Lecture 11

1 Definition 9.6: If R is a Dedekind domain, $\mathfrak{p} \subseteq R$ a non-zero prime ideal, we $v_{\mathfrak{p}}$ for the normalized valuation on $\mathrm{Frac}(R) = \mathrm{Frac}(R_{(\mathfrak{p})})$ corresponding to the DVR $R_{(\mathfrak{p})}$.

Eg. $R = \mathbb{Z}$, $\mathfrak{p} = (p)$, $v_{\mathfrak{p}}$ is the p-adic valuation.

Theorem 9.7: Let R be a Dedekind domain. Then every non-zero ideal $I \subseteq R$ can be written uniquely as a product of prime ideals:

$$I = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r} \quad (\mathfrak{p}_i \text{ distinct})$$

Remark: This clear for PID's (PID $\Rightarrow$ UFD)

Proof: (sketch) We quote the following properties of localization.

(i) If $I \subsetneq \mathfrak{p}$ then $I R_{(\mathfrak{p})} \subsetneq \mathfrak{p} R_{(\mathfrak{p})}$.

(ii) $I = J \Leftrightarrow I R_{(\mathfrak{p})} = J R_{(\mathfrak{p})}$, $\forall\, \mathfrak{p}$ prime ideals.

(iii) R Dedekind, $\mathfrak{p}_1, \mathfrak{p}_2$ non-zero prime ideals

$$\mathfrak{p}_1 R_{(\mathfrak{p}_2)} = \begin{cases} \mathfrak{p}_2 R_{(\mathfrak{p}_2)} & \text{if } \mathfrak{p}_1 = \mathfrak{p}_2 \\ R_{(\mathfrak{p}_2)} & \text{if } \mathfrak{p}_1 \neq \mathfrak{p}_2 \end{cases}$$

2 Let $I \subseteq R$ be a non-zero ideal.

Then by Lemma 9.3, there are distinct prime ideals

Then by Lemma [illegible], there are prime ideals

$\mathfrak{p}_1, \dots, \mathfrak{p}_r$ s.t. $\mathfrak{p}_1^{\beta_1} \cdots \mathfrak{p}_r^{\beta_r} \subseteq I$, where $\beta_i > 0$.

Let $\mathfrak{p}$ prime ideal, $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Then

(iii) $\Rightarrow$ $I R_{(\mathfrak{p})} = R_{(\mathfrak{p})}$

Corollary 9.5 $\Rightarrow$ $R_{(\mathfrak{p}_i)} = (\mathfrak{p}_i R_{(\mathfrak{p}_i)})^{\alpha_i} = \mathfrak{p}_i^{\alpha_i} R_{(\mathfrak{p}_i)}$

some $0 \leq \alpha_i \leq \beta_i$.

thus $I = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r}$ by property (ii).

For uniqueness, if $I = \mathfrak{p}_1^{\alpha_1} \cdots \mathfrak{p}_r^{\alpha_r} = \mathfrak{p}_1^{\gamma_1} \cdots \mathfrak{p}_r^{\gamma_r}$

then $\mathfrak{p}_i^{\alpha_i} R_{(\mathfrak{p}_i)} = \mathfrak{p}_i^{\gamma_i} R_{(\mathfrak{p}_i)} \Rightarrow \alpha_i = \gamma_i$ by

unique factorization in DVR's. $\square$

§ Dedekind domain + extensions

Let L/K be a finite extension. For $x \in L$

we write $\mathrm{Tr}_{L/K}(x) \in K$ for the trace of the

K-linear map $L \to L$, $y \mapsto xy$.

If L/K is separable, $[L:K] = n$ and $\sigma_1, \dots, \sigma_n : L \to \bar{K}$

denotes the set of embeddings of L into a

separable closure $\bar{K}$, then

$$\mathrm{Tr}_{L/K}(x) = \sum_{i=1}^{n} \sigma_i(x).$$

Lemma 10.1: Let L/K be a finite separable

extension of fields. Then the symmetric

bilinear pairing

$$(\, , \,) : L \times L \to K$$
$$(x, y) \mapsto \mathrm{Tr}_{L/K}(xy)$$

is non-degenerate

Proof: By the primitive element theorem, $L = K(\alpha)$ for some $\alpha \in L$. We consider the matrix A for $(\, , \,)$ in the K-basis for L given by $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$. Then

$$A_{ij} = \mathrm{Tr}_{L/K}(\alpha^{i+j}) = [BB^T]_{ij}$$

where B is the $n \times n$ matrix with

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1(\alpha) & \sigma_2(\alpha) & \cdots & \sigma_n(\alpha) \\ \vdots & \vdots & & \vdots \\ \sigma_1(\alpha^{n-1}) & \sigma_2(\alpha^{n-1}) & \cdots & \sigma_n(\alpha^{n-1}) \end{pmatrix}$$

