Transformation Groups, Vol. 2, No. 3, 1997, pp. 225–267 ©Birkhäuser Boston (1997)

EQUIVARIANT CHOW GROUPS FOR TORUS ACTIONS

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Abstract. We study Edidin and Graham's equivariant Chow groups in the case of torus actions. Our main results are: (i) a presentation of equivariant Chow groups in terms of invariant cycles, which shows how to recover usual Chow groups from equivariant ones; (ii) a precise form of the localization theorem for torus actions on projective, nonsingular varieties; (iii) a construction of equivariant multiplicities, as functionals on equivariant Chow groups; (iv) a construction of the action of operators of divided differences on the T-equivariant Chow group of any scheme with an action of a reductive group with maximal torus T. We apply these results to intersection theory on varieties with group actions, especially to Schubert calculus and its generalizations. In particular, we obtain a presentation of the Chow ring of any smooth, projective spherical variety.

Introduction and statement of the main results

Equivariant Chow groups for actions of linear algebraic groups on schemes have been studied by D. Edidin and W. Graham via Totaro's algebraic approximation of the classifying space. In the present paper, we develop further the theory of equivariant Chow groups in the case of torus actions, and we apply it to (usual) intersection theory on varieties with group actions, especially to Schubert calculus and its generalizations.

Indeed, equivariant Chow groups turn out to be modules over polynomial rings, and usual Chow groups are obtained from them by killing the action of all homogeneous polynomials. Moreover, the richer structure of equivariant Chow groups makes them easier to describe. This is what we do here for toric varieties, flag varieties and more generally for projective, nonsingular spherical varieties.

We consider linear algebraic group actions on schemes over an algebraically closed field k of arbitrary characteristic. These actions are as-

Received December 1, 1996. Accepted April 22, 1997.

sumed to be locally linear, i.e., the schemes that we consider are covered by invariant quasi-projective open subsets (and hence by invariant affine open subsets in the case of torus actions). This assumption is fulfilled, e.g., for normal schemes.

We use Edidin and Graham's definition of equivariant Chow groups (recalled in 2.1 below) and basic properties of these groups as well, see [15]. But except for §§6.6 and 6.7, the present paper is independent of Edidin and Graham's deepest results.

Let T be a torus. Denote by M the character group of T and by S the symmetric algebra over $\mathbf{Z}$ of the abelian group M; then S is the character ring of T. For a scheme X with an action of T, let $A_*^T(X)$ be the equivariant Chow group. The equivariant Chow group of a point identifies to S; more generally, $A_*^T(X)$ is an S-module. Our first result is a presentation of this module, which is reminiscent of the definition of usual Chow groups.

Theorem (2.1). *The S-module $A_*^T(X)$ is defined by generators $[Y]$ (where $Y \subset X$ is a T-invariant subvariety) and by relations $[\mathrm{div}_Y(f)] - \chi[Y]$ (where f is a rational function on Y which is an eigenvector of T of weight χ).*

(By convention, a variety is a reduced and irreducible scheme; a subvariety is a closed subscheme which is a variety). Another notion of equivariant Chow groups has been proposed by Nyenhuis, see [32] and [33]. He considers the abelian group generated by classes $[Y]$ as above, with relations $[\mathrm{div}_Y(f)]$ for f a *T-invariant* rational function on Y. A drawback of this notion is its noninvariance when X is replaced by $X \times M$ for a T-module M.

By the theorem above, Edidin and Graham's group is a quotient of Nyenhuis'. Moreover, the usual Chow group is a quotient of Edidin and Graham's. More precisely:

Corollary (2.3). *The (usual) Chow group $A_*(X)$ is the quotient of $A_*^T(X)$ by its subgroup $MA_*^T(X)$.*

As in equivariant cohomology, localization at fixed points is a powerful tool for studying equivariant Chow groups, see [16] §5 where an algebraic proof of Bott's residue formula is given. Here we give a very simple proof of the localization theorem for schemes with a torus action (see 2.3). Moreover, we obtain a description of the rational equivariant Chow ring in the case where X is projective and nonsingular. Under the latter assumption, we denote by $A_T^*(X)$ the equivariant Chow group, graded by codimension; it has a graded ring structure given by the intersection product, see [15]. We will have this multiplicative structure in mind whenever we use the notation $A_T^*(X)$.

Theorem (3.2), (3.3). *Let X be a projective, nonsingular variety with an action of T.*

(i) *The $S_{\mathbf{Q}}$-module $A_T^*(X)_{\mathbf{Q}}$ is free.*

(ii) *The pull-back by inclusion of fixed points* $i : X^T \to X$ *is an injective* $S_{\mathbf{Q}}$*-algebra homomorphism*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}} = S_{\mathbf{Q}} \otimes A^*(X^T)$$

which is surjective after inverting all nonzero elements of M.

(iii) *The image of* i^* *is the intersection of the images of* $i_{T'}^*$ *where* $T' \subset T$ *is any subtorus of codimension one, and where* $i_{T'} : X^T \to X^{T'}$ *is the inclusion.*

Properties (i) and (ii) are well-known for the rational equivariant cohomology ring of a compact symplectic manifold with a Hamiltonian action of a compact torus (see [26]). In our setting, they follow from the Bialynicki–Birula decomposition, recalled in 3.1. Statement (iii) will play an essential role in the description of the equivariant Chow ring of flag varieties (§6) and, more generally, of projective, smooth spherical varieties (§7). It has its origin in a recent result of Goresky, Kottwitz and MacPherson concerning equivariant cohomology of a topological space X with an action of a compact torus T, see [23] Theorem (6.3): They obtain an exact sequence

$$0 \to H_T^*(X, A) \to H_T^*(X^T, A) \to H_T^*(\cup_{T'} X^{T'}, X^T, A)$$

for A in the equivariant derived category of X, under certain assumptions on X and A. This result can be made more precise when X contains only finitely many T-invariant points and curves: then $H_T^*(X^T, \mathbf{R})$ consists of all m-tuples of polynomial functions on the Lie algebra of T, where m is the number of fixed points. Moreover, the image of $H_T^*(X, \mathbf{R})$ in $H_T^*(X^T, \mathbf{R})$ can be described by congruences involving pairs of fixed points, see [23] Theorem (1.2.2). In our algebraic situation, the arguments of [23] do not adapt; our approach is based on the Bialynicki–Birula decomposition and on an inductive description of the equivariant Chow ring, see 3.2. As a corollary, we obtain an algebraic version of Theorem (1.2.2) in [23]:

Theorem (3.4). *Let* X *be a projective, nonsingular variety where* T *acts with finitely many fixed points* $x_1, \ldots, x_m$ *and with finitely many invariant curves. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is the set of all $(f_1, \ldots, f_m) \in S_{\mathbf{Q}}^m$ *such that* $f_i \equiv f_j \pmod{\chi}$ *whenever* x_i *and* x_j *are connected by an irreducible invariant curve where* T *acts through the character* χ.

Together with Corollary 2.3, this gives a complete picture of the (usual) rational Chow ring of X.

To study possibly singular varieties (for example, toric and Schubert varieties), we develop in §4 a notion of equivariant multiplicity at a fixed point

x which is *nondegenerate*, that is, all weights of T in the tangent space T_xX are nonzero. Such a notion has already appeared for X a T-module with weights in an open half-space (see [25] and [8]) and, more generally, for nonsingular X in work of Rossmann; see [38]. A notion of equivariant multiplicity is studied in [32] when X is any T-module. Here we generalize Rossmann's results and we relate them to equivariant Chow groups, as follows.

Theorem (4.2), (4.5). *Let X be a scheme with an action of T, let $x \in X$ be a nondegenerate fixed point and let $\chi_1, \ldots, \chi_n$ be the weights of T_xX.*

(i) *There exists a unique S-linear map*

$$e_x : A_*^T(X) \to \frac{1}{\chi_1 \cdots \chi_n} S$$

such that $e_x[x] = 1$ and that $e_x[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x.

(ii) *The point x is nonsingular in X if and only if*

$$e_x[X] = \frac{1}{\chi_1 \cdots \chi_n}.$$

In this case, we have for any T-invariant subvariety $Y \subset X$:

$$e_x[X] = \frac{[Y]_x}{\chi_1 \cdots \chi_n}$$

where $[Y]_x$ denotes pull-back of $[Y]$ by inclusion of x into X. Moreover, $[Y]_x$ is Rossmann's equivariant multiplicity.

In §5 we show that equivariant multiplicities separate points in the equivariant Chow group of any toric variety X. In the case where X is simplicial (i.e., its fan Σ consists of simplicial cones; equivalently, it has quotient singularities by finite groups), this leads to a complete description of the S-module $A_*^T(X)$ in terms of piecewise polynomial functions on Σ, see 5.4. As a corollary, we obtain the following

Theorem (5.4). *Let X be a simplicial toric variety. Then $A_*^T(X)_{\mathbf{Q}}$ is isomorphic to the space of continuous, piecewise polynomial functions on the fan of X.*

The corresponding statement for equivariant cohomology was proved in [11] by another method.

In §6, we consider schemes X with an action of a connected reductive group G. Let B be a Borel subgroup of G, let T be a maximal torus of B, and let W be the Weyl group of (G,T). Then W acts on $A_*^T(X)$ compatibly with the S-module structure. It turns out (see 6.2 and 6.3) that this action extends to an action of the ring $\mathbf{D}$ of *operators of divided differences*, generated over S by the operators

$$D_\alpha = \frac{1}{\alpha}(id - s_\alpha)$$

where α is a simple root and $s_\alpha \in W$ is the corresponding reflection. These operators were introduced by Bernstein-Gelfand-Gelfand and Demazure for studying the cohomology ring of the flag variety G/B, see [2] and [13], [14]. Then Arabia, Kostant and Kumar described *equivariant* cohomology of G/B in terms of these operators, see [1] (and also [9]), [28], [29].

In 6.4 and 6.5, we obtain two descriptions of $A_*^T(G/B)$ by first using its $\mathbf{D}$-module structure, and then Theorem 3.4 and equivariant multiplicities. Both descriptions were known for $k = \mathbf{C}$ and equivariant cohomology (the first one being due to Arabia and the second one to Kostant and Kumar). Specifically, we show that the $\mathbf{D}$-module $A_*^T(G/B)$ is freely generated by the class of the B-fixed point, and we derive an inductive formula for equivariant multiplicities of Schubert varieties. As an application, we present in 6.5 a short, characteristic-free proof of Kumar's smoothness criterion for Schubert varieties (see [30] Theorem 5.5).

The equivariant Chow ring $A_T^*(G/B)$ is studied in 6.6; it turns out to be isomorphic to $A_G^*(G/B \times G/B)$ where G acts diagonally in $G/B \times G/B$. Moreover, the latter ring is isomorphic over the rationals to the tensor product $S \otimes S$ over the subring of invariants S^W. The action of $\mathbf{D}$ is then given by $D(u \otimes v) = D(u)v$, and the class of the B-fixed point in G/B identifies to the class of the diagonal in $G/B \times G/B$; these results are due to Kostant and Kumar for $k = \mathbf{C}$ and equivariant cohomology or K-theory (see [28] and [29]). Therefore, to relate both descriptions of $A_T^*(G/B)_{\mathbf{Q}}$, it is enough to find a representative in $S_{\mathbf{Q}} \otimes S_{\mathbf{Q}}$ of the class of the diagonal. For a classical group G, this has been done in a different formulation by Fulton (see [19] and [20]) and by Pragacz and Ratajski (see [35] and [36]). A formula for arbitrary G has been obtained by Graham, see [24].

For example, in the case where $G = \mathrm{GL}_n$, we have $S = \mathbf{Z}[x_1, \ldots, x_n]$ and the generators of $\mathbf{D}$ are the operators $D_1, \ldots, D_{n-1}$ such that

$$D_i f(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}.$$

The T-equivariant Chow ring of G/B is the quotient of the polynomial ring

$$\mathbf{Z}[x_1, \ldots, x_n, y_1 \ldots, y_n]$$

by the ideal generated by the

$$e_i(x_1, \ldots, x_n) - e_i(y_1, \ldots, y_n)$$

where $e_1, \ldots, e_n$ are the elementary symmetric functions. The class of the B-fixed point is represented by

$$\prod_{i+j \leq n} (x_i - y_j)$$

and the classes of the Schubert varieties are represented by the corresponding Schubert polynomials, see [19].

Back to the general case of a scheme X with an action of G, the rational G-equivariant Chow group $A_*^G(X)_{\mathbf{Q}}$ is isomorphic to the space of W-invariants $A_*^T(X)_{\mathbf{Q}}^W$, see [15] 3.2. In particular, the rational G-equivariant Chow group of the point is isomorphic to $S_{\mathbf{Q}}^W$. This connection between G- and T-equivariant Chow groups can be made precise:

Theorem (6.7). *Let X be a scheme with an action of a connected reductive group G.*

(i) *The map*

$$\begin{array}{rccc} \gamma: & S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*^G(X)_{\mathbf{Q}} & \to & A_*^T(X)_{\mathbf{Q}} \\ & u \otimes v & \mapsto & uv \end{array}$$

is an isomorphism of modules over $S_{\mathbf{Q}}$. Moreover, for all $D \in \mathbf{D}$, $u \in S$ and $v \in A_*^G(X)$, we have $D(uv) = D(u)v$.

(ii) *The rational Chow group $A_*(X)_{\mathbf{Q}}$ is the quotient of the rational equivariant Chow group $A_*^G(X)_{\mathbf{Q}}$ by its subgroup $S_+^W A_*^G(X)_{\mathbf{Q}}$, where S_+^W is the ideal of S^W generated by homogeneous elements of positive degree.*

(iii) *If moreover X is projective and nonsingular, then the $S_{\mathbf{Q}}^W$-module $A_*^G(X)_{\mathbf{Q}}$ is free.*

At this point, let us point out that although several results of the present paper are algebraic versions of known statements concerning equivariant cohomology, most proofs are completely different. In fact, the analogy between equivariant cohomology and equivariant intersection theory can be misleading: for example, the map $A_*^G(X) \to A_*(X)$ is always surjective over the rationals, whereas the corresponding statement in equivariant cohomology can fail, e.g. when $X = G$ where G acts by multiplication.

The final Section 7 contains applications of the previous theory to Chow groups of spherical varieties. Recall that a normal variety X with an action of a connected reductive group G is spherical if a Borel subgroup B of G has a dense orbit in X. Then it is known that G (and even B) has finitely many orbits in X. It follows that X contains only finitely many fixed points of a maximal torus $T \subset B$. If moreover X is projective and nonsingular, we describe the image of

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

by congruences involving pairs, triples or quadruples of T-fixed points (Theorem 7.3). Indeed, pairs of fixed points connected by a T-invariant curve give rise to a congruence as in Theorem 3.4. Moreover, invariant curves in a smooth, projective spherical variety are either isolated, or they occur in a one-parameter family, which sweeps out a projective plane (containing three fixed points) or a rational ruled surface (containing four fixed points).

