Part III Commutative Algebra

Example Sheet III, 2021, Solutions

1. (a) We note the given exact sequence is an exact sequence of graded S-modules, as multiplication by f takes $S(-d)_e = S_{e-d}$ to S_e. As length is additive in exact sequences, we have

$$F_{S/(f)}(j) = F_S(j) - F_{S(-d)}(j) = F_S(j) - F_S(j-d).$$

However, $F_S(j)$ is the dimension of the vector space of homogeneous polynomials of degree j in $n+1$ variables, which was calculated in lecture to be $\binom{j+n}{n}$. Thus for $j \geq d$, $F_{S/(f)}(j)$ agrees with the polynomial in j:

$$\binom{j+n}{n} - \binom{j+n-d}{n},$$

so the Hilbert polynomial is

$$f_{S/(f)} = \frac{(x+n)(x+n-1)\cdots(x+1)}{n!} - \frac{(x+n-d)(x+n-d-1)\cdots(x-d+1)}{n!}.$$

Note that when $n=2$, we obtain

$$\frac{1}{2}[(x+2)(x+1) - (x+2-d)(x+1-d)] = dx - \frac{d(d-3)}{2} = dx + \left(1 - \frac{(d-1)(d-2)}{2}\right).$$

You may or may not recognize $(d-1)(d-2)/2$ as the genus of a non-singular curve of degree d in the projective plane; this is not a coincidence. So some topology emerges from the Hilbert function!

(b) There is an exact sequence

$$0 \longrightarrow S(-d-e) \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} S(-d) \oplus S(-e) \xrightarrow{(f,g)} S \longrightarrow S/(f,g) \longrightarrow 0.$$

Here we use standard matrix notation for linear maps, e.g., the first non-trivial map is $h \mapsto (gh, -fh)$. We need to check exactness. Surjectivity on the right is obvious, and the image of the map (f,g) is the ideal generated by f and g, hence exactness at S is obvious. For exactness at $S(-d) \oplus S(-e)$, it's immediate that $\operatorname{im}\begin{pmatrix} g \\ -f \end{pmatrix} \subseteq \ker(f,g)$. Conversely, suppose that $(h,k) \in \ker(f,g)$. Then $hf + kg = 0$. But since f,g are coprime, this implies g divides h and f divides k, i.e., there are polynomials a,b such that $h = ag$, $k = bf$, so that $0 = hf + kg = agf + bfg = (a+b)fg$ so $a = -b$. Thus (h,k) is the image of $a \in S(-d-e)$, showing exactness.
Injectivity on the left is immediate because S is an integral domain.
Thus we get by additivity of lengths that

$$f_{S/(f,g)} = \binom{x+n}{n} - \binom{x+n-d}{n} - \binom{x+n-e}{n} + \binom{x+n-d-e}{n}.$$

2. (a) For $I = (f)$, note the top two degree terms in the expression for $f_{S/(f)}$ given are

$$\begin{aligned} & \frac{x^n}{n!} + \frac{n(n+1)/2}{n!}x^{n-1} - \left(\frac{x^n}{n!} + \frac{n(n+1)/2 - nd}{n!}x^{n-1}\right) \\ = \quad & \frac{dx^{n-1}}{(n-1)!}. \end{aligned}$$

Thus the degree is d. A similar slightly more tedious calculation for $I = (f,g)$ yields degree de, or we may use the result in (b).

(b) We have a short exact sequence

$$0 \longrightarrow (S/I)(-e) \xrightarrow{\cdot f} S/I \longrightarrow S/(I + (f)) \longrightarrow 0.$$

Note we need f not a zero-divisor in S/I for injectivity on the left. Then we get the identity on Hilbert polynomials

$$f_{S/(I+(f))}(x) = f_{S/I}(x) - f_{S/I}(x-e).$$

To see what the leading term of this difference is, it is enough to calculate, with $\delta = \deg f_{S/I}$,

$$\frac{dx^\delta}{\delta!} - \frac{d(x-e)^\delta}{\delta!} = \frac{dx^\delta}{\delta!} - \left(\frac{dx^\delta}{\delta!} - \frac{de\delta x^{\delta-1}}{\delta!} + \cdots\right) = \frac{dex^{\delta-1}}{(\delta-1)!} + \cdots,$$

where $\cdots$ represents lower order terms. Thus the degree is de.

