Toric Geometry: Example Sheet 1

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§ *Theory Problems*

EXERCISE 1. Given a cone $\sigma \subseteq N_{\mathbb{R}}$ prove that the double dual recovers the original cone:

$$(\sigma^\vee)^\vee = \sigma.$$

This justifies the use of the word "dual".

Proof: We provide two solutions to this problem.

(1) This is a rather inelegant solution which makes use of the identifications $V \cong V^\vee \cong (V^\vee)^\vee$ in the case that V is a finite dimensional vector space. It nonetheless reflects how one typically thinks of the dual cone $\sigma^\vee$ geometrically.

Recall that for any field K and any K-vector space V of dimension $n < \infty$, we can find a non-canonical isomorphism $V \cong V^\vee$. One typically constructs such an isomorphism as follows.

First, fix a basis $\{e_1, ..., e_n\}$ for V and define $e_i^\vee$ to be the K-linear functional $e_i^\vee\left(\sum_{i=1}^n a_i e_i\right) = a_i$. It is straightforward to check that $\{e_1^\vee, ..., e_n^\vee\}$ forms a basis for the dual space $V^\vee$. We may similarly define the basis $\{e_1^{\vee\vee}, ..., e_n^{\vee\vee}\}$ of the double dual $V^{\vee\vee}$.

The pairing $\langle -, - \rangle : V^\vee \times V \longrightarrow K$ appearing in the definition of $\sigma^\vee$ is the bilinear map defined $\langle \lambda, v \rangle = \lambda(v)$. Adopting the above notation in the case that $V = N_{\mathbb{R}}$, we see that this pairing is simply the standard Euclidean inner product. Indeed, letting $\{e_i\}$ denote the standard basis on $\mathbb{R}^n \cong N_{\mathbb{R}}$, given any $v \in N_{\mathbb{R}}$ and $m \in M_{\mathbb{R}}$ and choosing $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}$ such that $v = \sum a_i e_i$ and $m = \sum b_i e_i^\vee$, we see that

$$\begin{aligned}
\langle m, v \rangle &= m(v) \\
&= (b_1 e_1^\vee + ... + b_n e_n^\vee)(v) \\
&= b_1 e_1^\vee(v) + ... + b_n e_n^\vee(v) \\
&= b_1 \cdot a_1 + ... + b_1 \cdot a_1.
\end{aligned}$$

By identifying $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ via $e_i \leftrightarrow e_i^\vee$, we may in fact *define* $\langle m, v \rangle$ to be the Euclidean inner product. This is useful because the Euclidean inner product is symmetric, i.e. $\langle m, v \rangle = \langle v, m \rangle$. By further identifying $\mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R}) = M_{\mathbb{R}}^\vee$ with $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ by $e_i \leftrightarrow e_i^\vee \leftrightarrow e_i^{\vee\vee}$, we see that for $v \in M_{\mathbb{R}}^\vee$ and $m \in M_{\mathbb{R}}$,

$$\langle v, m \rangle \geq 0 \iff \langle m, v \rangle \geq \iff \langle m, v' \rangle \geq 0$$

where v' is the unique element in $N_{\mathbb{R}}$ corresponding to $v \in M_{\mathbb{R}}^\vee$. Thus, under these identifications, we quite literally have that $\left(\sigma^\vee\right)^\vee = \sigma$.

(2) After reading Fulton more closely, I realized that it is perhaps more natural to define $\left(\sigma^\vee\right)^\vee$ to be a subset of σ rather than a subset of $\mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$. Given a subset $A \subseteq M_{\mathbb{R}}$, we first define the *predual* cone $A^\vee \subseteq N_{\mathbb{R}}$ of A to be

$$A^\vee = \{v \in N_{\mathbb{R}} \mid \lambda(v) \geq 0, \text{ for all } \lambda \in A\},$$

and then define the double dual $\left(\sigma^\vee\right)^\vee$ to be the predual cone of $\sigma^\vee$. Showing that $\left(\sigma^\vee\right)^\vee = \sigma$ is therefore equivalent to showing that for any $v_0 \in N_\mathbb{R} \setminus \sigma$, there is some $\lambda \in \sigma^\vee$ such that $\lambda(v_0) < 0$.

To do this, we use a version of the Hahn-Banach theorem I came across on Wikipedia. I'm not entirely sure this works, as I'm taking for granted that $N_\mathbb{R} \cong \mathbb{R}^n$ as a *topological* vector space. Here is the theorem:

Theorem 0.1. *Let A and B be non-empty convex subsets of a real locally convex topological vector space X. If $\mathrm{Int}(A) \neq \emptyset$ and $B \cap \mathrm{Int}(A) = \emptyset$, then there exists a continuous linear functional $f : X \longrightarrow \mathbb{R}$ such that $\sup f(A) \leq \inf f(B)$ and $|f(a)| < \inf f(B)$ for all $a \in \mathrm{Int}(A)$.*

Let v_0 be any element of $N_\mathbb{R}$ not in σ. Let A be an open ball centered at v_0 such that $A \cap \sigma = \emptyset$. This exists because σ is a closed subset of $N_\mathbb{R}$ which does not contain v_0, meaning the distance from v_0 to σ is positive. By Hahn-Banach, there exists a linear functional $\lambda \in M_\mathbb{R}$ such that $\lambda(v_0) < M = \inf \lambda(B)$. We show that $M = v_0$, hence $\lambda \in \sigma^\vee$.