A presentation of the G-equivariant rational cohomology ring for a class ("regular embeddings") of spherical varieties has been obtained by Bifet, De Concini and Procesi, see [6]. For wonderful compactifications of symmetric spaces, a more precise result is due to Littelmann and Procesi, see [31]. Our approach is quite different; it leads to a less compact but more general description, which will be developed in a subsequent paper. Both descriptions coincide in the case of the canonical equivariant completion of a semisimple adjoint group, as shown in 7.3.

Finally, we express the action of operators of divided differences on the T-equivariant Chow group of any spherical variety X, in terms of the action of the Richardson-Springer monoid on the set of B-orbits in X, see [37] and [27].

2. A presentation of equivariant Chow groups for torus actions

2.1. Equivariant Chow groups.

First we recall Edidin and Graham's definition of these groups, based on a construction of Totaro (see [15] 2.2). Let X be a scheme with an action of a linear algebraic group G. Let V be a finite-dimensional rational G-module, and let $U \subset V$ a G-invariant open subset such that the quotient $U \to U/G$ exists and is a principal G-bundle. Then, for the diagonal action of G on $X \times U$, the quotient $X \times U \to (X \times U)/G$ exists and is a principal G-bundle. Set $n := \dim(X)$, $l := \dim(V)$ and $d := \dim(G)$. Define the i-th equivariant Chow group $A_i^G(X)$ as the $(i + l - d)$th Chow group of $(X \times U)/G$, if $\operatorname{codim}(V \setminus U) > n - i$ (such a pair (V, U) always exists, see [15]). Under this assumption, $A_i^G(X)$ is independent of the choice of (V, U). Finally, set $A_*^G(X) := \oplus_i A_i^G(X)$. Each invariant subvariety $Y \subset X$ defines a class $[Y]$ in $A_*^G(X)$ by setting $[Y] := [(Y \times U)/G]$.

In the case where $G = T$ is a torus, the graded abelian group $A_*^T(X)$ has the structure of an S-module, see [15]. To define it, it suffices to describe multiplication by $\chi \in M$. Let $k(\chi)$ be the one-dimensional T-module with weight χ. The first projection $U \times k(\chi) \to U$ descends to a map $(U \times k(\chi))/T \to U/T$ which defines a line bundle $L(\chi)$ over U/T. Then multiplication by χ is the first Chern class of the pull-back of $L(\chi)$ to $(X \times U)/T$. So M acts in $A_*^T(X)$ by homogeneous maps of degree -1.

Let X be a scheme with an action of T, let $Y \subset X$ be an invariant subvariety, and let f be a rational function on Y which is an eigenvector of T of some weight χ. Then the divisor of f defines a class $[\operatorname{div}_Y(f)]$ in

$A_*^T(X)$. Observe that the equality

$$[\mathrm{div}_Y(f)] = \chi[Y]$$

holds in the S-module $A_*^T(X)$. Indeed, f can be considered as a rational section of the pull-back of $L(\chi)$ to $(Y \times U)/T$, with divisor $[\mathrm{div}_Y(f)]$. This observation leads to the following description of $A_*^T(X)$.

Theorem. *The S-module $A_*^T(X)$ is defined by generators $[Y]$ where Y is an invariant subvariety of X and relations $[\mathrm{div}_Y(f)] - \chi[Y]$ where f is a rational function on Y which is an eigenvector of T of weight χ.*

2.2. Proof of Theorem 2.1.

Fix a nonnegative integer i and consider the i-th equivariant Chow group $A_i^T(X)$. We begin by constructing a T-module V and an open invariant subset $U \subset V$ such that the quotient $U \to U/T$ exists and is a principal T-bundle, and that $\mathrm{codim}(V \setminus U) > n - i$.

Choose a basis $(\chi_1, \ldots, \chi_d)$ of the free abelian group M of rank d. Set $\chi_0 := -\chi_1 - \cdots - \chi_d$. Consider the T-module k^{d+1} where T acts with weights $\chi_0, \chi_1, \ldots, \chi_d$ of multiplicity one. Choose a positive integer N and consider the T-module $V = (k^{d+1})^N$. Then $V = V_0 \times V_1 \times \cdots \times V_d$, each V_j being a N-dimensional vector space where T acts through the character χ_j. Set

$$U := \{(v_0, v_1, \ldots, v_d) \mid v_j \neq 0 \;\forall\, j\} = \prod_{j=0}^{d} (V_j \setminus \{0\}).$$

Then, for each $v \in U$, the orbit $T \cdot v$ is closed in V, and the isotropy group T_v is trivial. It follows that the quotient $U \to U/T$ exists and is a principal T-bundle. Moreover, the codimension of $V \setminus U$ is N. We choose N so that $N > n - i$.

Choose bases of $V_0, V_1, \ldots, V_d$ and denote by $T_j \subset GL(V_j)$ the corresponding tori of diagonal matrices. Then T embeds diagonally into the product torus

$$T_0 \times T_1 \times \ldots \times T_d := \tilde{T}.$$

Moreover, $\tilde{T}$ acts on V, and U is invariant under this action. This defines an action of $\tilde{T}$ on $(X \times U)/T$.

By [21] Theorem 1, the abelian group $A_i((X \times U)/T)$ is generated by the classes of i-dimensional $\tilde{T}$-invariant subvarieties of $(X \times U)/T$. Moreover, the relations between these classes are consequences of relations $[\mathrm{div}(f)] = 0$ where f is a rational function on an $(i+1)$-dimensional $\tilde{T}$-invariant subvariety of $(X \times U)/T$, which is an eigenvector of $\tilde{T}$. Translating these statements into the setting of equivariant Chow groups will lead to our result, as follows.

Let Y be an i-dimensional, $\tilde{T}$-invariant subvariety of $(X \times U)/T$. Let Z be the preimage of Y in $X \times U$. Then Z is invariant by the diagonal T-action and by the $\tilde{T}$-action on U. Therefore, Z is invariant by the action of $T \times \tilde{T}$

on $X \times U$, defined as follows: $(t, \tilde{t})(x, v) = (tx, \tilde{t}v)$. But $\tilde{T} = \prod T_j$ acts on $U = \prod(V_j \setminus \{0\})$ with finitely many orbits. So, by [21] Lemma 3, we have $Z = Z' \times \prod Z_j$ where Z' is a T-invariant subvariety of X, and where each Z_j is a T_j-invariant subvariety of $V_j \setminus \{0\}$. Denote by m_j the codimension of Z_j in $V_j \setminus \{0\}$; set $Y' := (Z' \times U)/T$. Then we claim that we have in the S-module $A_*^T(X)$

$$[Y] = \chi_0^{m_0} \cdots \chi_d^{m_d}[Y'].$$

To check this formula, recall that multiplication by χ_j in $A_*^T(X)$ is the first Chern class of the line bundle on $(X \times U)/T$, pull-back of the line bundle $L(\chi_j)$ on U/T associated to the character χ_j of T. But $L(\chi_j)$ corresponds to the Cartier divisor D_j in $U/T = (\prod(V_i \setminus \{0\}))/T$, image of the divisor $(\prod_{i \neq j}(V_i \setminus \{0\})) \times (H_j \setminus \{0\})$ where H_j is a hyperplane in V_j. It follows that

$$\chi_0^{m_0} \cdots \chi_d^{m_d}[Y'] = D_0^{m_0} \cdots D_d^{m_d}[(Z' \times \prod(V_j \setminus \{0\}))/T].$$

Now Z_j is the transversal intersection of m_j hyperplanes in V_j, and this proves our claim.

By the claim, the S-module $A_*^T(X)$ is generated by classes of invariant subvarieties of X. We now describe the relations between these classes. Let $Y \subset (X \times U)/T$ be a $\tilde{T}$-invariant subvariety of dimension $i+1$, and let f be a rational function on Y which is an eigenvector of $\tilde{T}$. Let $Z \subset X \times U$ be the preimage of Y. We consider f as a rational function on Z, invariant under the diagonal action of T; then f is an eigenvector of $T \times \tilde{T}$. We can write as above: $Z = Z' \times \prod Z_j$. Moreover, by [21] Lemma 3, we have $f = f' \prod f_j$ where $f' \in R(Z')$ is an eigenvector of T of some weight χ, each $f_j \in R(Z_j)$ is an eigenvector of T_j of some weight α_j, and $(\chi + \sum \alpha_j)|T = 0$; this expresses invariance of f under the diagonal action of T. Then the $\tilde{T}$-weight of f is $\sum \alpha_j$.

Now the preimage in Z of the cycle $\mathrm{div}_Y(f)$ is the cycle

$$\mathrm{div}_Z(f) = (\mathrm{div}_{Z'}(f') \times \prod Z_j) + (Z' \times \mathrm{div}_{\prod Z_j}(\prod f_j)).$$

Denoting by m_j the codimension of Z_j in $V_j \setminus \{0\}$, we then obtain in $A_*^T(X)$:

$$[\mathrm{div}_Y(f)] = \chi_0^{m_0} \cdots \chi_d^{m_d}([\mathrm{div}_{(Z' \times U)/T}(f') - \chi[(Z' \times U)/T]).$$

So $\mathrm{div}_Y(f)$ belongs to the S-module generated by our relations (the latter correspond to the case where $Z = Z' \times U$). □

2.3. Some applications.

An immediate consequence of Theorem 2.1 is the following relation between equivariant and usual Chow groups. For any scheme X with an action of T, the map $(X \times U)/T \to U/T$ is a locally trivial fibration for the Zariski topology, with fiber X. Because any two points in U/T are rationally equivalent, restriction to a fiber defines a canonical map $A_*^T(X) \to A_*(X)$.

Corollary 1. *The map $A_*^T(X) \to A_*(X)$ vanishes on $MA_*^T(X)$, and it induces an isomorphism*

$$A_*^T(X)/MA_*^T(X) \to A_*(X).$$

Proof. For any $i \geq 0$, the map $A_i^T(X) \to A_i(X)$ vanishes on $MA_{i+1}^T(X)$ by definition of the action of M. By Theorem 2.1, the abelian group $A_*^T(X)/MA_*^T(X)$ is defined by generators $[Y]$ (where $Y \subset X$ is a T-invariant subvariety) and relations $[\mathrm{div}_Y(f)]$ (where f is a rational function on Y which is an eigenvector of T). So the statement follows from [21] Theorem 1. $\square$

This result will be generalized to schemes with an action of any connected reductive group, in 6.6 below.

As another application, we give a simple proof of Edidin and Graham's localization theorem for equivariant Chow groups (see [16] for another proof, which works more generally for higher equivariant Chow groups).

Corollary 2. *Let $i : X^T \to X$ be the inclusion of the fixed point scheme. Then the S-linear map $i_* : A_*^T(X^T) \to A_*^T(X)$ is an isomorphism after inverting all nonzero elements of M.*

Proof. Let $Y \subset X$ be a T-invariant subvariety of positive dimension. If Y is not contained in X^T, then there exists a rational function f on Y which is an eigenvector of T for a nonzero weight χ. Indeed, Y contains a nonempty open affine T-stable subset U. The algebra of regular functions on U is a nontrivial rational T-module, and hence it contains an eigenvector of T with a nonzero weight. So we have after inverting χ: $[Y] = \chi^{-1}[\mathrm{div}_Y(f)]$. By induction on the dimension of Y, we obtain that i_* is surjective after inverting all nonzero elements of M.

For injectivity, assume that X is not fixed pointwise by T. Then, as before, we can find an irreducible component Y of X, and a nonconstant rational function f on Y which is an eigenvector of T of nonzero weight, say χ. Denote by $|D|$ the union of the support of the divisor of f in Y, and of the irreducible components of X which do not contain Y. Observe that $|D|$ contains all fixed points in X. Denote by $p : (X \times U)/T \to U/T$ the projection, and consider f as a rational section of $p^*L(\chi)$. More precisely, consider the pseudo-divisor (see [17] 2.2)

$$(p^*L(\chi), (|D| \times U)/T, f)$$

on $(X \times U)/T$. It defines a homogeneous map of degree -1

$$j^* : A_*^T(X) \to A_*^T(|D|).$$

Moreover, denoting by $j : |D| \to X$ the inclusion, the composition $j^* \circ j_*$ is multiplication by χ. Therefore, the map

$$j_* : A_*^T(|D|) \to A_*^T(X)$$

is injective after inverting χ. We conclude by Noetherian induction. $\square$

3. The equivariant Chow ring of a projective, nonsingular variety

3.1. The Bialynicki–Birula decomposition.

For a scheme X with an action of T, we denote by X^T its fixed point subscheme. Similarly, for a one-parameter subgroup λ of T, we have the fixed point subscheme $X^\lambda \supset X^T$. We call λ *generic* if $X^\lambda = X^T$. It follows easily from local linearity of X that generic one-parameter subgroups always exist.

For a subvariety Y of X^λ, we define subsets $X_+(Y,\lambda)$ and $X_-(Y,\lambda)$ by

$$X_\pm(Y,\lambda) = \{x \in X \mid \lim_{t\to 0} \lambda(t^{\pm 1})x \text{ exists and is in } Y\}.$$

We denote by $p_\pm : X_\pm(Y,\lambda) \to Y$ the maps $x \mapsto \lim_{t\to 0} \lambda(t^\pm)x$. Then $X_+(Y,\lambda)$, $X_-(Y,\lambda)$ are locally closed, T-invariant subvarieties of X, and p_+, p_- are T-equivariant morphisms.

Observe that any complete T-variety X is the disjoint union of locally closed subvarieties $X_+(Y,\lambda)$ where λ is a fixed generic one-parameter subgroup, and where Y runs over all connected components of X^T. If moreover X is nonsingular, a much stronger result holds, due to Bialynicki–Birula (see [3], [4], [5]). To state it, we introduce the following notation. For $x \in X^\lambda$, the tangent space T_xX is a module over the multiplicative group, via λ. We denote by $(T_xX)_0$ (resp. $(T_xX)_+$, $(T_xX)_-$) the sum of the weight subspaces of T_xX with zero (resp. positive, negative) weights. Then

$$T_xX = (T_xX)_- \oplus (T_xX)_0 \oplus (T_xX)_+.$$

Theorem. *Let X be a complete, nonsingular T-variety, and let λ be a generic one-parameter subgroup.*

(i) *The fixed point scheme $X^\lambda = X^T$ is nonsingular, and its tangent space at x is $(T_xX)_0$.*

(ii) *For any component Y of X^T, the maps $p_\pm : X_\pm(Y,\lambda) \to Y$ make $X_\pm(Y,\lambda)$ into an equivariant vector bundle over Y whose fiber at x is the T-module $(T_xX)_+$.*

In particular, the plus stratum $X_+(Y,\lambda)$ and the minus stratum $X_-(Y,\lambda)$ are nonsingular, and they intersect transversally along Y.