3. First, if A is any ring and $\mathfrak{p} \subseteq A$ a prime ideal, $S = A \setminus \mathfrak{p}$, note that by the fact localization preserves exactness, $S^{-1}(A/\mathfrak{p}) \cong S^{-1}A/S^{-1}\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. On the other hand, by Example Sheet 1, Q6(c), $S^{-1}(A/\mathfrak{p})$ can be identified with $((A/\mathfrak{p}) \setminus \{0\})^{-1}(A/\mathfrak{p})$, the field of fractions of $A/\mathfrak{p}$. In particular, in the case that $\mathfrak{p} = \mathfrak{m}$ is maximal, we have $A/\mathfrak{m}$ already a field and $A/\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$.

   Continuing in the case of a maximal ideal, let $\tilde{\mathfrak{m}} = \mathfrak{m}^e = \mathfrak{m}A_\mathfrak{m}$, and we claim $\mathfrak{m}^k/\mathfrak{m}^{k+1} \cong \tilde{\mathfrak{m}}^k/\tilde{\mathfrak{m}}^{k+1}$. To see this, first note that in general given a homomorphism $\varphi : A \to B$, ideals $I, J \subseteq A$, then $(IJ)^e = I^e J^e$ from the definition of extension and ideal product. Thus $\tilde{\mathfrak{m}}^k = (\mathfrak{m}^k)^e = S^{-1}\mathfrak{m}^k$, where $S = A \setminus \mathfrak{m}$. Now we have an exact sequence of A-modules

$$0 \to \mathfrak{m}^{k+1} \to \mathfrak{m}^k \to \mathfrak{m}^k/\mathfrak{m}^{k+1} \to 0.$$

   Localizing at S then gives the exact sequence

$$0 \to \tilde{\mathfrak{m}}^{k+1} \to \tilde{\mathfrak{m}}^k \to S^{-1}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \to 0.$$

   However, since $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a $A/\mathfrak{m} = A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$-vector space, it again follows easily from Example Sheet I, Q6, (c) that $S^{-1}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) \cong \mathfrak{m}^k/\mathfrak{m}^{k+1}$.

4. (a) Let $a_1, \ldots, a_n$ be a finite generating set for I. Then there is a surjective homomorphism $(A/I)[x_1, \ldots, x_n] \to \mathrm{gr}_I(A)$ taking $x_i \mapsto a_i \in I/I^2$. Thus $\mathrm{gr}_I(A)$ is a quotient of a polynomial ring over the Noetherian ring A/I, hence is Noetherian.

   (b) Suppose to the contrary that there exists non-zero $a, b \in A$ with $ab = 0$. By Krull's theorem, $\bigcap_{n=0}^\infty \mathfrak{m}^n = 0$. Thus there exists $n, m \geq 0$ such that $a \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$ and $b \in \mathfrak{m}^m \setminus \mathfrak{m}^{m+1}$. Let $\bar{a}, \bar{b}$ be the corresponding non-zero elements of $\mathrm{gr}_\mathfrak{m}(A)$ viewed as non-zero elements of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ and $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ respectively. Then $\bar{a}\bar{b} = 0$ as $ab = 0$, contradicting $\mathrm{gr}_\mathfrak{m}(A)$ being a domain.

5. Define a linear map $\theta : k[x_1, \ldots, x_n] \to k^n$ by $\theta(f) = ((\partial f/\partial x_1)(0), \ldots, (\partial f/\partial x_n)(0))$. It is immediate that $\theta(x_1), \ldots, \theta(x_n)$ form a basis for k^n and that $\theta(\mathfrak{m}^2) = 0$. Thus θ defines an isomorphism $\theta : \mathfrak{m}/\mathfrak{m}^2 \to k^n$.

   The rank of the Jacobian matrix may now be interpreted as the dimension of $\theta(I)$ under the map $\theta : \mathfrak{m} \to k^n$, or equivalently as the dimension of $\theta((I + \mathfrak{m}^2)/\mathfrak{m}^2)$, viewing $(I + \mathfrak{m}^2)/\mathfrak{m}^2$ as a subspace of $\mathfrak{m}/\mathfrak{m}^2$.

