Lecture 21

§ Construction of Art_K

K, non-arch. local field, π uniformiser of K. For $n \geq 1$, construct $K_{\pi,n}$ totally ramified Galois extension s.t.

(i) $K \subseteq \dots \subseteq K_{\pi,n} \subseteq K_{\pi,n+1} \subseteq \dots$

(ii) For $n \geq m \geq 1$ $\exists$ diagram

$$\begin{tikzcd} \mathrm{Gal}(K_{\pi,n}/K) \arrow[r, two heads] \arrow[d, "\varphi_n"', "\wr"] & \mathrm{Gal}(K_{\pi,m}/K) \arrow[d, "\varphi_m", "\wr"'] \\ \mathcal{O}_K^\times/U_K^{(n)} \arrow[r, two heads, "\text{canonical projection}"'] & \mathcal{O}_K^\times/U_K^{(m)} \end{tikzcd}$$

(iii) Setting $K_{\pi,\infty} := \bigcup_{n=1}^{\infty} K_{\pi,n}$, we have

$$K^{ab} = K^{ur} K_{\pi,\infty}.$$

Since $\mathcal{O}_K^\times = U_K^{(0)} \cong \varprojlim_n \mathcal{O}_K^\times/U_K^{(n)}$

Ex sheet 4 q1

(ii) $\Rightarrow$ $\exists$ iso. $\psi : \mathrm{Gal}(K_{\pi,\infty}/K) \cong \varprojlim_n \mathcal{O}_K^\times/U_K^{(n)} \cong \mathcal{O}_K^\times$

Define Art_K by

$$K^\times \cong \mathbb{Z} \times \mathcal{O}_K^\times \longrightarrow \mathrm{Gal}(K^{ur}/K) \times \mathrm{Gal}(K_{\pi,\infty}/K) \overset{(iii)}{\cong} \mathrm{Gal}(K^{ab}/K)$$

$$\pi^n u \leftrightarrow (\pi^n, u) \longmapsto \left((\mathrm{Frob}_{K^{ur}/K})^n, \psi^{-1}(u) \right).$$

2

$$\begin{tikzcd}[column sep=small] & \mathbb{Q}_p^{ab} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{Q}_p^{ur} & & \mathbb{Q}_p(\varepsilon_{p^\infty}) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & K^{ab} \arrow[dl, dash] \arrow[dr, dash] & \\ K^{ur} & & K_{\pi,\infty} \end{tikzcd}$$

$\mathbb{Z} \quad \mathbb{Q}_p \quad \mathbb{Z}_p \qquad\qquad \mathbb{Z} \quad K \quad \mathcal{O}_K$

Remark: Both $K_{\pi,\infty}$ and iso $K^\times \cong \mathbb{Z} \times \mathcal{O}_K^\times$ depend on π. These choices "cancel out" so Art_K is canonical.

Goal: Construct $K_{\pi,n}$.

VII Lubin-Tate theory

§ Formal group laws

R a ring, $R[\![X_1, \dots, X_n]\!] = \{ \sum_{k_1,\dots,k_n \geq 0} a_{k_1 \dots k_n} X_1^{k_1} \dots X_n^{k_n} \mid a_{k_1 \dots k_n} \in R \}$,

ring of formal power series in n-variables over R.

Definition 18.1: A (1-dimensional commutative) formal group law over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ satisfying:

(i) $F(X,Y) \equiv X + Y \mod \deg 2$

(ii) $F(X, F(Y,Z)) = F(F(X,Y), Z)$ (Associativity)

(iii) $F(X,Y) = F(Y,X)$ (commutativity)

Eg. • $\widehat{\mathbb{G}}_a(X,Y) = X + Y$, formal additive group

• $\widehat{\mathbb{G}}_m(X,Y) = X + Y + XY$, formal multiplicative group.

Lemma 18.2: F formal group law over R.

(i) $F(X,0) = X$, $F(0,Y) = Y$

(ii) $\exists$ a unique $i(X) \in X R[\![X]\!]$ s.t.

$F(X, i(X)) = 0$

$F(X, \iota(X)) = 0$.

Proof: Ex sheet 4. $\square$

K complete non-arch. valued field. F - formal group law over $\mathcal{O}_K$. Then $F(x,y)$ converges $\forall x, y \in m_K$ to an element in m_K

· Defining $x \cdot_F y = F(x,y)$, turns $(m, \cdot_F)$ into a commutative group.

