Appendix D

Maps from curves to projective space

Keynote Questions

(a) What is the smallest degree of a nonconstant map from a general curve of genus g to $\mathbb{P}^1$? (Answer on page 567.)
(b) In how many ways can a general curve C of genus 4 be expressed as a 3-sheeted cover of $\mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$? (Answer on page 577.)
(c) What is the smallest degree of a nondegenerate map from a general curve of genus g to $\mathbb{P}^2$? (Answer on page 567.)
(d) In how many ways can a general curve C of genus 6 be expressed as a curve of degree 6 in $\mathbb{P}^2$, up to automorphisms of $\mathbb{P}^2$? (Answer on page 578.)
(e) What is the smallest degree of an embedding of a general curve of genus g in $\mathbb{P}^3$? (Answer on page 567.)
(f) In how many ways can a general curve C of genus 8 be embedded as a curve of degree 9 in $\mathbb{P}^3$, up to automorphisms of $\mathbb{P}^3$? (Answer on page 578.)

In this book we have treated problems in enumerative geometry as interesting for the aspects of algebraic geometry that they illuminate, and for their own sake — there is a certain fascination with being able to enumerate solutions to a geometric problem, even when we cannot find those solutions explicitly. In this appendix we will see a striking example of another kind, where the methods of enumerative geometry are crucial in the analysis of a qualitative questions in geometry. We will describe and prove half of a foundational result in the theory of algebraic curves: the *Brill–Noether theorem*, which answers the question of when a general curve of given genus admits a map of given degree to projective space.

This material also illustrates the value of considering cycle theories other than the Chow ring of a variety. We will be working with Jacobians and symmetric powers of curves (see Section 10.3.1 for the definition and basic properties of symmetric powers in general, and Section D.2 below for a description of the Jacobian of a curve and its relation to the symmetric powers). These are spaces whose Chow rings remain opaque but whose cohomology rings are readily accessible (the Jacobian of a curve of genus g is homeomorphic to a product of $2g$ copies of S^1, so we can apply the Künneth formula to give a compact description of its cohomology ring). In particular, *Poincaré's formula* (Section D.5.1), which is crucial to our calculation, is readily verified in cohomology but difficult even to state in the Chow ring.

We will therefore assume for the duration of this appendix that we are working over the complex numbers, and use the results of Appendix C relating the Chow ring of a projective variety to its cohomology ring. The techniques developed can be extended to all characteristics using étale cohomology or numerical equivalence (Kleiman and Laksov [1972] and [1974], respectively).

D.1 What maps to projective space do curves have?

Until the 20th century, varieties were defined as subsets of projective space. In this respect, algebraic geometry was much like other fields in mathematics; for example, in the 19th century a group was by definition a subset of GL_n or S_n closed under the operations of composition and inversion; the modern definition of an abstract group did not appear until well into the 20th century. But in about 1860 Riemann's work introduced a way of talking about curves that crystallized, over the next hundred years, into our notion of an *abstract variety* — a geometric object defined independently of any particular embedding in projective space.

The basic problem of classifying all curves in projective space was thus broken down into two parts: the description of the family of abstract curves (the study of *moduli spaces* of curves), and the problem of describing all the ways in which a given curve C might be embedded in or, more generally, mapped to a projective space. To continue our analogy with group theory, the latter question is the analog of representation theory, that is, the study of the ways in which a given abstract group G can be mapped to GL_n.

Among the most basic questions we can pose along these lines is: "What maps to projective space do most curves of genus g have?" To focus on the objects of principal interest and avoid redundancies, we consider only *nondegenerate* maps $\varphi : C \to \mathbb{P}^r$, that is, maps whose image does not lie in any hyperplane. We define the *degree* of such a map to be the degree of the line bundle $\varphi^*\mathcal{O}_{\mathbb{P}^r}(1)$, or equivalently the cardinality of the preimage $\varphi^{-1}(H)$ of a general hyperplane $H \subset \mathbb{P}^r$.

There are really two parallel questions: First, "For which d, g and r is it the case that every curve of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$?" and second, "For which d, g and r is it the case that a *general* curve of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$?"

The second version of this question begs the further question of the meaning of the phrase "general curve of genus g." As explained in the introduction to this book, such a statement invokes the existence of a family of smooth projective curves. In this case we refer to the universal family of curves that exists over an open set of the *moduli space* parametrizing smooth, projective curves of genus g — a space whose points correspond naturally to isomorphism classes of such curves; this moduli space, denoted M_g, is irreducible. We will take this existence as given; details can be found in Harris and Morrison [1998].

Recall that maps $\varphi : C \to \mathbb{P}^r$ (modulo the group PGL_{r+1} of automorphisms of the target $\mathbb{P}^r$) correspond bijectively to pairs $(\mathcal{L}, V)$ with $\mathcal{L}$ a line bundle of degree d on C and $V \subset H^0(\mathcal{L})$ an $(r+1)$-dimensional vector space of sections without common zeros (base locus). If we drop the requirement that the sections of V have no common zeros, such an object is called a *linear series of degree d and dimension r*; classically, it was referred to as a g^r_d. (Note that if a g^r_d has a nonempty base locus then we can subtract that locus and get a map $\varphi : C \to \mathbb{P}^r$ with degree smaller than d.) The linear series is called *complete* if $V = H^0(\mathcal{L})$

It is easy to produce high-degree maps and high-degree embeddings: An application of the Riemann–Roch theorem shows that on a curve of genus g any line bundle of degree $\geq 2g$ defines a morphism, and any line bundle of degree $\geq 2g+1$ defines an embedding (see Hartshorne [1977, Section IV.3]); a slightly more refined argument shows that on any curve of genus g a general line bundle of degree $g+1$ defines a morphism and a general line bundle of degree $g+3$ defines an embedding.

If we are interested in the simplest representation of the curve, our primary question becomes: How low a degree line bundle can we find that gives a morphism, or an embedding? For the case of a general curve, these are Keynote Questions (a) and (e). The correct answers were given by Brill and Noether [1874] soon after Riemann's work, but the first complete proofs followed only about 100 years later! The numerical function

$$\rho(g, r, d) := g - (r+1)(g - d + r)$$

plays a central role. We will see how it arises in Section D.3.1. Here is the simplest version of the Brill–Noether theorem:

Theorem D.1. *(a) If $\rho(g, r, d) \geq 0$ then every curve of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$.*

(b) For a general curve this bound is sharp; that is, a general curve C of genus g admits a nondegenerate map of degree d or less to $\mathbb{P}^r$ if and only if $\rho(g, r, d) \geq 0$.

For example, we see that:

- All curves of genus g are expressible as covers of $\mathbb{P}^1$ of degree $\lceil (g+2)/2 \rceil$ or less, and for general curves this is sharp; thus all curves of genus 1 or 2 are expressible as 2-sheeted covers of $\mathbb{P}^1$, but general curves of genus 3 are not.

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- All curves of genus g admit maps to $\mathbb{P}^2$ of degree $\lceil (2g+6)/3 \rceil$ or less, and for general curves this is sharp; for example, curves of genus 2 or 3 admit maps of degree 4 to $\mathbb{P}^2$, but general curves of genus 4 do not.

Part (a) of Theorem D.1, the existence, was first proved by Kleiman and Laksov [1972] and Kempf [1971], while part (b) was first proved by Griffiths and Harris [1980]. The two statements require quite different methods, part (a) using enumerative geometry and part (b) involving specialization techniques. We give successively stronger forms of part (a) in Theorems D.1, D.9 and D.17, and we will give a proof of the strongest form; in Section D.3.2 we sketch some of the steps needed for part (b).

It is sometimes interesting to ask what happens for the complete linear series corresponding to a general line bundle $\mathcal{L}$ of degree d with $h^0(\mathcal{L}) \geq r+1$ on a specific curve C; again, this makes sense because the set $\mathrm{Pic}^d(C)$ of line bundles of degree d is a variety, isomorphic to the Jacobian variety of the curve, and the locus of $\mathcal{L} \in \mathrm{Pic}^d(C)$ with $h^0(\mathcal{L}) \geq r+1$ is a closed subset (we will explain both of these assertions below). It turns out that general linear series on general curves are as well-behaved as possible. The following result, together with the Brill–Noether theorem, answers Keynote Questions (c) and (e).

Theorem D.2. *Let C be a general curve of genus g.*

- $r = 1$*: The map $\varphi : C \to \mathbb{P}^1$ given by a general g^1_d is simply branched.*
- $r = 2$*: The map $\varphi : C \to \mathbb{P}^2$ given by a general g^2_d is birational onto a plane curve $C_0 \subset \mathbb{P}^2$ having only nodes as singularities.*
- $r \geq 3$*: The map $\varphi : C \to \mathbb{P}^r$ given by a general g^r_d is an embedding.*

See Eisenbud and Harris [1983a, Theorems 1,2] for the proof of parts (a) and (c), and Zariski [1982] or Caporaso and Harris [1998] for part (b).

Thus, for example, a general curve of genus g is birational to a plane curve of degree $\lceil (2g+6)/3 \rceil$, but no less, and is embeddable in projective space as a curve of degree $\lceil (3g+12)/4 \rceil$ but no less.

Before stating the more refined versions of the Brill–Noether theorem, we pause to describe three classic and more elementary results that provide limitations on what g^r_d's and embeddings in projective space a curve can possess: the theorems of Riemann and Roch, Clifford, and Castelnuovo.

D.1.1 The Riemann–Roch theorem

The Riemann–Roch theorem for curves (Chapter 14) says that for a line bundle $\mathcal{L}$ on a smooth curve of genus g

$$h^0(\mathcal{L}) = d - g + 1 + h^0(\omega_C \otimes \mathcal{L}^{-1}),$$

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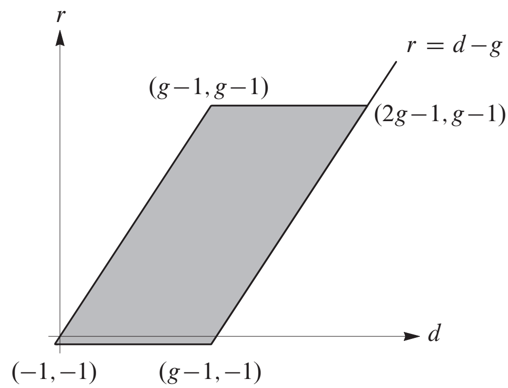

Figure D.1 Points in the shaded region correspond to degrees and dimension (d, r) of complete linear series whose existence is not excluded by the Riemann–Roch theorem.

where ω_C denotes the sheaf of differential forms on C. We will often exploit the equivalence between the notions of line bundles and divisors on a smooth curve, and write a divisor D in place of the line bundle $\mathcal{O}(D)$. Thus we allow ourselves to rewrite the Riemann–Roch theorem as

$$h^0(D) = d - g + 1 + h^0(K - D),$$

where K denotes a canonical divisor.