   Now let $\tilde{\mathfrak{m}}$ be the maximal ideal of A, the extension of the ideal $\bar{\mathfrak{m}} = \mathfrak{m}/I$ of $k[x_1, \ldots, x_n]/I$. Note then that by Question 4,

$$\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 = \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 = \mathfrak{m}/(I + \mathfrak{m}^2).$$

   Thus, noting $k = A/\tilde{\mathfrak{m}}$, we have

$$\dim_k \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 = \dim_k \mathfrak{m}/\mathfrak{m}^2 - \dim_k (I + \mathfrak{m}^2)/\mathfrak{m}^2 = n - \mathrm{rank}(\text{Jacobian matrix}).$$

   Thus A is regular if and only if the rank of the Jacobian matrix is $n - \dim A$, so that $\dim A = \dim_k \tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2$.

6. Let G be the subgroup of $\prod_{i=1}^\infty G_i$ defined by

$$G = \{(a_1, a_2, \ldots) \mid \phi_n(a_{n+1}) = a_n\}.$$

   Further, we take g_n to be the composition of the inclusion $G \hookrightarrow \prod_{i=1}^\infty G_i$ with the projection onto the ith factor. That this is a subgroup (subring, submodule) is immediate from the definition of homomorphism. We then show the universal property, so suppose given $h_n : H \to G_n$ satisfying the necessary compatibilities. Then since a map $h : H \to G$ must satisfy $g_n \circ h = h_n$, we have no choice but to define $h(a) = (h_1(a), h_2(a), \ldots)$. That this element lives in G follows from $\phi_n(h_{n+1}(a)) = h_n(a)$. Thus the map h exists and is unique.

7. First note from the construction of the inverse limit or from the universal property, the inverse limit is functorial, i.e., given maps $A_n \to B_n$ compatible with $\phi_n^A : A_{n+1} \to A_n$ and $\phi_n^B : B_{n+1} \to B_n$, we obtain a map between the inverse limits of the two systems.

   From the construction of the inverse limit in Question 6, injectivity of the left is immediate. Write $f_n : A_n \to B_n$, $g_n : B_n \to C_n$, and write A, B, C for the inverse limits and $f : A \to B$, $g : B \to C$ for the induced maps. Then $\mathrm{im} f \subseteq \ker g$ is immediate as $g \circ f = 0$. We need to prove the reverse inclusion. So let $(b_1, b_2, \ldots) \in \ker g$, so that $g_n(b_n) = 0$. Then there exists $a_n \in A_n$ with $f_n(a_n) = b_n$ by exactness of the sequence for A_n, B_n, C_n. Further, $f_n \circ \phi_n^A(a_{n+1}) = \phi_{n+1}^B(f_{n+1}(a_{n+1})) = \phi_{n+1}^B(b_{n+1}) = b_n$, so $\phi_n^A(a_{n+1}) = a_n$ by injectivity of f_n. Thus $(a_1, a_2, \ldots) \in A$. This shows the desired exactness.

For surjectivity on the right, let $c = (c_1, c_2, \ldots) \in C$. We may lift each c_n to $b_n \in B_n$ with $g_n(b_n) = c_n$. Now $g_n(\phi_n^B(b_{n+1}) - b_n) = \phi_n^C(c_{n+1}) - c_n = 0$, and thus there exists an $a_n \in A_n$ with $f_n(a_n) = \phi_n^B(b_{n+1}) - b_n$.

Using surjectivity of all ϕ_n^A, we may choose for each i a sequence $a_i = a_{i,i}, a_{i,i+1}, a_{i,i+2}, \ldots$ with $a_{i,j} \in A_j$ and $\phi_j^A(a_{i,j+1}) = a_{i,j}$. We then take $b'_n = b_n - f_n(\sum_{i=1}^{n-1} a_{i,n})$. Then

$$\begin{aligned}
\phi_n^B(b'_{n+1}) &= \phi_n^B(b_{n+1}) - f_n(\sum_{i=1}^{n-1} \phi_n^A(a_{i,n+1})) - f_n(\phi_n^A(a_{n,n+1})) \\
&= \phi_n^B(b_{n+1}) - f_n(\sum_{i=1}^{n-1} a_{i,n}) - (\phi_n^B(b_{n+1}) - b_n) \\
&= b_n - f_n(\sum_{i=1}^{n-1} a_{i,n}) = b'_n.
\end{aligned}$$

Thus $b' = (b'_1, b'_2, \ldots) \in B$ and $g(b') = c$, showing surjectivity.

