VIRTUAL FUNDAMENTAL CLASSES, GLOBAL NORMAL CONES AND FULTON'S CANONICAL CLASSES

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Introduction

This note, written in January 1997, grew out of an attempt to understand references [Be], [BeFa] and [LiTi]. In these papers two related but different methods are presented for the construction of a certain Chow class on moduli spaces of stable (parametrized) curves in a projective manifold V, called virtual fundamental class. This class replaces the usual fundamental class of these spaces in the definition of basic enumerative invariants of V involving curves, called Gromov-Witten (GW-) invariants. They are invariant under smooth deformations of V.

Both approaches are based on a globalization of the concept of normal cones of germs of the space under study inside some modelling space, that is $C_{U|M}$ for $U \subset X$ open with $\iota: U \hookrightarrow M$ and M smooth over k. The essential idea of using bundles of cones inside a vector bundle for globalizing virtual fundamental classes is due to Li and Tian. The data needed to glue differs however somewhat in the two constructions.

A proper understanding of the relationship between the two approaches seemed necessary for finding the natural framework for comparison of algebraic virtual fundamental classes with the author's definition in [Si1] of virtual fundamental classes in the symplectic context [Si3].

In a first step Behrend and Fantechi use a generalization of the concept of scheme, called Artin stacks, to make sense of the quotient $C_{U|M}/T_M|_U$. These quotients being unique up to canonical isomorphism they glue to an Artin (cone) stack \mathcal{C}_X intrinsically

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associated to any X. In a second step they need a morphism $\varphi^{\bullet}: [\mathcal{F}^{-1} \to \mathcal{F}^{0}] \to \mathcal{L}_{X}^{\bullet}$ (in the derived category) from a two-term complex of locally free sheaves to the cotangent complex, inducing an isomorphism in H^{0} and an epimorphism in H^{-1} , to cook up an ordinary cone $C(\varphi^{\bullet}) \subset F_{1}$, F_{1} the vector bundle associated to \mathcal{F}^{-1} . Intersection with the zero section finally produces the virtual fundamental class. So a priori the latter depends on the choice of φ^{\bullet} .

This is a very natural and mature approach, that clearly separates the globalization process of the normal cone from the construction of the virtual fundamental class. A possible disadvantage is that in dealing with Artin stacks some of the necessary verifications become rather technical, non-geometric in nature.

Li and Tian circumvent the morphism to the cotangent complex by introducing the notion of "perfect tangent obstruction complex". This is a morphism $\mathcal{F}^{-1} \to \mathcal{F}^0$ of locally free sheaves on X with kernel and cokernel being tangent and obstruction spaces for morphisms to X, compatible with base change. Using relative, formal, "Kuranishi families" as an intermediate object they construct a well-defined cone $C \subset F_1$. Another, less important difference to [BeFa] is the use of an absolute obstruction theory instead of one relative to the space of pre-stable curves.

In a previous version of [LiTi] the slightly stronger claim was made that already a presentation $\mathcal{F}^{-1} \to \mathcal{F}^0$ of Ω_X should suffice to construct the cone. In trying to understand this statement I was lead to the problem of reformulating [BeFa] from the point of view of gluing local cones. Since in the latter reference Artin stacks are used only as book-keeping device rather than as actual spaces it should not come as surprise that one can get along without them (this has already been indicated in op.cit.). Contrary to what I expected, things can be formulated in a rather elegant but direct way via some *yoga of cones bundles*. This part of the paper (Sections 2 and 3) is just a down-to-earth reformulation of (parts of) Sections 2–4 of [BeFa]. Section 1 presents the necessary notations concerning cones and linear spaces, the latter being a convenient way of looking at coherent sheaves for our purposes.

In Section 4 we establish a closed formula for virtual fundamental classes involving only the scheme-theoretic structure of X via Fulton's canonical class (Definition 4.3) and the Chern class of the virtual bundle $F_0 - F_1$, F_i the vector bundle associated to \mathcal{F}^{-i} (Theorem 4.6). This formula was actually found by the author in summer 1995 while searching for a purely algebraic definition of GW-invariants. It should be useful for computations, see the author's recent little survey [Si2].

A few remarks on GW-theory are in order. First, today I consider the yoga of cone bundles in Sections 2 and 3 as one ingredient for the most economic path to algebraic Gromov-Witten invariants. The other ingredients are going over to Deligne-Mumford stacks, and replacing the morphism to the cotangent complex by an obstruction theory. If the latter is defined similar to [Ar], 2.6, rather than in [LiTi], one can show [Si4] that it is locally nothing but a morphism to the cotangent complex as in [BeFa]. Hence the yoga of cone bundles applies to produce the virtual fundamental class.

Second, I would like to illustrate the perspective of the content of Section 4 by the following formula for virtual fundamental classes in Gromov-Witten theory.

Theorem 0.1. Let V be a projective variety, smooth over a field K of characteristic 0, and $R \in A_1(V)$, the first Chow group. If g = 0 or $C := C_{R,g,k}(V)$, the moduli space of stable curves $(C, \mathbf{x}, \varphi : C \to V)$ in V of genus g with k marked points $\mathbf{x} = (x_1, \ldots, x_k)$ and $\varphi_*[C] = R$, is embeddable into a space smooth over $\mathfrak{M}_{g,k}$, then the virtual fundamental class relevant for GW-invariants is

$$\llbracket \mathcal{C} \rrbracket = \left\{ c(\operatorname{ind}_{R,g,k}^V)^{-1} \cap c_F(\mathcal{C}/\mathfrak{M}_{g,k}) \right\}_{d(V,R,g,k)}.$$

Here $\{\,.\,\}_d$ denotes the d-dimensional part of a cycle, $\mathfrak{M}_{g,k}$ is the Artin stack of k-pointed pre-stable curves of genus g, $d(V,R,g,k)=c_1(V)\cdot R+(1-g)\dim V+3g-3$ is the expected dimension, and $c_F(\mathcal{C}/\mathfrak{M}_{g,k})$ is Fulton's canonical class for \mathcal{C} relative $\mathfrak{M}_{g,k}$.

Here $\operatorname{ind}_{R,g,k}^V = F_0 - F_1$ is the virtual vector bundle associated to the partial resolution $\varphi^{\bullet}: [\mathcal{F}^{-1} \to \mathcal{F}^0] \to \mathcal{L}_{\mathcal{C}/\mathfrak{M}_{g,k}}^{\bullet}$ mentioned in the introduction. It represents the (domain of the) perfect relative obstruction theory $(R\pi_*(f_V^T))^{\vee}$ of Behrend [Be] in $K^0(\mathcal{C})$.

A note on categories: To keep things simple we work here in the category of schemes of finite type over a field k, not necessarily algebraically closed or of characteristic 0. The extension to other base schemes is straightforward. For the purpose of GW-theory one also has to replace schemes by (generalizations of) orbifolds, that is Deligne-Mumford stacks in the algebraic category or analytic orbispaces in an analytic context. Again, our results can be easily adapted to these categories.

For GW-theory this is still not sufficient, because $\mathfrak{M}_{g,k}$ is an Artin stack rather than Deligne-Mumford. One can nevertheless give a construction of the relevant cone without ever really using Artin stacks. For instance, Fulton's canonical class relative $\mathfrak{M}_{g,k}$ has the following simple definition: Embed \mathcal{C} into a smooth Deligne-Mumford k-stack N. In the important case g=0 one could take $\mathcal{C}_{\iota_*R,g,k}(\mathbb{P}^N)$, if $\iota:V\hookrightarrow\mathbb{P}^N$ is a closed embedding. Let $q:\mathcal{U}\to\mathcal{C}$ be the universal curve. Then the pull-back of the (virtual) tangent bundle of $\mathfrak{M}_{g,k}$ is

$$T_{\mathfrak{M}} := \mathcal{E}xt_q^1(\omega_{\mathcal{U}/\mathcal{C}}, \mathcal{O}_{\mathcal{U}}) - \mathcal{E}xt_q^0(\omega_{\mathcal{U}/\mathcal{C}}, \mathcal{O}_{\mathcal{U}}),$$

as an element of $K^0(\mathcal{C})$ (the $\mathcal{E}\!\mathit{xt}^i_q$ are the derived functors of $q_* \circ \mathcal{H}\!\mathit{om}$) and

$$c_F(\mathcal{C}/\mathfrak{M}_{g,k}) = \left(c(T_{\mathfrak{M}})^{-1} \cup c(T_N)\right) \cap s(C_{\mathcal{C}/N})$$
$$= c(T_{\mathfrak{M}})^{-1} \cap c_F(\mathcal{C}).$$

Here all sheaves and classes have to be understood in the sense of Deligne-Mumford stacks. With these remarks understood the theorem is a special case of Theorem 4.6.

