# Toric Geometry:
# Theorems and Definitions

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# 1 Notation and Preliminaries

**Definition 1.1.**

- $\mathfrak{h} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is the upper half plane in $\mathbb{C}$
- $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) > 0\}$ is the set of all automorphisms of $\mathbb{R}^2$ with positive determinant
- $\mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) = 1\}$ is the standard special linear subgroup of $\mathrm{GL}_2(\mathbb{R})$
- $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; a,b,c,d \in \mathbb{Z} \right\}$ is the special linear subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

**Lemma 1.2.** The group $GL_2(\mathbb{R})^+$ acts transitively on $\mathfrak{h}$ by Möbius transformations (fractional linear transformations). That is, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and any $\tau = x + iy \in \mathfrak{h}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \in \mathfrak{h}$$

and

$$\tau = x + iy = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i.$$

**Lemma 1.3.** Suppose $k \in \mathbb{Z}$, $f : \mathfrak{h} \longrightarrow \mathbb{C}$ is a meromorphic function, and $g \in \mathrm{GL}_2(\mathbb{R})^+$. We define the **weight k action of** $g$ on $f$ to be the right action $(f \mapsto f|_k[g])$ where $f|_k[g] : \mathfrak{h} \longrightarrow \mathbb{C}$ is a meromorphic function defined

$$f|_k[g](\tau) = \det(g) f(g\tau) j(g,\tau)^k,$$

where if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $j(g,\tau) = (c\tau + d)$.

**Definition 1.4.** Let $k \in \mathbb{Z}$ and let $\Gamma \leq SL_2(\mathbb{Z})$ be a subgroup of finite index. A **weakly modular function** of weight $k$ and level $\Gamma$ is a meromorphic function $f$ on $\mathfrak{h}$ such that for all $\gamma \in \Gamma$, $f|_k[\gamma] = f$. A **modular function** (of weight $k$ and level $\Gamma$) is a weakly modular function which is meromorphic at $\infty$. A **modular form** is a weakly modular function which is holomorphic in $\mathfrak{h}$ and at $\infty$. A **cuspoidal modular form** is a modular form which vanishes at $\infty$.

# 2 Modular Forms on $SL_2(\mathbb{Z})$

**Theorem 2.1.** *We let* $\rho = e^{i\frac{2pi}{3}}$ *and define*

$$\mathcal{F} = \left\{ \tau \in \mathfrak{h} \;\middle|\; \mathrm{Re}(\tau) \leq \frac{1}{2}, |\tau| \geq 1 \right\}.$$

*Then*

*(a) Every* $\tau \in \mathfrak{h}$ *is in the* $\overline{\Gamma(1)}$ *orbit of an element of* $\mathcal{F}$

*(b) If* $\tau \in \mathfrak{h}$*, then* $\mathrm{Stab}_{\overline{\Gamma(1)}}(\tau) = \{1\}$*, except*

$$\mathrm{Stab}_{\overline{\Gamma(1)}}(i) = \{1, S\}, \quad \textit{and} \quad \mathrm{Stab}_{\overline{\Gamma(1)}}(\rho) = \{1, S, ST, (ST)^2\}.$$

*(c) S, T generate* $\overline{\Gamma(1)}$*.*

*Proof:*

(a) Suppose $\tau \in \mathfrak{h}$. If $\gamma \in \overline{\Gamma(1)}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

The set of numbers $c\tau + d$ form a subset of the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$ in $\mathbb{C}$, and therefore

$\square$