Preface

Shape is a fascinating and intriguing subject which has stimulated the imagination of many people. It suffices to look around to become curious. Euclid did just that and came up with the first pure creation. Relying on the common experience, he created an abstract world that had a life of its own. As the human knowledge progressed so did the ability of formulating and answering penetrating questions. In particular, mathematicians started wondering whether Euclid's "obvious" absolute postulates were indeed obvious and/or absolute. Scientists realized that Shape and Space are two closely related concepts and asked whether they really look the way our senses tell us. As Felix Klein pointed out in his Erlangen Program, there are many ways of looking at Shape and Space so that various points of view may produce different images. In particular, the most basic issue of "measuring the Shape" cannot have a clear cut answer. This is a book about Shape, Space and some particular ways of studying them.

Since its inception, the differential and integral calculus proved to be a very versatile tool in dealing with previously untouchable problems. It did not take long until it found uses in geometry in the hands of the Great Masters. This is the path we want to follow in the present book.

In the early days of geometry nobody worried about the natural context in which the methods of calculus "feel at home". There was no need to address this aspect since for the particular problems studied this was a non-issue. As mathematics progressed as a whole the "natural context" mentioned above crystallized in the minds of mathematicians and it was a notion so important that it had to be given a name. The geometric objects which can be studied using the methods of calculus were called smooth manifolds. Special cases of manifolds are the curves and the surfaces and these were quite well understood. B. Riemann was the first to note that the low dimensional ideas of his time were particular aspects of a higher dimensional world.

The first chapter of this book introduces the reader to the concept of smooth manifold through abstract definitions and, more importantly, through many we believe relevant examples. In particular, we introduce at this early stage the notion of Lie group. The main geometric and algebraic properties of these objects will be gradually described as we progress with our study of the geometry of manifolds. Besides their obvious usefulness in geometry, the Lie groups are academically very friendly. They provide a marvelous testing ground for abstract results. We have consistently taken advantage of this feature throughout this book. As a bonus, by the end of these lectures the reader will feel comfortable manipulating basic Lie theoretic concepts.

To apply the techniques of calculus we need things to derivate and integrate. These "things" are introduced in Chapter 2. The reason why smooth manifolds have many differentiable objects attached to them is that they can be locally very well approximated by linear spaces called tangent spaces . Locally, everything looks like traditional calculus. Each point has a tangent space attached to it so that we obtain a "bunch of tangent spaces" called the tangent bundle. We found it appropriate to introduce at this early point the notion of vector bundle. It helps in structuring both the language and the thinking.

Once we have "things to derivate and integrate" we need to know how to explicitly perform these operations. We devote the Chapter 3 to this purpose. This is perhaps one of the most unattractive aspects of differential geometry but is crucial for all further developments. To spice up the presentation, we have included many examples which will found applications in later chapters. In particular, we have included a whole section devoted to the representation theory of compact Lie groups essentially describing the equivalence between representations and their characters.

The study of Shape begins in earnest in Chapter 4 which deals with Riemann manifolds. We approach these objects gradually. The first section introduces the reader to the notion of geodesics which are defined using the Levi-Civita connection. Locally, the geodesics play the same role as the straight lines in an Euclidian space but globally new phenomena arise. We illustrate these aspects with many concrete examples. In the final part of this section we show how the Euclidian vector calculus generalizes to Riemann manifolds.

The second section of this chapter initiates the local study of Riemann manifolds. Up to first order these manifolds look like Euclidian spaces. The novelty arises when we study "second order approximations " of these spaces. The Riemann tensor provides the complete measure of how far is a Riemann manifold from being flat. This is a very involved object and, to enhance its understanding, we compute it in several instances: on surfaces (which can be easily visualized) and on Lie groups (which can be easily formalized). We have also included Cartan's moving frame technique which is extremely useful in concrete computations. As an application of this technique we prove the celebrated Theorema Egregium of Gauss. This section concludes with the first global result of the book, namely the Gauss-Bonnet theorem. We present a proof inspired from [25] relying on the fact that all Riemann surfaces are Einstein manifolds. The Gauss-Bonnet theorem will be a recurring theme in this book and we will provide several other proofs and generalizations.