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1. Cones

1.1. **Linear spaces.** For any algebraic k-scheme X, $\mathbb{A}^1_X = X \times \mathbb{A}^1_k$ has the structure of a ring over X: There are morphisms

$$\alpha: \mathbb{A}^1_X \times_X \mathbb{A}^1_X \longrightarrow \mathbb{A}^1_X \,, \quad \iota: \mathbb{A}^1_X \longrightarrow \mathbb{A}^1_X \,, \quad \mu: \mathbb{A}^1_X \times_X \mathbb{A}^1_X \longrightarrow \mathbb{A}^1_X \,,$$

and sections $n, e: X \to \mathbb{A}^1_X$ fulfilling the usual commutative ring axioms with α as addition, ι as additive inverse, μ as multiplication and n, e as neutral elements for α , μ .

A linear space over X is an \mathbb{A}^1_X -module of finite type over X, that is, an affine morphism $\pi: L \to X$ of finite type together with morphisms

$$a: L \times_X L \longrightarrow L, \quad m: \mathbb{A}^1_X \times_X L \longrightarrow L$$

and a zero section $z: X \to L$, fulfilling the usual module axioms relative X, that is $m \circ (\mu \times \operatorname{Id}_L) = m \circ (\operatorname{Id}_{\mathbb{A}^1_X} \times m)$ as maps from $\mathbb{A}^1_X \times_X \mathbb{A}^1_X \times_X L$ to L etc. By abuse of notation we just write L for the tuple (π, a, m) . In the sequel we will restrict to linear spaces that are *representable*, that is, which locally are closed subspaces of vector bundles with induced linear structure. With the obvious notion of homomorphism of linear spaces over X we get the category $\operatorname{Lin}(X)$ of representable linear spaces over X.

There is an anti-equivalence of categories

$$Lin(X) \longrightarrow Coh(X)$$

to the category of coherent \mathcal{O}_X -modules [EGA-II],§1.7: On objects this associates to $L \in \operatorname{Lin}(X)$ the sheaf $\operatorname{Hom}_{\operatorname{Lin}(X)}(L, \mathbb{A}^1_X)$; in the other direction, $\mathcal{F} \in \operatorname{Coh}(X)$ corresponds to

$$L(\mathcal{F}) := \operatorname{Spec}_{\mathcal{O}_{\mathbf{Y}}} S^{\bullet} \mathcal{F},$$

where $S^{\bullet}\mathcal{F}$ is the symmetric algebra over the \mathcal{O}_X -module \mathcal{F} . For example, the addition operation $a: L(\mathcal{F}) \times_X L(\mathcal{F}) \to L(\mathcal{F})$ comes from the diagonal morphism $\mathcal{F} \to \mathcal{F} \oplus \mathcal{F}$, $f \mapsto (f, f)$ by application of the functor $L = \operatorname{Spec}_{\mathcal{O}_X} \circ S^{\bullet}$. Note that a vector bundle E corresponds to the locally free sheaf $\mathcal{O}(E^{\vee})$, E^{\vee} the dual bundle.

Representable linear spaces are thus just another way to look at coherent sheaves. We will jump freely between both descriptions and use whichever seems more appropriate in a particular context. Note also that $\operatorname{Lin}(X)$ is an abelian category, so it makes sense to talk about monomorphisms, epimorphisms and exact sequences. A monomorphism $\Phi: E \to F$ of linear spaces corresponds to an epimorphism $\varphi: \mathcal{F} \to \mathcal{E}$ of sheaves and is thus a closed embedding of schemes. An epimorphism $\psi: F \to G$ of linear spaces, however, need not be a surjection of schemes (consider the inclusion $\psi: \mathcal{I} \hookrightarrow \mathcal{O}_X$ for any nontrivial ideal sheaf \mathcal{I}).

1.2. **Cones.** A cone C over X is a scheme of the form $\operatorname{Spec}_{\mathcal{O}_X} S^{\bullet}$ where $S^{\bullet} = \bigoplus_{d \geq 0} S^d$ is a graded \mathcal{O}_X -module with $S^0 = \mathcal{O}_X$ and S^{\bullet} generated by $S^1 \in \operatorname{Coh}(X)$. S^{\bullet} as graded algebra is not in general determined up to isomorphism by the scheme C over X. For the grading one needs to distinguish the generating submodule $S^1 = \mathcal{F}$, or, equivalently, a closed embedding $C \hookrightarrow L(\mathcal{F})$ into a linear space. Such datum could be called *polarization* of C. We will only deal with polarized cones in the sequel.

Example 1.1. If X is a closed subscheme of an algebraic k-scheme M with ideal sheaf \mathcal{I} then the cone

$$C_{X|M} = \operatorname{Spec}_{\mathcal{O}_X} (\oplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1})$$

over X is called *normal cone* to X in M. $C_{X|M}$ is naturally embedded into the *normal space* $N_{X|M} = L(\mathcal{I}/\mathcal{I}^2)$ of X in M (to avoid confusion with $(\mathcal{I}/\mathcal{I}^2)^{\vee}$, I would rather not call $N_{X|M}$ normal sheaf as in [BeFa]).

If C, C' are cones over X, then so is $C \oplus C' := C \times_X C'$.

To a polarized cone $C = \operatorname{Spec} S^{\bullet}$ is associated a Chow class on X, its Segre class

$$s(C) := \sum_{r>0} p_* (\xi^r \cap [\mathbb{P}(C)]),$$

where $p: \mathbb{P}(C) := \operatorname{Proj} S^{\bullet} \to X$ is the projection and $\xi = c_1(\mathcal{O}_{\mathbb{P}(C)}(1))$.

We propose the following formulation of the concept of exact sequence of cones [Fu, Expl.4.1.6].

¹"Linear space (over X)" or "linear fiber space" ("Linearer Faserraum") seem to be the classical notation for the "abelian cones" of [BeFa]

Definition 1.2. Let

$$0 \longrightarrow E \stackrel{\Phi}{\longrightarrow} F \stackrel{\Psi}{\longrightarrow} Q \longrightarrow 0 \tag{*}$$

be an exact sequence of linear spaces. Let $C \subset Q$ be a cone and set $\tilde{C} := \Psi^{-1}(C)$. Then (*) restricts to

$$0 \longrightarrow E \longrightarrow \tilde{C} \longrightarrow C \longrightarrow 0$$
.

Sequences of cones of this form will be called exact.

Remark 1.3. Exact sequences of cones might not be very useful unless (*) splits locally. In this case \tilde{C} is locally of the form $C \oplus E$, and as in [Fu, Expl.4.1.6] one can show $s(\tilde{C}) = s(C \oplus E)$. In the non-split case a convenient way to relate the Segre classes of \tilde{C} and C seem to be unknown.

But note that if E is a vector bundle (*) always splits locally, and so we retrieve the definition of exact sequences of cones as in [Fu].

For an exact sequence of cones as in the definition \tilde{C} is preserved by the additive action of E on F. In other words, \tilde{C} wears the structure of an E-module. More generally, if $\Phi: E \to F$ is a homomorphism of linear spaces and $C \subset F$ is a cone then C is called E-cone if C is an E-module via Φ , that is if C is preserved by the additive action of E on F induced by Φ .

