Algebraic Topology Homework 2

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§ *Problems from 1.1*

EXERCISE 2. Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof: Let X be a topological space with $x_0, x_1 \in X$ and suppose $h, g : [0,1] \longrightarrow X$ are homotopic paths such that $h(0) = g(0) = x_1$ and $h(1) = g(1) = x_0$. We would like to show that $\beta_h = \beta_g$, i.e. that h and g both induce the same homomorphism $\pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$. Change of basepoint homomorphisms are isomorphisms by Proposition 1.5 in Hatcher, so this is equivalent to showing that $\beta_h \circ \beta_g^{-1} = \beta_g^{-1} \circ \beta_h = \mathrm{id}$. But since $\beta_g^{-1} = \beta_{\overline{g}}$, this is a simple calculation. For any $[f] \in \pi_1(X, x_0)$, we have

$$\beta_h \beta_{\overline{g}}([f]) = \beta_h([\overline{g} \cdot f \cdot g]) = [h \cdot \overline{g} \cdot f \cdot g \cdot \overline{h}] = [f]$$

since $h \simeq g$, and similarly

$$\beta_{\overline{g}} \beta_h([f]) = \beta_{\overline{g}}([h \cdot f \cdot \overline{h}]) = [\overline{g} \cdot h \cdot f \cdot \overline{h} \cdot g] = [f].$$

This means $\beta_{\overline{g}}$ is the inverse of both β_g and β_h, and by the uniqueness of inverses, we conclude $\beta_h = \beta_g$. $\square$

EXERCISE 5. Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints.']

Proof: We prove the following chain of implications:

$(a) \implies (b):$ Suppose that for every map $S^1 \to X$ is nullhomotopic to a function $c_{x_0} : S^1 \to X$ where $c_{x_0}(x) = x_0$, i.e. suppose there exists a homotopy $f_t : S^1 \to X$ where $f_1 = c_{x_0}$ and where f_0 is any map $S^1 \to X$. Since (D^2, S^1) is a CW-pair, it has the homotopy extension property, and thus f extends to $f' : D^2 \times [0,1] \to X$ where $f'|_{S^1} = f$. Thus every map $S^1 \to X$ extends to $D^2 \to X$.

$(b) \implies (c):$ Suppose that $[f] \in \pi_1(X, x_0)$. $f(0) = f(1) = x_0$, meaning that we can interpret f instead as a function $f : S^1 \to X$. This f can be extended to a function $f' : D^2 \to X$, but since D^2 is contractible, f' is nullhomotopic. Thus, f is also nullhomotopic, and $[f] = [0]$. This is true of any arbitrary $[f] \in \pi_1(X, x_0)$, and so we know that $\pi_1(X, x_0)$ is trivial. Noticing that the choice of x_0 was arbitrary, we conclude that $\pi_1(X) = 0$.

$(c) \implies (a):$ This argument is very similar to the previous one. Each map $S^1 \to X$ can be interpreted as a loop in X, and since we assume $\pi_1(X) = 0$, every loop is homotopic to the trivial map. Thus every map $S^1 \to X$ is homotopic to a constant map in X.

A space X is "simply connected" if and only if it is path connected and $\pi_1(X) = 0$. As we just showed, this is equivalent to "X is path connected and every $S^1 \to X$ is nullhomotopic". If we interpret two maps $f, g : S^1 \to X$ as loops in X with basepoints x_0 and x_1 respectively, then f and g are homotopic in X by the homotopy that first shrinks f to the constant map c_{x_0}, moves x_0 along a path to x_1, and finally deforms c_{x_1} into g. Thus, any two maps $S^1 \to X$ are homotopic. We conclude that X is simply connected if and only if all maps $S^1 \to X$ are homotopic. $\square$

EXERCISE 6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof: We first show that Φ is onto. Let $[f]$ be a member of $[S^1, X]$. Since every map $S^1 \to X$ can be regarded as a loop in X, f is a loop based at some point $x_1 \in X$. Because X is path connected, there must be some path γ from x_1 to x_0. The map $g = \gamma \cdot f \cdot \overline{\gamma}$ is a continuous path that begins and ends at x_0, and is therefore a loop around x_0. Since f and g are homotopic, $[g] = [f]$. Since $[g] \in \pi_1(X, x_0)$, $\Phi([g]) = [f]$. We conclude that Φ is surjective.

