Topics in Commutative Algebra

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1. The Divisor Class Group

§ 1.1 *Preliminaries*

Let R be a domain and K be its fraction field.

Definition 1 *A ring R is integrally closed if whenever $x \in \mathsf{K}$ and $x^n + r_1 x^{n-1} + \cdots + r_n = 0$ for some r_i in R, then x is in R.*

Recall:
R_i: R_P is a regular local ring for every P in $\mathrm{Spec}(R)$ with $\mathrm{ht}(P) \leq i$.
S_i: $\mathrm{depth}(R_P) \geq \min\{i, \dim(R_P)\}$, for every P in $\mathrm{Spec}(R)$.

Theorem 1 (Serre's criterion) *A Noetherian domain R is integrally closed if and only if it satisfies R_1 and S_2.*

Remark 1 *S_2 means that principal ideals are unmixed i.e. any nonzero x in R has associated primes of height 1 only (i.e. no embedded primes).*

Proof: Let $Q \in \mathrm{Ass}(R/xR)$ be such that $\mathrm{ht}(Q) \geq 2$. Since x is a non-zerodivisor, $\mathrm{depth}(R_Q) = 1$ which is less than $\min\{2, \dim(R_Q)\}$. So R doesn't satisfy S_2.

Conversely assume that all principal ideals are unmixed, but R does not satisfy S_2. Then there is a Q in $\mathrm{Spec}(R)$ such that $\mathrm{depth}(R_Q) < \min\{2, \dim(R_Q)\}$. Then $\mathrm{depth}(R_Q) = 1$ i.e. Q is associated to any non-zero element in Q. But $\dim(R_Q) \geq 2$, which is a contradiction . $\square$

To Paraphrase: A ring R is integrally closed if and only if (a) R_P is a DVR for each prime P of height 1 in R and (b) principal ideals in R are unmixed .

Since $R_{\mathfrak{p}}$ is a DVR, there is an associated valuation $v_{\mathfrak{p}} : \mathsf{K} \longrightarrow \mathbb{Z}$ such that

$$R_{\mathfrak{p}} = \{\alpha \in \mathsf{K} : v_{\mathfrak{p}}(\alpha) \geq 0\}.$$

Since $R_{\mathfrak{p}}$ is a DVR, $\mathfrak{p}R_{\mathfrak{p}} = (tR_{\mathfrak{p}})$ (t is called the uniformizing parameter). Moreover, given any r in R, there is a unique n such that $r \in (t^n)_{\mathfrak{p}} \setminus (t^{n+1})_{\mathfrak{p}}$. In such a case, $v_{\mathfrak{p}}(r) = n$. For $\alpha = a/b$ in K, where a, b are in $R, v_{\mathfrak{p}}(\alpha) = v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b)$.
Note that $v_{\mathfrak{p}}(r) = \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/R_{\mathfrak{p}}r)$.

Primary Decomposition

We discuss the primary decompositions of principal ideals in Noetherian integrally closed domains R. Let x be a nonzero element in R. The ideal (x) has a primary decomposition:

$$(x) = q_1 \cap q_2 \cap \cdots \cap q_n$$

where q_i is $\mathfrak{p}_i$-primary and $\mathrm{Ass}(R/xR) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$.
It follows from the above discussion that

(i) All $\mathfrak{p}_i$ have height 1 (and hence are minimal over x).

(ii) $q_i = xR_{\mathfrak{p}_i} \cap R$ is unique.

Now, there is an n_i such that $xR_{\mathfrak{p}_i} = \mathfrak{p}_i^{n_i} R_{\mathfrak{p}_i}$ and hence $q_i = \mathfrak{p}_i^{(n_i)}$, which by definition is the contraction of $\mathfrak{p}_i^{n_i} R_{\mathfrak{p}_i}$ to R. But $n_i = v_{\mathfrak{p}_i}(x)$. Hence

$$(x) = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathfrak{p}^{(v_{\mathfrak{p}}(x))}.$$

This intersection makes sense as $v_{\mathfrak{p}}(x) = 0$ if x is not in $\mathfrak{p}$. $(*)$

Convention: $\mathfrak{p}^{(0)} = R$.

§ 1.2 *The Class Group*

Let R be an integrally closed Noetherian domain, $X^1(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}(\mathfrak{p}) = 1\}$. Let $X(R)$ be the free abelian group on $X^1(R)$. A typical element $z \in X(R)$ has the form

$$z = \sum_{\mathfrak{p} \in X^1(R)} n_{\mathfrak{p}} \mathfrak{p}$$

where $n_{\mathfrak{p}} \in \mathbb{Z}$, such that at most finitely many $n_{\mathfrak{p}}$ are non-zero.

An element in $X(R)$ is called a *divisor* (*effective* if $n_{\mathfrak{p}} \geq 0$ for every $\mathfrak{p}$). If $x \in R$, we define $\mathrm{div}(x) = \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(x)\mathfrak{p}$. (By $*$, this is a finite sum).
More generally, if I is an ideal in R, we may still speak of $v_{\mathfrak{p}}(I)$, as $IR_{\mathfrak{p}}$ will be a power of the maximal ideal of $R_{\mathfrak{p}}$, and we define

$$\mathrm{div}(I) = \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(I)\mathfrak{p}.$$

If the ring R needs to be specified, we write $\mathrm{div}_R(I)$ instead of $\mathrm{div}(I)$.

We can extend the definition of divisor to elements of K. Let $\alpha = a/b \in \mathsf{K}$, a, b in R. Define $\operatorname{div}(\alpha) = \operatorname{div}(a) - \operatorname{div}(b)$. We need to check that this is well-defined. Suppose that $a/b = c/d$. Then $ad = bc$. Hence for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $v_{\mathfrak{p}}(ad) = v_{\mathfrak{p}}(bc)$ i.e. $v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(d) = v_{\mathfrak{p}}(b) + v_{\mathfrak{p}}(c)$ i.e. $v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b) = v_{\mathfrak{p}}(c) - v_{\mathfrak{p}}(d)$. Hence, $\sum\limits_{\mathfrak{p} \in X^1(R)} [v_{\mathfrak{p}}(a) - v_{\mathfrak{p}}(b)]\mathfrak{p} = \sum_{\mathfrak{p} \in X^1(R)} [v_{\mathfrak{p}}(c) - v_{\mathfrak{p}}(d)]\mathfrak{p}$ and so

$$\operatorname{div}(a) - \operatorname{div}(b) = \operatorname{div}(c) - \operatorname{div}(d)$$

i.e. $\operatorname{div}(\alpha)$ is well-defined.

The set of all such principal divisors, denoted $P(R)$, is actually a subgroup. For $\operatorname{div}(\alpha) + \operatorname{div}(\beta) = \operatorname{div}(\alpha \cdot \beta)$, and $\operatorname{div}((\alpha)^{-1}) = -\operatorname{div}(\alpha)$.

Definition 2
1. The class group of R is defined to be $Cl(R) := X(R)/P(R)$.
2. Two divisors D_1 and D_2 are said to be linearly equivalent if $D_1 - D_2 \in P(R)$.

Lemma 2 *Any divisor is linearly equivalent to an effective divisor.*

Proof: Write $D = \sum n_{\mathfrak{p}}^+ \mathfrak{p} - \sum n_{\mathfrak{p}}^- \mathfrak{p}$, where $n_{\mathfrak{p}}^+ \geq 0$ and $n_{\mathfrak{p}}^- > 0$. Then $I = \bigcap \mathfrak{p}^{(n_{\mathfrak{p}}^-)}$ is not zero since $n_{\mathfrak{p}}^-$ is not zero for only finitely many $\mathfrak{p} \in X^1(R)$. Choose nonzero x in I. Then $\operatorname{div}(x) - \sum n_{\mathfrak{p}}^- \mathfrak{p}$ is effective since $v_{\mathfrak{p}}(x) \geq n_{\mathfrak{p}}^-$ for every $\mathfrak{p}$. Hence $D + \operatorname{div}(x)$ is effective. □

Proposition 3 (Localization sequence) *Let W be a multiplicatively closed subset of R. Then there is a short exact sequence*

$$0 \longrightarrow H \longrightarrow Cl(R) \longrightarrow Cl(R_W) \longrightarrow 0,$$

where $H = < [\mathfrak{p}] : \mathfrak{p} \cap W \neq \emptyset >$.

Proof: Define $X(R) \xrightarrow{\pi} X(R_W) \longrightarrow 0$ by

$$\mathfrak{p} \mapsto \begin{cases} 0 \text{ for } \mathfrak{p} \cap W \neq \emptyset \\ \mathfrak{p}_W \text{ for } \mathfrak{p} \cap W = \emptyset \end{cases}$$

Claim: $P(R) \hookrightarrow P(R_W)$.
To prove the claim, consider $\operatorname{div}(a)$, a in R. Write

$$(a) = \mathfrak{p}_1^{n_1} \cap \ldots \cap \mathfrak{p}_k^{n_k} \cap \mathfrak{q}_1^{m_1} \cap \ldots \cap \mathfrak{q}_l^{m_l} \qquad (*)$$

where $\mathfrak{p}_i \cap W \neq \emptyset$ and $\mathfrak{q}_i \cap W = \emptyset$. Then $\pi(\operatorname{div}(a)) = \sum_1^l m_i(\mathfrak{q}_i)_W$.

Localizing $*$ at W, we get $\operatorname{div}(\frac{a}{1}) = \pi(\operatorname{div}(a))$. Hence we get an induced map $\pi : Cl(R) \longrightarrow Cl(R_W)$. Obviously $\pi(H) = 0$, hence $H \subseteq \operatorname{Ker}(\pi)$. We want to prove $H \supseteq \operatorname{Ker}(\pi)$.

Consider a divisor D representing a class $[D] \in Cl(R)$ such that $\pi([D]) = 0$. We may assume by lemma 2 that D is an effective divisor. Let $D = \sum_{i=1}^{l} m_i\mathfrak{q}_i + \sum_{j=1}^{k} m_{l+j}\mathfrak{q}_{l+j}$, where $\mathfrak{q}_i \cap W = \emptyset$, $i = 1, 2, \ldots, l$ and $\mathfrak{q}_{l+j} \cap W \neq \emptyset$, $j = 1, 2, \ldots, k$. Then, $0 = \pi([D]) = \sum_1^l m_i[(\mathfrak{q}_i)_W]$. So there is an element $a \in R_W$ such that $\operatorname{div}_{R_W}(a) = \sum m_i(\mathfrak{q}_i)_W$.

For any unit u, $\operatorname{div}(ua) = \operatorname{div}(a)$. Hence without loss of generality we can assume that a is in R. Since $\operatorname{div}_{R_W}(a) = \sum m_i(\mathfrak{q}_i)_W$, $aR = (\bigcap_{i=1}^{l} \mathfrak{q}_i^{(m_i)}) \cap (\bigcap_{j=1}^{m} \mathfrak{p}_j^{(k_j)})$, where $\operatorname{ht}(\mathfrak{p}_j) = 1$ and $\mathfrak{p}_j \cap W \neq \emptyset$. Hence $[\operatorname{div}_R(a)] = \sum_{i=1}^{l} m_i[\mathfrak{q}_i] + \sum_{j=1}^{m} k_j[\mathfrak{p}_j]$. But $[\operatorname{div}_R(a)] = 0$ and $\sum_{j=1}^{m} k_j[\mathfrak{p}_j] \in H$. Hence $\sum_{i=1}^{l} m_i[\mathfrak{q}_i] \in H$ which implies that $[D] \in H$. $\square$

§ 1.3 *UFDs and Class Group:*

We are aiming for the following theorem

Theorem 4 *Let R be a Noetherian integrally closed domain. Then R is a UFD if and only if $Cl(R) = 0$.*

Definition 3 *A ring R is a UFD if every element r in R factors uniquely, up to order and units, into irreducibles.*

Aside: Let R be a domain. Then $\bigcap_{\operatorname{depth}(R_\mathfrak{p})=1} R_\mathfrak{p} = R$.
If R is integrally closed, $\operatorname{depth}(R_\mathfrak{p}) = 1$ if and only if $\dim(R_\mathfrak{p}) = 1$. So $\bigcap_{\mathfrak{p}\in X^1(R)} R_\mathfrak{p} = R$.

Proposition 5 *Let R be a Noetherian ring. Then R is a UFD if and only if every prime ideal of height 1 is principal.*

Proof: Note that in a Noetherian ring every nonzero prime ideal contains an irreducible element. This follows from the fact that every element is a product of irreducible elements.