Eg. $\widehat{\mathbb{G}}_m / \mathbb{Z}_p$, $x \cdot_{\widehat{\mathbb{G}}_m} y = x + y + xy$.

$$(p\mathbb{Z}_p, \cdot_{\widehat{\mathbb{G}}_m}) \cong (1 + p\mathbb{Z}, \times)$$
$$x \longmapsto 1 + x$$

Definition 18.3: Let F, G be formal group laws $/R$. A homomorphism $f : F \to G$ is an element $f(X) \in XR[[X]]$ s.t.

$$f(F(X,Y)) = G(f(X), f(Y)).$$

A hom. $f : F \to G$ is an isomorphism if $\exists$ hom $g : G \to F$ s.t. $f(g(X)) = X$, $g(f(X)) = X$.

Define $\mathrm{End}_R(F)$ set of homs. $f : F \to F$.

Proposition 18.4: R a $\mathbb{Q}$-algebra. There is an iso of formal group laws.

$$\exp : \widehat{\mathbb{G}}_a \longrightarrow \widehat{\mathbb{G}}_m.$$
$$\exp(X) = \sum_{n=1}^{\infty} \frac{X^n}{n!}.$$

Pf: Define $\log(X) = \sum^{\infty} (-1)^{n-1} X^n$ … Then $\exists$ equalities

Proof: Define $\log(X) = \sum_{n=1} (-1)^{n+1} \frac{X^n}{n}$. Then $\exists$ equalities of formal power series

$$\log(\exp(X)) = X \qquad \exp(\log(X)) = X.$$

$$\exp(\hat{G}_a(X,Y)) = \hat{G}_m(\exp(X,Y)) \qquad \square$$

Lemma 18.5: $\mathrm{End}_R(F)$ is a ring with addition

$$(f +_F g)(X) = F(f(X), g(X))$$

and multiplication given by composition.

Proof: $f, g \in \mathrm{End}_R(F)$

$$(f +_F g) \circ F(X,Y) = F(f(F(X,Y)), g(F(X,Y)))$$

$$= F(F(f(X), f(Y)), F(g(X), g(Y)))$$

Use ass. + comm. $\longrightarrow$ $= F(F(f(X), g(X)), F(f(Y), g(Y)))$

$$= F(f +_F g(X), f +_F g(Y)).$$

$\Rightarrow f +_F g \in \mathrm{End}_R(F)$.

$f \circ g \circ F = f \circ F \circ g = F \circ f \circ g$

$\Rightarrow f \circ g \in \mathrm{End}_R(F)$.

Ring axioms: Exercise. $\square$

§ Lubin-Tate formal groups.

K non-arch. local field. $|k| = q$.

Definition 19.1: A formal $\mathcal{O}_K$-module over $\mathcal{O}_K$ is a formal group law $F(X,Y) \in \mathcal{O}_K[[X,Y]]$

together with a ring hom

$$[\ ]_F : \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_K}(F) \text{ s.t.}$$

$\forall a \in \mathcal{O}_K \quad [a]_F(X) \equiv aX \mod X^2.$

A hom/iso $f : F \to G$ of formal $\mathcal{O}_K$-modules is a hom
6 of formal groups s.t. $f \circ [a]_F = [a]_G \circ f \ \forall a \in \mathcal{O}_K$

Definition 19.2: Let $\pi \in \mathcal{O}_K$ be a uniformizer. A Lubin-Tate series for π is a power series $f(X) \in \mathcal{O}_K[\![X]\!]$ s.t.

(a) $f(X) \equiv \pi X \mod X^2$

(b) $f(X) \equiv X^q \mod \pi.$

Eg. $K = \mathbb{Q}_p$. $f(X) = (X+1)^p - 1$ is a Lubin-Tate series for p.

1 Theorem 19.3: Let $f(X)$ be a Lubin-Tate series for π.

(i) $\exists$ a unique formal grouplaw F_f over $\mathcal{O}_K$ s.t. $f \in \mathrm{End}_{\mathcal{O}_K}(F_f)$.

(ii)

(ii) $\exists$ a ring. hom

$$[\ ]_{F_f} : \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_K}(F_f)$$

satisfying $\quad [\pi]_{F_f}(X) = f(X)$.

which endows F_f with structure of a formal $\mathcal{O}_K$-module over $\mathcal{O}_K$.