Example 1.4. In the situation of Example 1.1 $C_{X|M}$ is a $T_M|_X$ -cone via the natural homomorphism $\Phi: T_M|_X \to N_{X|M}$. On the sheaf level this action of $T_M|_X$ is

$$\bigoplus_{d} \mathcal{I}^{d}/\mathcal{I}^{d+1} \ \longrightarrow \ S^{\bullet}\Omega_{M}|_{X} \otimes \bigoplus_{d} \mathcal{I}^{d}/\mathcal{I}^{d+1},$$

where for $f_i \in \mathcal{I}/\mathcal{I}^2$ the image of $f_1 \cdot \ldots \cdot f_d$ in the direct summand $S^e \Omega_M|_X \otimes \mathcal{I}^{d-e}/\mathcal{I}^{d-e+1}$ of the target is the sum over all partitions $\{i_1, \ldots, i_e\}, \{j_1, \ldots, j_{d-e}\}$ of $\{1, \ldots, n\}$ of terms

$$\mathrm{d}f_{i_1}\cdot\ldots\cdot\mathrm{d}f_{i_e}\otimes f_{j_1}\cdot\ldots\cdot f_{j_{d-e}}.$$

There are many examples of morphisms of linear spaces $E \to F$ and E-cones $C \subset F$ that do not descend to the quotient Q = E/F, for instance the examples in Remark 2.14,3 and in Remark 3.5. However, there is one important class of morphisms where it is always possible, namely for locally split monomorphisms. We first treat the split case:

Lemma 1.5. Let \mathcal{E} , $\mathcal{F} \in \text{Coh}(X)$ and $E = L(\mathcal{E})$, $F = L(\mathcal{F})$ the corresponding linear spaces over X and $C \subset E \oplus F$ an E-invariant closed subscheme with respect to the action of E on the first summand.

Then C is of the form $E \oplus \bar{C}$ for some uniquely determined closed subscheme $\bar{C} \subset F$.

Proof. The statement is local in X, so we may assume $X = \operatorname{Spec} A$, $E = \operatorname{Spec} A[\mathbf{X}]/\langle \mathbf{e} \rangle$, $F = \operatorname{Spec} A[\mathbf{Y}]/\langle \mathbf{f} \rangle$ with $\mathbf{X} = (X_1, \dots, X_r)$, $\mathbf{Y} = (Y_1, \dots, Y_s)$ and $\mathbf{e} = (e_1, \dots, e_k)$, $\mathbf{f} = (f_1, \dots, f_l)$ tuples of linear forms with coefficients in A, $\langle \mathbf{e} \rangle$, $\langle \mathbf{f} \rangle$ the ideals generated by their entries. Then $C = \operatorname{Spec} A[\mathbf{X}, \mathbf{Y}]/I$ with I an ideal containing $\langle \mathbf{e} \rangle + \langle \mathbf{f} \rangle$.

The only possible candidate for \bar{C} is the intersection of C with $0 \oplus F$, that is $\bar{C} = \operatorname{Spec} A[\mathbf{X}, \mathbf{Y}]/(I + \langle \mathbf{X} \rangle) = \operatorname{Spec} A[\mathbf{Y}]/\bar{I}$, with $\bar{I} = \{f(0, \mathbf{Y}) \mid f(\mathbf{X}, \mathbf{Y}) \in I\}$. We have to show that $I = \langle \bar{I} \rangle + \langle \mathbf{e} \rangle$.

C to be E-invariant means that for any $f(\mathbf{X}, \mathbf{Y}) = \sum_{M,N} a_{MN} \mathbf{X}^M \mathbf{Y}^N \in I$

$$f(\mathbf{X} + \mathbf{X}', \mathbf{Y}) = \sum_{M,N} a_{MN} (\mathbf{X} + \mathbf{X}')^M \mathbf{Y}^N \in \langle I \rangle + \langle \varphi(\mathbf{e}) \rangle$$
 (*)

holds in $A[\mathbf{X}, \mathbf{X}', \mathbf{Y}]$, where $\varphi : A[\mathbf{X}] \to A[\mathbf{X}']$, $X_{\mu} \mapsto X'_{\mu}$. Modulo $\langle \mathbf{X} \rangle$ this says

$$f(\mathbf{X}', \mathbf{Y}) \in \langle \bar{I} \rangle + \langle \varphi(\mathbf{e}) \rangle$$

in $A[\mathbf{X}', \mathbf{Y}]$. Replacing \mathbf{X}' by \mathbf{X} we thus get $I \subset \langle \bar{I} \rangle + \langle \mathbf{e} \rangle$.

For the other direction we look at (*) modulo X + X' to conclude

$$f(0,\mathbf{Y}) = \sum_{N} a_{0N} \mathbf{Y}^{N} \in I + \mathbf{e} = I$$

for any $f \in I$, that is $\bar{I} \subset I$.

Proposition 1.6. Let

$$0 \longrightarrow F \longrightarrow E \stackrel{q}{\longrightarrow} Q \tag{*}$$

be an exact sequence of linear spaces with F a vector bundle, and let $C \subset E$ be an F-cone.

Then there exists a unique cone $\bar{C} \subset Q$ such that (*) induces an exact sequence of cones

$$0 \longrightarrow F \longrightarrow C \longrightarrow \bar{C} \longrightarrow 0.$$

In particular, C descends to Q: $C = q^{-1}(\bar{C})$.

Proof. By replacing Q by the closed subspace $E/F \subset Q$ we may assume q to be an epimorphism. Then, since F is a vector bundle, locally (*) splits and we may apply the previous lemma to construct $\bar{C} \subset Q$.

In other words, the proposition says that \bar{C} is the scheme-theoretic quotient of C by the free action of F. This is a convenient way to think about \bar{C} .

2. Going up and down for E-cones

In this section we investigate the behavior of E-cones under morphisms of twoterm complexes, that is commutative squares, in Lin(X). If $\Phi_{\bullet} = (\Phi_0, \Phi_1) : F_{\bullet} =$ $(F_0 \to F_1) \to (E_0 \to E_1)$ is such a morphism the corresponding morphism of coherent sheaves will be written $\varphi^{\bullet} = (\varphi^{-1}, \varphi^0) : \mathcal{E}^{\bullet} = (\mathcal{E}^{-1} \to \mathcal{E}^0) \to \mathcal{F}^{\bullet} = (\mathcal{F}^{-1} \to \mathcal{F}^0)$. Then $\Phi_i = L(\varphi^{-i}), E_i = L(\mathcal{E}^{-i}), F_i = L(\mathcal{F}^{-i})$ for i = 0, 1.

2.1. Going up.

Lemma 2.1. Let $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be a commutative square in Lin(X), and $C \hookrightarrow E_1$ an E_0 -cone. Then $\Phi_1^{-1}(C) \hookrightarrow F_1$ is an F_0 -cone.

Proof. Consider the diagram

$$\begin{array}{ccc}
F_0 \oplus F_1 & \xrightarrow{\alpha} & F_1 \\
 & & & \downarrow^{\Phi_1} \\
E_0 \oplus E_1 & \xrightarrow{\alpha'} & E_1
\end{array}$$

with horizontal arrows the morphisms defining the F_0 - and E_0 -module structures on F_1 and E_1 respectively. By hypothesis $E_0 \oplus C$ is a closed subscheme of $(\alpha')^{-1}(C)$. Thus $F_0 \oplus \Phi_1^{-1}(C) = (\Phi_0 \oplus \Phi_1)^{-1}(E_0 \oplus C)$ is a closed subscheme of $\alpha^{-1}(\Phi_1^{-1}(C))$. \square

By this lemma we are able to make the following definition.