We now show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugates. We show the forward implication first.

Assume that $\Phi([f]) = \Phi([g])$. This means that f and g are in the same equivalence class of $[S^1, X]$, so there must exist a homotopy $\varphi : S^1 \times [0, 1] \to X$, where $\varphi_0 = f$ and $\varphi_1 = g$. The induced homeomorphisms φ_{0*} and φ_{1*} then satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$, where $\beta_h : \pi_1(X, \varphi_1(s_0)) \to \pi_1(X, \varphi_0(s_0))$ and h is the loop $\varphi_t(s_0)$. This means

$$\varphi_{0*}([1]) = [g \cdot 1] = [g] = \beta_h \varphi_{1*}([1]) = \beta_h([f]) = [hf\overline{h}]$$

where $[1]$ is the equivalence class isomorphic to $1 \in \mathbb{Z}$.

We now show the reverse implication. Assume that $[g] = [h][f][\overline{h}]$ in $\pi_1(X, x_0)$, i.e. assume $[f]$ and $[g]$ are conjugates. We want to show that $[f] = [g]$, or that $[f] = [h][f][\overline{h}]$. Consider the following function:

$$F : [0, 1] \times S^1 \to X \qquad F_s(t) = \begin{cases} h(3t + s) & 0 \le t \le \frac{1-s}{3} \\ f\left(\frac{3}{1+2s}(t - \frac{1-s}{3})\right) & \frac{1-s}{3} \le t \le \frac{2+s}{3} \\ \overline{h}\left(3(t - \frac{2+s}{3})\right) & \frac{2+s}{3} \le t \le 1 \end{cases}$$

Since $F_0 = h \cdot f \cdot \overline{h}$, $F_1 = f$, and F is continuous by the pasting lemma, F is a homotopy between f and $h \cdot f \cdot \overline{h}$. Thus, $[f] = [h][f][\overline{h}]$ in $[S^1, X]$ and we conclude that $\Phi([g]) = \Phi([f])$.

$\square$

EXERCISE 10. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $Y \times \{x_0\}$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Proof: Let $[f]$ and $[g]$ be loops based at x_0 in X and y_0 in Y, respectively. Next consider the following homotopies:

$$f_t(s) = \begin{cases} x_0 & 0 \le s \le \frac{t}{2} \\ f(2s) & \frac{t}{2} \le s \le \frac{1+t}{2} \\ x_0 & \frac{1+t}{2} \le s \le 1 \end{cases}$$

and

$$g_t(s) = \begin{cases} y_0 & 0 \le s \le \frac{t}{2} \\ g(2s) & \frac{t}{2} \le s \le \frac{1+t}{2} \\ y_0 & \frac{1+t}{2} \le s \le 1 \end{cases}$$

Here f_0 is the path that transverses f and then stays at x_0. f_1 is the path that stays at x_0 for half the interval and then transverses f. g_0 is the path that stays at y_0 and then transverses g, and finally g_1 is the path that transverses g and then stays at y_0.

Next since $\pi_1(X, x_0) \times \pi_1(Y, y_0) \approx \pi(X \times Y, (x_0, y_0))$ and $h_t(s) = (f_t(s), g_t(s))$ is an element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$. However, since $f_0 \cdot g_0 \simeq f \cdot g$ and $f_1 \cdot g_1 \approx g \cdot f$, we conclude that $f \cdot g \simeq g \cdot f$. □

EXERCISE 15. Given a map $f : X \longrightarrow Y$ and a path $h : I \longrightarrow X$ from x_0 to x_1, show that $f_* \beta_h = \beta_{fh} f_*$.

Proof: Let $[\alpha] \in \pi_1(X, x_1)$ be an arbitrary equivalence class of loops based at x_1. By definition,

$$f_*(\beta_h([\alpha])) = f_*[h \cdot \alpha \cdot \overline{h}] = [f \circ (h \cdot \alpha \cdot \overline{h})]$$

and

$$\beta_{fh}(f_*([\alpha])) = \beta_{fh}([f \circ \beta]) = [(f \circ h) \cdot (f \circ \alpha) \cdot \overline{(f \circ h)}].$$

However, up to a possible reparameterization, $(f \circ h) \cdot (f \circ \alpha) \cdot \overline{(f \circ h)}$ and $f \circ (h \cdot \alpha \cdot \overline{h})$ are identical paths on Y. In fact, if we perform concatenation from consistently, then they are identical without *any* reparameterization.

