Lecture 15

I_L, I_K groups of fractional ideals

Prop. 9.7 $\Rightarrow I_K \cong \bigoplus_{\mathfrak{p} \text{ prime in } \mathcal{O}_K} \mathbb{Z}$, $I_L \cong \bigoplus_{\mathfrak{P} \text{ prime in } \mathcal{O}_L} \mathbb{Z}$

Define $N_{L/K} : I_L \to I_K$ group hom.

Determined by $\mathfrak{P} \mapsto \mathfrak{p}^f$, where $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$,

$f = f(\mathfrak{P}/\mathfrak{p})$ residue class degree.

Theorem 12.7: $N_{L/K}(\mathcal{D}_{L/K}) = d_{L/K}$.

Lemma 12.8: The diagram

$$\begin{tikzcd} L^\times \arrow[r] \arrow[d, "N_{L/K}"'] & I_L \arrow[d, "N_{L/K}"] \\ K^\times \arrow[r] & I_K \end{tikzcd}$$

commutes.

Proof: Note for L/K degree n ext. of complete discretely valued fields, v_L, v_K normalized valuations

$v_K = \frac{1}{e} v_L \Rightarrow v_K(N_{L/K}(x)) = f_{L/K}\, v_L(x)$.

Let $\mathfrak{p}$ prime of K. For $x \in L^\times$,

Corollary 10.10 $\Rightarrow N_{L/K}(x) = \prod_{\mathfrak{P} | \mathfrak{p}} N_{L_\mathfrak{P}/K_\mathfrak{p}}(x)$

Thus $v_\mathfrak{p}(N_{L/K}(x)) = \sum_{\mathfrak{P}|\mathfrak{p}} v_\mathfrak{p}(N_{L_\mathfrak{P}/K_\mathfrak{p}}(x))$.

$= \sum_{\mathfrak{P}|\mathfrak{p}} f(\mathfrak{P}/\mathfrak{p})\, v_\mathfrak{P}(x)$

$= v_\mathfrak{p}(N_{L/K}(x\mathcal{O}_L))$ by defn

Since true $\forall \mathfrak{p}$, have $N_{L/K}(x)\mathcal{O}_K$

$= N_{L/K}(x\mathcal{O}_L)$. □

Proof of Theorem 12.7: First assume $\mathcal{O}_K, \mathcal{O}_L$ PID's.

2 Let $x_1, \dots, x_n$ be an $\mathcal{O}_K$-basis for $\mathcal{O}_L$ and $y_1, \dots, y_n$ be dual basis w.r.t. Trace form

Let $\sigma_1, \dots, \sigma_n : L \to \overline{K}$ distinct embeddings (separable!)

$$\sum_{i=1}^{n} \sigma_i(x_j)\sigma_i(y_k) = Tr_{L/K}(x_j y_k) = \delta_{jk}.$$

But $\Delta(x_1, \dots, x_n) = \det(\sigma_i(x_j))^2$.

Thus $\Delta(x_1, \dots, x_n)\Delta(y_1, \dots, y_n) = 1$.

Write $\mathcal{D}_{L/K}^{-1} = \beta\mathcal{O}_L$, some $\beta \in L$. Then

$$\begin{aligned} d_{L/K}^{-1} &= (\Delta(x_1, \dots, x_n))^{-1} \\ &= (\Delta(y_1, \dots, y_n)) \\ &= (\Delta(\beta x_1, \dots, \beta x_n)) \quad \text{as } y_i, \beta x_i \text{ are } \mathcal{O}_K \text{ basis for } \mathcal{D}_{L/K}^{-1} \\ &= N_{L/K}(\beta)^2 \Delta(x_1, \dots, x_n) \end{aligned}$$

Thus $d_{L/K}^{-1} = N_{L/K}(\mathcal{D}_{L/K}^{-1})^2 d_{L/K}$

so $N_{L/K}(\mathcal{D}_{L/K}) = d_{L/K}$.

In general, localize at $S = \mathcal{O}_K \setminus \mathfrak{p}$, and use $S^{-1}\mathcal{D}_{L/K} = \mathcal{D}_{S^{-1}\mathcal{O}_L/S^{-1}\mathcal{O}_K}$ $S^{-1}d_{L/K} = d_{S^{-1}\mathcal{O}_L/S^{-1}\mathcal{O}_K}$. omit details

□

Theorem 12.9: If $\mathcal{O}_L = \mathcal{O}_K[\alpha]$ and α has min

3 polynomial $g(X) \in \mathcal{O}_K[X]$, then $\mathcal{D}_{L/K} = (g'(\alpha))$.

Proof. Let $\alpha = \alpha_1, \dots, \alpha_n$ roots of g.

Write $g(X)/X-\alpha = \beta_{n-1}X^{n-1} + \dots + \beta_1 X + \beta_0$, $\beta_i \in \mathcal{O}_L$, β_{n-1}

We claim $\displaystyle\sum_{i=1}^{n} \frac{g(X)}{X-\alpha_i} \cdot \frac{\alpha_i^r}{g'(\alpha_i)} = X^r$ for $0 \le r \le n-1$

Indeed difference is polynomial of degree $< n$

which vanishes for $X = \alpha_1, \dots, \alpha_n$.

Equating coefficients of $X^s \rightsquigarrow \mathrm{Tr}_{L/K}\left(\frac{\alpha^r \beta_s}{g'(\alpha)}\right) = \delta_{rs}$.

