Chapter 8

Compactifying parameter spaces

Keynote Questions

(a) (The five conic problem) Given five general plane conics $C_1, \dots, C_5 \subset \mathbb{P}^2$, how many smooth conics $C \subset \mathbb{P}^2$ are tangent to all five? (Answer on page 308.)

(b) Given 11 general points $p_1, \dots, p_{11} \in \mathbb{P}^2$ in the plane, how many rational quartic curves $C \subset \mathbb{P}^2$ contain them all? (Answer on page 321.)

All the applications of intersection theory to enumerative geometry exploit the fact that interesting classes of algebraic varieties — lines, hypersurfaces and so on — are themselves parametrized by the points of an algebraic variety, the *parameter space*, and our efforts have all been toward counting intersections on these spaces. But to use intersection theory to count something, the parameter space must be projective (or at least proper) so that we have a degree map, as defined in Chapter 1. In the first case we treated in this book, that of the family of planes of a certain dimension in projective space, the natural parameter space was the Grassmannian, and the fact that it is projective is what makes the Schubert calculus so useful for enumeration. When we studied the questions about linear spaces on hypersurfaces, we were similarly concerned with parameter spaces that were projective — the Grassmannian $\mathbb{G}(k, n)$ and, in connection with questions involving families of hypersurfaces, the projective space $\mathbb{P}^N$ of hypersurfaces itself. These spaces have an additional feature of importance: a universal family of the geometric objects we are studying, or (amounting to the same thing) the property of representing a functor we understand. This property is useful in many ways, first of all for understanding tangent spaces, and thus transversality questions.

In many interesting cases, however, the "natural" parameter space for a problem is *not* projective. To use the tools of intersection theory to count something, we must add points to the parameter space to complete it to a projective (or at least proper) variety. It is customary to call these new points the *boundary*, although this is not a topological

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boundary in any ordinary sense — the boundary points may look like any other point of the space — and (more reasonably) to call the enlarged space a *compactification* of the original space. If we are lucky, the boundary points of the compactification still parametrize some sort of geometric object we understand. In such cases we can use this structure to solve geometric problems. But as we shall see, the boundary can also get in the way, even when it seems quite natural. In such cases, we might look for a "better" compactification… but just how to do so is a matter of art rather than of science.

Perhaps the first problem in enumerative geometry where this tension became clear is the five conic problem, which was solved in a naive way, not taking the difficulty into account (and therefore getting the wrong answer) by Steiner [1848], and again, with the necessary subtlety (and correct answer!) by Chasles [1864]. In this case there is a very beautiful and classical construction of a good parameter space, the space of *complete conics*. In this chapter we will explore the construction, and briefly discuss two more general constructions: Hilbert schemes and Kontsevich spaces.

8.1 Approaches to the five conic problem

To reiterate the problem: Given five general plane conics $C_1, \dots, C_5$, how many smooth conics are tangent to all five? Here is a naive approach:

(a) The set of plane conics is parametrized by $\mathbb{P}^5$. The locus of conics tangent to each given C_i is an irreducible hypersurface $Z_i \subset \mathbb{P}^5$, as one sees by considering the incidence correspondence

$$\begin{tikzcd}[column sep=small] & \{(C, p) \in \mathbb{P}^5 \times C_i \mid C \text{ a conic tangent to } C_i \text{ at } p\} \arrow[dl] \arrow[dr, "\pi_2"] & \\ Z_i & & C_i \end{tikzcd}$$

and noting that the fibers of π_2 are linear subspaces of $\mathbb{P}^5$ of dimension 3. (Here, "tangent to C_i at p" means $m_p(C \cdot C_i) \geq 2$, that is, the restriction to C_i of the defining equation of C vanishes to order at least 2 at p.)

(b) The degree of Z_i is 6. To see this, we intersect Z_i with a general line in $\mathbb{P}^5$ — that is, we take a general pencil of conics and count how many are tangent to C_i. The conic C_i may be thought of as the embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ by the complete linear system of degree 2. Thus a general pencil of conics cuts out a general linear series on C_i of degree 4, and the degree of Z_i is the number of divisors in this family with fewer than four distinct points. The linear series defines a general map $C_i \to \mathbb{P}^1$ of degree 4 with distinct branch points, and by the Riemann–Hurwitz theorem (Section 7.7) the number of branch points of this map is six.

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(c) Thus the number of points of intersection of $Z_1, \dots, Z_5$, *assuming they intersect transversely,* will be $6^5 = 7776$.

Alas, 7776 is *not* the answer to the question we posed. The problem is not hard to spot: far from being transverse, the hypersurfaces Z_i do not even meet in a finite set! To be sure, the part of the intersection within the open set $U \subset \mathbb{P}^5$ of smooth conics (which is what we wanted to count) *is* finite, and even transverse, as we will verify below. The trouble is with the compactification: we used the space of all (possibly singular) conics, and "excess" intersection of the Z_i takes place along the boundary.

In more detail: the hypersurface Z_i is the *closure* in $\mathbb{P}^5$ of the locus of smooth conics C tangent to C_i. A smooth conic C is tangent to C_i exactly when the defining equation F of C, restricted to $C_i \cong \mathbb{P}^1$ and viewed as a quartic polynomial on $\mathbb{P}^1$, has a multiple root. When we extend this characterization to arbitrary conics C we see that *a double line is tangent to every conic*. Thus the five hypersurfaces $Z_1, \dots, Z_5 \subset \mathbb{P}^5$ will all contain the locus $S \subset \mathbb{P}^5$ of double lines, which is a Veronese surface in the $\mathbb{P}^5$ of conics. As we shall see, the intersection $\bigcap Z_i$ is the union of S and the finite set of smooth conics tangent to the five C_i. The presence of this extra component S means that the number we seek has little to do with the intersection product $\prod [Z_i] \in A^5(\mathbb{P}^5)$.

There are at least three successful approaches to dealing with this issue:

Blowing up the excess locus

Suppose we are interested in intersections inside some quasi-projective variety U and we have a compactification V of U; in the example above, U is the space of smooth conics and V the space of all conics. We could blow up some locus in the boundary $V \setminus U$ to obtain a new compactification. This is the classical way of separating subvarieties of a given variety that we do not want to meet. In the five conic problem, we would blow up the surface S in $\mathbb{P}^5$ and consider the proper transforms $\widetilde{Z}_i$ of the hypersurfaces Z_i in the blow-up $X = \mathrm{Bl}_S \mathbb{P}^5$. If we are lucky (and in this case we are), we will have eliminated the excess intersection — that is, the $\widetilde{Z}_i$ will not intersect anywhere in the exceptional divisor $E \subset X$ of the blow-up. (If this were not the case we would have to blow up again, along the common intersection $\bigcap \widetilde{Z}_i \cap E$.) In our case, the $\widetilde{Z}_i$ intersect transversely, and only inside U. To finish the argument, we could determine the Chow ring $A(X)$ of the blow-up, find the class $\zeta \in A^1(X)$ of the $\widetilde{Z}_i$ (as members of a family parametrized by an open subset of $\mathbb{P}^5$, they all have the same class) and evaluate the product $\zeta^5 \in A^5(X)$.

Readers who want to carry this out themselves can find a description of the Chow ring of a blow-up in Section 13.6; there is also a complete account of this approach in Griffiths and Harris [1994, Section 6.1].

This approach has the virtue of being universally applicable, at least in theory: Any component of any intersection of cycles can be eliminated by blowing up repeatedly. But often we cannot recognize the blow-up as the parameter space of any nice geometric

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objects, and this makes the computations less intuitive and sometimes unwieldy. For example, this approach to the problem of counting cubics satisfying nine tangency conditions (solved heuristically by Maillard and Zeuthen in the 19th century and rigorously in Aluffi [1990] and Kleiman and Speiser [1991]) requires multiple blow-ups of the space $\mathbb{P}^9$ of cubics and complex calculations.

Excess intersection formulas

Excess intersection problems were already considered by Salmon in 1847, and were generalized greatly by Cayley around 1868. The excess intersection formula of Fulton and MacPherson (see Fulton [1984, Chapter 9]) subsumes them all: It is a general formula that assigns to every connected component of an intersection $\bigcap Z_i \subset X$ a class in the appropriate dimension, in such a way that the sum of these classes (viewed as classes on the ambient variety X via the inclusion) equals the product of the classes of the intersecting cycles. This applies whenever all but at most one of the subvarieties Z_i are locally complete intersections in X; in our case all are hypersurfaces. We will give an exposition of the formula in Chapter 13, and show in Section 13.3.5 how it may be applied to the five conic problem, as was originally carried out in Fulton and MacPherson [1978].

As a general method, excess intersection formulas are often an improvement on blowing up. But, as with the blow-up approach, they require some knowledge of the normal bundles (or, more generally, normal cones) of the various loci involved.

Changing the parameter space

To understand what sort of compactification is "right" for a given problem is, as we have said, an art. In the case of the five conic problem, we can take a hint from the fact that the problem is about tangencies. The set of lines tangent to a nonsingular conic is again a conic in the dual space (we will identify it explicitly below). But when a conic degenerates to the union of two lines or a double line, the dual conic seems to disappear — the dual of a line is only a point! This leads us to ask for a compactification of the space of smooth conics that keeps track of information about limiting positions of tangents.

There are at least two ways to make a compactification that encodes the necessary information. One is to use the *Kontsevich space*. It parametrizes not subschemes of $\mathbb{P}^2$, but rather maps $f : C \to \mathbb{P}^2$ with C a nodal curve of arithmetic genus 0. This is an important construction, which generalizes to a parametrization of curves of any degree and genus in any variety. We will discuss it informally in the second half of this chapter. But proving even the existence of Kontsevich spaces requires a considerable development, and we will not take this route; the reader will find an exposition in Fulton and Pandharipande [1997].

The other way to describe a compactification of the space of smooth conics that preserves the tangency information is through the idea of *complete conics*. The space of complete conics is very well-behaved, and we will spend the first half of this chapter on this beautiful construction. It turns out that the space we will construct is isomorphic to the Kontsevich space for conics (and, for that matter, to the blow-up $\mathrm{Bl}_S \mathbb{P}^5$ of $\mathbb{P}^5$ along the surface of double lines), but generalizes in a different direction: There are analogs for quadric hypersurfaces of any dimension, for linear transformations ("complete colineations") and, more generally, for symmetric spaces (see De Concini and Procesi [1983; 1985], De Concini et al. [1988] and Bifet et al. [1990]), but not for curves of higher degree or genus. (There is an analogous construction but, as we will remark at the end of Section 8.2.2, in general the space constructed is highly singular and not well-understood.)

8.2 Complete conics

We begin with an informal discussion. Later in this section we will provide a rigorous foundation for what we describe. Recall that the *dual* of a smooth conic $C \subset \mathbb{P}^2$ is the set of lines tangent to C, regarded as a curve $C^* \subset \mathbb{P}^{2*}$. As we shall see, C^* is also a smooth conic (this would not be true in characteristic 2!).

8.2.1 Informal description

Degenerating the dual

Consider what happens to the dual conic as a smooth conic degenerates to a singular conic — either two distinct lines or a double line. That is, let $\mathcal{C} \to B$ be a one-parameter family of conics with parameter t, with C_t smooth for $t \neq 0$. Associating to each curve C_t the dual conic $C_t^* \subset \mathbb{P}^{2*}$ we get a regular map from the punctured disc $B \setminus \{0\}$ to the space $\mathbb{P}^{5*}$ of conics in $\mathbb{P}^{2*}$. (If $\mathbb{P}^2 = \mathbb{P}V$ and $\mathbb{P}^{2*} = \mathbb{P}V^*$, the space of conics on each are respectively $\mathbb{P}\,\mathrm{Sym}^2 V^*$ and $\mathbb{P}\,\mathrm{Sym}^2 V$ — in particular, they are naturally dual to one another, so if we write the former as $\mathbb{P}^5$ it makes sense to write the latter as $\mathbb{P}^{5*}$.) Since the space $\mathbb{P}^{5*}$ of all conics in $\mathbb{P}^{2*}$ is proper, this extends to a regular map on all of B — in other words, there is a well-defined conic $C_0^* = \lim_{t\to 0} C_t^*$. However, as we will see, this limit depends in general on the family $\mathcal{C}$ and not just on the curve C_0: in other words, the limit of the duals C_t^* is *not* determined by the limit of the curves C_t.