Suppose that R is a UFD and $\mathfrak{p}$ is a prime ideal in R of height 1. Then there is an irreducible element π in $\mathfrak{p}$. Since R is a UFD, (π) is a prime and hence is equal to $\mathfrak{p}$ since $\operatorname{ht}(\mathfrak{p}) = 1$. To prove the converse, it is enough to show that every irreducible element π generates a prime ideal. Let $\mathfrak{p}$ be a minimal prime containing π. Then, since $\operatorname{ht}(\mathfrak{p}) = 1$ and π is irreducible, $\mathfrak{p} = (\pi)$. $\square$

Lemma 6 *Let D be an effective divisor. Then $[D] = 0$ in Cl(R) if and only if $D = \operatorname{div}(a)$ for some a in R.*

Proof: If $D = \mathrm{div}(a)$, then by definition, $[D] = 0$ in $Cl(R)$. Conversely, suppose $[D] = 0$. Then $D = \mathrm{div}(\frac{a}{b})$ for some $a, b \in R$. Since $D = \mathrm{div}(a) - \mathrm{div}(b)$ is effective, for any $\mathfrak{p} \in X^1(R)$, $v_{\mathfrak{p}}(a) \geq v_{\mathfrak{p}}(b)$. Hence $a \in \mathfrak{p}^{(v_{\mathfrak{p}}(b))}$ for every $\mathfrak{p}$ which implies that $a \in (b)$, i.e. $a/b \in R$. $\square$

Proof of Theorem 4: Let us first assume that R is a UFD. If $\mathfrak{p}$ is a prime of height 1 in R, then there is an $a \in R$ such that $\mathfrak{p} = (a)$ i.e. $\mathfrak{p} = \mathrm{div}(a)$. This implies that $[\mathfrak{p}] = 0$ in $Cl(R)$ and hence $Cl(R) = 0$. For the converse, let $\mathfrak{p}$ be in $X^1(R)$. Then $[\mathfrak{p}] = 0$ in $Cl(R)$ and $\mathfrak{p}$ is an effective divisor. Hence by Lemma 6, there is an $a \in R$ such that $\mathfrak{p} = \mathrm{div}(a)$, i.e. $\mathfrak{p}$ is principal. $\square$

Corollary 7 (Nagata's Lemma) *Let R be an integrally closed domain, W be a multiplicatively closed subset of R generated by prime elements. Then $Cl(R) \simeq Cl(R_W)$. In particular, if R_W is a UFD, then R is a UFD.*

Proof: The second statement follows from the first by Theorem 4. We know that the sequence $0 \longrightarrow H \longrightarrow Cl(R) \longrightarrow Cl(R_W) \longrightarrow 0$ is exact, where $H =< [\mathfrak{p}] : \mathfrak{p} \cap W \neq \emptyset >$. Suppose $0 \neq w \in \mathfrak{p} \cap W$. Then $w = w_1 w_2 \ldots w_r$ where w_i are prime elements. Hence $w_i \in \mathfrak{p}$ for some i and hence $\mathfrak{p} = (w_i)$. This implies that $[\mathfrak{p}] = 0$ in $Cl(R)$ i.e. $H = 0$ which proves the isomorphism. $\square$

Corollary 8 *If R is a Noetherian domain, x a prime element in R such that R_x is a UFD, then R is a UFD.*

Proof: If R is integrally closed, then the result is immediate from Cor. 7. But R is integrally closed under the given hypothesis. $\square$

Example 1 If $n \geq 5$, then $\mathbb{C}[X_1, X_2, \ldots, X_n]/(X_1^2 + \cdots + X_n^2)$ is a UFD.
We can write $X_1^2 + X_2^2 = UV$ where $U = X_1 + iX_2$ and $V = X_1 - iX_2$. Then

$$R := \mathbb{C}[X_1, \ldots, X_n]/(X_1^2 + \cdots + X_n^2) \simeq \mathbb{C}[U, V, X_3, \ldots, X_n]/(UV + X_3^2 + \cdots + X_n^2).$$

We claim that U is a prime in R i.e. $R/(U)$ is a domain.
But $R/(U) \simeq \mathbb{C}[V, X_3, \ldots, X_n]/(\sum_{i=3}^n X_i^2) = \mathbb{C}[X_3, \ldots, X_n][V]/(\sum_{i=3}^n X_i^2)$ and for $n \geq 3$, $X_1^2 + \cdots X_n^2$ is a prime element. Hence, $R/(U)$ is a domain. Thus U is a prime element. Hence, in order to prove that R is a UFD, by Cor. 8, it is enough to prove that R_U is a UFD. Now,

$$R_U \simeq \mathbb{C}[U, U^{-1}, V, X_3, \ldots, X_n]/(V + U^{-1}(\sum_{i=3}^n X_i^2)) \simeq \mathbb{C}[U, U^{-1}, X_3, \ldots, X_n]$$

$= \mathbb{C}[U, X_3, \ldots, X_n]_U$ which is a UFD by Theorem 4 and Prop. 3. $\square$

Aside: We used the following facts:

1. For $n \geq 3, X_1^2 + \cdots + X_n^2$ is a prime element.
Proof: If $X_1^2 + \cdots + X_n^2 = l_1 l_2$, we may assume that both l_1 and l_2 are linear by homogeneity. Differentiating both sides partially with respect to X_i, we get $2X_i = (l_1)_{X_i} l_2 + l_1 (l_2)_{X_i}$ i.e. $(X_1, X_2, \ldots, X_n) \subseteq (l_1, l_2)$, which contradicts the fact that $\mathrm{ht}((l_1, l_2)) \leq 2$. $\square$

2. $\mathbb{C}[U, U^{-1}, V, X_3, \ldots, X_n]/(V + U^{-1}(X_3^2 + \cdots + X_n^2)) \simeq \mathbb{C}[U, U^{-1}, X_3, \ldots, X_n]$
Proof: More generally, $R[X_1, \ldots, X_n]/(X_1 - f(X_2, \ldots, X_n)) \simeq R[X_2, \ldots, X_n]$ since $R[X_1, \ldots, X_n] \simeq R[X_1', \ldots, X_n]$ where $X_1' = X_1 - f(X_2, \ldots, X_n)$. $\square$

Remark 2 If $n = 4$, then R is not a UFD. We can write $X_1^2 + X_2^2 = UV$, $X_3^2 + X_4^2 = -WZ$. Then $R \simeq \mathbb{C}[U, V, W, Z]/(UV - WZ)$. Then $\overline{UV} = \overline{WZ}$ where $\overline{U}, \overline{V}, \overline{W}, \overline{Z}$ are all irreducible.

Example 2 $Cl(\mathbb{C}[U, V, W, Z]/(UV - WZ)) \simeq \mathbb{Z}$. (Proof postponed)

Example 3 In general, if k is a field, $X = (X_{ij})_{r \times s}$ is an $r \times s$ matrix of variables and $I_k(X)$ be the ideal generated by the $k \times k$ minors of X for some $k \leq r$, $\mathsf{k}[X_{ij}]/I_k(X)$ is an integrally closed domain and its class group is $\mathbb{Z}$.

Theorem 9 *Let R be an integrally closed Noetherian domain. Then $R[X]$ is integrally closed and the map $\Phi : Cl(R) \longrightarrow Cl(R[X])$, defined as $\Phi([\mathfrak{p}]) = [\mathfrak{p}[X]]$ is an isomorphism.*

Proof: Define $\tilde{\Phi} : X(R) \longrightarrow X(R[X])$ by $\mathfrak{p} \mapsto \mathfrak{p}[X]$ for every prime $\mathfrak{p}$ of height 1 in R. We want to show that $\tilde{\Phi}$ defines a map $\Phi : Cl(R) \longrightarrow Cl(R[X])$. It is enough to show that $\tilde{\Phi}(\mathrm{div}_R(a)) = \mathrm{div}_{R[X]}(a)$. In order to show this, write $(a) = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} \mathfrak{p}^{(v_\mathfrak{p}(a))}$. Then $aR[X] = \bigcap_{\mathrm{ht}(\mathfrak{p}[X])=1} (\mathfrak{p}[X])^{(v_\mathfrak{p}(a))}$ since $R[X]$ is a free R-module. (If M is a free R-module, then $(\bigcap I_i)M = \bigcap (I_i M)$.) Hence $\mathrm{div}_{R[X]}(a) = \sum_{\mathrm{ht}(\mathfrak{p}[X])=1} v_\mathfrak{p}(a)(\mathfrak{p}[X]) = \tilde{\Phi}(\sum_{\mathfrak{p} \in X^1(R)} v_\mathfrak{p}(a)\mathfrak{p}) = \tilde{\Phi}(\mathrm{div}_R(a))$. Thus, $\tilde{\Phi}$ induces a map Φ on $Cl(R)$.

We want to show that Φ is surjective. In order to prove this, consider $W = R \setminus 0$, a multiplicatively closed subset of $R[X]$. We have the short exact sequence

$$0 \longrightarrow H \longrightarrow Cl(R[X]) \longrightarrow Cl(R[X]_W) \longrightarrow 0,$$

where $H =< Q : \mathrm{ht}(Q) = 1, Q \cap W \neq \emptyset >$. Now, $R[X]_W = R_W[X] = \mathsf{K}[X]$ which is a UFD. Hence $Cl(R[X]) = H$. Let Q be in $X^1(R[X])$. Then $Q \cap R = \mathfrak{q} \neq 0$ i.e. $\mathrm{ht}(\mathfrak{q}) = 1$ and since $\mathrm{ht}(Q) = 1$, $Q = \mathfrak{q}[X]$. So $Cl(R[X])$ is generated by $\Im(\Phi)$.

It remains to show that Φ is injective. Let $D \in X(R)$ represent $[D] \in Cl(R)$. Without loss of generality , we may assume that $D = \sum n_i \mathfrak{p}_i$ is effective. Suppose

$\Phi([D]) = 0$ i.e. $\sum n_i(\mathfrak{p}_i[X]) = \mathrm{div}_{R[X]}(f(X))$. By lemma 6 we can assume $f \in R[X]$. Hence we have $(f) = \bigcap(\mathfrak{p}_i[X])^{(n_i)}$. We want to show f is a constant. Let $a \in \bigcap \mathfrak{p}_i^{(n_i)}$. Then $a \in \bigcap(\mathfrak{p}_i[X])^{(n_i)}$ i.e. there is a $g \in R[X]$ such that $a = f(X)g(X)$. This implies that f is a constant, i.e. $D = \mathrm{div}_R(f(0))$. Hence $[D] = 0$. □

Thus we have the following

Corollary 10 *If R is integrally closed, $Cl(R) \simeq Cl(R[X_1, X_2, \ldots, X_n])$.*

§ 1.4 *Some Examples of Class Groups of Hypersurfaces*

We need the Jacobian criterion for the following discussion.

Theorem 11 (The Jacobian criterion) *Let k be a perfect field (i.e. $\mathrm{char}(\mathsf{k}) = 0$ or $\mathrm{char}(\mathsf{k}) = p$ and $\mathsf{k} = \mathsf{k}^{1/p}$), $S = \mathsf{k}[X_1, \ldots, X_n]$, $\mathfrak{p} = (f_1, \ldots, f_l)$ be a prime ideal of height c in S and $R := S/\mathfrak{p}$. Let $\mathfrak{q}$ be a prime in S containing $\mathfrak{p}$. Then the following are equivalent:*
(1) $R_{\mathfrak{q}}$ is a regular local ring.
(2) $I_c(\partial f_i/\partial X_j)_{l\times n}$ is not contained in $\mathfrak{q}$.
In particular, if $\mathrm{ht}(I_c(\partial f_i/\partial X_j)_{l\times n}) = c + m$, then R satisfies R_{m-1}.

Let $R = \mathsf{k}[X_1, \ldots, X_n]/(f)$, where f is irreducible and k is perfect. Since f is a non-zerodivisor, R is Cohen-Macaulay. Hence R satisfies S_2. In this case, by the Jacobian criterion and Serre's criterion, the following are equivalent:
(1) R is integrally closed.
(2) R is R_1.
(3) $\mathrm{ht}(I_1(\partial f/\partial X_i)) \geq 3$ i.e. $\mathrm{ht}((\partial f/\partial X_1, \ldots, \partial f/\partial X_n)) \geq 3$.

Example 4 Let $R = \mathbb{C}[X, Y, Z]/(Y^2 - XZ)$. Then $f = Y^2 - XZ$ and $I = (\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z) = (2Y, X, Z)$ has height 3. Hence R is integrally closed.

