Chapter 12

Porteous' formula

Keynote Questions

(a) Let $M \cong \mathbb{P}^{ef-1}$ be the space of $e \times f$ matrices and $M_k \subset M$ the locus of matrices of rank k or less. What is the degree of M_k? (Answer on page 433.)

(b) Let $S \subset \mathbb{P}^n$ be a smooth surface and $f : S \to \mathbb{P}^3$ the projection of S from a general plane $\Lambda \cong \mathbb{P}^{n-4} \subset \mathbb{P}^n$. At how many points $p \in S$ will the map f fail to be an immersion? (Answer on page 438.)

(c) Let $C \subset \mathbb{P}^3$ be a smooth rational curve of degree d. How many lines $L \subset \mathbb{P}^3$ meet C four times? (Answer on page 441.)

12.1 Degeneracy loci

We saw in Chapter 5 that the Chern class $c_i(\mathcal{F})$ of a vector bundle $\mathcal{F}$ of rank f on a smooth variety X can be characterized, when $\mathcal{F}$ is generated by global sections, as the class of the scheme where $e = f - i + 1$ general sections of $\mathcal{F}$ become dependent: Specifically, if the locus where a map

$$\varphi : \mathcal{O}_X^e \to \mathcal{F}$$

fails to have maximal rank has the expected codimension i, then $c_i(\mathcal{F})$ is the class of the scheme that is locally defined by the $e \times e$ minors of a matrix representing φ. A similar result holds for Segre classes.

We can substantially extend the usefulness of this characterization in two ways: by considering the rank-k locus of a map $\mathcal{O}_X^e \to \mathcal{F}$ for arbitrary $k \leq \min(e, f)$, and by replacing $\mathcal{O}_X^e$ with an arbitrary vector bundle $\mathcal{E}$. In this chapter we will do both: We henceforth consider the class of the scheme $M_k(\varphi)$ where a map of vector bundles $\varphi : \mathcal{E} \to \mathcal{F}$ has rank $\leq k$, locally defined by the ideal of $(k+1) \times (k+1)$ minors of a matrix representation of φ. Such loci are called *degeneracy loci*. We write e and f for the ranks of $\mathcal{E}$ and $\mathcal{F}$, respectively.

In the "generic" case, where X is an affine space of dimension ef and φ_{gen} is the map defined by an $f \times e$ matrix of variables, the codimension of the locus $M_k(\varphi_{\text{gen}})$ is $(e-k)(f-k)$ (Harris [1995, Proposition 12.2] or Eisenbud [1995, Exercise 10.10]).

In general we say that $(e-k)(f-k)$ is the *expected codimension* of $M_k(\varphi)$. In this chapter we will give a formula for the class of$M_k(\varphi)$ under the assumption that it has the expected dimension

Such a formula was first found by Giambelli in 1904, in the special case where $\mathcal{E}$ and $\mathcal{F}$ are both direct sums of line bundles. René Thom observed more generally in the context of differential geometry that when $M_k(\varphi)$ has the expected codimension its class (suitably construed) depends only on the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. This was made explicit by Porteous (see Porteous [1971], which reproduces notes from 1962), giving the expression now called Porteous' formula. (The formula might more properly be called the Giambelli–Thom–Porteous formula; we have chosen to call it the Porteous formula for brevity and because that is how it appears in much of the literature.) The result was proven (in a more general form, in which one specifies the ranks of the restriction of φ to a flag of subbundles of $\mathcal{E}$) in the context of algebraic geometry by Kempf and Laksov [1974].

The form of the expression is interesting in itself: Porteous' formula expresses $[M_k(\varphi)]$ as a polynomial in the components of the ratio $c(\mathcal{F})/c(\mathcal{E})$.

To get an idea of what is to come, consider the case $k = 0$, and suppose that the locus $M_0(\varphi)$, where the map φ induces the zero map on the fibers, has the expected codimension ef. The map φ may be regarded as a global section of the bundle $\mathcal{E}^* \otimes \mathcal{F}$, and the locus $M_0(\varphi)$ is the locus where this global section vanishes; thus its class is $c_{ef}(\mathcal{E}^* \otimes \mathcal{F})$.

The splitting principle makes it easy to understand $c_{ef}(\mathcal{E}^* \otimes \mathcal{F})$: If $\mathcal{E} = \bigoplus \mathcal{L}_i$ and $\mathcal{F} = \bigoplus \mathcal{M}_i$ were sums of line bundles, then $\mathcal{E}^* \otimes \mathcal{F}$ would be the sum of the $\mathcal{L}_i^* \otimes \mathcal{M}_j$. If we write $c(\mathcal{L}_i) = 1 + \alpha_i$ and $c(\mathcal{M}_j) = 1 + \beta_j$, then $c(\mathcal{L}_i^* \otimes \mathcal{M}_j) = 1 + \beta_j - \alpha_i$, so, by Whitney's formula,

$$c(\mathcal{E}^* \otimes \mathcal{F}) = \prod_{i,j} (1 + \beta_j - \alpha_i),$$

and in particular

$$c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \prod_{i,j} (\beta_j - \alpha_i).$$

This expression is symmetric in each of the two sets of variables α_i and β_j, so it can be written in terms of the elementary symmetric functions of these variables, which are the Chern classes of $\mathcal{E}$ and $\mathcal{F}$. If we think of the α_i as the roots of the *Chern polynomial* $c_t(\mathcal{E}) := 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots$, and similarly for $c_t(\mathcal{F})$, then $c_{ef}(\mathcal{E}^* \otimes \mathcal{F})$ is the classical *resultant* of $c_t(\mathcal{E})$ and $c_t(\mathcal{F})$, written $\operatorname{Res}_t(c_t(\mathcal{E}), c_t(\mathcal{F}))$. (See for example

Eisenbud [1995, Section 14.1] for more about resultants and their role in algebraic geometry.) By the splitting principle, the result we have obtained holds for all maps of vector bundles:

Proposition 12.1. *If $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks e and f on a smooth variety X, then*

$$c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \operatorname{Res}_t(c_t(\mathcal{E}), c_t(\mathcal{F})).$$

The polynomials $c_t(\mathcal{E})$ and $c_t(\mathcal{F})$ each have constant coefficient 1, and in this case we can express the resultant differently.

We first introduce some notation. For any sequence of elements $\gamma := (\gamma_0, \gamma_1, \dots)$ in a commutative ring and any natural numbers e, f, we set $\Delta^e_f(\gamma) = \det \mathbb{D}^e_f(\gamma)$, where

$$\mathbb{D}^e_f(\gamma) := \begin{pmatrix} \gamma_f & \gamma_{f+1} & \cdots & \cdots & \gamma_{e+f-1} \\ \gamma_{f-1} & \gamma_f & \cdots & \cdots & \gamma_{e+f-2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \gamma_{f-e+1} & \gamma_{f-e+2} & \cdots & \cdots & \gamma_f \end{pmatrix}.$$

Proposition 12.2. *If $a(t) = 1 + a_1 t + \cdots + a_e t^e$ and $b(t) = 1 + b_1 t + \cdots + b_f t^f$ are polynomials with constant coefficient 1, then*

$$\operatorname{Res}_t(a(t), b(t)) = \Delta^e_f\left[\frac{b(t)}{a(t)}\right] = (-1)^{ef}\,\Delta^f_e\left[\frac{a(t)}{b(t)}\right],$$

where $[b(t)/a(t)]$ denotes the sequence of coefficients $(1, c_1, c_2, \dots)$ of the formal power series $b(t)/a(t) = 1 + c_1 t + c_2 t^2 + \cdots$, and similarly for $[a(t)/b(t)]$.