For line bundles $\mathcal{L}$ of degree $d > 2g - 2$ the last term is zero, and so the Riemann–Roch theorem tells us the dimension precisely:

$$h^0(\mathcal{L}) = d - g + 1.$$

For line bundles of degree close to $2g - 2$ it gives us approximate information: For example, if $d = 2g - 2$, the Riemann–Roch theorem says that

$$h^0(\mathcal{L}) = \begin{cases} g & \text{if } \mathcal{L} = \omega_C, \\ g - 1 & \text{otherwise,} \end{cases} \tag{D.1}$$

and if $g > 0$ and $d = 2g - 3$ it says that

$$h^0(\mathcal{L}) = \begin{cases} g - 1 & \text{if } \mathcal{L} = \omega_C(-p) \text{ for some point } p \in C, \\ g - 2 & \text{otherwise.} \end{cases} \tag{D.2}$$

It also tells us that for any line bundle

$$d + 1 \geq h^0(\mathcal{L}) \geq d - g + 1.$$

Beyond these values the Riemann–Roch theorem gives less precise information.

We summarize what the Riemann–Roch theorem tells us about the possible existence of g^r_d's on curves of genus g in Figure D.1, which shows the possible pairs (d, r) where $\mathcal{L}$ is a line bundle of degree d on C and $r = h^0(\mathcal{L}) - 1$.

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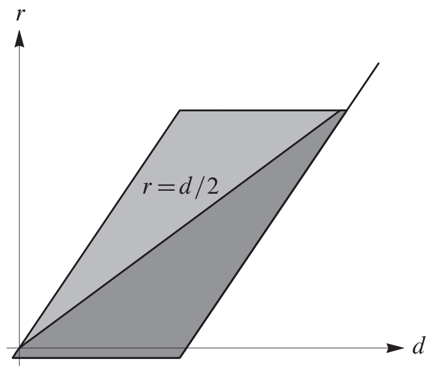

Figure D.2 Points in the shaded region correspond to degrees and dimension (d, r) of complete linear series whose existence is not excluded by Clifford's theorem. In fact, the result is sharp: every such degree and dimension occurs for some complete linear series on some curve.

D.1.2 Clifford's theorem

Clifford's theorem (Hartshorne [1977, Theorem IV.5.4]) says that if C is a curve of genus g and D is a divisor of degree $d \leq 2g - 2$ on C then

$$h^0(D) \leq d/2 + 1.$$

In the case when $h^0(K - D) = 0$, this inequality follows at once from the Riemann–Roch theorem, so its import is for effective divisors D such that $h^0(K - D) \neq 0$ — these are called *special divisors*. An extension of Clifford's theorem says that if, moreover, equality holds then either $\mathcal{L} = \mathcal{O}$, $\mathcal{L} = \omega_C$ or C is hyperelliptic and $\mathcal{L}$ is a multiple of the g^1_2 on C. If we exclude the cases of degree $d = 0$ and $d \geq 2g - 2$ (where after all Clifford tells us nothing new), we may state this as:

Theorem D.3 (Clifford)**.** *If C is a curve of genus g and $\mathcal{L}$ a line bundle of degree d on C with $0 < d < 2g - 2$, then*

$$h^0(\mathcal{L}) \leq d/2 + 1,$$

with equality holding only if C is hyperelliptic and $\mathcal{L}$ a multiple of the g^1_2.

This cuts the above graph of allowed values of d and r essentially in half, as shown in Figure D.2.

Clifford's theorem is sharp: For every d, g and r allowed by Theorem D.3, there exist curves of genus g and g^r_d's on them. This does not represent a satisfactory answer to our basic problem, however, for two distinct reasons:

(a) Given our motivation for studying g^r_d's on curves — the classification of curves in projective space — you may say that our real object of interest is not g^r_d's in general but those whose associated maps give generically one-to-one morphisms

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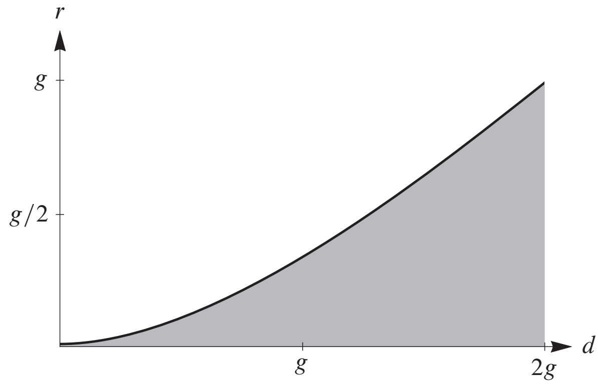

Figure D.3 Points in the shaded region correspond to degrees and dimension (d, r) of complete linear series defining birational maps whose existence is not excluded by Castelnuovo's theorem.

to $\mathbb{P}^r$. (When the morphism is generically one-to-one, or *birationally very ample* in more classical terminology, the source curve is the normalization of the target, so the image curves can be described as curves of geometric genus g — see Section 2.4.6.) The linear systems satisfying equality in Clifford's theorem — and, as we will see in a moment, those that are close to this — are *not* birationally very ample. Thus, we refine our original question and ask: "What birationally very ample linear series may exist on a curve of genus g?" — in other words, for which d, g and r do there exist irreducible, nondegenerate curves of degree d and geometric genus g in $\mathbb{P}^r$?

(b) As we have seen, interesting linear series that achieve equality in Clifford's theorem exist only on hyperelliptic curves, which are very special: they form a closed subset of codimension $g - 2$ in the space M_g of all smooth curves of genus g. Clifford's theorem thus leaves unanswered the question of what linear series exist on a *general curve* of genus g.

D.1.3 Castelnuovo's theorem

The issue of which linear series can embed a curve is dealt with in a theorem of Castelnuovo:

Theorem D.4 (Castelnuovo)**.** *Let $C \subset \mathbb{P}^r$ be an irreducible, nondegenerate curve of degree d and geometric genus g. Then*

$$g \leq \pi(d, r) := \binom{m}{2}(r - 1) + m\epsilon,$$

where $d = m(r - 1) + \epsilon + 1$ and $0 \leq \epsilon \leq r - 2$.

Castelnuovo showed that this bound is sharp, and using his analysis it is not hard to see that curves of all geometric genera between 0 and the bound do occur. Figure D.3 shows the values of d and r allowed by Castelnuovo's bound in the case $g = 30$. See Arbarello et al. [1985, Chapter 3] for a proof. It is worth pointing out, though, that a

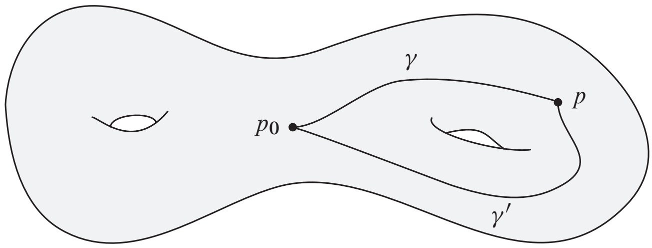

Figure D.4 The integral $\int_{p_0}^{p} \omega$ may depend on the choice of path.

slight variant of this question — "For which d, g and r do there exist *smooth*, irreducible and nondegenerate curves of degree d and genus g in $\mathbb{P}^r$?" remains open in general. It was solved for $r = 3$ in Gruson and Peskine [1982]; see Ciliberto [1987] and Harris and Eisenbud [1982, Chapter 3] for a discussion of some of the relevant issues in general.

It should be said that Castelnuovo's theorem answers question (a) in the preceding section (the question of what birationally very ample linear series a curve may have). It does not, however, address question (b); that is, it does not tell us what linear series are present on a general curve. To put it another way, inside the space M_g parametrizing smooth projective curves of genus g, the locus of "Castelnuovo curves" — that is, curves $C \subset \mathbb{P}^r$ of degree d with $g = \pi(d, r)$ — is contained in a subvariety of high codimension. Thus, the question remains of what linear series exist on all or most curves of genus g. This is the question addressed by the Brill–Noether theorem, of which Theorem D.1 is a weak version. To state it in a strong form and to prove it we will have to analyze the geometry of curves in greater depth.

D.2 Families of divisors

D.2.1 The Jacobian

A deeper study of linear series requires us to make sense of the set of linear systems as a variety. For this purpose we introduce the *Jacobian*.

One of the early motivations for studying algebraic curves came from calculus, specifically the desire to make sense of integrals of algebraic functions. In modern terms, this means integrals

$$\int_{p_0}^{p} \omega,$$

where ω is a holomorphic (or more generally meromorphic) differential on a smooth projective curve C over the complex numbers, p_0 and p are points of C and the integral is taken along a path from p_0 to p on C.

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One problem in trying to define such integrals is that if the curve in question has positive genus then the value of the integral depends on the choice of the path. A change of the path corresponds to adding a functional obtained by integration over a closed loop on C. Now, the integral of holomorphic (and hence closed) forms over a closed loop γ depends only on the homology class of γ; we thus have a map

$$H_1(C,\mathbb{Z}) \to H^0(\omega_C)^*$$

which embeds $H_1(C,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ as a discrete lattice in $H^1(\omega_C)^* \cong \mathbb{C}^g$ (see for example Griffiths and Harris [1994, p. 228]). Thus we may view the integral $\int_{p_0}^{p}$ as an element of the quotient

$$H^0(\omega_C)^*/H_1(C,\mathbb{Z}).$$

We define the *Jacobian* $\mathrm{Jac}(C)$ of the curve C to be this quotient. By our construction, $\mathrm{Jac}(C)$ is a complex torus, and in particular a compact complex manifold; in fact, it is a projective variety over $\mathbb{C}$, and the map $p \to \int_{p_0}^{p}$ is a map of projective varieties. We will use this often; see Griffiths and Harris [1994, §§2.2–2.3] for a treatment in the complex-analytic setting.

