Algebraic Topology Homework 2

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§ *Problems from 1.1*

EXERCISE 2. Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof: Let X be a topological space with $x_0, x_1 \in X$ and suppose $h, g : [0,1] \longrightarrow X$ are homotopic paths such that $h(0) = g(0) = x_1$ and $h(1) = g(1) = x_0$. We would like to show that $\beta_h = \beta_g$, i.e. that h and g both induce the same homomorphism $\pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$. Change of basepoint homomorphisms are isomorphisms by Proposition 1.5 in Hatcher, so this is equivalent to showing that $\beta_h \circ \beta_g^{-1} = \beta_g^{-1} \circ \beta_h = \text{id}$. But since $\beta_g^{-1} = \beta_{\overline{g}}$, this is a simple calculation. For any $[f] \in \pi_1(X, x_0)$, we have

$$\beta_h \beta_{\overline{g}}([f]) = \beta_h([\overline{g} \cdot f \cdot g]) = [h \cdot \overline{g} \cdot f \cdot g \cdot \overline{h}] = [f]$$

since $h \simeq g$, and similarly

$$\beta_{\overline{g}} \beta_h([f]) = \beta_{\overline{g}}([h \cdot f \cdot \overline{h}]) = [\overline{g} \cdot h \cdot f \cdot \overline{h} \cdot g] = [f].$$

This means $\beta_{\overline{g}}$ is the inverse of both β_g and β_h, and by the uniqueness of inverses, we conclude $\beta_h = \beta_g$. $\square$

EXERCISE 5. Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints.']

Proof: We prove the following chain of implications:

$(a) \implies (b)$: Suppose that for every map $S^1 \to X$ is nullhomotopic to a function $c_{x_0} : S^1 \to X$ where $c_{x_0}(x) = x_0$, i.e. suppose there exists a homotopy $f_t : S^1 \to X$ where $f_1 = c_{x_0}$ and where f_0 is any map $S^1 \to X$. Since (D^2, S^1) is a CW-pair, it has the homotopy extension property, and thus f extends to $f' : D^2 \times [0,1] \to X$ where $f'|_{S^1} = f$. Thus every map $S^1 \to X$ extends to $D^2 \to X$.

$(b) \implies (c)$: Suppose that $[f] \in \pi_1(X, x_0)$. $f(0) = f(1) = x_0$, meaning that we can interpret f instead as a function $f : S^1 \to X$. This f can be extended to a function $f' : D^2 \to X$, but since D^2 is contractible, f' is nullhomotopic. Thus, f is also nullhomotopic, and $[f] = [0]$. This is true of any arbitrary $[f] \in \pi_1(X, x_0)$, and so we know that $\pi_1(X, x_0)$ is trivial. Noticing that the choice of x_0 was arbitrary, we conclude that $\pi_1(X) = 0$.

$(c) \implies (a)$: This argument is very similar to the previous one. Each map $S^1 \to X$ can be interpreted as a loop in X, and since we assume $\pi_1(X) = 0$, every loop is homotopic to the trivial map. Thus every map $S^1 \to X$ is homotopic to a constant map in X.

A space X is "simply connected" if and only if it is path connected and $\pi_1(X) = 0$. As we just showed, this is equivalent to "X is path connected and every $S^1 \to X$ is nullhomotopic". If we interpret two maps $f, g : S^1 \to X$ as loops in X with basepoints x_0 and x_1 respectively, then f and g are homotopic in X by the homotopy that first shrinks f to the constant map c_{x_0}, moves x_0 along a path to x_1, and finally deforms c_{x_1} into g. Thus, any two maps $S^1 \to X$ are homotopic. We conclude that X is simply connected if and only if all maps $S^1 \to X$ are homotopic. $\square$

EXERCISE 6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof: We first show that Φ is onto. Let $[f]$ be a member of $[S^1, X]$. Since every map $S^1 \to X$ can be regarded as a loop in X, f is a loop based at some point $x_1 \in X$. Because X is path connected, there must be some path γ from x_1 to x_0. The map $g = \gamma \cdot f \cdot \overline{\gamma}$ is a continuous path that begins and ends at x_0, and is therefore a loop around x_0. Since f and g are homotopic, $[g] = [f]$. Since $[g] \in \pi_1(X, x_0)$, $\Phi([g]) = [f]$. We conclude that Φ is surjective.

We now show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugates. We show the forward implication first.