3.2. Attaching strata.

Definition. A scheme X with an action of T is called *filtrable* if it satisfies the following conditions:

(i) X is the union of its plus strata $X_+(Y,\lambda)$ for some generic one-parameter subgroup λ of T.

(ii) There is an indexing $\Sigma_1, \dots, \Sigma_n$ of the set of strata such that, for all indices i, the closure $\overline{\Sigma_i}$ is contained in the union of Σ_j with $j \geq i$.

By [5], any projective scheme is filtrable. We aim at an inductive description of the equivariant Chow ring of any nonsingular, filtrable scheme X with an action of T. By assumption, there exists a closed stratum $F = X_+(Y,\lambda)$, and moreover $X \setminus F$ is filtrable. We describe the ring $A_T^*(X)$ in terms of $A_T^*(F)$ and of $A_T^*(X \setminus F)$.

Denote by $j_F : F \to X$ and by $j_U : U := X \setminus F \to X$ the inclusion maps. Let d be the codimension of F in X, let N be the normal bundle to F in X, and let $c_d^T(N) \in A_T^d(F)$ be its top equivariant Chern class. Finally, let

$$q : A_T^*(F) \to A_T^*(F)/(c_d^T(N))$$

be the quotient map by the ideal generated by $c_d^T(N)$.

Proposition. (i) *Multiplication by $c_d^T(N)$ is injective in $A_T^*(F)_{\mathbf{Q}}$.*

(ii) *The maps $j_{F*} : A_*^T(F) \to A_*^T(X)$ and $j_U^* : A_*^T(X) \to A_*^T(U)$ fit up into an exact sequence*

$$0 \to A_*^T(F)_{\mathbf{Q}} \to A_*^T(X)_{\mathbf{Q}} \to A_*^T(U)_{\mathbf{Q}} \to 0.$$

(iii) *For any $\beta \in A_T^*(U)_{\mathbf{Q}}$, choose $\gamma \in A_T^*(X)_{\mathbf{Q}}$ such that $j_U^*\gamma = \beta$. Then the element $q(j_F^*(\gamma))$ of $A_Y^*(F)_{\mathbf{Q}}/(c_d^T(N))$ depends only on β, and the map*

$$\begin{array}{rccc} \pi : & A_T^*(U)_{\mathbf{Q}} & \to & A_T^*(F)_{\mathbf{Q}}/(c_d^T(N)) \\ & \beta & \mapsto & q(j_F^*(\gamma)) \end{array}$$

is a $S_{\mathbf{Q}}$-algebra homomorphism.

(iv) *The algebra homomorphisms*

$$(j_F^*, j_U^*) : A_T^*(X)_{\mathbf{Q}} \to A_T^*(F)_{\mathbf{Q}} \times A_T^*(U)_{\mathbf{Q}}$$

and

$$q - \pi : A_T^*(F)_{\mathbf{Q}} \times A_T^*(U)_{\mathbf{Q}} \to A_T^*(F)_{\mathbf{Q}}/(c_d^T(N))$$

define an exact sequence:

$$0 \to A_T^*(X)_{\mathbf{Q}} \to A_T^*(F)_{\mathbf{Q}} \times A_T^*(U)_{\mathbf{Q}} \to A_T^*(F)_{\mathbf{Q}}/(c_d^T(N)) \to 0.$$

If moreover the abelian group $A_*(Y)$ is torsion-free, then all statements above hold over $\mathbf{Z}$.

Proof. (i) We have $A_T^*(F) = A_T^*(Y)$ because F is an equivariant vector bundle over Y. Moreover, $A_T^*(Y) = S \otimes A^*(Y)$ by [16] Proposition 13 (alternatively, this follows from Theorem 2.1). Under the resulting identification of $A_T^*(F)$ to $S \otimes A^*(Y)$, $c_d^T(N)$ goes to

$$\prod_{i=1}^{d} (\chi_i \otimes 1 + 1 \otimes \alpha_i)$$

where $\chi_1, \ldots, \chi_d$ are the characters of $N_x = (T_xX)_-$ for any $x \in Y$, and where $\alpha_1, \ldots, \alpha_d$ are the corresponding Chern roots (see [17] 3.2.3; the formula makes sense as a symmetric function of the α_i's.) Therefore, we have

$$c_d^T(N) = (\prod_{i=1}^{d} \chi_i) \otimes 1 + \nu$$

for some nilpotent $\nu \in S \otimes A^*(Y)$. It follows that $c_d^T(N)$ is not a zero-divisor in $S \otimes A^*(Y)_{\mathbf{Q}}$.

(ii) It suffices to check that j_{F*} is injective over $\mathbf{Q}$. But composition

$$j_F^* \circ j_{F*} : A_*^T(F) \to A_{*+d}^T(F)$$

is multiplication by $c_d^T(N)$.

(iii) Let γ_1, γ_2 in $A_*^T(F)_{\mathbf{Q}}$ be such that $j_U^*(\gamma_1) = j_U^*(\gamma_2) = \beta$. Then $\gamma_1 - \gamma_2 \in j_{F*}A_*^T(F)_{\mathbf{Q}}$ and hence $j_F^*(\gamma_1) - j_F^*(\gamma_2) \in (c_d^T(N))$.

(iv) By construction, we have $(q - \pi) \circ (j_F^*, j_U^*) = 0$, (j_F^*, j_U^*) is injective, and $q - \pi$ is surjective. Let $(\alpha, \beta) \in A_T^*(F)_{\mathbf{Q}} \times A_T^*(U)_{\mathbf{Q}}$ be such that $(q - \pi)(\alpha, \beta) = 0$. Write $\beta = j_U^*(\gamma)$ for some $\gamma \in A_T^*(X)_{\mathbf{Q}}$. Then $q(\alpha) = \pi(\beta) = q(j_F^*(\gamma))$ and hence $\alpha = j_F^*(\gamma + j_{F*}(\delta))$ for some $\delta \in A_T^*(F)_{\mathbf{Q}}$. So $\beta = j_U^*(\gamma + j_{F*}(\delta))$ and (α, β) is in the image of (j_F^*, j_U^*). □

Corollary 1. *Let X be a nonsingular filtrable T-variety.*

(i) *The inclusion map $i : X^T \to X$ induces an injective S-algebra homomorphism*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

which is surjective over the quotient field of S.

(ii) *The $S_{\mathbf{Q}}$-module $A_*^T(X)_{\mathbf{Q}}$ is free. If moreover the $\mathbf{Z}$-module $A_*(X^T)$ is free, then the S-module $A_*^T(X)$ is free.*

(iii) *If X^T consists of finitely many points $x_1, \ldots, x_m$, then, for any generic one-parameter subgroup λ, the S-module $A_*^T(X)$ is freely generated by the classes of the closures of strata $X_+(x_i, \lambda)$ for $1 \leq i \leq m$.*

Proof. Let F and Y be as above. Recall that $A_*^T(F) = A_*^T(Y) = S \otimes A_*(Y)$. Combined with statement (ii) in the proposition, this implies our corollary, arguing by induction over the number of strata. □

Consider now a nonsingular *complex* algebraic variety X with an action of a complex algebraic torus T. Then there is a cycle map $cl_X : A^*(X) \to H^*(X, \mathbf{Z})$ which doubles the degree (see [17] Corollary 19.2). Similarly, there is a cycle map $cl_X^T : A_T^*(X) \to H_T^*(X, \mathbf{Z})$ where $H_T^*(X, \mathbf{Z})$ denotes equivariant cohomology with integral coefficients, see [15] 2.8.

Corollary 2. *Let X be a nonsingular, filtrable complex algebraic variety with an action of T. If the cycle map*

$$cl_{X^T} : A^*(X^T)_{\mathbf{Q}} \to H^*(X^T, \mathbf{Q})$$

is an isomorphism, then both cycle maps

$$cl_X^T : A_T^*(X)_{\mathbf{Q}} \to H_T^*(X, \mathbf{Q})$$

and

$$cl_X : A^*(X)_{\mathbf{Q}} \to H^*(X, \mathbf{Q})$$

are isomorphisms as well.

Proof. Observe that our inductive description of the equivariant Chow ring carries over to equivariant cohomology with little change. Moreover, our assumption implies that the cycle maps cl_Σ^T and cl_Σ are isomorphisms for any stratum Σ. Arguing by induction over the number of strata, it follows that cl_X^T is an isomorphism, and also that X has no rational cohomology in odd degree. Then the spectral sequence associated to the fibration $X \times_T ET \to BT$ collapses, where $ET \to BT$ is the universal T-bundle. So the $S_{\mathbf{Q}}$-module $H_T^*(X, \mathbf{Q})$ is free, and the map

$$H_T^*(X, \mathbf{Q})/MH_T^*(X, \mathbf{Q}) \to H^*(X, \mathbf{Q})$$

is an isomorphism. This implies that cl_X is an isomorphism. $\square$

3.3. The image of restriction to fixed points.

Let X be a nonsingular, filtrable T-variety and let $i : X^T \to X$ be the inclusion map. For any subtorus $T' \subset T$, let $i_{T'} : X^T \to X^{T'}$ be the inclusion map. Because i factors through $i_{T'}$, the image of $i^* : A_T^*(X) \to A_T^*(X^T)$ is contained in the image of $i_{T'}^*$. This observation leads to the following description of the image of i^* over the rationals.

Theorem. *Let X be a nonsingular, filtrable variety with an action of T. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is the intersection of the images of

$$i_{T'}^* : A_T^*(X^{T'})_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

where T' runs over all subtori of codimension one in T.

Proof. Observe that a codimension one subtorus of T is the kernel of a primitive character χ (that is, χ is not divisible in M); such a χ is uniquely determined up to sign.

We argue by induction over the number of strata. If X is a unique stratum, then X is isomorphic to the total space of a T-equivariant vector bundle over its fixed point set. Therefore, i^* and $i_{T'}^*$ are surjective. In the general case, let $F \subset X$ be a closed stratum and let Y be the fixed point set in F. Let d, N and $c_d^T(N)$ be as in 3.2. Recall that $A_T^*(F)$ is isomorphic to $A_T^*(Y) = S \otimes A^*(Y)$ via j_Y^*. Under this isomorphism, $c_d^T(N)$

goes to $\prod_{i=1}^{d}(\chi_i \otimes 1 + 1 \otimes \alpha_i)$ where $\chi_1, \dots, \chi_d$ are the weights of $(T_xX)_-$ for $x \in Y$, and where $\alpha_1, \dots, \alpha_d$ are the corresponding Chern roots of N. Decompose this product as

$$\prod_{i=1}^{d}(\chi_i \otimes 1 + 1 \otimes \alpha_i) = \prod_{\chi} c_\chi$$

where χ runs over all primitive characters of T, and where c_χ denotes the product of the $\chi_i \otimes 1 + 1 \otimes \alpha_i$ such that χ_i is a multiple of χ. For χ as above, the kernel of χ is a subtorus of codimension one of T, and c_χ is the top equivariant Chern class of the normal bundle to $F^{\ker(\chi)}$ in $X^{\ker(\chi)}$.

Let $\gamma \in A_T^*(X^T)_{\mathbf{Q}}$ be in the image of $i_{T'}^*$ for all subtori T' of codimension one. By the induction hypothesis applied to U, the class $j_{U^T}^*\gamma$ is in the image of the map $A_T^*(U)_{\mathbf{Q}} \to A_T^*(U^T)_{\mathbf{Q}}$. Recall that $U^T = X^T \setminus Y$. Because j_U^* : $A_T^*(X) \to A_T^*(U)$ is surjective, we can find $\alpha \in A_T^*(Y)_{\mathbf{Q}}$ and $\beta \in A_T^*(X)_{\mathbf{Q}}$ such that $\gamma = \alpha + i^*\beta$.

Let χ be a primitive character of T. Then $\alpha = \gamma - i^*\beta$ is in the image of $i_{\ker(\chi)}^*$. By Proposition 3.2 applied to the component of $X^{\ker(\chi)}$ which contains Y, it follows that α is divisible by c_χ in $A_T^*(F^{\ker(\chi)})_{\mathbf{Q}} = A_T^*(Y)_{\mathbf{Q}}$. Hence, by the easy lemma below (applied to $A = A^*(Y)$), α is divisible by $\prod_\chi c_\chi = c_d^T(N)$. So $\gamma - i^*\beta$ is in $c_d^T(N)A_T^*(Y)_{\mathbf{Q}} = i^*(j_{F*}A_*^T(F)_{\mathbf{Q}})$ and γ is in $i^*A_T^*(X)_{\mathbf{Q}}$. $\square$

Lemma. *Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded ring with $A_0 = \mathbf{Q}$ and $A_n = 0$ for n large enough. Set $B := A \otimes S_{\mathbf{Q}}$ and endow B with the grading $B = \oplus_{n=0}^{\infty} A_n \otimes S_{\mathbf{Q}}$. Let f, g, h in B such that:*

(i) *f is divisible by g and by h in B, and*

(ii) *g_0 and h_0 are nonzero and coprime in $S_{\mathbf{Q}}$.*

Then f is divisible by gh in B.

3.4. A structure theorem for the equivariant Chow ring.

We will deduce from Theorem 3.3 a complete description of the ring $A_T^*(X)_{\mathbf{Q}}$ in the case where X is projective, nonsingular and contains finitely many invariant points and curves. Other applications of Theorem 3.3 will be given in §7.

Theorem. *Let X be a nonsingular, filtrable variety where T acts with finitely many fixed points $x_1, \dots, x_m$ and with finitely many invariant curves. Then the image of*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is the set of all $(f_1, \dots, f_m) \in S_{\mathbf{Q}}^m$ such that $f_i \equiv f_j \pmod{\chi}$ whenever x_i and x_j lie in an invariant irreducible curve C and the kernel of the T-action on C is the kernel of the character χ. If moreover all such characters χ are primitive in M, then the statement holds over the integers.

Proof. Let π be a primitive character of T. Then the space $X^{\ker(\pi)}$ is at most one-dimensional, because X contains finitely many invariant curves.

Moreover, $X^{\ker(\pi)}$ is nonsingular, and hence it consists of a disjoint union of points and nonsingular, irreducible curves; let C be such a curve.

If C contains a unique fixed point x, then $i_x^* : A_T^*(C) \to A_T^*(x) = S$ is an isomorphism. Otherwise, C is isomorphic to projective line. It follows that C contains two distinct fixed points x, y. Moreover, the image of

$$i_C^* : A_T^*(C) \to A_T^*(C^T) = S \times S$$

consists of all pairs (f, g) such that $f \equiv g \pmod{\chi}$ where T acts on C through the weight χ (a multiple of π). Indeed, this image is the S-module generated by $i_C^*[x] = (\chi, 0)$, $i_C^*[y] = (0, -\chi)$ and $i_C^*[C] = (1, 1)$. Now apply Theorem 3.3 to obtain the statement over the rationals.

