Lecture 9

1 Theorem 7.11: (Ostrowski's Theorem) Any non-trivial absolute value on $\mathbb{Q}$ is equivalent to either the usual abs. value $|\cdot|_\infty$ or the p-adic abs. value $|\cdot|_p$ for some prime p.

Proof: Case 1: $|\cdot|$ is archimedean.

We fix $b>1$ an integer such that $|b|>1$ (exists by Lemma 7.9). Let $a>1$ be an integer and write b^n in base a:

$$b^n = c_m a^m + c_{m-1} a^{m-1} + \dots + c_0$$

$$0 \le c_i < a$$

Let $B = \max\limits_{0 \le c < a} (|c|)$, then we have

$$|b^n| \le (m+1) B \max(|a|^m, 1)$$

$$\Rightarrow |b| \le \underbrace{[(n(\log_a b)+1) B]^{1/n}}_{\to 1 \text{ as } n \to \infty} \max(|a|^{\log_a b}, 1)$$

$$\Rightarrow |b| \le \max(|a|^{\log_a b}, 1)$$

Then $|a|>1$ and $|b| \le |a|^{\log_a b}$ $(*)$

Switching the roles of a and b, we obtain

2 $$|a| \le |b|^{\log_b a} \quad (**)$$

$(*)+(**) \Rightarrow \log|a| = \log|b|$ (Use $\log_a b = \log$

$\frac{\log \quad}{\log a} = \frac{\log \quad}{\log b}$ ... $\frac{\log \quad}{\log a} = \frac{\quad}{\log}$

$= \lambda \in \mathbb{R}_{>0}$

$\Rightarrow |a| = a^{\lambda} \quad \forall a \in \mathbb{Z} \quad a > 1.$

$\Rightarrow |x| = |x|_{\infty}^{\lambda} \quad \forall x \in \mathbb{Q}.$

hence $|\cdot|$ equiv. to $|\cdot|_{\infty}$.

Case 2: $|\cdot|$ is non-archimedean.

As in Lemma 7.9, we have $|n| \leq 1 \ \forall n \in$

Since $|\cdot|$ is non-trivial, $\exists n \in \mathbb{Z}_{>1}$ s.t. $|n| < 1$.

Write $n = p_1^{e_1} \cdots p_r^{e_r}$ decomposition into prime factors. Then $|p| < 1$, some $p \in \{p_1, \ldots, p$

Suppose $|q| < 1$, some prime q, $q \neq p$.

Write $1 = rp + sq \quad r, s, \in \mathbb{Z}$.

Then $1 = |rp + sq|$

$\leq \max(|rp|, |sq|) < 1 \quad \#$

3 Thus $|p| = \alpha < 1$ and $|q| = 1 \ \forall$ primes $q \neq p$.

$\Rightarrow |\cdot|$ is equivalent to $|\cdot|_p$. $\square$

Theorem 7.12: Let $(K, |\cdot|)$ be a non-arch. local field of mixed char. Then K is a finite extension of $\mathbb{Q}_p$.

Pf. ... K mixed char $\Rightarrow$ char $K = 0$

Proof: K mixed char $\Rightarrow$ char $K = 0$

$\Rightarrow \mathbb{Q} \subseteq K$.

K non-arch. $\Rightarrow |\cdot||_{\mathbb{Q}} \sim |\cdot|_p$ for some p prime

K complete $\Rightarrow \mathbb{Q}_p \subseteq K$.

Suffices to show $\mathcal{O}_K$ finite as a $\mathbb{Z}_p$-module.

Let $\pi \in \mathcal{O}_K$ unif. v normalised val.

Let $v(p) = e$.

$$\mathcal{O}_K/p\mathbb{Z} = \mathcal{O}_K/\pi^e\mathcal{O}_K \supseteq \pi\mathcal{O}_K/\pi^e\mathcal{O}_K \supseteq \dots \pi^{e-1}\mathcal{O}_K/\pi^e\mathcal{O}$$

$\mathcal{O}_K/\pi\mathcal{O}_K$ finite $\Rightarrow$ successive quotients are finite

$\Rightarrow \mathcal{O}_K/p\mathbb{Z}$ finite

$\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{O}_K/p\mathbb{Z}$, so $\mathcal{O}_K/p\mathbb{Z}$ fin. dim $\mathbb{F}_p$. v.s.