(iii) If $g(X)$ another Lubin-Tate series, then $F_f \cong F_g$ as formal $\mathcal{O}_K$-modules.

F_f is Lubin-Tate formal group law for π - only depends on π up to iso.

f Lubin-Tate series for π.

$f(X) \equiv \pi X \mod X^2$

$f(X) \equiv X^q \mod \pi$ $\quad \rightsquigarrow F_f$ Lubin-Tate formal gp.

Eg. $K = \mathbb{Q}_p$, $f(X) = (X+1)^p - 1$

Lubin-Tate formal group F_f is $\widehat{\mathbb{G}}_m$.

STS $f \circ \widehat{\mathbb{G}}_m = \widehat{\mathbb{G}}_m \circ f$

$$f(\widehat{\mathbb{G}}_m(X,Y)) = (1+X)^p(1+Y)^p - 1$$
$$= \widehat{\mathbb{G}}_m(f(X), f(Y)).$$

Lemma 19.4: (Key Lemma)

$f(X), g(X)$ Lubin-Tate series for π. Let

$$L(X_1, \dots, X_n) = \sum_{i=1} a_i X_i, \quad a_i \in \mathcal{O}_K.$$

$\exists$ a unique power series

$$F(X_1, \dots, X_n) \in \mathcal{O}_K [\![X_1, \dots, X_n]\!] \quad \text{s.t.}$$

(i) $F(X_1, \dots, X_n) \equiv L(X_1, \dots, X_n) \mod \deg 2.$

(ii) $f(F(X_1, \dots, X_n)) = F(g(X_1), \dots, g(X_n)).$

Proof: We show by induction, $\exists$ unique $F_m \in \mathcal{O}_K[X_1, \dots, X_n]$ of total degree $\leq m$ s.t.

(a) $f(F_m(X_1, \dots, X_n)) \equiv F_m(g(X_1), \dots, g(X_n)) \mod \deg$

(b) $F_m(X_1, \dots, X_n) \equiv L(X_1, \dots, X_n) \mod \deg 2.$

(c) $F_m \equiv F_{m+1} \mod \deg m+1$

For $m=1$, take $F_1 = L$. (b) ✓

$$f(F(X_1, \dots X_n)) \equiv \pi L(X_1, \dots, X_n) \mod \deg 2$$
$$\equiv F(g(X_1), \dots, g(X_n)) \mod \deg 2$$

so (a) satisfied.

Suppose F_m constructed, $m \geq 1$.

Set $F_{m+1} = F_m + h$

$h \in \mathcal{O}_K[X_1, \dots, X_n]$ homogeneous of degree $m+1$. We have

$$f \circ (F_m + h) = f \circ F_m + \pi h \mod \deg m+2$$

since $f(X) \equiv \pi X \mod X^2$ (Eg. use $f(X+Y)$
$= f(X) + f'(X) Y$

$g(X) \equiv \pi X \mod X^2$

Similarly, $(F_m + h)\circ g \equiv F_m \circ g + h(\pi X_1, \dots, \pi X_n)$ mod deg

$$\equiv F_m \circ g + \pi^{m+1} h(X_1, \dots, X_n)$$

Thus (a) + (b) + (c) are satisfied iff

$$f \circ F_m - F_m \circ g \equiv (\pi - \pi^{m+1}) h \text{ mod deg } m+2$$

But $f(X) \equiv g(X) \equiv X^q \bmod \pi$

Thus $f \circ F_m - F_m \circ g \equiv F_m(X_1, \dots, X_n)^q$

$$- F_m(X_1^q, \dots, X_n^q) \bmod \pi.$$

$$\equiv 0 \bmod \pi.$$

Thus $f \circ F_m - F_m \circ g \in \pi\, \mathcal{O}_K[[X_1, \dots, X_n]]$

Let $r(X_1, \dots, X_n)$ be deg $m+1$ terms in $f \circ F_m - F_m \circ g$.

Then set $h := \dfrac{1}{\pi(1-\pi^m)} r \in \mathcal{O}_K[X_1, \dots, X_n]$

so that F_{m+1} satisfies (a) + (b) + (c).

Unique since h determined by property (a).

Set $F := \lim_{m \to \infty} F_m$, then $F(X_1, \dots, X_n)$ satisfies (i) + (ii).