Definition 2.2. (going up) Let $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be a commutative square in Lin(X) and $C \subset E_1$ an E_0 -cone. Then the F_0 -cone

$$\Phi_{\bullet}^!(C) := \Phi_1^{-1}(C)$$

in F_1 is called *pull-back* of C under Φ_{\bullet} .

The pull-back depends only on the homotopy class of φ^{\bullet} (or Φ_{\bullet}).

Proposition 2.3. Let $\varphi^{\bullet}, \psi^{\bullet} : \mathcal{E}^{\bullet} = [\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^{0}] \to \mathcal{F}^{\bullet}$ be homotopic commutative squares in Coh(X). Then for any E_0 -cone $C \subset E_1$

$$\Phi^!_{\bullet}(C) = \Psi^!_{\bullet}(C).$$

Proof. Let $k: \mathcal{E}^0 \to \mathcal{F}^{-1}$ be a homotopy: $\psi^{-1} = \varphi^{-1} + k \circ d$, $\psi^0 = \varphi^0 + d \circ k$. Writing K = L(k) and $\alpha : E_0 \oplus E_1 \to E_1$ for the structure map, Ψ_1 may be decomposed into

$$F_1 \stackrel{(K,\Phi_1)}{\longrightarrow} E_0 \oplus E_1 \stackrel{\alpha}{\longrightarrow} E_1$$
.

Since $E_0 \oplus C \subset \alpha^{-1}(C)$, $(K, \Phi_1)^{-1}(E_0 \oplus C) = \Phi_1^{-1}(C)$ is a closed subscheme of $\Psi_1^{-1}(C)$. But the claim is symmetric in Φ_{\bullet} , Ψ_{\bullet} , hence $\Phi_1^{-1}(C) = \Psi_1^{-1}(C)$.

The next result about functoriality of going up follows directly from the definition.

Proposition 2.4. Let $\Phi_{\bullet}: E_{\bullet} \to F_{\bullet}, \ \Psi_{\bullet}: F_{\bullet} \to G_{\bullet}$ be commutative squares of linear spaces and $C \subset G_1$ a G_0 -cone. Then

$$(\Psi_{\bullet} \circ \Phi_{\bullet})^!(C) = \Phi_{\bullet}^! \circ \Psi_{\bullet}^!(C).$$

2.2. Going down, or push-forward, of F_0 -cones in F_1 to E_1 is a little more subtle. The central tool will be Proposition 1.6. To make this proposition applicable we need a little lemma.

Lemma 2.5. Let $\varphi^{\bullet}: (\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^{0}) \to (\mathcal{F}^{-1} \xrightarrow{d'} \mathcal{F}^{0})$ be a commutative square in Coh(X). Then the complex

$$0 \longrightarrow \mathcal{E}^{-1} \overset{(d,\varphi^{-1})}{\longrightarrow} \mathcal{E}^0 \oplus \mathcal{F}^{-1} \overset{\varphi^0 \circ \operatorname{pr}_1 - d' \circ \operatorname{pr}_2}{\longrightarrow} \mathcal{F}^0 \longrightarrow 0$$

is exact at

- $\begin{array}{ll} i) & \mathcal{F}^0 & \textit{iff $H^0(\varphi^\bullet)$ is surjective} \\ ii) & \mathcal{E}^0 \oplus \mathcal{F}^{-1} & \textit{iff $H^0(\varphi^\bullet)$ is injective and $H^{-1}(\varphi^\bullet)$ is surjective} \\ iii) & \mathcal{E}^{-1} & \textit{iff $H^{-1}(\varphi^\bullet)$ is injective}. \end{array}$

Proof. Chase the diagram

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E}^{-1} \stackrel{d}{\longrightarrow} \mathcal{E}^{0} \longrightarrow \mathcal{Q} \longrightarrow 0$$

$$H^{-1}(\varphi^{\bullet}) \downarrow \qquad \qquad \qquad \downarrow \varphi^{0} \qquad \qquad \downarrow H^{0}(\varphi^{\bullet})$$

$$0 \longrightarrow \mathcal{K}' \longrightarrow \mathcal{F}^{-1} \stackrel{d'}{\longrightarrow} \mathcal{F}^{0} \longrightarrow \mathcal{Q}' \longrightarrow 0$$

If φ^{\bullet} is a quasi-isomorphism we thus get exactness of the stated complex. And φ^{\bullet} , viewed as a commutative square, is cartesian $(\mathcal{E}^{-1} = \mathcal{E}^0 \oplus_{\mathcal{F}^0} \mathcal{F}^{-1})$ iff $H^0(\varphi^{\bullet})$ is injective and $H^{-1}(\varphi^{\bullet})$ is an isomorphism, and it is cocartesian $(\mathcal{F}^0 = (\mathcal{E}^0 \oplus \mathcal{F}^{-1})/\mathcal{E}^{-1})$ iff $H^0(\varphi^{\bullet})$ is an isomorphism and $H^{-1}(\varphi^{\bullet})$ is surjective. Assume now that F_0 is a vector bundle and that $\Phi_{\bullet}: [F_0 \stackrel{D'}{\to} F_1] \to [E_0 \stackrel{D}{\to} E_1]$ induces an isomorphism on H^0 and a closed

embedding of linear spaces on H^1 . If these conditions are satisfied we say that going down is applicable to Φ_{\bullet} . Then

$$0 \longrightarrow F_0 \stackrel{(\Phi_0, -D')}{\longrightarrow} E_0 \oplus F_1 \stackrel{q}{\longrightarrow} E_1$$

is exact (Lemma 2.5, $q = D \circ \operatorname{pr}_1 + \Phi_1 \circ \operatorname{pr}_2$) and we may apply Proposition 1.6.

Definition 2.6. (going down) Let $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be a commutative square in Lin(X), to which going down is applicable (see above), and let $C \subset F_1$ be an F_0 -cone. The unique cone $\bar{C} \subset \text{im } q \subset E_1$ with $q^{-1}(\bar{C}) = E_0 \oplus C$, which exists by Proposition 1.6, is called *push-forward* of C by Φ_{\bullet} , denoted $(\Phi_{\bullet})_!(C)$.

Note that $(\Phi_{\bullet})_!(C)$ is actually an E_0 -cone because $E_0 \oplus C$ is one. And by Proposition 1.6:

Proposition 2.7. If going down is applicable to $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$, and $C \subset F_1$ is an F_0 -cone, there is an exact sequence of cones

$$0 \longrightarrow F_0 \longrightarrow E_0 \oplus C \longrightarrow (\Phi_{\bullet})_!(C) \longrightarrow 0.$$

Remark 2.8. Local freeness of F_0 (or local splittability of the relevant exact sequence of linear spaces) seems to be indispensable, since otherwise $E_0 \oplus C$ need not descend to E_1 . See Remark 2.14,3 for a related example.

As with going up, going down depends only on the homotopy class of Φ_{\bullet} .

Proposition 2.9. Let Φ_{\bullet} , $\Psi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be homotopic morphisms of commutative squares in Lin(X) and $C \subset F_1$ an F_0 -cone. If going down is applicable to Φ_{\bullet} (or, equivalently, to Ψ_{\bullet}) then

$$(\Phi_{\bullet})_!(C) = (\Psi_{\bullet})_!(C).$$

Proof. Let $K: F_1 \to E_0$ be a homotopy between Φ_{\bullet} and Ψ_{\bullet} , that is $\Psi_0 = \Phi_0 + K \circ D'$, $\Psi_1 = \Phi_1 + D \circ K$ $(D: E_0 \to E_1, D': F_0 \to F_1$ the differentials). Then the following diagram

$$0 \longrightarrow F_0 \stackrel{(\Phi_0, -D')}{\longrightarrow} E_0 \oplus F_1 \stackrel{q_\Phi}{\longrightarrow} E_1$$

$$\downarrow Id \downarrow \qquad \qquad \downarrow \downarrow \qquad \qquad \downarrow d \downarrow$$

$$0 \longrightarrow F_0 \stackrel{(\Psi_0, -D')}{\longrightarrow} E_0 \oplus F_1 \stackrel{q_\Psi}{\longrightarrow} E_1$$

with $\chi = (\operatorname{pr}_1 - K \circ \operatorname{pr}_2, \operatorname{pr}_2)$, $q_{\Phi} = D \circ \operatorname{pr}_1 + \Phi_1 \circ \operatorname{pr}_2$ and $q_{\Psi} = D \circ \operatorname{pr}_1 + \Psi_1 \circ \operatorname{pr}_2$, is commutative. Now $\chi^{-1}(E_0 \oplus C) = E_0 \oplus C$ and the conclusion follows from the definition of going down.