We will give the proof in Section 12.2.

For any element $\gamma \in A(X)$, we write $[\gamma]$ for the sequence $(\gamma_0, \gamma_1, \dots)$, where γ_i is the component of γ of degree i. The next corollary gives the expression of the top Chern class of a tensor product that we will use:

Corollary 12.3. *If $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks e and f on a smooth variety X, then*

$$c_{ef}(\mathcal{E}^* \otimes \mathcal{F}) = \operatorname{Res}_t(c_t(\mathcal{E}), c_t(\mathcal{F})) = \Delta^e_f\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right].$$

In particular, if $\varphi : \mathcal{E} \to \mathcal{F}$ is a homomorphism that vanishes in expected codimension ef, then

$$[M_0(\varphi)] = \Delta^e_f\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right].$$

Porteous' formula for the class of an arbitrary degeneracy locus follows the same pattern:

Theorem 12.4 (Porteous' formula)**.** *Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of ranks e and f on a smooth variety X. If the scheme $M_k(\varphi) \subset X$ has codimension $(e-k)(f-k)$, then its class is given by*

$$[M_k(\varphi)] = \Delta_{f-k}^{e-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right].$$

The formula is easiest to interpret in the case $k = e-1 < f$; in this case $\Delta_{e-k}^{f-k}(\gamma)$ is the determinant of the 1×1 matrix

$$\mathbb{D}_{f-e+1}^{1}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right] = \left\{\frac{c(\mathcal{F})}{c(\mathcal{E})}\right\}^{f-e+1},$$

where we write $\{\alpha\}^k$ for the codimension-k part of a Chow class $\alpha \in A(X)$. Specializing further, if $\mathcal{E} = \mathcal{O}_X^e$ then $\{c(\mathcal{F})/c(\mathcal{E})\}^{f-e+1} = c_{f-e+1}(\mathcal{F})$, so we recover the characterization of Chern classes as degeneracy loci (Theorem 5.3). If instead $\mathcal{F} = \mathcal{O}_X^f$, then $\{c(\mathcal{F})/c(\mathcal{E})]\}^{f-e+1} = \{1/c(\mathcal{E})\}^{f-e+1}$, the Segre class $s_{f-e+1}(\mathcal{E}^*)$, so we recover the characterization of the Segre class as degeneracy locus of a map $\mathcal{O}_X^f \to \mathcal{E}^*$.

More generally, $\Delta_{f-e+1}^1[c(\mathcal{F})/c(\mathcal{E})]$ represents an obstruction to the existence of an inclusion of vector bundles $\varphi : \mathcal{E} \to \mathcal{F}$: If φ were an inclusion of vector bundles, then $\mathcal{F}/\mathcal{E}$ would be a vector bundle of rank equal to $\operatorname{rank}\mathcal{F} - \operatorname{rank}\mathcal{E} = f - e$, so $\Delta_{f-e+1}^1[c(\mathcal{F})/c(\mathcal{E})] = \{c(\mathcal{F})/c(\mathcal{E})\}^{f-e+1} = 0$.

The proof of Theorem 12.4 will be given in Section 12.3: After a reduction to a "generic case," we will express $[M_k(\varphi)]$ as the image of $[M_0(\psi)]$, where ψ is a map from a bundle $\mathcal{S}$ of rank $e-k$ to a bundle $\mathcal{F}'$ of rank f on a Grassmannian bundle over X; under the pushforward to X, the entry $\{c(\mathcal{F}')/c(\mathcal{S})\}^i$ in the matrix $\mathbb{D}_f^{e-k}[c(\mathcal{F})/c(\mathcal{E})]$ will be replaced by $\{c(\mathcal{F})/c(\mathcal{E})\}^{i-k}$, yielding Porteous' formula.

12.2 Porteous' formula for $M_0(\varphi)$

In this section we will prove the resultant formula of Proposition 12.2, and thus complete the proof of Corollary 12.3.

Proof of Proposition 12.2: Since the polynomial $\prod(\beta_i - \alpha_j)$ has no repeated factors, it divides any polynomial in the α_i and β_j that vanishes when one of the α_i is equal to one of the β_j. We first show that $\Delta_f^e(b(t)/a(t))$ has this vanishing property. Indeed, if $a(t)$ and $b(t)$ have a common factor $1+\gamma$, then dividing $a(t)$ by this root gives a polynomial $\bar{a}(t)$ such that

$$\bar{a}(t)\frac{b(t)}{a(t)} = g(t)$$

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

is a polynomial of degree $< f$. If we write the ratio b/a as a power series

$$\frac{b(t)}{a(t)} = 1 + c_1 t + c_2 t^2 + \cdots,$$

and substitute the power series $1 + c_1 t + \cdots$ into this expression, we get a power series

$$g(t) = \bar{a}(t)c(t) = 1 + \bar{c}_1 t + \bar{c}_2 t^2 + \cdots$$

whose coefficients $\bar{c}_i = c_i + \bar{a}_1 c_{i-1} + \cdots + \bar{a}_{e-1} c_{i-e+1}$ vanish for $i \geq f$. It follows that the vector $(\bar{a}_{e-1}, \dots, \bar{a}_1, -1)$ is annihilated by $\mathbb{D}^e_f(b(t)/a(t))$, and thus $\Delta^e_f(b(t)/a(t)) = \det \mathbb{D}^e_f(b(t)/a(t)) = 0$.

It follows that

$$\Delta^e_f(b(t)/a(t)) = d \operatorname{Res}_t(a(t), b(t)) = d \prod_{i,j} (\beta_j - \alpha_i)$$

for some polynomial d in the α_i and β_j.

Writing $a(t) = \prod(1 + \alpha_i t)$ and $b(t) = \prod_j (1 + \beta_j t)$, we see that the coefficient of t^k in the power series $\sum c_k t^k = b(t)/a(t)$ is homogeneous of degree k in the variables α_i, β_j, and thus every term in the determinant $\Delta^e_f(b(t)/a(t))$ has degree ef, as does $\operatorname{Res}_t(a(t), b(t))$. It follows that d is a constant.