We have the short exact sequence

$$0 \longrightarrow H \longrightarrow Cl(R) \longrightarrow Cl(R_X) \longrightarrow 0,$$

$$\text{where } H =<[\mathfrak{p}] : X \in \mathfrak{p}, \mathrm{ht}(\mathfrak{p}) = 1> .$$

Now $R_X \simeq \mathbb{C}[X, X^{-1}, Y, Z]/(Y^2X^{-1} - Z) \simeq \mathbb{C}[X, X^{-1}, Y]$ which is a UFD. This implies that $Cl(R_X) = 0$ and $Cl(R) = H$. Hence in order to determine $Cl(R)$, we need to know the primes in R that contain X.

We have $R/XR \simeq \mathbb{C}[X, Y, Z]/(Y^2 - XZ, X) \simeq \mathbb{C}[Y, Z]/(Y^2)$. Thus there is a unique minimal prime containing X in R, that is $\mathfrak{p} = (X, Y)$. Hence

$$Cl(R) = \mathbb{Z}[\mathfrak{p}] \simeq \mathbb{Z}/n\mathbb{Z} \text{ for some } n \text{ (possibly 0).}$$

The primary decomposition of XR is of the form $\mathfrak{p}^{(l)}$ for some l. Hence we have $0 = [\mathrm{div}(x)] = l[\mathfrak{p}]$. Now l is the largest positive integer such that $X \in \mathfrak{p}^l R_\mathfrak{p} = (X,Y)^l R_{(X,Y)}$. But Z is a unit in $R_\mathfrak{p}$ and hence $X = Z^{-1}Y^2 \in \mathfrak{p}^2 R_\mathfrak{p}$. (Note: $\mathfrak{p}R_\mathfrak{p} = (Y)R_\mathfrak{p}$). So $l = 2$ and $Cl(R) = \mathbb{Z}/2\mathbb{Z}$. □

Example 5 Let R $= \mathbb{C}[X,Y,U,V]/(XY - UV)$. Then $f = XY - UV$ and $(I_1(\partial f/\partial X, Y, U, V)) = (Y, X, -U, -V)$ has height 4. Hence R is integrally closed. As before, by localizing at X, we see that

$$Cl(R) =< [\mathfrak{p}] : \mathrm{ht}(\mathfrak{p}) = 1, X \in \mathfrak{p} > .$$

Now $R/XR \simeq \mathbb{C}[Y,U,V]/(UV)$ i.e. $XR = (X,U) \cap (X,V)$ (since $(UV) = (U) \cap (V)$). Let $\mathfrak{p} = (X,U)$ and $\mathfrak{q} = (X,V)$. So $Cl(R) = \mathbb{Z}[\mathfrak{p}] + \mathbb{Z}[\mathfrak{q}]$. Now $\mathrm{div}(x) = \mathfrak{p} + \mathfrak{q}$ i.e. $[\mathfrak{q}] = -[\mathfrak{p}]$. Hence

$$Cl(R) = \mathbb{Z}[\mathfrak{p}] \simeq \mathbb{Z}/n\mathbb{Z} \text{ for some } n.$$

We claim that $n = 0$.

To prove the claim, we first identify R with the subring of the polynomial ring $S = \mathbb{C}[a,b,c,d]$ generated by ab, cd, ac, bd; we map $\mathbb{C}[X,Y,U,V]$ onto this subring by sending X to ab, Y to cd, U to ac and V to bd. Since f is clearly in the kernel, and both R and this subring have the same dimension, it follows they are isomorphic. Clearly $\mathfrak{p} = aS \cap R$. We claim that $\mathfrak{p}^{(n)} \subseteq a^n S \cap R \subseteq \mathfrak{p}^n$, which will prove that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$. As $\mathfrak{p}^n$ is clearly not principal, this will prove our claim.

Let $g \in \mathfrak{p}^{(n)}$. Then there exists an element $w \notin \mathfrak{p}$ such that $wg \in \mathfrak{p}^n$. Passing up to S this implies that $wg \in a^n S$. As a is prime in S and $w \notin aS$, this forces $g \in a^n S \cap R$. Note that under our identification of R as a subring, R has a $\mathbb{C}$-basis consisting of monomials $a^i b^j c^k d^l$ where $i + l = j + k$. If $h \in R$ and a^n divides h in S, it follows that h is a sum of monomials such that each monomial is divisible by $b^j c^k$ where $j + k \geq n$. Thus $a^n S \cap R \subseteq (aS \cap R)^n = \mathfrak{p}^n$. □

Theorem 12 (Andreotti-Salmon) *Let $(S, \mathfrak{m}, sk)$ be a regular local ring of dimension 3. Let $0 \neq f$ be an element of S such that $R = S/(f)$ is integrally closed. Then R is a UFD if and only if $f \neq \det(A)$ where A is an $n \times n$ matrix with coefficients in $\mathfrak{m}_s$ and $n \geq 2$.*

Proof:[Due to Eisenbud]
In order to prove this, we use the Hilbert-Burch Theorem in the following form:

Theorem 13 (Hilbert-Burch) *Let S be a regular local ring, I an ideal of height 2 in S such that S/I is Cohen-Macaulay. Let $I = (f_1, f_2, \ldots, f_n)$ (not necessarily minimal). Then there is an $n \times (n-1)$ matrix A with coefficients in S such that*

$\Delta_i = f_i$, where Δ_i is the i^{th} $(n-1)\times(n-1)$ minor of A (obtained by deleting the i^{th} row of A).

Conversely, if A is an $n\times(n-1)$ matrix with coefficients in S and $\mathrm{ht}(\Delta_1, \Delta_2, \ldots, \Delta_n) = 2$, then $S/(\Delta_1, \Delta_2, \ldots, \Delta_n)$ is Cohen-Macaulay.

Suppose that R is a UFD. Assume that there is an $n \times n$ matrix A with entries in $\mathfrak{m}_S$, $(n \geq 2)$ such that $f = \det(A)$. Let $A = [C_1 : C_2 : \cdots : C_n]$ where C_i is the ith column of A. Set $B := [C_1 : C_2 : \cdots : C_{n-1}]$ i.e. $A = [B : C_n]$. Let $I = I_{n-1}(B)$. By expanding $\det(A)$ along the last column, $f = \det(A) \in \mathfrak{m}I$. This forces $\mathrm{ht}(I) = 2$ [since $(f) \subseteq I$, $\mathrm{ht}(f) = 1$ and $(f) \neq I$ by NAK].

By the Hilbert-Burch Theorem, S/I is Cohen-Macaulay and in particular, I is unmixed of height 2. Therefore, IR has height 1 and is unmixed in R. Hence we can write $IR = \bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)}$, $\mathfrak{p}_i \in X^1(R)$. But $Cl(R) = 0$ implies there is an $a \in R$ such that $(a) = \bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)} = I$. Therefore $IS \subseteq (a, f)S \subseteq aS + \mathfrak{m}I$. This forces $I = aS$ by NAK which contradicts the fact that $\mathrm{ht}(I) = 2$. Thus $f \neq \det(A)$.

Conversely assume $f \neq \det(A)$ for any matrix A with entries in $\mathfrak{m}_S$. We need to prove that R is a UFD. It is enough to show that every prime $\mathfrak{q}$ of height 1 in R is principal.

Let Q be a prime in S corresponding to $\mathfrak{q}$. Then $\mathrm{ht}(Q) = 2$. Since $\dim(S/Q) = \dim(S) - \mathrm{ht}(Q) = 1$, S/Q is Cohen-Macaulay. [Fact: Every 1 dimensional domain is Cohen-Macaulay.] Write $Q = (f, g_2, \ldots, g_n)$. By the Hilbert-Burch Theorem, there is an $n \times (n-1)$ matrix A such that $\Delta_1 = f, \Delta_2 = g_2, \ldots, \Delta_n = g_n$, where Δ_i is the $(n-1)\times(n-1)$ minor of A obtained by deleting the ith row. If we show $n = 2$, then $\mathfrak{q}$ is principal.

Suppose $n \geq 3$. Then $(n-1) \geq 2$ and $f = \det(B)$ for the $(n-1)\times(n-1)$ matrix B obtained by deleting the first row of A. This is a contradiction unless B has an entry that is not in $\mathfrak{m}_S$ i.e. a unit. If A has a unit entry, by elementary row and column transformations we can transform A such that $a_{nn-1} = 1$. In fact we can further ensure that the other entries in the nth row and the $(n-1)$th column are zeroes. Let A' be the $(n-1)\times(n-2)$ matrix obtained by deleting the last row and last column of A (after the transformation). By the above observation, $I_{n-1}(A') = I_{n-1}(A)$ and $f = \det(B) = \det(B')$, where B' is the $(n-2)\times(n-2)$ matrix obtained by deleting the first row of A'. So if there is a unit entry in A, we can reduce n by 1 and therefore, can reduce down to $n = 2$. Hence $\mathfrak{q}$ is principal. $\square$

Some more examples:

Example 6 $\mathbb{C}[X, Y, Z]_{(X,Y,Z)}/(X^2 + Y^3 + Z^5)$.

Example 7 $\mathbb{C}[X, Y, Z]_{(X,Y,Z)}/(X^2 + Y^3 + Z^7)$.

Example 8 $\mathbb{R}[X, Y, Z]_{(X,Y,Z)}/(X^2 + Y^2 + Z^m)$ for any $m \geq 2$.

Example 9 $\mathbb{C}[|X,Y,Z|]/(X^2+Y^3+Z^5)$.

Example 10 $\mathbb{C}[|X,Y,Z|]/(X^2+Y^3+Z^7)$.

The first four are examples of UFDs whereas the last one is not.

We will prove that $R = \mathbb{C}[X,Y,Z]_{(X,Y,Z)}/(X^2+Y^3+Z^{2m-1})$ is a UFD, where m is any positive integer.

Let $S = \mathbb{C}[X,Y,Z]_{(X,Y,Z)}$ and $\mathfrak{m}_S = (X,Y,Z)$. By the Andreotti-Salmon Theorem it is enough to show that $f = X^2+Y^3+Z^{2m-1}$ is not $\det(A)$ for any $n \times n$ matrix A, $n \geq 2$, with entries in $\mathfrak{m}_S$. If there is such a matrix, then it must be a 2×2 matrix (due to the X^2 term) . Suppose $f = ad - bc$. Write

$$a = a_1X + a(X,Y,Z),\ b = b_1X + b(X,Y,Z),$$

$$c = c_1X + c(X,Y,Z) \text{ and } d = d_1X + d(X,Y,Z)$$

where $a(X,Y,Z)$, $b(X,Y,Z)$, $c(X,Y,Z)$ and $d(X,Y,Z)$ have no linear terms in X. Then, we have $a_1d_1 - b_1c_1 = 1$. Therefore, by elementary row and column operations, we can assume $a_1 = 1$, $d_1 = 1$, $b_1 = 0$ and $c_1 = 0$. Hence we have

$$a = X + a(X,Y,Z),\ b = b(X,Y,Z),\ c = c(X,Y,Z) \text{ and } d = X + d(X,Y,Z).$$

Using the X term in a, by elementary column operations, we can ensure that $b(X,Y,Z)$ is independent of X. This can be done because $b(X,Y,Z)$ has a finite degree in X. Similarly, by row operations, we may assume that $c(X,Y,Z)$ is also independent of X i.e. $b = b(Y,Z)$ and $c = c(Y,Z)$. Now since $ad - bc = f$, and the only term involving X in f is X^2, by comparing coefficients of higher powers of X, it is clear that $a(X,Y,Z)$ and $d(X,Y,Z)$ are also independent of X. Hence we now have

$$a = X + a(Y,Z),\ b = b(Y,Z),\ c = c(Y,Z) \text{ and } d = X + d(Y,Z).$$

Moreover, by comparing coefficients of X in $ad - bc = f$, we get $d(Y,Z) = -a(Y,Z)$. Therefore, by canceling the X^2 term, we are now left with

$$-b(Y,Z)c(Y,Z) - a(Y,Z)^2 = Y^3 + Z^{2m-1}.$$

By replacing $-b(Y,Z)$ by $b(Y,Z)$ and using reductions as above, we may assume that

$$b(Y,Z) = Y + b(Z),\ c(Y,Z) = Y^2 + c(Z) \text{ and } a(Y,Z) = a(Z).$$

Comparing coefficients of Y and Y^2 in

$$[Y + b(Z)][Y^2 + c(Z)] - [a(Z)]^2 = Y^3 + Z^{2m-1},$$

we get $b(Z) = 0$ and $c(Z) = 0$. Thus we end up having $-[a(Z)]^2 = Z^{2m-1}$, an obvious contradiction. □

Remark: If $f = X^2 + Y^3 + Z^{2m}$, where m is any positive integer, then f can be written as a determinant of a 2×2 matrix. We have $f = ad - bc$ where $a = X + iZ^m$, $d = X - iZ^m$, $b = Y$ and $c = -Y^2$.

