Notes for Tropical Geometry

Fall 2022

The University of Texas at Austin
Lectured by Bernd Seibert

Isaac Martin

Last Compiled: August 30, 2022

Contents

§ *Entry 1*

Written: 2022-Aug-23

0 Introduction/Motivation

Tropical geometry is the study of discrete structures appearing in limits of polynomial equations.
Course outline:

(1) Hypersurface amoebas, their skeleta, and tropical limits

(2)

1 Hypersurface amoebas, their skeleta, and tropical limits

1.1 Laurent polynomial ring

$\mathbb{C}[z_1^{\pm_1}, ..., z_n^{\pm}]$. Each such Laurent polynomial defines a holomorphic (algebraic) map $f : (\mathbb{C}^\times)^n \longrightarrow \mathbb{C}$ whose zero locus $V(f) \subseteq (\mathbb{C}^*)^n$ $f \neq 0$ is a **complex hypersurface.** The ring $\mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}]$ is a unique factorization domain which implies $f = f_1^{\alpha_1} \cdot ... \cdot f_m^{\alpha_m}$ where the f_i are ireducible, pairwise different, and hence $Z(f) = Z(f_1) \cup ... \cup Z(f_m)$. This locus is *always* a complex submanifold, even in the case of the nodal cubic for instance, of $\dim_{\mathbb{C}} = n - 1$ outside of a real codimension 2 subset $Z(f) \cap Z(\partial_1 f) \cap ... \cap Z(\partial_n f)$.

Example 1.1.

(a) $V(z + w) \subseteq (\mathbb{C}^\times)^2$ is isomorphic as a $\mathbb{C}$-manifold or as an algebraic variety to $\mathbb{C}^\times$. The map $\mathbb{C}^\times \mapsto V(z + w)$ given $u \mapsto (u, -u)$ parameterizes this curve.

(b) $V(z + w + 1) \subseteq (\mathbb{C}^\times)^2$ is isomorphic to $\mathbb{C}^\times \setminus \{0, 1\}$ via the map $u \mapsto (u, 1 - u)$.

1.2 The Log Map

Forget phases and use logarithmic coordinates.

$$\text{Log} : (\mathbb{C}^\times)^n \xrightarrow{1.1} \mathbb{R}^n_{>0} \xrightarrow{\log} \mathbb{R}^n$$

given by

$$(z_1, ..., z_n) \mapsto (|z_1|, ..., |z_n|) \mapsto (\log |z_1|, ..., \log |z_n|).$$

Definition 1.2. The **Hypersurface amoeba** of $f \in \mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}] \setminus \{0\}$ is

$$\mathcal{A}_f = \text{Log}(V(f)) \subseteq \mathbb{R}^n$$

(Gelfand, Vapranov, Zelevabsky)

Example 1.3.

(a) $f = z + w$

(b) $f = z + w + 1$

(c) $f = 1 + 5zw + w^2 - z^2 + 3z^2w - z^2w^2$

(add pictures later) careful to draw these such that the complements of the amoeba are all convex.

Observations:

- connected cusps of $\mathbb{R}^n \setminus \mathbb{C}_f$ are convex in $\dim = 2$. $\mathcal{A}_f$ looks like a thickened graph. We'll sketch a proof of a more general result.

Recall: $\mathcal{U} \subseteq \mathbb{C}$, $f : \mathcal{U} \setminus \{p_1, ..., p_r\} \longrightarrow \mathbb{C}$ are meromorphic with mkr poles $(p_1, ..., p_r)$ and s zeros with multiplicity. This implies

$$s - r = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

This is the argument principle from complex analysis. Appears in the derivative of $\frac{1}{2\pi i} \int_{S^1} \log |f| dz$. This appears in the Jensen formula: $\mathcal{U} \subseteq \mathbb{C}$ an open subset and assume it contains a closed disk of radius r $\{z \mid |z| \leq r\} = D$. Important that it includes the boundary. Then if we have a holomorphic function $f : \mathcal{U} \longrightarrow \mathbb{C}$ with zeros of f in D $a_1, ..., a_k$ such that $0 < |a_1| \leq |a_2| \leq ... \leq |a_k|$ (with multiplicity) then we have

$$\frac{1}{2\pi i} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|}.$$

This is the Jensen formula.

