Infinite Groups Example Sheet 1

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§ *Chapter 1*

EXERCISE 1. Show directly that every finite group is directly finite (without using the fact that sofic groups are directly finite)

Proof: Note first that G is a basis for $K[G]$ and therefore $K[G]$ has finite dimension exactly when G is a finite group.

Given an element $r \in K[G]$, define the map $\alpha_r : K[G] \longrightarrow K[G]$ to be left multiplication by r, i.e. $\alpha_r(s) = r \cdot s$. This map is K-linear due to the distributive and associative properties on $K[G]$.

Suppose we have $r, s \in K[G]$ such that $r \cdot s = 1$. Given any $x \in K[G]$ we have that $\alpha_r(\alpha_s(x)) = r(sx) = (rs)x = x$, hence the composition $\alpha_r \circ \alpha_s$ is the identity on $K[G]$, and is in particular both injective and surjective. Recall the following two general facts regarding compositions of functions:

- if $f \circ g$ is injective then g is injective
- if $f \circ g$ is surjective then f is surjective

Hence, α_r is surjective and α_s is injective. The rank-nullity theorem tells us that an endomorphism on a finite dimensional vector space is injective if and only if it is surjective, so both α_r and α_s are automorphisms of $K[G]$. Since inverses are unique and $\alpha_r \circ \alpha_s = \mathrm{id}_{K[G]}$, we conclude that α_r and α_s are inverses. Applying these maps to the identity yields that

$$rs = \alpha_r \circ \alpha_s(1) = 1 = \alpha_s \circ \alpha_r(1) = sr,$$

so we are done. □

EXERCISE 2. Let X_1 and X_2 be finite sets. Show that there is an injective homomorphism $\Phi : \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2) \longrightarrow \mathrm{Sym}(X_1 \times X_2)$, given by:

$$\Phi(\sigma_1, \sigma_2)(x_1, x_2) = \big(\sigma_1(x_1), \sigma_2(x_2)\big).$$

Deduce that the direct product of two sofic groups is sofic.

EXERCISE 3. Let X and Y be finite sets. Prove that $F(X) \cong F(Y)$ if and only if $|X| = |Y|$.

Proof: Suppose first that $|X| = |Y|$, which by definition means there is some bijection $f : X \longrightarrow Y$. Let $\iota : X \longrightarrow F(Y)$ be the function obtained by composing f with the inclusion $Y \hookrightarrow F(Y)$. We show that the pair $(\iota, F(Y))$ satisfy the universal property of a free group over X, and therefore $F(Y) \cong F(X)$.

Let G be an arbitrary group and fix a function $\phi : X \longrightarrow G$. Define the map $\Phi : F(X) \longrightarrow G$ by $y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_2} \mapsto \phi(x_1)^{\varepsilon_1} \phi(x_2)^{\varepsilon_2} \ldots \phi(x_n)^{\varepsilon_n}$, where $x_i = f^{-1}(y_i)$ and $\varepsilon_i \in \{\pm 1\}$. Note that for each $y \in Y$ there is a unique $x \in X$ such that $x = f^{-1}(y)$ since f is bijective.

The map Φ is a group morphism:

- $\Phi(1) = \Phi(yy^{-1}) = \phi(f^{-1}(y)) \cdot \phi(f^{-1}(y))^{-1} = 1_G$,
- $\Phi\left(y_1^{\varepsilon_1} ... y_m^{\varepsilon_m} z_1^{\delta_1} ... z_n^{\delta_n}\right) = \phi\left(f^{-1}(y_1)\right)^{\varepsilon_1} ... \phi\left(f^{-1}(y_m)\right)^{\varepsilon_m} \phi\left(f^{-1}(z_1)\right)^{\delta_1} ... \phi\left(f^{-1}(z_n)\right)^{\delta_n}$
$= \Phi(y_1^{\varepsilon_1} ... y_m^{\varepsilon_m} z_1^{\delta_1} ... z_n^{\delta_n})$.

Suppose we had another group homomorphism $\Psi : F(Y) \longrightarrow G$ extending ϕ. We then would have

$$\Psi(y^{-1}) = \Psi(y)^{-1} = \phi\left(f^{-1}(y)\right) = \Phi(y)^{-1} = \Phi(y^{-1})$$

for each $y \in Y$, and since

$$\Psi\left(y_1^{\varepsilon_1} ... y_n^{\varepsilon_n}\right) = \Psi(y_1)^{\varepsilon_1} \cdot ... \cdot \Psi(y_n)^{\varepsilon_n} = \phi\left(f^{-1}(y_1)\right)^{\varepsilon_1} ... \phi\left(f^{-1}(y_n)\right)^{\varepsilon_n} = \Phi\left(y_1^{\varepsilon_1} ... y_n^{\varepsilon_n}\right),$$

we get that $\Psi = \Phi$. Thus, Φ uniquely extends ϕ and therefore $F(Y)$ satisfies the universal property of a free group over X. This implies that $F(X) \cong F(Y)$.

Let us now consider the forward implication. One consequence of the universal property of free groups is that, for any group G, we have a one-to-one correspondence of functions $f : X \longrightarrow G$ and group homomorphisms $\Phi : F(X) \longrightarrow G$. Consider the case that $G \cong \mathbb{Z}/2\mathbb{Z}$. The map sending a function $f : X \longrightarrow G$ to $f^{-1}(0)$ yields a bijection between the possible functons $X \longrightarrow G$ and the subsets of X, so $\left|2^{|X|}\right| = |\mathrm{Hom}_{\mathsf{Set}}(X, \mathbb{Z}/2\mathbb{Z})|$. These facts together with the assumption that $F(X) \cong F(Y)$ give us

$$\begin{aligned} \left|2^{|X|}\right| &= |\mathrm{Hom}_{\mathsf{Set}}(X, \mathbb{Z}/2\mathbb{Z})| = \left|\mathrm{Hom}_{\mathsf{Grp}}(F(X), \mathbb{Z}/2\mathbb{Z})\right| \\ &= \left|\mathrm{Hom}_{\mathsf{Grp}}(Y, \mathbb{Z}/2\mathbb{Z})\right| = |\mathrm{Hom}_{\mathsf{Set}}(Y, \mathbb{Z}/2\mathbb{Z})| = \left|2^{|Y|}\right|. \end{aligned}$$

Up until now, we have not needed the finiteness of $|X|$ and $|Y|$. However, the statement "$\left|2^{|X|}\right| = \left|2^{|Y|}\right| \implies |X| = |Y|$" is true only when $|X| = |Y|$ (in fact, I think this statement is independent of ZFC when X and Y are infinite). In any case, by invoking the finiteness of X and Y, we are done. $\square$

EXERCISE 10 Another problem