We observe also that since $q^{-1}(\bar{C}) = E_0 \oplus C \subset E_0 \oplus F_1$ and $q|_{0 \oplus F_1} = \Phi_1$, $\Phi_1^{-1}(\bar{C}) = C$. In other words:

Proposition 2.10. Whenever going down is applicable to $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ then $\Phi_{\bullet}^{!}$ is a left inverse to $(\Phi_{\bullet})_{!}$, that is

$$\Phi_{\bullet}^{!}(\Phi_{\bullet})_{!}(C) = C$$

for any F_0 -cone $C \subset F_1$.

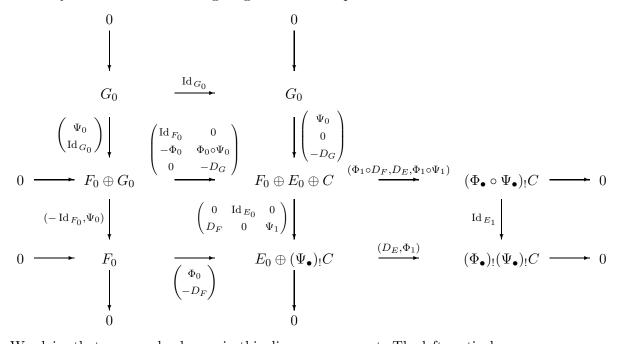
Note that $\Phi^!_{\bullet}$ is generally not right-inverse to $(\Phi_{\bullet})_!$. For example consider $\Phi_{\bullet} = (\mathrm{Id}, \iota) : (F_0 \to F_1) \to (F_0 \to F_1 \oplus N)$ for any linear space N over $X, \iota : F_1 \to F_1 \oplus N$ the inclusion of the first factor and F_0 acting trivially on N. Then for an F_0 -cone of the form $C \oplus N$ it holds $(\Phi_{\bullet})_!\Phi^!(C \oplus N) = C \oplus 0$. Compare however Proposition 2.12.

Going down is functorial:

Proposition 2.11. Let $\Psi_{\bullet}: G_{\bullet} \to F_{\bullet}$, $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be commutative squares of linear spaces to which going down is applicable, and let $C \subset G_1$ be a G_0 -cone. Then

$$(\Phi_{\bullet} \circ \Psi_{\bullet})_!(C) = (\Phi_{\bullet})_!(\Psi_{\bullet})_!(C).$$

Proof. Consider the following diagram of linear spaces and cones:



We claim that rows and columns in this diagram are exact. The left vertical sequence is trivially exact. The upper howizontal sequence is exact by Proposition 2.7 applied to $(\Phi_{\bullet})_!$ and the cone $(\Psi_{\bullet})_!(C) \subset F_1$. Exactness of the middle vertical sequence follows by adding a trivial E_0 -term to the analogous sequence for $(\Psi_{\bullet})_!$ and $C \subset G_1$. For the remaining middle horizontal sequence exactness of the enveloping sequence

$$0 \longrightarrow F_0 \oplus G_0 \longrightarrow F_0 \oplus E_0 \oplus G_1 \longrightarrow E_1 \longrightarrow 0$$

of linear spaces is easy to verify. Again by Proposition 2.7 the preimage $\tilde{C} \subset F_0 \oplus E_0 \oplus G_1$ of $(\Phi_{\bullet} \circ \Psi_{\bullet})_!(C) \subset E_1$ intersects $0 \oplus E_0 \oplus G_1$ in $0 \oplus E_0 \oplus C$, and it is invariant under the action of F_0 on the first factor. Hence $\tilde{C} = F_0 \oplus E_0 \oplus C$, proving exactness of the middle horizontal sequence.

Now exactness of the lower horizontal sequence and of the middle vertical sequence show that the preimage of $(\Phi_{\bullet})_!(\Psi_{\bullet})_!C$ under the composition of epimorphisms $F_0 \oplus E_0 \oplus G_1 \to E_0 \oplus F_1 \to E_1$ from the lower right square equals $F_0 \oplus E_0 \oplus C$. This is the same as the preimage of $(\Phi_{\bullet} \circ \Psi_{\bullet})_!C$. Therefore $(\Phi_{\bullet})_!(\Psi_{\bullet})_!C$ and $(\Phi_{\bullet} \circ \Psi_{\bullet})_!C$ are the same cones in E_1 .

2.3. The case of quasi-isomorphisms. By definition a morphism Φ_{\bullet} of two-term complexes is a quasi-isomorphism if $H^{i}(\Phi_{\bullet})$ is an isomorphism for i=0,1. This is equivalent to requiring that Φ_{\bullet} viewed as a commutative square is cartesian and cocartesian, see Lemma 2.5. Going up and down behaves well with respect to quasi-isomorphisms:

Proposition 2.12. Let $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be a quasi-isomorphism of two-term complexes of linear spaces with F_0 locally free. Then going up and down induces a functorial one-to-one correspondence between F_0 -cones $C \subset F_1$ and E_0 -cones $\bar{C} \subset E_1$.

Proof. In view of Proposition 2.10 it remains to show that if $\bar{C} \subset E_1$ is an E_0 -cone then $\bar{C} = (\Phi_{\bullet})_! \Phi_{\bullet}^!(\bar{C})$. This is a local problem. We may thus assume that there exists a local splitting $\sigma : E_0 \oplus F_1 \to F_0$ of the exact sequence

$$0 \longrightarrow F_0 \longrightarrow E_0 \oplus F_1 \stackrel{q}{\longrightarrow} E_1 \longrightarrow 0, \quad q = D \circ \operatorname{pr}_1 + \Phi_1 \circ \operatorname{pr}_2$$

from Lemma 2.5. Then $\chi = (\sigma, q) : E_0 \oplus F_1 \to F_0 \oplus E_1$ is an isomorphism mapping the diagonal F_0 -action on $E_0 \oplus F_1$ to the action on the first factor of $F_0 \oplus E_1$. Since σ is a splitting, χ induces an isomorphism $\ker(\sigma) \to E_1$. Therefore

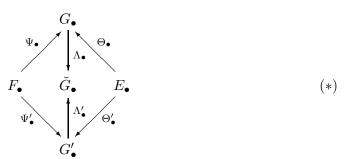
$$\chi(\ker(\sigma) \cap q^{-1}(\bar{C})) = \chi(q^{-1}(\bar{C})) \cap (0 \oplus E_1) = 0 \oplus \bar{C}.$$

But $\chi(q^{-1}(\bar{C}))$ is an F_0 -cone, and hence Proposition 1.6 implies $\chi(q^{-1}(\bar{C})) = F_0 \oplus \bar{C}$. By definition this says $\bar{C} = (\Phi_{\bullet})_! \Phi_{\bullet}^!(\bar{C})$.

Using the nice behavior under quasi-isomorphisms we may now define going down for morphisms in the derived category $D(\operatorname{Coh}(X))$ of the category of coherent sheaves. In the language of linear spaces a morphism of two-term complexes in the derived category $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ consists of

- (1) another two-term complex G_{\bullet}
- (2) a quasi-isomorphism $\Theta_{\bullet}: E_{\bullet} \to G_{\bullet}$
- (3) and a morphism $\Psi_{\bullet}: F_{\bullet} \to G_{\bullet}$.

