D-Modules Over Smooth Affine Varieties

Contents

Introduction

Write this later.

A word regarding references is in order. As this essay is entirely expository, all of the material presented here can be found in some form or another in another source. Direct citations are provided where feasible, but are not always possible since many facts exist as amalgamations of several different sources. Specific citations become especially quite difficult when a fact is presented in varying levels of generality in two different sources.

This essay focuses exclusively on D-modules over smooth varieties, with a heavy emphasis on the affine case. Such content lives somewhere between Coutinho's

The exposition on D-modules was largely informed by [Cou95] and [HTT08], while the online notes [Jef20] by Jack Jeffreys inspired the treatment of differential operators in section 1.

Acknowledgments

1 Differential Operators

One must first understand fields before one can define vectors spaces, and similarly one must first understand the ring of differential operators before one can study D-modules. In this section we do exactly that. We first define the ring of differential operators relative to an arbitrary ring homomorphism $A \longrightarrow R$ and discuss some of its basic properties before focusing on the case where A is a field and R a polynomial ring with coefficients in A. This latter object will provide a more explicit setting and will motivate arguments in the general case. We discuss several other examples, and conclude this section by defining the sheaf of differential operators over a smooth variety.

It is worth noting that there are several equivalent ways to define the ring of differential operators in characteristic zero. We discuss two such definitions in the case of a polynomial ring over a field and show that they are equivalent when $\mathrm{char}(K) = 0$. However, when $\mathrm{char}(K) > 0$, these two definitions will no longer coincide. It therefore becomes necessary to fix either the field characteristic or a particular definition for the ring of differential operators, and in these notes, we will do the latter.

1.1 The Ring of Differential Operators over an Arbitrary Ring

Let $A \longrightarrow R$ be a map of rings and let M and N be two R-modules. We may identify R with a subring of $\mathrm{End}_R(M)$ via the map which sends an element $f \in R$ to the R-linear map $\hat{f} : m \mapsto f \cdot m$ on M. We denote the image of $f \in R$ in $\mathrm{End}_R(M)$ by $\hat{f}_M$ when there is risk of confusing the domain of $\hat{f}$ with some other module. Given a morphism $\alpha \in \mathrm{Hom}_R(M, N)$, we will often abuse notation and write $[\alpha, \hat{f}]$ to mean $\alpha \circ \hat{f}_M - \hat{f}_N \circ \alpha$. This is no longer an abuse of notation when $M = N$, in which case $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$ is well-defined for any $\alpha, \beta \in \mathrm{End}_R(M)$.

Definition 1.1. With A, R, M and N as above, we inductively define the collection of differential operators of order $k \in \mathbb{Z}$, denoted $D^k_{R/A}(M, N)$, as follows:

- $D^k_{R/A}(M, N) = 0$ when $k < 0$
- $D^k_{R/A}(M, N) = \left\{\alpha \in \mathrm{Hom}_A(M, N) \;\middle|\; \left[\alpha, \hat{f}\right] \in D^{k-1}_{R/A}(M, N) \text{ for all } f \in R\right\}$ when $k \geq 0$.

We set $D_{R/A}(M, N) = \bigcup_{k \in \mathbb{Z}} D^k_{R/A}(M, N)$.

Remark 1.2. It is worth noting that $\alpha \in D_{R/A}(M, N)$ satisfies $[\alpha, \hat{f}] = 0 \in D^{-1}_{R/A}(M, N)$ exactly when α is R-linear, hence $D_{R/A}(M, N) = \mathrm{Hom}_R(M, N)$. Many sources, [Gin98] and [Ber] for instance, simply define $D^0_{R/A}(M, N) = \mathrm{Hom}_R(M, N)$ and proceed inductively from there.

Example 1.3. As a first example, suppose K is a field and R is a module finite K-algebra. Once we fix a basis for R, for any $f \in R$ the operator $\hat{f}$ is simply the diagonal matrix fI, where I is the identity matrix. Any other map $A \in \mathrm{Hom}_K(R, R)$ then satisfies

$$A \circ \hat{f} = A \cdot fI = fI \cdot A = \hat{f} \circ A,$$

hence $[A, \hat{f}] = 0$ and $A \in D^0_{R/K}$. It then follows that $D_{R/K} = \mathrm{Hom}_K(R, R)$.

We will see far more interesting examples later in section 1.2 and 1.3, but first we lay out some of the basis structure of rings of differential operators in general. The following lemma is elementary but nonetheless quite important:

Lemma 1.4. For each $k \in \mathbb{Z}$ we have an inclusion $D^{k-1}_{R/A}(M,N) \subseteq D^k_{R/A}(M,N)$. Furthermore, $D^k_{R/A}(M,N)$ is a left R-module under the action $f\alpha \mapsto \hat{f} \circ \alpha$ and a right R-module under the action $\alpha f \mapsto \alpha \circ \hat{f}$. This particularly implies that $R_{R/A}(M,N)$ is a left and right R-module under these same actions.

Proof: We prove both claims by induction. The first is clear: the base case follows from the simple fact that $D^{-1}_{R/A}(M,N) = 0 \subseteq D^0_{R/A}(M,N)$, and if $\alpha \in D^{k-1}_{R/A}(M,N)$ then $[\alpha, \hat{f}] \in D^{k-2}_{R/A}(M,N)$ for any $f \in R$ by definition. The inductive hypothesis then implies that $[\alpha, \hat{f}] \in D^{k-1}_{R/A}(M,N)$, and hence $\alpha \in D^k_{R/A}(M,N)$.

Note first that $\operatorname{Hom}_A(M,N)$ is an R-module by maps $R \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_A(M,N)$, and since $D^k_{R/A}(M,N) \subseteq \operatorname{Hom}_A(M,N)$, it suffices to show that $D^k_{R/A}(M,N)$ is closed under addition and multiplication by R. By Remark 1.2, $D^0_{R/A}(M,N) = \operatorname{Hom}_R(M,R)$, so our base case is done. Suppose then that $D^m_{R/A}(M,N)$ is a left R-module for each $m < k$ and note that for any two $f, g \in R$ the associated module endomorphisms commute by the commutativity of R, i.e. $\hat{f}\hat{g} = \hat{g}\hat{f}$. Fix $\alpha, \beta \in D^k_{R/A}(M,N)$ and $a, b \in R$. For any other $f \in R$ we have

$$
\begin{aligned}
[\hat{a}\alpha + \hat{b}\beta, \hat{f}] &= (\hat{a}\alpha + \hat{b}\beta)\hat{f} - \hat{f}(\hat{a}\alpha + \hat{b}\beta) \\
&= \hat{a}\alpha\hat{f} - \hat{a}\hat{f}\alpha + \hat{b}\beta\hat{f} - \hat{b}\hat{f}\beta \\
&= \hat{a}[\alpha, \hat{f}] + \hat{b}[\beta, \hat{f}].
\end{aligned}
$$

Both $\hat{a}[\alpha, \hat{f}]$ and $\hat{b}[\beta, \hat{f}]$ are elements of the left R-module $D^{k-1}_{R/A}(M,N)$, hence so is their sum. The proof that $D^k_{R/A}(M,N)$ is a right R-module is similar. $\square$

Notation 1.5.

- We write $D_{R/A}(M)$ for $D_{R/A}(M,M)$ when $M = N$. As we shall see in Corollay 1.8, $D_{R/A}(M)$ is a ring under pointwise-addition and composition and is called the *ring of differential operators over* M. Given two operators $\alpha, \beta \in D_{R/A}(M)$ we often drop the composition symbol and write $\alpha\beta$ to mean $\alpha \circ \beta$.

- When $R = M = N$, we simply write $D_{R/A}$.

- We write D_R for the ring of differential operators over R relative to the unique map $\mathbb{Z} \longrightarrow R$.

We will be primarily interested in $D_{R/K}$ for a K-algebra R.

It will be useful to establish some basic commutator relations. These have nothing to do with differential operators but will used extensively in later sections, often without comment.

Proposition 1.6. Let A be a (not necessarily commutative) ring, M a left A-module and $\alpha, \beta, \gamma \in \operatorname{End}_A(M)$ A-linear maps on M. Then

(a) $[\alpha, \beta + \gamma] = [\alpha, \beta] + [\alpha, \gamma]$ and $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$

(b) $[\hat{f}\alpha, \beta] = [\alpha, \hat{f}\beta] = \hat{f}[\alpha, \beta]$ for $f \in A$

(c) $[\alpha, \beta] = -[\beta, \alpha]$

(d) $[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta$ and $[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + \beta[\alpha, \gamma]$.

(e) $[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0$ (Jacobi identity).

Proof: These are all straightforward computations.

(a) We have that

$$[\alpha, \beta+\gamma] \;=\; \alpha(\beta+\gamma)-(\beta+\gamma)\alpha \;=\; \alpha\beta-\beta\alpha+\alpha\gamma-\gamma\alpha \;=\; [\alpha,\beta]+[\alpha,\gamma].$$

A nearly identical computation gives us the other identity.

(b) Fix an element $f \in A$. Every operator $\lambda \in \operatorname{End}_A(M)$ is A-linear and hence $\hat{f} \circ \lambda = \lambda \circ \hat{f}$, i.e. $\hat{f}$ is in the center of $\operatorname{End}_A(M)$. The desired identity follows immediately from this fact.

(c) $[\alpha,\beta] = \alpha\beta - \beta\alpha = -(\beta\alpha - \alpha\beta) = -[\beta,\alpha]$.

(d) This is more symbol pushing:

$$\begin{aligned}
[\alpha\beta,\gamma] &= \alpha\beta\gamma - \gamma\alpha\beta \\
&= \alpha\beta\gamma - \alpha\gamma\beta + \alpha\gamma\beta - \gamma\alpha\beta \\
&= \alpha[\beta,\gamma] + [\alpha,\gamma]\beta.
\end{aligned}$$

The other identity is proven nearly identically.

(e) The left hand side of this identity is

$$\alpha(\beta\gamma-\gamma\beta)-(\beta\gamma-\gamma\beta)\alpha+\beta(\gamma\alpha-\alpha\gamma)+(\gamma\alpha-\alpha\gamma)\beta+\gamma(\alpha\beta-\beta\alpha)-(\alpha\beta-\beta\alpha)\gamma.$$

All terms cancel one this expression is fully expanded.

□

1.1.1 Order of Differential Operators

Fix a commutative ring map $A \longrightarrow R$. A differential operator $D \in D_{R/A}(M)$ is said to be of *order* k if $D \in D^k_{R/A}(M)$ but $D \notin D^{k-1}_{R/A}(M)$ and we say $\operatorname{ord}(D) = k$. As the operator 0 is contained in $D^k_{R/A}$ for every $k \in \mathbb{Z}$, we say $\operatorname{ord}(0) = -\infty$. Here, we describe how order interacts with composition, addition, and commutation. Throughout this section $A \longrightarrow R$ is a map of commutative rings and M is an R-module.

Proposition 1.7. Suppose $\alpha \in D^m_{R/A}(M)$ and $\beta \in D^n_{R/A}(M)$. The following hold:

(a) $\alpha+\beta \in D^d_{R/A}(M)$ where $d = \max\{m,n\}$

(b) $\alpha\beta \in D^{m+n}_{R/A}(M)$

(c) $[\alpha,\beta] \in D^{m+n-1}_{R/A}(M)$.

Proof: Part (a) follows immediately from Lemma 1.4. We prove (b) and (c) simultaneously by induction on $m+n$. The base case is clear, for when $m+n=0$ we have $\alpha\beta \in Hom_R(R,R)$. Suppose then that both (b) and (c) hold for $m+n<k$ for some positive integer k. Fix $f \in R$ and let $m+n=k$. By the inductive hypothesis we then have that $\alpha[\beta,\hat{f}]$ and $[\alpha,\hat{f}]\beta$ are in $D^{m+n-1}_{R/A}(M)$, and hence

$$[\alpha\beta,\hat{f}] = \alpha[\beta,\hat{f}] + [\alpha,\hat{f}]\beta \in D^{m+n-1}_{R/A}(M)$$

by Proposition 1.6 (d). This proves (b).

Rearranging the terms of the Jacobi identity, we have that

$$\big[[\alpha,\beta],\hat{f}\big] = \big[\alpha,[\beta,\hat{f}]\big] + \big[\beta,[\hat{f},\alpha]\big].$$

The inductive hypothesis tells us that the rightmost terms are elements of $D^{m+n-2}_{R/A}(M)$, hence so is $\big[[\alpha,\beta],\hat{f}\big]$. This proves (c). $\square$

This proposition yields some basic facts regarding the structure of $D_{R/A}(M)$.

Corollary 1.8. Let $A \longrightarrow R$ be a map of commutative rings. Then $D_{R/A}(M)$ is a ring and the graded ring

$$S_{R/A}(M) := \bigoplus_{k\in\mathbb{N}} S^k_{R/A}(M); \qquad S^k_{R/A}(M) = D^k_{R/A}(M)/D^{k-1}_{R/A}(M)$$

is commutative. We call $S_{R/A}(M)$ the *graded ring associated to* $D_{R/A}(M)$ and discuss it further in Section 2.

Proof: For any two $\alpha,\beta \in D_{R/A}(M)$, $\alpha\beta \in D_{R/A}(M)$ by Proposition 1.7 (b), hence $D_{R/A}(M)$ is a subring of $\mathrm{End}_A(M)$.

We identify $S^k_{R/A}(M)$ with its image under inclusion $S^k_{R/A}(M) \longrightarrow S_{R/A}(M)$ and let $\overline{\alpha}$ denote the image of $\alpha \in D^k_{R/A}(M)$ in $S^k_{R/A}(M)$. For $\alpha \in D^m_{R/A}(M)$ and $\beta \in D^n_{R/A}(M)$, we have $[\alpha,\beta] \in D^{m+n-1}_{R/A}(M)$ by Proposition 1.7 (C), hence $\overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha} = \overline{[\alpha,\beta]} = 0$. Since every element of $S_{R/A}(M)$ can be written as a sum of finitely many $\overline{\alpha}$, we are done. $\square$

1.1.2 Derivations

As of yet there has been no reason to restrict our generality, but now, we focus our attention exclusively on rings of differential operators of the form $D_{R/A}$. We already understand operators of order 0; since $D^0_{R/A} = \mathrm{Hom}_R(R,R) \cong R$, they're simply the operators of the form $\hat{f}$ for some $f \in R$. In this section we seek to understand the operators of order 1 as well, i.e. the R-module $D^1_{R/A}$.

Recall that an A-derivation of R is an A-linear map $d : R \longrightarrow R$ such that $d(ab) = ad(b) + d(a)b$ for all $a, b \in R$. Note that $d(1) = d(1\cdot 1) = d(1) - d(1) = 0$. Further notice that for any derivation $d \in \mathrm{Der}_A(R)$ and $f, r \in R$,

$$[d,\hat{f}](r) = d\left(\hat{f}(r)\right) - \hat{f}\left(d(r)\right) = d(fr) - fd(r) = d(f)r.$$

This means that $[d,\hat{f}]$ is simply $\widehat{d(f)} \in D^0_{R/A}$ as a map on R, hence we have an inclusion $\iota : \mathrm{Der}_A(R) \hookrightarrow D^1_{R/A}$.