Proof. (Rudin, "Real and complex analysis")

(1) Assume f has no zeros and hence that $\log |f|$ harmonic. Using the mean value property for harmonic functions (go review Analysis) yields the Jensen Formula.

(2) For the general case, suppose we have $|a_1|, ..., |a_n| < r$, and then that $|a_{m+1}|, ..., |a_k| = r$. Consider $g(z) = f(z) \cdot \prod_{j=1}^{m} \frac{r^2 - \overline{a}_j z}{r(a_j - z)} \prod_{j=m+1}^{k} \frac{a_j}{a_j - z}$ with no zeros in $|z| \leq r$. This implies

$$g(0) = f(0) \cdot \prod_{j=1}^{m} \frac{r}{a_j}$$

by our first case.

(3) $|z| = r$, so on the boundary, we have

$$|\frac{r^2 - a_j z}{r(a_j - z)}| = \frac{1}{r} |\frac{r^2 \overline{z} - a_j |z|^2}{r(a_j - z)}| = \frac{r}{r} = 1$$

$$\implies \log |g(re^{i\theta})| = \log |f(re^{i\theta})| - \sum_{j=m+1}^{k} \log \overbrace{|1 - e^{i(\theta - \theta_j)}|}^{a_j = re^{i\theta_j}}$$

(4) Lemma: $\int_0^{2\pi} \log(1-e^{i\theta})d\theta = 0$. These four things together prove the Jensen formula.

$\square$

For $n > 1$we define something called the Ronkin function. We have $f \in \mathcal{O}(\mathrm{Log}^{-1}(\Omega)), \Omega \subseteq \mathbb{R}^n$ a (convex) open set. Then the **Ronkin Function** is defined

$$N_f(x) = (\frac{1}{2\pi i})^n \int_{\mathrm{Log}^{-1}(x)} \mathrm{Log}\,|f(z_1,...,z_n)| \frac{dz_1}{z_1} \vee ... \vee \frac{dz_n}{z_n}$$

Theorem 1.4. *(a) N_f is a convex $\mathcal{C}^0$-function*

(b) $\mathcal{A}_f = \mathrm{Log}(V(f)) \subseteq \Omega$ an Amoeba. For all $\mathcal{U} \subseteq \Omega$ open, connected $\mathcal{U} \cap \mathcal{A}_f = \emptyset \iff N_f|_{\mathcal{U}}$ affine linear.

(c) $x \in \Omega \setminus \mathcal{A}_f \implies \mathrm{grad}\, N_f(x) = (v_1, ..., v_n)$,

$$v_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \vee ... \vee \frac{dz_n}{z_n}.$$

Picture: $N_f(x) = \langle \alpha_1, x \rangle + c_1$

Proof. (sketch)

(a) $\log|f|$ is plurisubharmonic (i.e. is subharmonic (i.e. somehow less than harmonic functions on a circle) on each each holomorphic image of a disk). We have the following fact: if $h : \mathcal{U} \longrightarrow \mathbb{R}$ is subharmonic, $\mathcal{U} \subseteq \mathbb{C}$ a domain containing $\{|z| \leq R\}$, then $\varphi(r) = \int_{|z|=r=\exp(s)} h(x)dz$ is a convex function in $\log r = s$. Found this proof in a book of Runkin called "Introduction to the theory of entire functions," page 84.

