# Definitions and Theorems from Infinite Groups (Lent '22)

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Last compiled January 29, 2022

### § Free Groups and Presentations

**Definition 0.1** (Directly Finite). We say that a ring R (not necessarily commutative) with unity is *directly finite* (D.F.) if  $\forall a,b \in R, ab = 1 \implies ba = 1$ .

**Definition 0.2** (Group Algebra). Let G be a group, K a field. The *group algebra*, denoted K[G], is a K-algebra. As a set it consists of all finite linear combinations of elements in K and G:

$$K[G] = \left\{ \sum_{g \in G} \lambda_g \cdot g \;\middle|\; \lambda_g \in K, \lambda_g \neq 0 \text{ for only finitely many } g \;
ight\}.$$

Addition is defined pointwise:  $\lambda g + \lambda' g = (\lambda + \lambda')g$ . Multiplication is defined

$$(\lambda g) \cdot (\mu h) = (\lambda \mu)(gh)$$

and extended by distribution.

**Definition 0.3.** labeldefn:directly-finite-group A group G is said to be *directly finite* (D.F.) if K[G] is a directly finite ring for all fields K.

#### Example 0.4.

- (i) if G is abelian then K[G] is commutative, and therefore directly finite.
- (ii) if G is finite then it is also directly finite.

**Theorem 0.5** (Kaplansky). For any group G, the group algebra  $\mathbb{C}[G]$  is directly finite.

**Theorem 0.6** (Elek, Szako ('01')). Every sofic group is directly finite.

**Proposition 0.7.** Let G be a finite group. The map  $\rho: G \to \operatorname{Sym}(G)$  defined  $\rho(g): h \mapsto gh$  is an injective homomorphism. Moreover, for all  $e \neq g \in G$ ,  $\rho(g)$  has no fixed points.

**Definition 0.8** (Hamming Distance). Suppose  $\sigma, \tau \in \operatorname{Sym}(n)$  are two permutations. The *Hamming distance* from  $\sigma$  to  $\tau$  is

$$d_n(\sigma, \tau) = 1 - \frac{1}{n} |\{1 \le i \le n \mid \sigma(i) = \tau(i)\}|.$$

That is, it's a number between 0 and 1 and is equal to 0 if and only if  $\sigma = \tau$ .

**Definition 0.9** (Sofic). G is a sofic group if and only if  $\forall A \subseteq G, \ \forall \varepsilon > 0$  there exists  $n \in \mathbb{N}$  and a function  $\phi : A \to \operatorname{Sym}(n)$  such that

(i) for all  $g, h \in A$ , if  $gh \in A$ , then

$$d_n(\phi(gh), \phi(g)\phi(h)) \leq \varepsilon$$
,

i.e. the distance is "small"

(ii) for all  $e \neq g \in G$ ,

$$d_n(id_n, \phi(g)) \ge 1 - \varepsilon$$
,

i.e. the distance is "large".

Such a function  $\phi$  is a  $(A, \varepsilon)$ -representation.

**Example 0.10.** Every finite group is sofic.

**Theorem 0.11.** Every abelian group is sofic.

**Lemma 0.12.**  $\mathbb{Z}$  is sofic.

**Lemma 0.13.** A group G is sofic if and only if every finitely generated subgroup of G is sofic.

**Lemma 0.14.** If G and H are sofic groups, then  $G \times M$  is sofic.

**Theorem 0.11.** Given an abelian group G, we may assume it is finitely generated by lemma 0.13. By the structure theorem, we have

$$G\cong \mathbb{Z}^k\oplus rac{\mathbb{Z}}{p_1^{n_1}}\oplus ...\oplus rac{\mathbb{Z}}{p_i^{n_i}},$$

hence G is sofic by lemma 0.14.

## 1 2022 - 01 - 24: Free Groups

Throughout this section, *X* is a set and  $X^{-1} = \{x^{-1} \mid x \in X\}$  is the set of inverses of *X*.

**Definition 1.1.** A *word* in X is a finite sequence of symbols:  $y_1y_2...y_m$  with  $y_i \in X \cup X^{-1}$ . The empty word is a valid word. We denote the set of all words in X by W(X).

**Definition 1.2.** Concatenation of words is a map  $W(X) \times W(X) \to W(X)$  defined

$$(y_1...y_m, z_1...z_n) \mapsto y_1...y_mz_1...z_n.$$

This map gives W(X) the structure of a monoid where the empty word is the identity.

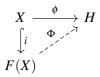
**Definition 1.3.** Given two words  $w, v \in W(X)$ , we say that  $w \sim v$  if it is possible to pass from one word to the other by means of a finite sequence of the following two operations:

- (a) insertion of an  $xx^{-1}$  or an  $x^{-1}x$  for  $x \in X$ , as consecutive elements of a word;
- (b) deletion of such an  $xx^{-1}$  or  $x^{-1}x$ .

The relation  $\sim$  is an equivalence relation on W(X) and we define the *free group on X* to be  $\frac{W(X)}{\sim}$ . The group operation on F(X) is induced by concatenation.

**Definition 1.4.** A word  $w \in W(X)$  is said to be *reduced* if there is no word v which can be obtained by operation (b) above; in other words, if w is the shortest word in the equivalence class  $[x] \in F(X)$ . For any other word  $v \in W(X)$ , there exists a unique reduced  $w \in W(X)$  such that [v] = [w].

**Theorem 1.5.** Let X be a set, H a group, and  $\phi: X \to H$  a function. Then there exists a unique group homomorphism  $\Phi: F(X) \to H$  such that



commutes.