For general information:

Theorem 14 (Grothendieck) *Let $R \simeq S/(f_1, f_2, \ldots, f_c)$ be a complete intersection (i.e. S is a regular local ring and $f_1, f_2, \ldots, f_c$ form a regular sequence in S). Suppose that $R_{\mathfrak{p}}$ is a UFD for every prime $\mathfrak{p}$ in R such that* $\mathrm{ht}(\mathfrak{p}) \leq 3$. *Then R is a UFD.*

Theorem 15 (Flenner) *Let $(R, \mathfrak{m})$ be a Noetherian graded ring over a field k ($\mathfrak{m}$ is the unique homogeneous maximal ideal). Let $\hat{R}$ denote the completion of R in the $\mathfrak{m}$ − adic topology. If R satisfies R_2 then the natural map $Cl(R) \longrightarrow Cl(\hat{R})$ is an isomorphism.*

Aside: Example 6 is a UFD whereas example 10, which is its completion, is not a UFD. This has got to do with 'rational singularities'. Also note that example 6 is R_1 but not R_2.

§ 1.5 *Finite Extensions and Class Groups*

Setup: Let $R \subseteq S$ be integrally closed domains where S is a finite R -module. Let K and L be their respective fraction fields and $n = [\mathsf{L} : \mathsf{K}]$ $(= [S : R])$.

The Going Up and Going Down theorems hold in this setup. In particular, if $\mathfrak{q}$ is a prime of height i in R and Q is a prime ideal in S minimal over $\mathfrak{q}S$, then $\mathrm{ht}(Q) = \mathrm{ht}(\mathfrak{q}) = i$ and $Q \cap R = \mathfrak{q}$.

Definition 4
(a) *If $Q \in \mathrm{Spec}(S)$, define the ramification index of Q over R as*

$$e_Q := \lambda(S_Q/(Q \cap R)S_Q).$$

(b) *Let $\kappa(Q)$ and $\kappa(\mathfrak{q})$ be the respective fraction fields of S_Q/QS_Q and $R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$ where $\mathfrak{q} = Q \cap R$. Then we define*

$$f_Q := [\kappa(Q) : \kappa(\mathfrak{q})].$$

Note that since S is a finite extension over R, S/Q is finite over $R/\mathfrak{q}$. Hence $\kappa(Q)$ is finite over $\kappa(\mathfrak{q})$ which implies that $f_Q < \infty$ for every prime ideal Q in S.

Theorem 16 (Ramification Theorem) *With notations as above, let $\mathfrak{q}$ be a prime ideal in R. Then*

$$\sum_{Q, Q \cap R = \mathfrak{q}} e_Q f_Q = n = [\mathsf{L} : \mathsf{K}].$$

Proof: Let $W = R \setminus \mathfrak{q}$. We can replace R by R_W and S by S_W and nothing changes. Hence without loss of generality , we may assume that R is a DVR. So S is semilocal and the maximal ideals are exactly the original Q's contracting to $\mathfrak{q}$. Better yet, S is a free R-module since S is torsion free (Fact: Over a DVR, a torsion free module is free). So $S \simeq R^n$ (since $\mathsf{K}^n \simeq \mathsf{L} \simeq S \otimes_R \mathsf{K}$). This gives us $\lambda_R(S/\mathfrak{q}S) = \lambda_R((R/\mathfrak{q})^n) = n$. Let $Q_1, Q_2, \ldots, Q_l$ be the maximal ideals of S (i.e. the primes in S contracting to $\mathfrak{q}$). By the Chinese Remainder Theorem, $S/\mathfrak{q}S \simeq \prod_{i=1}^{l} S_{Q_i}/\mathfrak{q}S_{Q_i}$. Hence we get

$$n = \sum \lambda_R(S_{Q_i}/\mathfrak{q}S_{Q_i}) = \sum \lambda_{S_{Q_i}}(S_{Q_i}/Q_i S_{Q_i})[\kappa(Q_i) : \kappa(\mathfrak{q})] = \sum_{Q, Q \cap R = \mathfrak{q}} e_Q f_Q,$$

which proves the theorem. $\square$

Some definitions: We want to define maps $i : X(R) \longrightarrow X(S)$ and $j : X(S) \longrightarrow X(R)$ such that $j \circ i = n\ 1_{X(R)}$. $(*)$

For $\mathfrak{q} \in X^1(R)$ define

$$i(\mathfrak{q}) := \sum_{Q \cap R = \mathfrak{q}} e_Q Q$$

and for $Q \in X^1(S)$ define

$$j(Q) := f_Q(Q \cap R).$$

Then for $\mathfrak{q}$ in $X^1(R)$,

$$j \circ i(\mathfrak{q}) = \sum_{Q, Q \cap R = \mathfrak{q}} e_Q f_Q \mathfrak{q} = n \cdot \mathfrak{q}$$

by the ramification theorem. Thus $j \circ i = n\ 1_{X(R)}$.

Theorem 17 *With notations as above, i and j induce maps $i : Cl(R) \longrightarrow Cl(S)$ and $j : Cl(S) \longrightarrow Cl(R)$ such that $j \circ i = n\ 1_{Cl(R)}$.*

Proof: The last statement follows from $(*)$. It suffices to prove that $i(\mathrm{div}(a)) \in P(S)$ for a in R and $j(\mathrm{div}(b)) \in P(R)$ for b in S. First we show that $i(\mathrm{div}_R(a)) = \mathrm{div}_S(a)$. Suppose $\mathrm{div}_R(a) = \sum_{\mathfrak{q} \in X^1(R)} v_{\mathfrak{q}}(a)\mathfrak{q}$. Then

$$i(\mathrm{div}_R(a)) = \sum_{\mathfrak{q} \in X^1(R)} \left[v_{\mathfrak{q}}(a) \sum_{Q \cap R = \mathfrak{q}} e_Q Q \right] = \sum_{Q \in X^1(S)} v_{Q \cap R}(a) e_Q Q.$$

On the other hand $\mathrm{div}_S(a) = \sum_{Q \in X^1(S)} v_Q(a)Q$. Hence it is enough to prove that $v_Q(a) = v_{Q\cap R}(a)e_Q$.

In order to prove this, consider t and u, the respective uniformizing parameters of $R_\mathfrak{q}$ and S_Q where $\mathfrak{q} = Q \cap R$. Since $\mathfrak{q}S_Q = tS_Q$, $v_Q(t) = \lambda_{S_Q}(S_Q/\mathfrak{q}S_Q) = e_Q$ i.e. $tS_Q = u^{e_Q}S_Q$. If $v_\mathfrak{q}(a) = l$, then $aS_Q = t^lS_Q = u^{e_Q l}S_Q$. Hence $v_Q(a) = e_Q v_{(Q\cap R)}(a)$ which proves that $i(\mathrm{div}_R(a)) = \mathrm{div}_S(a)$.

In order to show that $j : X(S) \longrightarrow X(R)$ induces a map on the class groups, we need to show that for any b in S, $j(\mathrm{div}_S(b))$ is a principal divisor in R i.e. $j(\mathrm{div}_S(b)) = \mathrm{div}_R(x)$ for some x in R. The question is: what is x?

Think of the multiplication by $b : \mathsf{L} \longrightarrow \mathsf{L}$ as a K-linear map. This gives an $n \times n$ matrix. Let us denote its determinant by $\det(b)$. We will now prove that x is $\det(b)$ i.e. we will show that (i) $\det(b)$ is in R and (ii) $j(\mathrm{div}_S(b)) = \mathrm{div}_R(\det(b))$. This will prove the theorem.

(i) In order to prove that $\det(b)$ is in R, consider $\mathfrak{q}$ in $X^1(R)$. Now $R_\mathfrak{q}$ is a DVR and $S_\mathfrak{q} = (R \setminus \mathfrak{q})^{-1}S$ is a torsion-free $R_\mathfrak{q}$ module which implies that $S_\mathfrak{q} \simeq R_\mathfrak{q}^{\oplus n}$. The map $\mathsf{L} \xrightarrow{b} \mathsf{L}$ is induced by the map $S \xrightarrow{b} S$ by tensoring with L. Choose a basis for $S_\mathfrak{q}$ over $R_\mathfrak{q}$; it forms a basis for Lover K. Computing the determinant using this basis shows that $\det(b)$ is in $R_\mathfrak{q}$. Hence $\det(b) \in \bigcap_{\mathfrak{q}\in X^1(R)} R_\mathfrak{q} = R$ since R is integrally closed.

(ii)Now we have $j(\mathrm{div}_S(b)) = j\left(\sum_{Q\in X^1(S)} v_Q(b)Q\right)$

$$= \sum_{Q\in X^1(S)} v_Q(b)[\kappa(Q) : \kappa(Q\cap R)](Q \cap R)$$

$$= \sum_{\mathfrak{q}\in X^1(R)} \left(\sum_{[Q\in X^1(S), Q\cap R = \mathfrak{q}]} v_Q(b)[\kappa(Q) : \kappa(\mathfrak{q})] \right) \mathfrak{q}.$$

In order to prove that $j(\mathrm{div}_S(b)) = \mathrm{div}_R(\det(b))$, we only need to show that $v_\mathfrak{q}(\det(b)) = \sum_{Q\cap R=\mathfrak{q}} v_Q(b)[\kappa(Q) : \kappa(\mathfrak{q})]$. Nothing changes upon passing to $R_\mathfrak{q}$ and $S_\mathfrak{q}$. Therefore without loss of generality we may assume that R is a DVR, S is semilocal and all the maximal ideals of S contract to $\mathfrak{m}_R(= \mathfrak{q}R_\mathfrak{q})$. As before, $S \simeq R^{\oplus n}$. We use the following lemma:

Lemma 18 *Let V be a DVR and $\Phi : V^n \longrightarrow V^n$ such that $\Delta = \det(\Phi) \neq 0$. Let $N = \mathrm{Coker}(\Phi)$. Then $\lambda_V(N) = \lambda_V(V/\Delta V)$.*

By the lemma applied to $0 \longrightarrow S \xrightarrow{\cdot b} S \longrightarrow S/bS \longrightarrow 0$, we get $\lambda_R(S/bS) = \lambda_R(R/(\det(b))) = v_\mathfrak{q}(\det(b))$. Now by the Chinese Remainder Theorem,

$$S/bS \simeq \prod_{Q\neq 0} (S_Q/bS_Q).$$

Hence we have

$$\lambda_R(S/bS) = \sum_{Q\neq 0} \lambda_R(S_Q/bS_Q)$$

$$= \sum_{Q\neq 0} \lambda_{S_Q}(S_Q/bS_Q)[\kappa(Q):\kappa(\mathfrak{q})] = \sum_{Q\cap R=\mathfrak{q}} v_Q(b)[\kappa(Q):\kappa(\mathfrak{q})]$$

which proves the result. □

Proof (of lemma 18): By the structure theorem for modules over PIDs, we can write N as the sum of cyclic modules. Let $N \simeq V/(d_1)\oplus\cdots\oplus V/(d_l)$. $(*)$
Now $\det(\Phi)N = 0$ by Cramer's Rule and hence N is torsion. Hence $d_j \neq 0$ for any j. So there is a generating set for N such that we have the short exact sequence $0 \longrightarrow V^l \xrightarrow{\Phi} V^l \longrightarrow N \longrightarrow 0$ where Φ is multiplication by the diagonal matrix $\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_l \end{pmatrix}$. So $\det(\Phi) = \det\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_l \end{pmatrix} = d_1d_2\ldots d_l$. (See Aside).
By $(*)$, we get that

$$\lambda_R(N) = \sum_{j=1}^{l} \lambda_R(V/d_j) = \lambda_R(V/(d_1\ldots d_l)) = \lambda_R(V/\det(\Phi))$$

which proves the lemma. □

Aside: In general, if $R^m \xrightarrow{\phi} R^n \longrightarrow N \longrightarrow 0$, with rank(N) $= r$, then $I_{n-r}(\phi)$ (called the Fitting ideal), does not depend on the presentation ϕ.