In the case where all such χ are primitive, the proof of Theorem 3.3 adapts to yield the statement over the integers. Indeed, Lemma 3.3 is replaced by the following observation: if $u \in S$ is divisible (in S) by pairwise distinct primitive characters $\chi_1, \ldots, \chi_n$, then u is divisible (in S) by $\prod_{i=1}^n \chi_i$. □

4. Equivariant multiplicities at nondegenerate fixed points

4.1. Nondegenerate fixed points.

Let X be a scheme with an action of T. Call a fixed point $x \in X$ *nondegenerate* if the tangent space $T_x X$ contains no nonzero fixed point. Equivalently, 0 is not a weight of the T-module $T_x X$. The set of weights (with multiplicities) of this module will be called the weights of x.

Observe that a fixed point in a nonsingular T-variety is nondegenerate if and only if it is isolated; indeed, we have $T_x(X^T) = (T_x X)_0$.

Proposition. *Let $x \in X$ be a nondegenerate fixed point with weights $\chi_1, \ldots$ $\ldots, \chi_n$. Then there exists an open affine T-invariant neighborhood U of x such that:*

(i) *The map $i_* : A_*^T(x) = S \to A_*^T(U)$ is injective, where i is inclusion of x in X.*

(ii) *The image of i_* contains $\chi_1 \cdots \chi_n A_*^T(U)$.*

Proof. We may assume that X is an invariant subvariety of a T-module. Then there exist regular functions $f_1, \ldots, f_n$ on X which are eigenvectors of T of weights $\chi_1, \ldots, \chi_n$, such that $f_1, \ldots, f_n$ vanish at x and the differentials $df_1(x), \ldots, df_n(x)$ are a basis of $T_x X$. We can assume furthermore that x is the unique common zero to $f_1, \ldots, f_n$; then x is the unique fixed point in X.

For any T-invariant subvariety $Y \subset X$, denote by $j(Y)$ the smallest integer j such that $f_j \neq 0$ on Y. We claim that

$$\Big(\prod_{j \geq j(Y)} \chi_j\Big)[Y] \in i_* A_*^T(X).$$

This claim is checked by induction on the dimension of Y. Indeed, if this dimension is zero, then $Y = \{x\}$ and there is nothing to prove. For positive-dimensional Y, we have

$$\chi_{j(Y)}[Y] = [\mathrm{div}_Y(f_j)]$$

and the latter is a combination of T-invariant subvarieties Z with $\dim(Z) = \dim(Y) - 1$ and $j(Z) > j(Y)$. So

$$(\prod_{j>j(Y)} \chi_j)[\mathrm{div}_Y(f_j)] \in i_* A_*^T(X)$$

by the induction hypothesis.

Now assertion (ii) follows from the claim, whereas (i) is a consequence of the localization theorem. □

4.2. Equivariant multiplicities.

Let Q be the quotient field of S; let $N = \mathrm{Hom}(M, \mathbf{Z})$ be the dual lattice to M, and let $N_{\mathbf{Q}} := N \otimes_{\mathbf{Z}} \mathbf{Q}$ be the associated rational vector space. Then Q is the field of rational functions on $N_{\mathbf{Q}}$ with rational coefficients.

Theorem. *Let $x \in X$ be a nondegenerate fixed point and let $\chi_1, \dots, \chi_n$ be its weights.*

(i) *There exists a unique S-linear map $e_{x,X} : A_*^T(X) \to Q$ such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x. Moreover, the image of $e_{x,X}$ is contained in $(1/\chi_1 \cdots \chi_n)S$.*

(ii) *For any T-invariant subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and it coincides with $e_{x,Y}[Y]$.*

(iii) *The point x is nonsingular in X if and only if $\chi_1 \cdots \chi_n e_{x,X} X = 1$.*

Proof. (i) Let $U \subset X$ as in Proposition 4.1. Denote by $j : U \to X$ the inclusion. By this proposition, for any $\alpha \in A_*^T(X)$, there exists a unique $\beta \in S$ such that

$$(\chi_1 \cdots \chi_n) j^* \alpha = i_* \beta.$$

Define $e_{x,X}$ by

$$e_{x,X}(\alpha) := \frac{\beta}{\chi_1 \cdots \chi_n}.$$

Then $e_{x,X}$ has the required properties.

Uniqueness of $e_{x,X}$ follows from the localization theorem.

(ii) The assertion on the degree of $e_{x,X}Y$ follows from the definition of $e_{x,X}$ given above. Denote by $i_Y : Y \to X$ the inclusion map. Then it follows from (i) that composition

$$e_{x,X} \circ (i_Y)_* : A_*^T(X) \to Q$$

coincides with $e_{x,Y}$.

(iii) If $e_{x,X}X = 1/\chi_1 \cdots \chi_n$ then $\dim_x(X) = n$ by (ii). But $\dim(T_xX) = n$ and hence x is nonsingular. Conversely, if x is nonsingular, then we can find rational functions $f_1, \ldots, f_n$ which are defined at x, eigenvectors of T of weights $\chi_1, \ldots, \chi_n$ and such that the divisors $\mathrm{div}(f_1), \ldots, \mathrm{div}(f_n)$ intersect transversally at x. Then we have

$$\chi_1 \cdots \chi_n[U] = [x]$$

in $A_*^T(U)$, and hence $\chi_1 \cdots \chi_n e_{x,X}[X] = 1$. □

For any T-invariant subvariety $Y \subset X$, we set $e_{x,X}[Y] := e_x[Y]$ (this makes sense because of (ii)) and we call $e_x[Y]$ the *equivariant multiplicity* of Y at x.

Corollary. *Let X be a scheme with an action of T such that all fixed points in X are nondegenerate, and let $\alpha \in A_*^T(X)$. Then we have in $A_*^T(X) \otimes_S Q$:*

$$\alpha = \sum_{x \in X^T} e_x(\alpha)[x].$$

Proof. By the localization theorem, i_* is surjective over Q. Therefore, we may assume that $\alpha = [x]$ for some $x \in X^T$. Then the statement follows from (i) above. □

4.3. The behavior of equivariant multiplicities under proper morphisms.

The following easy result will be used in §6 to compute equivariant multiplicities of Schubert varieties.

Proposition. *Let X, X' be schemes with an action of T and let $\pi : X' \to X$ be a proper surjective T-equivariant morphism which is generically finite of degree d. Let $x \in X$ be a nondegenerate fixed point such that all fixed points in the fiber $\pi^{-1}(x)$ are nondegenerate in X'. Then we have*

$$e_x[X] = \frac{1}{d} \sum_{x' \in X'^T, \pi(x') = x} e_{x'}[X'].$$

Proof. We may replace X by any T-invariant neighborhood of x. Therefore, we may assume that x is the unique fixed point in X. Then all fixed points in X' map to x by π. So we have by Corollary 4.2:

$$[X'] = \sum_{x' \in X'^T} e_{x'}[X'][x'].$$

Applying π_* to this equation, we obtain

$$d[X] = (\sum_{x' \in X'^T} e_{x'}[X'])[x].$$

On the other hand, we have $[X] = (e_x[X])[x]$. Together with Proposition 4.1 (i), this gives our formula. □

4.4. The case of an attractive fixed point.

Let X be a scheme with an action of T. Call a fixed point $x \in X$ *attractive* if all weights in the tangent space T_xX are contained in some open half-space of $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$. We denote by $\chi_1, \ldots, \chi_n$ these weights, and we set

$$\sigma_x := \{\lambda \in N_{\mathbf{R}} \mid \langle \lambda, \chi_i \rangle \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Then σ_x is a rational polyhedral convex cone in $N_{\mathbf{R}}$ with a nonempty interior σ_x^0. Any $\lambda \in \sigma_x^0$ defines a grading of the algebra of regular functions over X_x and hence a positive rational number: the multiplicity of this graded algebra (recall that the multiplicity of a finitely generated, graded k-algebra

$$A = \oplus_{n=0}^{\infty} A_n$$

is the unique number e such that

$$\sum_{m=0}^{n} \dim_k(A_m) \sim_{n \to \infty} e \frac{n^d}{d!}$$

where d is the dimension of A, see [39]).

Proposition. *Let $x \in X$ be an attractive fixed point with weights $\chi_1, \ldots, \chi_n$ and let $\lambda \in \sigma_x^0$.*

(i) *The set $X_x := X_+(x, \lambda)$ is independent of λ and this set is the unique affine T-invariant open neighborhood of x in X.*

(ii) *The rational function $e_x[X]$ is defined at λ and its value is the multiplicity of the algebra of regular functions on X_x graded via the action of λ.*

Proof. (i) Let U be an open T-invariant affine neighborhood of x in X. Because all weights of T_xX lie in the open half-space where $\lambda > 0$, the set U contains $X_+(x, \lambda)$ as an open T-invariant subset. Moreover, any T-invariant regular function on $X_+(x, \lambda)$ is constant, and hence the same holds for U. It follows that x is the unique closed orbit in U. If $U \neq X_+(x, \lambda)$ then the closed, T-invariant subset $U \setminus X_+(x, \lambda)$ contains a closed orbit and does not contain x, a contradiction.

(ii) Let $Y \subset X$ be a T-invariant subvariety. Denote by $(\varepsilon_x Y)(\lambda)$ the multiplicity of the algebra of regular functions on $Y \cap X_x$. Then it is easily checked that for any rational function f on $Y \cap X_x$ which is an eigenvector of T of weight χ, we have

$$\langle \chi, \lambda \rangle (\varepsilon_x Y)(\lambda) = \sum_D \mathrm{ord}_D(f)(\varepsilon_x D)(\lambda)$$

(sum over all irreducible, T-invariant divisors $D \subset Y$). By induction over the dimension of Y, it follows that the function

$$\begin{array}{ccc} \sigma_x^0 & \to & \mathbf{Q}, \\ \lambda & \mapsto & (\varepsilon_x Y)(\lambda) \end{array}$$

extends uniquely to a rational function on $N_{\mathbf{Q}}$, that is, to an element $\varepsilon_x Y$ of $\mathcal{Q}$. Moreover, for any rational function f on Y which is an eigenvector of T of weight χ, we have

$$\chi \, \varepsilon_x Y = \sum_D \operatorname{ord}_D(f) \, \varepsilon_x D.$$

So, by Theorem 2.1, the assignment $Y \to \varepsilon_x Y$ induces a S-linear map $\varepsilon_x : A_*^T(X) \to \mathcal{Q}$ such that $\varepsilon_x[x] = 1$. By Proposition 2.2, we conclude that $\varepsilon_x = e_x$. $\square$

4.5. The connection with Rossmann's equivariant multiplicity. Let X be a nonsingular variety with an action of T, let $x \in X$ be an isolated fixed point with weights $\chi_1, \ldots, \chi_n$ and let $Y \subset X$ be a T-invariant subvariety. An equivariant multiplicity $\rho_x Y$, with values in S, has been defined by Rossmann (see [38]; Rossmann's notation has been changed here). In fact, this notion is equivalent to ours, as shown by the following:

Theorem. *For any isolated fixed point x in a nonsingular T-variety X, and for any T-invariant subvariety $Y \subset X$, we have*

$$[Y]_x = \rho_x Y = \chi_1 \cdots \chi_n e_x[Y]$$

where $\chi_1, \ldots, \chi_n$ are the weights of x.

Proof. Restriction to x induces an S-linear map $\varepsilon_x : A_*^T(X) \to S$ such that $\varepsilon_x[x] = \chi_1 \ldots \chi_n$ and that $\varepsilon_x[Y] = 0$ for any T-invariant subvariety $Y \subset X$ which does not contain x. By Proposition 4.2 (i), we then have $\varepsilon_x = \chi_1 \ldots \chi_n e_x$.

So it suffices to prove that $\rho_x Y = \chi_1 \ldots \chi_n e_x[Y]$. For a T-invariant subvariety $Y \subset X$, denote by $C_x Y \subset T_x X$ its tangent cone at x. By Rossmann's definition, we have $\rho_x Y = \rho_0(C_x Y)$. We claim that $e_x[Y] = e_0[C_x Y]$. Indeed, let $\tilde{X}$ be the space obtained from the blow-up of $X \times \mathbf{A}^1$ (resp. $Y \times \mathbf{A}^1$) at $(x, 0)$ by removing the projectivization of $T_x X$ (resp. of $C_x Y$). Let T act on $X \times \mathbf{A}^1$ by acting trivially on $\mathbf{A}^1$. This defines an action of T on $\tilde{X}$, such that $\tilde{Y}$ is a T-invariant subvariety. Moreover, we have a flat, T-invariant morphism $p : \tilde{Y} \to \mathbf{A}^1$ such that $p^{-1}(0) \simeq C_x Y$ and that $p^{-1}(t) \simeq Y$ for all $t \neq 0$. It follows that $[Y] = [C_x Y]$ in $A_*^T(\tilde{Y})$. Intersecting in $\tilde{X}$ with the fixed point scheme (which is the strict transform of $x \times \mathbf{A}^1$), we obtain $[Y]_x = [C_x Y]_0$ which implies our claim.

So we may assume that X is a T-module with weights $\chi_1, \ldots, \chi_n$, that x is the origin, and that Y is invariant under scalar multiplication. Set $T' := T \times \mathbf{G}_m$, denote by θ the character $(t, u) \mapsto u$ of T', and let T' act on X with weights $\chi_1 + \theta, \ldots, \chi_n + \theta$. Then the origin is an attractive fixed point for this action. Using Proposition 4.4 (ii) and [R] p. 316, we then obtain

$$(\chi_1 + \theta) \cdots (\chi_n + \theta) e'_x[Y] = \rho'_x Y$$

where e', ρ' denote equivariant multiplicities with respect to T'. Now we conclude by the following easy consequence of Proposition 4.2.

Lemma. *Let $x \in X$ be a nondegenerate T-fixed point. Assume that the action of T on X extends to an action of a torus $T' \supset T$ which fixes x. Then x is nondegenerate for the T'-action. Moreover, for any T'-invariant subvariety $Y \subset X$, the T'-equivariant multiplicity $e'_x[Y]$ specializes to $e_x[Y]$ under the map $S' \to S$.*

5. Equivariant Chow groups of toric varieties

5.1. Toric varieties and fans.

Let X be a toric variety, that is, X is normal and T acts on X with a dense orbit isomorphic to T. Recall that X is determined by its fan Σ in $N_{\mathbf{R}}$, see, e.g., [18]. The cones of Σ parametrize the orbits in X; we denote by $\sigma \to \Omega_\sigma$ this parametrization, and by $V(\sigma)$ the closure of Ω_σ in X. Then $\Omega_\sigma = T/T_\sigma$ where T_σ is the subtorus of T with character lattice $M/M \cap \sigma^\perp$ and with lattice of one-parameter subgroups N_σ (the subgroup of N generated by $N \cap \sigma$). In particular, the dimension of Ω_σ is the codimension of σ.