To provide a compactification of the space U of smooth conics that captures this phenomenon, we realize U as a locally closed subset of $\mathbb{P}^5 \times \mathbb{P}^{5*}$: As we will see in the following section, the map $C \mapsto C^*$ is regular on smooth conics, so U is isomorphic to the graph of the map $U \to \mathbb{P}^{5*}$ sending a smooth conic C to its dual. That is, we set

$$U = \{(C, C^*) \in \mathbb{P}^5 \times \mathbb{P}^{5*} \mid C \text{ a smooth conic in } \mathbb{P}^2 \text{ and } C^* \subset \mathbb{P}^{2*} \text{ its dual}\}.$$

The desired compactification, the *variety of complete conics*, is the closure

$$X = \overline{U} \subset \mathbb{P}^5 \times \mathbb{P}^{5*}.$$

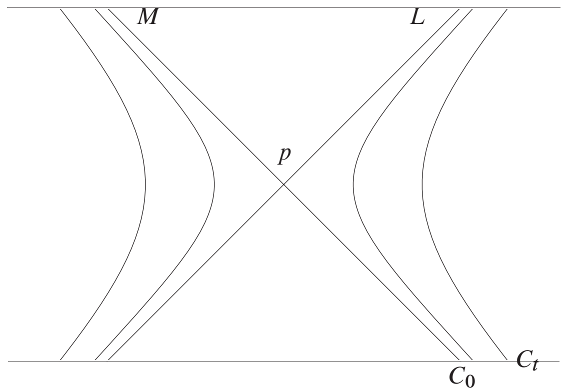

Figure 8.1 Conics specializing to a conic $C_0 = L \cup M$ of rank 2.

The dual of the dual of a smooth conic is the original conic, as we shall soon see (in fact, the same statement holds for varieties much more generally, and will be proven in Section 10.6), so the set U is symmetric under exchanging $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. It follows that X is symmetric too. (As one consequence of this symmetry, note that if $(C, C^*) \in X$ and either C or C^* is smooth, then the other is too.) The set $U \subset \mathbb{P}^5$ of smooth conics is by definition dense in X, and it follows that X is irreducible and of dimension 5 as well.

What happens to C^* when C becomes singular? Let us first consider the case of a family $\{C_t\}$ of smooth conics approaching a conic C_0 of rank 2, that is, $C_0 = L \cup M$ is the union of a pair of distinct lines; for example, the family given (in affine coordinates on $\mathbb{P}^2$) as

$$\mathcal{C} = \{(t, x, y) \in B \times \mathbb{P}^2 \mid y^2 = x^2 - t\},$$

as shown in Figure 8.1. The picture makes it easy to guess what happens: Any collection $\{L_t\}$ of lines with L_t tangent to C_t for $t \neq 0$ approaches a line L_0 through the point $p = L \cap M$, and conversely any line L_0 through p is a limit of lines L_t tangent to C_t. (Actually, the second statement follows from the first, given that the limit $C_0' = \lim_{t\to 0} C_t^*$ is one-dimensional.) Since C_0' is by definition a conic, it must be the double of the line in $\mathbb{P}^{2*}$ dual to the point p, irrespective of the family $\{C_t\}$ used to construct it or of the positions of the lines L and M.

Things are much more interesting when we consider a family of smooth conics $\{C_t \subset \mathbb{P}^2\}$ specializing to a double line $C_0 = 2L$, and ask what the limit $\lim_{t\to 0} C_t^*$ of the dual conics $C_t^* \subset \mathbb{P}^{2*}$ may be. One way to realize such a family of conics is as the images of a family of maps $\varphi_t : \mathbb{P}^1 \to \mathbb{P}^2$. Such a family of maps is given by a triple of polynomials $(f_t(x), g_t(x), h_t(x))$, homogeneous of degree 2 in $x = (x_0, x_1)$, whose coefficients are regular functions in t. In our present circumstances, our hypotheses say that for $t \neq 0$ the polynomials f_t, g_t and h_t are linearly independent (and so span $H^0(\mathcal{O}_{\mathbb{P}^1}(2))$), but for $t = 0$ they span only a two-dimensional vector space $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$. For now, we will make the additional assumption that the linear system $\mathcal{W} = (\mathcal{O}_{\mathbb{P}^1}(2), W)$ is base point free; the case where it is not will be dealt with below.

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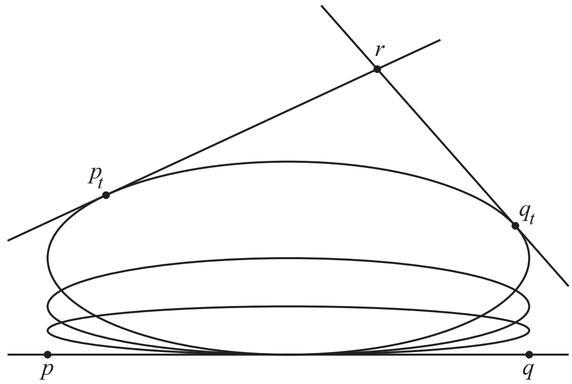

Figure 8.2 The family of conics $y^2 = t(x^2 - 2y)$.

To see what the limit of the dual conics C_t^* will be in this situation, let $u, v \in \mathbb{P}^1$ be the ramification points of the map $\varphi_W : \mathbb{P}^1 \to \mathbb{P}^1$ associated to W (note that the map φ_0 is just the composition of this map with the inclusion of the target $\mathbb{P}^1$ as the line $L \subset \mathbb{P}^2$), and let $p = \varphi_0(u)$ and $q = \varphi_0(v) \in L$ be their images. We claim that in this case the limit $\lim_{t\to 0} C_t^*$ of the dual conics is the conic $C_0^* = p^* + q^* \subset \mathbb{P}^{2*}$ consisting of lines through p and lines through q.

To prove this, let $r \in \mathbb{P}^2$ be any point not in L and not in any curve C_t, and let $\pi : \mathbb{P}^2 \to L$ be the projection from r to L. The composition $\pi \circ \varphi_t : \mathbb{P}^1 \to L \cong \mathbb{P}^1$ has degree 2; let $u_t, v_t \in \mathbb{P}^1$ be the ramification points of this map and $p_t, q_t \in L$ the corresponding branch points. Suppose that $\pi \circ \varphi_t$ is the map associated to the pencil $\mathcal{W}_t = (\mathcal{O}_{\mathbb{P}^1}(2), W_t)$ for a two-dimensional vector space $W_t \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$. As $t \to 0$, the linear system $\mathcal{W}_t$ approaches the linear system $\mathcal{W}$; correspondingly, the divisor $u_t + v_t$ approaches $u + v$ and $p_t + q_t$ approaches $p + q$. In other words, the tangent lines to C_t passing through r — which are exactly the lines $\overline{r, \varphi_t(u_t)} = \overline{r, p_t}$ and $\overline{r, \varphi_t(v_t)} = \overline{r, q_t}$ — approach the lines $\overline{r, p}$ and $\overline{r, q}$, *independently of* r. Thus every line through p or q is a limit of tangent lines to C_t, and conversely.

It is important to note that in this situation, unlike in the case where C_0 is the union of two distinct lines, the limit of the dual conics C_t^* is not determined by the conic C_0. As we will see in Section 8.2.2, the points p and q may be any pair of points of L, depending on the path along which C_t approaches C_0.

The remaining case to consider is when the branch points $p_t, q_t \in L$ of the maps $\pi \circ \varphi_t$ approach the same point $p \in L$. (Typically, this corresponds to the case where $\mathcal{W}$ has a base point: When $\mathcal{W}$ has a base point u, the ramification of $\mathcal{W}$ is concentrated at this point, which must then be the limit as $t \to 0$ of both the ramification points u_t and v_t of $\mathcal{W}_t$.) In this case, the same logic shows that the limit of the dual conics C_t^* will be the double $2p^*$ of the line $p^* \subset \mathbb{P}^{2*}$ dual to the image point $p = \varphi_0(u)$.

Types of complete conics

In conclusion, there are four types of complete conics, that is, points $(C, C') \in X$:

(a) $(C, C') \in U$; that is, C and C' are both smooth and $C' = C^*$. We will call these *smooth* complete conics.
(b) $C = L \cup M$ is of rank 2 and $C' = 2p^*$, where $p^* \subset \mathbb{P}^{2*}$ is the line dual to $p = L \cap M$.
(c) $C = 2L$ is of rank 1, and $C' = p^* \cup q^*$ is the union of the lines in $\mathbb{P}^{2*}$ dual to two points $p, q \in L$.
(d) $C = 2L$ is of rank 1, and $C' = 2p^*$ is the double of the line in $\mathbb{P}^{2*}$ dual to a point $p \in L$.

Note that the description is exactly the same if we reverse the roles of C and C', except that the second and third types are exchanged. Note also that the points of each type form a locally closed subset of X, with the first open and the last closed, and all four are orbits of the action of PGL_3 on $\mathbb{P}^5 \times \mathbb{P}^{5*}$.

As we have already explained, the locus of complete conics of type (a) is isomorphic to U; in particular, it has dimension 5. It is easy to see that those of type (b) are determined by the pair of lines L, M, and thus form a set of dimension 4. By symmetry (or inspection) the same is true for type (c). Finally, those of type (d) are determined by the line L and the point $p \in L$; thus these form a set of dimension 3, which is in fact the intersection of the closures of the sets of points described in (b) and (c).

8.2.2 Rigorous description

Let us now verify all these statements, using the equations defining the locus $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$. We could do this explicitly in coordinates, but it will save a great deal of ink if we use a little multilinear algebra. The reader to whom this is new will find more than enough background in Appendix 2 of Eisenbud [1995]. The multilinear algebra allows us to treat some basic properties in all dimensions with no extra effort, so we begin with some general results about duality for quadrics.

Duals of quadrics

Let V be a vector space. Recall that since we are assuming the characteristic of the ground field $\Bbbk$ is not 2 the following three notions are equivalent:

- A symmetric linear map $\varphi : V \to V^*$.
- A quadratic map $q : V \to \Bbbk$.
- An element $q' \in \mathrm{Sym}^2(V^*)$.

Explicitly, if we start with a symmetric map $\varphi : V \to V^*$ then we take $q(x) = \langle \varphi(x), x \rangle$, and the element $q' \in \mathrm{Sym}^2(V^*)$ comes about from the identification of $\mathrm{Sym}(V^*)$ with the ring of polynomial functions on V.

Any one of these objects, if nonzero, defines a quadric hypersurface $Q \subset \mathbb{P}V$, defined as the zero locus $Q = V(q)$ of q, or equivalently the locus

$$\{v \in \mathbb{P}V \mid \langle \varphi(v), v \rangle = 0\}.$$

(Here, and in the remainder of this discussion, we will abuse notation and use the same symbol v to denote both a nonzero vector $v \in V$ and the corresponding point in $\mathbb{P}V$.) The quadric $Q \subset \mathbb{P}V$ will be smooth if and only if φ is an isomorphism; more generally, the singular locus of Q will be the (projectivization of the) kernel of φ. The *rank* of Q is defined to be, equivalently, the rank of the linear map φ, or $n - \dim(Q_{\text{sing}})$ (where we adopt, for the present purposes only, the convention that $\dim(\varnothing) = -1$); another way to characterize it is to say that a quadric of rank k is the cone with vertex a linear space $Q_{\text{sing}} \cong \mathbb{P}^{n-k} \subset \mathbb{P}^n$ over a smooth quadric hypersurface $\overline{Q} \subset \mathbb{P}^{k-1}$.

Now, the dual of any variety $X \subset \mathbb{P}^n$ is defined to be the closure in $\mathbb{P}^{n*}$ of the locus of hyperplanes tangent to X (that is, containing the tangent space $\mathbb{T}_p X$ at a smooth point $p \in X$). (We will describe this construction in far more detail in Section 10.6.) Given the description in the last paragraph of a quadric Q of rank k as a cone, we see that the dual of a quadric of rank k has dimension $k - 2$. That said, we ask: what, in these terms, is the dual to Q?

