Part III Commutative Algebra

Example Sheet II, 2021, Solutions

1. If Z is not irreducible, we may write it as $Z = Z_1 \cup Z_2$ with Z_1, Z_2 proper closed subsets of Z. If both are irreducible, then we are done. Otherwise, we may, say, similarly write $Z_1 = Z_1' \cup Z_2'$. Repeating, either we eventually obtain a decomposition into irreducible closed subsets or we obtain an infinite strictly descending sequence of closed sets, contradicting X being Noetherian.

   To show uniqueness, suppose we have irredundant irreducible decompositions $Z = Z_1 \cup \cdots \cup Z_r = Z_1' \cup \cdots \cup Z_s'$. Then $Z_1' \subseteq Z = Z_1 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots (Z_r \cup Z_1')$. Since Z_1' is irreducible, we must have $Z_i \cup Z_1' = Z_1'$ for some i, i.e., $Z_1' \subseteq Z_i$. Similarly, $Z_i \subseteq Z_j'$ for some j. Thus $Z_1' \subseteq Z_j'$, so by irredundancy, $j = 1$ and $Z_i = Z_1'$. We may reorder and assume $i = 1$. Now take Z' to be the closure of $Z \setminus Z_1$. Then Z' has two irredundant irreducible decompositions $Z_2 \cup \cdots \cup Z_r$ and $Z_2' \cup \cdots \cup Z_r'$. Repeat inductively.

   Now suppose $I \subseteq A$ is a prime ideal and $V(I) = V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$ with $V(J_1), V(J_2)$ both strictly contained in $V(I)$. Now $I \in V(I)$ so $I \supseteq J_1 \cap J_2$. Thus $I \supseteq J_i$ for some i by statement on Handout, so $V(I) \subseteq V(J_i)$, contradicting the assumption. Thus $V(I)$ is irreducible.

   Note in any event, given an ideal $I \subseteq A$, $V(I) = V(\sqrt{I})$, and as $\sqrt{I}$ is the intersection of minimal primes in $V(I)$, $\sqrt{I}$ is completely determined by $V(I)$. Further, if $V(I) \subseteq V(J)$, then $\sqrt{J} \subseteq \sqrt{I}$. Now if A is Noetherian, suppose given a descending chain of closed subsets

   $$V(I_1) \supseteq V(I_2) \supseteq \cdots.$$

   We may replace each I_i with $\sqrt{I_i}$, and then we obtain an ascending chain of ideals

   $$\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \cdots$$

   which is necessarily stationary as A is Noetherian.

   Now the minimal associated primes of A/I are the minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $V(I)$, and certainly

   $$V(I) = V(\sqrt{I}) = V(\cap_i \mathfrak{p}_i) = \bigcup_i V(\mathfrak{p}_i),$$

   and the $V(\mathfrak{p}_i)$ are irreducible, so this gives an irreducible decomposition of $V(I)$. Since the $\mathfrak{p}_i$ are minimal, it is irredundant. The uniqueness proved above completes the argument.

2. We have
   $$I = (x, y) \cap (x, z) \cap (x, y, z)^2.$$

   The associated primes are (x, y), (x, z) and (x, y, z).

   How does one get this? I first identify the minimal primes by thinking geometrically. The zero set of I consists of the y-axis (where x, z are zero) and the z-axis, (where x, y are zero). This suggests $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$ as minimal primes. Finding the primary components corresponding to these two primes is easy: e.g., the $\mathfrak{p}_1$-primary component is the inverse image of $I^e = Ik[x, y, z]_{(x,y)}$. Note the extension of I in this localized ring is still generated by the same generating set, but since z is invertible in fact $x, y \in I^e$ and I^e is generated by x, y. Thus the inverse image of I^e is $\mathfrak{p}_1$. Similarly the $\mathfrak{p}_2$-primary component is (x, z). Now note that

   $$(x, y) \cap (x, z) = (x, yz) \neq I,$$

   so there must be an additional (embedded prime). At this point we guess that it is $\mathfrak{m} = (x, y, z)$ (in fact, I is what is known as a *monomial ideal*, i.e., is generated by monomials, and it isn't difficult to prove all associated primes are then monomial ideals. So (x, y, z) is a reasonably good guess: something interesting is happening at the origin.) We can confirm this guess by noting that $(x, y, z) = \mathrm{Ann}(x)$ when viewing $x \in k[x, y, z]/I$. At this point, we don't know how to do anything but guess a possible $\mathfrak{m}$-primary component of I.