(b) Prove this next time

(c) $x \in \Omega \setminus \mathcal{A}_f$. Note:

$$\frac{\partial}{\partial x_j} \log|f| = \frac{1}{2} \frac{\partial}{\partial x_j} \log(f\overline{f}) = \mathrm{Re}\left(z_j \frac{\partial}{\partial z_j} \log f\overline{f} \right) = \mathrm{Re}\left(\frac{z_j \partial_j f}{f} \right).$$

$x \in \Omega \setminus \mathcal{A}_f$ implies that

$$\frac{\partial}{\partial x_j} N_f(x) = \mathrm{Re}\left(\frac{1}{2\pi i}^n \int_{\mathrm{Log}^{-1}} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n} \right).$$

Note: for all j, we have

$$\gamma_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}.$$

This is a locally constant n-form on $\mathcal{U} \setminus A_f$ and is not defined on $\mathcal{A}_f$ since f is zero on $\mathcal{A}_f$. In fact, $\gamma_j \in \mathbb{Z} : \frac{1}{2\pi i} \int_{|z_j|=e^{x_j}} \frac{\partial_j f(z)}{f(z)} dz_j \in \mathbb{Z}$ by the argument principle.

Look at Passare, Rullgard "Amoebas, Monge – Ampere, measures and triangulations DMJ 2004" $\square$

§ ***Entry 2***

Written: 2022-Aug-25

Recall that last time we had $V(f) \subseteq (\mathbb{C}^\times)^n \xrightarrow{\text{Log}} \mathbb{R}^n$, and we took $f \in \mathbb{C}[z_1^\pm, ..., z_n^\pm]$. This map has image in $\mathcal{A}_f \subseteq \mathbb{R}^n$. Recall also that the complement of the amoeba decomposes as the following union of connected components.

$$\mathbb{R}^n \setminus \mathcal{A}_f = \Omega_1 \cup ... \cup \Omega_k.$$

These connected components correspond to integral points of the Newton polyhedron $\operatorname{conv}\{I \mid a_I \neq 0\}$ where $f = \sum_{\text{finite}} a_I z^I$. Ronkin function is

$$N_f(x) = \frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(x)} \text{Log}\,|f(x)| \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}$$

is convex on $\mathbb{R}^n$ and is **affine linear on each** Ω_i which then implies that each Ω_i is convex.

Note: $\mathcal{U} = \text{Log}^{-1}(\Omega)$, where Ω is open, connected is a **circular domain**, i.e. change the argument of an element in the set and you're still in the set. These are called **Reinhardt domains**.

It is a fact that $\mathcal{U}$ is a domain of holomorphy if and only if Ω is convex. Laurent series converge on $\text{Log}^{-1}(\Omega)$ since Ω is convex.

Corollary 1.5. $\text{Log}^{-1}(\Omega_i)$ are the domains of convergence of the Laurent series expansions of f.

1.3 The spine of a hypersurface amoeba

Let $\varphi_i = N_f|_{\text{Log}^{-1}(\Omega_i)} = \langle \alpha_i, \cdot \rangle + c_i$ with $\alpha_i \in (\mathbb{R}^n)^*$ and $c_i \in \mathbb{R}$ be the piecewise affine approximation of N_f. Define

$$\varphi = \max\{\varphi_i\}.$$

Note that whenever N_f is convex we get that $\varphi \leq N_f$. **CHECK THIS, SWAPPED FROM MIN TO MAX, CHECK THIS INEQUALITY REMAINS SAME**

Definition 1.6.

$$\begin{aligned} \varphi_f &:= \{x \in \mathbb{R}^n \mid \varphi \text{ not affine linear near } x\} \\ &= \{x \in \mathbb{R}^n \mid \varphi \text{ not differentiable at } x\} \\ &= \{x \in \mathbb{R}^n \mid \exists i \neq j \text{ s.t. } \varphi_i(x) = \varphi_j(x = \max_k\{\varphi_k(x)\})\} \end{aligned}$$

is called the **spine** of $\mathcal{A}_f$.