One of the most fascinating aspects of Riemann geometry is the intimate correlation "local-global". The Riemann tensor is a local object with global effects. There are currently many techniques of capturing this correlation. We have already described one in the proof of Gauss-Bonnet theorem. In Chapter 5 we describe another such technique which relies on the study of the global behavior of geodesics. We felt we had the moral obligation to present the natural setting of this technique and we briefly introduce the reader to the wonderful world of the calculus of variations. The ideas of the calculus of variations produce remarkable results when applied to Riemann manifolds. For example, we explain in rigorous terms why "very curved manifolds" cannot be "too long" .

In Chapter 6 we leave for a while the "differentiable realm" and we briefly discuss the fundamental group and covering spaces. These notions shed a new light on the results of Chapter 5. As a simple application we prove Weyl's theorem that the semisimple Lie groups with definite Killing form are compact and have finite fundamental group.

Chapter 7 is the topological core of the book. We discuss in detail the cohomology of smooth manifolds relying entirely on the methods of calculus. In writing this chapter we could not, and would not escape the influence of the beautiful monograph [17], and this explains the frequent overlaps. In the first section we introduce the DeRham cohomology and the Mayer-Vietoris technique. Section 2 is devoted to the Poincaré duality, a feature which sets the manifolds apart from many other types of topological spaces. The third section offers a glimpse at homology theory. We introduce the notion of (smooth) cycle and then present some applications: intersection theory, degree theory, Thom isomorphism and we prove a higher dimensional version of the Gauss-Bonnet theorem at the cohomological level. The fourth section analyzes the role of symmetry in restricting the topological type of a manifold. We prove Élie Cartan's old result that the cohomology of a symmetric space is given by the linear space of its bi-invariant forms. We use this technique to compute the lower degree cohomology of compact semisimple Lie groups. We conclude this section by computing the cohomology of complex Grassmannians relying on Weyl's integration formula and Schur polynomials. The chapter ends with a fifth section containing a concentrated description of Čech cohomology.

Chapter 8 is a natural extension of the previous one. We describe the Chern-Weil construction for arbitrary principal bundles and then we concretely describe the most important examples: Chern classes, Pontryagin classes and the Euler class. In the process, we compute the ring of invariant polynomials of many classical groups. Usually, the connections in principal bundles are defined in a global manner, as horizontal distributions. This approach is geometrically very intuitive but, at a first contact, it may look a bit unfriendly in concrete computations. We chose a local approach build on the reader's experience with connections on vector bundles which we hope will attenuate the formalism shock. In proving the various identities involving characteristic classes we adopt an invariant theoretic point of view. The chapter concludes with the general Gauss-Bonnet-Chern theorem. Our proof is a variation of Chern's proof.

Chapter 9 is the analytical core of the book.[1] Many objects in differential geometry are defined by differential equations and, among these, the elliptic ones play an important role. This chapter represents a minimal introduction to this subject. After presenting some basic notions concerning arbitrary partial differential operators we introduce the Sobolev spaces and describe their main functional analytic features. We then go straight to the core of elliptic theory. We provide an almost complete proof of the elliptic a priori estimates (we left out only the proof of the Calderon-Zygmund inequality). The regularity results are then deduced from the a priori estimates via a simple approximation technique. As a first application of these results we consider a Kazhdan-Warner type equation which recently found applications in solving the Seiberg-Witten equations on a Kähler manifold. We adopt a variational approach. The uniformization theorem for compact Riemann surfaces is then a nice bonus. This may not be the most direct proof but it has an academic advantage. It builds a circle of ideas with a wide range of applications. The last section of this chapter is devoted to Fredholm theory. We prove that the elliptic operators on compact manifolds are Fredholm and establish the homotopy invariance of the index. These are very general Hodge type theorems. The classical one follows immediately from these results. We conclude with a few facts about the spectral properties of elliptic operators.