To see this, concatenate from right to left without loss of generality. For $t \in [0, 1/2]$ we have

$$f \circ (h \cdot (\alpha \cdot \overline{h}))(t) = f(h(t)) = (f \circ h) \cdot \big((f \circ \alpha) \cdot \overline{(f \circ h)}\big)(t),$$

and we have something similar for $t \in [1/2, 3/4]$ and $t \in [3/4, 1]$. As the representatives of the resulting equivalence classes above are homotopic, the diagram shown by Hatcher commutes. □

EXERCISE 16.

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
(c) $X = S^1 \times D^2$ with A the interlocked circle in the solid torus.
(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.

(f) X the Möbius band and A its boundary circle.

Proof: (a) The space $\mathbb{R}^3$ contracts to the origin via the homotopy $F : \mathbb{R}^3 \times I \longrightarrow \mathbb{R}^3$ defined $F_t(x) = (1-t)x$, and hence has trivial fundamental group. By theorem 1.7, the circle S^1 has fundamental group isomorphic to $\mathbb{Z}$. Therefore there exists no inclusion $\pi_1(S^1, x_0) \longrightarrow \pi_1(\mathbb{R}^3, x_0)$, and hence by Proposition 1.17 in Hatcher, there exists no retraction of X onto a circle.

(b) In this case, we have that $\pi_1(X, x_0) \cong \mathbb{Z}$ by Proposition 1.12 and $\pi_1(A, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ by the same proposition. Note that because both of these spaces are path connected, the choice of basepoint does not matter. As any map $\mathbb{Z}$-linear morphism $\mathbb{Z}^2 \longrightarrow \mathbb{Z}$ will have nonzero kernel, there does not exist an injection $\mathbb{Z}^2 \longrightarrow \mathbb{Z}$, and hence there is no retraction $r : A \longrightarrow X$.

(c) The subset $A \subset X$ is contractible by pulling the two ends of the loops through each other. Thus, the map $\iota_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ induced by the inclusion $\iota : A \longrightarrow X$ takes every loop in A to something homotopic to the trivial loop in X. This means that the induced map ι_* is trivial, and in particular is not injective, hence by Proposition 1.17 X does not retract onto A.

(d) Suppose we did have a retraction $r : X \longrightarrow A$. In that case, the composition

$$D^2 \hookrightarrow X \xrightarrow{r} A \longrightarrow S^1$$

would also be a retraction. However, this is impossible, as D^2 is contractible while S^1 is not. Hence there is no such retraction r.

(e)

(f) Since X deformation retracts onto its central circle (a fact we used on the last homework) both X and A have fundamental group isomorphic to $\mathbb{Z}$. Let $x_0 \in A \subset X$ be a basepoint, and choose generators $\gamma \in \pi_1(A, x_0)$ and $\lambda \in \pi_1(X, x_0)$ for the fundamental groups. The image $\iota_*([\gamma])$ of $[\gamma]$ under the map induced by the inclusion $\iota : A \longrightarrow X$ is then equal to $2[\lambda]$ since traversing around the boundary circle corresponds to traversing around the central circle twice. Hence the induced map ι_* is really a map $\mathbb{Z} \longrightarrow \mathbb{Z}$ which sends $2 \mapsto 1$. This is impossible, as $2 \mapsto 1$ is not possible for such a group homomorphism as all $\mathbb{Z}$-linear maps $\mathbb{Z} \longrightarrow \mathbb{Z}$ are defined $1 \mapsto nz$ for some $n \in \mathbb{Z}$. Hence, there is no retraction of X onto the boundary A.

□

EXERCISE 20. Suppose $f_t : X \longrightarrow X$ is a homotopy such that f_0 and f_1 are each the identity map. Use Lemma 1.19 to show that for any $x_0 \in X$, the loop $f_t(x_0)$ represents an element of the center of $\pi_1(X, x_0)$. [One can interpret the result as saying that a loop represents an element of the center of $\pi_1(X)$ if it extends to a loop of maps $X \longrightarrow X$.]