Proposition. *Let X be a toric variety with fan Σ. Then the S-module $A_*^T(X)$ is defined by generators $F_\sigma = [V(\sigma)]$ (where $\sigma \in \Sigma$) and relations*

$$\chi F_\sigma - \sum_\tau \langle \chi, n_{\sigma\tau} \rangle F_\tau$$

where $\chi \in M \cap \sigma^\perp$; the summation is over all $\tau \in \Sigma$ which contain σ as a face of codimension one, and $n_{\sigma\tau} \in N/N_\sigma$ is the unique generator of the semigroup $(\tau \cap N)/N_\sigma$.

Proof. The T-invariant subvarieties in X are the orbit closures $V(\sigma)$. Moreover, any rational function f on $V(\sigma)$ which is an eigenvector of T is determined up to scalar multiplication by its weight χ: a character of T which vanishes identically on σ. By [18] p. 61, the divisor of f on $V(\sigma)$ is then $\sum_\tau \langle \chi, n_{\sigma\tau} \rangle F_\tau$. We conclude by Theorem 2.1. $\square$

5.2. Equivariant multiplicities of toric varieties.

For a closed convex cone σ in $N_{\mathbf{R}}$, we denote by

$$\sigma^\vee = \{x \in M_{\mathbf{R}} \mid \langle \lambda, x \rangle \geq 0 \ \forall \ \lambda \in \sigma\}$$

its dual cone. Moreover, for $\lambda \in N_{\mathbf{R}}$, we set

$$P_\sigma(\lambda) := \{x \in \sigma^\vee \mid \langle \lambda, x \rangle \leq 1 \}.$$

If λ is in σ^0, then $P_\sigma(\lambda)$ is a convex polytope.

Proposition. *Let X be a toric variety with a fixed point x, and let σ be the corresponding d-dimensional cone in Σ. Then, notation being as in 4.4,*

x is attractive, and $\sigma = \sigma_x$. Moreover, for any $\lambda \in \sigma^0$, the equivariant multiplicity $e_x[X](\lambda)$ is $d!$ times the volume of $P_\sigma(\lambda)$.

Proof. Recall that x is contained in a unique T-invariant open affine subset X_σ of X. Moreover, the set of weights of T in the algebra of regular functions $k[X_\sigma]$ is the intersection of M with $\sigma^\vee$. By Proposition 4.4 (i), it follows that x is attractive and that $\sigma_x = \sigma$. For any $\lambda \in \sigma^0$, we have for the grading of $k[X_\sigma]$ defined by λ:

$$\sum_{m=0}^{n} \dim k[X_\sigma]_m = \mathrm{card}\{\chi \in \sigma^\vee \mid \langle \chi, \lambda \rangle \leq n\}.$$

This function of n grows like n^d times the volume of $P_\sigma(\lambda)$. □

For x and σ as above, we denote $e_x : A_*^T(X) \to \mathcal{Q}$ by e_σ. More generally, for any $\sigma \in \Sigma$, we will define an S-linear map

$$e_\sigma : A_*^T(X) \to \mathcal{Q}_\sigma$$

where $\mathcal{Q}_\sigma$ is the field of rational functions on $(N_\sigma)_{\mathbf{Q}}$.

Let X_σ be the unique T-invariant open affine subset of X which contains Ω_σ as a closed subset. Then there exists a unique T_σ-toric variety S_σ such that X_σ is equivariantly isomorphic to $T \times_{T_\sigma} S_\sigma$. Moreover, S_σ is affine and contains a fixed point of T_σ. We define e_σ as composition

$$A_*^T(X) \to A_*^T(T \times_{T_\sigma} S_\sigma) = A_*^{T_\sigma}(S_\sigma) \to \mathcal{Q}_\sigma$$

where the first arrow is restriction to $T \times_{T_\sigma} S_\sigma$, and the second one is T_σ-equivariant multiplicity for S_σ.

Corollary. *Let σ, τ be cones in Σ.*

(i) *If τ is not contained in σ, then $e_\sigma F_\tau = 0$.*

(ii) *If τ is contained in σ, then, for any λ in the relative interior of σ, the value at λ of $e_\sigma F_\tau$ is $(\dim(\sigma) - \dim(\tau))!$ times the volume of the convex polytope in $\tau^\perp/\sigma^\perp$, image of the set $P_\sigma(\lambda) \cap \tau^\perp$. In particular, $e_\sigma F_\sigma = 1$.*

Proof. Apply the proposition above to the affine toric variety $S_\sigma \cap V(\tau)$ for the torus T_σ/T_τ. Then the associated cone is $(\sigma^\vee \cap \tau^\perp)/\sigma^\perp$ in the linear space $\tau^\perp/\sigma^\perp$. □

5.3. An embedding of the equivariant Chow group.

We show that the equivariant multiplicities constructed in 5.2 separate points in the equivariant Chow group of any toric variety. A complete description of this group will be given in 5.4 in the simplicial case; the general case is still open.

Proposition. *For any toric variety X with fan Σ, the S-linear map*

$$\prod_{\sigma\in\Sigma} e_\sigma : A_*^T(X) \to \prod_{\sigma\in\Sigma} \mathcal{Q}_\sigma$$

is injective. Moreover, this map induces an isomorphism over $\mathcal{Q}$.

Proof. By induction over the number of cones in Σ. Choose a maximal cone Σ and consider the commutative diagram

$$\begin{array}{ccccccccc} & & A_*^T(\Omega_\sigma) & \to & A_*^T(X) & \to & A_*^T(X\setminus\Omega_\sigma) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{Q}_\sigma & \to & \prod_{\tau\in\Sigma}\mathcal{Q}_\tau & \to & \prod_{\tau\in\Sigma\setminus\{\sigma\}}\mathcal{Q}_\tau & \to & 0 \end{array}$$

which induces a map $A_*^T(\Omega_\sigma) \to \mathcal{Q}_\sigma$. This map is S-linear and sends $[\Omega_\sigma] = F_\sigma$ to $e_\sigma F_\sigma = 1$. Moreover, $A_*^T(\Omega_\sigma) = A_*^T(T/T_\sigma)$ is the symmetric algebra over $M/M\cap\sigma^\perp$. Therefore, our map $A_T^*(\Omega_\sigma) \to \mathcal{Q}_\sigma$ identifies with inclusion of this symmetric algebra into its quotient field $\mathcal{Q}_\sigma$. $\square$

Remark. There exist projective toric surfaces X such that $A_1(X)$ has torsion, see [18] p. 65. For such X, the S-module $A_*^T(X)$ cannot be free: the assumption of nonsingularity is necessary in Corollary 3.2.1.

5.4. The case of simplicial toric varieties.

Recall that the toric variety X is *simplicial* if each cone of its fan is generated by linearly independent vectors; equivalently, X has quotient singularities by finite groups. In this case, we will describe the equivariant Chow group $A_*^T(X)$ in terms of piecewise polynomial functions.

Let $\Sigma(1)$ be the set of one-dimensional cones in Σ. For any $\rho \in \Sigma(1)$, the semigroup $\rho \cap N$ has a unique generator n_ρ. Because Σ is simplicial, any continuous and piecewise linear function on Σ is uniquely defined by its values at the n_ρ for $\rho \in \Sigma(1)$. In particular, there is a unique continuous piecewise linear function φ_ρ such that $\varphi_\rho(n_\rho) = 1$ and $\varphi_\rho(n_{\rho'}) = 0$ for all $\rho' \in \Sigma(1)$, $\rho' \neq \rho$.

Define the multiplicity of a cone $\sigma \in \Sigma$ by

$$\mathrm{mult}(\sigma) := [N_\sigma : \sum_{\rho\in\sigma(1)} \mathbf{Z} n_\rho]$$

and set

$$\varphi_\sigma := \mathrm{mult}(\sigma) \prod_{\rho\in\sigma(1)} \varphi_\rho.$$

Then φ_σ is a continuous, piecewise polynomial function on Σ which is homogeneous of degree $\dim(\sigma) = \mathrm{codim}_X V(\sigma)$ and vanishes outside the star of σ.

We denote by R_Σ the ring of continuous piecewise polynomial functions on Σ which take rational values on N. Then R_Σ is a $S_{\mathbf{Q}}$-algebra; let A_Σ be the S-submodule of R_Σ generated by the φ_σ, $\sigma \in \Sigma$. Observe that A_Σ is not a subalgebra of R_Σ in general; but A_Σ generates R_Σ as a $\mathbf{Q}$-vector space (see e.g. [10] Corollary 1.2).

Theorem. *Let X be a simplicial toric variety.*

(i) *For all cones σ, τ in Σ, restriction to σ^0 of $\varphi_\tau/\varphi_\sigma$ is defined and equal to $e_\sigma F_\tau$. In particular, $1/\varphi_\sigma$ coincides with $e_\sigma X$ on σ^0.*

(ii) *The S-linear map*

$$\begin{array}{ccc} A_*^T(X) & \to & \prod_{\sigma\in\Sigma} \mathcal{Q}_\sigma \\ u & \mapsto & (e_\sigma u/e_\sigma X)_{\sigma\in\Sigma} \end{array}$$

is injective, and its image is A_Σ.

Proof. (i) is a straightforward computation, each polytope $P_\sigma(\lambda)$ being a simplex.

(ii) By Proposition 5.3, our map is injective; by Proposition 5.1 and (i), its image is the S-module generated by the $(e_\sigma F_\tau/e_\sigma X)_{\sigma\in\Sigma} = \varphi_\tau$ $(\tau \in \Sigma)$. □

Remark. For X as above, the rational equivariant Chow group $A_T^*(X)_{\mathbf{Q}}$ carries an intersection product which makes it a graded algebra. Indeed, X is a quotient of a smooth toric variety with respect to another torus $\tilde{T}$, by a subtorus of $\tilde{T}$ which acts with finite isotropy groups (see e.g. [11] §1), so the assertion follows from [15] Theorem 4.

It can be shown as in [11] §3 that the map $A_T^*(X)_{\mathbf{Q}} \to R_\Sigma$ is an algebra isomorphism. As a consequence, the equivariant Riemann–Roch theorem for linearized coherent sheaves on X (proved in [11] for complex toric varieties, by using equivariant cohomology) can be obtained in a purely algebraic way.

6. Equivariant Chow groups for actions of connected reductive groups

6.1. A refined presentation of equivariant Chow groups.

We obtain a refinement of Theorem 2.1 for schemes with a torus action which extends to an action of a larger group.

Proposition. *Let X be a scheme with an action of a connected solvable linear algebraic group Γ, and let T be a maximal torus of Γ.*

(i) *The equivariant Chow group $A_*^T(X)$ is generated as an S-module by the classes $[Y]$ where $Y \subset X$ is a Γ-invariant subvariety.*

(ii) *If moreover the S-module $A_*^T(X)$ is free, then the S-module of relations between these classes is generated by the $[\mathrm{div}_Y(f)] - \chi[Y]$ where $Y \subset X$ is a Γ-invariant subvariety, and where f is a rational function on Y which is an eigenvector of Γ of weight χ.*

By Corollary 3.2.1, the assumption of (ii) is satisfied for projective, nonsingular X such that the abelian group $A_*(X^T)$ is free. We ignore whether (ii) holds in full generality.

Proof. Let $A_*^{(\Gamma)}(X)$ be the S-module defined by generators $[Y]$ and relations $[\mathrm{div}_Y(f)] - \chi[Y]$ as above. Then $A_*^{(\Gamma)}(X)$ is graded, where the degree of $[Y]$ is the dimension of Y. Consider the natural S-linear map

$$u : A_*^{(\Gamma)}(X) \to A_*^T(X)$$

which is homogeneous of degree zero. This induces a map

$$\overline{u} : A_*^{(\Gamma)}(X)/MA_*^{(\Gamma)}(X) \to A_*^T(X)/MA_*^T(X).$$

The right-hand side is $A_*(X)$ by Corollary 2.3.1, whereas the left-hand side is the abelian group defined by generators $[Y]$ and relations $[\mathrm{div}_Y(f)]$ for $Y \subset X$ invariant by Γ, and f a Γ-semi-invariant rational function on Y. By [21] Theorem 1, the map $\overline{u}$ is an isomorphism. We conclude by the graded Nakayama lemma, which can be applied because the degrees in $A_*^{(\Gamma)}(X)$ and $A_*^T(X)$ are at most the dimension of X. □

6.2. The action of the Weyl group on the equivariant Chow group.

Let G be a connected reductive group. Choose a Borel subgroup $B \subset G$ and a maximal torus T of B. Denote by W the Weyl group and by R the root system of (G, T). We have the set R_+ of positive roots and its subset Σ of simple roots. For $\alpha \in \Sigma$, we denote by $s_\alpha \in W$ the corresponding simple reflection and by $P_\alpha := B \cup Bs_\alpha B$ the corresponding minimal parabolic subgroup. Recall that the group W is generated by the s_α, $\alpha \in \Sigma$.

Let X be a scheme with an action of G. Then W acts on the equivariant Chow group $A_*^T(X)$ (this follows for example from the presentation of this group given in 2.1). To describe this action, it suffices to calculate the action of a simple reflection s_α on the class of a B-invariant subvariety $Y \subset X$. This is what we do in our next proposition after introducing more notation.

Let $P \subset G$ be a parabolic subgroup, and let $Y \subset X$ be a B-invariant subvariety. Denote by $P \times_B Y$ the quotient of $P \times Y$ by the B-action given by $b \cdot (p, y) := (pb^{-1}, by)$. Then the map $P \times Y \to X : (p, y) \mapsto py$ factors through a proper morphism $P \times_B Y \to PY$. In particular, PY is closed in X. If moreover the parabolic subgroup P is minimal and moves Y in X, then $\dim(P \times_B Y) = \dim(Y) + 1 = \dim(PY)$ and hence the morphism $P \times_B Y \to PY$ is generically finite.

Proposition. *Let X be a scheme with an action of G, let $Y \subset X$ be a B-invariant subvariety, and let α be a simple root with associated minimal parabolic subgroup $P = P_\alpha$.*

(i) *If Y is P-invariant, then $s_\alpha[Y] = [Y]$.*

(ii) *If Y is not P-invariant, then*

$$s_\alpha[Y] = [Y] - d(Y, \alpha)\alpha[PY]$$

where $d(Y, \alpha)$ is the degree of the morphism $P \times_B Y \to PY$.

(iii) *If moreover $k = \mathbf{C}$ and PY contains a dense B-orbit, then, denoting by P_Y the isotropy subgroup in P of a general point in PY, we have: $d(Y, \alpha) = 2$ if the image of P_Y in $\mathrm{Aut}(P/B) \simeq \mathrm{PGL}_2$ is the normalizer of a maximal torus in PGL_2, and $d(Y, \alpha) = 1$ otherwise.*

Proof. (i) Because $PY = Y$, we have $s_\alpha Y = Y$ and hence $s_\alpha[Y] = [Y]$.