If we take all the $a_i = 0$, we see that $b(t)/a(t) = b(t)$, and $\mathbb{D}^e_f(b(t)/a(t))$ becomes lower-triangular; in this case its determinant is $(b_f)^e = \left(\prod_j \beta_j\right)^e = \operatorname{Res}_t(a(t), b(t))$, so $d = 1$. □

12.3 Proof of Porteous' formula in general

12.3.1 Reduction to a generic case

We first explain how to reduce the proof to a case where a slightly stronger hypothesis holds:

(a) $M_k(\varphi)$ is of the expected dimension $(e-k)(f-k)$.
(b) $M_k(\varphi)$ is reduced.
(c) The points $x \in X$ where the map φ_x has rank exactly k are dense in $M_k(\varphi)$; equivalently, $M_{k-1}(\varphi)$ has codimension $> (e-k)(f-k)$.

To do this, consider the map

$$\psi : \mathcal{E} \xrightarrow{\binom{1}{\varphi}} \mathcal{E} \oplus \mathcal{F}$$

taking $\mathcal{E}$ onto the graph $\Gamma_\varphi \subset \mathcal{E} \oplus \mathcal{F}$ of φ. The original map φ is the composition of ψ with the projection to $\mathcal{F}$.

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

We now form the Grassmannian bundle $\pi : X' := G(e, \mathcal{E} \oplus \mathcal{F}) \to X$, and we write $\mathcal{S} \to \pi^*(\mathcal{E} \oplus \mathcal{F})$ for the tautological subbundle of rank e. Since ψ is an inclusion of bundles, the universal property of the Grassmannian guarantees that there is a unique map $u : X \to X'$ such that the pullback under u of the tautological inclusion map $\mathcal{S} \to \pi^*(\mathcal{E} \oplus \mathcal{F})$ on the Grassmannian is $\psi : \mathcal{E} \to \mathcal{E} \oplus \mathcal{F}$, and thus the pullback of the composite map $\varphi' : \mathcal{S} \to \pi^*(\mathcal{E} \oplus \mathcal{F}) \to \pi^*\mathcal{F}$ is φ. It follows that $M_k(\varphi) = u^{-1}(M_k(\varphi'))$.

Since $\mathcal{E} \subset \mathcal{E} \oplus \mathcal{F}$ is the kernel of the projection to $\mathcal{F}$, the points of $M_k(\varphi')$ are the points $x \in X'$ such that the fiber of $\mathcal{S}$ meets the fiber of $\pi^*\mathcal{E}$ in dimension at least $e-k$. With notation parallel to that of Chapter 4, this is the Schubert cycle $\Sigma := \Sigma_{(e-k)^{f-k}}(\mathcal{E})$: It is defined over any open subset of X where the bundles in question are trivial, by the same determinantal formula that defines the corresponding Schubert cycle in the case of vector spaces. A look at this formula shows that $\Sigma = M_k(\varphi)$ as schemes.

Over an open set in X where the bundles $\mathcal{E}$ and $\mathcal{F}$ are trivial, the Grassmannian $X' = G(e, \mathcal{E} \oplus \mathcal{F})$ is the product of X with the ordinary Grassmannian $G(e, e + f)$ and the Schubert cycle Σ is the product of X with the corresponding Schubert cycle in $G(e, e + f)$. As was mentioned in the discussion of the equations of Schubert varieties after Theorem 4.3, these varieties are reduced, irreducible and Cohen–Macaulay (Hochster [1973] or De Concini et al. [1982]). In particular, $M_k(\varphi') = \Sigma$ is reduced, irreducible and Cohen–Macaulay of codimension $(e-k)(f-k)$. Moreover, $M_{k-1}(\varphi') = \Sigma_{(e-k+1)^{f-k}}(\mathcal{E})$ has codimension $(e-k+1)(f-k) > \operatorname{codim} \Sigma$, so the points $x' \in X'$ where $\varphi'_{x'}$ has rank exactly k are dense in $M_k(\varphi')$.

Because $M_k(\varphi') = \Sigma$ is a Cohen–Macaulay subvariety, we can apply the Cohen–Macaulay case of Theorem 1.23 and conclude that $[M_k(\varphi)] = u^*[M_k(\varphi')]$. Since $u^* : A(X') \to A(X)$ is a ring homomorphism, and since the Chern classes of $\mathcal{S}$ and $\pi^*\mathcal{F}$ pull back to the Chern classes of $\mathcal{E}$ and $\mathcal{F}$ respectively, we see that it suffices to prove Porteous' formula for the map φ'.

12.3.2 Relation to the case $k = 0$

Replacing φ by φ' as above, we may assume that φ satisfies hypotheses (a), (b) and (c) of the previous section.

We next linearize the problem by introducing more data. To say that $x \in M_k(\varphi)$ means that there is some k-dimensional subspace of $\mathcal{F}_x$ that contains $\varphi_x(\mathcal{E}_x)$, and by assumptions (a) and (c) the subspace is equal to $\varphi_x(\mathcal{E}_x)$ when x is a general point of $M_k(\varphi)$. To make use of this idea, we introduce the Grassmannian $\pi : G(e-k, \mathcal{E}) \to X$. We write $\mathcal{S} \subset \pi^*\mathcal{E}$ for the tautological rank-k subbundle. Let $\mu : \mathcal{S} \to \pi^*\mathcal{E} \to \pi^*\mathcal{F}$ be the composite map. The locus in $G(e-k, \mathcal{F})$ where $\pi^*(\varphi) : \pi^*\mathcal{E} \to \pi^*\mathcal{F}$ factors through the tautological rank-k quotient $\pi^*(\mathcal{E})/\mathcal{S}$ may also be described as $M_0(\mu)$, the locus of points x where μ vanishes. It follows that the map from $M_0(\mu)$ to $M_k(\varphi)$ is

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

surjective and generically one-to-one. Since we have assumed that $M_k(\varphi)$ is reduced, we can compute the class $[M_k(\varphi)]$ as $\pi_*[M_0(\mu)]$.

From the fact that π is generically one-to-one on $M_0(\mu)$, we have that $\dim M_0(\mu) = \dim M_k(\varphi) = \dim X - (e-k)(f-k)$. Since $\dim G(e-k,\mathcal{E}) = \dim X + k(e-k)$, it follows that $M_0(\mu)$ has the expected codimension $(e-k)f$, and thus by Corollary 12.3

$$[M_k(\varphi)] = \pi_*[M_0(\mu)] = \pi_*\Delta_f^{e-k}\left[\frac{c(\pi^*\mathcal{F})}{c(\mathcal{S})}\right].$$

12.3.3 Pushforward from the Grassmannian bundle

Completion of the Proof of Theorem 12.4: It remains to compute

$$\pi_*\Delta_f^{e-k}\left[\frac{c(\pi^*\mathcal{F})}{c(\mathcal{S})}\right].$$

Let $\mathcal{Q} = (\pi^*\mathcal{E})/\mathcal{S}$. By Whitney's formula,

$$c(\mathcal{S}) = \frac{c(\pi^*\mathcal{E})}{c(\mathcal{Q})},$$

so

$$\frac{c(\pi^*\mathcal{F})}{c(\mathcal{S})} = \frac{c(\pi^*\mathcal{F})}{c(\pi^*\mathcal{E})}c(\mathcal{Q}) = \pi^*\left(\frac{c(\mathcal{F})}{c(\mathcal{E})}\right)c(\mathcal{Q}).$$

The point is that we have isolated the factors in the entries of the matrix

$$\mathbb{D}_f^{e-k}[\pi^*(c(\mathcal{F})/c(\mathcal{E}))c(\mathcal{Q})]$$

that are pullbacks from X. To take advantage of this, we expand the determinant $\Delta_f^{e-k}[\pi^*\big(c(\mathcal{F})/c(\mathcal{E})\big)c(\mathcal{Q})]$ into a sum of classes of the form $\pi^*(\delta)\epsilon$, where ϵ is a product of $e-k$ Chern classes of $\mathcal{Q}$. From the push-pull formula (Theorem 1.23), we see that π_* takes $\pi^*(\delta)\epsilon$ to $\delta\pi_*(\epsilon)$.