To state the result, recall that if $\varphi : V \to W$ is any map of vector spaces of dimension $n + 1$, then there is a *cofactor map* $\varphi^c : W \to V$, represented by a matrix whose entries are signed $n \times n$ minors of φ, satisfying $\varphi \circ \varphi^c = \det(\varphi)\, \mathrm{Id}_W$ and $\varphi^c \circ \varphi = \det(\varphi)\, \mathrm{Id}_V$. In invariant terms, φ^c is the composite

$$W \cong \textstyle\bigwedge^n W^* \xrightarrow{\bigwedge^n \varphi^*} \bigwedge^n V^* \cong V,$$

where the identifications $W \cong \bigwedge^n W^*$ and $\bigwedge^n V^* \cong V$ are defined by choices of nonzero vectors in the one-dimensional spaces $\bigwedge^{n+1} W^*$ and $\bigwedge^{n+1} V^*$ respectively. Note that when the rank of φ is $< n$ the map φ^c is zero.

Proposition 8.1. *Let $Q \subset \mathbb{P}(V) = \mathbb{P}^n$ be the quadric corresponding to the symmetric map $\varphi : V \to V^*$, and let $v \in V$ be a nonzero vector such that $\langle \varphi(v), v \rangle = 0$, so that $v \in Q$. The tangent hyperplane to Q at v is*

$$\mathbb{T}_v Q = \{w \in \mathbb{P}(V) \mid \langle \varphi(v), w \rangle = 0\}.$$

The dual of Q is thus

$$Q^* = \{\varphi(v) \in \mathbb{P}(V^*) \mid v \in Q \text{ and } \varphi(v) \neq 0\}.$$

In particular, if Q is nonsingular (that is, if the rank of φ is $n + 1$), then Q^* is the image $\varphi(Q)$ of Q under the induced map $\varphi : \mathbb{P}V \to \mathbb{P}V^*$, and Q^* is the quadric corresponding to the cofactor map φ^c.

On the other hand, if the rank of Q is n, and Q^c is the quadric corresponding to the cofactor map φ^c, then Q^c is the unique double hyperplane containing Q^*; that is, the support of Q^c is the hyperplane corresponding to the annihilator of the singular point of Q.

Proof: For any $w \in V$, the line $\overline{v,w} \subset \mathbb{P}V$ spanned by v and w is tangent to Q at v if and only if

$$\langle \varphi(v + \epsilon w), v + \epsilon w \rangle = 0 \bmod (\epsilon^2).$$

Expanding this out, we get

$$\langle \varphi(w), v \rangle + \langle \varphi(v), w \rangle = 0,$$

and, by the symmetry of φ and the assumption that we are not in characteristic 2, this is the case if and only if

$$\langle \varphi(v), w \rangle = 0,$$

proving the first statement and identifying the dual variety as $Q^* = \varphi(Q)$.

Suppose the rank of Q is n or $n+1$. Let φ^c be the matrix of cofactors of φ, so that $\varphi^c\varphi = \det\varphi \circ I$, where I is the identity map. Since $\operatorname{rank} Q = \operatorname{rank}\varphi \geq n$, the map φ^c is nonzero. The quadric Q^c is by definition the set of all $w \in V^*$ such that $\langle w, \varphi^c(w) = 0$. If $v \in Q$ then

$$\langle \varphi(v), \varphi^c\varphi(v) \rangle = (\det\varphi)\langle \varphi(v), v \rangle = 0,$$

so $\varphi(Q)$ is contained in Q^c.

If $\operatorname{rank}\varphi = n+1$, so that φ is an isomorphism, then $Q^* = \varphi(Q)$ is again a quadric hypersurface, and we must have $Q^* = Q^c$. If $\operatorname{rank}\varphi = n$, then since $\varphi^c\varphi = 0$ the rank of φ^c is 1, and the associated quadric is a double plane. On the other hand, Q is the cone over a nonsingular quadric in $\mathbb{P}^{n-1}$, and Q^* is the dual of that quadric inside a hyperplane (corresponding to the vertex of Q) in $\mathbb{P}^{n*}$. Thus Q^* spans the plane contained in Q^c. □

The following easy consequence will be useful for the five conic problem:

Corollary 8.2. *If Q and Q' are smooth quadrics, then Q and Q' have the same tangent hyperplane $l = 0$ at some point of intersection $v \in Q \cap Q'$ if and only if Q^* and Q'^* have the common tangent hyperplane $v = 0$ at the point of intersection $l \in Q^* \cap Q'^*$.*

In particular, it follows that if D is a smooth plane conic then the divisor $Z_D \subset X$, which is the closure of the set of complete conics (C, C') such that C is smooth and tangent to D, is equal to the divisor defined similarly starting from the dual conic D^*, that is, the closure of the set of (C, C') such that $C' \subset \mathbb{P}^{2*}$ is smooth and tangent to the dual conic D^*.

Proof: Suppose that Q and Q' correspond to symmetric maps φ and ψ. Since the tangent planes at v are the same, Proposition 8.1 shows that $\varphi(v) = \psi(v) \in Q^* \cap Q'^*$. Since $v = \varphi^{-1}(\varphi(v)) = \psi^{-1}(\psi(v)) \sim \psi^{-1}(\varphi(v))$, we see that Q^* and Q'^* are in fact tangent at $\varphi(v)$. (In addition to the fact that the duality interchanges points and planes, we are really proving that the dual of Q^* is Q, and similarly for Q'. Such a thing is actually true for any nondegenerate variety, as we will see in Section 10.6.) $\square$

Equations for the variety of complete conics

We now return to the case of conics in $\mathbb{P}^2$, and suppose that V is three-dimensional.

Proposition 8.3. *The variety*

$$X \subset \mathbb{P}(\mathrm{Sym}^2 V^*) \times \mathbb{P}(\mathrm{Sym}^2 V) = \mathbb{P}^5 \times \mathbb{P}^{5*}$$

of complete conics is smooth and irreducible. Thinking of $(\varphi, \psi) \in \mathbb{P}^5 \times \mathbb{P}^{5*}$ as coming from a pair of symmetric matrices $\varphi : V \to V^*$ and $\psi : V^* \to V$, the scheme X is defined by the ideal I generated by the eight bilinear equations specifying that the product $\psi \circ \varphi$ has its diagonal entries equal to one another (two equations) and its off-diagonal entries equal to zero (six equations).

(For the experts: it follows from the proposition that the ideal I has codimension 5, and that its saturation, in the bihomogeneous sense, is prime. Computation shows that the polynomial ring modulo I is Cohen–Macaulay. With the proposition, this implies that I is prime. In particular, I is preserved under the interchange of factors φ and ψ, which does not seem evident from the form given.)

Proof: Let Y be the subscheme defined by the given equations. We first show that Y agrees set-theoretically with X on at least the locus of those points (φ, ψ) where $\operatorname{rank} \varphi \geq 2$ or $\operatorname{rank} \psi \geq 2$, that is, where φ or ψ corresponds to a smooth conic or the union of two distinct lines. On the locus of smooth conics, φ has rank 3 and $(\varphi, \psi) \in Y$ if and only if $\psi = \varphi^{-1}$ up to scalars, so Proposition 8.1 shows that the dual conic is defined by ψ. Moreover, if the rank of φ is 2 and $(\varphi, \psi) \in Y$, then we see from the equations that $\psi \circ \varphi = 0$. Up to scalars, $\psi = \varphi^c$ is the unique possibility, and again Proposition 8.1 shows that the corresponding conic C' is the dual of C. To see the uniqueness (up to scalars) in terms of matrices, note that in suitable bases

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the symmetric matrices annihilating the image have the form

$$\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} = a\varphi^c.$$

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The same arguments show that when rank $\psi \geq 2$, $\varphi = \psi^c$ and again they correspond to dual conics. (Note that since rank $\psi^c = 1$ on this locus we do *not* have $\psi = \varphi^c$ there.)

Since X is defined as the closure in $\mathbb{P}^5 \times \mathbb{P}^{5*}$ of the locus U of pairs (C, C^*) with C smooth, we see now in particular that $X \subset Y$. We will show next that Y is smooth of dimension 5 locally at any point $(\varphi, \psi) \in Y$ where both φ and ψ have rank 1. We will use this to show that Y is everywhere smooth of dimension 5.

To this end, suppose that $(\varphi, \psi) \in Y$ and that both φ and ψ have rank 1. The tangent space to Y at the point (φ, ψ) may be described as the locus of pairs of symmetric matrices $\alpha : V \to V^*$, $\beta : V^* \to V$ such that

$$(\psi + \epsilon\beta) \circ (\varphi + \epsilon\alpha) \bmod (\epsilon^2)$$

has equal diagonal entries and zero off-diagonal entries. Since both φ and ψ have rank 1, the rank of $\psi \circ \alpha + \beta \circ \varphi$ is at most 2, so this is equivalent to saying that

$$\psi \circ \alpha + \beta \circ \varphi = 0.$$

We must show that this linear condition on the entries of the pair (α, β) is equivalent to five independent linear conditions. In suitable coordinates the maps φ, ψ will be represented by the matrices

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying out, we see that

$$\psi \circ \alpha = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \beta \circ \varphi = \begin{pmatrix} \beta_{1,1} & 0 & 0 \\ \beta_{2,1} & 0 & 0 \\ \beta_{3,1} & 0 & 0 \end{pmatrix}.$$

Thus the equation $\psi\alpha + \beta\varphi = 0$ is equivalent to the equations $\alpha_{2,1} + \beta_{2,1} = 0$ and $\alpha_{2,2} = \alpha_{2,3} = \beta_{1,1} = \beta_{3,1} = 0$: five independent linear conditions, as required.

To complete the proof of smoothness, note that Y is preserved scheme-theoretically by the action of the orthogonal group G. (Proof: If $(\varphi, \psi) \in Y$ and α is orthogonal, then $(\alpha\varphi\alpha^*, \alpha\psi\alpha^*) \in Y$ since $\alpha^*\alpha = 1$.) Any closed point on Y where rank $\varphi \geq 2$ degenerates under the action of G to a point where rank $\varphi = 1$. (Proof: If α is orthogonal, that is, $\alpha\alpha^* = 1$, then the matrix $\varphi\psi$ is diagonal if and only if $\alpha\varphi\alpha^*\alpha\psi\alpha^* = \alpha\varphi\psi\alpha^*$ is diagonal. Thus, in a basis for which φ is diagonal, stretching one of the coordinates will make the corresponding entry of φ approach zero, and $\psi = \varphi^c$ moves at the same time; a similar argument works when rank $\psi \geq 2$.) Consequently, if the singular locus of Y were not empty it would have to intersect the locus of pairs of matrices of rank 1, and we have seen that this is not the case.

Finally, to see Y is equal to X scheme-theoretically it is enough to show that the open subset U of Y is dense in Y. We use the fact that each point (φ, ψ) of Y corresponds to a unique pair of quadrics (Q, Q'). When φ has rank 2, Q corresponds to a pair of distinct lines, and Q' is uniquely determined. Thus this set is four-dimensional. The same goes for the case where ψ has rank 2. On the other hand, when both φ and ψ have rank 1, Q is the double of a line L and Q' is the double of a line corresponding to one of the points of L; thus this set is only three-dimensional. Since Y is everywhere smooth of dimension 5, any component of Y must intersect the set where φ and ψ have rank 3, as required. □

The classification of the points of X into the four types on page 296 follows from Proposition 8.3:

Corollary 8.4. *If $(\varphi, \psi) \in X$, then one of the following holds:*

(a) (Smooth complete conics) If φ is of rank 3, then ψ must be its inverse.
(b) If φ is of rank 2, then (since X is symmetric) the products $\psi \circ \varphi$ and $\varphi \circ \psi$ must both be zero; it follows that ψ is the unique (up to scalars) symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ.
(c) If φ is of rank 1, ψ may have rank 1 or 2; in the latter case, it may be any symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ.
(d) If φ and ψ both have rank 1, they simply have to satisfy the condition that the kernel of ψ contains the image of φ and vice versa.

Relations with the blow-up of $\mathbb{P}^5$

We mentioned at the beginning of this chapter that another approach to the problem of excess intersection in the five conic problem would be to blow up the excess component — that is, to pass to the blow-up $Z = \mathrm{Bl}_S \mathbb{P}^5$ of $\mathbb{P}^5$ along the surface $S \subset \mathbb{P}^5$ of double lines. It is natural to ask: what is the relation of the space X of complete conics to the blow-up Z?