Let's now consider an arbitrary element $\alpha \in D^1_{R/A}$. The map $\alpha' = \alpha - \widehat{\alpha(1)}$ is also an order 1 operator by Lemma 1.4; in fact, it's a derivation. Indeed, it is A-linear by virtue of its membership to $D^1_{R/A}$ and for any $r, s \in R$ we have

$$\alpha'(rs) = \alpha'\hat{r}(s) = (\hat{r}\alpha')(s) + \widehat{\alpha'(r)}(s) = r\alpha'(s) + \alpha'(r)s$$

since $[\alpha',\hat{r}] = \alpha'(r)$.

Consider then the map $\varphi : D^1_{R/A} \longrightarrow \mathrm{Der}_A(R)$ defined $\varphi(\alpha) = \alpha - \widehat{\alpha(1)}$. It is A-linear, and since $\alpha(1) = 0$ for any derivation α, $\varphi \circ \iota$ is the identity on $\mathrm{Der}_A(R)$. This means the short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow D^1_{R/A} \xrightarrow{\varphi} \mathrm{Der}_R(A) \longrightarrow 0$$

splits, giving us an isomorphism $D^1_{R/A} \cong \ker\varphi\varphi \operatorname{Der}_R(A)$. However, $\varphi(\alpha) = 0$ precisely when $\alpha = \widehat{\alpha(1)}$, i.e. when $\alpha \in D^0_{R/A} \cong \operatorname{Hom}_R(R, R) \cong R$. The results of this discussion are summarized in the proposition below.

Proposition 1.9. Let $A \longrightarrow R$ be a map of commutative rings. Then $D^1_{R/A} \cong R \oplus \operatorname{Der}_A(R)$ as A-modules via the map which sends $(f, d) \in R \oplus \operatorname{Der}_A(R)$ to $d + \hat{f}$. □

It is important to note that there is a more functorial way to define derivations. Given an A-algebra R, we first define the multiplication map $R \otimes_A R \longrightarrow R$ given by $x \otimes y \mapsto xy$. The kernel of this map is denoted $\Delta_{R/A}$ and is generated by elements of the form $r \otimes 1 - 1 \otimes r$:

$$\Delta_{R/A} = \langle \{r \otimes 1 - 1 \otimes r \mid r \in R\} \rangle = \ker(R \otimes_A R \xrightarrow{mult} R). \tag{1}$$

We use this to define the module of Kähler differentials:

Definition 1.10. Let R be an A-algebra. The module of A-linear Kähler differentials is

$$\Omega_{R/A} = \Delta_{R/A}/\Delta^2_{R/A}.$$

It comes equipped with a derivation $d : R \longrightarrow \Omega_{R/A}$ called the *universal derivation*:

$$d(r) = r \otimes 1 - 1 \otimes r \ + \ \Delta^2_{R/A}.$$

Hartshorne defines $\Omega_{R/A}$ to be the R-module, unique up to isomorphism, equipped with an A-derivation $d : R \longrightarrow \Omega_{R/A}$ such that for any other A-derivation $d' : R \longrightarrow M$ there exists a unique R-module map $f : \Omega_{R/A} \longrightarrow M$ with $d' = f \circ d$. This is equivalent to the definition given above. Hartshorne's definition does immediately make evident the following characterization of $\operatorname{Der}_A(R)$, however:

Proposition 1.11. Let M be an R-module. There exists an isomorphism of R-modules

$$\operatorname{Hom}_R(\Omega_{R/A}, M) \cong \operatorname{Der}_A(M)$$

given by precomposing a map $f : \Omega_{R/A} \longrightarrow M$ with the universal derivation $d : R \longrightarrow \Omega_{R/A}$. Hence the functor $M \mapsto \operatorname{Der}_A(M)$ is represented by $\Omega_{R/A}$.

1.1.3 Derivation Examples

Proposition 1.9 tells us that to understand $D^1_{R/A}$ it suffices to understand $\operatorname{Der}_A(R)$. Here, we explicitly describe the module $\operatorname{Der}_A(R)$ for specific rings R.

Example 1.12. Let K be a field of characteristic zero and $R = K[x_1, ..., x_n]$ a polynomial ring over K. By the product rule, the K-linear maps ∂_{x_i} $(1 \leq i \leq n)$ which send a polynomial f to its partial derivative in x_i are derivations. Any other derivation $\alpha \in \operatorname{Der}_K(R)$ satisfies

$$\alpha(x_i^k) = kx_i^{k-1}\alpha(x_i) = \partial_{x_i}(x_i^{k_i})\alpha(x_i).$$

This means that for a monomial $x_1^{k_1}...x_n^{k_n}$ we have

$$\begin{aligned}
\alpha\left(x_1^{k_1}...x_n^{k_n}\right) &= \alpha(x_1^{k_1})x_2^{k_2}...x_n^{k_n} \ + \ x_1^{k_1}\alpha(x_2^{k_2}...x_n^{k_n}) \\
&= \alpha(x_1)\partial_{x_1}(x_1^{k_1}...x_n^{k_n}) \ + \ x_1^{k_1}\left(\alpha(x_2^{k_2})x_3^{k_3}...x_n^{k_n} \ + \ x_2^{k_2}\alpha(x_3^{k_3}...x_n^{k_n})\right) \\
&\ \,\vdots \\
&= \alpha(x_1)\partial_{x_1}(x_1^{k_1}...x_n^{k_n}) \ + \ ... \ + \ \alpha(x_n)\partial_{x_1^{k_1}...x_n^{k_n}}.
\end{aligned}$$

Since monomials form a basis over K for R, we get that $\alpha = \alpha(x_1)\partial_{x_1} + ... + \alpha(x_n)\partial_{x_n}$. Hence $\{\partial_{x_1}, ..., \partial_{x_n}\}$ generates $\mathrm{Der}_K(R)$ as a R-module. In particular, $\mathrm{Der}_K(R)$ is a free-module over R of rank n.

Example 1.13. As before, let K be a field of characteristic zero. Consider the ring $R = K[t^2, t^3]$, noting that $R \cong K[x, y]/J$ for $J = (y^2 - x^3)$ via the map $x \mapsto t^2$ and $y \mapsto t^3$. As we will see, $\mathrm{Der}_K(R)$ is generated by $t\partial_t$ and $t^2\partial_t$.

First consider the derivations $D_1 = 2y\partial_x + 3x^2\partial_y$ and $D_2 = 3y\partial_y + 2x\partial_x$ on $K[x, y]$. They are also derivations on $K[x, y]/J$ since $D_1(J), D_2(J) \subseteq J$, and we will show they generate all of $\mathrm{Der}_K(K[x, y]/J)$. Any other derivation α on $K[x, y]/J$ can be written as $\alpha = f_1\partial_x + f_2\partial_y$ by the previous example with the extra condition that $\alpha(J) \subseteq J$. This is equivalent to the condition

$$-3x^2 f_1 \;+\; 2y f_2 = u(y^2 - x^3) \tag{2}$$

for some polynomial $u \in K[x, y]$. Notice that f_1 cannot have a constant term, if it did, the LHS of equation (2) would have a x^2 term while the RHS would not. This means f_1 may have terms of degree 1 or higher, hence we may write $f_1 = 2(yg + xh)$ for some $g, h \in K[x, y]$. Plugging this into equation (2) and rearranging yields

$$2yf_2 = u(y^2 - x^3) + 6x^2yg + 6x^3h$$

and substituting $u' = u - 6(y^2 - x^3)h$ gives

$$2yf_2 = u'(y^2 - x^3) + 6x^2yg + 6x^3h + 6(y^2 - x^3)h = u'(y^2 - x^3) + 6x^2yg + 6y^2h.$$

The LHS of this equation is divisible by y hence the R'S is too, implying $v = \frac{u'}{2y} \in K[x, y]$. Hence $f_2 = v(y^2 - x^3) + 3x^2g + 3yh$. We then get

$$\begin{aligned}
\alpha = f_1\partial_x + f_2\partial_y &= 2(yg + xh)\partial_x + \Big(v(y^2 - x^3) + 3x^2g + 3yh\Big)\,\partial_y \\
&= g(2y\partial_x + 3x^2\partial_y) + h(2x\partial_x + 3y\partial_y) + v(y^2 - x^3)\partial_y \\
&= gD_1 + hD_2 + v(y^2 - x^3)\partial_y.
\end{aligned}$$

Since $v(y^2 - x^3)\partial_y$ is the trivial derivation on $K[x, y]/J$, the above shows that α is in the $K[x, y]/J$-span of D_1 and D_2. Finally, for an arbitrary $f \in R$ we have

$$t\partial_t(f) = t \cdot \frac{\partial f}{\partial x}\frac{dx}{dt} + t \cdot \frac{\partial f}{\partial y}\frac{dy}{dt} = 2t^2\frac{\partial f}{\partial x} + 3t^3\frac{\partial f}{\partial_y} = (2x\partial_x + 3y\partial_y)(f) = D_2(f)$$

and

$$t^2\partial_t(f) = t^2 \cdot \frac{\partial f}{\partial x}\frac{dx}{dt} + t^2 \cdot \frac{\partial f}{\partial y}\frac{dy}{dt} = 2t^3\frac{\partial f}{\partial x} + 3t^4\frac{\partial f}{\partial y} = (2y\partial_x + 3x^2\partial_y)(f) = D_1(f)$$

by the chain rule.

1.2 The Weyl Algebras

Throughout this section $A = K$ and $R = K[x_1, ..., x_n]$, where K is a field. The ring $D_{R/K}$ in this case is called the n^{th} *Weyl Algebra*. The first Weyl algebra is an early example of a ring of differential operators. It first appeared as Dirac's *quantum algebra*, which consists of polynomial expressions in variables p and q subject to the relation $pq - qp = 1$. Weyl algebras admit tractable, explicit descriptions in terms of generators and relations and thereby serve as a fantastic source of examples. They also provide a good starting point for newcomers seeking to develop intuition (e.g. the author of this essay).

Our first aim in this section is to show the three main presentations of the n$^{\text{th}}$ Weyl algebra are equivalent.

Theorem 1.14. *(Definition) Let K be a field of characteristic 0 and let $R = K[x_1, ..., x_n]$. The following are isomorphic modules.*

- *The K-subalgebra $A_n(K) \subseteq \mathrm{End}_K(R)$ generated by the maps $\hat{x}_i$ and $\partial_{x_i} = \frac{\partial}{\partial x_i}$. We will often write simply A_n when there is no risk of ambiguity.*
- *The K-algebra D_n defined to be the free K-algebra in the $2n$-variables $y_1, ..., y_{2n}$ modulo the ideal J, where multiplication is given by concatenation on monomials and J is generated by all the elements of the form $[y_{i+n}, y_i] - 1$ for $1 \leq i \leq n$ or $[y_a, y_b]$ for $a \not\equiv b \mod n$, $1 \leq a, b \leq 2n$.*
- *The ring of differential operators $D_{R/K}$.*

Before we prove this we need to understand some basic facts about the module A_n.

Lemma 1.15. The generators of A_n satisfy the following relations:

$$[\partial_{x_i}, \hat{x}_j] = \delta_{ij}, \qquad [\partial_{x_i}, \partial_{x_j}] = [\hat{x}_i, \hat{x}_j] = 0$$

where δ_{ij} is the Kronecker delta function. Furthermore, for $f \in R$,

$$[\partial_{x_i}, \hat{f}] = \widehat{\frac{\partial f}{\partial x_i}}.$$

Proof: For any polynomial f (and more generally, any differentiable function) we have

$$\partial_{x_i}\hat{x}_j(f) = \partial_{x_i}(x_j \cdot f) = \partial_{x_i}(x_j) \cdot f \ + \ x_j \cdot \partial_{x_i}(f)$$

from the product rule in Calculus. Since $\partial_{x_i}(x_j) = \delta_{ij}$ and $x_j \cdot \partial_{x_i}(f) = \hat{x}_j\partial_{x_i}(f)$, rearranging the above yields the first relation.

Differentiation is K-linear, so it suffices to prove $\partial_{x_i}\partial_{x_j}(f) = \partial_{x_j}\partial_{x_j}(f)$ for a monomial f. This is clear from the power rule in Calculus. The fact $[\hat{x}_i, \hat{x}_j] = 0$ is a consequence of the commutativity of x_i and x_j in R.

Finally, it once again suffices to prove $[\partial_{x_i}, \hat{f}] = \frac{\partial f}{\partial_{x_i}}$ for monic monomials. We first show it holds for $f = x_i^m$. The relation $[\partial_{x_i}, \hat{x}_1] = 1$ serves as the base case, so suppose it holds for all $m < k$. Then

$$\partial_{x_i}\hat{x}_i^k = (\partial_{x_i}\hat{x}_i)\hat{x}_i^{k-1} = (1 + \hat{x}_i\partial_{x_i})\hat{x}_i^{k-1} = \hat{x}_i^{k-1} + \hat{x}_i\partial_{x_i}\hat{x}_i^k - 1.$$

The inductive hypothesis implies $\partial_{x_i}\hat{x}_i^{k-1} = (k-1)\hat{x}_i^{k-1} + \hat{x}_i^{k-1}\partial_{x_i}$, so after rearranging the above and combining like terms we have exactly that $[\partial_{x_i}, \hat{x}_i^k] = k\hat{x}_i^{k-1}$.

For an arbitrary monic monomial $x_1^{m_1}...x_n^{m_n}$ we have that

$$[\partial_{x_i}, \hat{x}_1^{m_1}...\hat{x}_n^{m_n}] = \hat{x}_1^{m_1}...\hat{x}_{i-1}^{m_{i-1}}[\partial_{x_i}, x_i^{m_i}]\hat{x}_{i+1}^{m+1}...\hat{x}_n^{m_n}$$

by repeated use of Proposition 1.6 (d). This reduces to

$$[\partial_{x_i}, \hat{x}_1^{m_1}...\hat{x}_n^{m_n}] = k \cdot \hat{x}_1^{m_1}...\hat{x}_i^{m_i-1}...\hat{x}_n^{m_n}$$

by what we have already proven. ☐

Remark 1.16. It is worth saying a few words about our choice of notation. Some authors suppress the notation $\hat{f}$ simply write "f" to refer interchangeably to $f \in R$ and its image in $D_{R/A}(M)$. This is reasonable, especially since the R-action on $D_{R/A}(M)$ is given by the inclusion $R \hookrightarrow D_{R/A}(M)$. Nonetheless, we prefer to differentiate between an element $f \in R$ and its image in $D_{R/A}(M)$ in this essay due to the notational similarity between $\partial_{x_i}\hat{f}$ and $\partial_{x_i}(f)$. These are two very different things; for example, $\partial_x(x) = 1 \in K[x]$ whereas $\partial_x \hat{x} = 1 + \hat{x}\partial_x \neq 1 \in A_1$.