Two morphisms defined by tuples $(G_{\bullet}, \Theta_{\bullet}, \Psi_{\bullet})$ and $(G'_{\bullet}, \Theta'_{\bullet}, \Psi'_{\bullet})$ are considered equivalent if there exists a two-term complex \tilde{G}_{\bullet} and quasi-isomorphisms $\Lambda_{\bullet}: G_{\bullet} \to \tilde{G}_{\bullet}$, $\Lambda'_{\bullet}: G'_{\bullet} \to \tilde{G}_{\bullet}$ making the following diagram commutative up to homotopy:



Definition 2.13. (going down in the derived category) Let $\Phi_{\bullet}: F_{\bullet} \to E_{\bullet}$ be a morphism of two-term complexes of linear spaces in the derived category, inducing an isomorphism on H^0 and a closed embedding on H^1 . Moreover, we require F_0 to be locally free. When these assumptions are satisfied we say that *going down is applicable* to Φ_{\bullet} . In this case the *push-forward* of an F_0 -cone $C \subset F_1$ is defined to be the E_0 -cone

$$(\Phi_{\bullet})_!(C) := (\Theta_{\bullet})^!(\Psi_{\bullet})_!(C) \subset E_1,$$

whenever $(G_{\bullet}, \Psi_{\bullet}, \Theta_{\bullet})$ is a representative of Φ_{\bullet} .

Using the previous results it is easy to check that this is well-defined.

Remark 2.14. One might wonder if there exists "going down" when being only given maps on the level of cohomology. There are three remarks I want to make on this.

(1) A map in cohomology is considerably weaker than a map of complexes, even for two-term complexes. For instance, let $E_{\bullet} = [E_0 \stackrel{D}{\to} E_1]$ be a non-split epimorphism of linear spaces and $K = \ker D$. Then $H^{\bullet}(E_{\bullet}) = H^{\bullet}([K_{\bullet} \to 0])$, but the identity map in cohomology is not induced by a morphism of complexes.

- (2) There is going down for "cones coming from cohomology": By such cones we mean cones of the form $C = p^{-1}(\bar{C})$ for some $\bar{C} \subset H^1(E_{\bullet})$, $p : E_1 \to H^1(E_{\bullet})$ the cokernel of E_{\bullet} . Namely, if $\rho : H^1(E_{\bullet}) \to H^1(F_{\bullet})$ is a closed embedding and $p' : F_1 \to H^1(F_{\bullet})$ is the cokernel of F_{\bullet} , one may set $\rho_!(C) := p'^{-1}(\rho(\bar{C}))$. In case $\rho = H^1(\Phi_{\bullet})$ with $\Phi_{\bullet} : E_{\bullet} \to F_{\bullet}$ a morphism to which going down is applicable, then $\rho_!(C)$ obviously coincides with $(\Phi_{\bullet})_!(C)$.
- (3) Not every E_0 -cone in E_1 comes from cohomology. As a simple example with $C \neq E_1$ but $H^1(E_{\bullet}) = 0$ take $X = \mathbb{A}^1_k = \operatorname{Spec} k[T]$, $E_0 = E_1 = L(\mathcal{O}_X) = \operatorname{Spec} k[T,U]$, $D: E_0 \to E_1$ corresponding to the homomorphism of k[T]-algebras sending U to TU, and $C = V(TU) \subset E_1$, the linear space corresponding to the structure sheaf of the origin.

See also Remark 3.5 for another, less artificial example.

3. Global normal cones

If $\iota: X \hookrightarrow M$ is a closed embedding of algebraic k-schemes the normal cone $C_{X|M} \subset N_{X|M}$ is a $TM|_{X}$ -cone (Example 1.4). With nonsingular M theses normal cones are essentially unique, namely up to vector bundle factors. In fact, if $\iota': X \hookrightarrow M'$ is another such embedding we may consider the diagonal $(\iota, \iota'): X \hookrightarrow M \times M'$ to reduce to the case where $\iota = \pi \circ \iota', \pi: M' \to M$ a smooth morphism. But then there is an exact sequence of cones

$$(3.1) 0 \longrightarrow (\iota')^* T_{M'|M} \longrightarrow C_{X|M'} \longrightarrow C_{X|M} \longrightarrow 0.$$

Based on this observation Behrend and Fantechi show that to any X there is associated a cone stack (a certain Artin stack) over X of pure relative dimension zero, the *intrinsic normal cone* \mathcal{C}_X . Locally, \mathcal{C}_X is nothing but the stack-theoretic quotient $C_{X|M}/T_M|_X$, and the above exact sequence of cones is responsible for the fact that these quotients glue.

One essential insight of Behrend and Fantechi is that one can retrieve an actual cone over X by giving a morphism $\varphi^{\bullet}: \mathcal{F}^{\bullet} \to \mathcal{L}_{X}^{\bullet}$ in the derived category inducing an isomorphism in H^{0} and an epimorphism in H^{-1} , and such that $\mathcal{F}^{\bullet} = [\mathcal{F}^{-1} \to \mathcal{F}^{0}]$ is a two-term complex of locally free sheaves. Here $\mathcal{L}_{X}^{\bullet}$ is the cotangent complex of X. (In the language of [BeFa], φ^{\bullet} is a "global resolution" of a "perfect obstruction theory" for X.)

The cotangent complex is a complicated and largely mysterious object canonically associated to any scheme, or even ringed topos [II]. However, here we will work exclusively with the truncated complex $\tau_{\geq -1} \mathcal{L}_X^{\bullet}$. This is simply an object of the derived category that has the following explicit local description: If $U \subset X$ is an open subscheme and $U \hookrightarrow M$ is a closed embedding into a smooth scheme M then

$$\tau_{\geq -1} \mathcal{L}_X^{\bullet} = [\mathcal{I}/\mathcal{I}^2 \to \Omega_M|_U],$$

where the complex on the right hand side has entries at -1 and 0 (this follows from the exact triangle for the cotangent complex, see below). In particular, if X is globally embedded into a smooth scheme we can avoid the cotangent complex at all.

Using our study of going up and down for E-cones we will see that the object needed is the following.

Definition 3.1. A global normal space for X is a morphism $\varphi^{\bullet}: \mathcal{F}^{\bullet} = [\mathcal{F}^{-1} \to \mathcal{F}^{0}] \to \tau_{\geq -1} \mathcal{L}_{X}^{\bullet}$ in the derived category with \mathcal{F}^{0} locally free and inducing an isomorphism in H^{0} and an epimorphism in H^{-1} .

Given a global normal space $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$, now written in terms of linear spaces $\tau_{\leq 1}(L_X)_{\bullet} = L(\tau_{\geq -1}\mathcal{L}_X^{\bullet})$ etc., we may construct a cone $C = C(\Phi_{\bullet}) \subset F_0$ as follows: Let $U \subset X$ be an open set embedded into a nonsingular $M, \iota: U \hookrightarrow M$. The exact triangle of relative cotangent complexes associated to $U \to M \to \operatorname{Spec} k$ yields a morphism in the derived category

$$\Lambda_{\bullet}: [T_M|_U \to N_{U|M}] \longrightarrow \tau_{\leq 1}(L_U)_{\bullet}$$

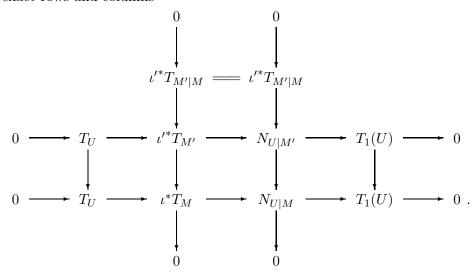
that induces isomorphisms in H^i , i=0,1. As $T_M|_U$ is a vector bundle the composition $\Phi_{\bullet}|_U \circ \Lambda_{\bullet}$ fulfills the assumption of Definition 2.13. We may thus define

$$(3.2) C|_{U} := (\Phi_{\bullet}|_{U} \circ \Lambda_{\bullet})_{!}(C_{U|M}).$$

It remains to show

Lemma 3.2. $(\Phi_{\bullet}|_{U} \circ \Lambda_{\bullet})_{!}(C_{U|M}) \subset F_{1}|_{U}$ is independent of choices.