4 $$\Rightarrow \det A = (\det B)^2 = \Big[\prod_{1 \le i < j \le n} \sigma_i(\alpha) - \sigma_j(\alpha)\Big]^2$$

(Vandermonde determinant)

$$\neq 0 \quad \text{since } \underset{\text{by separability}}{\sigma_i(\alpha) \neq \sigma_j(\alpha)}, \ i \neq j. \quad \square$$

Remark: In fact a finite extension of fields L/K is separable iff the trace form is non-degenerate. Ex Sheet 3.

Theorem 10.2: Let $\mathcal{O}_K$ be a Dedekind domain and L a finite separable extension of $K := \mathrm{Frac}($

Then the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L is

Then the integral closure $\mathcal{O}_L$ of $\mathcal{O}_K$ in L is a Dedekind domain.

Proof: Since $\mathcal{O}_L \subseteq L$, it is an integral domain.

We need to show that

(i) $\mathcal{O}_L$ is Noetherian

(ii) $\mathcal{O}_L$ is integrally closed in L

(iii) Every non-zero prime ideal P in $\mathcal{O}_L$ is
3 maximal.

(i) Let $e_1, \dots, e_n \in L$ be a K-basis for L. Upon scaling by K, we may assume $e_i \in \mathcal{O}_L, \forall i$. Let $f_i \in L$ be the dual basis w.r.t. the trace form $(\,,\,)$.

Let $x \in \mathcal{O}_L$, write $x = \sum_{i=1}^{n} \lambda_i f_i$, $\lambda_i \in K$.

Then $\lambda_i = Tr_{L/K}(x e_i) \in \mathcal{O}_K$

(For any $z \in \mathcal{O}_L$, $Tr_{L/K}(z)$ is a sum of elements which are integral over $\mathcal{O}_K$

$\Rightarrow Tr_{L/K}(z)$ is integral over $\mathcal{O}_K$

$\Rightarrow Tr_{L/K}(z) \in \mathcal{O}_K$).

Thus $\mathcal{O}_L \subseteq \mathcal{O}_K f_1 + \dots + \mathcal{O}_K f_n$

Since $\mathcal{O}_K$ is Noetherian, $\mathcal{O}_L$ is finitely generated as an $\mathcal{O}_K$-module, hence $\mathcal{O}_L$ is Noetherian.

(ii) Ex. sheet 2.

(iii) Let P be a non-zero prime ideal of $\mathcal{O}_L$, and define $p := P \cap \mathcal{O}_K$ a prime ideal of $\mathcal{O}_K$. Let $x \in P$, then x satisfies an equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0, \quad a_i \in \mathcal{O}_K$$

with $a_0 \neq 0$. Then $a_0 \in P \cap \mathcal{O}_K$ is a non-zero element of $p \Rightarrow p$ is non-zero

$\Rightarrow p$ is maximal.

We have $\mathcal{O}_K/p \hookrightarrow \mathcal{O}_L/P$, and $\mathcal{O}_L/P$ is a finite dimensional v.s. over $\mathcal{O}_K/p$.

Since $\mathcal{O}_L/P$ is an integral domain, it is a field (Eg. use rank-nullity theorem applied to map $y \mapsto zy$). $\square$

Remark: Theorem 10.2 holds without the assumption that L/K is separable.

Corollary 10.3: The ring of integers inside a number is a Dedekind domain. $\square$

Convention: $\mathcal{O}_K$ the ring of integers of a number field - $p \subseteq \mathcal{O}_K$ a non-zero prime ideal.

, We normalize $|\cdot|_\mathfrak{p}$ (abs. value associated to $v_\mathfrak{p}$) by $|x|_\mathfrak{p} = N\mathfrak{p}^{-v_\mathfrak{p}(x)}$ where $N\mathfrak{p} = \# \mathcal{O}_K/\mathfrak{p}$

Let $\mathcal{O}_K$ be a Dedekind domain.