8. Let $g, g' \in G$. We wish to find open neighbourhoods U, U' of g, g' which are disjoint. Choose an n such that $g' - g \notin G_n$. Then $(g + G_n) \cap (g' + G_n) = \emptyset$ as G_n is a subgroup. Thus $g + G_n$, $g' + G_n$ are disjoint open neighbourhoods of g, g' respectively. Thus G is Hausdorff.

   To show $\widehat{G}$ is an abelian group, we take addition of two Cauchy sequences g_n, g'_n to be the sequence $g_n + g'_n$. Note this is still Cauchy. In fact, g_n being Cauchy means for each $m \geq 1$, there exists an N such that for all $n, n' \geq N$, $g_n - g_{n'} \in G_m$. Thus for a given m, taking N large enough to work for both sequences g_n, g'_n, we have for $n, n' \geq N$ that $(g_n + g'_n) - (g_{n'} + g'_{n'}) \in G_m$. Thus $g_n + g'_n$ is Cauchy. We also need to show that addition is well-defined, but this is similarly straight-forward. The group axioms then follow immediately from the corresponding axioms for G, with 0 the constant sequence with value 0.

   The map $G \to \widehat{G}$ is given by $g \mapsto \{g_n = g\}$, the constant sequence. Note this is equivalent to zero iff for all m, there exists an N such that for all $n \geq N$, $g_n - 0 \in G_m$, i.e., $g \in G_m$ for all m. Thus the kernel is $\bigcap_{n=1}^{\infty} G_n$.

   We define a homomorphism $\lim_{\leftarrow} G/G_n \to \widehat{G}$. Let $(\bar{g}_1, \bar{g}_2, \ldots)$ be an element of the inverse limit. For each n, choose a lift $g_n \in G$ of $\bar{g}_n \in G/G_n$. I claim $\{g_n\}$ now forms a Cauchy sequence. Indeed, for a given $m > 0$, if $n, n' \geq m$, necessarily $g_n, g_{n'}$ have the same image in G/G_m, and hence $g_n - g_{n'} \in G_m$. Next we need to check that the equivalence class of Cauchy sequence is independent of the choice of lifting. But if g_n, g'_n are two choices of lifting, then $g_n - g'_n \in G_n$, so $\{g_n\}$, $\{g_{n'}\}$ give equivalent Cauchy sequences. That this map is a homomorphism is obvious.

   We now need to check injectivity and surjectivity. So suppose given $(\bar{g}_n)$ with a lift $\{g_n\}$ giving a Cauchy sequence equivalent to the constant sequence 0. Thus for each $m \geq 1$, there exists an N such that for all $n \geq N$, $g_n \in G_m$. Thus this tells us that $\bar{g}_i = 0$ for $i < m$. Since m is arbitrary, this shows all $\bar{g}_i = 0$. Hence the map is injective.

   For surjectivity, let $\{g_n\}$ be a Cauchy sequence. It follows from the definition of Cauchy sequence that the sequence of induced elements $\bar{g}_n \in G/G_m$ is eventually constant. Let $h_m \in G/G_m$ be this element. Then necessarily the image of h_{m+1} under the projection $G/G_{m+1} \to G/G_m$ is h_m, and $(h_1, h_2, \ldots)$ forms an element of the inverse limit. We thus just need to show the image of this in $\widehat{G}$ agrees with the eqiuvalence class of $\{g_n\}$. However, to compute the image, we may choose $m_1 < m_2 < m_3 < \cdots$ and lifts $\bar{h}_i = g_{m_i}$ of h_i to G; we just need to take m_i large enough so that $g_n = h_i \bmod G_i$ for $n \geq m_i$. But $\{g_{m_i}\}$ is a subsequence of $\{g_i\}$, hence gives an equivalent Cauchy sequence.

9. (a) Elements of the ring $k[[x_1, \ldots, x_n]]$ are formal series $\sum_{i_1,\ldots,i_n} c_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$. Products are defined using the usual product of power series. The ring $\mathbb{Z}_p$ consists of formal sums $\sum_{i=0}^{\infty} a_i p^i$ with $0 \leq a_i \leq p - 1$. These can be viewed as numbers in base p, but with the expansions unbounded to the left. The product can be thought of as the ordinary product in base p, again possibly going infinitely far to the left.