A generalization of the lemma:
Let R be a Noetherian local ring, $f : R^m \longrightarrow R^n$ $(m \geq n)$ be a map such that $\mathrm{grade}(I_n(f)) = m - n + 1$. (This is the maximal possible grade.) Assume that $N :=$ Coker(f) has finite length. Then $\lambda_R(N) = \lambda_R(R/I_n(f))$.

Example 11
1. Let $R = \mathbb{C}[X,Y,Z]/(Y^2 - XZ)$. Recall that we proved that $Cl(R) \simeq \mathbb{Z}/2\mathbb{Z}$. The main theorem of this section does not quite recover this, but does prove the class group is 2-torsion, as follows. Note that $R \simeq \mathbb{C}[S^2, ST, T^2]$. Let S be $\mathbb{C}[S,T]$. The respective quotient fields of S and R are $\mathsf{L} := \mathbb{C}(S,T)$ and $\mathsf{K} := \mathbb{C}(S/T, T^2)$. Then $[\mathsf{L}:\mathsf{K}] = 2$. We are in the setup of theorem 17. Hence we have maps $Cl(R) \xrightarrow{i} Cl(S)$ and $Cl(S) \xrightarrow{j} Cl(R)$ such that $j\circ i = 2\cdot Id_{Cl(R)}$. But S is a UFD and hence $Cl(S) = 0$.

Therefore we can conclude that $2 \cdot Cl(R) = 0$.

2. Let $R = \mathbb{C}[S^n, S^{n-1}T, \ldots, ST^{n-1}, T^n]$. Then as above, $n \cdot Cl(R) = 0$. In fact it can be proved that $Cl(R) \simeq \mathbb{Z}/n\mathbb{Z}$ (see the exercises).

§ 1.6 *Divisors Attached to Modules*

Definition 5 *The Grothendieck group of R denoted by $G_0(R)$ is the free abelian group on isomorphism classes of modules $\{M\}$ modulo the subgroup generated by the relations*

$$\{M_2\} = \{M_1\} + \{M_3\}$$

whenever there is a short exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$.

In this section, we want to attach divisors to modules over an integrally closed Noetherian domain R. This can be done by defining a map from $G_0(R)$ to $Cl(R)$. We will do this in two steps.
Step 1:
We define a map from the torsion modules over R to $Cl(R)$ as follows:

Let T be a torsion R-module, i.e. there is a nonzero x in R such that $x \cdot T = 0$. We will denote the image of T in $Cl(R)$ by $[T]$. Consider the map

$$T \mapsto [T] := \sum_{\mathfrak{p} \in X^1(R)} \lambda(T_{\mathfrak{p}})[\mathfrak{p}].$$

Since $\operatorname{ann}_R(T) \neq 0$, there are at most finitely many primes $\mathfrak{p}$ of height 1 containing $\operatorname{ann}_R(T)$. For such primes $\mathfrak{p}$, $\lambda(T_{\mathfrak{p}}) < \infty$. If $\mathfrak{p}$ is a height 1 prime not containing $\operatorname{ann}_R(T)$, then $T_{\mathfrak{p}} = 0$ and hence $\lambda(T_{\mathfrak{p}}) = 0$. Thus the sum is finite.

Remark 3
1. If $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ is a short exact sequence of torsion modules, then $[T_2] = [T_1] + [T_3]$. This follows from the fact that for any $\mathfrak{p} \in X^1(R)$, $0 \longrightarrow (T_1)_{\mathfrak{p}} \longrightarrow (T_2)_{\mathfrak{p}} \longrightarrow (T_3)_{\mathfrak{p}} \longrightarrow 0$ is exact and length is additive on short exact sequences.

2. Let T_1 and T_2 be torsion and $f : T_1 \longrightarrow T_2$ be a map such that $\operatorname{Ker}(f)$ and $\operatorname{Coker}(f)$ have annihilators of height at least 2. Then $[T_1] = [T_2]$. This is true since for any prime ideal $\mathfrak{p}$ of height 1, $\operatorname{Ker}(f)_{\mathfrak{p}} = 0 = \operatorname{Coker}(f)_{\mathfrak{p}}$.

3. If $\sum n_{\mathfrak{p}}[\mathfrak{p}]$ is an effective divisor and $I = \bigcap \mathfrak{p}^{(n_{\mathfrak{p}})}$, then note that $[R/I] = \sum n_{\mathfrak{p}}[\mathfrak{p}]$.

Step 2:
We now want to extend the definition of the divisor of a module to all finitely generated R-modules M. Let $r = \operatorname{rank}(M) := \dim_{\mathsf{K}}(M \otimes_R \mathsf{K})$ i.e. $(M \otimes_R \mathsf{K}) \simeq \mathsf{K}^r$, where

$\mathsf{K}$is the fraction field of R. Recall that $\mathrm{Hom}_R(R^r, M) \otimes_R \mathsf{K} \simeq \mathrm{Hom}_{\mathsf{K}}(\mathsf{K}^r, M \otimes_R \mathsf{K})$. This implies that there is a map $f : R^r \longrightarrow M$ which becomes an isomorphism after tensoring with K. Therefore $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are torsion and hence $\mathrm{Ker}(f)$ is 0 (since $\mathrm{Ker}(f)$ injects into R^r, which is free). This gives us a short exact sequence

$$0 \longrightarrow R^r \xrightarrow{f} M \longrightarrow T \longrightarrow 0 \qquad (*)$$

where $T \simeq \mathrm{Coker}(f)$ is torsion.

Definition 6
We define the map [ ]: free group isomorphism classes of R-modules $\longrightarrow Cl(R)$ *by* $M \mapsto [M] := [T]$ *where* T *is as in* $(*)$.

We need to show that $[M]$ is well-defined i.e. it is independent of the short exact sequence $(*)$. Let $F \simeq R^r$, $G \simeq R^r$ be free, T and L be torsion modules such that we have the two short exact sequences $0 \longrightarrow F \longrightarrow M \longrightarrow T \longrightarrow 0$ and $0 \longrightarrow G \longrightarrow M \longrightarrow L \longrightarrow 0$. We need to prove $[T] = [L]$ in $Cl(R)$. Since $F \otimes_R \mathsf{K} \simeq M \otimes_R \mathsf{K} \simeq G \otimes_R \mathsf{K}$, by clearing denominators, we can find a nonzero element x in R such that $xG \subseteq F$. But $xG \simeq G$ and M/xG is torsion, hence without loss of generality we may assume that $G \subseteq F$.
Justification: Consider

$$\begin{tikzcd}
0 \arrow[r] & xG \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & L' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & M \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

By the Snake Lemma we get $0 \longrightarrow G/xG \longrightarrow L' \longrightarrow L \longrightarrow 0$. Hence by remark 3.1 $[L'] = [L] + [G/xG]$. Now $[G/xG] = r[R/xR] = r[\mathrm{div}(x)]$ by remark 3.3. Thus $[G/xG] = 0$ and therefore $[L'] = [L]$. Hence we can assume $G \subseteq F$. Thus we have

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & M \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

Applying Snake Lemma again, if $K = F/G$, then we have the short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow T \longrightarrow 0$. Now all these are torsion and we have $[L] = [T] + [K]$. In order to prove $[L] = [T]$, we will show that $[K] = 0$ in $Cl(R)$.

We have the short exact sequence $0 \longrightarrow R^r \xrightarrow{\phi} R^r \longrightarrow K \longrightarrow 0$. We will prove that $[K] = 0$ by proving that the associated divisor to K is $\mathrm{div}_R(\det(\phi))$. Recall that $[K] = \sum_{\mathfrak{p} \in X^1(R)} \lambda(K_{\mathfrak{p}})[\mathfrak{p}]$. For $\mathfrak{p}$ in $X^1(R)$, $R_{\mathfrak{p}}$ is a DVR and therefore by lemma 18 applied to the short exact sequence

$$0 \longrightarrow R^r_{\mathfrak{p}} \xrightarrow{\phi} R^r_{\mathfrak{p}} \longrightarrow K_{\mathfrak{p}} \longrightarrow 0,$$

we get $\lambda(K_{\mathfrak{p}}) = \lambda((R^r/\det(\phi))_{\mathfrak{p}}) = v_{\mathfrak{p}}(\det(\phi))$. Hence $[K] = [\mathrm{div}_R(\det(\phi))] = 0$ in $Cl(R)$. $\square$

Corollary 19 *If F is a free R-module, then $[F] = 0$.*

Proposition 20 *If $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{\pi} M_3 \longrightarrow 0$ is a short exact sequence of finitely generated R-modules, then $[M_2] = [M_1] + [M_3]$.*

Proof: Choose free submodules $R^{r_i} =: F_i$ of M_i, $i = 1, 3$ such that $M_i/F_i \simeq T_i$ are both torsion. Let $x_1, x_2, \ldots, x_{r_1}$ be a basis for F_1, $y_1, \ldots, y_{r_3}$ be a basis for F_3. Choose z_j in M_2 such that $\pi(z_j) = y_j$, $j = 1, \ldots, r_3$. Then $F_2 := Rx_1 + \cdots + Rx_{r_1} + Rz_1 + \cdots + Rz_{r_3}$ is free of rank $r_1 + r_3$. Thus we have the commutaive diagram where each row is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0
\end{tikzcd}$$

Define $[T_2] = M_2/F_2$. Then by the Snake Lemma, we get the short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$. Since T_1 and T_3 are torsion, so is T_2. Hence we have $[M_2] = [T_2] = [T_1] + [T_3] = [M_1] + [M_3]$. $\square$

Remark 4 Let us compare $[\mathfrak{p}]$ as an R-module and $[\mathfrak{p}]$ as an element of $Cl(R)$ for any $\mathfrak{p} \in X^1(R)$. In order to avoid confusion, let us denote $[\mathfrak{p}]$ as an R-module by $[{}_R\mathfrak{p}]$.

Consider $0 \longrightarrow \mathfrak{p} \longrightarrow R \longrightarrow R/\mathfrak{p} \longrightarrow 0$. Then $[R] = [R/\mathfrak{p}] + [{}_R\mathfrak{p}]$. Since $[R] = 0$, we have $[{}_R\mathfrak{p}] = -[R/\mathfrak{p}]$. Now $R/\mathfrak{p}$ is torsion. Therefore, by definition, $[R/\mathfrak{p}] = \sum_{\mathfrak{q} \in X^1(R)} \lambda_{R_{\mathfrak{q}}}((R/\mathfrak{p})_{\mathfrak{q}})[\mathfrak{q}] = [\mathfrak{p}]$. This gives us

$$[{}_R\mathfrak{p}] = -[\mathfrak{p}].$$

Corollary 21 *If R is an integrally closed Noetherian domain in which every prime ideal $\mathfrak{p}$ of height 1 has a finite free resolution, then R is a UFD.*

Proof: Let $\mathfrak{p}$ be a prime ideal of height 1 in R. Let

$$\mathbf{F}_{\bullet}: \quad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathfrak{p} \longrightarrow 0$$

be a finite free resolution of $\mathfrak{p}$ where $F_i \simeq R^{r_i}$. By Prop. 20, $[{}_R\mathfrak{p}] = \sum_{j=0}^{n} (-1)^j [F_j] = 0$. By the previous remark, $[\mathfrak{p}] = 0$ which implies that $Cl(R) = 0$. Hence R is a UFD. $\square$

Corollary 22 *A regular local ring is a UFD.*

Proof: Every module over a regular local ring R has a finite free resolution. Therefore, by the previous corollary, R is a UFD. $\square$

§ 1.7 *Another View of the Class Group (leading to Lipman's Theorem)*

Let R be an integrally closed Noetherian domain with fraction field $Q(R) = \mathsf{K}$. Let $(\_)^* = \mathrm{Hom}_R(\_, R)$. Recall that if M is an R-module,

1. M is reflexive if $M \simeq M^{**}$ via the canonical map.

2. $\mathrm{rank}(M) = \dim_{\mathsf{K}}(M \otimes_R \mathsf{K})$.

Remark 5 *If $M \simeq M^{**}$ then the canonical map is an isomorphism (see the exercises).*

Definition 7 *We define $cl(R)$:= set of isomorphism classes of rank 1, finitely generated reflexive R-modules. This is an abelian group under the operation*

$$[M] \cdot [N] = [(M \otimes N)^{**}].$$

The multiplicative identity is $[R]$ and the inverse of $[M]$ is $[M^*]$.

We will prove that $cl(R) = Cl(R)$.