We now construct a basis for the Weyl algebra, a basis known as the *canonical basis*.

Lemma 1.17. The set $\mathbf{B} = \{\hat{x}^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$ is a basis for A_n as a K-vector space. By $\hat{x}^\alpha$ we mean the operator $\hat{x}_1^\alpha \cdot ... \cdot \hat{x}_n^\alpha$, and the degree of this monomial is the length of α defined $|\alpha| = \alpha_1 + ... + \alpha_n$.

Proof: By definition, A_n is generated by monomials in ∂_{x_i} and $\hat{x}_j$ for i and j ranging between 1 and n. Using the fact that $\partial_{x_i}\hat{x}_i - \hat{x}_i\partial_{x_i} = \widehat{\frac{\partial f}{\partial x_i}}$ from Lemma 1.15 we can move all $\hat{x}_j$ terms to the left of all ∂_i terms, so it is clear that $\mathbf{B}$ spans A_n.

We now show that $\mathbf{B}$ is linearly independent. Suppose that

$$D = \sum_{i=1}^{m} c_i \hat{x}^{\alpha_i} \partial^{\beta_i}.$$

We call this summation the *canonical form* of $D \in A_n$ and show that $D = 0$ if and only if $c_i = 0$ for each $1 \leq i \leq m$. Assume without loss of generality that $c_i \neq 0$ for all $1 \leq i \leq m$ and $(\alpha_i, \beta_j) = (\alpha_j, \beta_j)$ if and only if $i = j$; that is, make m as small as possible. Let β_ℓ be the multi-index such that $|\beta_\ell| = \min\{|\beta_1|, ..., |\beta_m|\}$. By repeated use of the power law we get that

$$\partial^{\beta_\ell}(x^{\beta_\ell}) = \beta_\ell! \neq 0$$

where $\beta! = \beta_1! \cdot ... \cdot \beta_n!$ for $\beta \in \mathbb{N}^n$, but that $\partial^{\beta_i}(x^{\beta_\ell}) = 0$ for all $|\beta_i| > |\beta_\ell|$. It is possible that ∂^{β_ℓ} appears multiple times in the above summation. For simplicity, set $\lambda = \beta_e ll!$ and let $\{\alpha'_1, ..., \alpha'_k\}$ be the (necessarily distinct) multi-indices such that $\hat{x}^{\alpha'_i}\partial^{\beta_\ell}$ appears with nonzero coefficient in the canonical form of D. Likewise let c'_i be the coefficient of $\hat{x}^{\alpha'_i}\partial^{\beta_\ell}$ appearing in the canonical form of D. Then

$$D(x^{\beta_\ell}) = \sum_{i=1}^{k} c'_i \hat{x}^{\alpha'_i} \partial^{\beta_\ell}(x^{\beta_\ell}) = \lambda \left(c'_1 x^{\alpha'_1} + ... + c'_k x^{\alpha'_k} \right).$$

Since the α'_i are pairwise distinct, the above polynomial is nonzero and $D \neq 0$. We conclude that $D = 0$ if and only if $c_i = 0$ and we conclude that $\mathbf{B}$ is linearly independent over K. □

To illuminate the details of the above proof, let's examine some examples of differential operators over a polynomial ring in canonical form.

Example 1.18. Consider the first Weyl algebra $D_{K[x]/K}$, which is generated by $\hat{x}$ and ∂. The following identities hold:

(a) $\partial^m \hat{x} = \hat{x}\partial^m + m \cdot \partial^{m-1}$ and

(b) $\partial^a \hat{x}^b = \sum_{j=0}^{d} j! \binom{a}{j} \binom{b}{j} \hat{x}^{b-j} \partial^{a-j}$.

These of course easily generalize to $D_{R/K}$ by replacing $\hat{x}$ with $\hat{x}_i$ and ∂ with ∂_i. They are both proven via induction and liberal use of the fact that $[\partial, \hat{x}^b] = b\hat{x}^{b-1}$, but neither proof is particularly enlightening. It is perhaps more useful to see an explicit computation for low values of a and b:

$$\begin{aligned}\partial^2 \hat{x}^3 &= \partial\left(\partial \hat{x}^3\right)\\ &= \partial\left(\hat{x}^3\partial \ + \ 3\hat{x}^2\right)\\ &= \hat{x}^3\partial^2 \ + \ 6\hat{x}^2\partial \ + \ 6\hat{x}\end{aligned}$$

and how (b) can be used to compute the canonical form of operators in larger Weyl algebras, for instance in $D_{K[x,y]/K}$:

$$\begin{aligned}\partial_x\partial_y^2\hat{x}^3\hat{y}^2 &= \partial_x^2\hat{x}^3 \cdot \partial_6^2\hat{y}^2\\ &= \left(\hat{x}^3\partial_x \ + \ 3\hat{x}^2\right)\left(\hat{y}^2\partial^2 \ + \ 4\hat{y}\partial_y \ + \ 2\right)\\ &= \hat{x}^3\hat{y}\partial_x\partial_y^2 \ + \ 3\hat{x}^2\hat{y}^2\partial_y^2 \ + \ 4\hat{x}^3\hat{y}\partial_x\partial_y \ + \ 12\hat{x}^2\hat{y}\partial_y \ + \ 2\hat{x}^3\partial_x \ + \ 6\hat{x}^2.\end{aligned}$$

In the general of setting of $D_{R/A}$ where $A \longrightarrow R$ is an arbitrary map of rings, we have a notion of order. For the ring of differential operators over a polynomial ring, the existence of the canonical basis gives us something something better: a notion of degree. This doesn't give us a graded structure, but it does recover some of the properties of degree in a polynomial ring.

Let $D \in A_n$ be an operator in canonical form. The degree of D, denoted $\deg(D)$, is the length $|(\alpha, \beta)|$ of the largest multindex $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ such that $x^\alpha\partial^\beta$ appears with nonzero coefficient in the canonical form of D. The following proposition should be compared to Proposition 1.7, and due to its similarity the proof is omitted (Hint: it suffices to check monomials).

Proposition 1.19 ([Cou95, Theorem 2.1.1.])**.** Let $D, D' \in A_n$ and assume $\mathrm{char}(K) = 0$.

(a) $\deg(DD') = \deg(D) + \deg(D')$

(b) $\deg(D + D') \leq \max\{\deg(D), \deg(D')\}$

(c) $\deg[D, D'] \leq \deg(D) + \deg(D') - 2$.

As $\deg(0) = -\infty$, an immediate corollary to part (a) of the above proposition is that A_n is a domain. We can also use the proposition to prove the following theorem:

Theorem 1.20. *The algebra A_n is simple.*

Proof: Let I be a nonzero two-sided ideal of A_n and suppose $D \in I$ is a nonzero operator. If $deg(D) = 0$, then $D \in K$ and $I = A_n$. If $\deg(D) = d > 0$, then there must be some summand $x^\alpha\partial^\beta$ with nonzero coefficient and for which either $\alpha \neq 0$ or $\alpha \neq 0$. In the former case, suppose the α_i component of α is nonzero. Then $[\partial_i, D] \neq 0$ and $\deg([\partial_i, D]) \leq d - 1$. Furthermore, since I is two-sided, $[\partial_i, D] \in I$. By replacing D with $[\partial_i, D]$ and repeating the above process, we can construct an element of degree 0 in I and hence conclude$I = A_n$. A similar argument in which we instead consider $[x_i, D]$ works in the case that $\beta \neq 0$. □

Note that while A_n does not have any proper nontrivial two-sided ideals, it has many left and right ideals and is by no means a division ring. Furthermore, the kernel of any map of nontrivial unital rings must necessarily be a two-sided ideal, hence we have the following corollary.

Corollary 1.21. If $\phi : A_n \longrightarrow B$ is a map of unital rings then it is injective. $\square$

We are now ready to prove Theorem 1.14.

Proof: (Theorem 1.14) We first show $A_n \cong D_n$. Let $K\{y_1, ..., y_{2n}\}$ denote the free algebra over K in $2n$ variables with multiplicative given by concatenation of monomials and let $J \subseteq K\{y_1, ..., y_{2n}\}$ be the ideal generated by all the elements of the form $[y_{i+n}, y_i] - 1$ for $1 \leq i \leq n$ or $[y_a, y_b]$ for $a \not\equiv b \mod n$, $1 \leq a, b \leq 2n$. Note $D_n = K\{y_1, ..., y_{2n}\}/J$ by definition.

Define a map $\psi : A_n \longrightarrow D_n$ by setting $\psi(x^\alpha \partial^\beta) = y^{(\alpha,\beta)} + J$, noting that it suffices to define ψ on monomials in canonical form. A quick check shows that each of the relations on the generators of A_n given in Lemma 1.15 are preserved by ψ, so it is indeed a map of rings. Using the relations given by J, the same proof used in Lemma 1.17 can be used to show $\{y^{\alpha,\beta} + J\}_{\alpha,\beta \subseteq \mathbb{N}^n}$ is a basis for D_n, so it is clear that ψ is surjective. Furthermore, ψ is a map of unital rings and is therefore injective by Corollary 1.21. Hence ψ is an isomorphism.

We now wish to prove $A_n \cong D_{R/K}$. Denote by C_k the subset of A_n consisting of operators of degree at most k. We use the following two facts without proof:

(i) If $P \in D_{R/K}$ and $[P, \hat{x}_i] = 0$ for each $1 \leq i \leq n$, then $P \in R$ ([Cou95, Lemma 3.2.1]).

(ii) Let $P_1, ..., P_n \in C_{r-1}$ and assume that $[P_i, x_j] = [P_j, x_i]$ for all $1 \leq i, j \leq n$. Then there exists $Q \in C_r$ such that $P_i = [Q, x_i]$, for $i = 1, ..., n$ ([Cou95, Lemma 3.2.2]).

From Proposition 1.19 it is clear that $C_k \subseteq D^k_{R/K}$, so it suffices to prove the reverse inclusion. We proceed by induction. Proposition 1.9 gives us the base case $k = 1$. Suppose then that $D^r_{R/K} = C_r$ for all $0 \leq r \leq k-1$ and that $P \in D^k_{R/K}$. Let $P_i = [P, \hat{x}_i]$ and note that $P_i \in D^{k-1}_{R/K}$ by definition. Since $\hat{x}_i$ and $\hat{x}_j$ commute for all $1 \leq i, j \leq n$ we have

$$[P_i, x_j] = [[P, x_i], x_j] = [[P, x_j], x_i] = [P_j, x_j]$$

by the Jacobi identity. By fact (ii) above, there exists some $Q \in C_k$ such that $[Q, x_i] = P_i$ for each $1 \leq i \leq n$ and hence $[Q - P, x_i] = 0$. Then $Q - P \in R$ by fact (i) above, so $P = Q + \hat{f}$ for some $f \in R$. This means $P \in C_k$, and we are done. $\square$

1.2.1 Difficulties in Prime Characteristic

Even at this early stage, we can see pieces of this theory break when $\operatorname{char} K = p > 0$. Consider $A_1 = K[x, \partial] \subseteq \operatorname{End}_K(K[x])$ for $K = \mathbb{F}_p$. Let k be any positive integer and consider the action ∂^p on $x^k \in K[x]$. If $k < p$, then $\partial^p(x^k) = 0$. If $k \geq p$, then at least one of the integers $k-p+1, k-p+2, ..., k-1, k$ is divisible by p, and hence

$$\partial^p(x^k) = k(k-1)(k-2)...(k-p+1)x^{k-p} = 0.$$

Since ∂^p is zero on a basis for $K[x]$, it is identically zero on all of $K[x]$. This means ∂ is a nilpotent element and hence A_1 is not a domain.

Now consider D_1, the free algebra in x and ∂ over K modulo the relation $[\partial, x] = 1$. In contrast to A_1, this ring is a domain since Proposition 1.19 still holds, so we no longer have $A_1 \cong D_1$. It is not clear that D_1

ought to be our choice of definition for the Weyl algebra however, for there is another major departure from the characteristic zero world: D_n is not simple. For example,

$$[\partial, x^p] = px^{p-1} = 0,$$

from which it follows that D_1 has a nontrivial center, a two-sided ideal.

Furthermore, in characteristic zero, not all operators can be written as R-linear combinations of compositions of derivations. Take for instance the operator $\alpha \in D_{R/\mathbb{F}_p}$ when $R = \mathbb{F}_p[x]$ defined

$$x^n \mapsto \begin{cases} \binom{n}{p} x^{n-p} & \text{if } n \geq p \\ 0 & \text{otherwise} \end{cases}.$$

In characteristic zero, this operator is simply $\frac{1}{p!}\partial^p$, but in characteristic $p > 0$ it cannot be written as the composition of smaller order operators.

To summarize, when working with rings of differential operators $D_{R/K}$, it is necessary to fix either the characteristic of K or the choice of definition for $D_{R/K}$. In this document we have chosen to do the later – in situations of prime characteristic, we use the definition for $D_{R/K}$ given in Section 1.1.

1.3 Differential Operators on a Smooth Variety

It seems natural to ask whether there exist nice descriptions of $D_{R/K}$ comparable to those given by Theorem 1.14 and Lemma 1.17 when R is "nearly a polynomial ring". When "nearly a polynomial ring" is interpreted to mean "a regular K-algebra of finite type, the answer turns out to be "yes". The regular hypothesis is quite necessary, as we shall see. Regular finitely-generated K-algebras are also precisely the local version of smooth algebraic varieties, which we introduce in the context of differential operators here. Throughout this section K is still a field of characteristic zero.

1.3.1 Regular *K*-Algebras of Finite Type

Theorem 1.22. *Let R be a regular K-algebra of finite type. Then $D^m_{R/K}$ is generated as an R-module by all products of up to m many K-derivations of R. In particular, $D_{R/K}$ is generated by R and $\mathrm{Der}_K(R)$ as an R-module.*

Proof: The case in which R is a domain is handled by [MR01, Theorem 15.5.5]. Here is a rough outline of the ideas used. Suppose $L = \mathrm{Frac}(R)$ and $\{x_1, ..., x_n\}$ is a transcendence basis for L over K. One can pass to the polynomial ring $K[x_1, ..., x_n]$ and use the fact that $\mathrm{Der}_K(L) = \sum L \cdot \partial/\partial x_i$ to show that $D_{L/K}$ is spanned by L and $\mathrm{Der}_K(L)$ by mimicking the proof of the polynomial case. It then only remains to prove $D_{R/K} = \{\alpha \in D_{L/K} \mid \alpha(R) \subseteq R\}$.

The general case is given by [Muh88, Theorem 1.15]. Every regular ring is reduced, hence the intersection of all minimal primes in R is 0. The ring R can therefore be written as a product of domains by the Chinese Remainder Theorem. Muhasky uses the fact that $D_{(R_1 \times R_2)/K} \cong D_{R_1/K} \times D_{R_2/K}$ to conclude. □

Not only do derivations generate $D_{R/K}$, but each operator can be expressed in a way reminiscent of the canonical form for operators in the Weyl algebra (see Lemma 1.17). Namely, if R is a regular K-algebra of Krull dimension n, then any $P \in D^k_{R/K}$ can be written as a finite sum

$$P = \sum_\alpha \hat{f}_\alpha \partial^\alpha$$

where each $\alpha \in \mathbb{N}^n$, $f_\alpha \in R$ and $\{\partial_1, ..., \partial_n\}$ generate $\operatorname{Der}_K(R)$. This fact is slightly stronger than Theorem 1.22 however. See Theorem 1.26 below.

