Algebraic Topology Homework 2

Isaac Martin

Last compiled September 11, 2022

§ ***Problems from 1.2***

EXERCISE 1.2.1. Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Proof: Recall that two elements of $G * H$ are equal if and only if their reductions are identical. We use this fact without comment.

Suppose that $g \in G$ and $h \in H$ are both nontrivial elements. Then both ghg^{-1} and h are reduced in $G * H$, and hence are not equal as they are of different lengths. This means $gh \neq hg$ for all nontrivial elements $g \in G$ and $h \in H$.

Now suppose we have some reduced word $w_1 w_2 ... w_n \in G * H$ where $w_i \in G \cup H$ for $1 \leq i \leq n$ and $n \geq 2$. Again let $g \in G$ and $h \in H$ be reduced words. We have four cases to consider.

(1) If $w_1, w_k \in G$, then hw and wh are both reduced and are hence not equal.

(2) If $w_1, w_k \in H$, then gw and wg are both reduced and are hence not equal.

(3) If $w_1 \in G$ and $w_k \in H$, then $w_2 \in H$ by the assumption that w is reduced. Hence both $gw_2...w_k$ and wg are reduced, and since $k \geq 2$, we have that $gw \neq wg$.

(4) If $w_1 \in H$ and $w_k \in G$, then $w_2 \in G$ and we get $hw \neq wh$ by the same argument as above.

Thus, every nontrivial element of $G * H$ fails to commute with some other element, meaning the center of $G * H$ is trivial.

We now show that the only elements of $G * H$ are the conjugates of finite-order elements of G and H. Let $w \in G * H$ be finite order, i.e. assume $w^k = 1$ where 1 is the empty word for some $k \in$.

First, notice that w must have an odd number of letters. If $w = w_1 ... w_{2n}$ is reduced, then w_1 and w_{2n} belong to different groups, and therefore $w^2 = w_1 ... w2nw_1 ... w2n$ is also reduced. Successive multiplication of w with itself will only make the word longer. w must therefore have an odd number of elements in order to reduce upon successive multiplication. Thus the reduced form of w is $w_1 ... w_{2n+1}$.

As previously noted, we need w to shrink upon successive products. This means that w_1 and w_{2k+1} must multiply to 1 in either H or G, i.e. $w_1 = w_{2n+1}^{-1}$. Similarly, $w_2 = w_{2n}^{-1}$, $w_3 = w_{2n-1}^{-1}$, and $w_n = w_{n+2}^{-1}$. This observation means that

$$(w_1 ... w_n)^{-1} = w_n^{-1} ... w_1 - 1 = w_{n+2} ... w_{2n+1}.$$

Therefore

$$w = (w_1 ... w_n) w_{n+1} (w_{n+2} ... w_{2n+1})$$

and finally,

$$w^k = (w_1 ... w_n) w_{n+1}^k (w_{n+2} ... w_{2n+1}) = 1 \implies w_{n+1}^k = 1$$

And since w_{n+1} must be an element in either H or G, we conclude that w is the conjugate of some finite order element in G or H.

$\square$

EXERCISE 1.2.2. Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, ..., X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.

Proof: $\square$

EXERCISE 1.2.11. The **mapping torus** T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. We consider first the case where $X = S^1 \vee S^1$. We can express X as a CW-complex with one 0-cell and two 1-cells through the following construction. Let x_0 be a 0-cell. Attach the ends of two 1-cells to x_0, and we have X.

Now, because f is basepoint preserving, if we take x_0 to be our basepoint, $x_0 \mapsto x_0$ which means that under the equivalence relation, $(x_0, 0) \mapsto (x_0, 1)$. As stated in Hatcher, we can regard T_f as the construction of $X \vee S^1$ with appropriate cells attached, i.e. as the space obtained by taking every k cell in X and attaching a $k + 1$ cell. This is visualized in the diagram below. By Proposition 1.26, we therefore have that $\pi_1(T_f) \cong \pi_1(X \vee S^1)/N$. However, this is precisely the fundamental group from question (8). Thus,

$$\pi_1(T_f) \approx (**)/\langle aba^{-1}b^{-1}, cdc^{-1}d^{-1}\rangle$$

Where $a = f_*(a)$, etc.

We now consider the case where $X = S^1 \times S^1$. This is a torus. We once again regard T_f as the space obtained by attaching appropriate cells to $X \vee S^1$. This time we attach one 3-cell (for the 2-cell of the torus) and two two-cells (for the two 1-cells of the torus). One again, the wedge with S^1 is the result of attaching one 1-cell to the basepoint of X.

From part (b) of Proposition 1.26, we know that the 3-cell is simply connected and therefore doesn't affect $\pi_1(T_f)$. We therefore obtain almost exactly the same fundamental group as before, except that we have an extra 1-cell. This extra cell causes a and b to commute. Therefore,

$$\pi_1(T_f) \approx (**)/\langle aba^{-1}b^{-1}, cdc^{-1}d^{-1} \mid ab = ba\rangle$$

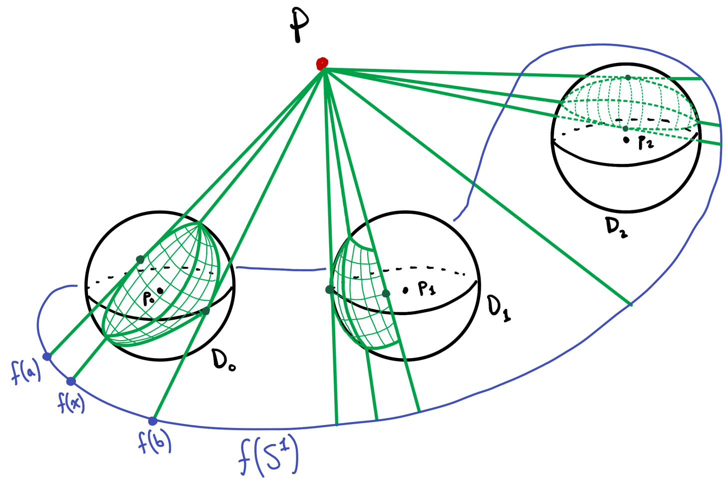

Figure 1: The homotopy in Case 1