   (b) Note we have for each n that $M_3/I^n M_3 \cong M_2/(M_1 + I^n M_2)$, and an exact sequence

   $$0 \to (M_1 + I^n M_2)/I^n M_2 = M_1/(M_1 \cap I^n M_2) \to M_2/I^n M_2 \to M_2/(M_1 + I^n M_2) \to 0$$

   Noting $M_1/(M_1 \cap I^{n+1} M_2) \to M_1/(M_1 \cap I^n M_2)$ is always surjective, we then get from Question 7 an exact sequence

   $$0 \to \varprojlim M_1/(M_1 \cap I^n M_2) \to \widehat{M_2} \to \widehat{M_3} \to 0,$$

   and we just need to show the first inverse limit is isomorphic to $\widehat{M_1}$.

   By Question 8, it is sufficient to show that the topologies induced by the filtrations $I^n M_1$ and $M_1 \cap I^n M_2$ are the same. To show this, one needs to show that for each $n \gg 0$ there exists an n' such that

$I^{n'}M_1 \subseteq M_1 \cap I^n M_2$ (which is trivial since we may take $n = n'$) and similarly for each $n \gg 0$ there exists an n' such that $M_1 \cap I^{n'} M_2 \subseteq I^n M_1$. However, this is essentially the content of the Artin-Rees theorem, which is why we need the Noetherian and finitely generated hypotheses. There is an r such that $M_1 \cap I^{n'} M_2 = I^{n'-r}(M_1 \cap I^r M_2) \subseteq I^{n'-r} M_1$ for $n' > r$. Thus we may take $n' = n + r$.

(c) The canonical homomorphism is $m \mapsto (m + IM, m + I^2 M, \ldots)$. The kernel is precisely $\bigcap_{n=1}^\infty I^n M$. Alternatively, use Question 8.

(d) For the existence of the homomorphism, we just need to show that there is a map $\widehat{A} \otimes_A \widehat{M} \to \widehat{A} \otimes_{\widehat{A}} \widehat{M}$. But this comes from the existence of an A-bilinear map $\widehat{A} \times \widehat{M} \to \widehat{A} \otimes_{\widehat{A}} \widehat{M}$ given by $(a, m) \mapsto a \otimes m$.
It is easy to see I-adic completion commutes with direct sums as $(M_1 \oplus M_2)/I^n(M_1 \oplus M_2) = (M_1/I^n M_1) \oplus (M_2/I^n M_2)$.
Thus if $M = A^n$ is free, in fact $M \otimes_A \widehat{A} = \widehat{A}^n = \widehat{M}$, and we have an isomorphism. If M is finitely generated, then we have an exact sequence

$$0 \to K \to F \to M \to 0$$

with $F = A^n$ for some n, and hence a diagram (ignore the $\widehat{K}$ for the moment)

$$\begin{tikzcd}
 & \widehat{A} \otimes_A K \arrow[r] \arrow[d, "\gamma"] & \widehat{A} \otimes_A F \arrow[r] \arrow[d, "\beta"] & \widehat{A} \otimes_A M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{K} \arrow[r] & \widehat{F} \arrow[r, "\delta"'] & \widehat{M} \arrow[r] & 0
\end{tikzcd}$$

with the top line exact. Since we aren't assuming A is Noetherian, we don't know the bottom row is exact, but we do know δ is surjective by Question 7. Indeed, we have an exact sequence

$$0 \to K/(K \cap I^n F) \to F/I^n F \to M/I^n M = F/(K + I^n F) \to 0.$$

Thus by Question 7 we do get an exact sequence of inverse limits with surjectivity on the right, but not necessarily with $\widehat{K}$ on the left. Since β is an isomorphism, it follows then that $\widehat{A} \otimes_A M \to \widehat{M}$ is surjective. If A is Noetherian, then the bottom row is also exact by (b), and since K is then also finitely generated, γ is surjective. A simple diagram chase now shows $\widehat{A} \otimes_A M \to \widehat{M}$ is injective.

10. (a) Since A is Noetherian, I^n is finitely generated, and hence $\widehat{I^n} \cong \widehat{A} \otimes_A I^n$ by Question 8, (d). On the other hand, the inclusion $I^n \hookrightarrow A$ then gives an inclusion $\widehat{A} \otimes_A I^n \hookrightarrow \widehat{A} \otimes_A A \cong \widehat{A}$, and the image of this inclusion is clearly $(I^n)^e$. Applying for $n = 1$ gives the result.