(ii) Set $Z := P \times_B Y$ with inclusion map $i : Y \to Z$ and projection $\pi : Z \to P/B$. Then P/B is isomorphic to projective line where B acts by the character $-\alpha$. So π can be seen as a B-semi-invariant rational function on Z with divisor $-[i(Y)] + [s_\alpha i(Y)]$. Therefore, we have in $A_*^T(Z)$

$$-\alpha[Z] = -[i(Y)] + s_\alpha[i(Y)].$$

Now consider the proper, surjective morphism

$$f : Z = P \times_B Y \to PY$$

and apply f_* to the identity above. Then we obtain our formula, because f is P-equivariant and maps $i(Y)$ isomorphically to Y.

(iii) is implicit in [37] §4 and in [27] §3; it can be checked as follows. Choose $y \in Y$ such that By is dense in Y. Then the space $P \times_B Y$ contains $P \times_B By$ as a dense P-orbit. This orbit is mapped by f onto $Py = P/P_y$. Therefore, the degree of f is $[P_y : B_y] = [P_y : P_y \cap B]$. Now $P_y = P_Y$ acts on P/B with a dense orbit (because B has a dense orbit in Py) and with a finite orbit $P_y/P_y \cap B$. So our statement follows by inspection of groups acting on projective line with a dense orbit. □

Corollary. *Let X, Y and α be as above, and let $x \in X$ be a nondegenerate fixed point of T. Then $s_\alpha x$ is nondegenerate, and we have*

$$e_{s_\alpha x}[Y] = \begin{cases} s_\alpha e_x[Y] & \text{if } PY = Y, \\ s_\alpha(e_x[Y] - d(Y,\alpha)\alpha e_x[PY]) & \text{otherwise.} \end{cases}$$

Proof. By uniqueness of $e_x : A_*^T(X) \to \mathcal{Q}$, we have

$$e_{s_\alpha x}[Y] = s_\alpha e_x[s_\alpha Y].$$

This formula implies both statements. Alternatively, one may apply apply Proposition 4.3 to the morphism $f : P \times_B Y \to PY$. Then the fixed points above x are $i(x)$ and $s_\alpha i(x)$. Both are nondegenerate, and we have

$$e_{i(x)}[P \times_B Y] = -\alpha^{-1} e_x[Y], \; e_{s_\alpha i(x)}[P \times_B Y] = \alpha^{-1} s_\alpha(e_x[Y]).$$ □

6.3. The action of operators of divided differences.

Let $\mathcal{Q}[W]$ be the twisted group ring of W with coefficients in $\mathcal{Q}$ (the fraction field of S), that is, $\mathcal{Q}[W]$ is the $\mathcal{Q}$-vector space with basis W and multiplication

$$(\sum_{u \in W} f_u\, u)(\sum_{v \in W} g_v\, v) = \sum_{w \in W} (\sum_{w = uv} f_u u(g_v))\, w.$$

Let α be a simple root. Following [13], define an *operator of divided differences* $D_\alpha \in \mathcal{Q}[W]$ by

$$D_\alpha = \frac{1}{\alpha}(id - s_\alpha).$$

Then D_α acts on $\mathcal{Q}$. Observe that

$$D_\alpha(uv) = uD_\alpha(v) + D_\alpha(u)s_\alpha(v) \quad \forall u, v \in \mathcal{Q}$$

and that

$$D_\alpha(\chi) = \langle \chi, \alpha^\vee \rangle \quad \forall \chi \in M.$$

It follows that D_α leaves S invariant.

Theorem. *Let X be a scheme with an action of G. Then there exists a unique action of D_α on $A_*^T(X)$ such that:*

(i) *For all $u \in S$ and $v \in A_*^T(X)$, we have*

$$D_\alpha(uv) = uD_\alpha(v) + D_\alpha(u)s_\alpha(v).$$

(ii) *For any B-invariant subvariety Y of X, we have*

$$D_\alpha[Y] = \begin{cases} 0 & \text{if } P_\alpha Y = Y, \\ d(Y,\alpha)[P_\alpha Y] & \text{if } P_\alpha Y \neq Y \end{cases}$$

where $d(Y,\alpha)$ denotes the degree of the map $P_\alpha \times_B Y \to P_\alpha Y$, see 6.2.

Moreover, we have for all $u \in A_*^T(X)$:

$$\alpha D_\alpha(u) = u - s_\alpha(u).$$

Finally, D_α commutes with G-equivariant proper push-forwards, flat pull-backs and Gysin morphisms associated to l.c.i. morphisms.

Proof. Uniqueness of D_α follows from Proposition 6.1. For existence, we fix an index i and we construct an operator $D_\alpha : A_i^T(X) \to A_{i+1}^T(X)$ as follows. Let V and U be as in 2.1, and set $Z := X \times U$. Then the quotient map $Z \to Z/G$ factors through $p : Z/T \to Z/B$ followed by $q : Z/B \to Z/P$ where $P := P_\alpha$. Since $p : Z/T \to Z/B$ is flat with fibers isomorphic to affine space, we can identify $A_*(Z/T)$ with $A_*(Z/B)$ via p^* (see [22] Theorem 8.3). Define

$$\begin{array}{cccc} D_\alpha : & A_i(Z/B) & \to & A_{i+1}(Z/B), \\ & u & \mapsto & q^*(q_* u). \end{array}$$

Arguing as in the proof of Proposition 1 in [15], we see that D_α is independent of the choice of V. To prove that (i) holds, observe that $q : Z/B \to Z/P$ is the projective line bundle associated to the action of P on P/B. It follows that

$$A_*(Z/B) = q^* A_*(Z/P) \oplus c \cap q^* A_*(Z/P)$$

where $c \in A^1(Z/B)$ identifies to multiplication by α. Moreover, $q^* A_*(Z/P)$ consists of fixed points of s_α in $A_*(Z/B)$. So, using the projection formula, it suffices to check (i) for $v = [Z/B]$ and for $v = \alpha[Z/B]$. But both cases reduce to the well-known formula

$$q^* q_* u = \frac{u - s_\alpha(u)}{\alpha}$$

in S (a consequence of Propositions 3 and 4 in [14]).

For (ii), let $Y \subset X$ be a B-stable subvariety and set $Y' := (Y \times U)/B$. If $PY = Y$, then $\dim(q(Y')) = \dim(Y') + 1$. Therefore, $q_*[Y'] = 0$ and $D_\alpha[Y] = 0$. On the other hand, if $PY \neq Y$, then $q|_{Y'} : Y' \to q(Y')$ is generically finite of degree $d(Y, \alpha)$. Indeed, we have $d(Y, \alpha) = [P_y : B_y]$ for general $y \in Y$ (see the proof of Proposition 6.2) and hence $d(Y, \alpha)$ is the cardinality of the set

$$\{p \in P \mid py \in By\}/B.$$

But this set identifies with the fiber of q at $(y, v)B$ for any $v \in U$. It follows that

$$D_\alpha[Y] = d(Y, \alpha)[q^{-1}(q(Y'))] = d(Y, \alpha)[PY].$$

By Proposition 6.2, we have $[Y] - s_\alpha[Y] = \alpha D_\alpha[Y]$ for any B-invariant subvariety Y of X. Using Proposition 6.1 and (i), it follows that $u - s_\alpha u = \alpha D_\alpha(u)$ for any $u \in A_*^T(X)$.

Finally, we check that D_α commutes with G-equivariant proper push-forward. Let X' be a scheme with a G-action and let $f : X' \to X$ be a proper, G-equivariant morphism. Set $Z' := X' \times U$; consider the induced maps $f_B : Z'/B \to Z/B$, $f_P : Z'/P \to Z/P$ and $q' : Z'/B \to Z'/P$. Then we have a Cartesian square

$$\begin{array}{ccc} Z'/B & \rightarrow & Z'/P \\ \downarrow & & \downarrow \\ Z/B & \rightarrow & Z/P \end{array}$$

with flat horizontal arrows. It follows that $(f_P)_* q'^* = q^*(f_B)_*$ and hence that $(f_B)_* q'^* q'_* = q^* q_* (f_B)_*$ as required. The proofs of the other assertions are similar. $\square$

6.4. The ring of operators of divided differences.

Following [13], denote by $\mathbf{D}$ the subring of $Q[W]$ generated by $S[W]$ and by the operators D_α for all simple roots α. We call $\mathbf{D}$ the *ring of operators of divided differences.* We have

$$S[W] \subset \mathbf{D} \subset Q[W].$$

Moreover, $\mathbf{D}$ can be seen as the ring of endomorphisms of the abelian group S generated by the operators D_α and by arbitrary multiplications by elements of S. Observe that $\mathbf{D}$ consists of S^W-linear endomorphisms, where S^W denotes the ring of W-invariants in S.

For any scheme X with an action of G, the ring $S[W]$ acts on the equivariant Chow group $A_*^T(X)$. By Theorem 6.3, this action extends to an action of the ring $\mathbf{D}$. We will describe the latter action in the case where $X = G/B$ is the flag variety of G. For this, we introduce the following notation.

For any $w \in W$, we choose a reduced decomposition

$$w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$$

where $l = l(w)$ is the length of w. We define $D_w \in \mathbf{D}$ by

$$D_w := D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_l}.$$

By [13], the operator D_w is independent of the choice of the reduced decomposition of w. Moreover, the D_w ($w \in W$) are a basis of $\mathbf{D}$ as a left S-module.

The following result and its proof are algebraic versions of the work of Arabia for T-equivariant cohomology (see [1]). They can be deduced from this work when $k = \mathbf{C}$, using the isomorphism $A_T^*(G/B) \simeq H_T^*(G/B)$ (a consequence of Corollary 3.2.2).

Proposition. *The* $\mathbf{D}$*-module* $A_*^T(G/B)$ *is freely generated by the class of the* B*-fixed point* x*. Moreover,* $D_w[x]$ *is the class of the Schubert variety* $\overline{BwB}/B$ *in* $A_*^T(G/B)$*. Finally, denoting by* $\int_{G/B} : A_T^*(G/B) \to S$ *the push-forward for the structural morphism, we have for all* $D \in \mathbf{D}$*:*

$$\int_{G/B} D[x] = D(1).$$

Proof. First observe that the S-module $A_*^T(G/B)$ is free, e.g. by Corollary 3.2. We construct a basis of this module as follows. For $w \in W$, denote by $C(w)$ the Schubert cell BwB/B and by $X(w)$ the closure of $C(w)$ in G/B. Then, by the Bruhat decomposition, each $C(w)$ is an affine space of dimension $l(w)$, and G/B is the disjoint union of the $C(w)$ ($w \in W$). In fact, the $C(w)$ are the Bialynicki–Birula cells associated to a one-parameter subgroup in the interior of the positive Weyl chamber. So, by Corollary 3.2, the classes $[X(w)]$ are a basis of the S-module $A_*^T(G/B)$.

Let w be a nontrivial element of W. Write $w = s_\alpha \tau$ with α simple and $l(\tau) = l(w) - 1$. Then the map $P_\alpha \times_B X(\tau) \to X(w)$ is birational. Using Theorem 6.3, it follows that $D_\alpha[X(\tau)] = [X(w)]$ and hence that $[X(w)] = D_w[x]$ in $A_*^T(G/B)$. Therefore, we have $A_*^T(G/B) = \mathbf{D}[x]$. Furthermore, the S-module $\mathbf{D}$ is torsion-free of rank $|W|$, which is the rank of the S-module $A_*^T(G/B)$. It follows that the map

$$\begin{array}{rcl} \mathbf{D} & \to & A_*^T(G/B), \\ D & \mapsto & D[x] \end{array}$$

is an isomorphism.

Finally, for any $w \in W$, we have

$$\int_{G/B} D_w[x] = \int_{G/B} [X(w)] = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, our formula for $\int_{G/B}$ holds when $D = D_w$. By linearity, it holds for all $D \in \mathbf{D}$. □

Remark. The proof above shows by geometric arguments that the D_w are independent of the choice of reduced decompositions, and that they form a basis of the left S-module $\mathbf{D}$.

6.5. Equivariant multiplicities of Schubert varieties.
Denote by $i : (G/B)^T \to G/B$ the inclusion of the fixed point set, and identify $(G/B)^T$ with W. Then $A_*^T((G/B)^T)$ identifies with the ring $S[W]$ as a S-algebra with a compatible action of W.

Proposition. (i) *The image of*

$$i^* : A_T^*(G/B) \to S[W]$$

consists of all $\sum_{w \in W} f_w\, w$ *such that* $f_w \equiv f_{s_\alpha w}$ *(mod* α*) whenever* $w \in W$ *and* $\alpha \in R^+$.

(ii) *We have*

$$i^*[X(w)] = (-1)^N (\prod_{\alpha \in R^+} \alpha) \sum_{\tau \leq w} \det(\tau)\, e_\tau[X(w)]\, \tau$$

where N *is the number of positive roots, and where* e_τ *denotes equivariant multiplicity at* τ *in* G/B*, see §4. Moreover,* e_τ *is uniquely defined by*

$$e_1[x] = 1, \;\; e_\tau[x] = 0 \text{ for all } \tau \neq 1$$

where x *is the* B*-fixed point in* G/B*, and by the recursive formula*

$$e_\tau[X(s_\alpha w)] = \frac{e_\tau[X(w)] - s_\alpha(e_{s_\alpha \tau}[X(w)])}{\alpha}$$

for all simple roots α *such that* $l(s_\alpha w) = l(w) + 1$.

Proof. (i) follows from 3.4 combined with the description of all T-invariant curves in G/B. This description can be found in [12]; we recall it for completeness. Let $C \subset G/B$ be a T-invariant curve with weight $\alpha \in M$. Then the kernel of α is a singular torus in the sense of [7] (13.2), and hence α is a root. Let $G_\alpha \subset G$ be the centralizer of the kernel of α. Then G_α is a reductive group of semisimple rank one. Moreover, C is equal to $G_\alpha w B/B$ for some $w \in W$. So $C^T = \{w, s_\alpha w\}$. Now we conclude by 3.4.

(ii) Observe that any $\tau \in W$ is an attractive fixed point in G/B with weights $-w(\alpha)$ ($\alpha \in R^+$). Using 4.5, it follows that we have for all $u \in A_T^*(G/B)$:

$$i^*u = \sum_{\tau \in W} u_\tau\, \tau = \sum_{\tau \in W} e_\tau(u) \,(\prod_{\alpha \in R^+} -\tau(\alpha))\, \tau$$
$$= (-1)^N (\prod_{\alpha \in R^+} \alpha) \sum_{\tau \in W} \det(\tau) e_\tau(u) \tau.$$

Finally, the recursive formula follows from Corollary 6.2. ☐

The image of i^* is also described in work of Kostant and Kumar for $k = \mathbf{C}$ and equivariant cohomology (see [28] and [29]), and termed the *nil*

Hecke ring: it consists of the $u \in S[W]$ such that $D_w u \in S[W]$ for all $w \in W$. This description can also be deduced from Proposition 6.4 (its equivalence with the one given above is not clear a priori.) Statement (ii) is due to Kostant and Kumar in this setting, whereas a closed formula for equivariant multiplicities of Schubert varieties has been given by Rossmann (see [38] 3.2.)