The last chapter is entirely devoted to a very important class of elliptic operators namely the Dirac operators. The important role played by these operators was singled out in the works of Atiyah and Singer and, since then, they continue to be involved in the most dramatic advances of modern geometry. We begin by first describing a general notion of Dirac operators and their natural geometric environment, much like in [11]. We then isolate a special subclass we called *geometric Dirac operators*. Associated to each such operator is a very concrete Weitzenböck formula which can be viewed as a bridge between geometry and analysis, and which is often the source of many interesting applications. The abstract considerations are backed by a full section describing many important concrete examples.

In writing this book we had in mind the beginning graduate student who wants to specialize in global geometric analysis in general and gauge theory in particular. The second half of the book is an extended version of a graduate course in differential geometry we taught at the University of Michigan during the winter semester of 1996.

The minimal background needed to successfully go through this book is a good knowledge of vector calculus and real analysis, some basic elements of point set topology and linear algebra. A familiarity with some basic facts about the differential geometry of curves of surfaces would ease the understanding of the general theory, but this is not a must. Some parts of the chapter on elliptic equations may require a more advanced background in functional analysis.

[1] In the new edition, this chapter has become Chapter 10.

The theory is complemented by a large list of exercises. Quite a few of them contain technical results we did not prove so we would not obscure the main arguments. There are however many non-technical results which contain additional information about the subjects discussed in a particular section. We left hints whenever we believed the solution is not straightforward.

Personal note It has been a great personal experience writing this book, and I sincerely hope I could convey some of the magic of the subject. Having access to the remarkable science library of the University of Michigan and its computer facilities certainly made my job a lot easier and improved the quality of the final product.

I learned differential equations from Professor Viorel Barbu, a very generous and enthusiastic person who guided my first steps in this field of research. He stimulated my curiosity by his remarkable ability of unveiling the hidden beauty of this highly technical subject. My thesis advisor, Professor Tom Parker, introduced me to more than the fundamentals of modern geometry. He played a key role in shaping the manner in which I regard mathematics. In particular, he convinced me that behind each formalism there must be a picture, and uncovering it, is a very important part of the creation process. Although I did not directly acknowledge it, their influence is present throughout this book. I only hope the filter of my mind captured the full richness of the ideas they so generously shared with me.

My friends Louis Funar and *Gheorghe Ionesei*[2] read parts of the manuscript. I am grateful to them for their effort, their suggestions and for their friendship. I want to thank Arthur Greenspoon for his advice, enthusiasm and relentless curiosity which boosted my spirits when I most needed it. Also, I appreciate very much the input I received from the graduate students of my "Special topics in differential geometry" course at the University of Michigan which had a beneficial impact on the style and content of this book.

At last, but not the least, I want to thank my family who supported me from the beginning to the completion of this project.

Ann Arbor, 1996.

Preface to the second edition

Rarely in life is a man given the chance to revisit his "youthful indiscretions". With this second edition I have been given this opportunity, and I have tried to make the best of it.

The first edition was generously sprinkled with many typos, which I can only attribute to the impatience of youth. In spite of this problem, I have received very

[2]He passed away while I was preparing this new edition. He was the ultimate poet of mathematics.

good feedback from a very indulgent and helpful audience, from all over the world.

In preparing the new edition, I have been engaged on a massive typo hunting, supported by the wisdom of time, and the useful comments that I have received over the years from many readers. I can only say that the number of typos is substantially reduced. However, experience tells me that Murphy's Law is still at work, and there are still typos out there which will become obvious only in the printed version.

The passage of time has only strengthened my conviction that, in the words of Isaac Newton, "in learning the sciences examples are of more use than precepts". The new edition continues to be guided by this principle. I have not changed the old examples, but I have polished many of my old arguments, and I have added quite a large number of new examples and exercises.

The only major addition to the contents is a new chapter (Chapter 9) on classical integral geometry. This is a subject that captured my imagination over the last few years, and since the first edition developed all the tools needed to understand some of the juiciest results in this area of geometry, I could not pass the chance to share with a curious reader my excitement about this line of thought.

One novel feature in our presentation of the classical results of integral geometry is the use of tame geometry. This is a recent extension of the better know area of real algebraic geometry which allowed us to avoid many heavy analytical arguments, and present the geometric ideas in as clear a light as possible.

Notre Dame, 2007.

Contents