Having defined the Jacobian, we see that, after choosing a base point $p_0 \in C$, integration defines a map $C \to \mathrm{Jac}(C)$ and, more generally, maps

$$u = u_d : C_d \to \mathrm{Jac}(C)$$

from the symmetric powers $C_d = C^d/\mathfrak{S}_d$ of C to its Jacobian, defined by

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i}.$$

These maps are called *Abel–Jacobi* maps.

D.2.2 Abel’s theorem

Theorem D.5 (Abel)**.** *Let $u : C_d \to \mathrm{Jac}(C)$ be the Abel–Jacobi map. Divisors D, E on C of the same degree are linearly equivalent if and only if $u(D) = u(E)$.*

One direction of Abel’s theorem — the “if” half — is relatively easy to prove: If D and E are linearly equivalent, there is a pencil of divisors, parametrized by $\mathbb{P}^1$, interpolating between them. But if $f : \mathbb{P}^1 \to A$ is any map from $\mathbb{P}^1$ to a torus, the pullbacks $f^*\eta$ of holomorphic 1-forms on A vanish identically. Since these 1-forms generate the cotangent space at every point of A, it follows that the differential df is identically zero and hence that f is constant; thus $u(D) = u(E)$. The hard part (which was in fact proved by Clebsch) is the converse. See Griffiths and Harris [1994, p. 235] for a treatment.

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The import of Abel's theorem is that we may, for each d, identify the set of linear equivalence classes of effective divisors of degree d on C with the Jacobian $\mathrm{Jac}(C)$. The identification is not canonical; it depends on the choice of a base point $p_0 \in C$. In Section D.2.3 we will use this correspondence to show that there exists a *fine moduli space* $\mathrm{Pic}^d(C)$ for line bundles of degree d on C — that is, a space together with a universal family — and that $\mathrm{Pic}^d(C)$ is isomorphic (again, non-canonically) to $\mathrm{Jac}(C)$.

Abel's theorem tells us that the fiber $u^{-1}(u(D))$ of u through a point $D \in C_d$ is the complete linear system $|D| = \{E \in C_d \mid E \sim D\}$ — set-theoretically, at least, a projective space. (We will see in Theorem D.6 that it is indeed isomorphic to $\mathbb{P}^{h^0(D)-1}$.) Beyond this, the behavior of the map u depends very much on d.

If $p_1, \dots, p_d \in C$ are general points with $d \leq g$, then the conditions of vanishing at the p_i are independent linear conditions on differential forms. Writing $D = p_1 + \cdots + p_d$, we get $h^0(\omega - D) = g - d$. From the Riemann–Roch theorem, we see that $h^0(D) = 1$; that is, no other effective divisor of degree d is linearly equivalent to D. It follows from Abel's theorem that the map $u : C_d \to \mathrm{Jac}(C)$ is birational onto its image, and in particular that the image $W_d := u(C_d) \subset \mathrm{Jac}(C)$ is again d-dimensional.

In particular, the map $u : C_g \to \mathrm{Jac}(C)$ is birational (this statement is called the *Jacobi inversion theorem*; see Exercise D.18 for a more classical version). Further, the image of $u : C_{g-1} \to \mathrm{Jac}(C)$ is a divisor in $\mathrm{Jac}(C)$, called the *theta divisor* and written Θ.

When $d \geq g$ the same argument shows that the map $u : C_d \to \mathrm{Jac}(C)$ is surjective. When $g \leq d \leq 2g - 2$ the dimensions of the fibers will vary, but when $d > 2g - 2$ the picture becomes regular: the fibers of $u : C_d \to \mathrm{Jac}(C)$ are all of dimension $d - g$. We will see in this case that $u : C_d \to \mathrm{Jac}(C)$ is in fact a projective bundle, and in Section D.5.2 we will identify the vector bundle $\mathcal{E}$ on $\mathrm{Jac}(C)$ such that $C_d \cong \mathbb{P}\mathcal{E}$.

Note that the subset of line bundles $\mathcal{L}$ of degree d such that $h^0(\mathcal{L}) \geq r + 1$ is Zariski closed in $\mathrm{Pic}^d(C)$: it is the locus where the fiber dimension of u is r or greater. We define the *Brill–Noether locus* $W^r_d(C)$ to be the set

$$W^r_d(C) = \{\mathcal{L} \in \mathrm{Pic}^d(C) \mid h^0(\mathcal{L}) \geq r + 1\}.$$

This Zariski closed subset of $\mathrm{Pic}^d(C)$ has a natural scheme structure, which we will explain in Section D.4.2 below.

Here is the first step toward proving that for large d the Abel–Jacobi map is the projection from a projectivized vector bundle on $\mathrm{Jac}(C)$:

Theorem D.6. *For any d, the scheme-theoretic fiber of the Abel–Jacobi map $u : C_d \to \mathrm{Jac}(C)$ through a point $D \in C_d$ is the projective space $|D| = \mathbb{P}H^0(\mathcal{O}_C(D))$. For $d > 2g - 2$, the map u is a submersion, that is, the differential $du : T_D C_d \to T_{u(D)} \mathrm{Jac}(C)$ is surjective everywhere.*

Proof: From Abel's theorem above, we see that the projective space

$$|D| \cong \{(\sigma, D') \in \mathbb{P}H^0(\mathcal{O}_C(D)) \times C_d \mid \sigma \text{ vanishes on } D\}$$

projects onto the fiber $u^{-1}u(D) \subset C_n$, so it is enough to show that $u^{-1}u(D)$ is smooth and of dimension equal to $h^0(\mathcal{O}_C(D)) - 1$.

We start with the case $d > 2g - 2$, and consider a point $D \in C_d$ corresponding to a reduced divisor, that is, $D = p_1 + \cdots + p_d \in C_d$ with the points p_i distinct. From the definition of the Jacobian as $H^0(\omega_C)^*/H_1(C, \mathbb{Z})$ we see that the cotangent space at any point is

$$T^*_{u(D)} \operatorname{Jac}(C) = H^0(\omega_C),$$

the space of regular differentials on C. We may similarly identify the cotangent space to C_d at D: because the p_i are distinct we have

$$T^*_D C_d = \bigoplus T^*_{p_i} C = \bigoplus H^0((\omega_C)_p) = H^0(\omega_C/\omega_C(-D)).$$

Differentiating the Abel–Jacobi map

$$D = \sum p_i \mapsto \sum \int_{p_0}^{p_i}$$

with respect to the points p_i, we see that, in terms of these identifications, the differential du_D of u at a point $D \in C_d$ is given as the transpose of the evaluation map

$$\begin{aligned} H^0(\omega_C) &\to \bigoplus T^*_{p_i} C, \\ \omega &\mapsto (\omega(p_1), \ldots, \omega(p_d)); \end{aligned}$$

in particular, the cokernel of the differential du_D of u at a point $D \in C_d$ is the annihilator of the subspace $H^0(\omega_C(-D)) \subset H^0(\omega_C)$ of differentials vanishing along D. Since we are working in the range $d > 2g - 2 = \deg \omega_C$, there are no such differentials, and we are done.

In fact, the identification $T^*_D C_n = H^0(\omega_C/\omega_C(-D))$ extends to all divisors $D \in C_d$ and in these terms the differential is again the transpose of the evaluation map

$$H^0(\omega_C) \to H^0(\omega_C/\omega_C(-D))$$

(see for example Arbarello et al. [1985, §IV.1]); so the same logic applies.

Finally, in the case $d \leq 2g - 2$ the Riemann–Roch theorem tells us that the dimension of the kernel of the differential du at any point D — that is, the dimension of the cokernel of the evaluation map $H^0(\omega_C) \to H^0(\omega_C/\omega_C(-D))$ — is exactly the dimension $r(D) = h^0(\mathcal{O}_C(D)) - 1$ of the fiber of u through D; thus the fibers of u are smooth in this case as well, even though they are not all of the same dimension. □

D.2.3 Moduli spaces of divisors and line bundles

Abel's theorem tells us that when $d \geq g$ the fibers of the Abel–Jacobi map $C_d \to \mathrm{Jac}$ are in one-to-one correspondence with the line bundles $\mathcal{L}$ of degree d on C, making the set of bundles $\mathrm{Pic}^d(C)$ into an analytic variety; the same follows for every d via the isomorphisms $\mathrm{Pic}^d(C) \cong \mathrm{Pic}^e(C)$. The defining isomorphism $\mathrm{Pic}^d(C) \cong \mathrm{Jac}(C)$ is not canonical: it depends on the choice of a base point p_0. If we chose a different point p_0' then the identifications would take $\mathcal{L}$ to $\mathcal{L}(d(p_0' - p_0))$. For clarity, it is usually best to think of the $\mathrm{Pic}^d(C)$ as distinct schemes.

The variety $\mathrm{Pic}^d(C)$ is actually a *fine moduli space*, in the sense that $\mathrm{Pic}^d(C) \times C$ carries a *universal line bundle* $\mathcal{P}$. The key property of $\mathcal{P}$ is that its restriction to each fiber $\{\mathcal{L}\} \times C$ is isomorphic to the corresponding line bundle $\mathcal{L}$. In fact, it satisfies a stronger functorial characterization:

Proposition D.7. *Let C be a smooth, projective curve of genus g, and d any integer. Let $p_0 \in C$ be a point. There exists a projective scheme* $\mathrm{Pic}^d(C)$ *and a line bundle $\mathcal{P}$ on* $\mathrm{Pic}^d(C) \times C$ *such that:*

(a) $\mathcal{P}$ is trivial on $\mathrm{Pic}^d(C) \times \{p_0\}$*; and*

(b) for any scheme B and any line bundle $\mathcal{M}$ on $B \times C$ of relative degree d that is trivial on $B \times \{p_0\}$, there exists a unique map $\varphi : B \to$ $\mathrm{Pic}^d(C)$ *such that*

$$\mathcal{M} = (\varphi \times \mathrm{Id}_C)^*\mathcal{P}.$$

(The condition $\mathcal{P}|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$ is the special case of part (b) where B is a point.) The "universal line bundle" $\mathcal{P}$ on $\mathrm{Pic}^d(C) \times C$ is called the *Poincaré bundle* (with respect to p_0). We postpone its construction to Section D.4.1.

We have constructed $\mathrm{Pic}^d(C)$ as an analytic variety, but it has in fact the structure of an algebraic variety, and can be constructed for curves over any field. This was first done by André Weil. He observed that via the birational map $u : C_g \to \mathrm{Jac}(C)$ an open subset of $\mathrm{Jac}(C)$ was isomorphic to an open subset of C_g. Composing u with translations, we see that $\mathrm{Jac}(C)$ may be covered by such open sets, and these can be glued together to construct $\mathrm{Jac}(C)$. (Indeed, it was the desire to carry out this construction that led Weil to the definition of an abstract variety.) In even greater generality, Grothendieck applied his theory of *étale equivalence relations* to construct $\mathrm{Pic}^d(C)$ as the quotient of C_d by linear equivalence for large d; see Milne [2008] for a description.