As an application, here is a short proof of a smoothness criterion for Schubert varieties, due to S. Kumar for $k = \mathbf{C}$ (see [30] Theorem 5.5). Let $\tau, w \in W$ such that $\tau \leq w$. Then τ is a nonsingular point of $X(w)$ if and only if

$$(K) \qquad e_\tau[X(w)] = (-1)^{l(w)} \prod_{\alpha \in R^+,\, s_\alpha \tau \leq w} \alpha^{-1}.$$

Indeed, recall that the T-invariant curves through τ in G/B are the $G_\alpha \tau B/B$ where α is a positive root and where $G_\alpha \subset G$ is the corresponding reductive subgroup of semisimple rank one. Moreover, $G_\alpha \tau B/B$ is contained in $X(w)$ if and only if $s_\alpha \tau \leq w$.

If τ is a nonsingular point of $X(w)$, then the weights of $T_\tau X(w)$ are the weights of the T-invariant curves through τ, i.e., the opposites of positive roots α such that $s_\alpha \tau \leq w$. The number of such weights is dim $X(w) = l(w)$. So (K) follows from 4.2.

Conversely, assume that (K) holds. Consider the unique open affine T-invariant neighborhood $X(w)_\tau$ of τ in $X(w)$ defined in 4.4, and denote by A the algebra of regular functions on $X(w)_\tau$. By the proof of Proposition 2.2 in [34], or by the proof of Proposition 5.2 in [30], for any $\alpha \in R^+$ such that $s_\alpha \tau \leq w$, there exists $f_\alpha \in A$ which is an eigenvector of T of weight $-\alpha$. Moreover, the unique common zero of the f_α's is τ.

Because the degree of the rational function $e_\tau[X(w)]$ is $-\dim X(w) = -l(w)$, equation (K) implies that the number of f_α's is the dimension of $X(w)$. Therefore, the f_α's generate a polynomial subring R of A and then A is a finite R-module. Moreover, because the equivariant multiplicities of R and of A are equal, the rank of the R-module A is one. It follows that $A = R$, i.e., $X(w)_\tau$ is an affine space. □

6.6. The rational equivariant Chow ring of the flag variety.

The results in 6.4 and 6.5 give a picture of the equivariant Chow group $A_*^T(G/B)$ as an S-module. In this section, we describe the rational equivariant Chow ring $A_T^*(G/B)_{\mathbf{Q}}$, the action of $\mathbf{D}$ on this ring, and its relation to the previous picture and to the rational G-equivariant Chow ring $A_G^*(G/B \times G/B)_{\mathbf{Q}}$ as well, obtaining equivariant versions of classical results on Chow groups of flag varieties (see [2], [13] and [14]). Similar results for equivariant K-theory are due to Kostant and Kumar, see [29].

For any $\chi \in M$, consider the G-equivariant line bundle $G \times_B k(\chi)$ over G/B and denote by $c^T(\chi)$ its T-equivariant Chern class. Then $c^T(\chi)$ is in

$A^1_T(G/B)$. The additive map

$$\begin{array}{rcl} M & \to & A^1_T(G/B), \\ \chi & \mapsto & c^T(\chi) \end{array}$$

extends to a ring homomorphism: the *characteristic homomorphism*

$$c^T : S \to A^*_T(G/B).$$

Proposition. (i) *The map*

$$\begin{array}{rcl} S \times S & \to & A^*_T(G/B), \\ (f,g) & \mapsto & f c^T(g) \end{array}$$

is S^W-bilinear.

(ii) *The induced map*

$$\gamma : S \otimes_{S^W} S \to A^*_T(G/B)$$

is an isomorphism over the rationals. If moreover G is special, then γ is an isomorphism.

(iii) *For all $D \in \mathbf{D}$ and f, g in S, we have*

$$D(fc^T(g)) = D(f)c^T(g)$$

and moreover

$$\int_{G/B} f\, c^T(g) = (-1)^N (\prod_{\alpha \in R^+} \alpha^{-1})\, f \sum_{w \in W} det(w)\, w(g).$$

(iv) *For any character χ of T, we have in $A^*_T(G/B)$:*

$$c^T(\chi)[X(w)] = w(\chi)[X(w)] + \sum_\beta \langle \chi, \check{\beta} \rangle [X(ws_\beta)]$$

(sum over all positive roots β such that $l(ws_\beta) = l(w) - 1$). In particular,

$$c^T(\chi) = w_0(\chi)[G/B] + \sum_\alpha \langle \chi, \check{\alpha} \rangle [X(w_0 s_\alpha)]$$

(sum over all simple roots α), where w_0 is the longest element in W.

Proof. (i) Observe that $i^*(fc^T(g)) = \sum_{w \in W} f\, w(g)\, w$. It follows that the map

$$\begin{array}{rcl} S \times S & \to & S[W], \\ (f,g) & \mapsto & i^*(fc^T(g)) \end{array}$$

is S^W-bilinear. Now (i) follows from injectivity of i^*.

(ii) The map γ induces a map

$$\overline{\gamma} : S \otimes_{S^W} S/M(S \otimes_{S^W} S) = S/S^W_+ S \to A^*_T(G/B)/MA^*_T(G/B) = A^*(G/B)$$

where S^W_+ denotes the ideal of S^W generated by homogeneous elements of positive degree. By [13], the map $\overline{\gamma}$ is an isomorphism over the rationals; if moreover G is special, then $\overline{\gamma}$ is an isomorphism. Therefore, γ is surjective over the rationals, by Nakayama's lemma. But $S_{\mathbf{Q}}$ is a free module over $S^W_{\mathbf{Q}}$ and hence $S_{\mathbf{Q}} \otimes_{S^W_{\mathbf{Q}}} S_{\mathbf{Q}}$ is a free module over $S_{\mathbf{Q}}$. It follows that γ is an isomorphism over the rationals (resp. an isomorphism if G is special).

(iii) For any simple root α, we have

$$(id - s_\alpha)(fc^T(g)) = (f - s_\alpha(f))c^T(g)$$

because $c^T(g)$ is W-invariant. Moreover, the S-module $A^*_T(G/B)$ is free, and therefore

$$D_\alpha(fc^T(g)) = D_\alpha(f)c^T(g).$$

Finally, we have by Bott's residue formula (see [16] §5, or use Corollary 4.2):

$$\int_{G/B} f\, c^T(g) = \sum_{w \in W} f\, w(g)\, e_w[G/B] = f \sum_{w \in W} w(g) \prod_{\alpha \in R^+} (-w(\alpha))$$

which implies our second formula.

(iv) Let $u \in A^T_*(G/B)$. Using the formula

$$D_\alpha(\chi u) = s_\alpha(\chi)D_\alpha(u) + \langle \chi, \check{\alpha} \rangle u$$

(valid for any simple root α) and induction over the length of w, we obtain

$$D_w(\chi u) = w(\chi)D_w(u) + \sum_\beta \langle \chi, \check{\beta} \rangle D_{ws_\beta}(u).$$

In particular, taking $u = [x]$, we obtain

$$D_w(\chi[x]) = w(\chi)[X(w)] + \sum_\beta \langle \chi, \check{\beta} \rangle [X(ws_\beta)].$$

But $\chi[x] = c^T(\chi)[x]$ (as can be checked by restriction to fixed points). Moreover, by (iii), multiplication by $c^T(\chi)$ commutes with D_w. So we obtain

$$D_w(\chi[X]) = c^T(\chi)D_w[x] = c^T(\chi)[X(w)]$$

which proves the first formula. The second one follows by taking $w = w_0$; then the positive roots β such that $l(ws_\beta) = l(w) - 1$ are exactly the simple roots. □

Statement (iv) is a version of Proposition (4.30) in [28]. It implies readily an equivariant version of the Chevalley formula which describes multiplication by the class of a Schubert subvariety of codimension one in G/B. Recall that these varieties are the $X(w_0 s_\alpha)$ where α is a simple root, and that their classes generate the S-algebra $A^T_*(G/B)$. So the following result describes (in theory) multiplication of classes of all Schubert varieties; see [28] Proposition (4.31) for more on this topic.

Corollary. *For any simple root α, we have in $A_T^*(G/B)$:*

$$[X(w_0 s_\alpha)][X(w)] = (w(\omega_\alpha) - w_0(\omega_\alpha))[X(w)] + \sum_\beta \langle \omega_\alpha, \check{\beta} \rangle [X(ws_\beta)]$$

(sum over all positive roots β such that $l(ws_\beta) = l(w) - 1$), where ω_α is the fundamental weight which is not orthogonal to α.

Finally, we interpret the T-equivariant Chow group of G/B as the G-equivariant Chow group of $G/B \times G/B$. Let V be a G-module, and let $U \subset V$ be an open G-invariant subset such that the quotient $U \to U/G$ exists and is a principal G-bundle. For any scheme X with an action of B, denote by $G \times_B X$ the quotient of $G \times X$ by the diagonal action of B. Then the maps

$$(X \times U)/T \to (X \times U)/B \simeq ((G \times_B X) \times U)/G$$

induce an isomorphism of degree N

$$A_*^G(G \times_B X) \to A_*^T(X).$$

If moreover $X = G/B$, then the map

$$\begin{array}{ccc} G \times_B X & \to & G/B \times G/B, \\ (g, u)B & \mapsto & (gB, guB) \end{array}$$

is a G-equivariant isomorphism, where G acts diagonally on $G/B \times G/B$. It follows that there is an isomorphism

$$A_T^*(G/B) \to A_G^*(G/B \times G/B)$$

which maps each $[X(w)]$ to $[G(x, X(w))]$. In particular, the class of the B-fixed point x is mapped to the class of the diagonal in $G/B \times G/B$. Moreover, γ is identified to the characteristic homomorphism of $G/B \times G/B$.

Similarly, taking for X a point, we obtain that the characteristic homomorphism $S = A_T^*(pt) \to A_G^*(G/B)$ is an isomorphism.

6.7. The module structure of equivariant Chow groups.
By the results of 6.6, there is an isomorphism

$$S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*^G(G/B)_{\mathbf{Q}} \to A_*^T(G/B)_{\mathbf{Q}}.$$

Moreover, the rational Chow group $A_*(G/B)_{\mathbf{Q}}$ is the quotient of $A_*^G(G/B)_{\mathbf{Q}}$ by its subgroup $S_+^W A_*^G(G/B)_{\mathbf{Q}}$. In this section, we will show that both results extend to any scheme X with an action of G.

Recall the isomorphism (see [15] Proposition 6)

$$A_*^G(X)_{\mathbf{Q}} \simeq A_*^T(X)_{\mathbf{Q}}^W$$

(if moreover G is special, then this statement holds over the integers.) In particular, the rational G-equivariant Chow group of the point is isomorphic to $S_{\mathbf{Q}}^W$. Therefore, the S-module structure on $A_*^T(X)$ restricts to the structure of a $S_{\mathbf{Q}}^W$-module on $A_*^G(X)_{\mathbf{Q}}$ together with a map

$$\gamma : S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*^G(X)_{\mathbf{Q}} \to A_*^T(X)_{\mathbf{Q}}.$$

Observe that the left-hand side is a $\mathbf{D}$-module via $D(u \otimes v) = D(u) \otimes v$ (this makes sense because $\mathbf{D}$ consists of S^W-linear endomorphisms of S). By Theorem 6.3, the right-hand side is a $\mathbf{D}$-module as well.

Theorem. *Let X be a scheme with an action of a connected reductive group G with maximal torus T. Then the map*

$$\gamma : S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*^G(X)_{\mathbf{Q}} \to A_*^T(X)_{\mathbf{Q}}$$

is an isomorphism of $\mathbf{D}$*-modules. If moreover G is special, then the statement holds over the integers.*

Proof. As in the proof of Proposition 6.3, it is enough to check that the map

$$\begin{array}{rccc} \gamma: & S_{\mathbf{Q}} \otimes_{S_{\mathbf{Q}}^W} A_*(Z/G)_{\mathbf{Q}} & \to & A_*(Z/B)_{\mathbf{Q}}, \\ & u \otimes v & \mapsto & c(u) \cap \pi^* v \end{array}$$

is an isomorphism of $\mathbf{D}$-modules. Here $Z \to Z/G$ is a principal G-bundle, $\pi : Z/B \to Z/G$ is the associated complete flag bundle, and c is the characteristic homomorphism, defined as follows: a character $\chi \in M$ acts by multiplication by the first Chern class of the line bundle $Z \times_B k(\chi)$ over Z/B.

By [41] Theorem 2.3, the map γ is an isomorphism. To conclude the proof, it suffices to check that γ is D_α-linear for each simple root α. By the proof of Proposition 6.3, we have $D_\alpha = q^* q_*$ where $q : Z/B \to Z/P_\alpha$ is induced by $\pi : Z \to Z/G$. But $q_*(\pi^* v) = 0$ for all $v \in A_*(Z/G)$ and hence we have for all $u \in S$:

$$D_\alpha(c(u) \cap \pi^* v) = (q^* q_* c(u)) \cap \pi^* v = c(D_\alpha u) \cap \pi^* v. \qquad \square$$

Now we combine this theorem with our previous results for tori, to study the module structure of $A_*^G(X)_{\mathbf{Q}}$ over the polynomial ring $S_{\mathbf{Q}}^W$.

Corollary. *Let X be a scheme with an action of G.*

(i) *The rational Chow group $A_*(X)_{\mathbf{Q}}$ is the quotient of the rational equivariant Chow group $A_*^G(X)_{\mathbf{Q}}$ by its subgroup $S_+^W A_*^G(X)_{\mathbf{Q}}$ where S_+^W denotes the ideal of S^W generated by all homogeneous elements of positive degree.*

(ii) *If moreover X is projective and nonsingular, then the $S_{\mathbf{Q}}^W$-module $A_*^G(X)_{\mathbf{Q}}$ is free.*

Proof. (i) follows immediately from the theorem above, together with Corollary 2.3.