The answer is that they are in fact the same space. To see this, it is helpful to recall the characterization of blow-ups given in Eisenbud and Harris [2000, Proposition IV-22]: For an affine scheme Y and subscheme $A \subset Y$ with ideal $(f_1, \dots, f_k)$, the blow-up $\mathrm{Bl}_A Y \to Y$ of Y along A is the closure in $Y \times \mathbb{P}^{k-1}$ of the graph of the map $Y \setminus A \to \mathbb{P}^{k-1}$ given by $[f_1, \dots, f_k]$. We can globalize this: Let Y be any scheme and $A \subset Y$ a subscheme. If $\mathcal{L}$ is a line bundle on Y and $\sigma_1, \dots, s_k \in H^0(\mathcal{L})$ sections generating the subsheaf $\mathcal{I}_{A/Y} \otimes \mathcal{L}$, then the closure of the graph of the map $Y \setminus A \to \mathbb{P}^{k-1}$ given by $[f_1, \dots, f_k]$ is the blow-up $\mathrm{Bl}_A Y \to Y$ of Y along A.

This is exactly what we have in the construction of the space X of complete conics. Again, we think of the space $\mathbb{P}^5$ of conics as the space of symmetric 3×3 matrices M, and the Veronese surface $S \subset \mathbb{P}^5$ of double lines as the locus of matrices of rank 1. The six minors σ_i of M are then sections of $\mathcal{O}_{\mathbb{P}^5}(2)$ generating $\mathcal{I}_{S/\mathbb{P}^5}(2)$, so that the blow-up

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$\mathrm{Bl}_S \mathbb{P}^5$ is the closure of the graph of the map $[\sigma_1, \dots, \sigma_6] : \mathbb{P}^5 \setminus S \to \mathbb{P}^5$. But as we have just seen this is the map sending a conic to its dual, so the closure of the graph is the variety X of complete conics.

One note: We could construct an analogous compactification of the space $U \subset \mathbb{P}^N$ of smooth plane curves of any degree d by associating to each smooth $C \subset \mathbb{P}^2$ its dual curve. This defines a regular map $U \to \mathbb{P}^M$, where $\mathbb{P}^M$ is the space of plane curves of degree $d(d-1)$, and we can compactify U by taking the closure in $\mathbb{P}^N \times \mathbb{P}^M$ of the graph of this map. The resulting spaces are highly singular — already in the case $d = 3$, Aluffi [1990] showed it takes five blow-ups of $\mathbb{P}^9$ to resolve the singularities — so in general this is not a useful approach.

8.2.3 Solution to the five conic problem

Now that we have established that the space X of complete conics is smooth and projective, we will show how to solve the five conic problem. To any smooth conic $D \subset \mathbb{P}^2$ we associate a divisor $Z = Z_D \subset X$, which we define to be the closure in X of the locus of pairs $(C, C^*) \in X$ with C smooth and tangent to D, and let $\zeta = [Z_D] \in A^1(X)$ be its class. We will address each of the following issues:

- We have to show that in passing from the "naive" compactification $\mathbb{P}^5$ of the space U of smooth conics to the more sensitive compactification X, we have in fact eliminated the problem of extraneous intersection; in other words, we have to show that for five general conics C_i the corresponding divisors $Z_{C_i} \subset X$ intersect only in points $(C, C') \in X$ with C and $C' = C^*$ smooth.
- We have to show that the five divisors Z_{C_i} are transverse at each point where they intersect.
- We have to determine the Chow ring of the space X, or at least the structure of a subring of $A(X)$ containing the class ζ of the hypersurfaces Z_{C_i} we wish to intersect.
- We have to identify the class ζ in this ring and find the degree of the fifth power $\zeta^5 \in A^5(X)$.

Complete conics tangent to five general conics are smooth

We begin by recalling that X is symmetric under the operation of interchanging the factors $\mathbb{P}^5$ and $\mathbb{P}^{5*}$.

Let us start by showing that no complete conic (C, C') of type (b) lies in the intersection of the divisors associated to five general conics. The first thing we need to do is to describe the points (C, C') of type (b) lying in Z_D for a smooth conic D. This is straightforward: If $C = L \cup M$ is a conic of rank 2 which is a limit of smooth conics tangent to D, then C also must have a point of intersection multiplicity 2 or more with D; thus either L or M is a tangent line to D, or the point $p = L \cap M$ lies on D. (Note that by symmetry a similar description holds for the points of type (c): the complete conic $(2L, p^* + q^*)$ will lie on Z_D only if L is tangent to D, or p or q lie on D.)

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Now, suppose that (C, C') is a complete conic of type (b) lying in the intersection of the divisors $Z_i = Z_{C_i}$ associated to five general conics C_i. Write $C = L \cup M$, and set $p = L \cap M$. We note that since the C_i are general, no three are concurrent; thus p can lie on at most two of the conics C_i. We will proceed by considering three cases in turn:

- *p lies on none of the conics C_i.* This is the most immediate case: Since the conics C_i^* are also general, it is likewise the case that no three of them are concurrent. In other words, no line in the plane is tangent to more than two of the C_i, and correspondingly $(L \cup M, p) \in Z_{C_i}$ for at most four of the C_i.
- *p lies on two of the conics C_i*, say C_1 and C_2. Since C_3, C_4 and C_5 are general with respect to C_1 and C_2, none of the finitely many lines tangent to two of them passes through a point of $C_1 \cap C_2$; thus L and M can each be tangent to at most one of the conics C_3, C_4 and C_5, and again we see that $(L \cup M, p) \in Z_{C_i}$ for at most four of the C_i.
- *p lies on exactly one of the conics C_i*, say C_1. Now, since C_1 is general with respect to C_2, C_3, C_4 and C_5, it will not contain any of the finitely many points of pairwise intersection of lines tangent to two of them. Thus L and M cannot each be tangent to two of the conics $C_2, \dots, C_5$, and once more we see that $(L \cup M, p) \in Z_{C_i}$ for at most four of the C_i.

Thus no complete conic of type (b) can lie in the intersection of the Z_{C_i}; by symmetry, no complete conic of type (c) can either.

It remains to verify that no complete conic (C, C') of type (d) can lie in the intersection $\bigcap Z_{C_i}$, and again we have to start by characterizing the intersection of a cycle $Z = Z_D$ with the locus of complete conics of type (d).

To do this, write an arbitrary complete conic of type (d) as $(2M, 2q^*)$, with $q \in M$. If $(2M, 2q^*) \in Z_D$, then there is a one-parameter family $(C_t, C_t') \in Z_D$ with C_t smooth, $C_t' = C_t^*$ for $t \neq 0$ and $(C_0, C_0') = (2M, 2q^*)$; let $p_t \in C_t \cap D$ be the point of tangency of C_t with D, and set $p = \lim_{t \to 0} p_t \in M$. The tangent line $\mathbb{T}_{p_t} C_t = \mathbb{T}_{p_t} D$ to C_t at p_t will have as limit the tangent line L to D at p, so $L \in q^*$. Thus both p and q are in both L and M. If $p = q$ then in particular $q \in D$. On the other hand, if $p \neq q$, then we must have $M = \overline{p, q} = L$, so $M \in D^*$. We conclude, therefore, that *a complete conic $(2M, 2q^*)$ of type (d) can lie in Z_D only if either $q \in D$ or $M \in D^*$.*

Given this, we see that the first condition ($q \in C_i$) can be satisfied for at most two of the C_i, and the latter ($M \in C_i^*$) likewise for at most two; thus no complete conic $(2M, 2q^*)$ of type (d) can lie in Z_{C_i} for all $i = 1, \dots, 5$.

Transversality

In order to prove that the cycles $Z_{C_i} \subset X$ intersect transversely when the conics $C_1, \dots, C_5$ are general, we need a description of the tangent spaces to the Z_{C_i} at points of $\bigcap Z_{C_i}$. We have just shown that such points are represented by smooth conics, and the open subscheme parametrizing smooth conics is the same whether we are working

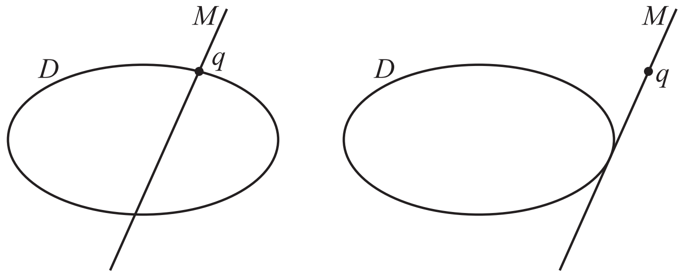

Figure 8.3 The two types of complete conics $(2M, 2q^*)$ of type (d) tangent to D.

in $\mathbb{P}^5$ or in the space of complete conics, so we may express the answer in terms of the geometry of $\mathbb{P}^5$.

Lemma 8.5. *Let $D \subset \mathbb{P}^2$ be a smooth conic curve and $Z_D^\circ \subset \mathbb{P}^5$ the variety of smooth plane conics C tangent to D.*

(a) If C has a point p of simple tangency with D and is otherwise transverse to D, then Z_D° is smooth at $[C]$.
(b) In this case, the projective tangent plane $\mathbb{T}_{[C]}Z_D^\circ$ to Z_D° at $[C]$ is the hyperplane $H_p \subset \mathbb{P}^5$ of conics containing the point p.

Proof: First, identify D with $\mathbb{P}^1$, and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathcal{O}_D(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(4)).$$

This map is surjective, with kernel the one-dimensional subspace spanned by the section representing D itself. In terms of projective spaces, the restriction induces a rational map

$$\pi_D : \mathbb{P}^5 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(4)) = \mathbb{P}^4$$

(this is just the linear projection of $\mathbb{P}^5$ from the point $D \in \mathbb{P}^5$ to $\mathbb{P}^4$). The closure Z_D° in $\mathbb{P}^5$ is thus the cone with vertex $D \in \mathbb{P}^5$ over the hypersurface $\mathcal{D} \subset \mathbb{P}^4$ of singular divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$; Lemma 8.5 will follow directly from the next result:

Proposition 8.6. *Let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree d on $\mathbb{P}^1$ and $\mathcal{D} \subset \mathbb{P}^d$ the discriminant hypersurface, that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d-2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p.*

Proof: Note that we have already seen this statement: it is the content of Proposition 7.21 (stated in Section 7.7.3 as a consequence of the topological Hurwitz formula). For another proof, this time in local coordinates, we can introduce the incidence correspondence

$$\Psi = \{(F, p) \in \mathbb{P}^d \times \mathbb{P}^1 \mid \text{ord}_p(F) \geq 2\},$$

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and write down its equations in local coordinates (a, x) in $\mathbb{P}^d \times \mathbb{P}^1$: Ψ is the zero locus of the polynomials

$$R(a, t) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

and

$$S(a, t) = d a_d x^{d-1} + (d-1) a_{d-1} x^{d-2} + \cdots + 2 a_2 x + a_1.$$

Evaluated at a general point (a, x) where $a_1 = a_0 = x = 0$, all the partial derivatives of R and S vanish except

$$\begin{pmatrix} \dfrac{\partial R}{\partial a_1} & \dfrac{\partial R}{\partial a_0} & \dfrac{\partial R}{\partial x} \\ \dfrac{\partial S}{\partial a_1} & \dfrac{\partial S}{\partial a_0} & \dfrac{\partial S}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2a_2 \end{pmatrix}.$$

The fact that the first 2×2 minor is nonzero assures us that Ψ is smooth at the point, and the fact that $a_2 \neq 0$ and the characteristic is not 2 assures us that the differential $d\pi : T_{(a,0)}\Psi \to T_a\mathbb{P}^d$ of the projection $\pi : \mathcal{D} \to \mathbb{P}^d$ is injective, with image the plane $a_0 = 0$. Finally, the fact that π is one-to-one at such a point tells us the image $\mathcal{D} = \pi(\Psi)$ is smooth at the image point. $\square$

Getting back to the statement of Lemma 8.5, if $C \subset \mathbb{P}^2$ is a conic with a point p of simple tangency with D and is otherwise transverse to D, then, by Proposition 8.6, $\mathcal{D}$ is smooth at the image point in $\mathbb{P}^4$, with tangent space the space of polynomials vanishing at p. Since Z_D is the cone over $\mathcal{D}$ it follows that Z_D is smooth at C; the tangent space statement follows as well. $\square$

In order to apply Lemma 8.5, we need to establish some more facts about a conic tangent to five general conics:

Lemma 8.7. *Let $C_1, \dots, C_5 \subset \mathbb{P}^2$ be general conics, and $C \subset \mathbb{P}^2$ any smooth conic tangent to all five. Each conic C_i is simply tangent to C at a point p_i and is otherwise transverse to C, and the points $p_i \in C$ are distinct.*

Proof: Let U be the set of smooth conics, and consider incidence correspondences

$$\Phi = \{(C_1, \dots, C_5; C) \in (U^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\}$$
$$\subset \Phi' = \{(C_1, \dots, C_5; C) \in ((\mathbb{P}^5)^5 \times U) \mid \text{each } C_i \text{ is tangent to } C\}.$$

The set Φ is an open subset of the set Φ'. Since U is irreducible of dimension 5 and the projection map $\Phi' \to U$ on the last factor has irreducible fibers $(Z_C)^5$ of dimension 20, we see that Φ', and thus also Φ, is irreducible of dimension 25.

