Algebraic Topo logy Homework 2

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§ ***Problems from 1.2***

EXERCISE 1.9

EXERCISE 1.10

EXERCISE 1.12

EXERCISE 1.14

EXERCISE 1.21 Show that the join $X * Y$ of two nonempty space X and Y is simply-connected if X is path-connected.

Proof. Let $(x, y, t), (u, v, s) \in X * Y$. We regard $X * Y$ as the space $X \times Y \times I/\sim$ where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.

We first show that $X * Y$ is path connected. There exists a path from (x, y, t) to $(x, y, 0)$, call it γ_1. Because X is path connected, there exists a path from $(x, y, 0)$ to $(u, y, 0)$ which is identified with $(u, v, 0)$ in $X * Y$. We call the path from $(x, y, 0)$ to $(u, v, 0)$ γ_2. Finally, call the path from $(u, v, 0)$ to (u, v, s) γ_3. The path $\gamma_3\gamma_2\gamma_1$ starts at (x, y, t) and ends at (u, v, s). Therefore, there is path connecting any two points in $X * Y$, so $X * Y$ is path connected.

In order to apply Van Kampen's theorem, we must make a choice of open sets with which to express $X * Y$. These must be open sets whose intersection has a familiar fundamental group. Let $A = X \times Y \times [0, 1)/\sim$ and $B = X \times Y \times (0, 1]/\sim$. These sets are open, and their intersection is precisely $X \times Y \times (0, 1)$ *without* the equivalence relation. Since we can continuously retract the copies of I extending from X down onto X, we can deformation retract A onto X. Similarly, we can deformation retract B onto Y. This is similar to the process in problem 0.6(a) in Hatcher. Finally, by retracting either end of the intervals onto the point {1/2}, we can deformation retract $A \cap B$ to $X \times Y \times \{\frac{1}{2}\}$, which is homeomorphic to $X \times Y$.

This means $\pi_1(A \cap B) \approx \pi_1(X) \times \pi_1(Y)$, $\pi_1(A) \approx \pi_1(X)$, and $\pi_1(B) \approx \pi_1(Y)$. Van Kampen tells us that

$$\pi_1(X * Y) \approx \pi_1(A) * \pi_1(B)/N$$

where N is generated by elements of the form $\iota_A(\omega)\iota_B(\omega)^{-1}$ where $\iota_A : \pi_1(A \cap B) \hookrightarrow \pi_1(A)$ and $\iota_B : \pi_1(A \cap B) \hookrightarrow \pi_1(B)$ are the homomorphisms induced by the inclusions. However, as we concluded above, $\pi_1(A \cap B) \approx \pi_1(X) \times \pi_1(Y)$. This means that ι_A and ι_B are actually surjective projections, and therefore the elements of the form $\iota_A(\omega)\iota_B(\omega)^{-1}$ are equivalently ab^-1, $a \in \pi_1(A)$ and $b \in \pi_1(B)$. By modding out by N we are thus actually identifying *all* points. Thus,

$$\pi_1(X * Y) \approx \pi_1(A) * \pi_1(B)/N = 1$$

□