The symmetric power C_d that is the source of the Abel–Jacobi map is also a fine moduli space, supporting a universal family of divisors; equivalently, we may regard C_d as the Hilbert scheme of subschemes of degree d on C:

Proposition D.8. *There exists a divisor $\mathcal{D} \subset C_d \times C$ such that, for any scheme B and any effective divisor $\Delta \subset B \times C$ finite over B of relative degree d, there exists a unique map $\varphi : B \to C_d$ such that*

$$\Delta = (\varphi \times \mathrm{Id}_C)^{-1}\mathcal{D}.$$

It is easy to make the "universal divisor" explicit: it is just the reduced divisor

$$\mathcal{D} = \{(D, p) \in C_d \times C \mid p \in D\},$$

which we will encounter repeatedly in what follows.

For proofs of Propositions D.7 and D.8, see, e.g., Arbarello et al. [1985, §IV.2].

D.3 The Brill–Noether theorem

Here is our second version of the Brill–Noether theorem. Recall that we have defined $\rho(g, r, d) := g - (r + 1)(g - d + r)$.

Theorem D.9 (Brill–Noether)**.** *(a) For every curve C of genus g*

$$\dim W^r_d(C) \geq \rho(g, r, d).$$

(b) If C is a general curve of genus g then equality holds.

The appearance of ρ in this theorem can be understood as follows: g is the dimension of the Jacobian of C, which may be thought of as the space $\mathrm{Pic}^d(C)$ of all line bundles of degree d on C. Furthermore, if a line bundle $\mathcal{L} \in \mathrm{Pic}^d(C)$ has $h^0(\mathcal{L}) = r+1$, so that $\mathcal{L} \in W^r_d(C) \setminus W^{r+1}_d(C)$, the Riemann–Roch theorem asserts that $h^1(\mathcal{L}) = g - d + r$. Thus the Brill–Noether theorem asserts that if C is a general curve of genus g then the codimension of $W^r_d \subset \mathrm{Pic}^d(C)$ is $h^0(\mathcal{L})h^1(\mathcal{L})$. Indeed, as we shall see, W^r_d can be thought of as the rank-k locus of a map between vector bundles of ranks $k + h^0(\mathcal{L})$ and $k + h^1(\mathcal{L})$ (for any large k!), so this is the "expected" codimension in the sense of Chapter 12.

In general, the formula of Theorem D.9 is far more restrictive than Castelnuovo's bound. For example, Figure D.5 shows the values of d and r allowed in the case $g = 100$; we see that only a tiny fraction of the complete linear series allowed by Castelnuovo actually occur on a general curve.

In the final section of this appendix we will give an enumerative proof of the existence half of Theorem D.9, namely:

Theorem D.10. *If $\rho = g - (r + 1)(g - d + r) \geq 0$, then every smooth curve C of genus g has*

$$\dim W^r_d(C) \geq \rho.$$

In particular, there exist linear systems on C of degree d and dimension r.

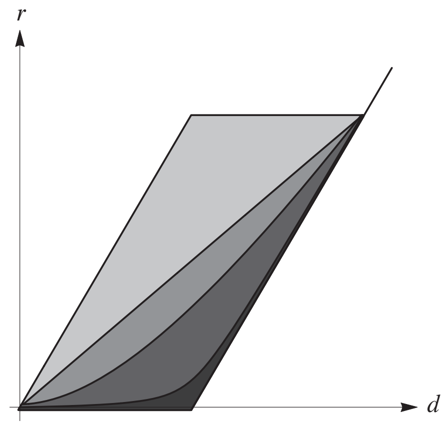

Figure D.5 The shaded regions correspond to points (d, r) such that there exists:

- a complete linear series whose existence is not contradicted by the Riemann–Roch theorem;
- a complete linear series that actually exists on some curve of genus 100;
- a complete linear series on some curve of genus 100, which defines a birational map; and
- a complete linear series on a general curve of genus 100 (this necessarily defines a birational map when $r > 1$).

The heart of our proof of Theorem D.10 is an enumerative formula for the class of $W_d^r(C)$, given in the stronger Theorem D.17. When $g = (r+1)(g-d+r)$ — that is, when $\rho = 0$ — this formula becomes a number:

Corollary D.11. *If C is a smooth curve of genus $g = (r+1)(g-d+r)$ and if W_d^r is finite, then C possesses*

$$g! \prod_{i=1}^{r} \frac{i!}{(g-d+r+i)!}$$

linear series of degree d and dimension r, counted with multiplicity. When $r = 1$ and $g = 2(g-d+1) = 2k$, this number is the Catalan number

$$\frac{1}{k+1}\binom{2k}{k}.$$

Proof: By Poincaré's formula (Proposition D.13 below), $\deg(\theta^g) = g!$; substituting in the formula of Theorem D.17 yields the corollary. Note that the multiplicity with which a given linear series occurs is equal to the multiplicity of the scheme W_d^r at the corresponding point. $\square$

For example, in the first case that is not answered by the Riemann–Roch theorem, we can ask if a general curve of genus 4 is expressible as a 3-sheeted cover of $\mathbb{P}^1$, and if so in how many ways; this is the content of Keynote Question (b). Corollary D.11 gives an answer: It says that C will admit two such maps. Indeed, we can see directly

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that there are two: the canonical model of a non-hyperelliptic curve of genus 4 is the complete intersection of a quadric and a cubic surface in $\mathbb{P}^3$, and if the quadric is smooth its two rulings will each cut out a g^1_3 on C. (Note that we see in this example a case where multiplicities may arise: If the quadric surface in question is a cone, the curve C will possess only one g^1_3, counted with multiplicity 2.)

Similarly, Corollary D.11 says that a general curve C of genus 6 will possess five g^1_4's, and dually five g^2_6's. Again, we can see these linear series explicitly: By the extension below, any one of the g^2_6's will give a birational embedding of C as a plane sextic $C_0 \subset \mathbb{P}^2$ with four nodes as singularities. The five g^1_4's will then be the pencils cut out on C by the pencil of lines through each node and the pencil cut by conics passing through all four. (See Exercises D.19 and D.20 for a proof that these are all the g^1_4's on C, and Exercise D.21 for another example.)

There are various extensions of this theorem for general curves C of genus g: Fulton and Lazarsfeld [1981] showed that if $\rho > 0$ then $W^r_d(C)$ is irreducible; Gieseker proved that the singular locus of $W^r_d(C)$ is exactly $W^{r+1}_d(C)$, and hence (given Exercise D.23) that $W^r_d(C)$ is reduced (see Harris and Morrison [1998] for a discussion of this theorem and its proofs). In particular, we see that for a general curve there are in fact no multiplicities in the formula in Corollary D.11.

D.3.1 How to guess the Brill–Noether theorem and prove existence

Here is one way to describe the locus $W^r_d(C)$. Fix a divisor

$$D = p_1 + \cdots + p_m$$

consisting of m distinct points of C. For any line bundle $\mathcal{L}$ on C, there is an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(D) \xrightarrow{b_{\mathcal{L}}} \bigoplus_{i=1}^{m} \mathcal{L}(D)_{p_i} \cong \bigoplus_{i=1}^{m} \mathcal{O}_{p_i} \longrightarrow 0,$$

and taking cohomology we see that

$$H^0(\mathcal{L}) = \mathrm{Ker}\left(H^0(\mathcal{L}(D)) \xrightarrow{h^0 b_{\mathcal{L}}} \bigoplus_{i=1}^{m} \mathbb{C} \right).$$

If the number m is large — say $m > 2g-2-d$ — then the Riemann–Roch formula tells us that $h^0(\mathcal{L}(D)) = m + d - g + 1$, independently of $\mathcal{L}$. Thus, as $\mathcal{L}$ varies over the set $\mathrm{Pic}^d(C)$ of line bundles of degree d, the locus $W^r_d(C) \subset \mathrm{Pic}^d(C)$ is the locus where the $m \times (m + d - g + 1)$ matrix $h^0 b_{\mathcal{L}}$ has rank at most $(m + d - g + 1) - (r + 1)$. The expected codimension of the locus where an $s \times t$ matrix with $s \leq t$ has rank u, in the sense of Chapter 12, is $(s - u)(t - u)$. Thus the "expected" codimension of W^r_d in

$\mathrm{Pic}^d(C)$ is $(r+1)(m-(m+d-g+1-r-1)) = (r+1)(g-d+r)$, exactly the codimension predicted for a general curve by the Brill–Noether theorem.

As we will see, the maps $h^0 b_{\mathcal{L}}$ vary algebraically with $\mathcal{L} \in \mathrm{Pic}^d(C)$. It follows that *if* $W^r_d(C)$ *is nonempty* for a given curve C then its dimension is at least $\rho(g,r,d)$, and, given the existence of one curve C_0 for which W^r_d is really nonempty and of dimension $\rho(g,r,d)$, it would follow that this is true for an open set of curves in any family containing C_0. Brill and Noether must have known many cases where these conditions were all satisfied, but they lacked the tools to give a proof of the theorem.

In Section D.4.2 we will identify the map $b_{\mathcal{L}}$ as the fiber of a map of vector bundles $b : \mathcal{F} \to \mathcal{G}$ over $\mathrm{Pic}^d(C)$, which is isomorphic to the Jacobian $\mathrm{Jac}(C)$ of C (this implies that the map $h^0 b_{\mathcal{L}}$ varies algebraically with $\mathcal{L}$). As remarked above, this implies that $W^r_d(C)$ has dimension at least $\rho(g,r,d)$ provided that it is nonempty.

To prove that $W^r_d(C)$ is nonempty when $\rho(g,r,d) \geq 0$, we will compute the Chern classes of the vector bundles $\mathcal{F}$ and $\mathcal{G}$. Porteous' formula allows us to compute the class $\alpha \in H^{2(r+1)(g-d+r)}(\mathrm{Jac}(C))$ that the locus W^r_d *would* have *if* it had dimension $\rho(g,r,d)$. We will show that this class is nonzero when $0 \leq (r+1)(g-d+r) \leq g$, and this suffices to prove the desired existence.