Let us examine two examples, one in which the hypotheses of Theorem 1.22 hold and one in which they do not.

Example 1.23. Let K be a field of characteristic zero and set $R = K[x, y]/(f)$ where $f = x^3 - x - y^2$. As the matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 3x_0^2 - 1 & -2y_0 \end{bmatrix}$$

is rank 1 for all points $(x_0, y_0) \in K^2$ in the graph of f, R is easily seen to be regular by the Jacobian criterion. Hence, to understand $D_{R/K}$ it suffices to understand the derivations on R.

It isn't terribly difficult to see that the set of derivations on R is given by

$$\operatorname{Der}_K(R) = \frac{\{\theta \in \operatorname{Der}_K(K[x, y]) \mid \theta((f)) \subseteq (f)\}}{(f) \operatorname{Der}_K(K[x, y])}.$$

We know $\operatorname{Der}_K(R)$ is a one-dimensional K-vector space since $\operatorname{Der}_K(R)$ is two-dimensional by example 1.12. It therefore suffices to find one derivation $\theta : K[x, y] \longrightarrow K[x, y]$ which fixes (f) to compute $\operatorname{Der}_K(R)$. Furthermore, since $\theta(f \cdot g) = f\theta(g) + g\theta(f)$, θ fixes (f) if and only if $\theta(f) \in (f)$, reducing our task of calculating $\operatorname{Der}_K(R)$ to finding a single derivation θ on $K[x, y]$ which sends f to a multiple of itself. But this is exceptionally easy; the derivation $\theta = \partial_x(f)\partial_y - \partial_y(f)\partial_x$ maps f to zero.

We conclude that $D_{R/K} = \bigoplus_{k=0}^{\infty} R \cdot \theta^k$ where $\theta = (3x^2 - 1)\partial_y + 2y\partial_x$.

Example 1.24. We return to the curve $f = y^2 - x^3$, which has a singularity at the origin. Let $R = K[t^2, t^3]$ and recall from Example 1.13 that $K[x, y]/(f) \cong K[t^2, t^3]$.

Consider the operator $\alpha = t\partial_t^2 - \partial_t$ in $D_{K[t]/K}$. Since $\alpha(t^2) = 0$ and $\alpha(t^3) = 3t^2$, $\alpha(R) \subseteq R$ and therefore $\alpha|_R \in D_{R/K}$. However, $\operatorname{Der}_K(R)$ is generated as a vector space by $t\partial_t$ and $t^2\partial$, and by considering these to be operators on $K[t]$ it is clear that α is outside the subring of $D_{K[t]/K}$ generated by $t^2\partial_t$ and $t\partial_t$. Therefore $D_{R/K}$ is strictly larger than the ring generated by $\operatorname{Der}_K(R)$ and R, highlighting the need for the regular hypothesis in

1.3.2 Smooth Varieties

We now define the sheaf of differential operators on a smooth variety, the primary setting of [HTT08]. The definitions given here are precisely those found in section 1.1 of [HTT08] contextualized within the discussion up to this point.

Definition 1.25. Let X be a smooth variety over a field K of characteristic zero and $\mathcal{O}_X$ be its structure sheaf. We denote by $\mathcal{E}nd_K\, \mathcal{O}_X$ the sheaf of K-linear endomorphisms of $\mathcal{O}_X$. We say that a section $\theta \in (\mathcal{E}nd_K\, \mathcal{O}_X)(X)$ is a *vector field on* X if $\theta(U) = \theta|_U$ is a K-derivation on $\mathcal{O}_X(U)$ for each open subset $U \subseteq X$. For any open subset $U \subseteq X$, the set of vector fields on U is denoted $\Theta(U)$. Then $\Theta(U)$ is an $\mathcal{O}_X(U)$-module, and the assignment $U \mapsto \Theta(U)$ is a sheaf of $\mathcal{O}_X$-modules. We denote this sheaf by Θ_X and note that when X is affine, $\Theta_X \cong \widetilde{\operatorname{Der}_K}(\mathcal{O}_X(X))$.

We then have the following theorem.

Theorem 1.26. *Let X be a smooth algebraic variety of dimension n over an algebraically closed field K. Then for each point $p \in X$, there exist an affine open neighborhood V of p, regular functions $x_i \in K[V] = \mathcal{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$ for $1 \leq i \leq n$ satisfying the conditions*

$$\begin{cases} [\partial_i, \partial_j] = 0, \qquad \partial_i(x_j) = \delta_{ij} \ (1 \leq i, j \leq n) \\ \Theta_V = \bigoplus_{i=1}^n \mathcal{O}_V \partial_i \end{cases}.$$

Moreover, we can choose the functions $x_1, ..., x_n$ so that they generate the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_{X,p}$. We call the set $\{x_i, \partial_i\}_{1 \leq i \leq n}$ a local coordinate system *of p on U.*

Proof: [HTT08, Theorem A.5.1]. $\square$

Note that the elements x_i appearing in the local coordinate system above are regular functions $x_i : V \longrightarrow K$, not elements of $\mathrm{End}_K(\mathcal{O}_X(V))$.

It follows from Theorem 1.22 that for any affine open $U \subseteq X$, the ring of differential operators of $\mathcal{O}_X(U)$ is generated by $\mathcal{O}_X(U)$ and $\Theta_X(U)$. This justifies the following definition:

Definition 1.27. Let X be a smooth variety over a field K of characteristic zero. We define the sheaf D_X of *differential operators on* X to be the K-subalgebra of $\mathcal{E}nd_K(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

For any point $p \in X$, we may find an affine open $U \subseteq X$ containing p and a local coordinate system $\{x_i, \partial_i\}_{1 \leq i \leq n}$ such that

$$D_U = D_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_X(U)\partial^\alpha$$

by combining Theorems 1.22 and 1.26. When X is not smooth, it is instead necessary to consider the sheaf given locally on open affines by $U \mapsto D_{\mathcal{O}_X(U)/K}$, where $D_{\mathcal{O}_X(U)/K}$ is defined as in Definition 1.1. This definition agrees with the one above by the theory we have developed thus far, and as we are only concerned with smooth varieties in these notes, we will always have access to a system of local coordinates.

Alternatively, one can define D_X by gluing on open affines. One does this by setting $\Gamma(U, D_X) = D_{\Gamma(U, \mathcal{O}_X)/K}$ for each open affine $U \subseteq X$. For this to work, we need the following compatibility result:

Proposition 1.28. Let R be a finitely generated regular K-algebra of dimension n. For nonzero $f \in R$, denote by R_f the localization of R at the set $\{1, f, f^2, ...\}$. Then

$$D_{R_f/K} \cong R_f \otimes_R D_{R/K} \quad \text{and} \quad D^i_{R_f/K} \cong R_f \otimes_R D^i_{R/K}.$$

Proof: Let $\varphi : R \longrightarrow R_f$ be the canonical map. The prime ideals of R_f correspond to the primes in R which avoid f, hence we have an isomorphism $(R_f)_\mathfrak{p} \cong R_{\varphi^{-1}(\mathfrak{p})}$ for each prime $\mathfrak{p} \subseteq R_f$. The local ring $R_{\varphi^{-1}(\mathfrak{p})}$ is regular, hence R_f is regular.

Set $W_f = \{1, f, f^2, ...\} \subseteq R$ so that $R_f = W_f^{-1} R$. By the isomorphism $\Omega_{W_f^{-1}R/K} \cong W_f^{-1}\Omega_{R/K}$ [Har77, Proposition 2.8.3], we have

$$\mathrm{Der}_K(R_f) \cong \mathrm{Hom}_{R_f}(W_f^{-1}\Omega_{R/K}, R_f) \cong W_f^{-1} \mathrm{Hom}_R(\Omega_{R/K}, R) \cong W_f^{-1} \mathrm{Der}_K(R) \cong R_f \otimes_R \mathrm{Der}_K(R).$$

The R_f-module is generated by R_f and $\mathrm{Der}_K(R_f)$ by Theorem 1.22, hence the above isomorphism extends to an isomorphism $D^i_{R_f/K} \cong R_f \otimes_R D^i_{R/K}$. $\square$

This means that an operator in $D_{R_f/K}$ extends to an operator in $D_{R/K}$ once we multiply by a large enough power of f.

It is worth noting that $\Gamma(X, D_X)$ generally fails to embed in $\mathrm{End}_K(\mathcal{O}_X(X))$ when X is not affine, which explains why we must define differential operators locally on affine opens. We conclude this section on differential operators with an example demonstrating this failure.

Example 1.29. Let $X = \mathbb{P}^1_K$ and let $U_0 = \mathbb{A}^1_K$ and $U_1 = \mathbb{A}^1_K$ denote the standard affine opens of X. If x_0 is the coordinate on U_0 and x_1 the coordinate on U_1, then $\Gamma(U_0, D_X)$ is the Weyl algebra generated by $\hat{x}_0, \partial_0$ and $\Gamma(U_1, D_X)$ is the Weyl algebra generated by $\hat{x}_1, \partial_1$. We may view the sheaf D_X to be the sheaf obtained by gluing $D_X|_{U_0}$ and $D_X|_{U_1}$ over $U_0 \cap U_1$, and hence a global differential operator $\theta \in \Gamma(X, D_X)$ is fully specified by a pair (θ_0, θ_1) of two elements $\theta_0 \in \Gamma(U_0, D_X)$ and $\theta_1 \in \Gamma(U_1, D_X)$ such that $\theta_0 = \theta_1$ on $U_0 \cap U_1$.

We change coordinates from U_0 to U_1 via $x_0 \mapsto x_1^{-1}$. To express ∂_1 in terms of $\hat{x}_0, \partial_0$ on the open set $U_0 \cap U_1$ we use the chain rule:

$$\partial_1 = \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_0}\frac{d\hat{x}_0}{dx_1} = -\hat{x}_1^{-2}\partial_0 = -\hat{x}_0^2\partial_0.$$

Two differential operators

$$\theta_0 = \sum_{i=1}^{n} a_i \hat{x}_0^{b_i}\partial_0^{c_i} \quad \text{and} \quad \theta_1 = \sum_{j=1}^{m} \alpha_j \hat{x}_1^{\beta_j}\partial_1^{\gamma_j}$$

are therefore equal on $U_0 \cap U_1$ if and only if

$$\sum_{i=1}^{n} a_i \hat{x}_0^{b_i}\partial_0^{c_i} \;=\; \sum_{j=1}^{m} \alpha_j \hat{x}_0^{-\beta_j}\left(-\hat{x}_0^2\partial_1\right)^{\gamma_j}.$$

Determining whether two such arbitrary operators agree on $U_0 \cap U_1$ is quite difficult in general, as it involves expanding multiple terms of the form $(-\hat{x}_0^2\partial_0)^\gamma$ at once. However, we can use this restriction criterion to easily construct an infinite set of K-linearly independent global differential operators. Define $\delta = -\hat{x}_0^2\partial \in \Gamma(U_0, D_X)$. Then δ^n is equal to ∂_1^n for any $n \in \mathbb{N}$, and so the set $\{(\delta^n, \partial_1^n)\}$ is a K-linearly independent set of global differential operators. This means $\Gamma(X, D_X)$ is infinite dimensional as a K-vector space.

Since $\mathrm{End}_K(\mathcal{O}_X(X)) = \mathrm{End}_K(K) = K$ is a 1-dimensional K-vector space, there is no embedding $\Gamma(X, D_X) \longrightarrow \mathrm{End}_K(\mathcal{O}_X(X))$.

1.4 A Word Regarding Non-Regular K-Algebras

To conclude our discussion of the ring of differential operators, we say a brief word about the singular case. There is still an "R-linear" way to compute the modules $D^i_{R/K}$ even when R is not regular. We loosen our assumptions on R and once again take R to be an algebra over another commutative ring A. Taking cues from the characterization of $\mathrm{Der}_K(R)$ in terms of Kähler differentials, we define

$$P^i_{R/A} = \frac{R \otimes_A R}{\Delta^{i+1}_{R/A}} \tag{3}$$

to be the module of ith principal parts of R over A, with $\Delta_{R/A}$ as in Definition 1.10. One can then prove that

$$D^i_{R/A} \cong \mathrm{Hom}_R(P^i_{R/A}, R).$$

Thus the functor $R \mapsto D^i_{R/A}$ is represented by $P^i_{R/A}$. This construction can be found in [Moo04], who attributes it to Grothendieck.

2 *D*-Modules: Basic Definitions and Facts

We start with the definition of a D-module.

Definition 2.1. Let X be a smooth variety over a field K. A left (or right) D-module over X, or a D_X-module, is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ together with a left (or right) action by D_X. We say that $\mathcal{M}$ is a *coherent* D_X-module if it is locally finitely generated over D_X.

In the affine case, a D-module corresponds to a module M over a ring of differential operators, i.e. a left or right $D_{R/A}$-module, via $M \mapsto \tilde{M}$ (see Example 2.8). When working over an affine variety $\operatorname{Spec} R = X$, it therefore suffices (and is typically more convenient) to study $M = \Gamma(X, \mathcal{M})$ rather than $\mathcal{M}$ itself.

Note that a coherent D_X-module is *not* necessarily coherent as an $\mathcal{O}_X$-module. For instance, $A_n(K) = \Gamma(X, D_X)$ is the nth Weyl algebra when $X = \operatorname{Spec} K[x_1, ..., x_n]$, and though $A_n(K)$ is trivially finitely generated as a module over itself, it is certainly not finitely generated as a $\Gamma(X, \mathcal{O}_X) = K[x_1, ..., x_n]$-module as there is no way to increase the degree of an operator via the action of a polynomial.

We start this section with several examples before discussing the basic theory relating to the structure of D-modules.

2.1 Examples of *D*-modules

Let R be a regular finitely generated K-algebra. We start with a trivial example.

Example 2.2. Every ring is a module over itself, so $D_{R/K}$ is a left $D_{R/K}$-module as are all of its left ideals. The polynomial ring R is also a left $D_{R/K}$-module, where the left action of an operator $\alpha \in D_{R/K}$ on $f \in R$ is given by applying α to f, i.e. $\alpha \cdot f = \alpha(f)$.

This is quite unremarkable, so we quickly move on to some more interesting examples.

Example 2.3. Let $I = D_{R/K}\partial$ and $J = D_{R/K}\hat{x}$ be the left ideals of $D_{R/K}$ generated by ∂ and $\hat{x}$ respectively and let $M = D_{R/K}/I$ and $N = D_{R/K}/J$. These are quotients of left $D_{R/K}$-modules and are therefore themselves $D_{R/K}$-modules. As K-vector spaces, it is clear that $M \cong K[\hat{x}]$ and $M \cong K[\partial]$.