3. Show the following statements about the support of an A-module:

   (a) Localizing the exact sequence at a prime $\mathfrak{p}$ the sequence remains exact and we see that $(M_2)_\mathfrak{p} \neq 0$ if and only if either $(M_1)_\mathfrak{p} \neq 0$ or $(M_3)_\mathfrak{p} \neq 0$.

   (b) Certainly because $M_i \subseteq M$, $\mathrm{Supp}(M_i) \subseteq \mathrm{Supp}(M)$, giving the inclusion in one direction. Conversely, suppose that $\mathfrak{p} \in \mathrm{Supp}(M)$. By assumption, the obvious map $\bigoplus_i M_i \to M$ is surjective. Localizing at $\mathfrak{p}$, we see that $\bigoplus_i (M_i)_\mathfrak{p} \neq 0$, so $\mathfrak{p} \in \mathrm{Supp}(M_i)$ for some i.

(c) Note that as M is finitely generated, so is $M_{\mathfrak{p}}$, and

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$$

is zero if and only $M_{\mathfrak{p}} = 0$ by Nakayama's lemma. On the other hand,

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) \cong (M \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) \cong M \otimes_A k(\mathfrak{p}),$$

hence the result.

(d) For this, first note the following sequence of isomorphisms:

$$\begin{aligned} (M \otimes_A N) \otimes_A k(\mathfrak{p}) &\cong M \otimes_A (N \otimes_A k(\mathfrak{p})) \\ &\cong (M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} (N \otimes_A k(\mathfrak{p})). \end{aligned}$$

Here we use successively use associativity of the tensor product and the identity mentioned in the problem. Since $k(\mathfrak{p})$ is a field, $M \otimes_A k(\mathfrak{p})$, $N \otimes_A k(\mathfrak{p})$ are vector spaces. The tensor product of two vector spaces is non-zero if and only if the two vector spaces are non-zero. Thus from (c) we obtain $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$.

(e) Let $\mathfrak{p} \in \operatorname{Spec}(M)$, and let $\mathfrak{q} \in \operatorname{Spec} B$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Then

$$(M \otimes_A B) \otimes_B k(\mathfrak{q}) \cong M \otimes_A k(\mathfrak{q}) \cong (M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{q}).$$

Note for the last isomorphism, we use the fact that f induces a field homomorphism $k(\mathfrak{p}) \to k(\mathfrak{q})$. Indeed, the composed map $A \to B \to B_{\mathfrak{q}}$ induces a map $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ as if $a \in A \setminus \mathfrak{p}$, $f(a) \notin \mathfrak{q}$ and hence is invertible in $B_{\mathfrak{q}}$. Further, as $f(\mathfrak{p}) \subseteq \mathfrak{q}$, we obtain a well-defined map $k(\mathfrak{p}) \to k(\mathfrak{q})$.

Now $\mathfrak{p} \in \operatorname{Supp}(M)$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$ if and only if $(M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} k(\mathfrak{q}) \neq 0$, i.e., $\mathfrak{q} \in \operatorname{Supp}(M \otimes_A B)$.

4. Let A be a ring and let $A[x]$ be the polynomial ring. If $I \subseteq A$ is an ideal, denote by $I[x]$ the set of all polynomials in $A[x]$ with coefficients in I.

(a) By definition I^e is the ideal generated by I in $A[x]$, which is clearly $I[x]$.

(b) Note that in general that $A[x]/I[x] \cong (A/I)[x]$, so $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$, which is an integral domain. Hence $\mathfrak{p}[x]$ is a prime ideal.