D.3.2 How the other half is proven

The proof of the other half of the Brill–Noether theorem — the statement that for a general curve C the dimension $\dim W^r_d(C)$ is at most $\rho(g,r,d)$, and in particular that C possesses no g^r_d's when $\rho < 0$ — requires very different ideas. One could prove it by exhibiting for each g, r and d a smooth curve C of genus g with $\dim W^r_d(C) = \rho(g,r,d)$ (or with $W^r_d(C) = \varnothing$ if $\rho < 0$), but no one has ever succeeded in doing this explicitly for large g. The known proofs fall into two families:

Degeneration to singular curves

One approach to this problem is to consider a one-parameter family of curves $\{C_t\}$ specializing from a smooth curve to a singular one, C_0. What needs to be done in this setting is first of all to describe the limit as $t \to 0$ of a g^r_d on C_t, and then to prove that such limits do not exist on C_0 when $\rho < 0$. This was done in the original proof, with C_0 a general g-nodal curve (that is, $\mathbb{P}^1$ with g pairs of general points identified); the possible limits of a g^r_d on C_t were identified in Kleiman [1976] and the proof that no such limit exists when $\rho < 0$ was given in Griffiths and Harris [1980]. Another proof (Eisenbud and Harris [1983a]) used a g-cuspidal curve as C_0, and in Eisenbud and Harris [1983b] the role of C_0 was played by a curve consisting of a copy of $\mathbb{P}^1$ with g elliptic tails attached. See Figure D.6 for examples of these curves. Much more recently, a proof was given using the methods of tropical geometry in Cools et al. [2012].

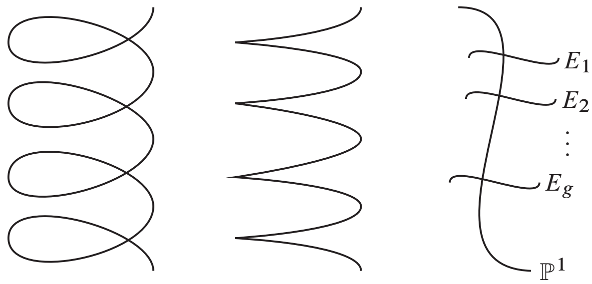

Figure D.6 Three singular curves used in specialization arguments for the nonexistence half of Brill–Noether.

Curves on a very general K3 surface

A completely different proof was given by Lazarsfeld [1986], who showed that a smooth curve C embeddable in a very general K3 surface — specifically, one whose Picard group was generated by the class of the curve C — necessarily satisfied the statement of the basic Brill–Noether theorem. The theorem was thus proved by specializing to a smooth curve, rather than a singular one (though the smooth curve in question could still not be explicitly given, inasmuch as we have no way to explicitly produce K3 surfaces with Picard number 1).

D.4 W^r_d as a degeneracy locus

In the remainder of this appendix we will deal with a fixed curve C. To simplify notation, we will write Jac for the Jacobian $\mathrm{Jac}(C)$ and Pic^d for the Picard variety $\mathrm{Pic}^d(C)$ parametrizing line bundles of degree d on C.

In this section we will explain how to construct the family of all line bundles of a given degree, and how to put the maps $b_{\mathcal{L}}$ of Section D.3.1 together into a map of bundles. To do this, we first need to construct the *Poincaré bundle*, a fundamental object in the theory.

D.4.1 The universal line bundle

Choose a base point $p_0 \in C$. The Poincaré bundle is a line bundle on the product $\mathrm{Pic}^d \times C$ whose restriction to the fiber $\{\mathcal{L}\} \times C$ over $\mathcal{L} \in \mathrm{Pic}^d$ is isomorphic to $\mathcal{L}$ and whose restriction to the cross-section $\mathrm{Pic}^d \times \{p_0\}$ is trivial.

Without the normalizing condition of triviality on $\mathrm{Pic}^d \times \{p_0\}$ the bundle $\mathcal{P}$ would not be determined uniquely: We could tensor with the pullback of any line bundle on $\mathrm{Pic}^d(C)$ and get another. But with the normalizing condition, Corollary B.6(b) shows that the Poincaré bundle is unique — if it exists.

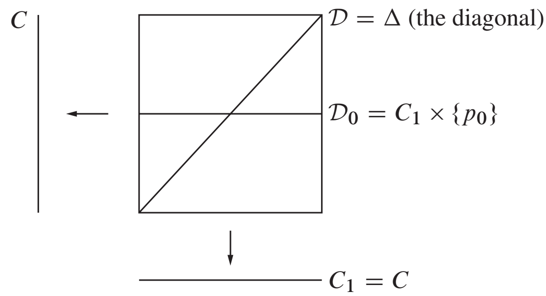

Figure D.7 The divisors $\mathcal{D}$ and $\mathcal{D}_0$ in the case $d = 1$.

We will construct the Poincaré bundle as the direct image of a line bundle $\mathcal{M}$ on $C_d \times C$ under the map

$$\eta = u \times \mathrm{Id} : C_d \times C \to \mathrm{Pic}^d \times C.$$

For $p \in C$, let $X_p \subset C_d$ be the divisor that is the image of $\{p\} \times C^{d-1}$ in C_d, that is, the set of divisors containing p. To describe $\mathcal{M}$, we write $\mathcal{D}_0$ for the divisor $X_{p_0} \times C \subset C_d \times C$, and let $\mathcal{D} \subset C_d \times C$ be the *universal divisor* of degree d as in Proposition D.8, that is,

$$\mathcal{D} = \{(D, p) \in C_d \times C \mid D - p \geq 0\}.$$

Thus the restriction of $\mathcal{D}$ to a fiber $\{D\} \times C$ of the projection to C_d is the divisor D, and the restriction of $\mathcal{D}$ to the fiber $C_d \times \{p\}$ of the projection to C is the divisor X_p. Finally, define

$$\mathcal{M} = \mathcal{O}_{C_d \times C}(\mathcal{D} - \mathcal{D}_0),$$

and set $\mathcal{P} = \eta_* \mathcal{M}$.

Proposition D.12. $\mathcal{P} = \eta_*(\mathcal{M})$ *is a Poincaré bundle on* $\mathrm{Pic}^d \times C$*; in particular,*

$$\mathcal{P}|_{\{\mathcal{L}\} \times C} \cong \mathcal{L}$$

for any point $\mathcal{L} \in \mathrm{Pic}^d$ *and*

$$\mathcal{P}_{\mathrm{Pic}^d \times \{p_0\}} \cong \mathcal{O}_{\mathrm{Pic}^d}.$$

Proof: Since the restriction of $\mathcal{M}$ to any fiber $\mathbb{P}^r$ of η is trivial (both divisors $\mathcal{D}$ and $\mathcal{D}_0$ intersect $\mathbb{P}^r$ in a hyperplane), the theorem on cohomology and base change (Theorem B.9) shows that the direct image $\eta_* \mathcal{M}$ is a line bundle and the formation of this direct image commutes with base change.

The proof that $\mathcal{P}_{\mathrm{Pic}^d \times \{p_0\}} \cong \mathcal{O}_{\mathrm{Pic}^d}$ is immediate: If we restrict to the preimage

$$\eta^{-1}(\mathrm{Pic}^d \times \{p_0\}),$$

the divisors $\mathcal{D}$ and $\mathcal{D}_0$ agree, so that $\mathcal{M}|_{\eta^{-1}(\mathrm{Pic}^d \times \{p_0\})}$ is trivial, and so is its direct image.

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To prove that $\mathcal{P}|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$ we use the theorem on cohomology and base change. It implies that the formation of the direct image $\eta_*(\mathcal{M})$ commutes with base change, so we can first restrict to the preimage

$$|\mathcal{L}| \times C = \eta^{-1}(\{\mathcal{L}\} \times C),$$

where $|\mathcal{L}| \cong \mathbb{P}^r \subset C_d$ is the linear system of effective divisors D on C with $\mathcal{O}_C(D) \cong \mathcal{L}$. The restriction of η to $|\mathcal{L}| \times C$ is projection on the second factor.

As observed, the line bundle $\mathcal{M} = \mathcal{O}_{C_d \times C}(\mathcal{D} - \mathcal{D}_0)$ is trivial on each fiber of $\eta : |\mathcal{L}| \times C \to C$, so that the restriction $\mathcal{M}|_{|\mathcal{L}|\times C}$ must be a pullback of some line bundle on C; to prove that $\eta_*(\mathcal{M})|_{\{\mathcal{L}\}\times C} \cong \mathcal{L}$ amounts to showing that this line bundle is $\mathcal{L}$. Thus, it suffices to prove that

$$\mathcal{M}|_{|\mathcal{L}|\times C} = \eta^*\mathcal{L}. \tag{D.3}$$

For this it is enough to show that, for a general divisor D,

$$\mathcal{M}|_{\{D\}\times C} \cong \mathcal{L}.$$

This is immediate if D does not contain the point p_0: By definition, the divisor $\mathcal{D}_0 = X_{p_0} \times C \subset C_d \times C$ is disjoint from $\{D\} \times C$, while the divisor $\mathcal{D}$ intersects $\{D\} \times C$ in the divisor D. □

D.4.2 The evaluation map

Fix a reduced divisor $D = p_1 + \cdots + p_m$ of degree $m \geq 2g - 1 - d$ on C, and set $n = m + d$. Choosing D gives us an identification of Pic^d with Pic^n. We will describe the locus $W_d^r + D \subset \mathrm{Pic}^n$ as the degeneracy locus of an evaluation map.

On the product $\mathrm{Pic}^n \times C$, consider the evaluation map

$$\mathcal{P} \to \mathcal{P}|_\Gamma,$$

where

$$\Gamma = \mathrm{Pic}^n \times D = \bigcup_{i=1}^m (\mathrm{Pic}^n \times \{p_i\})$$

is the union of the horizontal sections of $\mathrm{Pic}^n \times C$ over Pic^n corresponding to the points p_i. Taking the direct image of this map under the projection $\pi : \mathrm{Pic}^n \times C \to \mathrm{Pic}^n$, we have a map of vector bundles

$$\rho : \mathcal{E} := \pi_*(\mathcal{P}) \to \mathcal{F} = \bigoplus_{i=1}^m \mathcal{L}_i,$$

where $\mathcal{L}_i$ is the restriction of $\mathcal{P}$ to the cross-section $\mathrm{Pic}^n \times \{p_i\}$. For each point $\mathcal{L} \in \mathrm{Pic}^n$, this is the map

$$\mathcal{E}_\mathcal{L} = H^0(\mathcal{L}) \to \bigoplus \mathcal{L}_{p_i}$$

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obtained by evaluating sections of $\mathcal{L}$ at the points p_i. In particular, the kernel of this map is the vector space $H^0(\mathcal{L}(-D)) \subset H^0(\mathcal{L})$ of sections vanishing along D. We have now proven that *the locus $W_d^r + D \subset \mathrm{Pic}^n$ is the locus where the map ρ has rank $n-g-r$ or less*.