To understand the $D_{R/K}$-action on M, it suffices to understand the action of $\hat{x}$ and ∂ on the basis $\{1 + I, \hat{x} + I, \hat{x}^2 + I, ...\}$ of M. The action of $\hat{x}$ is multiplication; it's an infinite Jordan block with one's along the upper diagonal and zeros elsewhere. Since $\partial\hat{x} = 1 + \hat{x}\partial$ and $\hat{x}\partial \in I$, we have that $\partial(\hat{x} + I) = 1 + I$. Similarly, $\partial(\hat{x}^k + I) = \partial(\hat{x}^k) + I = \hat{x}^{k-1}$, so as a K-linear map, ∂.

NOT FINISHED FINISH THIS YOU PIECE OF SHIT

Example 2.4. Let $K = \mathbb{C}$, denote by A the Weyl algebra over $\mathbb{C}$, and fix a subset $U \subseteq \mathbb{C}$ open with respect to the Euclidean topology. Every holomorphic function is analytic, and therefore the set $\mathcal{H}(U)$ of holomorphic functions on U is a left A-module. Somewhat more surprising is the fact that it is not a torsion module, one can show that the function $h(x) = \exp(\exp(z))$ is not killed by any element of A for instance. See [Cou95, Chapter 5.3] for details.

Example 2.5 (Module Associated to a Differential Equation)**.** Let $K = \mathbb{R}$, denote by A_n the nth Weyl algebra and fix a set $U \subseteq \mathbb{R}^n$. The set $\mathcal{C}^\infty(U)$ of infinitely differentiable functions in $x_1, ..., x_n$ is then an A_n module.

Consider now an arbitrary operator $P = \sum_{i=1}^m g_{\alpha_i}\partial^{\alpha_i} \in A_n$ where $\alpha_i \in \mathbb{N}^n$ is a multi-index for each

$1 \le i \le n$. This operator gives us a differential equation:

$$P(f) = \sum_{i=1}^{m} g_{\alpha_i} \partial^{\alpha_i}(f) = 0$$

where $f \in C^\infty(U)$. We can similarly define a system of differential equations

$$P_1(f) = ... = P_k(f) = 0 \tag{4}$$

given $P_1, ..., P_k \in A_n$. The $\mathbb{R}$-vector space of solutions to this system is certainly not an A_n-module, if f satisfies the system there is no expectation that $\partial_{x_i}(f)$ does as well for instance, but it does nonetheless admit a nice description via the theory of A_n-modules.

Let $J = \sum_{i=1}^{k} A_n P_k$ be the left ideal generated by $P_1, ..., P_k$ and set $M = A_n/J$. We say that M is the A_n-module associated to the system (4). We will show that the set of polynomial solutions to (4) is isomorphic to $\mathrm{Hom}_{A_n}(M, \mathbb{R}[x_1, ..., x_n])$ as a $\mathbb{R}$-vector space.

First, consider a polynomial solution $f \in \mathbb{R}[x_1, ..., x_n]$ to (4), and associate to f the A_n-module homomorphism $\varphi_f : A_n \longrightarrow \mathbb{R}[x_1, ..., x_n]$ defined by $1 \mapsto f$. If $Q \in J$, then $Q(f) = 0$, so $\varphi_f(Q) = 0$ and hence φ_f induces a map $\overline{\varphi_f} : M \longrightarrow \mathbb{R}[x_1, ..., x_n]$.

Consider now the $\mathbb{R}$-linear map $f \mapsto \overline{\varphi_f}$ taking a polynomial solution of (4) to its associated A_n-module homomorphism. It's inverse is the map $\sigma \mapsto \sigma(1)$ which sends a homomorphism $\sigma : M \longrightarrow \mathbb{R}[x_1, ..., x_n]$ to its evaluation at $1 \in M$.

These examples have all been of left A_n-modules, but we can turn left modules into right modules and vice versa.

Example 2.6. (Construction)

Now let X be a smooth variety over K.

Example 2.7. A necessary and sufficient condition for a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules to be affine is that for any affine $U \subseteq X$, $\mathcal{F}|_U \cong \tilde{M}$ where $M = \Gamma(U, \mathcal{F})$ (see [Har77, Chapter 2.5]). If $U \subseteq X$ is affine and $f \in \mathcal{O}_X(U)$, then

$$\Gamma(D(f), D_X|_U) = D_{\mathcal{O}_X(U)_f/K} \cong \mathcal{O}_X(U)_f \otimes_{\mathcal{O}_X(U)} D_{\mathcal{O}_X(U)/K}$$

by Proposition 4.5, so $D_X|_U \cong \Gamma(\widetilde{U, D_X})$. This implies that D_X is itself a left D-module, and we can similarly see that $\mathcal{O}_X$ is a left D_X-module. Indeed, for any open affine U, the algebra $D_X(U)$ acts on $\Gamma(U, D_X)$ and $\Gamma(U, \mathcal{O}_X)$ by the construction in Section 1.3.2.

Example 2.8. When $X = \operatorname{Spec} R$ is affine, every left D_X-module $\mathcal{M}$ corresponds to a left $D_{R/K}$-module via $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$. Examples 2.3 through 2.6 are therefore all examples of D-modules over $\mathbb{A}_K^n$ once we pass to the associated sheaf.

ADD ONE MORE EXAMPLE YOU SHIT

2.2 Filtrations

We would like to define invariants such as dimension and multiplicity for D-modules. Commutative algebra provides a concrete theory of exactly this for graded modules over graded commutative rings, but we have neither commutativity nor a graded structure. One possible solution is to associate a graded commutative ring to $D_{R/K}$ and a compatible graded module to a $D_{R/K}$-module M. We accomplish exactly this via filtrations.

This is a brief overview of some definitions concerning filtered K-algebras, tailored to the purposes of this essay. We are primarily interested in good filtrations of finitely generated A_n-modules, as these provide us with sufficient conditions to discuss dimension. A more general treatment suitable to the case of $\mathcal{D}_X$-modules over a scheme X can be found in Chapter 1 of [Gin98], which largely serves as the inspiration for this section. Though all of our statements deal with left modules, everything holds if we replace "left" with "right" and make the obvious, necessary changes.

Definition 2.9. Let R be a K-algebra. We say R is a *filtered K-algebra* if it comes equipped with a collection $\{F_i\}_{i\in\mathbb{N}}$ of K-vector spaces such that

(1) $K = F_0 \subset F_1 \subset F_2 \subset ... \subset R$

(2) $F_i \cdot F_j \subseteq F_{i+j}$.

(3) $R = \bigcup_{i\geq 0} F_i$, (we say the filtration is *exhausting*)

When equipped with a filtration, R is said to be a *filtered K-algebra*. We often write this as a pair $(R, F_\bullet)$. We often set $F_{-1} = \{0\}$ and iterate over $\mathbb{Z}$ rather than $\mathbb{N}$.

Remark 2.10 (Definition Ext.)**.** Let $(R, F_\bullet)$ be as in the above definition. The collection of sets $\{F^i + r\}_{i\in\mathbb{Z}, r\in R}$ form the basis of a topology on R. With this in mind, it is often convenient to impose two additional conditions:

(4) $\bigcap_{i\geq -1} F_i = \{0\}$, which is equivalent to say that the topology induced by $F_\bullet$ is separating,

(5) R is complete with respect to this topology.

We also have a notion of a filtered ring in which we replace the K-vector spaces with abelian groups, but in this essay we will only be concerned with filtered K-algebras.

Example 2.11. The collection $D^\bullet_{R/K} = \left\{D^k_{R/K}\right\}_{k\in\mathbb{N}}$ is a filtration of $D_{R/K}$. Requirement (1) holds by Lemma 1.4, requirement (2) by Proposition 1.7 (be) and requirement (3) by definition of $D_{R/K}$. This is called the *order filtration* on $D_{R/K}$. The order filtration on the n^{th} Weyl algebra A_n is given a special name: the *Bernstein filtration*. We denote this filtration $\mathcal{B} = \{B_k\}_{k\geq 0}$ where $B_k = \{D \in A_n \mid \deg(D) \leq k\}$.

Example 2.12. Suppose $R = \bigoplus_{i\in\mathbb{N}} R_i$ is a graded ring. Then $(R, F_\bullet)$ is a filtered K-algebra with respect to the filtration $F_k = \bigoplus_{i=0}^k R_i$.

Definition 2.13. Let $(R, F_\bullet)$ be a filtered K-algebra. The *associated graded K-algebra*, $\operatorname{gr}^{F_\bullet} R$, is defined

$$\operatorname{gr}^{F_\bullet} R = \bigoplus_{i=0}^{\infty} F_i/F_{i-1}.$$

When the filtration is known, we will often suppress it from the notation and simply write $\operatorname{gr} R$. For any $r \in F_i$, we denote by $\sigma_i(r)$ its image in F_i/F_{i-1} and say $\sigma_i(r)$ is the i$^{\text{th}}$ *principal symbol* of r. The associated graded ring to the filtration given in Example 2.12 recovers the original graded ring, as one might hope.

We use the principal symbol maps σ_i to define an algebra structure on $\operatorname{gr}^{F_\bullet} R$. A *homogeneous element* of $\operatorname{gr}^{F_\bullet} R$ is any operator $d \in \operatorname{gr}^{F_\bullet} R$ such that $d = \sigma_k(a)$ for some $a \in F_k$. Given two homogeneous elements $\sigma_i(a)$ and $\sigma_j(b)$, we define their product by

$$\sigma_i(a) \cdot \sigma_j(b) = \sigma_{i+j}(a \cdot b).$$

Extending this multiplication to all of $\operatorname{gr}^{F_\bullet} R$ by distributivity makes $\operatorname{gr}^{F_\bullet} R$ into a graded K-algebra whose homogeneous components are the individual summands F_k/F_{k-1}.

Example 2.14. Let $S_n = \operatorname{gr}^{\mathcal{B}} A_n$. Then the graded algebra S_n is isomorphic to $K[y_1, ..., y_{2n}]$.

The conceptual sketch of this statement is perhaps more enlightening than the full proof. Since we have surjective maps $\pi_k : A_n \longrightarrow B_k \xrightarrow{\sigma_k} B_k/B_{k-1}$, S_n is generated as an algebra by the images of elements $x_1, ..., x_n, \partial_1, ..., \partial_n \in A_n$. The only thing preventing us from defining a isomorphism $K[y_1, ..., y_{2n}] \longrightarrow S_n$ sending $y_i \mapsto x_i$ and $y_{i+n} \mapsto \partial_i$ for $1 \leq i \leq n$ is commutativity, however, we see that

$$\pi_1(\partial_i x_i) = \pi_1(x_i\partial_i + 1) = \pi_1(x_i\partial_i) + \pi_1(1) = \pi_1(x_i\partial_i),$$

so $[\partial_i, x_i] = 0$ in B_1/B_0. This gives us commutativity in S_n and allows us to define a surjective homomorphism $K[y_1, ..., y_{2n}] \longrightarrow S_n$. Since there are no additional relations between the generators $x_1, ..., x_n, \partial_1, ..., \partial_n$, this is an isomorphism. See [Cou95, pg. 58] for all the full detail.

Definition 2.15. Let $(R, F_\bullet)$ be a filtered K-algebra and M a left R-module. A *filtration of M* compatible with $F_\bullet$ is a family $\Gamma = \{\Gamma_0\}_{i\geq 0}$ of K-vector spaces satisfying

(1) $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq ... \subseteq M$,

(2) $F_i\Gamma_j \subseteq \Gamma_{i+j}$.

(3) $M = \bigcup_{i\geq 0} \Gamma_i$

Such a module is said to be *filtered*, and as with algebras, we set $\Gamma_{-1} = 0$. In this section, we additionally adopt the convention that

(4) Γ_i is a finite-dimensional K-algebra for each $i \geq 0$,

which will become important in our discussion of dimension. The *associated graded module* to M is

$$\operatorname{gr}^{\Gamma} M = \bigoplus_{i=0}^{\infty} \Gamma_i/\Gamma_{i-1}$$

and is a graded $\operatorname{gr} R$ module.

The associated grading can tell us something about its filtered module.

Theorem 2.16. *Suppose that R is a filtered K-algebra with filtration $F_\bullet$ such that $S = \operatorname{gr}^{F_\bullet} R$ is Noetherian. Let M be a left R-module with filtration $\Gamma = \{\Gamma_i\}_{i\geq 0}$. If $\operatorname{gr}^{\Gamma} M$ is a Noetherian then so is M.*

Proof: Let $N \subseteq M$ be a R-submodule of M. We prove that it is finitely generated. Define $\Gamma'_i = N \cap \Gamma_i$ for $i \geq 0$. The collection $\Gamma' = \{\Gamma'_i\}$ is then a filtration of N, which we call the *induced filtration of N by Γ*. The inclusions $\Gamma'_i \subseteq \Gamma_i$ give us an inclusion $\operatorname{gr}^{\Gamma'} N \subseteq \operatorname{gr}^{\Gamma} M$, and since $\operatorname{gr}^{\Gamma} M$ is Noetherian, $\operatorname{gr}^{\Gamma'} N$ must be a finitely generated as an S-module.

Let $\{c_1, ..., c_r\}$ be a generating set for $\mathrm{gr}^{\Gamma'} M$. We assume that each c_i is homogeneous without loss of generality; each c_i is a linear sum of finitely many homogeneous elements and we can therefore replace each c_i by its homogeneous components without compromising the finiteness of our generating set. For each c_i we can therefore find some integer k_i and some $u_i \in \Gamma'_{k_i}$ such that $\mu_{k_i}(u_i) = c_i$. Let $m = \max\{k_1, ..., k_r\}$, and note that $u_i \in \Gamma'_m$ for each $1 \le i \le r$. We show that Γ'_m generates N.

Suppose $v \in \Gamma_\ell$. If $\ell \le m$ then $v \in \Gamma'_\ell \subseteq \Gamma'_m$, and hence v is in the R-submodule of M generated by Γ'_m. Suppose now that $\ell > m$ and $\Gamma_{\ell-1}$ is contained in the R-linear span of Γ'_m. Because $\{\mu_{k_1}(u_1), ..., \mu_{k_r}(u_r)\}$ generates $\mathrm{gr}^{\Gamma'} N$ as an S-module, there exist $a_1, ..., a_r$ such that

$$\mu_\ell(v) = \sum_{i=1}^{r} \sigma_{\ell-k_i}(a_i)\mu_{k_i}(u_i).$$

Hence

$$\mu_\ell\left(v - \sum_{i=1}^{r} a_i u_i\right) = 0$$

$$v' = v - \sum_{i=1}^{r} a_i u_i \in \Gamma'_{\ell-1}.$$

The element v is a linear sum of elements in Γ'_m if and only if v' is too. However, $v' \in \Gamma'_{\ell-1}$ and is therefore in the R-linear span of Γ'_m by the inductive hypothesis. Hence $v \in R \cdot \Gamma'_m$, and since every element of N is contained in Γ'_ℓ, Γ'_m generates N.