(c) Again, $A[x]/\mathfrak{q}[x] \cong (A/\mathfrak{q})[x]$. To prove $\mathfrak{q}[x]$ is primary, it is enough to show that any zero divisor is nilpotent in $(A/\mathfrak{q})[x]$. So let $f \in (A/\mathfrak{q})[x]$ be a zero-divisor. There is some $g \in A[x] \setminus \mathfrak{q}[x]$ with $f \cdot g = 0$. I first claim that I can take $g \in A \setminus \mathfrak{q}$. First, choose g of least degree m so that $f = 0$. Write $f = a_0 + \cdots + a_n x^n$, $g = b_0 + \cdots + b_m x^m$. Then $a_n b_m = 0$, so $a_n g = 0$, as $a_n g f = 0$ but $\deg a_n g < m$. If k is a positive integer such that $a_j g = 0$ for $j \geq k$, then $(f - (a_k x^k + \cdots + a_n x^n))g = 0$, which similarly implies $a_{k-1} g = 0$. Thus inductively we see that all coefficient of f kill g, and thus we have shown $a_i b_j = 0$ for all i, j. Thus $b_m f = 0$, and all coefficients of f are zero-divisors in $A/\mathfrak{q}$. Thus they are all nilpotent since $\mathfrak{q}$ is primary, and hence f is nilpotent. Indeed, if $a_i^{n_i} = 0$, then $f^{\sum n_i} = 0$.

(d) The fact that each $\mathfrak{q}_i[x]$ is primary follows from (c). The statement that $I[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is obvious. If the decomposition is not irredundant, say $I[x] = \bigcap_{i=2}^n \mathfrak{q}_i[x]$, then by looking at constant terms, we see $I = \bigcap_{i=2}^n \mathfrak{q}_i$, contradicting irredundancy of the decomposition for I.

(e) Suppose $I[x] \subseteq \mathfrak{q} \subsetneq \mathfrak{p}[x]$ with $\mathfrak{q} \subset A[x]$ prime. Then intersecting these ideals with A, we obtain $I \subseteq \mathfrak{q} \cap A \subset \mathfrak{p}$, and thus by minimality of $\mathfrak{p}$, $\mathfrak{q} \cap A = \mathfrak{p}$, so $\mathfrak{p} \subseteq \mathfrak{q}$. But then $\mathfrak{p}[x] \subseteq \mathfrak{q}$, as $\mathfrak{q}$ is an ideal, contradicting $\mathfrak{q} \subsetneq \mathfrak{p}[x]$. Thus $\mathfrak{p}[x]$ is a minimal prime over $I[x]$.

5. (a) If $x \in A$ is a zero-divisor, then $x \in \operatorname{Ann}(a)$ for some non-zero $a \in A$. If $\mathfrak{p}$ is a minimal prime over $\operatorname{Ann}(a)$, then $x \in \mathfrak{p}$ and $\mathfrak{p} \in D(A)$.

Conversely, suppose $x \in \mathfrak{p} \in D(A)$. We need to show x is a zero-divisor. So $\mathfrak{p}$ is a minimal prime over $\operatorname{Ann}(a)$ for some $a \in A$. As a consequence, x is nilpotent in the ring $(A/\operatorname{Ann}(a))_{\mathfrak{p}}$ as $\mathfrak{p}^e$ is the only prime ideal of this ring. So there exists an $s \in A \setminus \mathfrak{p}$ and $n > 0$ such that $x^n s \in \operatorname{Ann}(a)$. As $s \notin \mathfrak{p}$, $s \notin \operatorname{Ann}(a)$ so $sa \neq 0$. Thus we may take n minimal so that $x^{n-1} sa \neq 0$, $x^n sa = x \cdot (x^{n-1} sa) = 0$. Thus x is a zero-divisor.

(b) Now let $S \subseteq A$ be a multiplicatively closed subset. Now $S^{-1}\mathfrak{p} \in D(S^{-1}A)$ if and only if $S^{-1}\mathfrak{p}$ is a minimal prime over $\operatorname{Ann}(a/s)$ for some $a \in A, s \in S$. Now I claim that $\operatorname{Ann}(a)^e = \operatorname{Ann}(a/s)$. For the inclusion $\subseteq$, it suffices to note that any element of A which annihilates a also annihilates a/s. For the converse, suppose $b/t \in S^{-1}A$ annihilates a/s. Then there exists $t' \in S$ such that $abt' = 0$. Thus $bt' \in \operatorname{Ann}(a)$, and $b/t = (bt')/(tt') \in \operatorname{Ann}(a)^e$.

Thus under the inclusion $\operatorname{Spec} S^{-1}A \subseteq \operatorname{Spec} A$, the minimal primes over $\operatorname{Ann}(a/s)$ are precisely the minimal primes over $\operatorname{Ann}(a)$ which are disjoint from S. This shows that $D(S^{-1}A) = D(A) \cap \operatorname{Spec} S^{-1}A$.