The fibers of the morphism π are all isomorphic to the Grassmannian $G(e-k,e)$, which has dimension $(e-k)k$, so any class of codimension $< (e-k)k$ pushes forward to 0. Since $\mathcal{Q}$ has rank k, the only product of $e-k$ Chern classes of $\mathcal{Q}$ that has nonzero pushforward is $c_k(\mathcal{Q})^{e-k}$, and this class will push forward to some multiple $d[X]$ of the fundamental class of X. The coefficient d is the intersection number of $c_k(\mathcal{Q})^{e-k}$ with the general fiber of π.

We can compute d by first restricting $\mathcal{Q}$ to a general fiber of π, obtaining the tautological quotient bundle $\overline{\mathcal{Q}}$ on $G(e-k,e)$. The number d is the degree of $c_k(\overline{\mathcal{Q}})^{e-k}$, which is 1 by Corollary 4.2.

It follows that if ϵ is any k-fold product of Chern classes of $\mathcal{Q}$ then

$$\pi_*(\pi^*(\delta)\epsilon) = \begin{cases} \delta & \text{if } \epsilon = c_k(\mathcal{Q})^{e-k}, \\ 0 & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

In particular, since each term in the usual expansion of an $(e-k)\times(e-k)$ determinant is the product of $e-k$ factors,

$$\pi_*\Delta_f^{e-k}\left[\frac{c(\pi^*\mathcal{F})}{c(\mathcal{S})}\right] = \pi_* \det\begin{pmatrix} c_f & c_{f+1} & \cdots & c_{e-k+f-1} \\ c_{f-1} & c_f & \cdots & c_{e-k+f-2} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{f-e+k+1} & \cdots & \cdots & c_f \end{pmatrix},$$

where

$$c_j = \pi^*\left\{\frac{c(\mathcal{F})}{c(\mathcal{E})}\right\}^{j-k} c_k(\mathcal{Q}).$$

Since π^* is a ring homomorphism, the coefficient of $c_k(\mathcal{Q})^{e-k}$ in this expansion is

$$\pi^*\Delta_{f-k}^{e-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right],$$

and we see that

$$\pi_*\Delta_f^{e-k}\left[\frac{c(\pi^*\mathcal{F})}{c(\mathcal{S})}\right] = \Delta_{f-k}^{e-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right],$$

as required. □

12.4 Geometric applications

12.4.1 Degrees of determinantal varieties

As a direct application, we use Porteous' formula to determine the degrees of the varieties of $e\times f$ matrices of rank $\leq k$. Another approach is to form an explicit basis for the component of each degree in the homogeneous coordinate ring of this variety, and thus to compute the Hilbert function. This seems first to have been done in Hodge [1943]; for a modern treatment related to his ideas, see De Concini et al. [1982]. For other approaches, generalizations and references, see Abhyankar [1984] and Herzog and Trung [1992].

Theorem 12.5. *Let A be an $e\times f$ matrix of linear forms on $\mathbb{P}^r$, and let $M_k := M_k(A)\subset\mathbb{P}^r$ be the scheme defined by its $(k+1)\times(k+1)$ minors. If M_k has the expected codimension $(e-k)(f-k)$ in $\mathbb{P}^r$, then its degree is*

$$\deg(M_k) = \prod_{i=0}^{e-k-1}\frac{i!(f+i)!}{(k+i)!(f-k+i)!}.$$

Note that as a special case we could take $\mathbb{P}^r = \mathbb{P}^{ef-1}$ the space of all nonzero $e \times f$ matrices modulo scalars and A the matrix of homogeneous coordinates on $\mathbb{P}^{ef-1}$. Indeed, the general case follows from this one: Any matrix A as in the statement of the theorem corresponds to a linear map $\mathbb{P}^r \to \mathbb{P}^{ef-1}$, and if the preimage M_k of the locus $\Phi_k \subset \mathbb{P}^{ef-1}$ of matrices of rank k or less has the expected codimension its degree must be equal to the degree of Φ_k.

The formula simplifies in the case $k = e-1$, with $f \geq e$. Here we see that

$$\deg(M_{e-1}) = \frac{0!(f)!}{(e-1)!(f-(e-1))!} = \binom{f}{e-1};$$

if in addition $e = f$, the degree is f, the degree of the determinant.

On the other hand, when $k = 1$ the formula telescopes: We have

$$\prod_{i=0}^{e-2} \frac{i!}{(i+1)!} = \frac{1}{(e-1)!}$$

and

$$\prod_{i=0}^{e-2} \frac{(f+i)!}{(f-1+i)!} = \frac{(e+f-2)!}{(f-1)!},$$

so we get

$$\deg(M_1) = \frac{1}{(e-1)!} \cdot \frac{(e+f-2)!}{(f-1)!} = \binom{e+f-2}{e-1}.$$

Note that when $\mathbb{P}^r = \mathbb{P}^{ef-1}$ is the space of all nonzero $e \times f$ matrices modulo scalars and A the matrix of homogeneous coordinates on $\mathbb{P}^{ef-1}$, the scheme $M_1(A)$ is the Segre variety $\mathbb{P}^{e-1} \times \mathbb{P}^{f-1} \subset \mathbb{P}^{ef-1}$ and the formula gives the degree of that variety, agreeing with the computation made by other means in Section 2.1.5.