Theorem 1.7. *[(Passare, Rullgard)]*

(a) *φ_f is the $(n-1)$-skeleton of a face-fitting decomposition of $\mathbb{R}^n$ into convex (with integrally defined facets) polyhedra.*

(b) *$\mathcal{A}_f$ deformation retracts onto φ_f.*

This notation is slightly confusing to me – φ_f is a subset of the graph of φ_f, it is not itself a function.

1.4 Tropical Limits and Maslov "dequantization"

$(\mathbb{R}_{>0}, +, \cdot) \xrightarrow{h\cdot\log = \log_t} (\mathbb{R}, \oplus_h, \odot_h)$ is a semiring isomorphism. The inverse is $(\mathbb{R}_{>0}, +, \cdot) \xleftarrow{\exp(x/h) \leftarrow x} (\mathbb{R}, \oplus_h, \odot)$ with

$$x \oplus_h y = h \cdot \log\left(\exp\left(\frac{x}{h}\right) + \exp\left(\frac{y}{h}\right)\right) \xrightarrow{h\to 0} \max\{x, y\}$$
$$x \odot y = h \cdot \log\left(\exp\left(\frac{x}{h}\right) \cdot \exp\left(\frac{y}{h}\right)\right) = x + y.$$

Now consider $f_h \in \mathbb{C}(h)[z_1^{\pm}, ..., z_n^{\pm}]$ e.g. $\frac{h^2+1}{h} z_1^2 + (h^3 - h^2) z_1 z_2^{-1}$. For all h we have that

$$\mathcal{A}_n(f_n) = \mathrm{Log}_t\,(V(f_h)) = h \cdot \mathcal{A}(f_h) \subseteq \mathbb{R}^n$$

are the amoeba for the rescaled Log-map $\mathrm{Log}_t = h\,\mathrm{Log}$. Here's a theorem from a paper prior to tropical geometry truly kicking off.

Theorem 1.8. *$\mathcal{A}_h(f_h)$ converges for $h \longrightarrow 0$ in the Hausdorff distance to the tropical hypersurface $V(\mathrm{trop}(f_h))$.*

$$f_h = \alpha_1 z^{\underline{u}_1} + ... + a_r z^{\underline{u}_r},\ a_i \in \alpha_i \in \mathbb{C}(h)$$

then

$$\mathrm{trop}\, f_h = \max\{\langle \underline{u}_1, -\rangle + c_1, ..., \langle \underline{u}_r, -\rangle + c_r\}$$

where $c_i = \mathrm{val}_0(\alpha_i)$, order of $\alpha_i(h)$ at $h = 0$.

$$\mathrm{val}_0(\frac{h^2+1}{h}) = -1, \mathrm{val}_0(h^3 - h^2)) = 2.$$

INCLUDE BOARD WITH HAUSDORFF DISTANCE

2 Tropical Arithmetic

2.1 Tropical semiring

Definition 2.1. $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is the tropical semiring or the min-plus algebra. We set

- $x \oplus y := \min\{x, y\}$
- $x \odot y := x + y.$

Both operations are commutative, associative, and are together distributive.

We have the following identities:

- $x \odot (y \oplus z) = x \odot y \oplus x \odot z$
- $x \oplus \infty = x$

- $x \oplus 0 = \begin{cases} 0 & x \geq 0 \\ x & x < 0 \end{cases}$
- $x \odot 0 = x$
- $x \odot \infty := \infty$

Explanation:

$$\begin{aligned}(x \oplus y)^3 &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= 3 \min\{x, y\} \\ &= \min\{3x, 3y\} = x^3 \oplus y^3 \\ &= \min\{3x, 2x + y, x + 2y, 3y\} = x^3 \oplus x^2 y \oplus xy^2 \oplus y^3\end{aligned}$$

Noting that $x^3 = 0 \odot x^3$, $x^2 y = 0 \odot x^2 y$, etc. we see that these are the coefficients of Pascal's triangle in tropical land, and that the coefficients are all 0. Hence the tropical Pascal triangle is just a bunch of 0's.