It is left to show that there is a finite subset of Γ'_m which generates N. However, Γ'_m is a finite dimensional K-vector space. Any K-basis for Γ'_m will generate all of Γ'_m and will therefore serve as a set of generators for N. □

Note that the set $\{u_1, ..., u_r\}$ in the above proof is not necessarily a generating set for N. The induction step gives us an algorithm for writing any $v \in \Gamma'_\ell$ in terms of the u_i only in the case that $\ell > \max\{\deg(u_1), ..., \deg(u_r)\}$.

We have the following immediate corollary.

Corollary 2.17. The nth Weyl algebra A_n is left Noetherian.

Proof: The associated graded ring of A_n with respect to the Bernstein filtration is the polynomial ring in two variables by Example 2.14, which is Noetherian. □

As mentioned in the introduction to this section, these statements hold if we replace "left" by "right" and make the necessary adjustments, meaning A_n is also right Noetherian. This is quite convenient, for it means any finitely generated left or right A_n-module is automatically Noetherian.

The converse of Theorem 2.16 need not always hold, that is, it need not be the case that $\mathrm{gr}^{\Gamma} M$ is finitely generated even if M is finitely generated. We therefore distinguish filtrations which produce finitely generated associated graded modules.

Definition 2.18. Let M be a left module over a filtered K-algebra $(R, F_\bullet)$. A filtration Γ of M is said to a *good filtration* with respect to $F_\bullet$ if $\operatorname{gr}^\Gamma M$ is finitely generated. Good filtrations provide a framework to discuss the dimension of modules over the Weyl algebra.

Good filtrations always exist for finitely generated modules.

Proposition 2.19. Let $(R, F_\bullet)$ be a filtered K-algebra and M be a finitely generated left R-module. Then there exists a good filtration Γ of M compatible with $F_\bullet$.

Proof: Let $u_1, ..., u_r$ be a generating set for M over R and define $\Gamma_k = \sum_{i=0}^r F_k u_i$. Then $\operatorname{gr}^\Gamma M$ is finitely generated over $\operatorname{gr}^{F_\bullet} R$ by the images of $u_1, ..., u_k$ in Γ_k. □

We end the section on filtrations by stating two propositions, both of which are included primarily for convenient use in the discussion of holonomic modules over A_n. One provides a criterion for easily checking whether a filtration is good, and the other allows us to compare two good filtrations of a module.

Proposition 2.20. Let M be a left module over a filtered K-algebra $R, F_\bullet)$. A filtration Γ of M with respect to $F_\bullet$ is good if and only if there exists an integer k_0 such that $\Gamma_{i+k} = F_i \Gamma_k$ for all $k \geq k_0$.

Proof: [Cou95, Proposition 8.3.1] □

This criterion is useful for determining both good and bad filtrations.

Example 2.21. Consider the Bernstein filtration $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ on the Weyl algebra A_n. Set $\Gamma_i = B_{2i}$. We then have that $B_i \Gamma_k = B_{i+2k} \neq B_{2(i+k)} = \Gamma_{i+k}$, so Γ_i is not a good filtration of A_n with respect to the Bernstein filtration.

Proposition 2.22. Let M be a left module of the filtered K-algebra $(R, F_\bullet)$. Suppose that Γ and Λ are two filtrations of M with respect to $F_\bullet$. The following statements are true.

(a) If Γ is good with respect to $F_\bullet$ then there exists some k_0 such that $\Lambda_i \subseteq \Lambda_{i+k_0}$ for all $i \in \mathbb{N}$.

(b) If both Γ and Λ are good with respect to $F_\bullet$, then there exists some k_1 such that $\Lambda_{i-k_1} \subseteq \Gamma_i \subseteq \Lambda_{i+k_1}$.

Proof: [Cou95, Proposition 8.3.2] □

2.3 Modules over the Weyl algebra

Throughout this section K is a field of characteristic 0, $A_n = D_{K[x_1,...,x_n]/K}$ is the nth Weyl algebra, S_n is the associated graded ring to A_n with respect to the Bernstein filtration $\mathcal{B}$, and M is a finitely generated left A_n module.

We work almost entirely with left A_n-modules in this section, but all results in this section hold if "left" is replaced with "right" and the obvious modifications are made.

2.3.1 Dimension

The primary goal of this section is a proof of Bernstein's Inequality, a striking example of how the theory of D-modules can drastically differ from that of modules over commutative rings. To accomplish this, it is necessary to discuss several basic facts regarding the dimension of modules over the Weyl algebra, theory which relies on dimension theory from commutative algebra. We brazenly omit proofs and discussion of these facts in eternal deference to Atiyah-Macdonald [AM16].

Recall that if $M = \oplus_{i\geq 0} M_i$ is a finitely generated graded module over a polynomial ring $K[x_1, ..., x_m]$, then there exists a polynomial $\chi(t) \in \mathbb{Q}[t]$ and a positive integer N such that

$$\sum_{i=0}^{t} \dim_K(M_i) = \chi(t)$$

for all $t \geq N$. We typically suppress N from our notation and simply write "for all $t \gg 0$" to mean "for all t sufficiently large". The polynomial $\chi(t)$ is called the Hilbert polynomial of M.

If M is a finitely generated left A_n-module then there exists a filtration Γ of M which is good with respect to the Bernstein filtration by Proposition 2.19. The associated graded module $\operatorname{gr}^\Gamma M$ is then seen to be Noetherian since it is finitely generated over $S_n = \operatorname{gr}^\Gamma A_n$, a Noetherian ring. This means the Hilbert polynomial for $\operatorname{gr}^\Gamma M$ exists, and we denote it by $\chi(t, \Gamma, M) \in \mathbb{Q}[t]$. This discussion leads us to the following definition.

Definition 2.23. Let M be a finitely generated left A_n-module equipped with a good filtration Γ with respect to the Bernstein filtration. Denote by $\chi(t, \Gamma, M)$ the Hilbert polynomial of $\operatorname{gr}^\Gamma M$. Let a be the leading coefficient of $\chi(t, \Gamma, M)$ and let d be its degree. The *dimension* $d(M)$ of M is d and the *multiplicity* $m(M)$ of M is $d! \cdot a$. Both of these are nonnegative integers.

See [AM16] for details, or [Cou95, Chapter 9] for a discussion tailored specifically to modules over the Weyl algebra. The latter sources also provides a brief argument demonstrating that the definitions of dimension and multiplicity do not depend on the choice of good filtration.

Example 2.24. It is well known that the Hilbert polynomial of the polynomial rink $K[x_1, ..., x_m]$ is degree m. Hence the Hilbert polynomial of $S_n = K[y_1, ..., y_{2n}]$ is degree $2n$ and $d(A_n) = 2n$. By this same argument, $d(K[x_1, ..., x_n]) = n$.

Proposition 2.25. Let M be a finitely-generated left A_n-module and $N \subseteq M$ a submodule. Then

(a) $\dim(M) = \max\{d(N), d(M/N)\}$

(b) If $\dim(N) = \dim(M/N)$ then $m(M) = m(N) + m(M/N)$.

Proof:

(a) Let us first see how the Hilbert polynomials of M, N and M/N related. Denote by S_n the associated graded ring of A_n, and let Γ be a good filtration of M with respect to $\mathcal{B}$. LetΓ' and Γ'' be the induced filtrations for N and M/N. We then obtain the following short exact sequence of associated graded S_n-modules:

$$0 \longrightarrow \operatorname{gr}^{\Gamma'} N \longrightarrow \operatorname{gr}^{\Gamma} M \longrightarrow \operatorname{gr}^{\Gamma''} M/N \longrightarrow 0.$$

We know $\operatorname{gr}^\Gamma M$ is a finitely generated S_n-module since Γ is good, hence $\operatorname{gr}^{\Gamma''} M/N$ is also finitely generated since it is isomorphic to a quotient of $\operatorname{gr}^\Gamma M$. Likewise, since S_n is Noetherian and $\operatorname{gr}^{\Gamma'} N$ is isomorphic to a submodule of $\operatorname{gr}^\Gamma M$, $\operatorname{gr}^{\Gamma'} N$ is finitely generated. This tells us that Γ' and Γ'' are both good filtrations.

Now consider the short exact sequence of vector spaces

$$0 \longrightarrow \Gamma'_k/\Gamma'_{k-1} \longrightarrow \Gamma_k/\Gamma_{k-1} \longrightarrow \Gamma''_k/\Gamma''_{k-1} \longrightarrow 0$$

for $0 \leq k$. By the rank-nullity theorem, $\dim_K \Gamma_k/\Gamma_{k-1} = \dim_K \Gamma'_k/\Gamma'_{k-1} + \dim_K \Gamma''_k/\Gamma''_{k-1}$, so

$$\sum_{k=0}^{\infty} (\dim_K \Gamma_k/\Gamma_{k-1}) = \sum_{k=0}^{\infty} \left(\dim_K \Gamma'_k/\Gamma'_{k-1} + \dim_K \Gamma''_k/\Gamma''_{k-1}\right)$$

and thus for $s >> 0$ we get

$$\chi(s, \Gamma, M) = \chi(s, \Gamma', M) + \chi(s, \Gamma'', N).$$

As all of the above are polynomials with positive leading coefficients by CITE THEOREM, we get that $\deg\left(\chi(s, \Gamma', M) + \chi(s, \Gamma'', N)\right) = \deg\left(\chi(s, \Gamma', M)\right) + \left(\chi(s, \Gamma'', N)\right)$ and hence

$$\dim(M) = \max\left\{\dim(N), \dim(M)\right\}.$$

(b) If $\dim(M/N) = \dim(N)$ then the polynomials $\chi(s, \Gamma, M)$, $\chi(s, \Gamma', M)$ and $\chi(s, \Gamma'', N)$ all have the same degree. This then implies that the leading term of $\chi(s, \Gamma, M)$ is equal to the sum of the leading terms of $\chi(s, \Gamma', N)$ and $\chi(s, \Gamma'', M/N)$.

□

Corollary 2.26. Let M be a finitely generated A_n-module. Then $d(M) \leq 2n$.

Proof: Let $\{u_1, ..., u_r\}$ be a generating set over A_n for M. There then exists a surjective homomorphism $\phi : A_n^{\oplus r} \longrightarrow M$. Proposition 2.25 then tells us that $d(A_n^{\oplus r}) = \max\{d(M), d(\ker \phi)\}$.

We claim that $d(A_n^{\oplus r}) = 2n$. Indeed, we have seen that $d(A_n) = 2n$, and there exists an exact sequence

$$0 \longrightarrow A_n \longrightarrow A_n^{\oplus r} \longrightarrow A_n^{\oplus (r-1)} \longrightarrow 0$$

from which we get that $(A_n^{\oplus r}) = \max\{d(A_n), d(A_n^{\oplus (r-1)})\}$. Induction on r then gives us the desired result, hence $\max\{d(M), d(\ker \phi)\} \leq 2n$. We conclude $d(M) \leq 2n$. □

2.3.2 Bernstein's Inequality

Theorem 2.27 (Bernstein's Inequality)**.** *If M is a finitely-generated left $A_n(K)$-module, then either $n \leq \dim(M)$ or $M = 0$.*

Proof: Let $\mathcal{B} = \{B_k\}_{k \geq 0}$ be the Bernstein filtration. Fix a generating set $u_1, ..., u_r$ for M over A_n and let Γ be the good filtration obtained by setting $\Gamma_k = \sum_{i=1}^{r} B_k u_i$, as in the proof of Proposition 2.19. Finally, let $\chi(t) = \chi(t, \Gamma, M)$ be the Hilbert polynomial of M.

We first show that the K-vector space B_i embeds in $\mathrm{Hom}_K(\Gamma_i, \Gamma_{2i})$ for each $i \geq 0$. Define $\phi_a : \Gamma_i \longrightarrow \Gamma_{2i}$ by $u \mapsto au$ for $a \in B_i$ and let $\phi : B_i \longrightarrow \mathrm{Hom}_K(\Gamma_i, \Gamma_{2i})$ be the K-linear map $a \mapsto \phi_a$, noting that ϕ is injective exactly when $a\Gamma_i \neq 0$ for any $0 \neq a \in B_i$. We prove that ϕ is injective by induction on i.

For $i = 0$ we have $B_0 = K$, and hence ϕ is injective exactly when $\Gamma_0 \neq 0$. Since $u_1, ..., u_r \in \Gamma_0$, this is satisfied.

Assume now that ϕ is injective for all $1 \leq j < i$, that is, if $0 \neq b \in B_j$ then $b\Gamma_j \neq 0$. Fix some nonzero $a \in B_i$. The canonical form of a must then include a nonzero term which is a product of either $\hat{x}_\ell$ or ∂_{x_ℓ} for some $1 \leq \ell \leq n$. In particular,

$$[a, D] \neq 0, \quad \text{for some } \; D \in \{\hat{x}_1, ..., \hat{x}_n, \partial_{x_1}, ..., \partial_{x_n}\}.$$

Suppose that $a\Gamma_i = 0$. Since $\deg(D) = 1$, $D\Gamma_{i-1} \subseteq \Gamma_i$, so $a(D\Gamma_{i-1}) \subseteq \Gamma_i$. We then have that

$$[a, D]\Gamma_{i-1} = a(D\Gamma_{i-1}) - D(a\Gamma_{i-1}) = 0. \tag{$*$}$$

However, $\deg([a, D]) \leq \deg(a) - 1$ by Proposition 1.19 (c), so $[a, D]$ is a nonzero element of B_{i-1}. Hence $(*)$ is contradicts the inductive hypothesis and $a\Gamma_i \neq 0$. This proves that ϕ is injective for all values $i \geq 0$.

We now prove that $d(M) \geq n$. That ϕ is injective implies

$$\dim_K(B_i) \leq \dim_K(\mathrm{Hom}_K(\Gamma_i, \Gamma_{2i}))$$

for all $i \geq 0$. Let's examine the RHS of this inequality. It is a fact of elementary linear algebra that $\dim_K(\mathrm{Hom}_K(\Gamma_i, \Gamma_{2i}) = \dim_K(\Gamma_i) \dim_K(\Gamma_{2i})$, hence for $i \gg 0$, $\dim_K(\mathrm{Hom}_K(\Gamma_i, \Gamma_{2i}) = \chi(i)\chi(2i)$.

Now consider the LHS. By definition, the set of all elements of the form $\hat{x}^\alpha \partial^\beta$ with $\alpha, \beta \in \mathbb{N}^n$ satisfying $|\alpha| + |\beta| \leq 2i$ forms a basis for B_i as a K-vector space. A combinatorial argument shows that the number of monomials in k variables of degree at least d is $\binom{k+d}{k}$. Hence $\dim_K(B_i) = \binom{i+2n}{2n}$. Expanding, we see that

$$\binom{i+2n}{2n} = \frac{(i+2n)!}{i!(2n)!} = \frac{1}{(2n)!}(i+2n)(1+2n-1)...(1+2n-(2n-1))$$

is a polynomial in i of degree $2n$. In order for the above inequality to hold for all values of i, $\chi(i)\chi(2i)$ must likewise be at least degree $2n$. However, $\deg(\chi(i)\chi(2i)) = 2\deg(\chi(i)) = 2d(M)$. This means $2d(M) \geq 2n$, or $d(M) \geq n$ as desired. $\square$

We have already seen examples of left A_n-modules whose dimensions are $2n$ and n, these were A_n and $K[x_1, ..., x_n]$ respectively. There also exist A_n-modules of dimension k for each integer $n \leq k \leq 2n$.