(c) Now suppose that (0) has a primary decomposition. Then the primes associated to the primary components are the associated primes of A. If $\mathfrak{p}$ is such an associated prime, then there exists an $a \in A$ such that $\mathfrak{p} = \mathrm{Ann}(a)$, and in particular $\mathfrak{p} \in D(A)$.

Conversely, suppose that $\mathfrak{p} \in D(A)$ is a minimal prime over $\mathrm{Ann}(a)$. Then $\mathfrak{p}A_\mathfrak{p} \in D(A_\mathfrak{p})$ by (b), with $\mathfrak{p}A_\mathfrak{p}$ a minimal prime over $\mathrm{Ann}(a/1)$ by the argument in (b). Since $A_\mathfrak{p}$ is a local ring, certainly $\mathfrak{p}A_\mathfrak{p}$ is then the unique minimal prime over $\mathrm{Ann}(a/1)$, and hence $\mathrm{Ann}(a/1) = \mathrm{Ann}(a)^e$ is primary in $S^{-1}A$.

Now in general the contraction of a primary ideal is primary: if $\varphi : A \to B$, $\mathfrak{q} \subseteq B$ primary, and $a \cdot b \in \varphi^{-1}(\mathfrak{q})$, then $\varphi(a) \cdot \varphi(b) \in \mathfrak{q}$ and thus either $\varphi(a) \in \mathfrak{q}$, i,e., $a \in \mathfrak{q}^c$, or $\varphi(b)^n \in \mathfrak{q}$, i.e., $b^n \in \mathfrak{q}^c$. Thus $\mathrm{Ann}(a)^{ec}$ is primary. But

$$\mathrm{Ann}(a)^{ec} = \mathrm{Ann}(a/1)^c,$$

and if $b \in A$ such that $ab/1 = 0$ in $A_\mathfrak{p}$, there exists an $s \notin \mathfrak{p}$ such that $abs = 0$. But then $bs \in \mathrm{Ann}(a) \subseteq \mathfrak{p}$, so $b \in \mathrm{Ann}(a)$. Thus $\mathrm{Ann}(a/1)^c = \mathrm{Ann}(a)$, so we see that $\mathrm{Ann}(a)$ is $\mathfrak{p}$-primary. This means in particular that $\mathfrak{p}$ is the unique minimal prime over $\mathrm{Ann}(a)$, and hence $\mathfrak{p}$ is an associated prime of $A/\mathrm{Ann}(a)$, i.e., there exists a $b \in A/\mathrm{Ann}(a)$ with $\mathfrak{p} = \mathrm{Ann}(b)$. However, $A/\mathrm{Ann}(a)$ is isomorphic to the submodule of A generated by a, so b may be identified with an element of A and we see $\mathfrak{p}$ is an associated prime of A.

6. (a) If $a \in A$ is in the kernel of $A \to A_\mathfrak{p}$, then there exists $s \in A \setminus \mathfrak{p}$ such that $as = 0$. But $a \notin \mathfrak{p}$ implies that $a + \mathfrak{p}$ is a zero-divisor in the integral domain $A/\mathfrak{p}$, hence $a \in \mathfrak{p}$.

   (b) If $a \in S_\mathfrak{p}(0)$, there exists an $s \in A \setminus \mathfrak{p}$ such that $sa = 0$. But if $\mathfrak{p} \supseteq \mathfrak{p}'$, $s \in A \setminus \mathfrak{p}'$ also and $a \in S_{\mathfrak{p}'}(0)$.

   (c) If $\mathfrak{p}$ is minimal, then in particular it is the unique minimal prime over $S_\mathfrak{p}(0)$, and thus $\sqrt{S_\mathfrak{p}(0)} = \mathfrak{p}$. Conversely, suppose $\sqrt{S_\mathfrak{p}(0)} = \mathfrak{p}$, so that $\mathfrak{p}$ is then necessarily the unique minimal prime over $S_\mathfrak{p}(0)$. If $\mathfrak{p}$ is not minimal, then there exists minimal $\mathfrak{p}' \subsetneq \mathfrak{p}$, and then $S_\mathfrak{p}(0) \subseteq S_{\mathfrak{p}'}(0)$, so $\sqrt{S_\mathfrak{p}(0)} \subseteq \sqrt{S_{\mathfrak{p}'}(0)} = \mathfrak{p}'$ by the first sentence of this argument. Thus $\sqrt{S_\mathfrak{p}(0)} \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$, so we obtain a contradiction and $\mathfrak{p}$ is minimal.