Assume that $\Phi([f]) = \Phi([g])$. This means that f and g are in the same equivalence class of $[S^1, X]$, so there must exist a homotopy $\varphi : S^1 \times [0, 1] \to X$, where $\varphi_0 = f$ and $\varphi_1 = g$. The induced homeomorphisms φ_{0*} and φ_{1*} then satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$, where $\beta_h : \pi_1(X, \varphi_1(s_0)) \to \pi_1(X, \varphi_0(s_0))$ and h is the loop $\varphi_t(s_0)$. This means

$$\varphi_{0*}([1]) = [g \cdot 1] = [g] = \beta_h \varphi_{1*}([1]) = \beta_h([f]) = [hf\overline{h}]$$

where $[1]$ is the equivalence class isomorphic to $1 \in \mathbb{Z}$.

We now show the reverse implication. Assume that $[g] = [h][f][\overline{h}]$ in $\pi_1(X, x_0)$, i.e. assume $[f]$ and $[g]$ are conjugates. We want to show that $[f] = [g]$, or that $[f] = [h][f][\overline{h}]$. Consider the following function:

$$F : [0, 1] \times S^1 \to X \qquad F_s(t) = \begin{cases} h(3t + s) & 0 \le t \le \frac{1-s}{3} \\ f\left(\frac{3}{1+2s}(t - \frac{1-s}{3})\right) & \frac{1-s}{3} \le t \le \frac{2+s}{3} \\ \overline{h}\left(3(t - \frac{2+s}{3})\right) & \frac{2+s}{3} \le t \le 1 \end{cases}$$

Since $F_0 = h \cdot f \cdot \overline{h}$, $F_1 = f$, and F is continuous by the pasting lemma, F is a homotopy between f and $h \cdot f \cdot \overline{h}$. Thus, $[f] = [h][f][\overline{h}]$ in $[S^1, X]$ and we conclude that $\Phi([g]) = \Phi([f])$.

$\square$

EXERCISE 10. From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $Y \times \{x_0\}$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Proof: Let $[f]$ and $[g]$ be loops based at x_0 in X and y_0 in Y, respectively. Next consider the following homotopies:

$$f_t(s) = \begin{cases} x_0 & 0 \leq s \leq \frac{t}{2} \\ f(2s) & \frac{t}{2} \leq s \leq \frac{1+t}{2} \\ x_0 & \frac{1+t}{2} \leq s \leq 1 \end{cases}$$

and

$$g_t(s) = \begin{cases} y_0 & 0 \leq s \leq \frac{t}{2} \\ g(2s) & \frac{t}{2} \leq s \leq \frac{1+t}{2} \\ y_0 & \frac{1+t}{2} \leq s \leq 1 \end{cases}$$

Here f_0 is the path that transverses f and then stays at x_0. f_1 is the path that stays at x_0 for half the interval and then transverses f. g_0 is the path that stays at y_0 and then transverses g, and finally g_1 is the path that transverses g and then stays at y_0.

Next since $\pi_1(X, x_0) \times \pi_1(Y, y_0) \approx \pi(X \times Y, (x_0, y_0))$ and $h_t(s) = (f_t(s), g_t(s))$ is an element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$. However, since $f_0 \cdot g_0 \simeq f \cdot g$ and $f_1 \cdot g_1 \approx g \cdot f$, we conclude that $f \cdot g \simeq g \cdot f$. $\square$

EXERCISE 15. Given a map $f : X \longrightarrow Y$ and a path $h : I \longrightarrow X$ from x_0 to x_1, show that $f_*\beta_h = \beta_{fh} f_*$.

Proof: Let $[\alpha] \in \pi_1(X, x_1)$ be an arbitrary equivalence class of loops based at x_1. By definition,

$$f_*(\beta_h([\alpha])) = f_*[h \cdot \alpha \cdot \overline{h}] = [f \circ (h \cdot \alpha \cdot \overline{h})]$$

and

$$\beta_{fh}(f_*([\alpha])) = \beta_{fh}([f \circ \beta]) = [(f \circ h) \cdot (f \circ \alpha) \cdot \overline{(f \circ h)}].$$

However, up to a possible reparameterization, $(f \circ h) \cdot (f \circ \alpha) \cdot \overline{(f \circ h)}$ and $f \circ (h \cdot \alpha \cdot \overline{h})$ are identical paths on Y. In fact, if we perform concatenation from consistently, then they are identical without *any* reparameterization.

To see this, concatenate from right to left without loss of generality. For $t \in [0, 1/2]$ we have

$$f \circ (h \cdot (\alpha \cdot \overline{h}))(t) = f(h(t)) = (f \circ h) \cdot \big((f \circ \alpha) \cdot \overline{(f \circ h)}\big)(t),$$

and we have something similar for $t \in [1/2, 3/4]$ and $t \in [3/4, 1]$. As the representatives of the resulting equivalence classes above are homotopic, the diagram shown by Hatcher commutes. $\square$

EXERCISE 16.

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

(c) $X = S^1 \times D^2$ with A the interlocked circle in the solid torus.