In particular, the determinantal ideal defines a scheme structure on W_d^r. Though this structure appears to depend on the choice of the divisor D, in fact it does not: for any choice of D one can prove that the scheme W_d^r has a universal property independent of D that characterizes it. This is done explicitly, for example, in Arbarello et al. [1985, Chapter 4].

It remains to show that $W_d^r(C)$ is nonempty. To do so we will compute the Chern classes of the bundles $\mathcal{E}$ and $\mathcal{F}$ and apply Porteous' theorem. Before we do so, however, we must develop some basic information about the cohomology ring of the Jacobian, where these Chern classes live, and we must also identify the bundle $\mathcal{E}$ in a more useful way.

D.5 Natural classes in the cohomology ring of the Jacobian

Why are we working with the cohomology ring of the Jacobian rather than with the Chow ring? For one thing, it is computable: Since the Jacobian of a curve of genus g is topologically the product of $2g$ copies of the circle, its cohomology ring is an exterior algebra on $2g$ generators of degree 1, whereas the Chow groups of an abelian variety of dimension ≥ 2 are largely unknown. In particular, since any deformation of a cycle along a continuous path preserves the homology class of the cycle, the cohomology class of a cycle does not change under translation. The same is not true modulo rational equivalence, and this is very much an issue here: Most of the cycles whose classes we might hope to determine are in fact only defined after a choice of base point — in effect, only up to translation.

That said, our first goal will be to identify the classes in $H^*(\mathrm{Jac}, \mathbb{Z})$ of certain basic cycles. To start, the Jacobian is the quotient of the contractible space $H^0(\omega_C)^* \cong \mathbb{C}^g$ by the subgroup $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, so there is a natural identification

$$H_1(\mathrm{Jac}, \mathbb{Z}) = H_1(C, \mathbb{Z}).$$

The first cohomology $H^1(\mathrm{Jac}, \mathbb{Z})$ is similarly identified with $H^1(C, \mathbb{Z})$. These identifications are induced by the Abel–Jacobi map $u : C \to \mathrm{Jac}$ because the integral takes a closed path $\gamma : [0, 1] \to \mathbb{C}$ to the path $\tilde{\gamma} : [0, 1] \to H^0(\omega_C)^*$ defined by

$$\tilde{\gamma}(t) = \int_{\gamma([0,t])} \in H^0(\omega_C)^*,$$

which joins the origin to the lattice point corresponding to the homology class of γ.

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We choose a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for $H^1(C, \mathbb{Z})$, normalized so that the cup product has the form

$$(\alpha_i \cup \beta_j)[C] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0 \quad \text{for all } i, j.$$

By abuse of notation, we will also use the symbols α_i and β_i to denote the corresponding cohomology classes in $H^1(\mathrm{Jac}, \mathbb{Z})$.

Since the cohomology ring of the Jacobian is the exterior algebra generated by the α_i and β_j, we can compute arbitrary products of these elements. Given a multi-index $I = (i_1, \ldots, i_k)$ with $i_1, \ldots, i_k \in \{1, \ldots, g\}$, we will write α_I and β_I for the classes

$$\alpha_I = \alpha_{i_1} \cup \cdots \cup \alpha_{i_k} \quad \text{and} \quad \beta_I = \beta_{i_1} \cup \cdots \cup \beta_{i_k}$$

in $H^k(\mathrm{Jac}, \mathbb{Z})$. The classes

$$\{\alpha_I \cup \beta_J \mid I, J \subset \{1, \ldots, g\}\}$$

form a basis for $H^*(\mathrm{Jac}, \mathbb{Z})$. The cup product in complementary dimension is given (via the identification $H^{2g}(\mathrm{Jac}, \mathbb{Z}) \cong \mathbb{Z}$) for $I, J, K, L \subset \{1, \ldots, g\}$ by

$$\big((\alpha_I \cup \beta_J) \cup (\alpha_K \cup \beta_L)\big)[\mathrm{Jac}] = \begin{cases} \pm 1 & \text{if } K = I' \text{ and } L = J', \\ 0 & \text{otherwise,} \end{cases} \tag{D.4}$$

where I' denotes the complement of I. To determine the signs, we can pull back to the direct product of g copies of C, and use (D.10) below to prove that

$$(\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2 \cup \cdots \cup \alpha_g \cup b_g)[\mathrm{Jac}] = +1;$$

the signs of the other expressions in (D.4) are determined by skew-symmetry.

Of special interest are the classes $\eta_i \in H^2(\mathrm{Jac}, \mathbb{Z})$ defined as

$$\eta_i = \alpha_i \cup \beta_i \quad \text{for } i = 1, \ldots, g.$$

For a multi-index $I = (i_1, \ldots, i_k)$ with $i_1, \ldots, i_k \in \{1, \ldots, g\}$, we will write η_I for the class

$$\eta_I = \eta_{i_1} \cup \cdots \cup \eta_{i_k} \in H^{2k}(\mathrm{Jac}, \mathbb{Z}).$$

For example, by what we have just said $\eta_{(1,\ldots,g)} = 1 \in H^{2g}(\mathrm{Jac}, \mathbb{Z}) = \mathbb{Z}$. Rearranging the terms, we see that in general

$$\eta_I = (-1)^{\binom{k}{2}} \alpha_I \cup \beta_I.$$

The cup product in complementary dimension is easy to compute: For $I, J \subset \{1, \ldots, g\}$ we have

$$(\eta_I \cup \eta_J)[\mathrm{Jac}] = \begin{cases} 1 & \text{if } J = I', \\ 0 & \text{otherwise.} \end{cases} \tag{D.5}$$

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D.5.1 Poincaré's formula

The objects of primary interest to us are the classes of the subvarieties $W_d \subset \mathrm{Pic}^d$ parametrizing effective divisor classes of degree d, that is, the images of the maps $u = u_d : C_d \to \mathrm{Jac} \cong \mathrm{Pic}^d$. Like most of the objects in our treatment, the map u depends on the choice of base point $p_0 \in C$, so the subvarieties $W_d \subset \mathrm{Jac}$ are really only defined up to translation, though their classes in $H^*(\mathrm{Jac}, \mathbb{Z})$ are well-defined. Here is the basic result:

Proposition D.13 (Poincaré's formula)**.**

$$[W_d] = \sum_{\substack{I \subset \{1,\dots,g\} \\ |I| = g-d}} \eta_I \;\in\; H^{2g-2d}(\mathrm{Jac}, \mathbb{Z}).$$

The divisor $W_{g-1} \subset \mathrm{Jac}$ occurs often, and is usually called the *theta divisor* on Jac and denoted by Θ; its class is denoted by $\theta \in H^2(\mathrm{Jac}, \mathbb{Z})$. By Proposition D.13 we have

$$\theta = \eta_1 + \cdots + \eta_g.$$

With this notation we can restate the proposition as

$$[W_d] = \frac{\theta^{g-d}}{(g-d)!}.$$

The formula $[W_d] = [W_{g-1}]^{g-d}/(g-d)!$ makes sense in the ring of cycles on Jac modulo numerical equivalence — we do not need to introduce the topological cohomology of Jac to state it — and indeed it was proven in this numerical form for curves and their Jacobians over arbitrary fields in Kleiman and Laksov [1974]. We do not know if there is an analogous formula in a finer cycle theory such as the group of cycles modulo rational or algebraic equivalence.

Proof of Proposition D.13: By Poincaré duality, it suffices to take the product of both sides of the formula with an arbitrary element $\alpha_I \cup \beta_J$ with $|I| = |J| = d$, evaluate on the fundamental class of Jac and show that they are the same. In view of (D.4) and (D.5) above, in the case $I \neq J$ we have $\alpha_I \cup \beta_J \cup \eta_K = 0$ for any η_K of complementary degree, while if $I = J$ we have $(\alpha_I \cup \beta_I \cup \eta_K)[\mathrm{Jac}] = (-1)^{\binom{d}{2}}$ if $K = I'$ and 0 otherwise. Equivalently, since the map $C_d \to W_d$ is generically one-to-one for $d \leq g$, Proposition D.13 will be proven if we show that

$$(u^*(\alpha_I \cup \beta_J))[C_d] = \begin{cases} (-1)^{\binom{d}{2}} & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases} \tag{D.6}$$

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To evaluate the expression on the left, it is useful to pull back from the symmetric power C_d of the curve to C^d, the ordinary d-fold product. Let $\pi : C^d \to C_d$ be the quotient map, and let $\nu = u \circ \pi : C^d \to \mathrm{Jac}$ be the composition. Since π is a $d!$-fold cover,

$$\pi_*[C^d] = d! \cdot [C_d],$$

so (D.6) is equivalent to

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = \begin{cases} (-1)^{\binom{d}{2}} d! & \text{if } I = J, \\ 0 & \text{otherwise;} \end{cases} \tag{D.7}$$

that is,

$$(\nu^*(\alpha_I \cup \beta_J))[C^d] = 0 \quad \text{if } I \neq J \tag{D.8}$$

and

$$(\nu^* \eta_I)[C^d] = d! \quad \text{for all } I. \tag{D.9}$$

Let $\rho_k : C^d \to C$ be projection on the k-th factor, and set $\alpha_i^k = \rho_k^* \alpha_i$ and $\beta_i^k = \rho_k^* \beta_i$. By the Künneth formula, $H^1(C^d) = \bigoplus_k \rho_k^*(H^1(C))$. Writing $\iota_k : C \to C^d$ for the inclusion sending C to

$$\{p_0\} \times \cdots \times \{p_0\} \times C \times \{p_0\} \times \cdots \times \{p_0\},$$

with C in the k-th position, we see that $\rho_k \iota_k : C \to C$ is the identity, while $\rho_j \iota_k$ is the constant map when $j \neq k$. It follows that if $\gamma \in H^1(C^d)$ then $\gamma = \sum_k \rho_k^*(\iota_k^*(\gamma))$. Applying this to $\nu^* \alpha_i$ and $\nu^* \beta_i$, we see that

$$\nu^* \alpha_i = \alpha_i^1 + \cdots + \alpha_i^d \quad \text{and} \quad \nu^* \beta_i = \beta_i^1 + \cdots + \beta_i^d. \tag{D.10}$$