(ii) By Corollary 3.2, the $S_{\mathbf{Q}}$-module $A_*^T(X)_{\mathbf{Q}}$ is free. Moreover, the $S_{\mathbf{Q}}^W$-module $S_{\mathbf{Q}}$ is free, and hence the $S_{\mathbf{Q}}^W$-module $A_*^T(X)_{\mathbf{Q}}$ is free too. Now the $S_{\mathbf{Q}}^W$-module $A_*^T(X)_{\mathbf{Q}}^W$ is a direct summand of $A_*^T(X)_{\mathbf{Q}}$, and hence it is projective. Moreover, $A_*^T(X)_{\mathbf{Q}}^W$ is graded, with degrees bounded from above. Therefore, it is free over $S_{\mathbf{Q}}^W$. □

7. Equivariant Chow groups of spherical varieties

7.1. Fixed points of codimension one tori in spherical varieties.

Let G be a connected reductive group, $B \subset G$ a Borel subgoup and $T \subset B$ a maximal torus. Let X be a projective, nonsingular G-variety. Assuming that X is spherical (i.e., B has a dense orbit in X), we will apply Theorem 3.3 to the description of the rational equivariant Chow ring $A_T^*(X)_{\mathbf{Q}}$. For this, we study fixed points of codimension one subtori of T.

Recall that a subtorus $T' \subset T$ is *regular* if its centralizer $C_G(T')$ is equal to T; otherwise T' is *singular.* A subtorus of codimension one is singular if and only if it is the kernel of some positive root α. Then α is unique, and the group $C_G(T')$ is the product of T' with a subgroup Γ isomorphic to SL_2 or to PSL_2. Observe that the fixed point set of T' in any G-variety inherits an action of the group $C_G(T')/T'$, a quotient of Γ.

Proposition. *Let X be a spherical G-variety, and let $T' \subset T$ be a subtorus of codimension one.*

(i) *If T' is regular, then $X^{T'}$ is at most one-dimensional.*

(ii) *If T' is singular, then $X^{T'}$ is at most two-dimensional. If moreover X is complete and nonsingular, then any two-dimensional connected component of $X^{T'}$ is (up to a finite, purely inseparable equivariant morphism) either a rational ruled surface*

$$\mathbf{F}_n = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$$

where $C_G(T')$ acts through the natural action of SL_2*, or the projective plane where $C_G(T')$ acts through the projectivization of a nontrivial* SL_2*-module of dimension three.*

Proof. Let Y be an irreducible component of $X^{T'}$. By a result of Luna (personal communication), the $C_G(T')$-variety Y is spherical. Luna's proof is as follows: Let λ be a generic one-parameter subgroup of T'; then $C_G(T') = C_G(\lambda)$. Denote by $G(\lambda)$ the set of all $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t^{-1})$ exists. Recall that $G(\lambda)$ is a parabolic subgroup of G, with Levi subgroup $C_G(\lambda)$. Moreover, the Bialynicki–Birula stratum $X_+(Y,\lambda)$ is invariant under $G(\lambda)$, and the map $p_+ : X_+(Y,\lambda) \to Y$ is a $G(\lambda)$-equivariant retraction, where $G(\lambda)$ acts on Y through its quotient $C_G(\lambda)$. Because X is spherical, it contains only finitely many orbits of any Borel subgroup of G. Therefore, a Borel subgroup of $G(\lambda)$ has finitely many orbits in $X_+(Y,\lambda)$, and finally a Borel subgroup of $C_G(\lambda)$ has finitely many orbits in Y.

If T' is regular, then $C_G(T')$ acts on $X^{T'}$ through the one-dimensional torus T/T', whence (i). If T' is singular, then Y is a spherical Γ-variety. So the dimension of Y is at most the dimension of a Borel subgroup of Γ, whence the first statement of (ii).

If moreover X is complete and nonsingular, then the same holds for Y. Choose a point y such that the orbit $\Gamma \cdot y$ is open in Y, and denote by H the preimage in SL_2 of the isotropy group Γ_y. Then the map

$$\begin{array}{ccc} \mathrm{SL}_2/H & \to & Y, \\ gH & \mapsto & g \cdot y \end{array}$$

is dominant and purely inseparable.

Since H is a spherical subgroup of SL_2, three cases can occur:

(1) H is a one-dimensional torus. Then the homogeneous space SL_2/H admits $\mathbf{P}^1 \times \mathbf{P}^1$ as an equivariant completion, with boundary the diagonal Δ. The rational Γ-equivariant map

$$f : \mathbf{P}^1 \times \mathbf{P}^1 - - \to Y$$

is defined at some point of Δ, and hence everywhere because Δ is a unique Γ-orbit. Moreover, f cannot contract Δ, and therefore f is finite.

(2) H is the normalizer of a one-dimensional torus. Then SL_2/H admits $\mathbf{P}(sl_2)$ (the projectivization of the Lie algebra of SL_2) as an equivariant completion, with boundary the conic of nilpotent matrices. By the argument above, the rational equivariant map

$$f : Y - - \to \mathbf{P}(sl_2)$$

is everywhere defined and finite.

(3) H is the semi-direct product of the subgroup $U \subset \mathrm{SL}_2$ of unipotent matrices, by the cyclic group Z_n of diagonal matrices with eigenvalues (ζ, ζ^{-1}) where ζ is a n-th root of unity.

First consider the case where $n = 1$. Then $\mathrm{SL}_2/H = k^2 \setminus \{0\}$ as an SL_2-variety. Arguing as before, we see that any nonsingular completion of SL_2/H is isomorphic to $\mathbf{P}(k^2 \oplus k)$ or to its blow-up at the origin, i.e. to $\mathbf{F}_1$.

Finally, for any integer n, the homogeneous space SL_2/UZ_n admits $\mathbf{F}_n$ as an equivariant completion. Moreover, the boundary consists of the curves

$\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus 0)$ and $\mathbf{P}(0 \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. Both curves are homogeneous, and therefore the inclusion $\mathrm{SL}_2/H \to Y$ extends to a morphism $f : \mathbf{F}_n \to Y$. For $n > 1$, no boundary curve can be contracted to yield a nonsingular surface, and hence f is finite. □

7.2. Equivariant Chow rings of rational ruled surfaces.

Let D be the torus of diagonal matrices in SL_2 and let α be the character of D given by

$$\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2.$$

We will identify the rational character ring of D with $\mathbf{Q}[\alpha]$.

Consider a rational ruled surface $\mathbf{F}_n$ with ruling $\pi : \mathbf{F}_n \to \mathbf{P}^1$. Observe that $\mathbf{F}_n$ contains exactly four fixed points x, y, z, t of D where x, y (resp. z, t) are mapped to 0 (resp. ∞) by π. Moreover, we may assume that x and z lie in one G-invariant section of π, and that y and t lie in the other G-invariant section. This ordering of the fixed points identifies $A_T^*(\mathbf{F}_n^D)$ with $\mathbf{Q}[\alpha]^4$.

On the other hand, denote by $\mathbf{P}(V)$ the projectivization of a nontrivial SL_2-module V of dimension three. The weights of D in V are either $-\alpha$, 0, α (in the case where $V = sl_2$) or $-\frac{\alpha}{2}$, 0, $\frac{\alpha}{2}$ (in the case where $V = k^2 \oplus k$). We denote by x, y, z the corresponding fixed points of D in $\mathbf{P}(V)$, and we identify $A_D^*(\mathbf{P}(V)^T)$ with $\mathbf{Q}[\alpha]^3$.

Proposition. *Notation being as above, the image of*

$$i^* : A_D^*(\mathbf{F}_n)_{\mathbf{Q}} \to S_{\mathbf{Q}}^4$$

consists of all (f_x, f_y, f_z, f_t) such that $f_x \equiv f_y \equiv f_z \equiv f_t \pmod{\alpha}$ and $f_x - f_y + f_z - f_t \equiv 0 \pmod{\alpha^2}$. Moreover, the image of

$$i^* : A_D^*(\mathbf{P}(V))_{\mathbf{Q}} \to S_{\mathbf{Q}}^3$$

consists of all (f_x, f_y, f_z) such that $f_x \equiv f_y \equiv f_z \pmod{\alpha}$ and $f_x - 2f_y + f_z \equiv 0 \pmod{\alpha^2}$.

Proof. First we consider the case of $\mathbf{P}(V)$. The closures of the Bialynicki–Birula cells are then: the point z, the line (yz) and the whole $\mathbf{P}(V)$. The classes of these closures are mapped by i^* to

$$(0, 0, 2\alpha^2),\ (0, \alpha, 2\alpha),\ (1, 1, 1)$$

in the case where $V = sl_2$, and to

$$(0, 0, \frac{\alpha^2}{2}),\ (0, \frac{\alpha}{2}, \alpha),\ (1, 1, 1)$$

in the case where $V = k^2 \oplus k$. By Corollary 3.2 (iii), the image of i^* is generated as an S-module by images of closures of cells. This easily implies our statement.

The proof for $\mathbf{F}_n$ is similar; it is enough to check the result for $\mathbf{F}_0$ and $\mathbf{F}_1$ (indeed, for any positive n, the surface $\mathbf{F}_n$ is the quotient of $\mathbf{F}_1$ by the action of a cyclic group of order n which commutes with the action of D). □

7.3. Equivariant Chow rings of projective nonsingular spherical varieties.

Recall that any spherical G-variety contains only finitely many orbits of G, and therefore only finitely many fixed points of T. Combining Theorem 3.3 with the results of 7.1 and 7.2, we obtain immediately the following

Theorem. *For any projective nonsingular spherical G-variety X, the map*

$$i^* : A_T^*(X)_{\mathbf{Q}} \to A_T^*(X^T)_{\mathbf{Q}}$$

is injective. Moreover, the image of i^* consists of all families $(f_x)_{x\in X^T}$ such that:

(i) *$f_x \equiv f_y \pmod{\chi}$ whenever x, y are connected by a T-invariant curve with weight χ.*

(ii) *$f_x - 2f_y + f_z \equiv 0 \pmod{\alpha^2}$ whenever α is a positive root, x, y, z lie in a component of $X^{\ker(\alpha)}$ isomorphic to $\mathbf{P}^2$, and x, y, z are ordered as in 7.2.*

(iii) *$f_x - f_y + f_z - f_t \equiv 0 \pmod{\alpha^2}$ whenever α is a positive root, x, y, z, t lie in a component of $X^{\ker(\alpha)}$ isomorphic to a rational ruled surface, and x, y, z, t are ordered as in 7.2.*

This approach to equivariant Chow rings of spherical varieties will be pursued in a subsequent paper. Here we observe that case (ii) occurs e.g. when X is the space of complete conics; then $X^{\ker(\alpha)}$ is either the space of pairs of lines through a given point, or the space of pairs of points on a given line. An example where case (iii) occurs is the blow-up of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$; then $X^{\ker(\alpha)}$ is the strict transform of $\ell \times \ell$ where ℓ is a line in $\mathbf{P}^2$. Finally, here is an example where cases (ii) and (iii) do not occur.

Let G be a connected semisimple adjoint group of rank r. Consider G as a $G \times G$-variety for the action given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$ (this variety is spherical by the Bruhat decomposition). There exists a canonical smooth equivariant completion $G \subset \overline{G}$; its boundary $\overline{G} \setminus G$ consists of r smooth irreducible divisors intersecting transversally along an orbit of $G \times G$. The construction of $\overline{G}$ is due to De Concini and Procesi over $\mathbf{C}$ as a special case of their construction of canonical compactifications of adjoint symmetric spaces; it was extended by Strickland to arbitrary characteristic, see [40].

Let T be a maximal torus of G, with normalizer N and Weyl group $W = N/T$. Then the closure $\overline{N}$ of N in $\overline{G}$ is smooth, and is the disjoint union of $|W|$ copies of $\overline{T}$; moreover, $\overline{N}$ contains all $T \times T$-fixed points in X, see [31] 4.1. It is easy to see that $\overline{G}$ contains only finitely many $T \times T$-invariant curves, and that all such curves are contained either in $\overline{N}$ or in closed $(G \times G)$-orbits. Therefore, using Theorem 3.4, we obtain that the restriction map

$$A_{T\times T}^*(\overline{G})_{\mathbf{Q}} \to A_{T\times T}^*(\overline{N})_{\mathbf{Q}}$$

is injective, and that the composition

$$A_{G\times G}^*(\overline{G})_{\mathbf{Q}} = A_{T\times T}^*(\overline{G})_{\mathbf{Q}}^{W\times W} \to A_{T\times T}^*(\overline{N})_{\mathbf{Q}}^{W\times W} = (S_{\mathbf{Q}} \otimes A_T^*(\overline{T}))^W$$

is an isomorphism. This was proved in [31] 2.3 for equivariant cohomology and $k = \mathbf{C}$.

7.4. The action of operators of divided differences.

Let X be a spherical G-variety. Then X contains only finitely many B-orbits. Equivalently, the set $\mathcal{B}(X)$ of B-invariant subvarieties of X is finite. A short proof of this result was given by Knop (see [27] Corollary 2.6), based on the action on $\mathcal{B}(X)$ of a monoid W^* defined as follows: W^* is the set W endowed with the product $*$ such that

$$\overline{BwB} * \overline{B\tau B} = \overline{BwB\tau B}$$

in G. This monoid had already appeared in Richardson and Springer's work on B-orbits in symmetric spaces, see [37]. Its action on $\mathcal{B}(X)$ is defined by

$$w * Y := \overline{BwY}.$$

We will relate this action to the action of $\mathbf{D}$ on $A_*^T(X)$. For this, we associate to Y and w as above, an integer $d(Y,w)$: If the map $BwB \times_B Y \to BwY$ is generically finite (i.e., if $\dim(BwY) = \dim(Y) + l(w)$), then $d(Y,w)$ is its degree; otherwise, $d(Y,w) = 0$. Observe that $d(Y,s_\alpha) = d(Y,\alpha)$ with notation as in 6.1.

Proposition. *Let X be a spherical variety. Then, for any $w \in W$ and $Y \in \mathcal{B}(X)$, we have in $A_*^T(X)$:*

$$D_w[Y] = d(Y,w)[w * Y].$$

If moreover $k = \mathbf{C}$, then $d(Y,w)$ is 0 or a power of 2.

Proof. Choose a reduced decomposition $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$. Then the map

$$\overline{Bs_{\alpha_1}B} \times_B \overline{Bs_{\alpha_2}B} \times_B \cdots \times_B \overline{Bs_{\alpha_l}B} \to \overline{BwB}$$

is birational. It follows that

$$d(Y,w) = d(Y,\alpha_l)\, d(P_{\alpha_l}Y, \alpha_{l-1}) \cdots d(P_{\alpha_2} \cdots P_{\alpha_l}Y, \alpha_1)$$

which implies the first statement, using Proposition 6.2 (iii). The second statement follows from Theorem 6.3 (ii). $\square$

Remark. Call $Y \in \mathcal{B}(X)$ *induced* if there exists $w \in W$ and $Z \in \mathcal{B}(X)$, such that $Y = w * Z$ and $Z \neq Y$. If Y is not induced, call it *cuspidal.* By the proposition above, the $\mathbf{D}$-module $A_*^T(X)$ is generated by classes of cuspidal B-orbit closures. This raises the question of their description. In the case where X is a unique orbit of G, observe that any closed B-orbit is cuspidal. The converse holds in G/B by the Bruhat decomposition, and also in symmetric spaces by [37] Theorem 4.6, but not in general.

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