Proof: Let $h(t) = f_t(x_0)$ be the loop taken by x_0 over f_t. Lemma 1.19 says that

$$f_{0*} = \beta_h f_{1*}.$$

For any $[g] \in \pi_1(X, x_0)$, we get that

$$[g] = f_{0*}[g] = \beta_h f_{1*}[g] = \beta_h[g] = [h] * [g] * [\overline{h}]$$

since both f_0 and f_1 are identity maps. This means that $[g] \cdot [h] = [h] \cdot [g]$. Since g was chosen arbitrarily, we have that $[h]$ is in the center of $\pi_1(X, x_0)$. $\square$

§ *Problems from 1.2*

EXERCISE 3. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

Proof: *Without Van Kampen:* Let $X = \mathbb{R}^n - \{p_0, ..., p_m\}$ be the space obtained by removing m-many points from $\mathbb{R}^n$, where $n \geq 3$, and let $f : S^1 \longrightarrow X$ be a continuous map. Denote by x_0 the basepoint of S^1. We construct an explicit homotopy between f and a constant map $c : S^1 \longrightarrow X$, which by problem 5, is sufficient to conclude X is simply-connected.

Our homotopy will essentially be a linear interpolation with the extra stipulation that we never come within some fixed distance of any point p_i. To accomplish this, we first set r to be half the minimum distance between the p_i and between the path $f(S^1)$ and the p_i, that is,

$$r = \inf \left\{ \|x - p_i\| \;\middle|\; x \in f(S^1) \cup \{p_0, ..., p_m\}, 1 \leq i \leq n \right\} \Big/ 2.$$

The set $f(S^1)$ is the image of a compact set under a continuous function and is hence itself compact, so the minimum of $\|x - p_i\|$ for $x \in f(S^1)$ is obtained for some i and some x. This implies that r is positive.

We now imagine placing closed balls of radius r centered at each p_i. It will be useful to enumerate these, so we define $D_i = D_r^n(p_i) \subseteq \mathbb{R}^n$, where $D_r(p_i)$ is the closed n-dimensional disk of radius r centered at p_i. These open balls are pairwise disjoint and do not intersect the path $f(S^1)$ by the definition of r. In particular, it means that points within distance r of p_i are closer to p_i than to p_j for any $j \neq i$. Fix a point $P \in X \setminus \bigcup_{i=0}^m D_i$, and additionally assume that P does not lie on the line between x_0 and p_i for any $0 \leq i \leq n$.

Choose an element $x \in S^1$ and consider the line segment between $f(x)$ and P parameterized by the function $\ell_x : [0,1] \longrightarrow \mathbb{R}^n$, $\ell_x(t) = (1-t)f(x) + tP$. The homotopy $L : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$ defined $L(x,t) = \ell_x(t)$ is then a homotopy between f and the constant map $c_P : S^1 \longrightarrow \mathbb{R}^n$ defined $c_p(x) = P$. Ideally, this would also be a homotopy of f to c_P in X, but the collection of line segments $\{F_x([0,1])\}_{x \in S^1}$ may intersect some of the points p_i. We find a homotopy from L to a map F whose image in $\mathbb{R}^n$ is at least distance r from all the p_i.

Suppose the line segment between $x \in S^1$ and P passes through the interior of D_i. We have two cases:

1. there is some largest interval $[a,b] \in S^1$ such that $\ell_a([0,1])$ and $\ell_b([0,1])$ are both tangent to D_i, $a < x < b$ and $\ell_y([0,1]) \cap D_i \neq \text{Ø}$ (see figure 1) or
2. $\ell_y([0,1]) \cap D_i \neq \text{Ø}$ for all $y \in S^1$ (see figure 2).