4 Let $x_1, \dots, x_n \in \mathcal{O}_K$ coset reps for $\mathbb{F}_p$ basis of $\mathcal{O}_K/p\mathbb{Z}$.

$y \in \mathcal{O}$, Prop 3.5 (ii) $\Rightarrow y = \sum_{i=0}^{\infty}\left(\sum_{j=1}^{n} a_{ij}x_j\right)p^i$

$$= \sum_{j=1}^{n}\underbrace{\left(\sum_{i=0}^{\infty} a_{ij}p^i\right)}_{\in \mathbb{Z}_p} x_j \quad \text{some } a_{ij} \in \mathbb{Z}_p$$

$\Rightarrow \mathcal{O}_K$ finite over $\mathbb{Z}_p$. $\square$

Example sheet 2: K complete archimedean field. Then $K \cong \mathbb{R}$ or $\mathbb{C}$.

In summary: K a local field. Then either

(i) $K \cong \mathbb{R}$, $\mathbb{C}$ ...

(i) $K = \mathbb{R}, \mathbb{C}$ - arch.

(ii) $K \cong \mathbb{F}_{p^n}((t))$ - non-arch. equal char.

(iii) K is a finite extension of $\mathbb{Q}_p$ - non-arch mixed char.

~~§ Global fields~~

Definition 8.1

A global field is a field which is either:

(i) An algebraic number field

(ii) A global function field, i.e. a finite extension of $\mathbb{F}_p(t)$.

Lemma 8.2: Let $(K, |\cdot|)$ be a complete discretely valued field, L/K a finite Galois extension with abs. value $|\cdot|_L$ extending $|\cdot|$. Then for $x \in L$ and $\sigma \in Gal(L/K)$, we have

$$|\sigma(x)|_L = |x|_L$$

Proof: Since $x \mapsto |\sigma(x)|_L$ is another abs. value on L extending K, Lemma follows from uniqueness of $|\cdot|_L$. $\square$

Lemma 8.3: (Krasner's Lemma)

Let $(K, |\cdot|)$ a complete discretely valued field. Let $f(X) \in K[X]$ be a separable

irreducible ...

irreducible polynomial with roots $\alpha_1, \dots, \alpha_n \in \overline{K}$ ← sep. closure of K

Suppose $\beta \in \overline{K}$ with
$$|\beta - \alpha_1| < |\beta - \alpha_i| \text{ for } i = 2, \dots, n.$$
Then $\alpha_1 \in K(\beta)$.

Proof: Let $L = K(\beta)$, $L' = L(\alpha_1, \dots, \alpha_n)$

Then L'/L is a Galois extension. Let $\sigma \in \mathrm{Gal}(L'/L)$

We have $|\beta - \sigma(\alpha_1)| = |\sigma(\beta - \alpha_1)|$
$$= |\beta - \alpha_1| \quad \text{(Lemma 8.2)}$$
$\Rightarrow \sigma(\alpha_1) = \alpha_1 \Rightarrow \alpha_1 \in K(\beta)$. □

Proposition 8.4: Let $(K, |\cdot|)$ be complete discretely valued field and $f(X) = \sum_{i=0}^{n} a_i X^i \in \mathcal{O}_K[X]$ be a separable irreducible monic polynomial. Let $\alpha \in \overline{K}$ be a root of f. Then $\exists \varepsilon > 0$ s.t. for any $g(X) = \sum_{i=0}^{n} b_i X^i \in \mathcal{O}_K[X]$ monic with $|a_i - b_i| < \varepsilon$, there exists a root β of $g(X)$ s.t. $K(\alpha) = K(\beta)$.

Proof: Let $\alpha = \alpha_1, \dots, \alpha_n \in \overline{K}$ be the roots of f which are nec. distinct. Then $f'(\alpha) \neq 0$.

We choose ε sufficiently small s.t.
$$|g(\alpha_1)| < |f'(\alpha_1)|^2$$

and $|f'(\alpha_1) - g'(\alpha_1)| < |f'(\alpha_1)|$

Then we have $|g(\alpha_1)| < |f'(\alpha_1)|^2 \overset{\text{Lemma 1.5}}{=} |g'(\alpha_1)|^2$

By Hensel's Lemma applied to field $K(\alpha_1)$, there exists $\beta \in K(\alpha_1)$ s.t. $g(\beta) = 0$ and $|\beta - \alpha_1| < |g'(\alpha_1)|$.

$$|g'(\alpha_1)| = |f'(\alpha_1)| = \prod_{i=2}^{n} |\alpha_1 - \alpha_i| \leq |\alpha_1 - \alpha_i|$$

for $i = 2, \dots, n$

(Use $|\alpha_1 - \alpha_i| \leq 1$ since α_i integral)

Since $|\beta - \alpha_1| < |g'(\alpha_1)| = |f'(\alpha_1)|$

$$\leq |\alpha_1 - \alpha_i| \overset{\text{Lemma 1.5}}{=} |\beta - \alpha_i|$$

$i = 2 \dots n$.

Klasner's Lemma $\Rightarrow \alpha \in K(\beta)$

$\Rightarrow K(\alpha) = K(\beta)$. $\square$