There are certainly points in Φ where the conditions of the lemma are satisfied: simply choose a conic C and five general conics C_i tangent to it. Thus the set of $(C_1, \ldots, C_5; C) \in \Phi$ where the conditions of the lemma are not satisfied is a proper closed subset, and as such it can have dimension at most 24, and cannot dominate U^5 under the projection to the first factor. This proves the lemma. $\square$

To complete the argument for transversality, let $[C] \in \bigcap Z_{C_i}$ be a point corresponding to the conic $C \subset \mathbb{P}^2$. By Lemma 8.7 the points p_i of tangency of C with the C_i are distinct points on C. Since C is the unique conic through these five points, the intersection of the tangent spaces to Z_{C_i} at $[C]$

$$\bigcap \mathbb{T}_{[C]} Z_{C_i} = \bigcap H_{p_i} = \{[C]\}$$

is zero-dimensional, proving transversality.

8.2.4 Chow ring of the space of complete conics

Having confirmed that the intersection $\bigcap Z_{C_i}$ indeed behaves well, let us turn now to computing the intersection number. We start by describing the relevant subgroup of the Chow group $A(X)$.

First, let $\alpha, \beta \in A^1(X)$ be the pullbacks to $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ of the hyperplane classes on $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are respectively represented by the divisors

$$A_p = \{(C, C^*) \mid p \in C\}$$

(for any point $p \in \mathbb{P}^2$) and

$$B_L = \{(C, C^*) \mid L \in C^*\}$$

(for any point $L \in \mathbb{P}^{2*}$).

Also, let $\gamma, \varphi \in A^4(X)$ be the classes of the curves Γ and Φ that are the pullbacks to X of general lines in $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are, respectively, the classes of the loci of complete conics (C, C^*) such that C contains four general points in the plane, and such that C^* contains four points $L_i \in \mathbb{P}^{2*}$ (that is, C is tangent to four lines in $\mathbb{P}^2$).

Lemma 8.8. *The group $A^1(X)$ of divisor classes on X has rank 2, and is generated over the rationals by α and β. The intersection number of these classes with γ and φ are given by the table*

$$\begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \gamma \\ \varphi \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

Proof: We first show that the rank of $A^1(X)$ is at most 2. The open subset $U \subset X$ of pairs (C, C^*) with C and C^* smooth is isomorphic to the complement of a hypersurface in $\mathbb{P}^5$, and hence has torsion Picard group: Any line bundle on U extends to a line bundle on $\mathbb{P}^5$, a power of which is represented by a divisor supported on the complement $\mathbb{P}^5 \setminus U$.

Thus, if L is any line bundle on X, a power of L is trivial on U and so is represented by a divisor supported on the complement $X \setminus U$. But the complement of U in X has just two irreducible components: the closures D_2 and D_3 of the loci of complete conics of type (b) and (c). Any divisor class on X is thus a rational linear combination of the classes of D_2 and D_3, from which we see that the rank of the Picard group of X is at most 2.

Since passing through a point is one linear condition on a quadric, we have $\deg(\alpha\gamma) = 1$ and dually $\deg(\beta\varphi) = 1$. Similarly, since a general pencil of conics will cut out on a line $L \subset \mathbb{P}^2$ a pencil of degree 2, which will have two branch points, we see that $\deg(\alpha\varphi) = 2$ and again by duality $\deg(\beta\gamma) = 2$. Since the matrix of intersections between α, β and γ, φ is nonsingular, we conclude that α and β generate $\mathrm{Pic}(X) \otimes \mathbb{Q}$. □

In fact, α and β generate $A^1(X)$ over $\mathbb{Z}$ as well, as we could see from the description of X as a blow-up of $\mathbb{P}^5$.

The class of the divisor of complete conics tangent to C

It follows from Lemma 8.8 that we can write $\zeta = p\alpha + q\beta \in A^1(X) \otimes \mathbb{Q}$ for some $p, q \in \mathbb{Q}$. To compute p and q, we use the fact that, restricted to the open set $U \subset X$, the divisor Z is a sextic hypersurface; it follows that $\deg \zeta\gamma = p + 2q = 6$, and since ζ is symmetric in α and β we get $\deg \zeta\varphi = q + 2p = 6$ as well. Thus

$$\zeta = 2\alpha + 2\beta \in A^1(X) \otimes \mathbb{Q}.$$

From this we see that $\deg(\zeta^5) = 32 \deg(\alpha + \beta)^5$, and it suffices to evaluate the degree of the class $\alpha^{5-i}\beta^i \in A^5(X)$ for $i = 0, \dots, 5$. By symmetry, it is enough to do this for $i = 0$, 1 and 2.

To do this, observe first that the projection of $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ onto the first factor is an isomorphism on the set U_1 of pairs (C, C') such that $\operatorname{rank} C \geq 2$ (the map $U \to \mathbb{P}^{5*}$ sending a smooth conic C to its dual in fact extends to a regular map on U_1 sending a conic $C = L \cup M$ of rank 2 to the double line $2p^* \in \mathbb{P}^{5*}$, where $p = L \cap M$). Since all conics passing through three given general points have rank ≥ 2, the intersections needed will occur only in U_1. Since the degree of a zero-dimensional intersection is equal to the degree of the intersection scheme, this implies that we can make the computations on $\mathbb{P}^5$ instead of on X. For this we will use Bézout's theorem:

- $i = 0$: Passing through a point is a linear condition on quadrics. There is a unique quadric through five general points, and the intersection of five hyperplanes in $\mathbb{P}^5$ has degree 1, so $\deg(\alpha^5) = 1$.
- $i = 1$: The quadrics tangent to a given line form a quadric hypersurface in $\mathbb{P}^5$. Since not all conics in the one-dimensional linear space of conics through four general points will be tangent to a general line, $\deg(\alpha^4\beta) = 2$.

- $i = 2$: Similarly, we see that the conics passing through three given general points and tangent to a general line form a conic curve in $U_1 \subset \mathbb{P}^5$. Not all these conics are tangent to another given general line. (For example, after fixing coordinates we may think of circles as the conics passing through the points $\pm\sqrt{-1}$ on the line at ∞. Certainly there are circles through a given point and tangent to a given line that are not tangent to another given line.) It follows that $\deg(\alpha^3\beta^2)$ is the degree of the zero-dimensional intersection of a plane with two quadrics, that is, 4.

Thus

$$\begin{aligned}\deg((\alpha+\beta)^5) &= \binom{5}{0} + 2\binom{5}{1} + 4\binom{5}{2} + 4\binom{5}{3} + 2\binom{5}{4} + \binom{5}{5} \\ &= 1 + 10 + 40 + 40 + 10 + 1 \\ &= 102\end{aligned}$$

and, correspondingly,

$$\zeta^5 = 2^5 \cdot 102 = 3264.$$

This proves:

Theorem 8.9. *There are 3264 plane conics tangent to five general plane conics.*

Of course, the fact that we are imposing the condition of being tangent to a conic is arbitrary; we can use the space of complete conics to count conics tangent to five general plane curves of any degree, as Exercises 8.11 and 8.12 show, and indeed we can extend this to the condition of tangency with singular curves, as Exercises 8.14–8.16 indicate. See Fulton et al. [1983] for a general formula enumerating members of a k-dimensional families of varieties tangent to k given varieties.

Other divisor classes on the space of complete conics

We will take a moment here to describe as well the classes of two other important divisors on the space X of complete conics: the closures E and G of the strata of complete conics of types (b) and (c). As we mentioned in the initial section of this chapter, the space X can also be realized, via the projection map $X \subset \mathbb{P}^5 \times \mathbb{P}^{5*} \to \mathbb{P}^5$, as the blow-up of $\mathbb{P}^5$ along the Veronese surface $S \subset \mathbb{P}^5$ of double lines, or dually as a blow-up of $\mathbb{P}^{5*}$; in these descriptions of X, the divisors G and E are the exceptional divisors of the blow-up maps.

We can describe the classes ϵ and ξ of E and G by the same method we used to determine the class of Z_d, that is, by calculating their intersection numbers with the curves Γ and Φ. For E, we see that a general pencil of plane conics will have three singular elements, so that $\deg(\epsilon\gamma) = 3$ (that is, the image of E in $\mathbb{P}^5$ is a cubic hypersurface), while the image of E in $\mathbb{P}^{5*}$ has codimension 3, and so will not meet a general line in $\mathbb{P}^{5*}$ at all, so that $\deg(\epsilon\varphi) = 0$; solving, we obtain

$$\epsilon = 2\alpha - \beta, \quad \text{and dually} \quad \xi = 2\beta - \alpha.$$

Alternatively, we can argue that in the space $\mathbb{P}^5$ of conics the closure B_L of the locus of smooth conics tangent to a given line $L \subset \mathbb{P}^2$ is a quadric hypersurface containing the Veronese surface of double lines. It necessarily has multiplicity 1 along S (the singular locus of a quadric hypersurface in $\mathbb{P}^n$ is contained in a proper linear subspace of $\mathbb{P}^n$), so that its proper transform has class $\beta = 2\alpha - \xi$, and the relations above follow.

8.3 Complete quadrics

There are beautiful generalizations of the construction of the space of complete conics to the case of quadrics in $\mathbb{P}^n$ and more general bilinear forms or homomorphisms. The paper Laksov [1987] gives an excellent account and many references. Here is a sketch of the beginning of the story. As usual we restrict ourselves to characteristic 0, though the description holds more generally as long as the ground field has characteristic $\neq 2$.

As in the case of conics, we represent a quadric in $\mathbb{P}^n = \mathbb{P}V$ by a symmetric transformation $\varphi : V \to V^*$, or equivalently a symmetric bilinear form in $\mathrm{Sym}^2 V^*$. To this transformation we associate the sequence of symmetric transformations

$$\varphi_i : \textstyle\bigwedge^i V \to \bigwedge^i (V^*) = (\bigwedge^i V)^* \quad \text{for } i = 1, \dots, n.$$

Here the identification $\bigwedge^i (V^*) = (\bigwedge^i V)^*$ is canonical — see for example Eisenbud [1995, Section A2.3].

We think of φ_i as an element of $\mathrm{Sym}^2(\bigwedge^i V^*)$, and we define *the variety of complete quadrics in* $\mathbb{P}^n$, which we will denote by Φ, to be the closure in

$$\prod_{i=1}^{n} \mathbb{P}(\mathrm{Sym}^2(\textstyle\bigwedge^i V^*))$$

of the image of the set of smooth quadrics under the map $\varphi \mapsto (\varphi_1, \dots, \varphi_n)$.

The space $\mathbb{P}(\bigwedge^i V^*)$ in which the quadric corresponding to φ_i lies is the ambient space of the Grassmannian $G_i = \mathbb{G}(i-1, n)$ of $(i-1)$-planes in $\mathbb{P}^n$, and in fact an $(i-1)$-plane $\Lambda \subset \mathbb{P}^n$ is tangent to Q if the point $[\Lambda] \in G_i$ lies in this quadric.