Example 2.28.

2.3.3 Holonomic Modules

The Bernstein inequality tells us that a nonzero finitely-generated left $A_n(K)$-module M must have dimension at least n. Those modules of minimal dimension are called *holonomic modules*. Holonomic modules turn out to have particularly nice properties; for instance, they are preserved under inverse and direct images, as we shall see in a later section.

Definition 2.29. A finitely generated left $A_n(K)$-module M is said to be *holonomic* if either $M = 0$ or $\dim(M) = n$.

Examples are easy to identify thanks to Bernstein. We know that $R = K[x_1, ..., x_n]$ is holonomic since $\dim K[x_1, ..., x_n] = n$, and furthermore, both I and R/I are holonomic when I is any proper ideal of R by Proposition 2.25. As another example, in the case that $n = 1$, for any nonzero ideal $I \subseteq A_1$ we have that $\dim(A_1/I) \leq 1$ by Proposition 2.25. We know A_n/I is nonzero since I is proper, hence $\dim(A_1/I) = 1$ by

Bernstein's inequality.

Proposition 2.30. The following are true.

(a) Submodules and quotients of holonomic A_n-modules are holonomic.

(b) Direct sums of holonomic A_n-modules are holonomic.

Proof: Statement (a) follows from Bernstein's inequality and the fact that for any finitely generated A_n-module M and submodule $N \subseteq M$, $d(M) = \max\{d(N), d(M/N)\}$.

Suppose $M_1, ..., M_k$ are all holonomic A_n-modules. Statement (b) follows by applying the above reasoning to the short exact sequence

$$0 \longrightarrow M_k \longrightarrow M_1 \oplus ... \oplus M_k \longrightarrow M_1 \oplus ... \oplus M_{k-1} \longrightarrow 0$$

and induction on k. □

Proposition 2.31. Holonomic modules are Artinian. Furthermore, their length is finite and bounded by their multiplicity.

Proof: Here we use the additivity of multiplicity from Proposition 2.25 (b). Let M be a holonomic left A_n-module and suppose we have a descending chain of proper submodules

$$M = N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_k. \tag{$*$}$$

By Proposition 2.30, N_i and N_i/N_{i+1} are holonomic for each i. Together with the properness of the above inclusions, this implies $d(N_i) = d(N_i/N_{i+1}) = n$. We then have

$$m(M) = \sum_{i=0}^{k-1} m(N_i/N_{i+1}) \;+\; m(N_k).$$

Multiplicity is a nonnegative integer, and since the multiplicity of a nonzero module is by definition nonzero, $d(M) \geq k$ (allowing for the case that $N_k = 0$). However, $m(M)$ is itself a finite integer, so we cannot find a chain ($*$) of length greater than $m(M)$. In particular, any infinite chain must either stabilize, in which case $m(N_i/N_{i+1}) = 0$ for all $i \gg 0$, or terminate with $N_i = 0$ for all $i \gg 0$. □

2.3.4 Lemma on B-Functions

Let f be a polynomial in $K[x_1, ..., x_n]$ and let s be a new variable. We will consider the Weyl algebra $A_n(K(s))$ over the field of rational functions in s and the $A_n(K(s))$-module generated by the formal symbol f^s, upon which a rational function $p \in K(s)$ acts in the obvious way and the operator ∂_i acts by the formula

$$\partial_j \left(f^s\right) = \frac{s}{f} \cdot \frac{\partial f}{\partial x_i}. \tag{5}$$

Note that when we write f^{s+k} for some integer k, we mean $f^k \cdot f^s$. When s is an integer and f^s is treated not as a formal symbol but as a power, this action agrees with the existing action of ∂_j. The above formula means that $A_n(K(s))f^s$ is an $A_n(K(s))$-submodule of $K(s)[x_1, ..., x_n, f^{-1}]f^s$.

Lemma 2.32. Suppose M is a left A_n-module with a filtration Γ. If there exists a polynomial $q \in K[y]$ of degree n such that $\dim_K(\Gamma_i) \leq q(i)$ for sufficiently large i, then M is finitely generated and holonomic. In addition, if a is the leading coefficient of q, then $m(M) \leq n!a$.

Proof: Suppose first that $0 \neq N \subseteq M$ is a finitely generated submodule. We then have a good filtration Λ of N with respect to the Bernstein filtration on A_n by Proposition 2.19 as well as an induced filtration on N given by $\Gamma_i \cap N$. By Proposition 2.22, there exists some positive integer k_0 such that $\Lambda_i \subseteq \Gamma_{i+k_0} \cap N$ for all $i \in \mathbb{N}$, and hence $\dim_K(\Lambda_i) \leq \dim_K(\Gamma_{i+k_0} \cap N) \leq q(i + k_0)$.

Let $\chi(t) = \chi(t, \Lambda, N)$ be the Hilbert polynomial for N with respect to Λ. For $i \gg 0$, we have

$$\chi(i) = \sum_{j=0}^{i} \dim_K(\Lambda_i/\Lambda_{i-1}) = \dim_K(\Gamma_i) \leq q(i + k_0).$$

This means $\deg(\chi) \leq \deg(q) = n$, and therefore N is holonomic by Bernstein's inequality. Since a polynomial converges to its largest term in the limit $t \longrightarrow \infty$, this also implies that $m(M) \leq n!a$, where a denotes the leading coefficient of q.

Consider now an ascending chain of finitely generated modules

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq ... \subsetneq N_k$$

where $N_i \subseteq M$. Each of these is holonomic by what we have just proven. Repeating the argument from the proof of Proposition 2.31, we have that

$$m(N_k) = \sum_{i=1}^{k} m(N_i/N_{i-1}) \ + \ m(N_0) \geq k.$$

However, we also have that $m(N_k) \leq n!a$ by what we have already shown. This means that $n!a$ is an upper bound on the length on an ascending chain in M, and therefore M itself is finitely generated. Repeating the above argument for M, we get that $d(M) = n$ and $m(M) \leq n!a$. □

The following corollary is crucial to the proof of Theorem 2.34.

Corollary 2.33. Fix a polynomial $f \in K[x_1, ..., x_n]$. The left $A_n(K(s))$-module $M = K(s)[x_1, ..., x_n, f^{-1}]f^s$ defined above is holonomic.

Proof: Let $m = \deg(f)$ in $K[x_1, ..., x_n]$. Define

$$\Gamma_k = \left\{ qf^{-k} \cdot f^s \;\middle|\; \deg(q) \leq (m+1)k \right\}.$$

We write $qf^{-k} \cdot f^s$ rather than qf^{s-k} to emphasize that qf^{-k} is an element in $K[x_1, ..., x_n f^{-1}]$ acting on f^s. Using the conventions of Definition 2.15, we show in detail that Γ is a filtration of M with $\dim_K(\Gamma_k) \leq \binom{n+k(m+1)}{k(m+1)}$. The holonomy of M then follows immediately from the previous lemma. Note that $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ is the Bernstein filtration on $A_n(K(s))$, as per usual.

(1) Clearly, if $qf^{-k} \cdot f^s \in \Gamma_k$, then

$$\deg(q \cdot f) = \deg(q) + \deg(f) \leq (m+1)k + m \leq mk + k + m + 1 = (m+1)(k+1).$$

Hence $qf^{-k} \cdot f^s = (qf)f^{-(k+1)} \cdot f^s \in \Gamma_{k+1}$, and therefore $\{\Gamma_i\}_{i \in \mathbb{N}}$ is an upward nested sequence of K-vector spaces.

(2) Fix $1 \leq i \leq n$. The left action of $\hat{x}_i \in A_n(K(s))$ on $qf^{-k} \cdot f^s \in \Gamma_k$ increases the degree of q by 1, so $\hat{x}_i \left(qf^{-k} \cdot f^s\right) \in \Gamma_{k+1}$. The left action of ∂_{x_i} on $qf^{-k} \cdot f^s$ is given by

$$\begin{aligned}\partial_{x_i}(qf^{-k} \cdot f^s) &= \partial_{x_i}(q)p^{-k} \cdot p^s \; - \; kp^{-(k+1)}\partial_{x_i}(f)q \cdot f^s + qf^s \frac{s}{f} f^s \partial_{x_i}(f) \\ &= \left(\partial_{x_i}(q)f \; + \; (s-k)q\partial_{x_i}(f)\right) f^{-(k+1)} \cdot p^s.\end{aligned}$$

Both terms inside the parentheses have degree at most $\deg(q) + m - 1$, which is less than $(m+1)(k+1)$ because $\deg(q) \leq (m+1)k$, so $\partial_{x_i}(qf^{-k} \cdot f^s) \in \Gamma_{k+1}$.

The set $\{\hat{x}_1, ..., \hat{x}_n, \partial_{x_1}, ...\partial_{x_n}\}$ forms a basis for B_1, hence $B_1 \cdot \Gamma_k \subseteq \Gamma_{k+1}$. Furthermore, $B_i\Gamma_k \subseteq \Gamma_{k+i}$ since $B_i = B^i$.

(3) Choose an arbitrary element $p \in K(s)[x_1, ..., x_n, f^{-1}]$ so that $p \cdot f^s$ represents an arbitrary element of M. Set $k \leq \deg(p)$ and $q = pf^k$. Then

$$p \cdot f^s = qf^{-k} \cdot f^s \;\text{ and }\; \deg(q) = \deg(f) + km \leq k + km = (m+1)k,$$

so $f \cdot p^s \in \Gamma_k$. Every element of M is in Γ_k for some k, hence $\bigcup_{i=0}^{\infty} \Gamma_k = M$.

(4) The set of elements of the form $uf^{-k} \cdot f^s$ where u is a monomial of $K[x_1, ..., x_n]$ with degree at most $(m+1)k$ generates Γ_k as a K-vector space, so each Γ_k is finite dimensional.

As discussed in the proof of Bernstein's theorem, there are $\binom{n+k(m+1)}{k(m+1)}$ many monomials in $K[x_1, ..., x_n]$ of degree at most $(m+1)k$, so $\dim_K(\Gamma_k) \leq \binom{n+k(m+1)}{k(m+1)}$ by (4) above. This binomial coefficient is a degree n polynomial in k, hence M is holonomic by Lemma 2.32. □

We can now prove the Lemma on b-functions. Like many other named lemmas in mathematics, it is listed not as a lemma but as a theorem.

Theorem 2.34. *Fix $f \in K[x_1, ..., x_n]$. There exists a polynomial $B(s) \in K[s]$ and a differential operator $D(s) \in A_n(K)[s]$ such that*

$$B(s)f^s = D(s)f^{s+1}.$$

The set of all such $B(s)$ form an ideal in $K[s]$, the monic generator of which is called the Bernstein polynomial *of f and is denoted by $b_f(s)$.*

Proof: The case in which $f = 0$ is trivial, so assume $f \neq 0$. Since $A_n(K(s))f^s$ is a submodules of $K(s)[x_1, ..., x_n f^{-1}]f^s$, it too is holonomic and consequently of finite length. The descending sequence

$$A_n(K(s)) \cdot f^s \supseteq A_n(K(s)) \cdot f^{s+1} \supseteq A_n(K(s)) \cdot f^{s+2} \supseteq ...$$

must therefore terminate. In particular, there must exist some positive integer k such that

$$A_n(K(s))f^k \cdot f^s = A_n(K(s))f^{k+1} \cdot f^s.$$

This implies that

$$f^{s+k} = D(s)f^{s+k+1}$$

for some $D(s) \in A_n(K(s))$. As s is simply a dummy variable, we can send $s \mapsto s-k$ to get $f^s = D(s-k)f^{s+1}$. Note that $D(s-k)$ is simply a polynomial in $\hat{x}_1, ..., \hat{x}_n, \partial_{x_1}, ..., \partial_{x_n}$ with coefficients in $K(s)$, so we may multiply by an appropriate $B(s) \in K[s]$ to clear denominators and get that $B(s)D \in A_n(K)[s]$. Setting $D'(s) = B(s)D(s-k)$ yields

$$B(s)f^s = D'(s)f^{s+1}$$

as desired. □

Example 2.35. Let $f = x_1^2 + ... + x_n^2$. Notice that

$$\partial_{x_i}^2 f^{s+1} = 4x_i^2(s+1)sf^{s-1} + 2(s+1)f^s.$$

Letting $D = \partial_{x_1}^2 + ... + \partial_{x_n}^2$, we get that

$$\begin{aligned} D(f^{s+1}) &= \sum_{i=0}^{n} \left(4x_i^2(s+1)sf^{s-1} \ + \ 2(s+1)f^s\right) \\ &= 4(s+1)s(x_1^2 + ... + x_n^2)f^{s-1} + 2n(s+1)f^s \\ &= 2(s+1)(2s+n)f^s, \end{aligned}$$

hence $b_f(s) = 2(s+1)(2s+n)f^s$.

2.4 Algebraic *D*-Modules

We now wish to extend the results of this section to the setting of algebraic D-modules. Throughout this section, X is a smooth variety over K and $\mathcal{M}$ is a D_X-module.

2.4.1 Filtrations and the Characteristic Variety

Just as in the affine case, we study D-modules through filtrations. The sheaf D_X is given locally on affines $\operatorname{Spec} R = U \subseteq X$ by operators $f \in D_{R/K}$, and likewise, for $k \in \mathbb{N}$ we may consider the coherent sheaf of $\mathcal{O}_X$-modules D_X^k given locally by order k operators $f \in D_{R/K}^k$. The collection $D_X^\bullet$ of coherent $\mathcal{O}_X$-modules is then a filtration of D_X. A filtration of $\mathcal{M}$ is then an increasing family of coherent submodules $\mathcal{F}_\bullet = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ each of which satisfies

$$D_X^i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}.$$

This collection is also required to be exhausting, i.e.

$$\bigcup_{i\in\mathbb{Z}} \mathcal{F}_i(U) = \mathcal{M}(U) \quad \text{for each open } U \subseteq X.$$

We say that a filtration $\mathcal{F}_\bullet$ of $\mathcal{M}$ is a *good filtration* if the associated graded $\operatorname{gr}^{D_X^\bullet} D_X$-module

$$\operatorname{gr}^{\mathcal{F}_\bullet} \mathcal{M} \ = \ \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_i/\mathcal{F}_{i-1}.$$

Every coherent D_X-module $\mathcal{M}$ has a good filtration locally by Proposition 2.19. Somewhat more surprising is the fact that good filtrations exist globally as well.