We must have that $M \leq 0$ since $\lambda(0) = 0$ and $0 \in \sigma$. If $M < 0$, then there would necessarily be some $x \in \sigma$ such that $\lambda(x) < 0$. Assuming this to be the case, set $a = \frac{2\lambda(v_0)}{\lambda(x)}$, noting that $a > 0$ since $\lambda(x), \lambda(v_0) < 0$. This means that $ax \in \sigma$. However, recalling that $\lambda(v_0) < 0$, we have that

$$\lambda(ax) = a\lambda(x) = 2\lambda(v_0) < \lambda(v_0),$$

which is impossible since $\lambda(v_0) < \lambda(u)$ for all $u \in \sigma$. Hence, by contradiction, $M = 0$ and λ is nonnegative on all of σ. This means $\lambda \in \sigma^\vee$, so we are done.

I sincerely hope there is another proof besides the two provided here. The first feels highly unnatural and the second seems non-trivial. Given that both Cox-Little-Schneck and Fulton omit a proof of this fact in their book and that neither includes this problem as an exercise, I expect there exists a more natural, obvious proof of this fact that I am missing. $\square$

EXERCISE 2. Given a cone $\sigma \subseteq N_\mathbb{R}$ prove that σ is full-dimensional if and only if $\sigma^\vee$ is strictly convex.

Proof: Recall that a cone σ is said to be strictly convex if it does not contain a line. It turns out that it suffices to require that it does not contain a line through the origin, i.e. a one-dimensional subspace:

Lemma 0.2. A rational polyhedral cone σ is strictly convex if and only if it does not contain a line through the origin.

Proof. The forward implication is clear. Suppose then that σ is not strictly convex and contains the line $\ell : v_0 + tv$. For any $m \in \sigma^\vee$ we have $m(v_0 + tv) \geq 0$ for all $t \in \mathbb{R}$. Since $m(v_0 + tv) = m(v_0) + tm(v)$, this implies that $-tm(v) \leq m(v_0)$ for all $t \in \mathbb{R}$. However, $m(v)$ and $m(v_0)$ are constants, so the only way this is possible is if $m(v) = m(v_0) = 0$. Thus $m(tv) = 0$ for all $m \in \sigma^\vee$ and $t \in \mathbb{R}$, and therefore $tv \in \left(\sigma^\vee\right)^\vee = \sigma$. $\square$

We now proceed to the main problem.

($\Longleftarrow$) Suppose that σ is not full-dimensional. Then σ is contained in a hyperplane $V \subsetneq N_\mathbb{R}$, and $V^\perp$ is a one-dimensional subspace. For any $u \in V^\perp$ and any $v \in \sigma$, we have that

$$\langle u, v \rangle = 0,$$

and thus $u \in \sigma^{\perp}$. This implies that $\sigma^{\vee}$ contains a copy of $V^{\perp}$ and hence is not strictly convex.

($\Longrightarrow$) Suppose now that σ is full dimensional, so $\sigma + (-\sigma) = N_{\mathbb{R}}$. Fix some $m \in \sigma^{\vee}$ such that $m \neq 0$, and thus $\dim(\ker m) < \dim(N_{\mathbb{R}})$. The smallest subspace of $N_{\mathbb{R}}$ containing σ is $N_{\mathbb{R}}$ itself, hence $\sigma \setminus \ker m$ is nonempty.

Fix some $v \in \sigma \setminus \ker m$, noting that $v \neq 0$. Since $m(v) \geq 0$ and $m(v) \neq 0$, we must have that $m(v) > 0$ and $-m(v) < 0$. The element $-m \in M_{\mathbb{R}}$ lies on the line spanned by m but is not strictly nonnegative on σ; therefore, $\sigma^{\vee}$ does not contain the line spanned by m. Since m was chosen arbitrarily, $\sigma^{\vee}$ contains no line through the origin and so $\sigma^{\vee}$ is strictly convex by Lemma (0.2). □

EXERCISE 3. Let Σ and Σ' be fans in vector spaces $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$. Work out for yourself the correct definition of the product fan $\Sigma \times \Sigma'$ in $N_{\mathbb{R}} \oplus N'_{\mathbb{R}}$. Show that there is a natural isomorphism:

$$X_{\Sigma \times \Sigma'} \cong X_{\Sigma} \times_{\operatorname{Spec} \mathbb{C}} X_{\Sigma'}.$$

Slogan: "The construction of a toric variety from a fan commutes with products."