From the definition we see that Φ has an open set U isomorphic to the open set corresponding to quadrics in the projective space of quadratic forms on $\mathbb{P}^n$. As with the case of complete conics, there is a beautiful description of the points that are not in U.

To start, let $\mathbb{P}^n = \mathbb{P}V$ and consider a flag $\mathcal{V}$ of subspaces of V of arbitrary length r and dimensions $k = \{k_1 < \cdots < k_r\}$:

$$0 \subset V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_r} \subset V.$$

Now consider the variety F_k of pairs $(\mathcal{V}, Q)$, where $\mathcal{V}$ is a flag as above and $Q = (Q_1, \dots, Q_{r+1})$, where the Q_i are smooth quadric hypersurfaces in the projective space $\mathbb{P}(V_{k_i}/V_{k_{i-1}})$; this is an open subset of a product of projective bundles over the variety of flags $\mathcal{V}$. We then have:

Proposition 8.10. *There is a stratification of* Φ *whose strata are the varieties* F_k*, where* k *ranges over all strictly increasing sequences* $0 < k_1 < \cdots < k_r < r$.

One can also describe the limit of a family of smooth quadrics $q_t \in U = F_\varnothing$ when the family approaches a quadric q_0 of rank $n+1-k$, as in

$$\varphi_t := \begin{pmatrix} t \cdot I_k & 0 \\ 0 & I_{n+1-k} \end{pmatrix}.$$

The limit lies in the stratum $F_{\{k\}}$, where the flag consists of one intermediate space $0 \subset V_k \subset V$; the k-plane V_k will be the kernel of φ_0, the quadric Q_2 on $\mathbb{P}(V/V_k)$ will be the quadric induced by Q_0 on the quotient, and Q_1 will be the quadric on $\mathbb{P}V_k$ associated to the limit

$$\lim_{t\to 0} \frac{\varphi_t|_{V_k}}{t}.$$

In general, the stratum F_k lies in the closure of F_l exactly when $l \subset k$; the specialization relations can be defined inductively, using the above example.

8.4 Parameter spaces of curves

So far in this chapter we have been studying compactifications of parameter spaces of smooth conics. The most obvious is perhaps $\mathbb{P}^5$, which we can identify as the space of all subschemes of $\mathbb{P}^2$ having pure dimension 1 and degree 2 (and thus arithmetic genus 0), and we have shown how the compactification by complete conics was more useful for dealing with problems involving tangencies. Here we have used the fact that the dual of a smooth conic is again a smooth conic. It would have been a different story if the problem had involved twisted cubics in $\mathbb{P}^3$ rather than conics in $\mathbb{P}^2$ — if we had asked, for example, for the number of twisted cubic curves meeting each of 12 lines, or tangent to each of 12 planes, or, as in one classical example, the number of twisted cubic curves tangent to each of 12 quadrics. In that case it is not so clear how to make any parameter space and compactification at all!

In this section, we will discuss two general approaches to constructing parameter spaces for curves in general: the Hilbert scheme of curves and the Kontsevich space of stable maps. (In specific cases, other approaches may be possible as well; for example, see Cavazzani [2016] in the case of twisted cubics.) The Hilbert scheme and the Kontsevich space each have advantages and disadvantages, as we will see.

8.4.1 Hilbert schemes

Recall from Section 6.3 that the Hilbert scheme $\mathcal{H}_P(\mathbb{P}^n)$ is a parameter space for subschemes of $\mathbb{P}^n$ with Hilbert polynomial P; in the case of curves (one-dimensional subschemes), this means all subschemes with fixed degree and arithmetic genus. We start by describing the Hilbert schemes parametrizing conic and cubic curves in $\mathbb{P}^2$ and $\mathbb{P}^3$; when we come to Kontsevich spaces, we will describe these cases in that setting for contrast.

The Hilbert schemes of conics and cubics in $\mathbb{P}^2$

As we have observed, these are just the projective spaces $\mathbb{P}^5$ and $\mathbb{P}^9$ associated to the vector spaces of homogeneous quadratic and cubic polynomials on $\mathbb{P}^2$; they parametrize subschemes of $\mathbb{P}^2$ with Hilbert polynomial $2m + 1$ and $3m$ respectively.

The Hilbert scheme of plane conics in $\mathbb{P}^3$

We will discuss this space at much greater length in the following chapter (where, in particular, we will prove all the assertions made here!). Briefly, the story is this: Any subscheme of $\mathbb{P}^3$ with Hilbert polynomial $2m+1$ is necessarily the complete intersection of a plane and a quadric surface; the plane, naturally, is unique. This means that the Hilbert scheme admits a map to the dual projective space $\mathbb{P}^{3*}$; the fiber over a point $H \in \mathbb{P}^{3*}$ is the $\mathbb{P}^5$ of conics in $H \cong \mathbb{P}^2$. (This $\mathbb{P}^5$-bundle structure is what allows us to calculate its Chow ring; we will use this information to solve the enumerative problem of counting the conics meeting each of eight general lines in $\mathbb{P}^3$.) In any event, the Hilbert scheme of plane conics in $\mathbb{P}^3$ is irreducible and smooth of dimension 8.

The Hilbert scheme of twisted cubics

In the case of the Hilbert scheme parametrizing twisted cubic curves in $\mathbb{P}^3$ (that is, parametrizing subschemes of $\mathbb{P}^3$ with Hilbert polynomial $3m + 1$) we begin to see some of the pathologies that affect Hilbert schemes in general. It has one component of dimension 12 whose general point corresponds to a twisted cubic curve. But it also has a second component, whose general point corresponds to the union of a plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ and a point $p \in \mathbb{P}^3$. Moreover, this second component has dimension 15 (the choice of plane has three degrees of freedom, the cubic inside the plane nine more, and the point gives an additional three). These two components meet along the 11-dimensional subscheme of singular plane cubics C with an embedded point at the singularity, not contained in the plane spanned by C (see Piene and Schlessinger [1985]).

8.4.2 Report card for the Hilbert scheme

The Hilbert scheme is from some points of view the most natural parameter space that is generally available for projective schemes. Among its advantages: As shown in Section 6.3, it represents a functor that is easy to understand. There is a useful cohomological description of the tangent spaces to the Hilbert scheme, and, beyond that, a deformation theory that in some cases can describe its local structure. It was shown

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to be connected in characteristic 0 by Hartshorne [1966] and in finite characteristic by Pardue [1996] (see Peeva and Stillman [2005] for another proof). Reeves [1995] showed that the radius of the graph of its irreducible components is at most one more than the dimension of the varieties being parametrized. And, of course, associated to a point on the Hilbert scheme is all the rich structure of a homogeneous ideal in the ring $\mathbb{k}[x_0, \dots, x_n]$ and its free resolution.

However, as a compactification of the space of smooth curves, the Hilbert scheme has drawbacks that sometimes make it difficult to use:

(a) *It has extraneous components, often of differing dimensions.* We see this phenomenon already in the case of twisted cubics, above. Of course we could take just the closure $\mathcal{H}^\circ$ in the Hilbert scheme of the locus of smooth curves, but we would lose some of the nice properties, like the description of the tangent space. (Thus while it is relatively easy to describe the singular locus of $\mathcal{H}$, we do not know in general how to describe the singular locus of $\mathcal{H}^\circ$ along the locus where it intersects other components; in the case of twisted cubics it was not known until Piene and Schlessinger [1985] that $\mathcal{H}^\circ$ is smooth.)

   In fact, we do not know for curves of higher degree how many such extraneous components there are, nor their dimensions: For $r \geq 3$ and large d, the Hilbert scheme of zero-dimensional subschemes of degree d in $\mathbb{P}^r$ will have an unknown number of extraneous components of unknown dimensions, and this creates even more extraneous components in the Hilbert schemes of curves.

(b) *No one knows what is in the closure of the locus of smooth curves*. If we do choose to deal with the closure $\mathcal{H}^\circ$ of the locus of smooth curves rather than the whole Hilbert scheme — as it seems we must — we face another problem: except in a few special cases, we cannot tell if a given point in the Hilbert scheme is in this closure. That is, we may not know how to tell whether a given singular one-dimensional scheme $C \subset \mathbb{P}^r$ is smoothable.

(c) *It has many singularities*. The singularities of the Hilbert scheme are, in a precise sense, arbitrarily bad: Vakil [2006b] has shown that the completion of every affine local $\mathbb{k}$-algebra appears (up to adding variables) as the completion of a local ring on a Hilbert scheme of curves.

8.4.3 The Kontsevich space

In the case of curves in a variety, the *Kontsevich space* is an alternative compactification. A precise treatment of this object is given in Fulton and Pandharipande [1997]; here we will treat it informally, sketch some of its properties, and indicate how it is used, with the hope that this will give the interested reader a taste of what to expect.

The Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ parametrizes what are called *stable maps* of degree d and genus g to $\mathbb{P}^r$. These are morphisms

$$f : C \to \mathbb{P}^r,$$

with C a connected curve of arithmetic genus g having only nodes as singularities, such that the image $f_*[C]$ of the fundamental class of C is equal to d times the class of a line in $A_1(\mathbb{P}^r)$, and satisfying the one additional condition that the automorphism group of the map f — that is, automorphisms φ of C such that $f \circ \varphi = f$ — is finite. (This last condition is automatically satisfied if the map f is finite; it is relevant only for maps that are constant on an irreducible component of C, and amounts to saying that any smooth, rational component C_0 of C on which f is constant must intersect the rest of the curve C in at least three points.) Two such maps $f : C \to \mathbb{P}^r$ and $f' : C' \to \mathbb{P}^r$ are said to be the same if there exists an isomorphism $\alpha : C \to C'$ with $f' \circ \alpha = f$. There is an analogous notion of a family of stable maps, and the Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ is a coarse moduli space for the functor of families of stable maps. Note that we are taking the quotient by automorphisms of the source, but not of the target, so that $\overline{M}_{g,0}(\mathbb{P}^r, d)$ shares with the Hilbert scheme $\mathcal{H}_{dm-g+1}(\mathbb{P}^r)$ a common open subset parametrizing smooth curves $C \subset \mathbb{P}^r$ of degree d and genus g.

There are natural variants of this: the space $\overline{M}_{g,n}(\mathbb{P}^r, d)$ parametrizes maps $f : C \to \mathbb{P}^r$ with C a nodal curve having n marked distinct smooth points $p_1, \dots, p_n \in C$. (Here an automorphism of f is an automorphism of C fixing the points p_i and commuting with f; the condition of stability is thus that any smooth, rational component C_0 of C on which f is constant must have at least three distinguished points, counting both marked points and points of intersection with the rest of the curve C.) More generally, for any projective variety X and numerical equivalence class $\beta \in \mathrm{Num}_1(X)$, we have a space $\overline{M}_{g,n}(X, \beta)$ parametrizing maps $f : C \to X$ with fundamental class $f_*[C] = \beta$, again with C nodal and f having finite automorphism group.

It is a remarkable fact that the Kontsevich space is proper: In other words, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^r$ is a flat family of subschemes of $\mathbb{P}^r$ parametrized by a smooth, one-dimensional base $\mathcal{D}$, and the fiber C_t is a smooth curve for $t \neq 0$, then no matter what the singularities of C_0 are there is a unique stable map $f : \widetilde{C}_0 \to \mathbb{P}^r$ which is the limit of the inclusions $\iota_t : C_t \hookrightarrow \mathbb{P}^r$. Note that this limiting stable map $f : \widetilde{C}_0 \to \mathbb{P}^r$ depends on the family, not just on the scheme C_0; the import of this in practice is that the Kontsevich space is often locally a blow-up of the Hilbert scheme along loci of curves with singularities worse than nodes. (This is not to say we have in general a regular map from the Kontsevich space to the Hilbert scheme; as we will see in the examples below, the limiting stable map $f : \widetilde{C}_0 \to \mathbb{P}^r$ does not determine the flat limit C_0 either.) We will see how this plays out in the four relatively simple cases discussed above in connection with the Hilbert scheme:

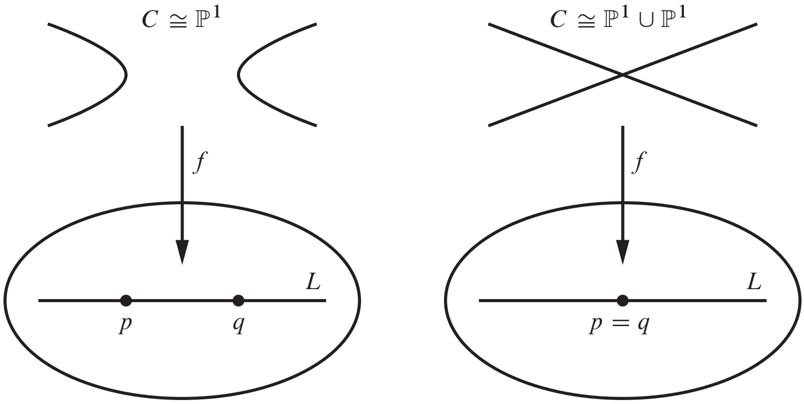

Figure 8.4 Stable maps of degree 2 with image a line.