Lemma 2.36. Let $\mathcal{M}$ be a coherent D_X-module. Then there exists a good filtration $F_\bullet\mathcal{M}$ of $\mathcal{M}$ by coherent $\mathcal{O}_X$-modules.

Proof: $\square$

Definition 2.37. The *characteristic variety* $\mathrm{Ct}(\mathcal{M})$ of $\mathcal{M}$ is the closed algebraic subset of T^*X given by $\widetilde{\mathrm{gr}^{\mathcal{F}_\bullet}\mathcal{M}}$, the sheaf associated to $\mathrm{gr}^{\mathcal{F}_\bullet}\mathcal{M}$, with reduced scheme structure.

3 Inverse Images, Direct Images and Kashiwara's Theorem

Given a morphism of smooth varieties $\varphi : X \longrightarrow Y$, we have functors $\varphi_* : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$ and $\varphi^* : \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X)$, which are the direct image and inverse image functors respectively. These are the primary way in which we obtain sheaves on Y from sheaves on X and vice versa. We would like to define similar operations in the categories of left (and right) D-modules.

Unfortunately, a complete account of these topics requires the use of derived categories. The problem is homological: the full statement of Kathiawar's theorem establishes an equivalence of categories via the direct image functor on the derived category of D-modules, but the candidates for this functor are not necessarily exact on the category of D-modules themselves. We nonetheless can provide a meaningful discussion if one accepts several limitations. In particular, the direct image functor associated to a closed embedding $\iota : Y \longrightarrow X$ *is* exact. We focus primarily on this case.

As should be expected by now, many algebraic constructions are brushed under the rug, the tensor product of bimodules perhaps chief among them. The reader may wish to visit [Cou95], [HTT08] or [Gin98] if this is unfamiliar. Throughout this section, K is a field of characteristic zero and both X and Y are smooth algebraic varieties over K.

3.1 Inverse Images

Suppose $\varphi : X \longrightarrow Y$ is a morphism of smooth algebraic varieties over K and M is a left D_Y-module. We wish to build a left D_X-module from M in a meaningful way. The inverse image of M

$$\varphi^* M = \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1} M$$

is a left $\mathcal{O}_X$-module, and we can endow it with a D_X-module structure in the following way.

Fix a point $p \in Y$, an affine neighborhood U of p, a local coordinate system $\{y_i, \partial_{y_i}\}_{1 \le i \le n}$ of p on U, and set $V = \varphi^{-1}(U)$. It suffices to define the $\mathcal{O}_X(V)$ and $\Theta_X(V)$ action on elements of the form $r \otimes u \in \mathcal{O}_X(V) \otimes_{\varphi^{-1}\mathcal{O}_Y(V)} \varphi^{-1} M(V)$, as such elements generate $\varphi^{-1}M(V)$ and $\mathcal{O}_X$ and Θ_X generate D_X. We define the action of $a \in \mathcal{O}_X(V)$ on $r \otimes u$ by $a \cdot (r \otimes u) = ar \otimes u$ and the action of a vector field $\theta \in \Theta_X(V)$ on $r \otimes u$ by

$$\theta(r \otimes u) = \theta(r) \otimes u \;+\; r \sum_{i=1}^{n} \theta(y_i \circ \varphi) \otimes \partial_{y_i}(u). \tag{$*$}$$

To check that this does indeed produce a D_X-action on $\varphi^* M$, we need to verify that it satisfies the relations

$$[\partial_{x_i}, \hat{x}_j] = \delta_{ij}$$
$$[\hat{x}_i, \hat{x}_j] = [\partial_{x_i}, \partial_{x_j}] = 0$$

in an affine neighborhood $U' \subseteq X$ of $\varphi^{-1}(p)$ with a local coordinate system $\{x_i, \partial_{x_i}\}_{1 \le i \le m}$. We check the first relation on and claim the others follows similarly. For $r \otimes u \in \varphi^* M$, we have

$$\begin{aligned}
\partial_{x_i} \hat{x}_j (r \otimes u) &= \partial_{x_i}(x_j r \otimes u) \\
&= \partial_{x_i}(x_j r) \otimes u \;+\; x_j r \sum_{k=1}^{n} \partial_{x_i}(y_k \circ \varphi) \otimes \partial_{y_k}(u) \\
&= r\delta_{ij} \otimes u \;+\; x_j \partial_{x_i}(r) \otimes u \;+\; x_j r \sum_{k=1}^{n} \partial_{x_i}(y_k \circ \varphi) \otimes \partial_{y_k}(u) \\
&= \delta_{ij}(r \otimes u) \;+\; x_j \left(\partial_{x_i}(r) \otimes u \;+\; r \sum_{k=1}^{n} \partial_{x_i}(y_k \circ \varphi) \otimes \partial_{y_k}(u) \right) \\
&= \delta_{ij}(r \otimes u) \;+\; \hat{x}_j \partial_{x_i}(r \otimes u),
\end{aligned}$$

hence $[\partial_{x_i}, \hat{x}_j](r \otimes u) = \delta_{ij}(r \otimes u)$. It holds on arbitrary elements of $\varphi^* M$ by the linearity of the commutator.

This discussion is summarized by the following definition.

Definition 3.1. Let $\varphi : X \longrightarrow Y$ be a morphism of smooth algebraic varieties and let M be a D_Y-module. Then the inverse image $\varphi^* M$ of M endowed with the action defined in $*$ is D_X-module, the *inverse image* of M.

As a sanity check, let's ensure the inverse image works as expected when φ is the identity map.

Example 3.2. Let $\varphi : X \longrightarrow X$ be the identity morphism on a smooth variety X and M a D_X-module. Note that the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} M(U)$ is a sheaf. We have $\varphi^{-1}(\mathcal{F})(U) = \mathcal{F}(U)$ for any sheaf $\mathcal{F}$ on X since φ is the identity, hence for any open set $V \subseteq X$,

$$\varphi^* M(V) = \mathcal{O}_X(V) \otimes_{\varphi^{-1}\mathcal{O}_X(V)} \varphi^{-1} M(V) \cong \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(V)} M(V) \cong M(V).$$

Fix a point $p \in X$, an affine open neighborhood $U \subseteq X$ of p, and a local coordinate system $\{x_i, \partial_{x_i}\}_{1 \le i \le n}$ at p on U. Let $\theta \in \Theta_X(U)$ be a vector field on U and let $\theta = \sum_{i=1}^n a_i \partial_{x_i}$ be θ expressed in local coordinates (here, $a_i \in \mathcal{O}_X(U)$). For any $u \in M$, we have that

$$\begin{aligned}
\theta(1 \otimes u) &= \theta(1) \otimes u \;+\; \sum_{i=1}^{n} \theta(x_i \circ \varphi) \otimes \partial_{x_i}(u) \\
&= \sum_{i=1}^{n} \theta(x_i) \otimes \partial_{x_i}(u) \\
&= \sum_{i=1}^{n} a_i \otimes \partial_{x_i}(u)
\end{aligned}$$

$$1 \otimes \left(\sum_{i=1}^{n} a_i \partial_{x_i}(u) \right) = 1 \otimes \theta(u),$$

so $\varphi^* M \cong M$ via the isomorphism $1 \otimes u \mapsto u$.

However, inverse images can behave badly even for relatively simple morphisms $\varphi : X \longrightarrow Y$. For instance, the inverse image of a coherent module need not itself be coherent.

Example 3.3. Suppose $X = Y = \mathbb{A}_K^1$, so that $D_X = D_y = \tilde{A}_1$, the first Weyl algebra (review Theorem 1.14 for our Weyl algebra notation). Though X and Y are two copies of the same variety, we distinguish the coordinate systems of X and Y by $\{x, \partial_x\}$ and $\{y, \partial_x\}$, noting that these are valid coordinate systems for any $p \in X$ or $p \in Y$ respectively.

Consider the morphism $\varphi : X \longrightarrow Y$ defined $\varphi(x) = x^2$ and note that the induced map on global sections $\varphi^\sharp : K[y] \longrightarrow K[x]$ sends a polynomial $f(y)$ to $f(x^2)$. Finally, let $M = A_1$, so that $\tilde{M}$ is Weyl algebra considered as a module over itself.

Hartshorne tells us that $\varphi^*(M) \cong (K[x] \otimes_{K[y]} M)^\sim$ [Har77, Proposition 5.2], so the global sections of $\varphi^*(M)$ are generated by elements of the form $f \otimes u$ for $f \in K[x]$ and $u \in M$. Though $\tilde{M}$ is coherent as a D_Y-module, we will see that $\varphi^* \tilde{M}$ is not a coherent D_X module.

It suffices to check that $\Gamma(X, \varphi^*(\tilde{M})) = K[x] \otimes_K [y] M$ is not finitely generated as a $\Gamma(X, D_X) = A_1$-module. Suppose we have some finite set of elements $B \subseteq K[x] \otimes_K [y] M$. The span of an element $f \otimes u \, + \, f' \otimes u'$ is contained in the span of $\{f \otimes u, f' \otimes u'\}$, so we assume that B is comprised entirely of elements of the form $f \otimes u$ for $f \in K[x]$ and $u \in M$. Furthermore, by writing u in its canonical form (see Lemma 1.17) we may assume that u is of the form $\hat{y}^a \partial_y^b$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

Suppose b is the largest natural number such that $f \otimes \hat{y}^a \partial_y^b$ is an element of B for some $a \in \mathbb{N}$ and $f \in K[x]$. From the $K[y]$-action on $K[x]$, we get that $f \otimes \hat{y}^a \partial_y^b = x^{2a} f \otimes \partial_y^b$. Noting that $x \circ \varphi = x^2$, we have

$$\begin{aligned}
\partial_x(f \otimes \hat{y}^a \partial_y^b) &= \partial_x(x^{2a} f \otimes \partial_y^b) \\
&= \partial(x^{2a} f) \otimes \partial_y^b \, + \, x^{2a} f \partial_x(x^2) \otimes \partial_y(\partial_y^b) \\
&= \Big(2a x^{2a-1} f(x) + x^{2a} f'\Big) \otimes \partial_y^b \, + \, 2x^{2a+1} f \otimes \partial_y^{b+1}.
\end{aligned}$$

Thus, the action of ∂_x will increase the degree of both the first and second component of $x^{2a} f \otimes \partial_y^b$ by 1. This means the A_1-span of $K[x] \otimes_{K[y]} M$ avoids elements such as $1 \otimes \partial_y^{b+1}$, as 1 has degree 0 and ∂_y^{b+1} has degree larger than b, the largest power of ∂_y appearing in the set B. Therefore, the span of any finite subset of $K[x] \otimes_{K[y]} M$ will be a proper subset, so $\varphi^*(\tilde{M})$ is not a coherent D_X-module.

Given a morphism $\varphi : X \longrightarrow Y$, it of course makes sense to take the inverse image of D_Y itself. This is the module $\varphi^* D_Y = \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1} D_Y$, and in addition to the left D_X-action endowed by the inverse image it comes equipped with an obvious $\varphi^{-1} D_Y$ action. These actions are compatible, and therefore $\varphi^* D_Y$ is a $(D_X, \varphi^{-1} D_Y)$-bimodule. It plays an important role in direct images and in Kashiwara's equivalence, so we give it a special name.

Definition 3.4. Suppose $\varphi : X \longrightarrow Y$ is a morphism of smooth varieties. We define the *transfer module* $D_{X \to Y}$ to be the $(D_X, \varphi^{-1} D_Y)$-bimodule $\varphi^* D_Y = \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1} D_Y$.

3.2 Direct Images

3.3 Kashiwara's Equivalence

Theorem 3.5. *Let $\iota : Y \hookrightarrow X$ be a closed embedding. The functor ι_* is an equivalence of categories between the category of coherent right D_Y-modules and the category of coherent right D_X-modules with support contained in Y.*

4 Applications of *D*-Modules

4.1 *D*-Modules in Positive Characteristic

In this section we'll discuss the theory of D-Modules in positive characteristic.

4.2 The Structure of Differential Operators in Positive Characteristic

Definition 4.1. Let $A \longrightarrow R$ be a map of commutative rings and let M and N be (two-sided) R-modules. We define

$$D^0_{R/A}(M,N) = \operatorname{Hom}_R(M,N)$$

and then inductively define

$$D^i_{R/A}(M,N) = \{\varphi \in \operatorname{Hom}_A(M,N) \mid [\varphi, f] \in D^{i-1}_{R/A}(M,N) \text{ for all} f \in R\}.$$

Note that we may identify R with its image in $\operatorname{End}_R(M)$ by the map $f \mapsto \overline{f}_M$ where $\overline{f}_M : m \mapsto f \cdot m$. By $[\varphi, f]$ we mean $\varphi \circ \overline{f}_M - \overline{f}_N \circ \varphi$.

We let $D_{R/A}(M,N) = \bigcup_{i=0}^{\infty} D^i_{R/A}(M,N)$. In the case that $M = N$ we write $D_{R/A}(M)$ and when $R = M = N$ we write $D_{R/A}$, sometimes omitting R and A when the context is clear.

It is often convenient to let $D^i_{R/A}(M,N) = 0$ for $i < 0$. In fact, since φ is R-linear if and only if $[\varphi, f] = 0$ for each $f \in R$, we could instead declare $D^i_{R/A}(M,N) = 0$ before proceeding with the inductive definition above.

The following is our first theorem regarding differential operators in characteristic $p > 0$.

Theorem 4.2 ([Yek92, Lemma 1.4.8])**.** *Let K be a perfect field with $\operatorname{char} K = p > 0$ and R be essentially of finite type as a K-algebra. Then*

$$D_{R/K} = \bigcup_{e \in \mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R,R).$$

Proof: The proof of this statement in [Yek92] is slightly more general than this statement, likely opportunity to condense it. □

Definition 4.3. Let $A \longrightarrow R$. A *D-ideal* of R is a $D_{R/A}$-submodule of R. Since $R \hookrightarrow D_{R/A}$, any such submodule is closed under multiplication by R and is therefore an ideal of R, justifying the name.

Example 4.4. If $I \subseteq R$ is a D-ideal, then R/I is a $D - module.$

Proposition 4.5. If $W \subseteq R$ is a multiplicatively closed set and M is a D-module, then $W^{-1}M$ is a D-module by the rule

$$\alpha \cdot \frac{m}{\omega} = \sum_{i=0}^{\operatorname{ord}(\alpha)} \frac{\alpha^{(i)} \cdot m}{\omega^{i+1}}$$

where $\alpha^{(0)} = \alpha$ and $\alpha^{(i+1)} = [\alpha^{(i)}, \overline{\omega}]$.

Definition 4.6. Let $A \longrightarrow R$ be a map of rings. We say that R is *D-module simple* if it is a simple D-module. We say R is *D-algebra simple* if it is a simple D-algebra.

The ring of differential operators in prime characteristic can detect singularities.

Theorem 4.7 ([Smi95, Theorem 2.2 (4)]). *Let $\operatorname{char}(R) = p > 0$ and suppose R is F-finite. Then R is strongly F-regular if and only if R if F-split and is a finite product of D-simple rings.*

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