By symmetry we may assume that $I = \{1, \ldots, d\}$. Applying Formula (D.10) to

$$\nu^* \eta_I = \nu^* \alpha_1 \cup \nu^* \beta_1 \cup \cdots \cup \nu^* \alpha_d \cup \nu^* \beta_d,$$

we get the sum of all products of the form

$$\alpha_1^{j_1} \cup \beta_1^{k_1} \cup \cdots \cup \alpha_d^{j_d} \cup \beta_d^{k_d}.$$

This product is zero unless each $j_i = k_i$ and the set $\{j_1 \ldots, j_d\}$ is equal to $\{1, \ldots, d\}$. For these $d!$ terms,

$$\alpha_1^{j_1} \cup \beta_1^{j_1} \cup \cdots \cup \alpha_d^{j_d} \cup \beta_d^{j_d}[C^d] = 1,$$

because $\alpha_i^{j_i} \cup \beta_i^{j_i}$ is the pullback of the class of a point under ρ_{j_i}. □

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D.5.2 Symmetric powers as projective bundles

We now return to the argument of Section D.4.2. Recall that we have chosen a base point p_0, and that $\mathcal{P}$ denotes the Poincaré bundle on $\mathrm{Pic}^n \times C$ for some $n > 2g-2$, normalized so that $\mathcal{P}$ is trivial on $\mathrm{Pic}^n \times \{p_0\}$. To complete the argument of Theorem D.10 we need to compute the Chern classes of $\pi_*\mathcal{P}$, where $\pi : \mathrm{Pic}^n \times C \to \mathrm{Pic}^n$ is the projection. To do this we will identify $\pi_*\mathcal{P}$ in another way:

Theorem D.14. *With notation as above, the map $u : C_n \to \mathrm{Pic}^n$ is isomorphic to the projective bundle $\mathbb{P}(\pi_*\mathcal{P}) \to \mathrm{Pic}^n$, via an isomorphism $\mathbb{P}(\pi_*\mathcal{P}) \to C_n$ sending $\mathcal{O}_{\mathbb{P}(\pi_*\mathcal{P})}(1)$ to $\mathcal{O}_{C_n}(X_{p_0})$.*

Note that there is a natural identification of the fibers of the Abel–Jacobi map $u : C_n \to \mathrm{Pic}^n$ with the fibers of the projective bundle: By Theorem D.6, the scheme-theoretic fiber of u over a point $\mathcal{L} \in \mathrm{Pic}^n$ is the projective space $|\mathcal{L}|$. On the other hand, the restriction of $\mathcal{P}$ to the fiber $C \cong C \times \{\mathcal{L}\}$ is $\mathcal{L}$. Since the degree n of $\mathcal{L}$ is large, its higher cohomology vanishes, and the theorem on cohomology and base change (Theorem B.9) shows that the fiber of $\pi_*\mathcal{P}$ at a point $\mathcal{L} \in \mathrm{Pic}^n$ is the vector space $H^0(\mathcal{L})$ of sections of the line bundle $\mathcal{L}$. The projectivization of this space is again the projective space $|\mathcal{L}| = u^{-1}\{\mathcal{L}\}$. This fiber-by-fiber argument does not constitute a proof, but it suggests how we might go about giving one.

The divisor $X_{p_0} \subset C_n$, consisting of divisors containing p_0, cuts out the hyperplane section in each fiber of u, so, by Proposition 9.4, $C_n \to \mathrm{Pic}^n$ is a projective bundle in the Zariski topology. That is, u is the projection $\mathbb{P}\mathcal{G} \to \mathrm{Pic}^n$, where $\mathcal{G} := (u_*\mathcal{O}_{C_n}(X_{p_0}))^*$.

To prove Theorem D.14, accordingly, it suffices to show that the direct image $\pi_*\mathcal{P}$ is isomorphic to the dual of $u_*\mathcal{O}_{C_n}(X_{p_0})$. The key is to consider the direct image $\pi_*\mathcal{P}$ as the direct image of the line bundle $\mathcal{M} = \mathcal{O}_{C_n \times C}(\mathcal{D} - \mathcal{D}_0)$ in two ways: by definition $\pi_*\mathcal{P} = \pi_*(\eta_*\mathcal{M})$, and since $\pi \circ \eta = u \circ \pi_1$ we can also write it as $u_*(\pi_{1*}\mathcal{M})$. It may be helpful to have a diagram of the relevant objects:

$$\begin{tikzcd}
 & & \mathcal{M} = \mathcal{O}_{C_n \times C}(\mathcal{D} - \mathcal{D}_0) & & \\
 & & C_n \times C \arrow[dl, "\eta"'] \arrow[dr, "\pi_1"] & & \\
\begin{matrix}\pi_{1*}\mathcal{M} \\ \mathcal{O}_{C_n}(X_{p_0})\end{matrix} & C_n \arrow[dr, "u"'] & & \mathrm{Pic}^n \times C \arrow[dl, "\pi"] & \eta_*\mathcal{M} = \mathcal{P} \\
 & & \mathrm{Pic}^n & & \\
 & & \begin{matrix} u_*\pi_{1*}\mathcal{M} = \pi_*\mathcal{P} \\ u_*\mathcal{O}_{C_n}(X_p)\end{matrix} & &
\end{tikzcd}$$

It will also be helpful to have the following lemma:

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Lemma D.15. *With notation as above,*

$$\pi_{1*}\mathcal{M} \cong u^*(\pi_*\mathcal{P}).$$

Proof: There is a natural evaluation map

$$u^*u_*(\pi_{1*}\mathcal{M}) \to \pi_{1*}\mathcal{M};$$

since $u_*\pi_{1*}\mathcal{M} = \pi_*\mathcal{P}$, this gives a map

$$u^*(\pi_*\mathcal{P}) \to \pi_{1*}\mathcal{M}.$$

We claim this map is an isomorphism. This follows by looking at the map on fibers at a point $D \in C_n$. Since the sheaves involved have no higher cohomology of the fibers of the morphisms, the theorem on cohomology and base change allows us to identify the fibers of $u^*(\pi_*\mathcal{P})$ and $\pi_{1*}\mathcal{M}$ at D with the spaces

$$u^*(\pi_*\mathcal{P})_D = H^0(\mathcal{M}|_{|D|\times C}) \quad \text{and} \quad (\pi_{1*}\mathcal{M})_D = H^0(\mathcal{M}|_{\{D\}\times C}),$$

and in these terms the induced map $u^*(\pi_*\mathcal{P})_D \to (\pi_{1*}\mathcal{M})_D$ is the restriction map $H^0(\mathcal{M}|_{|D|\times C}) \to H^0(\mathcal{M}|_{\{D\}\times C})$. By equation (D.3), the bundle $\mathcal{M}|_{|D|\times C}$ on $|D|\times C$ is the pullback of a bundle on C, and so the restriction map is an isomorphism on global sections. □

We now proceed with the proof of Theorem D.14:

Proof of Theorem D.14: Since $\mathcal{D}_0 = \pi_1^*(X_{p_0})$, we can write

$$\mathcal{M} = \mathcal{O}_{C_n\times C}(\mathcal{D} - \mathcal{D}_0) = \mathcal{O}_{C_n\times C}(\mathcal{D}) \otimes \pi_1^*\mathcal{O}_{C_n}(-X_{p_0}),$$

and so

$$\pi_{1*}\mathcal{M} = \pi_{1*}(\mathcal{O}_{C_n\times C}(\mathcal{D})) \otimes \mathcal{O}_{C_n}(-X_{p_0}).$$

We have an inclusion of sheaves $\mathcal{O}_{C_n\times C} \hookrightarrow \mathcal{O}_{C_n\times C}(\mathcal{D})$ coming from the effective divisor $\mathcal{D}$; taking the direct image under π_1 gives an inclusion

$$\mathcal{O}_{C_n} \hookrightarrow \pi_{1*}\mathcal{O}_{C_n\times C}(\mathcal{D}).$$

Note that this is actually an inclusion of vector bundles. Indeed, for any point $D \in C_n$ we have $\mathcal{D}|_{\{D\}\times C} = D$, so by the theorem on cohomology and base change the fiber of $\pi_{1*}\mathcal{O}_{C_n\times C}(\mathcal{D})$ at D is just the vector space $H^0(\mathcal{O}_C(D))$, and the image of the inclusion is the one-dimensional subspace of sections vanishing on D. Tensoring with $\mathcal{O}_{C_n}(-X_{p_0})$, we get an inclusion of bundles

$$\mathcal{O}_{C_n}(-X_{p_0}) \hookrightarrow \pi_{1*}\mathcal{M}$$

and, dualizing, a surjection

$$\rho : (\pi_{1*}\mathcal{M})^* \to \mathcal{O}_{C_n}(X_{p_0}).$$

To finish, we claim that the pushforward of ρ gives an isomorphism

$$u_*\rho : u_*((\pi_{1*}\mathcal{M})^*) \longrightarrow u_*\mathcal{O}_{C_n}(X_{p_0});$$

given the identification of Lemma D.15, this will complete the proof of Theorem D.14. Once more invoking the theorem on cohomology and base change, it is enough to prove that ρ induces an isomorphism on global sections on each fiber $|\mathcal{L}| = u^{-1}(\mathcal{L}) \subset C_n$ of u; so, let us consider the restriction of the sheaves in question to $|\mathcal{L}| = \mathbb{P}H^0(\mathcal{L})$. To begin with, by Lemma D.15 the restriction of the bundle $\pi_{1*}\mathcal{M} = u^*(\pi_*\mathcal{P})$ to $|\mathcal{L}|$ is the trivial vector bundle with fiber $V = H^0(\mathcal{L})$, and so

$$(\pi_{1*}\mathcal{M})^*|_{|\mathcal{L}|} = V^* \otimes \mathcal{O}_{\mathbb{P}V}.$$

Next, the restriction of $\mathcal{O}_{C_n}(X_{p_0})$ to $|\mathcal{L}|$ is given by

$$\mathcal{O}_{C_n}(X_{p_0})|_{|\mathcal{L}|} \cong \mathcal{O}_{\mathbb{P}V}(1),$$

and in these terms the map $\rho|_{|\mathcal{L}|}$ is just the quotient map

$$V^* \otimes \mathcal{O}_{\mathbb{P}V} \to \mathcal{O}_{\mathbb{P}V}(1)$$

on the projective space $\mathbb{P}V$. This induces an isomorphism on global sections, and we are done. □

D.5.3 Chern classes from the symmetric power

We can now calculate the Chern class of $\mathcal{E} := \pi_*\mathcal{P}$:

Theorem D.16. *For $n \geq 2g - 1$, the pushforward $\mathcal{E}$ of the Poincaré bundle $\mathcal{P}$ from* $\mathrm{Pic}^n \times C$ *to* Pic^n *has Chern class $c(\mathcal{E}) = e^{-\theta}$; that is, $c_i(\mathcal{E}) = (-1)^i\theta^i/i!$ for each i.*

Proof: Computing the Chern class is equivalent to computing the Segre class $s(\mathcal{E}) = \sum s_i(\mathcal{E})$, since $c(\mathcal{E}) = 1/s(\mathcal{E})$ by Proposition 10.3. Recall that

$$s_k(\mathcal{E}) = u_*(\zeta^{k+n-g}),$$

where $\zeta = [X_p] \in H^2(C_n)$.