Proof. It suffices to treat the case of another embedding $\iota': U \to M'$ s.th. $\iota = \pi \circ \iota'$ for some smooth morphism $\pi: M' \to M$, see above. We have a commutative diagram with exact rows and columns



Here $T_1(U)$ is the linear space associated to the first higher cotangent sheaf of U. This shows that $D\pi$ induces a quasi-isomorphism $\Theta_{\bullet}: [T_{M'}|_{U} \to N_{U|M'}] \to [T_{M}|_{U} \to N_{U|M}]$ with $\Lambda'_{\bullet} = \Lambda_{\bullet} \circ \Theta_{\bullet}$. Moreover, from the exact sequence of cones (3.1)

$$C_{U|M'} = \Theta_1^{-1}(C_{U|M}) = \Theta_{\bullet}^!(C_{U|M}).$$

Thus by Proposition 2.12 we conclude

$$(\Lambda'_{\bullet})_!(C_{U|M'}) = (\Lambda_{\bullet})_!(\Theta_{\bullet})_!\Theta_{\bullet}^!(C_{U|M}) = (\Lambda_{\bullet})_!(C_{U|M}).$$

Here is the first of the two main results of this paper.

Theorem 3.3. Let X be an algebraic k-scheme. To any global normal space Φ_{\bullet} : $\tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ for X is associated an F_0 -cone $C(\Phi_{\bullet}) \subset F_1$, locally of the form (3.2), and of pure dimension equal to $\operatorname{rk} F_0$.

Proof. It remains to check the statement on the dimension. Locally, we may choose a representation of $\Phi_{\bullet} \circ \Lambda_{\bullet} : [T_M|_U \to N_{U|M}] \to F_{\bullet}$, where $\iota : U \hookrightarrow M$ is a closed embedding of an open $U \subset X$, by $(G_{\bullet}, \Theta_{\bullet}, \Psi_{\bullet})$ with

•
$$\Psi_{\bullet}: [T_M|_U \to N_{U|M}] \to G_{\bullet}$$

- $\Theta_{\bullet}: F_{\bullet} \to G_{\bullet}$ a quasi-isomorphism
- $G_{\bullet} = [G_0 \to G_1]$ with G_0 a vector bundle (!).

We get two exact sequences of cones (Proposition 2.7)

The first one shows that $(\Psi_{\bullet})_!C_{U|M}$ is pure dimensional of dimension equal to $\operatorname{rk} G_0$, and then by the second one $C(\Phi_{\bullet})$ is pure $(\operatorname{rk} F_0)$ -dimensional.

Definition 3.4. $C(\Phi_{\bullet})$ is called the *global normal cone* associated to the global normal space Φ_{\bullet} .

Remark 3.5. 1) The picture would be especially simple if for any closed embedding $\iota: X \hookrightarrow M$ into a nonsingular M, $C_{X|M}$ came from a cone in the intrinsically defined linear space $T_1(X)$. This is, however, generally wrong:

Consider the fat point $X = \operatorname{Spec} R$, $R = k[X,Y]/(X^2,XY,Y^2)$, with its embedding into $M = \mathbb{A}^2_k = \operatorname{Spec} k[X,Y]$. Letting A,B,C correspond to the generators X^2,XY,Y^2 of the ideal, $C_{X|M} = \operatorname{Spec} R[A,B,C]/(B^2 - AC,XC - YB,XB - YA)$. $T_1(X)$ is the linear space corresponding to the kernel I_T of

$$R[A, B, C]/(XC - YB, XB - YA) \longrightarrow R[dX, dY],$$

 $A \mapsto 2XdX, B \longmapsto YdX + XdY, C \mapsto 2YdY,$

which is $(XA, YA, XB, YB, XC, YC) = (X, Y) \cdot (A, B, C)$. A cone in $N_{X|M} = \operatorname{Spec} R[A, B, C]/(XC - YB, XB - YA)$ comes from $T_1(X)$ iff its ideal is generated by polynomials in XA, YA, XB, YB, XC, YC. This is not the case for $B^2 - AC$, and so $C_{X|M}$ does not come from a cone in $T_1(X)$.

2) By uniqueness of minimal free resolutions of modules over a local ring (see e.g. [Ei, Thm.20.2]) it is not hard to show that for any, not necessarily closed point $x \in X$ there is a minimal germ of global normal spaces at x. This is constructed by embedding an étale neighborhood of x in X into a smooth k-scheme M of dimension embdim $_x X = \dim_{k(x)} \Omega_x \otimes k(x)$. We assume k perfect here to assure that regular k-schemes are smooth over k. A germ of global normal space at x can then be defined by selecting a minimal set of generators for the ideal defining $X \hookrightarrow M$. This germ of global normal space is minimal in the sense that any other germ of global normal space at x can be obtained by adding trivial factors.

As a consequence, the germ at x of any global normal cone is isomorphic to $C_{X|M}$ plus a vector bundle factor. Morally speaking, the "nonlinear parts" of global normal cones are locally unique.

4. VIRTUAL FUNDAMENTAL CLASS AND FULTON'S CANONICAL CLASS

4.1. Virtual fundamental classes. If X is an algebraic k-scheme and $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ is a global normal space for X with also F_1 a vector bundle we speak of a free global normal space of $\operatorname{rank} \operatorname{rk}(\Phi_{\bullet}) = \operatorname{rk} F_0 - \operatorname{rk} F_1$. We may then intersect the zero section $s: X \to F_1$ of F_1 with the global normal cone $C(\Phi_{\bullet}) \subset F_1$ to produce a class on X.

Definition 4.1. Let $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ be a free global normal space. The Chow class

$$[X, \Phi_{\bullet}] := s^! [C(\Phi_{\bullet})] \in A_{\mathrm{rk}(\Phi_{\bullet})}(X)$$

is called virtual fundamental class of X with respect to Φ_{\bullet} .

Note that $[X, \Phi_{\bullet}]$ contains as much information as $[C(\Phi_{\bullet})] \in A_{\mathrm{rk}(F_0)}(F_1)$, for

$$[C(\Phi_{\bullet})] = p! s! [C(\Phi_{\bullet})] = p! [X, \Phi_{\bullet}],$$

where $p: F_1 \to X$ is the projection.

One of the most important property of such classes is their compatibility with specializations. In the application to the construction of invariants from moduli spaces associated to a projective manifold V, say (as in Gromov-Witten or Donaldson-theory), this property implies invariance under smooth deformations of V. There are two versions of the specialization theorem, one involving global normal spaces of the total space of a family, and the other working with relative global normal spaces (that is, a morphism $\Phi_{\bullet}: \tau_{\leq 1}(L_{\mathcal{X}|S})_{\bullet} \to F_{\bullet}$, where $\mathcal{X} \to S$ is the family of algebraic k-schemes under consideration). We do not have anything to add to the presentation in [BeFa, Proposition 5.10 and Proposition 7.2], which translates almost literally into our language.