Plane conics

One indication of how useful the Kontsevich space can be is that, in the case of $\overline{M}_0(\mathbb{P}^2, 2)$ (that is, plane conics), the Kontsevich space is actually equal to the space of complete conics.

To begin with, if $C \subset \mathbb{P}^2$ is a conic of rank 2 or 3 — that is, anything but a double line — then the inclusion map $\iota : C \hookrightarrow \mathbb{P}^2$ is a stable map; thus the open set $W \subset \mathbb{P}^5$ of such conics is likewise an open subset of the Kontsevich space $\overline{M}_0(\mathbb{P}^2, 2)$.

But when a one-parameter family $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^2$ of conics specializes to a double line $C_0 = 2L$, the limiting stable map is a finite, degree-2 map $f : C \to L$, with C isomorphic to either $\mathbb{P}^1$ or two copies of $\mathbb{P}^1$ meeting at a point. Such a map is characterized, up to automorphisms of the source curve, by its branch divisor $B \subset L$, a divisor of degree 2. (If B consists of two distinct points, then $C \cong \mathbb{P}^1$, while if $B = 2p$ for some $p \in L$, the curve C is reducible.) Thus the data of the limiting stable map is equivalent to specifying the limiting dual curve.

This suggests what is in fact the case: The identification of the common open subset W of the Kontsevich space $\overline{M}_0(\mathbb{P}^2, 2)$ and the Hilbert scheme $\mathcal{H}_{2m+1}(\mathbb{P}^2) = \mathbb{P}^5$ extends to a regular morphism, and to a biregular isomorphism of $\overline{M}_0(\mathbb{P}^2, 2)$ with the space X of complete conics, commuting with the projection $X \to \mathbb{P}^5$:

$$\begin{tikzcd} \overline{M}_0(\mathbb{P}^2, 2) \arrow[rr, "\cong"] \arrow[dr] & & X \arrow[dl] \\ & \mathcal{H}_{2m+1}(\mathbb{P}^2) = \mathbb{P}^5 & \end{tikzcd}$$

We will not verify these assertions, but they are not hard to prove given the analysis of limits of conics and their duals in Section 8.2.1.

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Plane conics in $\mathbb{P}^3$

By contrast, there is not a natural regular map in either direction between the Hilbert scheme of conics in space and the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 2)$. Of course there is a common open set U: its points correspond to reduced connected curves of degree 2 embedded in $\mathbb{P}^3$ (such a curve is either a smooth conic in a plane or the union of two coplanar lines). To see that the identification of this open set does not extend to a regular map in either direction, note first that, as before, if $\mathcal{C} \subset \mathcal{D} \times \mathbb{P}^3$ is a family of conics specializing to a double line C_0, then the limiting stable map is a finite, degree-2 cover $f : C \to L$, and this cover is not determined by the flat limit C_0 of the schemes $C_t \subset \mathbb{P}^3$. Thus the identity map on U does not extend to a regular map from the Hilbert scheme to the Kontsevich space. On the other hand, the scheme C_0 is contained in a plane — the limit of the unique planes containing the C_t. Since it has degree 2, the plane containing it is unique. But this plane is not determined by the data of the map f. Thus the identity map on U does not extend to a regular map from the Kontsevich space to the Hilbert scheme either.

The birational equivalence between the Hilbert scheme and the Kontsevich space is of a type that appears often in higher-dimensional birational geometry: the Kontsevich space is obtained from the Hilbert scheme $\mathcal{H}$ by blowing up the locus of double lines, and then blowing down the exceptional divisor along another ruling. (The blow-up of $\mathcal{H}$ along the double line locus is isomorphic to the blow-up of $\overline{M}_0(\mathbb{P}^3, 2)$ along the locus of stable maps of degree 2 onto a line; both can be described as the space of pairs $(H; (C, C^*))$, where $H \subset \mathbb{P}^3$ is a plane and (C, C^*) a complete conic in $H \cong \mathbb{P}^2$.) The birational isomorphism between the Hilbert scheme and Kontsevich space in this case is an example of what is known as a *flip* in higher-dimensional birational geometry.

Plane cubics

Here, we do have a regular map from the Kontsevich space $\overline{M}_1(\mathbb{P}^2, 3)$ to the Hilbert scheme $\mathcal{H}_{3m}(\mathbb{P}^2) \cong \mathbb{P}^9$, and it does some interesting things: It blows up the locus of triple lines, much as in the example of plane conics, and the locus of cubics consisting of a double line and a line as well. But it also blows up the locus of cubics with a cusp, and cubics consisting of a conic and a tangent line, and these are trickier: The blow-up along the locus of cuspidal cubics, for example, can be obtained either by three blow-ups with smooth centers or by one blow-up along a nonreduced scheme supported on this locus.

But what we really want to illustrate here is that the Kontsevich space $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ is not irreducible — in fact, it is not even nine-dimensional! For example, maps of the form $f : C \to \mathbb{P}^2$ with C consisting of the union of an elliptic curve E and a copy of $\mathbb{P}^1$, where f maps $\mathbb{P}^1$ to a nodal plane cubic C_0 and maps E to a smooth point of C_0, form a 10-dimensional family of stable maps; in fact, these form an open subset of a second irreducible component of $\overline{M}_1(\mathbb{P}^2, 3)$, as illustrated in Figure 8.5.

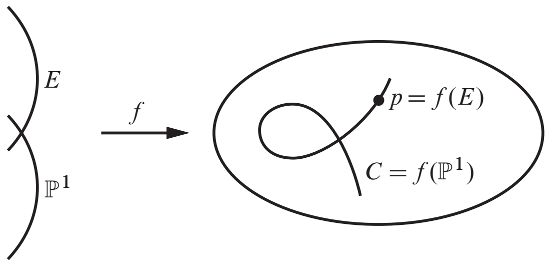

Figure 8.5 A typical point in the 10-dimensional component of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$.

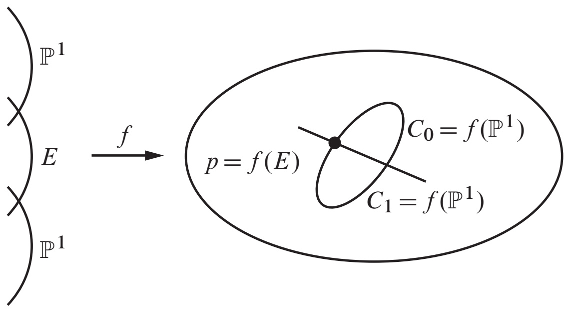

Figure 8.6 General member of a third component of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$.

And there is also a third component, whose general member is depicted in Figure 8.6.

Twisted cubics

Here the shoe is on the other foot. The Hilbert scheme $\mathcal{H} = \mathcal{H}_{3m+1}$ has, as we saw, a second irreducible component besides the closure $\mathcal{H}_0$ of the locus of actual twisted cubics, and the presence of this component makes it difficult to work with. For example, it takes quite a bit of analysis to see that $\mathcal{H}_0$ is smooth, since we have no simple way of describing its tangent space; see Piene and Schlessinger [1985] for details. By contrast, the Kontsevich space is irreducible, and has only relatively mild (finite quotient) singularities.

8.4.4 Report card for the Kontsevich space

As with the Hilbert scheme, there are difficulties in using the Kontsevich space:

(a) *It has extraneous components*. These arise in a completely different way from the extraneous components of the Hilbert scheme, but they are there. A typical example of an extraneous component of the Kontsevich space $\overline{M}_g(\mathbb{P}^r, d)$ consists of maps $f : C \to \mathbb{P}^r$ in which C was the union of a rational curve $C_0 \cong \mathbb{P}^1$, mapping to a rational curve of degree d in $\mathbb{P}^r$, and C_1 an arbitrary curve of genus g meeting C_0 in one point and on which f was constant; if the curve C_1 does not itself admit a nondegenerate map of degree d to $\mathbb{P}^r$, this map cannot be smoothed.

So, using the Kontsevich space rather than the Hilbert scheme does not solve this problem, but it does provide a frequently useful alternative: There are situations where the Kontsevich space has extraneous components and the Hilbert scheme does not — like the case of plane cubics described above — and also situations where the reverse is true, such as the case of twisted cubics.

(b) *No one knows what is in the closure of the locus of smooth curves*. This, unfortunately, remains an issue with the Kontsevich space. Even in the case of the space $\overline{M}_g(\mathbb{P}^2, d)$ parametrizing plane curves, where it might be hoped that the Kontsevich space would provide a better compactification of the Severi variety parametrizing reduced and irreducible plane curves of degree d and geometric genus g than simply its closure in the space $\mathbb{P}^N$ of all plane curves of degree d, the fact that we do not know which stable maps are smoothable represents a real obstacle to its use.

(c) *It has points corresponding to highly singular schemes, and these tend to be in turn highly singular points of $\overline{M}_g(\mathbb{P}^r, d)$*. Still true, but in this respect, at least, it might be said that the Kontsevich space represents an improvement over the Hilbert scheme: Even when the image $f(C)$ of a stable map $f : C \to \mathbb{P}^r$ is highly singular, the fact that the source of the map is at worst nodal makes the deformation theory of the map relatively tractable.

Finally, we mention one other virtue of the Kontsevich space: It allows us to work with tangency conditions, without modifying the space and without excess intersection. The reason is simple: If $X \subset \mathbb{P}^r$ is a smooth hypersurface, the closure Z_X in $\overline{M}_g(\mathbb{P}^r, d)$ of the locus of embedded curves tangent to X is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(X)$ is nonreduced or positive-dimensional. Thus, for example, a point in $\overline{M}_g(\mathbb{P}^2, d)$ corresponding to a multiple curve — that is, a map $f : C \to \mathbb{P}^2$ that is multiple-to-one onto its image — is not necessarily in Z_X.

8.5 How the Kontsevich space is used: rational plane curves

One case in which the Kontsevich space is truly well-behaved is the case $g = 0$. Here the space $\overline{M}_0(\mathbb{P}^r, d)$ is irreducible — it has no extraneous components — and, moreover, its singularities are at worst finite quotient singularities (in fact, it is the coarse moduli space of a smooth Deligne–Mumford stack). Indeed, the use of the Kontsevich space has been phenomenally successful in answering enumerative questions about rational curves in projective space. We will close out this chapter with an example of this; specifically, we will answer the second keynote question, and, more generally, the question of how many rational curves $C \subset \mathbb{P}^2$ of degree d are there passing through $3d - 1$ general points in the plane.

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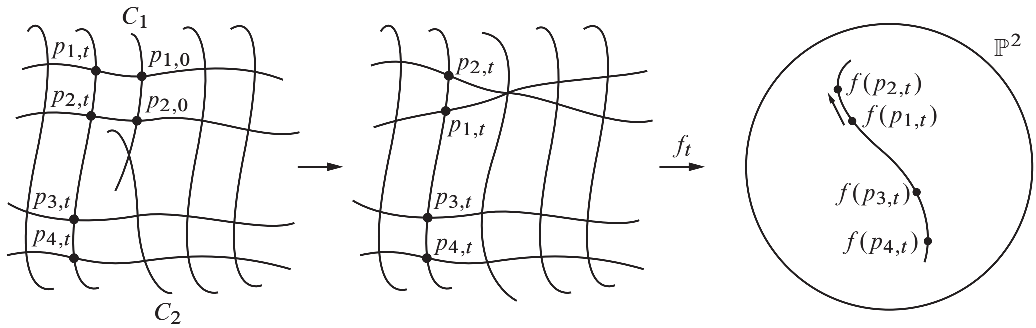

Figure 8.7 A family of maps that blows down C_1.