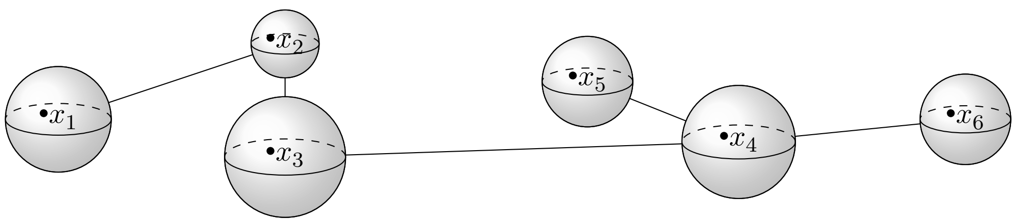
In case (1), the intersection $L([a,b] \times [0,1]) \cap D_i$ is homotopic to a plane passing through D_i, and we may homotope L to a new map L' which runs along a sector of $S^{n-1} \cong \partial D_i$ in such a way that fixes the endpoints of each path $\ell_y[0,1] \cap D_i$ for $y \in [a,b]$. In Case (2), the intersection $L(S^1 \times [0,1])$ is homotopic to a cylinder $S^1 \times [0,1]$ (see figure 2 again) and can similarly be homotoped to the boundary ∂D_i via a homotopy which fixes the endpoints of the paths $\ell_y([0,1]) \cap D_i$ for all $y \in S^1$.

We may do this for each $x \in S^1$ and each $0 \leq i \leq m$. The end result is a homotopy $F : S^1 \times [0,1] \longrightarrow \mathbb{R}^n$ which, for each $x \in S^1$, is a concatenation of linear paths in bR^n and arcs along the surface of n-dimensional disks. Call this homotopy F. It is a homotopy between f and c_p since $F(x,0) = f(x)$ and $F(x,1) = P$, and it is a homotopy in X since it avoids all points $p_0, ..., p_m$.

With Van Kampen: I wrote a solution to this problem using Van Kampen before realizing that wasn't allowed. Here's that solution too – I couldn't bear deleting it because it includes a pretty tikz picture. Let

$\{x_1, ..., x_k\}$ be a finite collection of points in $\mathbb{R}^n$. We can place a $n-1$ sphere around every point removed from $\mathbb{R}^n$, ensuring that the radius is small enough to contain only one "hole", and connect every sphere with another via a straight path. Call this space X. Just as $\mathbb{R}^n$ with a single point removed deformation retracts to S^{n-1}, I claim that $\mathbb{R}^n - \{x_1, ..., x_k\}$ deformation retracts onto X. Every point inside of one of the spheres retracts onto the sphere, and choosing to map every point in $\mathbb{R}^n$ that is not in X onto the nearest point in X retains continuity.

Every path-connected open set on this surface has the trivial fundamental group, so by Van Kampen's theorem, the fundamental group of X is also trivial. Since $\mathbb{R}^n - \{x_1, ..., x_k\}$ deformation retracts onto X, we conclude that $\pi_1(\mathbb{R}^n - \{x_1, ..., x_k\}) = [1]$. □



