Solutions to Hungerford Chapter 5.1

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EXERCISE 1.

(a) Show that $[F : K] = 1$ if and only if $F = K$.

(b) Show that if $[F : K]$ is prime, then there are no intermediate fields between F and K.

(c) If $u \in F$ has degree n over K, then n divides $[F : K]$.

Proof:

(a) If $[F : K] = 1$ then F is a 1-dimensional vector space over K. Hence there exists some element $x \in F$ such that for all other $y \in F$ there is some $k \in K$ such that $y = kx$. Define a map $\varphi : K \longrightarrow F$ by $\varphi(k) = kx$. This is surjective by the previous observation and is injective as it is a map of fields. Hence it is an isomorphic.

(b) If E is a field such that $F/E/K$, then $[F : E][E : K] = [F : K]$. Since $[F : K]$ is prime, either $[E : K] = 1$ or $[F : E] = 1$. In the first case $E \cong K$ and in the second $E \cong F$ by part (a). Hence there are no proper intermediate field extensions.

(c) If $u \in F$ has degree n over K, then the field extension $K(u)$ of K is degree n. Since $K(u) \subseteq F$, $K(u)$ is an intermediate extension and hence $[F : K(u)][K(u) : K] = [F : K] \implies [F : K(u)]n = [F : K]$, giving us the result.

□