2.2 Linear algebra

The usual operations (formally) make sense over $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, e.g.

$$\begin{aligned}(u_1, u_2, u_3) \cdot (v_1, v_2, v_3)^T &= u_1 \odot v_1 \oplus u_2 \odot v_2 \oplus u_3 \odot v_3 \\ &= \min\{u_1 + v_1, u_2 + v_2, u_3 + v_3\}.\end{aligned}$$

$$(u_1, u_2, u_3)^T \odot (v_1, v_2, v_3) = \begin{pmatrix} u_1 \odot v_1 & u_1 \odot v_2 & \dots \\ u_2 \odot v_2 & \dots & \\ & & u_3 \ddots v_3 \end{pmatrix}$$

Definition 2.2. Matrices that can be written as $u^t \odot v$ have **tropical rank** 1.

Definition 2.3. The Barvihok rank of $A \in M(m \times n, \mathbb{R})$ is $\min\{k \mid \exists u_1, ..., u_k, v_1, ..., v_k, A = u_1^T \odot v_1 \oplus ... \oplus u_k^T \odot v_k\}$.

There are other notions of rank: Kapronov rank, tropical rank [MLS, S.5.3].

Looking at **tropical linear systems** $A \odot x = b$ has applications in engineering, dynamic programming (optimization via recursive structures, e.g. Find a shortest (weighted) path through a directed graph) etc. More on this in section 3.

2.3 Tropical Polynomials

Definition 2.4. A **Tropical polynomial** is a Laurent polynomial over $x_1, ..., x_n$, i.e. is a function on $\mathbb{R}, \oplus, \odot)^n$. A monomial is

$$x_1^{u_1} \odot x_2^{u_2} \cdot ... \cdot x_n^{u_n}$$

§ *Entry 3*

Recall that a tropical polynomial $f = a_1 \odot x^{\underline{u}_1} \oplus ... \oplus a_n \odot x^{\underline{u}_n}$ is a concave piecewise affine function

$$p(x) = \min\{\langle \underline{u_1} \rangle + a_1, ..., \langle \underline{u_n}, - \rangle + a_n\}.$$

Example 2.5. $p = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d = \min\{3x + a, 2x + b, x + c, d\}$. We say that the linear breaks of this graph are the vanishing points of p.

Lemma 2.6. For any concave, piecewise affine function with $\mathbb{Z}$-derivatives $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ there exists a tropical polynomial f with $p(x) = (x \mapsto f(x)$ in $(\mathbb{R}, \oplus, \odot))$.

Note: $f = \oplus a_I \odot x^I$ is only unique if we assume that for each I with $a_I \neq \infty$ we have that the map $x \mapsto \langle I, x \rangle + a_I$ agrees with h in a neighborhood of $x \in \mathbb{R}^n$.

Exercise 2.7 (Tropical Fundamental Theorem of Algebra)**.** Every PA function $p : \mathbb{R} \longrightarrow \mathbb{R}$ with integral derivatives (constant derivatives which are integers) can be written uniquely as a minimal product of tropical linear functions $a \odot x$.

Example 2.8 (Example of Tropical FTA Decomposition)**.** Take $f = x^2 \oplus 17 \odot x \oplus 2$. We then have

$$\begin{aligned} f &= x^2 \oplus 17 \odot \oplus x \oplus 2 \\ &= \min\{2x, x + 17, 2\} \\ &= \min\{2x, x + 1, 2\} \\ &= (x \oplus 1) \odot (x \oplus 1) \end{aligned}$$

Unique factorization fails for $n > 1$.