Proof: (a) The space $\mathbb{R}^3$ contracts to the origin via the homotopy $F : \mathbb{R}^3 \times I \longrightarrow \mathbb{R}^3$ defined $F_t(x) =$

$(1-t)x$, and hence has trivial fundamental group. By theorem 1.7, the circle S^1 has fundamental group isomorphic to $\mathbb{Z}$. Therefore there exists no inclusion $\pi_1(S^1, x_0) \longrightarrow \pi_1(\mathbb{R}^3, x_0)$, and hence by Proposition 1.17 in Hatcher, there exists no retraction of X onto a circle.

(b) In this case, we have that $\pi_1(X, x_0) \cong \mathbb{Z}$ by Proposition 1.12 and $\pi_1(A, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ by the same proposition. Note that because both of these spaces are path connected, the choice of basepoint does not matter. As any map $\mathbb{Z}$-linear morphism $\mathbb{Z}^2 \longrightarrow \mathbb{Z}$ will have nonzero kernel, there does not exist an injection $\mathbb{Z}^2 \longrightarrow \mathbb{Z}$, and hence there is no retraction $r : A \longrightarrow X$.

(c) The subset $A \subset X$ is contractible by pulling the two ends of the loops through each other. Thus, the map $\iota_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ induced by the inclusion $\iota : A \longrightarrow X$ takes every loop in A to something homotopic to the trivial loop in X. This means that the induced map ι_* is trivial, and in particular is not injective, hence by Proposition 1.17 X does not retract onto A.

(d) Suppose we did have a retraction $r : X \longrightarrow A$. In that case, the composition

$$D^2 \hookrightarrow X \xrightarrow{r} A \longrightarrow S^1$$

would also be a retraction. However, this is impossible, as D^2 is contractible while S^1 is not. Hence there is no such retraction r.

(e)

(f) Since X deformation retracts onto its central circle (a fact we used on the last homework) both X and A have fundamental group isomorphic to $\mathbb{Z}$. Let $x_0 \in A \subset X$ be a basepoint, and choose generators $\gamma \in \pi_1(A, x_0)$ and $\lambda \in \pi_1(X, x_0)$ for the fundamental groups. The image $\iota_*([\gamma])$ of $[\gamma]$ under the map induced by the inclusion $\iota : A \longrightarrow X$ is then equal to $2[\lambda]$ since traversing around the boundary circle corresponds to traversing around the central circle twice. Hence the induced map ι_* is really a map $\mathbb{Z} \longrightarrow \mathbb{Z}$ which sends $2 \mapsto 1$. This is impossible, as $2 \mapsto 1$ is not possible for such a group homomorphism as all $\mathbb{Z}$-linear maps $\mathbb{Z} \longrightarrow \mathbb{Z}$ are defined $1 \mapsto nz$ for some $n \in \mathbb{Z}$. Hence, there is no retraction of X onto the boundary A.

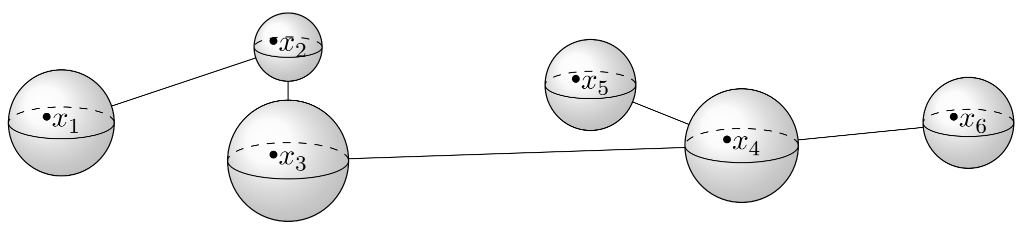
$\square$

§ ***Problems from 1.2***

EXERCISE 3. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

Proof: Let $\{x_1, ..., x_k\}$ be a finite collection of points in $\mathbb{R}^n$. We can place a $n-1$ sphere around every point removed from $\mathbb{R}^n$, ensuring that the radius is small enough to contain only one "hole", and connect every sphere with another via a straight path. Call this space X. Just as $\mathbb{R}^n$ with a single point removed deformation retracts to S^{n-1}, I claim that $\mathbb{R}^n - \{x_1, ..., x_k\}$ deformation retracts onto X. Every point inside of one of the spheres retracts onto the sphere, and choosing to map every point in $\mathbb{R}^n$ that is not in X onto the nearest point in X retains continuity.

Every path-connected open set on this surface has the trivial fundamental group, so by Van Kampen's theorem, the fundamental group of X is also trivial. Since $\mathbb{R}^n - \{x_1, ..., x_k\}$ deformation retracts onto X, we conclude that $\pi_1(\mathbb{R}^n - \{x_1, ..., x_k\}) = [1]$. □



The space X defined above