Before turning to an explicit formula for the computation of $[X, \Phi_{\bullet}]$ in hopefully more accessible terms, we want to add the following point of view: For any puredimensional cone C in a vector bundle F there is a formula for the intersection with the zero locus in terms of the Segre class of C and the total Chern class of F [Fu, Expl. 4.1.8]. Applied to $C(\Phi_{\bullet}) \subset F_1$ it says

$$[X, \Phi_{\bullet}] = \{c(F_1) \cap s(C(\Phi_{\bullet}))\}_{\mathrm{rk}(\Phi_{\bullet})},$$

where $\{\cdot\}_d: A_*(X) \to A_d(X)$ denotes the projection to the d-dimensional part. Now for any r > 0, the image of $[C(\Phi_{\bullet})]$ under the monomorphism $\iota_r: F_1 \hookrightarrow F_1 \oplus \mathbb{A}^r_X$ becomes rationally trivial, while $c(F_1 \oplus \mathbb{A}^r_X) = c(F_1)$. Thus letting $\tilde{s}^r: X \hookrightarrow F_1 \oplus \mathbb{A}^r_X$ be the zero section we see

$$\{c(F_1) \cap s(C(\Phi_{\bullet}))\}_{\mathrm{rk}(\Phi_{\bullet})-r} = (\tilde{s}^r)! (\iota_r)_* [C(\Phi_{\bullet})] = 0.$$

This teaches us two things: First, if F_1 splits off a trivial factor $F_1 = \bar{F}_1 \oplus \mathbb{A}^1_X$ with $\operatorname{im} \Phi_1 \subset \bar{F}_1$ then $[X, \Phi_{\bullet}] = 0$. So the result is trivial if F_1 is not chosen small enough. And second, if $[X, \Phi_{\bullet}] \neq 0$ then $\operatorname{rk}(\Phi_{\bullet})$ is the smallest number d such that

$$\{c(F_1) \cap s(C(\Phi_{\bullet}))\}_d \neq 0.$$

Example 4.2. Let X be smooth of dimension n. Then the cotangent complex of X is exact at \mathcal{L}_X^{-1} . So the natural morphism $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to [T_X \to O], \ O = X$ the trivial linear space over X, is an isomorphism in H^0 and H^1 . Then $C(\Phi_{\bullet}) = X = O$ and $c(F_1) \cap s(C(\Phi_{\bullet})) = s(C(\Phi_{\bullet})) = [X]$ has vanishing components in dimensions smaller than n.

4.2. Fulton's canonical class. If an algebraic k-scheme X is globally embeddable into a smooth k-scheme M (e.g. X quasi-projective) then

$$c_F(X) := c(T_M|_X) \cap s(C_{X|M}) \in A_*(X)$$

is a Chow-class on X that is independent of the choice of embedding [Fu, Expl. 4.2.6].

Definition 4.3.
$$c_F(X)$$
 is called Fulton's canonical class.

Note that if X is smooth one may choose X = M and so $c_F(X) = c(T_X) \cap [X]$. For comparison of $c_F(X)$ with Mather's and MacPherson's Chern classes see [A1].

Given a (not necessarily free) global normal space $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ for X, $c_F(X)$ can also be expressed as follows:

Proposition 4.4. Let $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ be a global normal space for a quasi-projective X. Then

$$c_F(X) = c(F_0) \cap s(C(\Phi_{\bullet})).$$

Proof. By quasi-projectivity there exists a global closed embedding $\iota: X \hookrightarrow M$ of X into a smooth M. This yields a globally defined morphism in the derived category $\Lambda_{\bullet}: [T_M|_X \to N_{X|M}] \to \tau_{\leq 1}(L_X)_{\bullet}$. Also by quasi-projectivity any sheaf is the quotient of a locally free sheaf. Hence there is a global representative $(G_{\bullet}, \Theta_{\bullet}, \Psi_{\bullet})$ of $\Phi_{\bullet} \circ \Lambda_{\bullet}$ in the construction of Theorem 3.3, that is, with G_0 a vector bundle, $\Theta_{\bullet}: F_{\bullet} \to G_{\bullet}$ a quasi-isomorphism, $\Psi_{\bullet}: [T_M|_X \to N_{X|M}] \to G_{\bullet}$. We get two exact sequences of cones with vector bundle kernels (see Proposition 2.7)

$$0 \longrightarrow T_M|_X \longrightarrow G_0 \oplus C_{X|M} \longrightarrow (\Psi_{\bullet})_!(C_{X|M}) \longrightarrow 0$$

$$0 \longrightarrow F_0 \longrightarrow G_0 \oplus C(\Phi_{\bullet}) \longrightarrow (\Psi_{\bullet})_!(C_{X|M}) \longrightarrow 0,$$

which by the multiplicativity of Segre classes in exact sequences of cones with vector bundle kernels imply

$$c(T_M|_X) \cap s(C_{X|M}) = c(G_0) \cap s\left((\Psi_{\bullet})!(C_{X|M})\right) = c(F_0) \cap s(C(\Phi_{\bullet})). \qquad \Box$$

Remark 4.5. If X is any algebraic k-scheme with global normal spaces one could take the right-hand side of the formula in the proposition as definition for a generalization of Fulton's canonical class on projective schemes. However, I was not able to prove independence of this class from the choice of Φ_{\bullet} . And in case X is not quasi-projective but embeddable into a smooth scheme, in the construction of Theorem 3.3 we might not be able to choose G_0 locally free. Then the coincidence of this class with $c_F(X)$ is not clear either. The problem is that on one hand the globally defined complex linking two global normal spaces $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$, $\Phi'_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F'_{\bullet}$ is the cotangent complex, which need not be globally representable by a complex L_{\bullet} with L_0 a vector bundle, while on the other hand Segre classes do not behave well in exact sequences unless the kernels are vector bundles.

We are now ready to deduce the announced formula for the virtual fundamental class.

Theorem 4.6. Let X be a projective k-scheme and $\Phi_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F_{\bullet}$ a free global normal space for X of constant rank d. Then

$$[X, \Phi_{\bullet}] = \left\{ c(\operatorname{ind} F_{\bullet})^{-1} \cap c_F(X) \right\}_d,$$

where ind F_{\bullet} is the virtual bundle $F_0 - F_1 \in K^0(X)$.

Proof. As remarked at the end of the last subsection the virtual fundamental class can be computed by the formula

$$s^![C(\Phi_{\bullet})] = \left\{ c(F_1) \cap s(C(\Phi_{\bullet})) \right\}_d.$$

Now just insert $c(F_0)^{-1} \cup c(F_0)$ and use Proposition 4.4

Remark 4.7. 1) This formula enlightens the dependence of virtual fundamental classes on the choice of global normal spaces: Interestingly, $[X, \Phi_{\bullet}]$ depends only on the index bundle of F_{\bullet} rather than on any of the finer data used to construct $C(\Phi_{\bullet})$. But note also that for another choice $\Phi'_{\bullet}: \tau_{\leq 1}(L_X)_{\bullet} \to F'_{\bullet}$ of global normal space, $[X, \Phi'_{\bullet}]$ can not in general be computed from $[X, \Phi_{\bullet}]$ and $\inf F_{\bullet}$, $\inf F'_{\bullet}$ alone.

2) One can take this formula as definition of the virtual fundamental class of X without

knowing anything about the more sophisticated theory of global normal cones in the non-projective case. This was the point of view of the author in summer 1995 in an attempt to define GW-invariants in algebraic geometry, when I observed it from formal considerations. Unfortunately, I was not aware of Vistoli's rational equivalence [Vi], from which the crucial independence of the invariants under smooth deformations can be derived. I learned also that the same formula has independently been discovered by Brussee for complex spaces constructed as zero locus of holomorphic Fredholm sections of holomorphic Banach bundles over complex Banach manifolds, as occurring for example in Seiberg-Witten theory [Bs] (the interpretation of Brussee's $c_*(X)$ as Fulton's canonical class is not quite clear, though).

3) At the beginning of Section 3 we mentioned Behrend and Fantechi's intrinsic normal cone C_X , which locally was the stack-theoretic quotient of $C_{X|M}$ by the action of $T_M|_X$ for some embedding $X \hookrightarrow M$ into a smooth space.

Now if X is globally embedded into a smooth scheme M, \mathcal{C}_X is globally the quotient of $C_{X|M}$ by $T_M|_X$. Hence in view of the multiplicative behavior of Segre classes in exact sequences of cones with vector bundle kernels, $c_F(X)$ could with some right considered as Segre class of \mathcal{C}_X .

Conversely, if there was a theory of Segre classes for cone stacks, the Segre class of \mathcal{C}_X would generalize Fulton's canonical class to arbitrary algebraic k-schemes. \square

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