Example 2.9. Take $f(x, y) = (x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0) = (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0)$. The Newton polygon of $p = a_1 x^{\underline{u}_1} \oplus ... \oplus a_r x^{\underline{u}_r}$ gives us

$$\operatorname{Newt}(p) = \operatorname{conv}\{\underline{u}_i \mid a_i \neq \infty\}$$

Here: $f = x^2 y^2 \oplus x^2 y \oplus x y^2 \oplus xy \oplus x \oplus y \oplus 0$.

3 Dynamic Programming

3.1 Shortest paths in graphs

If G is a directed graph with n nodes $1, 2, ..., n$ and directed edges (i, j) have a weight $d_{ij} \in \mathbb{R}_{\geq 0}$ with $d_{ii} = 0$. We say $d_{ij} = \infty$ if there is no edge from i to j. We can conveniently present these distances in an $n \times n$ **adjacency matrix** in the extended reals, i.e

$$D_G = (d_{ij})_{i,j=1,...,n} \in M(n \times n, \mathbb{R} \cup \{\infty\}).$$

Example 3.1. <picture>

$$D_G = \begin{pmatrix} 0 & 3 & 1 & \infty \\ 1 & 0 & \infty & 3 \\ 1 & 2 & 0 & 0 \\ \infty & 1 & 1 & 0 \end{pmatrix}.$$

Proposition 3.2. The shortest length of a path from i to j is

$$(ij)\text{-entry of } D_G^{\otimes(n-1)} = \overbrace{D_G \odot ... \odot D_G}^{(n-1)-times}.$$

Proof: We have that

$$d_{ij}^r := \min\{\text{(weighted) length of a path from } i \text{ to } j \text{ with} \leq r \text{ edges}\}.$$

We have that $d_{ij}^{(1)} = d_{ij}$. If $d_{ij} \geq 0$, then a shortest path in the number of edges runs through each node at most once (otherwise, reverse the loop from i to i to arrive at a shorter path).

This implies that $d_{ij^{(n-1)}}$ = length of shortest weighted path from i to j. Recursively this gives

$$\begin{aligned} d_{ij}^{(r)} &= \min\{d_{ik}^{(}r-1) + d_{kj} \mid k \in \{1,...,n\}\} \\ &= d_{i_1}^{(r-1)} \odot d_{1j} \oplus ... \oplus d_{in}^{(r-1)} \odot d_{nj} \\ &= \begin{pmatrix} d_{i1}^{(r-1)} & \dots & d_{in}^{(r-n)} \end{pmatrix} \odot \begin{pmatrix} d_{1j} \\ \vdots \\ d_{nj} \end{pmatrix} \\ &= d_{ij}^{(r)} = (i,j)\text{-th entry of } D_G^{\odot r}. \end{aligned}$$

□

This can also be viewed as a limit of a quantum computation (Maslov's dequantization). Replace D_G with a matrix $A_G(\epsilon)$ where $A_G(\epsilon)_{ij} = \epsilon^{D_G(i,j)}$.

3.2 Integer Linear Programming

Given $A = (a_{ij}) \in M(d \times n, \mathbb{N})$ with $w \in \mathbb{R}^n$ and $b \in \mathbb{N}^d$. We'd like to solve the optimization problem $w \cdot u$ for $u \in \mathbb{N}^n$ subject to $Au \leq b$ or $Au = b$.

We can simplify this in the following way. For all j, take $\sum_i a_{ij} = \alpha$. Column sums are equal. We then have $b_1 + ... + b_d = m\alpha$, for $m \in \mathbb{N}$.

Then: $Au = b \implies u_1 + ... + u_n = m$. Indeed, $m\alpha = b_1 + ... + b_d = \sum_{i,j} \alpha_{ij} u_j = \sum_j (\sum_i a_{ij}) u_j = \alpha(u_1 + ... + u_n)$.

Proposition 3.3.

$$\min\{w \cdot u \mid Au = b\} = \text{coeff of } x_1^{b_1} \oplus ... \oplus x_d^{b_d}$$

in $(w \odot x_1^{a_{12}} \odot)$

3.3 The assignment problem and the tropical determinant