Proof of Theorem 12.5: Multiplication by the matrix A defines a vector bundle map

$$(\mathcal{O}_{\mathbb{P}^r})^{\oplus e} \to (\mathcal{O}_{\mathbb{P}^r}(1))^{\oplus f},$$

and we are asking for the class of the locus where this map has rank k or less. Letting $\zeta \in A^1(\mathbb{P}^r)$ be the hyperplane class, we have

$$c(\mathcal{F}) = (1+\zeta)^f = \sum_{r=0}^{f} \binom{f}{r} \zeta^r,$$

from which we conclude that the class of M_k is given by

$$[M_k] = \begin{vmatrix} \binom{f}{f-k}\zeta^{f-k} & \cdots & \binom{f}{f+e-2k-1}\zeta^{f+e-2k-1} \\ \vdots & & \vdots \\ \binom{f}{f-e+1}\zeta^{f-e+1} & \cdots & \binom{f}{f-k}\zeta^{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{f}{f-k} & \cdots & \binom{f}{f+e-2k-1} \\ \vdots & & \vdots \\ \binom{f}{f-e+1} & \cdots & \binom{f}{f-k} \end{vmatrix} \zeta^{(e-k)(f-k)}.$$

The degree of M_k is thus the last determinant, which may be simplified as follows. To begin with, we make a series of column operations: First, we replace each column, starting with the second, with the sum of it and the column to its left, arriving at

$$\begin{vmatrix} \binom{f}{f-k} & \binom{f}{f-k+1} & \cdots & \binom{f}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f}{f-e+2} & \cdots & \binom{f}{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \binom{f}{f-k} & \binom{f+1}{f-k+1} & \cdots & \binom{f+1}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f+1}{f-e+2} & \cdots & \binom{f+1}{f-k} \end{vmatrix}.$$

Now we do the same thing again, this time starting with the third column, then again, starting with the fourth, and so on, obtaining the determinant

$$\begin{vmatrix} \binom{f}{f-k} & \binom{f+1}{f-k+1} & \cdots & \binom{f+e-k-1}{f+e-2k-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \binom{f}{f-e+1} & \binom{f+1}{f-e+2} & \cdots & \binom{f+e-k-1}{f-k} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{f!}{k!(f-k)!} & \frac{(f+1)!}{k!(f-k+1)!} & \cdots & \frac{(f+e-k-1)!}{k!(f+e-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{f!}{(e-1)!(f-e+1)!} & \frac{(f+1)!}{(e-1)!(f-e+2)!} & \cdots & \frac{(f+e-k-1)!}{(e-1)!(f-k)!} \end{vmatrix}.$$

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

We can pull a factor of $f!$ from the first column, $(f+1)!$ from the second, and so on; similarly, we can pull a $k!$ from the denominators in the first row, a $(k+1)!$ from the denominators in the second row, and so on. We arrive at the product

$$\prod_{i=0}^{e-k-1} \frac{(f+i)!}{(k+i)!} \begin{vmatrix} \frac{1}{(f-k)!} & \frac{1}{(f-k+1)!} & \cdots & \frac{1}{(f+e-2k-1)!} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(f-e+1)!} & \frac{1}{(f-e+2)!} & \cdots & \frac{1}{(f-k)!} \end{vmatrix}.$$

Next, we multiply the first column by $(f-k)!$, the second by $(f-k+1)!$, and so on, obtaining the expression

$$\prod_{i=0}^{e-k-1} \frac{(f+i)!}{(k+i)!(f-k+i)!} \begin{vmatrix} 1 & 1 & \cdots \\ f-k & f-k+1 & \cdots \\ (f-k)(f-k-1) & (f-k+1)(f-k) & \cdots \\ \vdots & \vdots & \end{vmatrix}.$$

Finally, we can recognize the columns of this matrix as the series of monic polynomials $1, x, x(x-1), x(x-1)(x-2), \ldots$ of degrees $0, 1, 2, \ldots, m-k-1$, applied to the integers $f-k, f-k+1, \ldots, f+e-2k-1$. Its determinant is thus equal to the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots \\ f-k & f-k+1 & f-k+2 & \cdots \\ (f-k)^2 & (f-k+1)^2 & (f-k+2)^2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix},$$

which is equal to $\prod_{i=0}^{e-k-1} i!$. Putting this all together, we have established Theorem 12.5. $\square$

12.4.2 Pinch points of surfaces

Let $C \subset \mathbb{P}^n$ be a smooth curve. A classical theorem (Exercise 3.34) describes the projection $\pi = \pi_\Lambda : C \to \mathbb{P}^2$ from a general $(n-3)$-plane: it is an immersion whose image has only ordinary nodes as singularities. We would now like to describe in similar fashion the geometry of the projection $\pi : S \to \mathbb{P}^3$ of a smooth surface $S \subset \mathbb{P}^n$ from a general plane $L \cong \mathbb{P}^{n-4}$.

In general, Mather [1971; 1973] gave normal forms for the singularities of general projections of a smooth variety of dimension ≤ 7 to a hypersurface. For the case of a smooth surface $S \subset \mathbb{P}^n$ and a general projection $\pi : S \to \mathbb{P}^3$ this is classical and easy to describe; see for example Griffiths and Harris [1994, p. 611]. There are precisely three local analytic types of singular points of the image $\pi(S) \subset \mathbb{P}^3$:

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

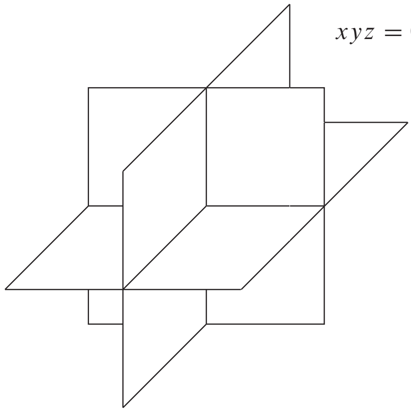

Figure 12.1 Three branches of the double curve meeting in a triple point.

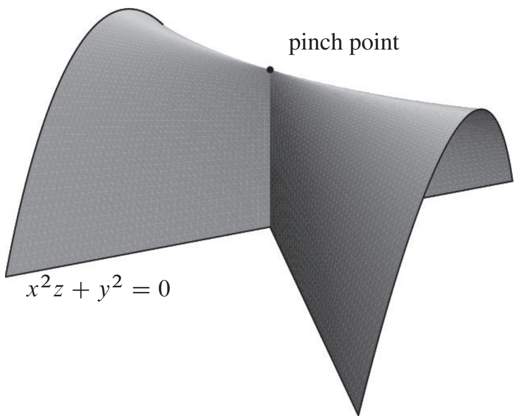

Figure 12.2 Double curve passing through a pinch point.

- There is a curve in $\pi(S)$ over which the map is two-to-one, and in an analytic neighborhood of a general point on this curve the surface S_0 is the union of two smooth sheets crossing transversely.
- There are a finite number of points in $\pi(S)$ with preimage of length 3, and at each such point $\pi(S)$ is the union of three smooth sheets intersecting transversely (Figure 12.1).
- There are finitely many points of S where the differential of π is not injective, and in suitable local coordinates near such a point the map is $(s,t) \mapsto (s, st, t^2)$, so that the image satisfies the equation $x^2z = y^2$. Such points in the image are called *pinch points* (Figure 12.2). In the given coordinates the double curve is the z-axis, and the two sheets at the point $(0, 0, z_0)$ will have tangent planes $y = \pm\sqrt{z_0} \cdot x$.

The geometry of the map, and of the image, is beautiful: pinch points are points of the double curve of $\pi(S)$ where the local monodromy interchanges the two sheets.