Proposition 23 *Every rank 1 reflexive module M is isomorphic to a height 1 unmixed ideal I in R. Conversely every unmixed ideal I of height 1 is a rank 1 reflexive R-module.*

Discussion/Remarks: Let M be a finitely generated torsion-free module of rank 1 over R. Then $M \otimes_R \mathsf{K} \simeq \mathsf{K}$. Since M is torsion-free, $M \hookrightarrow M \otimes_R \mathsf{K} \simeq \mathsf{K}$.

Aside: Classically (in algebraic number theory) finitely generated R-submodules of $\mathsf{K}$are called "fractionary ideals" or "orders". Let $M =< r_i/x_i : i = 1, \ldots, n >$ as an R-module, where $r_i, x_i \in R$ for all i. By clearing denominators, we can find an x in R such that $xM = I$, an ideal in R. Thus every rank 1, finitely generated torsion-free R-module is just an ideal in R up to isomorphism.

In view of the above discussion, to prove the correspondence of Prop. 23, it is enough to show that I is an unmixed ideal of height 1 if and only if $I \simeq I^{**}$. This is due to the fact that every reflexive module is torsion-free.

Aside: Any submodule of a free module is torsion-free. So, for any R-module M, M^* is torsion-free.

We first prove the following lemma

Lemma 24 *Let x be a non-zero element in an ideal I of R. Then $I^* \simeq (x) :_R I$.*

Proof: Define $\Phi : I^* \longrightarrow (x) :_R I$ by $f \mapsto f(x)$ for any f in I^*. Note that if $i \in I$, then $if(x) = xf(i) \in (x)$. Hence $f(x)$ is an element of $(x) :_R I$. We want to prove that Φ is an isomorphism. To prove Φ is injective, consider $f \in I^*$ such that $f(x) = 0$. Then $xf(i) = if(x) = 0$ for each i in I. But x is a non-zerodivisor and hence $f = 0$ in I^*. Now it remains to show that Φ is surjective. Let a be in $(x) :_R I$. Define $f_a : I \longrightarrow R$ by multiplication by a/x. If $i \in I$, then $a/x \cdot i \in R$ and $f_a(x) = a$. Therefore $\Phi(f_a) = a$ and hence Φ is surjective. $\square$

Aside: Note that $if(x) = xf(i)$ for all i in I gives us
1. $f(i)/i = f(x)/x$ for every non-zero i in I and
2. each f in I^* is given by $i \mapsto i \cdot f(x)/x$.

Corollary 25 *Let $x, y \in I$ be nonzero. Then $I^{**} \simeq (y) :_R ((x) :_R I)$. In particular, if x is non-zero then $I^{**} \simeq (x) :_R ((x) :_R I)$.*

By the corollary, to prove Prop. 23, it is enough to prove the following statement:
Let I be an ideal in R and $x \in I$ be non-zero. Then I is an unmixed ideal of height 1 if and only if $I = (x) :_R ((x) :_R I)$.
Proof: Any ideal of the form $(x) :_R J$ $(\neq R)$ is unmixed of height 1. This follows by primary decomposition. In general, every associated prime of $K :_R J$ is associated to K. As R satisfies S_2, (x) is an unmixed ideal of height 1.
Hence if $I = (x) :_R ((x) :_R I)$, then I is an unmixed ideal of height 1. (Note: If $(x) :_R J = R$, then $J = (x)$ and hence $(x) :_R J = R \simeq (x)$.)

Conversely suppose I is a height 1, unmixed ideal. Clearly $I \subseteq (x) :_R ((x) :_R I)$. To prove equality, since I is unmixed, it is enough to prove equality after localizing at an arbitrary prime minimal over I.
(Fact: If $I \subseteq J$, then $I = J$ if and only if $I_\mathfrak{p} = J_\mathfrak{p}$ for every prime $\mathfrak{p}$ associated to I.)

Hence without loss of generality we may assume that R is a DVR with maximal ideal $\mathfrak{p}$. Let $(t) = \mathfrak{p}$. Then $I = (t^k)$ where $k = v_\mathfrak{p}(I)$. Since $x \in I$, $l := v_\mathfrak{p}(x) \geq k$. Then $(x) = (t^l)$ gives us $(x) :_R I = (t^{l-k})$ and hence $(x) :_R ((x) :_R I) = (t^l) :_R (t^{l-k}) = (t^k) = I$. Thus if I is an unmixed ideal of height 1, then $I = (x) :_R ((x) :_R I)$. $\square$

Remark 6 *I^* can be identified with $\{\alpha \in \mathsf{K} : \alpha I \subseteq R\} = I^{-1}$.*

Comment about Height 1 Unmixed Ideals
Recall that if I is unmixed, all the associated primes of I are minimal over I. If $\mathrm{ht}(I) = 1$, then $\mathrm{ht}(\mathfrak{p}) = 1$ where $\mathfrak{p}$ is any prime minimal over I. Therefore, if I is an unmixed ideal of height 1, then all its associated primes are also of height 1. This implies that

$$I = \bigcap_{\mathfrak{p} \in X^1(R)} \mathfrak{p}^{(v_{\mathfrak{p}}(I))}.$$

Lemma 26 *Let R be an integrally closed Noetherian domain, I, J ideals in R. Consider the following statements:*
(1) $I \simeq J$ as R-modules.
(2) There exist non-zero elements a and b in R such that $aI = bJ$.
(3) $[\mathrm{div}(I)] = [\mathrm{div}(J)]$ where $\mathrm{div}(I)$ is defined as in §2.
Then (1) and (2) are equivalent and they imply (3). If I and J are unmixed ideals of height 1, then (3) implies (1) and (2).

Proof: (1) $\Rightarrow$ (2): Let $\Phi : I \xrightarrow{\simeq} J$ be an isomorphism. Then $I \overset{\Phi}{\hookrightarrow} R$. Hence Φ is given by multiplication by some element $\alpha = a/b$ of K. Therefore $J = \Phi(I) = a/b \cdot I$ which gives us $aI = bJ$.
(2) $\Rightarrow$ (1): We have $I \simeq aI$ and $bJ \simeq J$. Hence $aI = bJ$ implies $I \simeq J$.
(2) $\Rightarrow$ (3): Let $\mathfrak{p} \in X^1(R)$. Then $aI = bJ$ gives us $v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(I) = v_{\mathfrak{p}}(aI) = v_{\mathfrak{p}}(bJ) = v_{\mathfrak{p}}(b) + v_{\mathfrak{p}}(J)$.
Then

$$\sum_{\mathfrak{p} \in X^1(R)} (v_{\mathfrak{p}}(a) + v_{\mathfrak{p}}(I))\mathfrak{p} = \sum_{\mathfrak{p} \in X^1(R)} (v_{\mathfrak{p}}(b) + v_{\mathfrak{p}}(J))\mathfrak{p}$$

and hence $[\mathrm{div}(I)] = [\mathrm{div}(J)]$.

Let us assume that I and J are unmixed ideals of height 1 and that $[\mathrm{div}(I)] = [\mathrm{div}(J)]$. We want to show that there are elements a and b in R such that $aI = bJ$. Since $[\mathrm{div}(I)] = [\mathrm{div}(J)]$, there is an element $\alpha = a/b$ in Ksuch that $\mathrm{div}(I) - \mathrm{div}(J) = \mathrm{div}(\alpha) = \mathrm{div}(a) - \mathrm{div}(b)$. Hence $\mathrm{div}(aI) = \mathrm{div}(bJ)$. If we can show that both aI and bJ are unmixed of height 1, $\mathrm{div}(aI) = \mathrm{div}(bJ)$ implies that their primary components are the same and hence $aI = bJ$. Thus we have reduced the problem to proving: If I is an unmixed ideal of height 1, then so is aI.

Let Q be in $Ass_R(R/aI)$. Then there is a y in R, not in aI, such that $Q = (aI :_R y)$. Now $(aI :_R y)$ is a subset of $(a :_R y)$ as well as $(I :_R y)$ which are both of height 1 unless they are the whole ring. If $\mathrm{ht}(Q) \geq 2$, then $(a :_R y) = R$. Hence there is a $z \in R \setminus I$ such that $y = az$. This implies that $Q = (aI :_R az) = (I :_R z)$. Therefore $Q \in Ass_R(R/I)$ which is not possible since I is unmixed. This proves that aI is unmixed. $\square$

Let M be a rank 1, reflexive R-module. Then there is an unmixed ideal I of height 1 in R such that $M \simeq I$. This gives us a set map $cl(R) \xrightarrow{\theta} Cl(R)$ defined as $[M] \mapsto [\mathrm{div}(I)]$. This is well-defined by lemma 26. We will prove that θ is an isomorphism between the two groups.

Lemma 27 *Let R be an integrally closed Noetherian domain, x be a non-zero element in an ideal I of R. Write $I = \bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)} \cap J$ where $\mathrm{ht}(\mathfrak{p}_i) = 1$ and $\mathrm{ht}(J) \geq 2$. Then*

$$(x) :_R ((x)_R : I) = \begin{cases} \bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)} & \text{if } l \geq 1 \\ R & \text{if } l = 0 \end{cases}$$

Proof: By Corollary 25, it suffices to prove that $(x) :_R I = (x) :_R (\bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)})$. Set $q = \bigcap_{i=1}^{l} \mathfrak{p}_i^{(n_i)}$. We have inclusions $Jq \subseteq I \subseteq q$ which induce containments $(x) : q \subseteq (x) : I \subseteq (x) : qJ$. However, $(x) : qJ = (x) : q) : J$, and $(x) : q$ is an unmixed ideal of height one. Since J has height at least two (and therefore is not in any prime of height one), it follows that $(x) :_R q = (x) :_R qJ$. $\square$

Theorem 28 *Let R be an integrally closed Noetherian domain. Then $cl(R) \xrightarrow{\theta} Cl(R)$ is an isomorphism of groups.*

Proof: We first show that θ is a group homomorphism i.e. we need to show that if M and N are rank 1, reflexive R-modules, then $\theta([M][N]) = \theta([M]) + \theta([N])$.

Choose ideals I and J in R such that $I \simeq M$ and $J \simeq N$. Then $[M][N] = [I][J] = [(I \otimes_R J)^{**}]$. Hence we need to find an ideal $L \simeq (I \otimes_R J)^{**}$ and show that $[\mathrm{div}(L)] = [\mathrm{div}(I)] + [\mathrm{div}(J)]$. We claim that $(I \otimes_R J)^{**} \simeq (IJ)^{**}$.

Consider the surjective homomorphism $i \otimes_R j \mapsto ij$ from $I \otimes_R J$ to IJ. Let T be its kernel. Since $I \otimes_R J \otimes_R \mathsf{K} \simeq IJ \otimes_R \mathsf{K}$, $T \otimes_R \mathsf{K} = 0$. Therefore T is a torsion submodule. Hence we get $(IJ)^* \simeq (I \otimes_R J)^*$ since any torsion module maps to 0 in R. Dualizing again gives us $(I \otimes_R J)^{**} \simeq (IJ)^{**}$.

Now, by lemma 27 $(IJ)^{**} =$ height 1 unmixed component of $IJ = \bigcap_{\mathfrak{p} \in X^1(R)} \mathfrak{p}^{(v_{\mathfrak{p}}(IJ))}$. This implies that

$$\mathrm{div}((IJ)^{**}) = \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(IJ)\mathfrak{p} = \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(I)\mathfrak{p} + \sum_{\mathfrak{p} \in X^1(R)} v_{\mathfrak{p}}(J)\mathfrak{p} = \mathrm{div}(I) + \mathrm{div}(J)$$

which proves that θ is a group homomorphism.

To prove that θ is onto, consider a class represented by $\sum n_{\mathfrak{p}}\mathfrak{p}$. Let $I = \bigcap \mathfrak{p}^{(n_{\mathfrak{p}})}$. By lemma 27, I is reflexive. Hence by Prop. 23, I is unmixed and $\mathrm{ht}(I) = 1$. Then $[I] \mapsto [\mathrm{div}(I)] = \sum n_{\mathfrak{p}}[\mathfrak{p}]$. Hence θ is surjective.

We want to show that θ is injective. Consider an unmixed ideal I, of height 1 such that $[\mathrm{div}(I)] = 0$ in $Cl(R)$. Then there is an element $x \in R$ such that $\mathrm{div}(I) = \mathrm{div}(x)$.