The space X used with Van Kampen

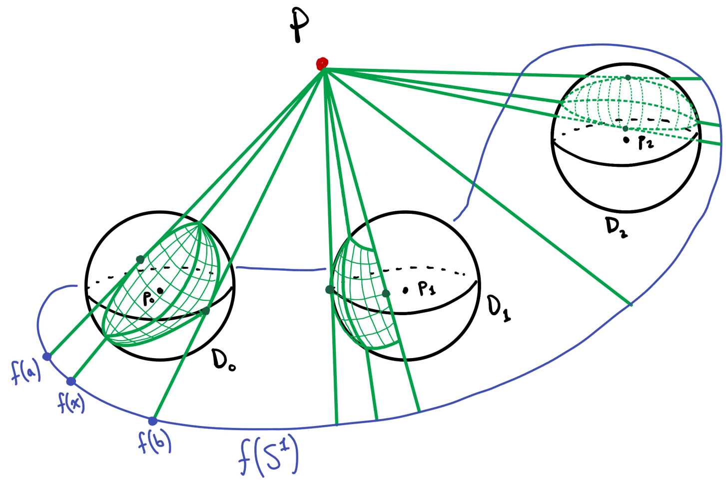


Figure 1: The homotopy in Case 1

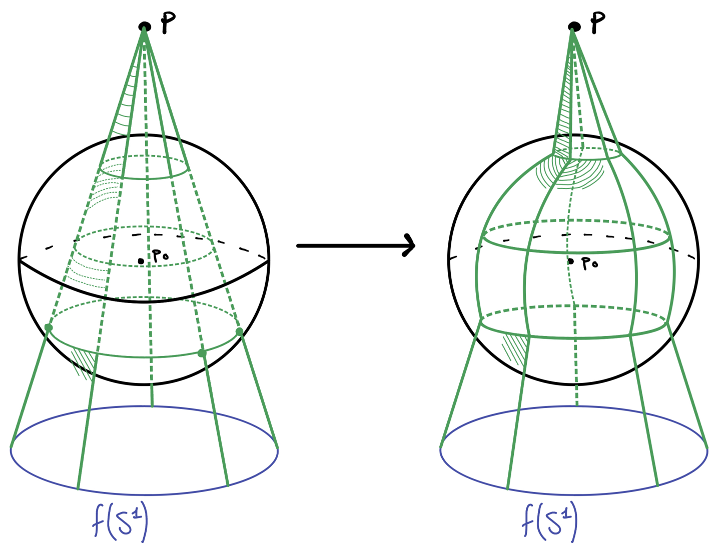


Figure 2: The homotopy in Case 2

Bonus Exercise: A "Bad" Group Action. Let $X = \mathbb{R}^2 - \{0\}$. Let G be the group of homeomorphisms of X generated by the transformation $(x, y) \mapsto (2x, y/2)$. Let Y be the quotient space X/G.

(a) Prove that every orbit is discrete. This is meant as a stepping stone to the more general result (b).

(b) Prove that $G's$ action on S is what Hatcher calls a covering space action (pg. 72).

(c) Prove that Y is a manifold, except for the fact that it is *not* Hausdorff.

Proof: (a) Fix a point $p = (x, y) \in X$. For any $h \in \mathbb{Z}$, we have that $g^n(x, y) = (2^n x, y/x^n)$, and hence

$$\begin{aligned} \|p - g^n p\|^2 &= (x - 2^n x)^2 + (y - \frac{1}{2}y)^2 \\ &= x^2 (1 - 2^n)^2 + y^2 \left(1 - \frac{1}{2^n}\right)^2 \\ &> \frac{x^2}{4} + \frac{y^2}{4} = \frac{\|p\|^2}{4}. \end{aligned}$$

If we set $r = \min \left\{ \frac{\|p\|^2}{4}, \|p\| \right\}$ then $g^n p \notin B_r(p)$. Thus, for any point in X, no point of its orbit comes within distance r, and hence the orbits of X under G are discrete.

(b) Note that, because g is a homeomorphism, $g^n(U) \cap g^m(U) = \text{Ø}$ if and only if $g^{n-m}(U) \cap U = \text{Ø}$. Let $r_p = \min \left\{ \frac{\|p\|^2}{8}, \|p\| \right\}$. For $p \in X$ and any two points $x, y \in B_r(p)$, we have that $\|x - y\| < \frac{\|p\|}{4}$ by the triangle inequality. Hence, by part (a), all the images $g \cdot B_r(p)$ are disjoint from $B_r(p)$, and the action of G upon X satisfies what Hatcher calls a covering space action.

(c) Let $[p] \in X/G$ and let $B_r(p)$ be the ball of radius r centered at $p \in X$ from part (b). Because $g^n B_r(p) \cap g^m B_r(p) = \text{Ø}$ for each $n, m \in \mathbb{Z}$, the projection map $\pi : X \longrightarrow X/G$ is a homeomorphism on $B_r(p)$ and $U = \pi(B_r(p))$ is an open neighborhood of $[p]$ in X/G. Since

$$\|p - 0\| = \|p\| \geq \min \left\{ \frac{\|p\|^2}{8}, \|p\| \right\} = r,$$

$0 \notin B_r(p)$ and hence $B_r(p)$ is an open disk in $\mathbb{R}^2$. The inverse of the restriction $\pi|_{B_r(p)}$ then yields a homeomorphism $g : U \longrightarrow B_r(p)$, so X/G is locally homeomorphic to $\mathbb{R}^2$.

I do not know why X/G fails to be Hausdorff, but I imagine it has something to do with the fact that the origin is a limit point of the orbit of any point of the form $(x, 0), (0, y) \in X$.

$\square$