We now ask the enumerative question: In terms of the standard invariants of the surface $S \subset \mathbb{P}^n$, *how many pinch points will S_0 have?* This is Keynote Question (b), and we can answer it with Porteous' formula. The differential of the map π is a vector bundle map

$$d\pi : \mathcal{T}_S \to \pi^*\mathcal{T}_{\mathbb{P}^3},$$

and the formula tells us that the number of points where this map fails to be injective, counted with multiplicities, is the degree-2 piece of the quotient

$$\frac{c(\pi^*\mathcal{T}_{\mathbb{P}^3})}{c(\mathcal{T}_S)}.$$

Denote by $\zeta = c_1(\mathcal{O}_S(1))$ the pullback to S of the hyperplane class on $\mathbb{P}^3$ (equivalently, the restriction to S of the hyperplane class on $\mathbb{P}^n$). Pulling back the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^4 \longrightarrow \mathcal{T}_{\mathbb{P}^3} \longrightarrow 0$$

to S, we see that $c(\pi^*\mathcal{T}_{\mathbb{P}^3}) = 1 + 4\zeta + 6\zeta^2$. Writing c_1 and c_2 for $c_1(\mathcal{T}_S^*)$ and $c_2(\mathcal{T}_S^*)$, we have

$$\frac{c(\pi^*\mathcal{T}_{\mathbb{P}^3})}{c(\mathcal{T}_S)} = \frac{1 + 4\zeta + 6\zeta^2}{1 - c_1 + c_2} = (1 + 4\zeta + 6\zeta^2)(1 + c_1 + (c_1^2 - c_2)).$$

Since $\deg(\zeta^2) = \deg S$, the degree-2 part of this expression is

$$6\deg(S) + \deg(4\zeta c_1 + c_1^2 - c_2).$$

Proposition 12.6. *The number of pinch points of a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$ is*

$$6\deg(S) + \deg(4\zeta c_1 + c_1^2 - c_2),$$

where the $c_i = c_i(\mathcal{T}_S^*)$ are the Chern classes of the cotangent bundle of S.

Proof: The map π fails to be an immersion at a point $s \in S \subset \mathbb{P}^n$ if and only if the projection center $\Lambda \cong \mathbb{P}^{n-4} \subset \mathbb{P}^n$ meets the projective tangent plane $\mathbb{T}_p S$ to S at p. The union of the tangent planes to S is clearly at most four-dimensional, so for general Λ the number of such points is finite.

It remains to prove that each pinch point counts with multiplicity 1 in the degeneracy locus of the differential $d\pi : \mathcal{T}_S \to \pi^*\mathcal{T}_{\mathbb{P}^3}$. This is equivalent to saying that at each point where the differential drops rank the 2×2 minors of the Jacobian matrix representing it generate the maximal ideal. From the local form of the map given above, we see that the Jacobian is

$$\begin{array}{c c} & \begin{array}{cc} \partial/\partial s & \partial/\partial t \end{array} \\ \begin{array}{c} s \\ st \\ t^2 \end{array} & \begin{pmatrix} 1 & 0 \\ t & s \\ 0 & 2t \end{pmatrix} \end{array},$$

with ideal of 2×2 minors (s, t) as required. $\square$

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

A beautiful example of Proposition 12.6 is the projection $\pi : S \to \mathbb{P}^3$ of the Veronese surface $S = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from a general line $L \subset \mathbb{P}^5$. We will sketch the geometry of the map and its image briefly.

To begin with, the secant variety of the Veronese surface is a hypersurface $X \subset \mathbb{P}^5$. This is a reflection of the fact that a point $x \in \mathbb{P}^5$ lying on a secant line $\overline{p,q}$ of S in fact lies on a one-dimensional family of secants: the line in $\mathbb{P}^2$ through the points $p, q \in S = \mathbb{P}^2$ is carried, under the Veronese map, to a conic curve $C \subset S \subset \mathbb{P}^5$, and, since the point x lies in the plane spanned by C, every line through x in that plane will be a secant line to S.

Now, since X is in fact a cubic hypersurface, a general line $L \subset \mathbb{P}^5$ will meet X in three points, corresponding to three lines in $\mathbb{P}^2$. Each of these lines will be carried into a conic in $\mathbb{P}^5$ and then under projection from L will be mapped onto a line with degree 2 (and two branch points). Thus the double curve of $\pi(S)$ will consist of three lines, meeting in a single point in $\mathbb{P}^3$ (the unique triple point of $\pi(S)$), and each line will have two pinch points, accounting for the six pinch points predicted by Proposition 12.6.

12.4.3 Pinch points and the tangential variety of S

There is an alternative derivation of the formula of Proposition 12.6, based on the observation that the number of pinch points of a general projection of a smooth surface $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$ is related to the degree of the *tangential surface* $T(S)$ of S, that is, the union

$$T(S) = \bigcup_{p \in S} \mathbb{T}_p S \subset \mathbb{P}^n$$

of the projective tangent planes to S. More precisely, the number of pinch points of a general projection of $S \subset \mathbb{P}^n$ to $\mathbb{P}^3$ is the degree of $T(S)$ times the number d of tangent planes $\mathbb{T}_p S$ containing a general point of $T(S)$.

To carry this out, recall from Section 7.4.3 that the *Gauss map* $\mathcal{G} : S \to \mathbb{G}(2, n)$ is the map sending each point $p \in S$ to its projective tangent plane $\mathbb{T}_p S$; assuming $T(S)$ does have the expected dimension 4, we can apply Proposition 10.4 on the degree of a variety swept out by linear spaces to express this degree as the second Segre class of the pullback $\mathcal{G}^*(\mathcal{S})$ via $\mathcal{G}$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(2, n)$. To express this, we recall also from Section 7.4.3 that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathcal{G}^*\mathcal{S} \longrightarrow \mathcal{T}_X(-1) \longrightarrow 0$$

relating $\mathcal{G}^*(\mathcal{S})$ to the hyperplane bundle of $S \subset \mathbb{P}^n$ and a twist of the tangent bundle. Applying the formula for the Chern class of a tensor product with a line bundle (Proposition 5.17) and the Whitney formula, we arrive at the conclusion of Proposition 12.6 again.

There is one advantage of this approach: Once we show that the tangential variety $T(S)$ is indeed four-dimensional, we may conclude that *if $S \subset \mathbb{P}^n$ is any smooth, irreducible nondegenerate surface and $n \geq 4$, the number of pinch points of a general projection of S to $\mathbb{P}^3$ is positive*. We will sketch a proof that $T(S)$ is indeed four-dimensional in Exercises 12.16–12.18.

On the other hand, the derivation of Proposition 12.6 via Porteous carried out here has an advantage as well: It applies to any map of a surface S to $\mathbb{P}^3$, assuming only that the singularities of the map and its image are as described on page 436 for a general projection. Indeed, the same method may be applied to a map of a surface S to any smooth threefold X, again assuming the singularities of the map are as described.