Since I is an unmixed ideal of height 1, $I = (x) \simeq R$. Hence θ is injective. $\square$

As a consequence of the above theorem, if R is an integrally closed Noetherian domain, t a non-zero element in R such that R/tR is also integrally closed, then we can define a map $j : Cl(R) \longrightarrow Cl(R/tR)$ by mapping the isomorphism class of a rank 1 reflexive module M to the class of $(M/tM)^{**}$. (It needs to be checked that this map makes sense and is well-defined).

§ 1.8 *Class Groups of Graded Rings*

We are aiming for the following theorems

Theorem 29 *If R is an integrally closed Noetherian domain which is graded over a field, then Cl(R) is generated by $[\mathfrak{p}]$, where $\mathfrak{p}$ ranges over all homogeneous prime ideals of height 1.*

Theorem 30 *With the same assumptions as in Theorem 29, let $\mathfrak{m}$ be the unique maximal homogeneous ideal in R. Then the natural map $Cl(R) \longrightarrow Cl(R_{\mathfrak{m}})$ is an isomorphism.*

General Remarks and Lemmas on Graded Rings

We say R is graded over a field if $R = \bigoplus_{i \geq 0} R_i$ as an abelian group where R_0 is a field and $R_i R_j \subseteq R_{i+j}$ for all i and j.

An ideal I in R is said to be homogeneous if I is generated by homogeneous elements or equivalently $I = \bigoplus_{i \geq 0}(I \cap R_i)$.
If I is any ideal in R, then we can define

$$\tilde{I} = < r \in I : r \text{ is homogeneous} >$$

which is the ideal generated by the elements in $\bigcup_{i \geq 0}(R_i \cap I)$.

Notation: By $X^1_{\text{hom}}(R)$, we mean $\{\mathfrak{p} : \mathfrak{p} \text{ is a homogeneous prime of height } 1\}$.

Remark 7 Let R be a graded ring over a field and M be a finitely generated graded R-module. Let $\mathfrak{m}$ be the unique maximal homogeneous ideal in R. Then
(a) (Graded NAK) If $L \subseteq M$ is a graded submodule and $M = L + \mathfrak{m}M$, then $L = M$.
(b) If $M_{\mathfrak{m}} = 0$, then $M = 0$.
(c) If R is Noetherian and $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module, then M is a free R-module.

Proof: (a) Passing to M/L, we may assume that $L = 0$. Suppose $M \neq 0$. Choose a non-zero homogeneous element x in M of least degree. This is possible since M is finitely generated and R is $\mathbb{N}$-graded. Then $M \neq \mathfrak{m}M$ since x cannot be in $\mathfrak{m}M$.

(b) Since M is finitely generated and $M_{\mathfrak{m}} = 0$, there is an element $r = r_0 + m$ in R, r_0 a non-zero element in R_0 and $m \in \mathfrak{m}$ such that $rM = 0$ i.e. $r_0M = mM$. Since r_0 is a unit, we have $M = mM$ and hence by graded NAK, $M = 0$.

(c) Let $M_{\mathfrak{m}} \simeq R_{\mathfrak{m}}^n$. Since $R_{\mathfrak{m}}$ is local, any set of generators for $M_{\mathfrak{m}}$ contains a free basis. So, without loss of generality we may assume that there are homogeneous elements $z_1, z_2, \dots, z_n$ in M which form a basis after localizing.
Consider the map from R^n to M mapping the standard basis elements e_i to z_i, $1 \leq i \leq n$. Let K and C be the kernel and the cokernel respectively. Then K and C are finitely generated graded R-modules such that $K_{\mathfrak{m}} = 0$ and $C_{\mathfrak{m}} = 0$. Hence by (b), $K = 0$ and $C = 0$.

Lemma 31 *If R is graded and $\mathfrak{p}$ is a prime ideal in R, then so is $\tilde{\mathfrak{p}}$.*

Proof: Let $a = \sum_{i=d}^{e} a_i$, $b = \sum_{j=f}^{g} b_j$, a_i, b_j homogeneous elements for all i, j, be elements of R such that $ab \in \tilde{\mathfrak{p}}$, $b \notin \tilde{\mathfrak{p}}$. Assume $b_f \notin \tilde{\mathfrak{p}}$. Then as $\tilde{\mathfrak{p}}$ is homogeneous $a_d b_f \in \tilde{\mathfrak{p}} \subseteq \mathfrak{p}$. This implies that $a_d \in \mathfrak{p}$ and hence is in $\tilde{\mathfrak{p}}$. Continuing this way, $a_i \in \tilde{\mathfrak{p}}$, for every i. Thus, $a \in \tilde{\mathfrak{p}}$ i.e. $\tilde{\mathfrak{p}}$ is a prime ideal. $\square$

Lemma 32 *Let R be a graded domain over a field. Let W be the multiplicatively closed subset of all homogeneous non-zero elements. Then*
(1) *R_W is graded (with possibly negative degrees).*
(2) *$L := (R_W)_{(0)}$ is a field.*
(3) *Let t be a non-zero homogeneous element in R_W of least positive degree, then $R_W = L[t, t^{-1}]$ and t is transcendental over L.*

Proof: (1) Set $\deg(a/b) = \deg(a) - \deg(b)$ for non-zero a, b in $\bigcup R_i$. It is easy to show that this gives a grading on R_W.

(2) Let $\alpha = a/b$ be a non-zero element in $(R_W)_0$. Then $\deg(\alpha) = 0$ implies that $\deg(a) = \deg(b)$. Therefore $\alpha^{-1} = b/a$ is also in $(R_W)_0$ i.e. $L = (R_W)_0$ is a field.

(3) Let α be a non-zero homogeneous element algebraic over L. i.e. there are $l_1, \dots, l_n$ in L such that $\alpha^n + l_1\alpha^{n-1} + \cdots + l_n = 0$. If $\deg(t)$ is not 0, by looking at the piece in $n \cdot \deg(\alpha)$, we get $\alpha^n = 0$, which is not possible. Thus if α is a homogeneous element of non-zero degree, it is transcendental over L.
Suppose that t is a non-zero homogeneous element of least positive degree in R_W. By the above observation, $L[t, t^{-1}] \subseteq R_W$. In order to prove $L[t, t^{-1}] \supseteq R_W$, consider

$a/b \in R_W$. Let $\deg(a/b) = n$ and $\deg(t) = r$. We have $\deg(a) = n + \deg(b)$. By the choice of t, $\deg(a) \geq r$. By the Division Algorithm, we can find integers q and s such that $\deg(a) = qr + s$, where $0 \leq s < r$. Then the degree of $at^{-q} \in R_W$ is $\deg(a) - qr = s$. Since $s < r$, by the choice of t, $s = 0$. Thus $a \in L[t, t^{-1}]$. □

Proof of Theorem 29: Let $W = \bigcup R_i \setminus 0$. We have the short exact sequence $0 \longrightarrow H \longrightarrow Cl(R) \longrightarrow Cl(R_W) \longrightarrow 0$, where $H =< [\mathfrak{p}] : \mathfrak{p} \in X^1(R), \mathfrak{p} \cap w \neq \emptyset >$. By lemma 32, R_W is a UFD. Hence $Cl(R) =< [\mathfrak{p}] : \mathfrak{p} \in X^1(R), \mathfrak{p} \cap w \neq \emptyset >$ i.e.

$$Cl(R) =< [\mathfrak{p}] : \mathfrak{p} \in X^1(R), \mathfrak{p} \text{ contains a non-zero homogeneous element} > .$$

Now if $\mathfrak{p}$ contains a non-zero homogeneous element, $\tilde{\mathfrak{p}}$ is not zero . Since, by lemma 31, $\tilde{\mathfrak{p}}$ is prime and $\mathrm{ht}(\mathfrak{p}) = 1$, we have $\mathfrak{p} = \tilde{\mathfrak{p}}$ i.e.

$$Cl(R) =< [\mathfrak{p}] : \mathfrak{p} \in X^1(R), \mathfrak{p} \text{ is homogeneous} > .$$

Thus $Cl(R)$ is generated by $[\mathfrak{p}]$, where $\mathfrak{p}$ ranges over all homogeneous prime ideals of height 1. □

Proof of Theorem 30 By the localization sequence, the map $Cl(R) \xrightarrow{\pi} Cl(R_{\mathfrak{m}})$ is always surjective. We want to show that π is injective.

Let D be a divisor representing a class $[D]$ in $Cl(R)$ such that $\pi([D]) = 0$. Without loss of generality we can assume D is effective and moreover $D = \sum n_{\mathfrak{p}}\mathfrak{p}$ where $\mathfrak{p}$ is a homogeneous prime of height 1. Set $I = \bigcap_{\mathfrak{p} \in X^1_{\mathrm{hom}}(R)} \mathfrak{p}^{(n_{\mathfrak{p}})}$. Then as an exercise one can check that I is homogeneous. Now $\mathrm{div}(I) = D$. Since $\pi([D]) = 0$, $I_{\mathfrak{m}}$ is a principal ideal in $R_{\mathfrak{m}}$ and hence is free. Therefore by remark 7(c), I is free i.e. I is principal. This implies that D is a principal divisor i.e. $[D] = 0$ in $Cl(R)$. □

§ 1.9 *Faithfully Flat Extensions*

We want to prove

Theorem 33 *Let R be an integrally closed Noetherian local domain. Suppose that $\phi : R \longrightarrow S$ is a faithfully flat homomorphism, where S is an integrally closed Noetherian domain. Then there is a "natural map" $Cl(R) \xrightarrow{\tilde{\phi}} Cl(S)$ which is injective.*

Lemma 34 *Assume the conditions of Theorem 33. Then*
(1) *For every finitely generated R-module M, $M^{**} \otimes_R S \simeq (M \otimes_R S)^{**}$.*
(2) *If M is a rank r R-module, then $M \otimes_R S$ is a rank r S-module.*

Proof: (1) Recall that $\mathrm{Hom}_R(M,N)\otimes_R S \simeq \mathrm{Hom}_S(M\otimes_R S, N\otimes_R S)$ as long as $R \longrightarrow S$ is flat and M is finitely presented. Then

$$\mathrm{Hom}_S(\mathrm{Hom}_S(M\otimes_R S, S), S) \simeq \mathrm{Hom}_S(\mathrm{Hom}_R(M,R)\otimes_R S, S)$$

$$\simeq \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)\otimes_R S$$

i.e. $M^{**}\otimes_R S \simeq (M\otimes_R S)^{**}$.

(2) Let K and L be the fraction fields of R and S respectively. By faithful flatness, $\mathsf{K}\subseteq\mathsf{L}$. Then

$$(M\otimes_R S)\otimes_S \mathsf{L} \simeq M\otimes_R \mathsf{L} \simeq (M\otimes_R \mathsf{K})\otimes_{\mathsf{K}} \mathsf{L} \simeq \mathsf{K}^{\oplus r}\otimes_{\mathsf{K}}\mathsf{L} \simeq \mathsf{L}^{\oplus r}.$$

Thus if M is a rank r R-module, then $M\otimes_R S$ is a rank r S-module.

Proof of Theorem 33: Define $\tilde{\phi} : Cl(R) \longrightarrow Cl(S)$ by sending the isomorphism class of a rank 1, reflexive R-module M to the class of $M\otimes_R S$. By the above lemma, $M\otimes_R S$ is a rank 1, reflexive S-module. We need to check that $\tilde{\phi}$ is a group homomorphism, i.e. $\tilde{\phi}([M][N]) = \tilde{\phi}([M])\tilde{\phi}([N])$.
Now $[M][N] = [(M\otimes_R N)^{**}]$. Hence

$$\tilde{\phi}([M][N]) = [(M\otimes_R N)^{**}\otimes_R S]$$

$$= [((M\otimes_R S)\otimes_S (N\otimes_R S))^{**}] = \tilde{\phi}([M])\tilde{\phi}([N]).$$

To prove injectivity, it is enough to prove that if M is a reflexive R-module of rank 1 such that $M\otimes_R S\simeq S$, then $M\simeq R$.

Let $R^m \xrightarrow{[a_{ij}]} R^n \longrightarrow M \longrightarrow 0$ be a finite presentation for M over R (i.e. $a_{ij}\in\mathfrak{m}_R$). This induces the presentation $S^m \xrightarrow{[a_{ij}]} S^n \longrightarrow M\otimes_R S \longrightarrow 0$ of $M\otimes_R S$ over S. Since $R\longrightarrow S$ is faithfully flat, there is a maximal ideal $\mathfrak{n}$ in S containing $\mathfrak{m}_R S$. If $\mathfrak{l}\simeq S/\mathfrak{n}$, then tensoring with $\mathfrak{l}$ over S, we get $\mathfrak{l}^m \xrightarrow{[0]} \mathfrak{l}^n \longrightarrow M\otimes_R \mathfrak{l} \longrightarrow 0$ i.e. $M\otimes_R\mathfrak{l}\simeq\mathfrak{l}^n$. Since $M\otimes_R S\simeq S$, we must have $n=1$. Therefore $M\simeq R/J$. But M is reflexive and hence torsion-free i.e. $J=0$, which completes the proof. $\square$

Example 12
(1) If $(R,\mathfrak{m})$ is an integrally closed Noetherian local domain such that $\hat{R}$ is integrally closed, then $Cl(R)\hookrightarrow Cl(\hat{R})$.

