for Z. The closed imbedding $\pi^{-1}(U \cap Y) \to \pi^{-1}(U)$ is given by a homomorphism of homogeneous rings $\bigoplus_{d \geq 0} (I_Y + I_Z)^d \to \bigoplus_{d \geq 0} ((I_Y + I_Z)(A/I_Y))^d$ and similarly for Z. Clearly the kernel of this ring homomorphism is the homogeneous ideal $\bigoplus_{d \geq 0} I_Y^d$ and similarly for Z. Now if the two closed subschemes intersect as we have supposed then there is a homogeneous prime ideal of $\bigoplus_{d \geq 0} (I_Y + I_Z)^d$ that contains both of these homogeneous ideals. But $\bigoplus_{d \geq 0} I_Y^d$ and $\bigoplus_{d \geq 0} I_Z^d$ generate $\bigoplus_{d \geq 0} (I_Y + I_Z)^d$ so there can be no proper homogeneous prime ideal containing them both. Hence, the intersection is trivial.