12.4.4 Quadrisecants to rational curves

As a final application of Porteous' formula we will count the number of *quadrisecant lines* to a rational space curve $C \subset \mathbb{P}^3$ (that is, lines meeting C in four points). This will conclude our discussion, began in Section 10.5, of special secant planes to rational curves; other cases that can similarly be dealt with using the Porteous formula are suggested in Exercise 12.23.

The question we will address here is: Is there an enumerative formula for the number of quadrisecant lines to a curve $C \subset \mathbb{P}^3$, say in terms of the degree d and genus g of C? We discuss a formula in the general case at the end of the subsection, but first we treat the rational case, and we suppose that $C \cong \mathbb{P}^1$.

Instead of looking at all lines in $\mathbb{P}^3$ and imposing the condition of meeting C four times, we will look at 4-tuples of points on C and impose the condition that they span only a line. We will use the set-up of Section 10.4.2: We identify the space of subschemes Γ of degree 4 in $C \cong \mathbb{P}^1$ with the symmetric power $C^{(4)} \cong \mathbb{P}^4$ of C, and introduce the bundle $\mathcal{E}^*$ on $C^{(4)}$ with fibers

$$\mathcal{E}^*_\Gamma = H^0(\mathcal{O}_\Gamma(d)).$$

As in Section 10.4.2, a global section σ of $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(d)$ gives rise to a global section of $\mathcal{E}^*$ by restriction to each subscheme of C in turn. The restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \hookrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \to H^0(\mathcal{O}_\Gamma(d))$$

gives us a map $\varphi : \mathcal{F} \to \mathcal{E}^*$ of vector bundles on $C^{(4)}$, where $\mathcal{F}$ is the trivial bundle with fiber $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$, and we see that *the locus $M_2(\varphi) \subset \mathbb{P}^4$ of subschemes $\Gamma \subset \mathbb{P}^1$ of degree 4 with* $\dim \overline{\Gamma} = 1$ *is the locus where the map φ has rank 2*. (Note that φ can never have rank < 2.) Porteous' formula will then give us an expression for the class of this locus, valid in the case it has the expected dimension $4 - 2 \times 2 = 0$.

To carry this out, recall from Theorem 10.16 that the Chern classes of $\mathcal{E}^*$ are

$$c_i(\mathcal{E}^*) = \binom{d-4+i}{i}\zeta^i \in A^i(\mathbb{P}^4),$$

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

where $\zeta \in A^1(\mathbb{P}^4)$ is as always the hyperplane class. Now we apply Porteous' formula, which tells us that the class of the locus of subschemes $\Gamma \subset \mathbb{P}^1$ contained in a line is

$$\begin{aligned}
[M_2(\varphi)] &= \begin{vmatrix} c_2(\mathcal{E}^*) & c_3(\mathcal{E}^*) \\ c_1(\mathcal{E}^*) & c_2(\mathcal{E}^*) \end{vmatrix} \\
&= \begin{vmatrix} \binom{d-2}{2}\zeta^2 & \binom{d-1}{3}\zeta^3 \\ \binom{d-3}{1}\zeta & \binom{d-2}{2}\zeta^2 \end{vmatrix} \\
&= \begin{vmatrix} \binom{d-2}{2} & \binom{d-1}{3} \\ \binom{d-3}{1} & \binom{d-2}{2} \end{vmatrix} \zeta^4 \\
&= \left(\tfrac{1}{4}(d-2)^2(d-3)^2 - \tfrac{1}{6}(d-1)(d-2)(d-3)^2\right)\zeta^4 \\
&= \tfrac{1}{12}(d-3)^2(d-2)(3(d-2) - 2(d-1))\zeta^4 \\
&= \tfrac{1}{12}(d-2)(d-3)^2(d-4)\zeta^4.
\end{aligned}$$

This gives us the enumerative formula:

Proposition 12.7. *If $C \subset \mathbb{P}^3$ is a rational space curve of degree d possessing only finitely many quadrisecant lines then the number of such lines, counted with multiplicities, is*

$$\tfrac{1}{12}(d-2)(d-3)^2(d-4).$$

Note as a check that this number is 0 in the cases $d = 2$, 3 and 4, as it should be.

We will see in Exercise 12.24 a condition for a given quadrisecant line to be simple, that is, to count with multiplicity 1. We will also see in Exercise 12.25 that for $C \subset \mathbb{P}^3$ a *general* rational curve of degree d — that is, a general projection of a rational normal curve from $\mathbb{P}^d$ to $\mathbb{P}^3$ — all quadrisecants are simple, so this is the actual number of quadrisecant lines.

Quadrisecants to curves of higher genus

If we try to generalize the arguments above to the case where C has higher genus, a new issue arises. The Hilbert scheme parametrizing subschemes of degree 4 in $\mathbb{P}^1$ is $\mathbb{P}^4$, whose Chow ring we know. But the space of subschemes of degree 4 of a smooth curve C of higher genus — again, the fourth symmetric power $C^{(4)}$ of C — is more complex; in particular, its Chow ring is much harder to determine explicitly (see Collino [1975]). The most general formula, for the number of d-secant $(d-r-1)$-planes to a curve of degree n and genus g in $\mathbb{P}^s$, is derived (using topological cohomology groups rather than Chow groups) in Chapter 8 of Arbarello et al. [1985]; the formula for the number q of quadrisecant lines to a curve $C \subset \mathbb{P}^3$ of degree d and genus g is

$$q = \tfrac{1}{12}(d-2)(d-3)^2(d-4) - \tfrac{1}{2}g(d^2 - 7d + 13 - g).$$

We could also approach the problem of counting quadrisecant lines to a space curve of arbitrary genus via the classical theory of correspondences. This is described in Chapter 2 of Griffiths and Harris [1994].

12.5 Exercises

Exercise 12.8. Let $A = (P_{i,j})$ be a 2×3 matrix whose entries $P_{i,j}$ are general polynomials of degree $a_{i,j}$ on $\mathbb{P}^3$. Assuming that $a_{1,j} + a_{2,k} = a_{1,k} + a_{2,j}$ for all j and k — so that the minors of A are homogeneous — what is the degree of the curve $M_1(A)$ where A has rank 1?

Exercise 12.9. In Exercise 2.32, we introduced the variety of triples of collinear points, that is,

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

Calculate the class $\psi = [\Psi] \in A^{n-1}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ by applying Porteous to the evaluation map $\mathcal{E} \to \mathcal{F}$, where $\mathcal{E}$ is the trivial bundle on $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ with fiber $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and

$$\mathcal{F} = \pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^n}(1) \oplus \pi_3^*\mathcal{O}_{\mathbb{P}^n}(1),$$

with $\pi_i : \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ projection on the i-th factor.

Exercise 12.10. In Exercise 9.37, we introduced

$$\Phi_r = \{(L, M) \in \mathbb{G}(1, r) \times \mathbb{G}(1, r) \mid L \cap M \neq \varnothing\}.$$

Find the class of Φ_r in $A(\mathbb{G}(1, r) \times \mathbb{G}(1, r))$ using Porteous' formula.

The following exercise uses Porteous to generalize the result of Exercise 9.44.