Since we have not even defined the Kontsevich space, this analysis will be far from complete. The paper Fulton and Pandharipande [1997] provides enough background to complete it.

Before starting the calculation, let us check that we do in fact expect a finite number. Maps of degree d from $\mathbb{P}^1$ to $\mathbb{P}^2$ are given by triples $[F, G, H]$ of homogeneous polynomials of degree d on $\mathbb{P}^1$ with no common zeros; since the vector space of polynomials of degree d on $\mathbb{P}^1$ has dimension $d + 1$, the space U of all such triples has dimension $3d + 3$. Now look at the map $U \to \mathbb{P}^N$ from U to the space $\mathbb{P}^N$ of plane curves of degree d, sending such a triple to the image (as divisor) of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^2$. This has four-dimensional fibers (we can multiply F, G and H by a common scalar, or compose the map with an automorphism of $\mathbb{P}^1$), so we conclude that the image has dimension $3d - 1$. In particular, we see that there are no rational curves of degree d passing through $3d$ general points of $\mathbb{P}^2$, and we expect a finite number (possibly 0) through $3d - 1$. We will denote the number by $N(d)$.

We will work on the space $M_d := \overline{M}_{0,4}(\mathbb{P}^2, d)$ of stable maps from curves with four marked points. This is convenient, since on M_d we have a rational function φ, given by the *cross-ratio*: at a point of M_d corresponding to a map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ with $C \cong \mathbb{P}^1$ irreducible, it is the cross-ratio of the points $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$; that is, in terms of an affine coordinate z on $\mathbb{P}^1$,

$$\varphi = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)},$$

where $z_i = z(p_i)$. The cross-ratio takes on the values 0, 1 and ∞ only when two of the points coincide, which in our setting corresponds to when the curve C is reducible: For example, if C has two components C_1 and C_2, with $p_1, p_2 \in C_1$ and $p_3, p_4 \in C_2$, then by blowing down the curve C_1 in the total space of the source family, we can realize $(C, p_1, \ldots, p_4)$ as a limit of pointed curves $(C_t, p_1(t), \ldots, p_4(t))$ with C_t irreducible and $\lim_{t\to 0} p_1(t) = \lim_{t\to 0} p_2(t)$ (see Figure 8.7). Thus φ has a zero at such a point. Similarly, if three of the p_i, or all four, lie on one component of C, then φ will be equal to the cross-ratio of four distinct points on $\mathbb{P}^1$, and so will not be 0, 1 or ∞.

We now introduce a curve $B \subset M_d$ on which we will make the calculation. Fix a point $p \in \mathbb{P}^2$ and two lines $L, M \subset \mathbb{P}^2$ passing through p; fix two more general points $q, r \in \mathbb{P}^2$ and a collection $\Gamma \subset \mathbb{P}^2$ of $3d-4$ general points. We consider the locus

$$B = \left\{ f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2 \;\middle|\; \begin{array}{c} f(p_1) = q,\ f(p_2) = r, \\ f(p_3) \in L,\ f(p_4) \in M, \\ \Gamma \subset f(C) \end{array} \right\} \subset M_d.$$

Since, as we said, the space of rational curves of degree d in $\mathbb{P}^2$ has dimension $3d-1$, and we are requiring the curves in our family to pass through $3d-2$ points (the points q and r, and the $3d-4$ points of Γ), our locus B will be a curve.

There may be points in B for which the source C of the corresponding map $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ is reducible. But in these cases C *will have no more than two components*. To see this, note that if the image of C has components $D_1, \ldots, D_k$ of degrees $d_1, \ldots, d_k$, by the above the curve D_i can contain at most $3d_i - 1$ of the $3d-2$ points $\Gamma \cup \{q, r\}$. Thus

$$3d - 2 \geq \sum_{i=1}^{k} (3d_i - 1) = 3d - k,$$

whence $k \leq 2$. As a consequence, we see that the map f cannot be constant on any component: By the stability condition, if f were constant on a component C_0, then C_0 would have to meet at least three other components — but f can be nonconstant on only two, and it follows that the stability condition is violated on some component.

This argument also shows that there are only finitely many points in B for which the source C is reducible: If $D = D_1 \cup D_2 \subset \mathbb{P}^2$, with D_i a rational curve of degree d_i, and $\Gamma \cup \{q, r\} \subset D$, then by the above D_i must contain exactly $3d_i - 1$ of the $3d-2$ points $\Gamma \cup \{q, r\}$. The number of such plane curves D is thus

$$\binom{3d-2}{3d_1-1} N(d_1) N(d_2).$$

Moreover, for each such plane curve D there are $d_1 d_2$ stable maps $f : C \to \mathbb{P}^2$ with image D: We can take C the normalization of D at all but any one of the points of intersection $D_1 \cap D_2$. (By Exercise 8.18, D_1 and D_2 will intersect transversely.)

On with the calculation! We equate the number of zeros and the number of poles of φ on B. To begin with, we consider points $f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$ of B with C irreducible. Since $f(p_1) = q$ and $f(p_2) = r$ are fixed and lie off the lines L and M, the only way any two of the points p_i can coincide on such a curve is if

$$f(p_3) = f(p_4) = p, \quad \text{where } L \cap M = \{p\}.$$

Such points are zeros of φ; the number of these zeros is the number of rational plane curves of degree d through the $3d-1$ points p, q, r and Γ, that is to say, $N(d)$. (Of course, to make a rigorous argument we would have to determine the multiplicities of these zeros; since we are just sketching the calculation, we will omit the verification that all multiplicities are 1, here and in the following.)

What about zeros and poles of φ coming from points

$$f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$$

in B with $C = C_1 \cup C_2$ reducible? As we have observed, we get a zero of φ at such a point if and only if the points p_1 and p_2 lie on one component of C and p_3 and p_4 lie on the other. How many such points are there? Well, letting d_1 be the degree of the component C_1 of C containing p_1 and p_2, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q, r and $3d_1 - 3$ of the points of Γ, while C_2 contains the remaining $3d - 4 - (3d_1 - 3) = 3d_2 - 1$ points of Γ. For any subset of $3d_1 - 3$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_2 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as $d_1 d_2$ choices of the point $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1 d_2^3 \binom{3d-4}{3d_1-3} N(d_1)N(d_2)$$

zeros of φ arising in this way.

The poles of φ are counted similarly. These can occur only at points

$$f : (C; p_1, p_2, p_3, p_4) \to \mathbb{P}^2$$

in B with $C = C_1 \cup C_2$ reducible, specifically with the points p_1 and p_3 lying on one component of C and p_2 and p_4 on the other. Again letting d_1 be the degree of the component C_1 of C containing p_1 and p_3, and $d_2 = d - d_1$ the degree of the other component C_2, we see that $f(C_1)$ must contain q and $3d_1 - 2$ points of Γ, and $f(C_2)$ the remaining $3d - 4 - (3d_1 - 2) = 3d_2 - 2$ points of Γ, plus r. For any subset of $3d_1 - 2$ points of Γ, the number of such plane curves is $N(d_1)N(d_2)$, and for each such plane curve there are d_1 choices of the point $p_3 \in C_2 \cap f^{-1}(L)$ and d_2 choices of the point $p_4 \in C_2 \cap f^{-1}(M)$, as well as $d_1 d_2$ choices of the point $f(C_1 \cap C_2) \in f(C_1) \cap f(C_2)$. We thus have a total of

$$\sum_{d_1=1}^{d-1} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N(d_1)N(d_2)$$

poles of φ arising in this way. Now, adding up the poles and zeros, we conclude that

$$N(d) = \sum_{d_1=1}^{d-1} d_1 d_2 \left[d_1 d_2 \binom{3d-4}{3d_1-2} - d_2^2 \binom{3d-4}{3d_1-3} \right] N(d_1)N(d_2),$$

a recursive formula that allows us to determine $N(d)$ if we know $N(d')$ for $d' < d$.

To see how this works, we start with the fact that there is a unique line through two points, and a unique conic through five general points, so $N(d_1) = N(d_2) = 1$. Next, if we take $d = 3$ we see that

$$N(3) = 2\left[2\binom{5}{1} - 4\binom{5}{0}\right] + 2\left[2\binom{5}{4} - \binom{5}{3}\right] = 12.$$

In fact, we have already seen this in Proposition 7.4: The set of all cubics containing eight general points $p_1, \dots, p_8 \in \mathbb{P}^2$ is a general pencil, and we are counting the number of singular elements of that pencil.

Continuing to $d = 4$, we have

$$N(4) = 3 \cdot 12\left[3\binom{8}{1} - 9\binom{8}{0}\right] + 4\left[4\binom{8}{4} - 4\binom{8}{3}\right] + 3 \cdot 12\left[3\binom{8}{7} - \binom{8}{6}\right] = 620.$$

Always ignoring the question of multiplicity, this answers Keynote Question (b): There are 620 rational quartic curves through 11 general points of $\mathbb{P}^2$.

Exercises 8.19 and 8.20 suggest some additional problems that can be solved using spaces of stable maps.

8.6 Exercises

Exercise 8.11. Let $D \subset \mathbb{P}^2$ be a smooth curve of degree d, and let $Z_D \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z_D.

Exercise 8.12. Now let $D_1, \dots, D_5 \subset \mathbb{P}^2$ be general curves of degrees $d_1, \dots, d_5$. Show that the corresponding cycles $Z_{D_i} \subset X$ intersect transversely, and that the intersection is contained in the open set U of smooth conics.

Exercise 8.13. Combining the preceding two exercises, find the number of smooth conics tangent to each of five general curves $D_i \subset \mathbb{P}^2$.

Exercise 8.14. Let $D \subset \mathbb{P}^2$ be a curve of degree d with δ nodes and κ ordinary cusps (for a definition of cusps, see Section 11.4.1), and smooth otherwise. Let $Z_D \subset X$ be the closure, in the space X of complete conics, of the locus of smooth conics tangent to D at a smooth point of D. Find the class $[Z_D] \in A^1(X)$ of the cycle Z_D.

Exercise 8.15. Let $\{D_t\}$ be a family of plane curves of degree d, with D_t smooth for $t \neq 0$ and D_0 having a node at a point p. What is the limit of the cycles Z_{D_t} as $t \to 0$?

Exercise 8.16. Here is a very 19th century way of deriving the result of Exercise 8.11 above. Let $\{D_t\}$ be a pencil of plane curves of degree d, with D_t smooth for general t and D_0 consisting of the union of d general lines in the plane. Using the description of the limit of the cycles Z_{D_t} as $t \to 0$ in the preceding exercise, find the class of the cycle Z_{D_t} for t general.

Exercise 8.17. True or false: There are only finitely many PGL_4-orbits in the Kontsevich space $\overline{M}_0(\mathbb{P}^3, 3)$.

Exercise 8.18. Let Γ_1 and Γ_2 be collections of $3d_1 - 1$ and $3d_2 - 1$ general points in $\mathbb{P}^2$, and $D_i \subset \mathbb{P}^2$ any of the finitely many rational curves of degree d_i passing through Γ_i. Show that D_1 and D_2 intersect transversely.

Exercise 8.19. Let $p_1, \dots, p_7 \in \mathbb{P}^2$ be general points and $L \subset \mathbb{P}^2$ a general line. How many rational cubics pass through $p_1, \dots, p_7$ and are tangent to L?

Exercise 8.20. (a) Let $M = \overline{M}_0(\mathbb{P}^2, d)$ be the Kontsevich space of rational plane curves of degree d, and let $U \subset M$ be the open set of immersions $f : \mathbb{P}^1 \to \mathbb{P}^2$ that are birational onto their images. For $D \subset \mathbb{P}^2$ a smooth curve, let $Z_D^\circ \subset U$ be the locus of maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(\mathbb{P}^1)$ is tangent to D at a smooth point of $f(\mathbb{P}^1)$, and $Z_D \subset M$ its closure. Show that Z_D is contained in the locus of maps $f : C \to \mathbb{P}^r$ such that the preimage $f^{-1}(D)$ is nonreduced or positive-dimensional.

(b) Given this, show that for $D_1, \dots, D_{3d-1}$ general curves the intersection $\bigcap Z_{D_i}$ is contained in U.