   (d) Let $x \in \bigcap_{\mathfrak{p} \in D(A)} S_\mathfrak{p}(0)$, and let $\mathfrak{p}$ be a minimal prime over $\mathrm{Ann}(x)$. Then $\mathfrak{p} \in D(A)$. Thus $x \in S_\mathfrak{p}(0)$. If $x \neq 0$, there exists an $s \notin \mathfrak{p}$ such that $sx = 0$, so $s \in \mathrm{Ann}(x) \subseteq \mathfrak{p}$, which is a contradiction. Thus $x = 0$.

7. Let $I \neq 0$ be an ideal of A. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideal containing I; by assumption there is indeed a finite set of such. Choose $x_0 \in I$ non-zero, and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_{r+s}$ be the maximal ideals containing x_0. Since $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_{r+s}$ do not contain I there exists $x_j \in I$ such that $x_j \notin \mathfrak{m}_{r+j}$. Since each $A_{\mathfrak{m}_i}$ is Noetherian, the extension of I in $A_{\mathfrak{m}_i}$ is finitely generated. Hence there exists $x_{s+1}, \ldots, x_t \in I$ whose image in $A_{\mathfrak{m}_i}$ generated $A_{\mathfrak{m}_i}I$ for $i = 1, \ldots, r$. Let $I_0 = (x_0, \ldots, x_t)$. Then certainly by construction the extension of I_0 agrees with the extension of I for each $\mathfrak{m}_1, \ldots, \mathfrak{m}_{r+s}$. On the other hand, if $\mathfrak{m}$ is a maximal ideal of A not containing x_0, then $x_0/1$ is invertible in $A_\mathfrak{m}$ and thus $A_\mathfrak{m}I_0 = A_\mathfrak{m}I = (1)$ for each such maximal ideal. Thus the inclusion $I_0 \hookrightarrow I$ becomes an isomorphism after localizing at each maximal ideal, and being an isomorphism is a local property, so $I_0 = I$. Thus I is finitely generated.

8. (a) If $f, g \in S$ and $f \cdot g \notin S$, then $f \cdot g \in \mathfrak{p}_i$ for some i. Thus $f \in \mathfrak{p}_i$ or $g \in \mathfrak{p}_i$, in which case either $f \notin S$ or $g \notin S$.

   (b) First we show that $S^{-1}\mathfrak{p}_i$ is maximal. Let $a/s \notin S^{-1}\mathfrak{p}_i$, and suppose $a \notin k$. In particular, $a \notin \mathfrak{p}_i$, and hence a contains some monomial which does not involve $x_{m_i+1}, \ldots, x_{m_{i+1}}$. Remove all monomials of a which lie in $\mathfrak{p}_i$; the new a still satisfies $a \notin \mathfrak{p}_i$. Then $a + x_{m_i+1}$ does not lie in any of the $\mathfrak{p}_i$, so $a + x_{m_i+1} \in S$, i.e., is a unit in $S^{-1}A$. Thus every non-zero element of $S^{-1}A/S^{-1}\mathfrak{p}_i$ is invertible, and so $S^{-1}\mathfrak{p}_i$ is a maximal ideal.

   We now need to show these are the only maximal ideals. The prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of A disjoint from S, so in particular, it will be enough to show that for any ideal $I \subseteq \bigcup_i \mathfrak{p}_i$, $I \subseteq \mathfrak{p}_i$ for some i. This statement was given on the beginning of term handout in the case of finite unions, so some further work is required here.

   So suppose $I \subseteq \bigcup_{i \in L} \mathfrak{p}_i$ for some index set L. If L is finite, then we can use the result from the handout and conclude that $L \subseteq \mathfrak{p}_i$ for some i. Otherwise, assume that L is countable and that L is the minimal subset of $\{1, 2, 3, \ldots\}$ for which $I \subseteq \bigcup_{i \in L} \mathfrak{p}_i$. For $a \in A$, set