Since $\mathcal{O}_L$ has $\mathcal{O}_K$-basis $1, \alpha, \dots, \alpha^{n-1}$, $\mathcal{D}_{L/K}^{-1}$ has

$\mathcal{O}_K$-basis $\frac{\beta_0}{g'(\alpha)}, \frac{\beta_1}{g'(\alpha)}, \dots, \frac{\beta_{n-1}}{g'(\alpha)} = \frac{1}{g'(\alpha)}$

$\Rightarrow \mathcal{D}_{L/K}^{-1} = (g'(\alpha))^{-1} \Rightarrow \mathcal{D}_{L/K} = (g'(\alpha))$ $\square$

$\mathfrak{P}$ prime of $\mathcal{O}_L$, $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$. Can define

$\mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}}$ using $\mathcal{O}_{K_\mathfrak{p}}, \mathcal{O}_{L_\mathfrak{P}}$.

We identify $\mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}}$ with a power of $\mathfrak{P}$

Theorem 12.7: $\mathcal{D}_{L/K} = \prod_{\mathfrak{P}} \mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}}$.

Proof: Let $x \in L$, $\mathfrak{p} \subseteq \mathcal{O}_K$ prime. Then

(*) $\mathrm{Tr}_{L/K}(x) = \sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{Tr}_{L_\mathfrak{P}/K_\mathfrak{p}}(x)$ (cf. Cassels-Fröhlich / Cardley 10.10)

4 Let $r(\mathfrak{P}) = v_\mathfrak{P}(\mathcal{D}_{L/K})$, $s(\mathfrak{P}) = v_\mathfrak{P}(\mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}})$

NTS $r(\mathfrak{P}) = s(\mathfrak{P})$.

$r(\mathfrak{P}) \ge s(\mathfrak{P})$: $x \in L$ with $v_\mathfrak{P}(x) \ge -s(\mathfrak{P})$ $\forall \mathfrak{P}$

Then $\mathrm{Tr}_{L_\mathfrak{P}/K_\mathfrak{p}}(xy) \in \mathcal{O}_{K_\mathfrak{p}}$, $\forall y \in \mathcal{O}_L$ and $\forall \mathfrak{P}$.

(*) $\Rightarrow \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_{K_\mathfrak{p}}$, $\forall y \in \mathcal{O}_L$, $\forall \mathfrak{p}$.

$\Rightarrow \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_K$ $\forall y \in \mathcal{O}_L$; $x \in \mathcal{D}^{-1}$

$\Rightarrow \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_K,\ \forall y \in \mathcal{O}_L$, i.e. $x \in \mathcal{D}^{-1}_{L/K}$.

Thus $\mathcal{D}_{L/K} \subseteq \prod_{L_\mathfrak{P}/K_\mathfrak{p}} \mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}}$ (finite product) see later.

$\Rightarrow\ r(\mathfrak{P}) \geq s(\mathfrak{P})\ \forall \mathfrak{P}$.

$r(\mathfrak{P}) \leq s(\mathfrak{P})$: Fix $\mathfrak{P}$ and let $x \in \mathfrak{P}^{-r(\mathfrak{P})} \setminus \mathfrak{P}^{-r(\mathfrak{P})+1}$.

Then $v_\mathfrak{P}(x) = -r(\mathfrak{P})$, $v_{\mathfrak{P}'}(x) \geq 0\ \forall \mathfrak{P}' \neq \mathfrak{P}$.

By (*): $\mathrm{Tr}_{L_\mathfrak{P}/K_\mathfrak{p}}(xy) = \underbrace{\mathrm{Tr}_{L/K}(xy)}_{\in\,\mathcal{O}_K} - \sum_{\substack{\mathfrak{P}'|\mathfrak{p} \\ \mathfrak{P}' \neq \mathfrak{P}}} \underbrace{\mathrm{Tr}_{L_{\mathfrak{P}'}/K_\mathfrak{p}}(xy)}_{\in\,\mathcal{O}_{K_\mathfrak{p}}}\ \forall y \in \mathcal{O}_L$

$\Rightarrow \mathrm{Tr}_{L_\mathfrak{P}/K_\mathfrak{p}}(xy) \in \mathcal{O}_{K_\mathfrak{p}}\quad \forall y \in \mathcal{O}_{L_\mathfrak{P}}$ (by continuity)

$\Rightarrow x \in \mathcal{D}^{-1}_{L_\mathfrak{P}/K_\mathfrak{p}}$

$\Rightarrow -v_\mathfrak{P}(x) = r(\mathfrak{P}) \leq s(\mathfrak{P})$ □

Corollary 12.8: $d_{L/K} = \prod_{\mathfrak{P}|\mathfrak{p}} d_{\mathfrak{P}/K_\mathfrak{p}}$.

Proof: Apply $N_{L/K}$ to $\mathcal{D}_{L/K} = \prod_{\mathfrak{P}} \mathcal{D}_{L_\mathfrak{P}/K_\mathfrak{p}}$. □

§13 Unramified and totally ramified extensions of local fields

Let L/K finite separable extension of non-arch. local fields.

Corollary 11.6 implies

(*) $[L:K] = e_{L/K} f_{L/K}$

Lemma 13.1: Let $M/L/K$ be finite sep. extensions of local fields. Then

(i) $f_{M/K} = f_{M/L} f_{L/K}$

(ii) $e_{M/K} = e_{M/L} e_{L/K}$

Proof: (i) $f_{M/K} = [k_M : k] = [k_M : k_L][k_L : k]$

$$= f_{M/L} f_{L/K} .$$

(ii) (i) + (*) $\square$

Definition 13.1: The extension L/K is said to be

$$\begin{cases} \text{unramified} & \text{if } e_{L/K} = 1 \iff f_{L/K} = [L:K] \\ \text{ramified} & \text{if } e_{L/K} > 1 \iff f_{L/K} < [L:K] \\ \text{totally ramified} & \text{if } e_{L/K} = [L:K] \iff f_{L/K} = 1 \end{cases}$$

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