Uniqueness of F follows from uniqueness of F_m. $\square$

Proof of theorem 19.3

(i) By Lemma 19.4, there exists a unique

$F_f(X,Y) \in \mathcal{O}_K[[X,Y]]$ s.t.

[illegible]

- $F_f(X,Y) \equiv X+Y \mod \deg 2$
- $f(F_f(X,Y)) = F_f(f(X), f(Y))$.

F_f is a formal group law:

- Associativity:

$$F_f(X, F_f(Y,Z)) \equiv X+Y+Z \mod \deg 2$$
$$\equiv F_f(F_f(X,Y),Z) \mod \deg 2.$$

and
$$f \circ F_f(X, F_f(Y,Z)) = F_f(f(X), f(F_f(Y,Z)))$$
$$= F_f(f(X), F_f(f(Y), f(Z)))$$

Similarly $f \circ F_f(F_f(X,Y),Z) = F_f(F_f(f(X),f(Y)), f(Z$

Thus $F_f(X, F_f(Y,Z)) = F_f(F_f(X,Y),Z)$

by uniqueness in Lemma 19.4.

- Commutativity: Similar (uniqueness)
- $F_f(X,0) = X$, $F_f(0,Y) = Y$ (uniqueness).

(ii) By Lemma 19.4, for $a \in \mathcal{O}_K$,

$\exists! [a]_{F_f} \in \mathcal{O}_K[[X]]$ s.t.

$$[a]_{F_f} \equiv aX \mod X^2$$

$$f \circ [a]_{F_f} = [a]_{F_f} \circ f.$$

Then $[a]_{F_f} \circ F_f = F_f \circ [a]_{F_f}$ (uniqueness)

i.e. $[a]_{F_f} \in \mathrm{End}_{\mathcal{O}_K}(F_f)$

- The map $[\ ]_{F_f} : \mathcal{O}_K \to \mathrm{End}_{\mathcal{O}_K}(F_f)$ is a

[illegible]

ring hom. (uniqueness).

$\Rightarrow F_f$ is a formal $\mathcal{O}_K$-module (over $\mathcal{O}_K$).

· $[\pi]_{F_f} = f$ (uniqueness).

(iii) If $g(x)$ another Lubin-Tate series for π.

Let $\theta(x) \in \mathcal{O}_K[[X]]$ unique power series s.t.

$$\theta(x) \equiv X \mod X^2$$

and $\theta \circ f(x) = g \circ \theta(x)$

Then $\theta \circ F_f = F_g(\theta(x), \theta(y))$ (uniqueness)

Thus $\theta \in \mathrm{Hom}_{\mathcal{O}_K}(F_f, F_g)$

reversing roles of f and g; obtain

$\theta^{-1} \in \mathcal{O}_K[[X]]$, $\theta^{-1} \in \mathrm{Hom}_{\mathcal{O}_K}(F_g, F_f)$

with $f \circ \theta^{-1}(x) = \theta^{-1} \circ g(x)$.

Then $\theta^{-1} \circ \theta(x) = X$, $\theta \circ \theta^{-1}(x) = X$ (uniqueness

$\Rightarrow \theta$ iso.

(Uniqueness) $\Rightarrow$ $\theta \circ [a]_{F_f}(x) = [a]_{F_g} \circ \theta(x)$ $\forall a \in \mathcal{O}_K$

and hence θ is an iso. of formal $\mathcal{O}_K$-modules. $\square$

> § Lubin-Tate extensions

$\bar{K}$ alg. closure of K, $\bar{m} \subseteq \mathcal{O}_{\bar{K}}$ max. ideal.

Lemma 20.1: F a formal $\mathcal{O}_K$-module. Then

$\bar{m}$ becomes a (genuine) $\mathcal{O}_K$-module with operations

$$x +_F y = F(x, y), \quad x, y \in \bar{m}$$

$a \cdot_F x = [a]_F(x) \quad a \in \mathcal{O}_K, x \in \bar{m}.$

Proof: Note: $\bar{K}$ not complete, so cannot apply convergence arguments directly.

$x \in \bar{m} \Rightarrow x \in m_L$ for some L/K finite.

$[a]_F \in \mathcal{O}_K[[X]] \Rightarrow [a]_F(x)$ converges in L, and since m_L closed, $[a]_F(x) \in m_L \subseteq \bar{m}$.

Similarly $x +_F y \in \bar{m}$.

Module structure: follows from definitions. $\square$