   $$L(a) := \{i \,|\, a \in \mathfrak{p}_i\}.$$

   Note that $L(a)$ is always finite. Let $a \in I$. If there does not exist $b \in I$ with $L(a) \cap L(b) = \emptyset$, then necessarily $I \subseteq \bigcup_{i \in L(a)} \mathfrak{p}_i$, and we can apply the finite result. Thus we may assume there exists $a, b \in I$ with $L(a) \cap L(b) = \emptyset$. Note that we never have $L(a) = \emptyset$ as then a does not lie in $\bigcup \mathfrak{p}_i$. So we have $L(a), L(b) \neq \emptyset$. Let $i \in L(b)$, d the degree of the polynomial a. Then $0 \neq a + x_{m_i+1}^{d+1} b \in I$. However I claim that $L(a + x_{m_i+1}^{d+1} b) = \emptyset$, which provides a contradiction. Note $L(x_{m_i+1}^{d+1} b) = L(b)$, and since $L(a) \cap L(b) = \emptyset$, $a + x_{m_i+1}^{d+1} b$ does not lie in any $\mathfrak{p}_j$ for $j \in L(a) \cup L(b)$. However, the lowest degree term of $x_{m_i+1}^{d+1} b$ is of greater degree than the term of highest degree in a, so there can be no cancellation between these two terms. Thus if $a \notin \mathfrak{p}_j$, a has a nonzero monomial not in $\mathfrak{p}_j$, and that monomial survives in $a + x_{m_i+1}^{d+1} b$. Thus $a + x_{m_i+1}^{d+1} b$ does not lie in $\mathfrak{p}_j$. This allows us to conclude this expression does not lie in any of the $\mathfrak{p}_j$, as desired.

(c) Note the intersection of an infinite number of the ideals $S^{-1}\mathfrak{p}_i$ is the zero ideal, so any non-zero $x \in S^{-1}A$ is contained in only a finite number of maximal ideals. On the other hand, the localization $(S^{-1}A)_{S^{-1}\mathfrak{p}_i}$ is isomorphic to

$$k(X_i)[x_{m_i+1}, \ldots, x_{m_{i+1}}]_{(x_{m_i+1}, \ldots, x_{m_{i+1}})}$$

where X_i denotes the set of all variables x_i save $x_{m_i+1}, \ldots, x_{m_{i+1}}$. Here $k(X_i)$ is the field of rational functions in these countably many variables. However, this ring is Noetherian.

(d) We note that we have for each i a chain of prime ideals in $S^{-1}A$ obtained by localizing the chain of ideals

$$0 \subseteq (x_{m_i+1}) \subseteq (x_{m_i+1}, x_{m_i+2}) \subseteq \cdots \subseteq \mathfrak{p}_i,$$

of length $m_{i+1} - m_i$. Since these differences are assume to be increasing, the height of $S^{-1}\mathfrak{p}_i$ is unbounded as $i \to \infty$, and hence $\dim S^{-1}A = \infty$.

9. To obtain the lower bound, let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_d$ be a chain of prime ideals of A with $d = \dim A$. Then we obtain a chain of ideals

$$\mathfrak{p}_0[x] \subset \cdots \subset \mathfrak{p}_d[x] \subset \mathfrak{p}_d[x] + (x).$$

All ideals but the last are prime by Q4 (b). For the last, note $A[x]/(\mathfrak{p}_d[x] + (x)) \cong A/\mathfrak{p}_d$ is an integral domain, so $\mathfrak{p}_d[x] + (x)$ is also prime. This gives a chain of primes of length $d + 1$.

For the upper bound, let

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_e$$

be a chain of prime ideals in $A[x]$. Let $\mathfrak{q}_e = \mathfrak{p}_e \cap A$, i.e., $\mathfrak{q}_e = f^*(\mathfrak{p}_e)$ in the notation of the hint. Suppose $\mathfrak{q}_i = \mathfrak{q}_{i+1} = \mathfrak{q}_{i+2}$. Then $\mathfrak{p}_i, \mathfrak{p}_{i+1}, \mathfrak{p}_{i+2}$ lie in $(f^*)^{-1}(\mathfrak{q}_i)$, and this set is in one-to-one correspondence with primes of $k(\mathfrak{q}_i)[x]$. However, as the latter ring is dimension one, we obtain a contradiction. Thus we can have at most chains $\mathfrak{p}_i \subset \mathfrak{p}_{i+1}$ of length two with $\mathfrak{q}_i = \mathfrak{q}_{i+1}$, from which follows the upper bound.