EXERCISE 4. In lectures, we claimed that the toric variety X_{σ} is smooth if and only if σ is generated by a subset of a $\mathbb{Z}$-basis for N. Complete the proof of this statement. Give an example to show that if σ is generated by a subset of a $\mathbb{Q}-basis$ then X_{σ} need not be smooth.

EXERCISE 5. Let X be a *not-necessarily-normal* toric variety with dense torus T. Recall that we partitioned the lattice $N = \operatorname{Hom}_{\mathrm{AlgGrp}}(\mathbb{C}^*, T)$ of T based on the limits of one-parameter subgroups of T inside X. If X were normal, this would give the fan of X and therefore determine X uniquely. Give examples to show that, without the normality assumption, this data does not uniquely determine X.

PROBLEM 6 Let $S \subseteq M$ be an affine semigroup. The *saturation* of S is defined to be:

$$S^{\mathrm{sat}} = \{m \in M \ : \ cm \in S \text{ for some } c \in \mathbb{Z}_{\geq 1}\}.$$

Clearly S^{sat} is saturated, and S is saturated if and only if $S = S^{\mathrm{sat}}$. Consider the inclusion

$$\mathbb{C}[S] \subseteq \mathbb{C}[S^{\mathrm{sat}}].$$

Look up "integral closure" of an integral domain, and prove that $\mathbb{C}[S^{\mathrm{sat}}]$ is the integral closure of $\mathbb{C}[S]$. The dual morphism is known as the normalization of $\operatorname{Spec} \mathbb{C}[S]$. In each of the following examples, write down equations in affine space for both $\operatorname{Spec} \mathbb{C}[S]$ and its normalization $\operatorname{Spec} \mathbb{C}[S^{\mathrm{sat}}]$, and study the morphism between them:

(a) $S = 2\mathbb{N} + 3\mathbb{N} \subseteq \mathbb{Z}$,

(b) $S = (1,1)\mathbb{N} + (1,0)\mathbb{N} + (0,2)\mathbb{N} \subseteq \mathbb{Z}^2$.

Proof: Recall from class (or Cox-Little-Schenck Theorem 1.3.5) that since S^{sat} is saturated, $\mathbb{C}[S^{\mathrm{sat}}]$ is integrally closed. The integral closure of a domain R is the smallest integrally closed subring $R' \subseteq \operatorname{Frac}(R)$ such that $R \subseteq R'$, and because $\mathbb{C}[S] \subseteq \mathbb{C}[S^{sat}]$, we need only show that every element of $\mathbb{C}[S^{\mathrm{sat}}]$is integral over $\mathbb{C}[S]$.

Certainly an element $az^m \in \mathbb{C}[S^{\text{sat}}]$ where $a \in \mathbb{C}[S]$ is integral over $\mathbb{C}[S]$, since

$$\begin{aligned} az^m \in \mathbb{C}[S^{\text{sat}}] &\implies m \in S^{\text{sat}} \implies cm \in S \text{ for some } c \in \mathbb{Z}_{\geq 1} \\ &\implies (az^m)^c = a^c z^{cm} \in \mathbb{C}[S] \implies f(x) = x^c - a^c z^{cm} \in \mathbb{C}[S][x] \text{ vanishes at } az^m. \end{aligned}$$

But then an arbitrary element $\sum a_i z^{m_i}$ of $\mathbb{C}[S^{\text{sat}}]$ is simply a sum of elements integral over $\mathbb{C}[S]$ and is therefore itself integral over $\mathbb{C}[S]$. If this argument is unsatisfactory, write $A = \mathbb{C}[S]$ and instead consider the chain of inclusions

$$A \hookrightarrow A\left[z_1^{m_1}\right] \hookrightarrow A\left[z_1^{m_1}, z_2^{m_2}\right] \hookrightarrow \ldots \hookrightarrow A\left[z_1^{m_1}, \ldots, z_k^{m_k}\right].$$

The ring $A[z_i^{m_i}]$ is finitely generated as an A-module for each $1 \leq i \leq k$ since $z_i^{m_i}$ is integral over A, hence $A[z_1^{m_1}, \ldots, z_i^{m_i}]$ is also finitely generated as an A -module by the transitivity of finite generation.

Let's compute some examples.

(a) We get that $S^{\text{sat}} = \mathbb{N}$ since $2k \in S$ for any $k \in \mathbb{N}$. We also have that $\mathbb{C}[S] = \mathbb{C}[t^2, t^3] \cong \mathbb{C}[x,y]/(x^3 - y^2)$ and $\mathbb{C}[S^{\text{sat}}] = \mathbb{C}[t]$, so the inclusion $\mathbb{C}[S] \hookrightarrow \mathbb{C}[S]$ induces a projection $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\left(\mathbb{C}[x,y]/(x^3 - y^2)\right)$ from the affine line to the singular cubic $y^2 = x^3$.

TODO: FINISH WRITE UP OF (a) AND DO PART (b) □

§ *Practice Problems*

EXERCISE 1. First problem

EXERCISE 10