Since we are working in $H^*(C_n)$ rather than $A(C_n)$, the class ζ is also the class of the divisor $X_q = C_{n-1} + q \subset C_d$ for any point $q \in C$. To represent the class ζ^{k+n-g} we can just choose distinct points $p_1, \ldots, p_{k+n-g} \in C$ and consider the intersection

$$\bigcap X_{p_i} = \{D \in C_n \mid D - p_i \geq 0 \text{ for all } i\} = C_{g-k} + E \subset C_d,$$

where $E = p_1 + \cdots + p_{k+n-g}$. This intersection is generically transverse — it is visibly transverse at a point $E + D'$, where D' consists of $g - k$ distinct points distinct from $p_1, \ldots, p_{k+n-g}$, and no component of the intersection is contained in the complement of this locus — and so we have

$$\zeta^{k+n-g} = [C_{g-k} + E] \in H^{2k+2n-2g}(C_d).$$

We have

$$s_k(\mathcal{E}) = u_*[C_{g-k} + E] = [W_{g-k}] \in H^{2k}(\mathrm{Jac}).$$

Applying Poincaré's formula, this yields

$$s_k(\mathcal{E}) = \frac{\theta^k}{k!}.$$

We can express this compactly as

$$s(\mathcal{E}) = e^{\theta},$$

from which the theorem follows. □

Theorem D.16 allows us to give a description of the cohomology ring of C_d; though we will not use it in what follows, we state it here. By the analog of Theorem 9.6 for topological cohomology, we have for $n \geq 2g - 1$ that

$$H^*(C_n) = H^*(\mathrm{Jac})[\zeta] \Big/ \left(\zeta^{n-g+1} - \theta\zeta^{n-g} + \frac{\theta^2}{2}\zeta^{n-g-1} - \cdots\right).$$

Finally, we remark that there is an alternative way to derive the Chern classes of $\mathcal{E}$: Given that $\mathcal{E} = \eta_*\mathcal{M}$, we can apply Grothendieck–Riemann–Roch to the morphism η to arrive at Theorem D.16. This approach involves a larger initial investment — we have to have more knowledge of products in $H^*(C_d)$ than we currently do — but is also much more broadly applicable. This approach is carried out in Arbarello et al. [1985, Chapter 8], where many other applications are given.

D.5.4 The class of W^r_d

Here is our third and final version of the Brill–Noether theorem. It sharpens Theorem D.9 by giving the class of $W^r_d(C)$ in case this locus has the expected dimension. This enumerative statement is the key to the proof of the qualitative existence statement, which, surprisingly, does not depend on an a priori knowledge of the dimension.

Theorem D.17 (Enumerative Brill–Noether)**.** *(a) For every curve C of genus g, the locus $W^r_d(C)$ is nonempty of dimension*

$$\dim W^r_d(C) \geq \rho(g, r, d).$$

(b) If C is a curve of genus g such that $\dim W^r_d(C) = \rho(g, r, d)$, *then the class of $W^r_d(C)$ in the cohomology ring of the Jacobian* $\mathrm{Jac}(C)$ *is*

$$[W^r_d] = \prod_{i=1}^{r} \frac{i!}{(g-d+r+i)!}\theta^{(r+1)(g-d+r)}.$$

If $\rho \geq 0$ then this class is nonzero.

(c) If C is a general curve of genus g then $\dim W^r_d(C) = \rho(g, r, d)$.

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Proof of parts (a) and (b): Part (a) follows from part (b) as in the discussion of the content of enumerative formulas generally (Section 3.1) because, first, a determinantal locus is either empty or of dimension at least the "expected dimension" (see Lemma 5.2), and second, if it were empty, then we could consider it to have been of the correct dimension, and thus it would have to have the nonzero homology class of part (b), a contradiction.

We will use Porteous' formula to calculate the class of W^r_d as a degeneracy locus of the map $\mathcal{E} \to \mathcal{F}$ of vector bundles on Pic^n obtained by pushing forward the evaluation map $\mathcal{P} \to \bigoplus \mathcal{L}_i$.

The necessary ingredients are the Chern class of $\mathcal{E}$, computed in the previous section, and the Chern class of $\mathcal{F}$. The line bundles $\mathcal{L}_i$ can all be continuously deformed to the bundle $\mathcal{P}_{p_0} = \mathcal{P}|_{\mathrm{Pic}^n \times \{p_0\}}$, which is trivial by our normalization of $\mathcal{P}$. The Chern classes $c_1(\mathcal{L}_i) \in H^2(\mathrm{Pic}^n)$ are thus all 0, so that

$$c(\mathcal{F}) = 1 \in H^*(\mathrm{Pic}^n).$$

We have

$$\frac{c(\mathcal{F})}{c(\mathcal{E})} = \frac{1}{e^{-\theta}} = e^{\theta},$$

and so Porteous' formula tells us that if W^r_d has pure dimension $\rho = g-(r+1)(g-d+r)$ then its class is the determinant

$$[W^r_d] = \begin{vmatrix} \frac{\theta^{g-d+r}}{(g-d+r)!} & \frac{\theta^{g-d+r+1}}{(g-d+r+1)!} & \cdots & \frac{\theta^{g-d+2r}}{(g-d+2r)!} \\ \frac{\theta^{g-d+r-1}}{(g-d+r-1)!} & \frac{\theta^{g-d+r}}{(g-d+r)!} & \cdots & \frac{\theta^{g-d+2r-1}}{(g-d+2r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\theta^{g-d}}{(g-d)!} & \frac{\theta^{g-d+1}}{(g-d+1)!} & \cdots & \frac{\theta^{g-d+r}}{(g-d+r)!} \end{vmatrix}.$$

In other words,

$$[W^r_d] = D_{a,r} \cdot \theta^{(r+1)(g-d+r)},$$

where $D_{a,r}$ is the $(r+1) \times (r+1)$ determinant

$$D_{a,r} = \begin{vmatrix} \frac{1}{a!} & \frac{1}{(a+1)!} & \cdots & \frac{1}{(a+r)!} \\ \frac{1}{(a-1)!} & \frac{1}{a!} & \cdots & \frac{1}{(a+r-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(a-r)!} & \frac{1}{(a-r+1)!} & \cdots & \frac{1}{a!} \end{vmatrix}.$$

It remains to evaluate $D_{a,r}$. To do this, we clear denominators by multiplying the first column by $a!$, the second column by $(a+1)!$, and so on; we arrive at the expression

$$D_{a,r} = \prod_{i=0}^{r} \frac{1}{(a+i)!} \cdot M,$$

where M is the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+r \\ a(a-1) & (a+1)a & \cdots & (a+r)(a+r-1) \\ \vdots & \vdots & \ddots & \vdots \\ a\cdots(a-r+1) & (a+1)\cdots(a-r+2) & \cdots & (a+r)\cdots(a+1) \end{vmatrix}.$$

Since the columns of M all consist of the same sequence of monic polynomials, applied to the arguments $a, \ldots, a+r$, the determinant is equivalent to the Vandermonde determinant, and thus has value

$$\prod_{0 \le i < j \le r} (j-i) = \prod_{i=0}^{r} i!.$$

Thus

$$D_{a,r} = \prod_{i=1}^{r} \frac{i!}{(a+i)!}.$$

It follows that if the dimension of $W_d^r(C)$ is ρ then it has the class given in the theorem. In particular, if it were empty then it would have this class, which is nonzero, a contradiction. Thus it must be nonempty. Since it is defined as a degeneracy locus, it must have dimension at least the "expected dimension" locally at each of its points. This completes the proof of both Theorems D.17 and D.10. □

D.6 Exercises

Exercise D.18. Use the statements of Section D.2.2 to prove the original form of Jacobi inversion: Given two g-tuples of points $p_1, \ldots, p_g$ and $q_1, \ldots, q_g \in C$ on a smooth curve C of genus g, there exists a g-tuple of points $r_1, \ldots, r_g \in C$, whose coordinates are rational functions of the coordinates of the p_i and q_i, such that

$$\sum \int_{p_0}^{p_i} + \sum \int_{p_0}^{q_i} = \sum \int_{p_0}^{r_i}.$$

Exercise D.19. Let $C_0 \subset \mathbb{P}^2$ be a plane sextic with four nodes as singularities, whose normalization is a general curve of genus 6. Show that no three of the nodes are collinear. *Hint:* Use a dimension count.

Exercise D.20. Let $C_0 \subset \mathbb{P}^2$ be a plane sextic with four nodes as singularities, whose normalization is a general curve of genus 6. We have seen that there are five g^1_4's on C: the pencils cut out on C by the pencil of lines through each node and the pencil cut by conics passing through all four. Show that there are no others.

Exercise D.21. Let C be a curve of genus 8, embedded in $\mathbb{P}^3$ by one of the g^3_8's on C. Show that if C is general then the image curve $C_0 \subset \mathbb{P}^3$ does not lie on a cubic surface. In case it does, can you locate the 14 g^1_5's on C?

Exercise D.22. Let C be a general curve of genus 9. How many plane octic curves $C_0 \subset \mathbb{P}^2$ are birational to C?

Exercise D.23. Show that $W^r_d(C) \setminus W^{r+1}_d(C)$ is dense in $W^r_d(C)$.
Hint: For any point $\mathcal{L} \in W^{r+1}_d(C)$, consider the line bundle $\mathcal{L}(p-q)$ for general points $p, q \in C$.