Exercise 12.11. Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle of rank r on X and $\mathcal{F}, \mathcal{G} \subset \mathcal{E}$ subbundles of ranks a and b. For any k, find the class of the locus

$$\Sigma = \{p \in X \mid \dim(\mathcal{F}_p \cap \mathcal{G}_p) \geq k\},$$

assuming this locus has the expected (positive) codimension.

Exercise 12.12. Verify Proposition 12.6 directly in case S is a smooth surface in $\mathbb{P}^3$ to begin with.

Exercise 12.13. Verify Proposition 12.6 directly in case $S = S(1, 2) \subset \mathbb{P}^4$ is a cubic scroll (Section 9.1.1). What is the double curve of the image S_0?

Exercise 12.14. Verify Proposition 12.6 directly in case $S = S(2, 2) \subset \mathbb{P}^5$ is a rational normal surface scroll. What does the double curve of S_0 look like in this case, and how many triple points will S_0 have?

Exercise 12.15. Let $S \subset \mathbb{P}^n$ be a smooth surface.

(a) Show that we have a map from the projective bundle $\mathbb{P}(\mathcal{T}_S \oplus \mathcal{O}_S)$ to $\mathbb{P}^n$ with image the tangential variety X of S (specifically, carrying the fiber over p to the tangent plane $\mathbb{T}_p(S)$).
(b) Show that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ under this map is the line bundle

$$\mathcal{O}_{\mathbb{P}(\mathcal{T}_S \oplus \mathcal{O}_S)}(1) \otimes \mathcal{O}_S(1).$$

(c) Use this and our description of the Chow ring of the projective bundle $\mathbb{P}(\mathcal{T}_S \oplus \mathcal{O}_S)$ to re-derive the formula of Proposition 12.6 for the degree of X.

Exercises 12.16–12.18 suggest a proof of the assertion made in Section 12.4.3 that if $S \subset \mathbb{P}^n$ is a nondegenerate surface and $n \geq 4$ then the union of the tangent planes to S is a fourfold.

Exercise 12.16. Let $B \subset \mathbb{G}(2, n)$ be an irreducible surface, and let

$$X = \bigcup_{\Lambda \in B} \Lambda \subset \mathbb{P}^n$$

be the variety swept out by the corresponding 2-planes. Show that if X is three-dimensional then

(a) a general point $p \in X$ lies on a one-dimensional family of planes $\Lambda \in B$, and hence
(b) any two planes $\Lambda, \Lambda' \in B$ meet in a line.

Exercise 12.17. Let $B \subset \mathbb{G}(2, n)$ be an irreducible surface such that any pair of planes $\Lambda, \Lambda' \in B$ meet in a line. Show that either all the planes Λ lie in a fixed 3-plane or all the planes Λ contain a fixed line.

Exercise 12.18. Using the preceding two exercises, conclude that if $S \subset \mathbb{P}^n$ is a nondegenerate surface and $n \geq 4$ then the union of the tangent planes to S is a fourfold.

Exercise 12.19. Let $X \subset \mathbb{P}^n$ be a smooth sixfold and $\pi : X \to \mathbb{P}^7$ a general projection. Find the number of points where the differential $d\pi$ has rank 4 or less.

Exercise 12.20. Let $S \subset \mathbb{P}^n$ be a smooth surface of degree d whose general hyperplane section is a curve of genus g; let e and f be the degrees of the classes $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S) \in A^2(S)$. Find the class of the cycle $T_1(S) \subset \mathbb{G}(1, n)$ of lines tangent to S in terms of d, e, f and g. From Exercise 4.21, we need only the intersection number $[T_1(S)]\cdot\sigma_3$; find it as the number of pinch points of a projection of S from a general $\mathbb{P}^{n-4}$.

Exercise 12.21. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve. Show that the general secant line to C is not trisecant, and deduce that the locus of trisecant lines to C, if nonempty, has dimension 1. (For extra credit, show that it is empty only in case C is a twisted cubic or an elliptic quartic.)

https://doi.org/10.1017/CBO9781139062046.014 Published online by Cambridge University Press

Exercise 12.22. Check the conclusion of Proposition 12.7 for general rational curves of degrees $d = 5$ and 6 by independently counting the number of quadrisecant lines to such curves.
Hint: Show that such a curve C will lie on a smooth cubic surface S, and observe that the quadrisecant lines to C will be contained in S.

Exercise 12.23. Use Porteous' formula to find the expected number of:

(a) Trisecant lines to a rational curve $C \subset \mathbb{P}^4$.
(b) 6-secant 2-planes to a rational curve $C \subset \mathbb{P}^4$.
(c) 8-secant 3-planes to a rational curve $C \subset \mathbb{P}^5$.
(d) 4-secant 2-planes to a rational curve $C \subset \mathbb{P}^6$.

Exercise 12.24. Let $C \subset \mathbb{P}^3$ be a smooth curve and $L \subset \mathbb{P}^3$ a line meeting C in exactly four points $p_1, \dots, p_4$ and not tangent to C at any of them. Suppose that the tangent lines $\mathbb{T}_{p_i}(C)$ to C at the p_i are pairwise independent mod L (that is, they span distinct planes with L), and that the cross-ratio of the four points $p_1, \dots, p_4 \in L$ is *not* equal to the cross-ratio of the four planes $\mathbb{T}_{p_1}(C) + L, \dots, \mathbb{T}_{p_4}(C) + L$. Show that $\Gamma = p_1 + \cdots + p_4 \in \mathbb{P}^4$ counts as a quadrisecant line with multiplicity 1 (that is, the 3×3 minors of a matrix representative of φ near Γ generate the maximal ideal $\mathfrak{m}_\Gamma \subset \mathcal{O}_{\mathbb{P}^4,\Gamma}$).

Exercise 12.25. Let C now be a general rational curve of degree d in $\mathbb{P}^3$.

(a) Show that C has no 5-secant lines.
(b) Show that if $L \subset \mathbb{P}^3$ is any quadrisecant line to C, then L meets C in four distinct points.
(c) Finally, show that every quadrisecant line to C satisfies the conditions of the preceding exercise, and deduce that the number of quadrisecant lines to C is exactly $\frac{1}{12}(d-2)(d-3)^2(d-4)$. (Note: The two preceding parts are straightforward dimension counts; this one is a little more subtle.)

Exercise 12.26. Let $\{Q_\mu \subset \mathbb{P}^3\}_{\mu \in \mathbb{P}^3}$ be a general web of quadrics in $\mathbb{P}^3$, that is, the three-dimensional linear series corresponding to a general four-dimensional vector space $V \subset H^0(\mathcal{O}_{\mathbb{P}^3}(2))$.

(a) Find the number of 2-planes $\Lambda \subset \mathbb{P}^3$ that are contained in some quadric of the net.
(b) A line $L \subset \mathbb{P}^3$ is said to be a *special line* for the web if it lies on a pencil of quadrics in the web. Find the class of the locus $\Sigma \subset \mathbb{G}(1,3)$ of special lines.