Notes for Tropical Geometry

Fall 2022

The University of Texas at Austin
Lectured by Bernd Seibert

Isaac Martin

Last Compiled: August 26, 2022

Contents

§ *Entry 1*

Written: 2022-Aug-26

Left off last time by defining cell complexes aka CW complexes. Recall that that it is a space X which can be constructed ass a discrete set X^0 and then for all $n > 0$ we take $X^n = X^{n-1} \cup D^n_\alpha$ where α index in an index set indexes on n-cells.

We're given maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$ with $X^{(n)}$ is $X^{n-1} \cup \coprod D^n_\alpha$ with identification $x \in S^{n-1}_\alpha$ is identified with $\varphi_\alpha(x) \in X^{n-1}$.

The φ_α's need not be injective, but $D^n_\alpha - S^{n-1}_\alpha$ does inject into X^n.

We write e_α for the interior of D^n_α.

Either $X = X^n$ **or** $X = \bigcup_n X^n$ with the "weak topology": a subset of X is closed if and only if its intersection with each X^n is open.

Example 0.1. Basic Examples, all ways to write S^n:

(i) $\{\text{norm 1 vectors } in \mathbb{R}^{n+1}\}$

(ii) $D^n/\partial D^n$ meaning crush $S^{n-1} = \partial D^n$

(iii) cell complex with $X^n = 1$ pt with an n-disk attached (by a unique map $\partial D^n \longrightarrow \{pt\}$).

(iv) 1-pt compactification of $\mathbb{R}^n$

(v) Cell complex: $S^0 = S^{n-1}$ with 2 disks attached.

(5) allows for the definition of $S^\infty = \bigcup_n S^n$, which is not locally compact but is contractible.

Key properties of cell complexes: (approx)

- normal
- locally contractible
- every subset is deformation retractable of some neighborhood of itself
- ($\star$) every compact set lies in the union of finitely many cells.
- ($\star$) a function on X is continuous if and only if its restriction to each cell is continuous, i.e. $\Phi_\alpha : D^n_\alpha \longrightarrow X$ followed by f is continuous. Recall that this Φ_α is the characteristic map of cell e_α.

Example 0.2. We may define $\mathbb{RP}^n = S^{n+1}/\{\pm 1\}$. The cell complex structure is a "quotient of 2-cells-of-each-dimension" version of S^n. $\mathbb{RP}^n$ has one cell of each dimension, the ataching map $\partial D^n = S^{n-1}$ being the canonical map $S^{n-1} \longrightarrow S^{n-1}/\{\pm 1\} = \mathbb{RP}^n$.

Each cell has its boundary wrapped twice around $\mathbb{RP}^{n-1}$ in $\mathbb{RP}^2$.

The following is a severely important/useful tool. If X a space and $A \subseteq X$, then (X, A) has the homotopy extension property (HEP) if for all maps $F : X \longrightarrow Y$ and every homotopy $F : A \times I \longrightarrow Y$, $F|_A$ to some other map $A \longrightarrow Y$, there exists an extension to $\tilde{F} : X \times I \longrightarrow Y$. The idea is this: if you're given a motion of A inside Y, then you can drag along with it the points of X.

Theorem 0.3. *If X is a CW complex and $A \subseteq X$ is a sub-CW complex then (X, A) has the HEP property.*

Theorem 0.4. *If (X, A) has the HEP and A is contractible, then $X \xrightarrow{q} X/A$ is a homotopy equivalence.*

Proof: The clever part is writing down a homotopy inverse g. Suppose $f_t : A \times I \longrightarrow A$ is a contraction $(f_0 = \mathrm{id}_A, f_1 = \text{ const })$. We think of the contraction as a map $f_t : A \times I \longrightarrow X$, and then use HEP to get $f_t : X \times I\ otX$. Observe $f_1 : X \longrightarrow X$ sends A to a point, inducing a map $g : X/A \longrightarrow X$ as X/A is exactly all of A collapsed to a point.

We need to check that g is an inverse. Consider $X \xrightarrow{q} X/A \xrightarrow{g} X$ is $f_1 \simeq \mathrm{id}_X$ and

$$X/A \xrightarrow{g} X \xrightarrow{q} X/A$$

is $\overline{f}_1$ (function $X/A \longrightarrow X/A$) induced by $f : X \longrightarrow X$ $\overline{f}_1 \simeq \overline{f}_0$ since f_t sends A into $A \subseteq X$, hence induces $X/A \longrightarrow X/A$. The $\overline{f}_t$ are a homotopy between $\overline{f}_0 - \mathrm{id}_{X/A}$ and $\overline{f}_1$ = q$\circ g$. $\square$