(b) By the argument just given,

$$\widehat{I^n} = (I^n)^e = (I^e)^n = (\widehat{I})^n.$$

(c) By (b) and Question 9(b), $\widehat{A/I^n} \cong \widehat{A}/\widehat{I^n} \cong \widehat{A}/\widehat{I}^n$. On the other hand, $\widehat{A/I^n}$ is the inverse limit of modules

$$M_m = \begin{cases} A/I^m & m \leq n \\ A/I^n & n \geq m \end{cases}$$

As this inverse system becomes stationary, it is clear that the inverse limit agrees with A/I^n. Hence $A/I^n \cong \widehat{A}/\widehat{I}^n$.
The desired result follows by taking successive quotients.

(d) For any $x \in \widehat{I}$, suppose there is a maximal ideal of $\widehat{A}$ not containing x. Then $\mathfrak{m}$ and x generate the unit ideal of $\widehat{A}$, so there exists a $y \in \widehat{A}$, $a \in \mathfrak{m}$ such that $a + xy = 1$. However, we note that $a = 1 - xy$ is in fact invertible in $\widehat{A}$ via the formal power series expansion

$$(1 - xy)^{-1} = 1 + xy + (xy)^2 + \cdots.$$

This contradicts $x \notin \mathfrak{m}$, so x lies in the Jacobson radical of $\widehat{A}$.

Now assuming $(A, \mathfrak{m})$ is local and $I = \mathfrak{m}$, by (c) we have $\widehat{A}/\widehat{\mathfrak{m}} \cong A/\mathfrak{m}$ a field, so $\widehat{\mathfrak{m}}$ is a maximal ideal. By (d), $\widehat{\mathfrak{m}}$ is then contained in the intersection of all maximal ideals, and hence is the unique maximal ideal. Thus $\widehat{A}$ is local.

11. Let $a_1 \in A/\mathfrak{m}$ be a simple root of $\bar{f}$, i.e., $\bar{f}(a_1) = 0$, $\bar{f}'(a_1) \neq 0$. We will construct inductively $a_n \in A/\mathfrak{m}^{n+1}$ such that $f(a_n) = 0 \bmod \mathfrak{m}^{n+1}$ and $a_n = a_{n-1} \bmod \mathfrak{m}^n$.

Denote by $f_n \in (A/\mathfrak{m}^{n+1})[x]$ the image of f. Assume we have constructed a_n, and choose a lift $b \in A/\mathfrak{m}^{n+1}$ of a_n. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f'_{n+1}(b).$$

Note that $f'_{n+1}(b)$ is invertible as by assumption it does not lie in $\mathfrak{m}/\mathfrak{m}^{n+2}$, the unique maximal ideal of $A/\mathfrak{m}^{n+2}$. Further, $\alpha = f_{n+1}(b)/f'_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$, so has square zero. But then $(x+\alpha)^p = x^p + p\alpha x^{p-1}$ for any $p \geq 0$, from which it follows that

$$f_{n+1}(a_{n+1}) = f_{n+1}(b-\alpha) = f_{n+1}(b) - \alpha f'_{n+1}(b) = f_{n+1}(b) - f_{n+1}(b) = 0.$$

Thus we get $a = (a_1, a_2, \ldots)$ an element of $\widehat{A}$ with $f(a) = 0$.

12. (a) Note that $y^2 - x^2(1+x)$ is irreducible. This can be proved by trying to write it as a product fg and getting your hands dirty; I will leave out the details. Thus the ideal generated by this polynomial is prime since $k[x,y]$ is a UFD, and hence we have an integral domain.

    On the other hand, in $k[[x,y]]$, we may factor

    $$y^2 - x^2(1+x) = (y - x\sqrt{1+x})(y + x\sqrt{1+x}),$$

    where we use the Taylor series expansion

    $$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots.$$

    Thus the two factors are zero-divisors in the ring.

    (b) To see that $\operatorname{gr}_{\mathfrak{m}} A$ is not an integral domain, note that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ defines a non-zero element of degree 1 in $\operatorname{gr}_{\mathfrak{m}} A$. But $x^2 = y^3 \in \mathfrak{m}^3$, so the product is zero.