(2) If $(R,\mathfrak{m})$ is an integrally closed Noetherian local domain, then $R[[T_1,\ldots,T_n]]$ is integrally closed (see the exercises). Moreover $Cl(R)\hookrightarrow Cl(R[[T_1,\ldots,T_n]])$.

§ 1.10 *Lipman's Theorem*

Definition 8 *An integrally closed Noetherian domain R is said to have a discrete divisor class group (DCG) if the map $Cl(R) \longrightarrow Cl(R[[T]])$ is an isomorphism.*

Some background lemmas

Lemma 35 *Let R be a Noetherian ring, M,N finitely generated R-modules. Then for any $i < \mathrm{depth}_{\mathrm{ann}_R(M)} N$, $\mathrm{Ext}^i_R(M,N) = 0$.*

Recall: In general, $\mathrm{depth}_I(N)$ = length of the longest regular sequence on N which is in I.

Proof: Induct on $\mathrm{depth}_{\mathrm{ann}_R(M)}(N) = n$. The case $n = 0$ is vacuously true. If $n = 1$, we need to prove that $\mathrm{Hom}_R(M,N) = 0$.

Choose $x \in \mathrm{ann}_R(M)$ which is a non-zerodivisor on N. Let $f \in \mathrm{Hom}_R(M,N)$ and m be any element in M. Then $0 = f(xm) = xf(m)$. Hence $f(m) = 0$ for every m in M which implies $f = 0$ in $\mathrm{Hom}_R(M,N)$ i.e. $\mathrm{Hom}_R(M,N) = 0$.

Now suppose that $n \geq 2$. Choose $x \in \mathrm{ann}_R(M)$ which is a non-zerodivisor on N. Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot x} N \longrightarrow N/xN \longrightarrow 0.$$

By applying $\mathrm{Hom}_R(M, \_)$, we get

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \xrightarrow{\cdot x} \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N/xN) \longrightarrow$$

$$\cdots \longrightarrow \mathrm{Ext}^i_R(M,N) \xrightarrow{\cdot x} \mathrm{Ext}^i_R(M,N) \longrightarrow \mathrm{Ext}^i_R(M,N/xN) \longrightarrow$$

But $\mathrm{depth}_{\mathrm{ann}_R(M)}(N/xN) = \mathrm{depth}_{\mathrm{ann}_R(M)}(N) - 1 = n-1$. By induction, $\mathrm{Ext}^i_R(M,N) = 0$ for all $i < n-1$. Now multiplication by x is zero on M and hence on $\mathrm{Ext}^i_R(M,N)$ (since $\mathrm{ann}_R(M) \cup \mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(\mathrm{Ext}^i_R(M,N))$ for all i). Therefore the long exact sequence breaks up into short exact sequences

$$0 \longrightarrow \mathrm{Ext}^i_R(M,N) \longrightarrow \mathrm{Ext}^i_R(M,N/xN) \longrightarrow \mathrm{Ext}^{i-1}_R(M,N) \longrightarrow 0.$$

This implies that $\mathrm{Ext}^i_R(M,N) = 0$ for any $i < \mathrm{depth}_{\mathrm{ann}_R(M)}(N)$. □

Lemma 36 (Ext Shifting) *Let R be a Noetherian ring, M,N be finitely generated R-modules. Assume that $xN = 0$ and x is a non-zerodivisor on both R and M. Set $\overline{R} := R/xR$ and $\overline{M} := M/xM$. Then for every $i \geq 0$,*

$$Ext^i_R(M,N) = Ext^i_{\overline{R}}(\overline{M},N).$$

Proof: Let $\mathbf{F}_\bullet$ be a free resolution of M. Since x is a non-zerodivisor on M, $\overline{\mathbf{F}_\bullet} := \mathbf{F}_\bullet \otimes_R \overline{R}$ is a free resolution of $\overline{M}$. Therefore

$$\mathrm{Ext}^i_{\overline{R}}(\overline{M}, N) \simeq H^i(\mathrm{Hom}_{\overline{R}}(\overline{\mathbf{F}_\bullet}, N))$$

$$\simeq H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet, N))$$

(since $xN = 0$). But $H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet, N)) \simeq Ext^i_R(M, N)$ which proves the result. $\square$

An aside about M^{**} and reflexive modules

We have to be careful in assuming M^{**} is reflexive. The first step is to know that it is non-trivial. Here is an example to illustrate this fact.

Example 13 Let $R = \mathsf{k}[X, Y]/(X, Y)^2$ where $\mathsf{k}$is a field. Let $M = \mathsf{k} = R/(x, y)$ where x and y are the respective images of X and Y modulo $(X, Y)^2$. We have $\mathsf{k}^* = \mathrm{Hom}_R(\mathsf{k}, R) = 0 :_R (x, y) = (x, y) \simeq \mathsf{k}^2$. Then $\mathsf{k}^{**} \simeq \mathsf{k}^4$. In fact $\mathsf{k}^{*n} \simeq \mathsf{k}^{2^n}$ for every positive integer n. Hence k^{*n} is not reflexive for any n.

Aside:
Some questions
1. Does M^{**} need to be reflexive over a one dimensional domain?

2. Let R be an Artinian local ring that is not Gorenstein and M an R-module that is not reflexive. Is it true that
 (a) M^{*^n} is not reflexive for any n?
 (b) $\mathsf{k} | M^{*^n}$, $n >> 0$?

Some Facts
1. Over a Gorenstein local Artinian ring, every module is reflexive.

2. Over an Artinian local ring that's not Gorenstein, $\mathsf{k}^* \simeq \mathsf{k}^t$ for some $t > 1$ i.e. k is never reflexive.

Remark 8 If R is an integrally closed Noetherian domain and M is any finitely generated R-module, then M^{**} is reflexive. In fact we will show that the first syzygy of any torsion-free module is reflexive. This will imply that M^* itself is reflexive.

Suppose M is torsion-free over R. Let F be a free R-module mapping onto M with kernel N i.e. we have a short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0.$$

By dualizing we get

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow N^* \longrightarrow \mathrm{Ext}^1_R(M,R) \longrightarrow 0.$$

Let K be the cokernel F^*/M^*. We have the short exact sequence

$$0 \longrightarrow K \longrightarrow N^* \longrightarrow \mathrm{Ext}^1_R(M,R) \longrightarrow 0.$$

By dualizing we get

$$0 \longrightarrow N^{**} \longrightarrow K^* \longrightarrow \mathrm{Ext}^1_R(\mathrm{Ext}^1_R(M,R),R) \longrightarrow \cdots .$$

Now $\mathrm{depth}_{\mathrm{ann}(\mathrm{Ext}^1_R(M,R))}(R) \geq 2$ and hence $\mathrm{Ext}^1_R(\mathrm{Ext}^1_R(M,R),R) = 0$ by lemma 35. This implies that $N^{**} \simeq K^*$ which gives us

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "Mt.f."] & 0 \\
0 \arrow[r] & K^* \arrow[r] & F^{**} \arrow[r] & M^{**} \arrow[r] & \mathrm{Ext}^1_R(K,R) \arrow[r] & 0
\end{tikzcd}$$

where the second short exact sequence is obtained by dualizing $0 \longrightarrow M^* \longrightarrow F^* \longrightarrow K \longrightarrow 0$.

Applying the Snake Lemma, we get that $N \simeq N^{**}$ i.e. any first syzygy of a torsion-free module is reflexive. We use the fact that the map from M to M^{**} is injective since M is torsion free.

Now let M be any arbitrary R-module. Then if F is a free module mapping onto M with kernel N, by dualizing we get $0 \longrightarrow M^* \longrightarrow F^* \longrightarrow N^*$. If C is the cokernel F^*/M^*, then $C \hookrightarrow N^*$ and hence is torsion-free. Since M^* is the first syzygy of C, this implies that M^* is reflexive.

Note that since a reflexive module is torsion-free, every syzygy of a torsion-free module is reflexive.

Justification for $\mathrm{depth}_{\mathrm{ann}(\mathrm{Ext}^1_R(M,R))}(R) \geq 2$:

Since M is torsion-free, $M_\mathfrak{p}$ is free over the DVR $R_\mathfrak{p}$ for every $\mathfrak{p}$ in $X^1(R)$. Hence $(\mathrm{Ext}^1_R(M,R))_\mathfrak{p} = 0$ for every $\mathfrak{p}$ in $X^1(R)$. This implies that $\mathrm{ht}(\mathrm{ann}(\mathrm{Ext}^1_R(M,R))) \geq 2$. Therefore $\mathrm{depth}_{\mathrm{ann}(\mathrm{Ext}^1_R(M,R))}(R) \geq 2$ since R satisfies S_2.

Setup for Lipman's Theorem

Let R be an integrally closed Noetherian domain, t a non-zero element in R such that R/tR is also an integrally closed domain. (eg. $R = A[[T]]$, where A is integrally closed, then $A \simeq R/TR$). By abuse of language, we write M^* to mean the dual of M as usual, but we also write $(M/tM)^*$ to mean the $\mathrm{Hom}_{R/tR}(M/tM, R/tR)$.

We define $j : Cl(R) \longrightarrow Cl(R/tR)$ as follows: If M is a reflexive R-module of rank 1, then $j([M]) := [(M/tM)^{**}]$. By remark 8, $(M/tM)^{**}$ is a reflexive R/tR-module. We need to show that it has rank 1.

It is enough to prove that M/tM has rank 1 i.e. it is enough to show that $(M/tM) \otimes_{\frac{R}{tR}} \mathsf{L} \simeq \mathsf{L}$ where L is the fraction field of R/tR.
But we have

$$M/tM \otimes_{\frac{R}{tR}} \mathsf{L} \simeq M \otimes_{\frac{R}{tR}} \mathsf{L} \simeq M_{(t)} \otimes_{\frac{R}{tR}} \mathsf{L} \simeq R_{(t)} \otimes_{\frac{R}{tR}} \mathsf{L}$$

since $M_{(t)} \simeq R_{(t)}$ as $M_{(t)}$ is a torsion-free module of rank one over the DVR $R_{(t)}$.
Therefore, we now have

$$M/tM \otimes_{\frac{R}{tR}} \mathsf{L} \simeq R_{(t)} \otimes_{\frac{R}{tR}} \mathsf{L} \simeq R_{(t)} \otimes_R \mathsf{L} \simeq \mathsf{L} = R_{(t)}/tR_{(t)}.$$

Exercise: Let $\mathfrak{p}$ be a prime ideal in R and M be an R-module. Then show that $\operatorname{rank}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = \mu(M_{\mathfrak{p}})$.

Theorem 37 (Lipman) *Let S be an integrally closed Noetherian domain, t be a non-zero element in* Jac*(S). Suppose that $R := S/tS$ is also an integrally closed domain and that the map $j_{\mathfrak{p}} : Cl(S_{\mathfrak{p}}) \longrightarrow Cl(R_{\mathfrak{p}})$ is injective for every $\mathfrak{p}$ in the set*

$$\{\mathfrak{p} \in \operatorname{Spec}(S) : t \in \mathfrak{p}, \operatorname{depth}(R_{\mathfrak{p}}) \leq 2\}.$$

Then $j : Cl(S) \longrightarrow Cl(R)$ is injective.

We will first prove a couple of lemmas.

Lemma 38 *Let S and R be as in Theorem 37, M is a finitely generated S-module and $t \in S$ a non-zerodivisor on M. Suppose $\overline{M} := M/tM$ has the property that $\overline{M}^{**}$ is free and $\overline{M}_{\mathfrak{p}}$ is reflexive for every $\mathfrak{p}$ in the set $\{\mathfrak{p} \in \operatorname{Spec}(S) : t \in \mathfrak{p}, \operatorname{depth}(R_{\mathfrak{p}}) \leq 2\}$. Then,*
(1) $\operatorname{Ext}^1_S(M, S) = 0$.
(2) $\overline{M^*} \simeq \overline{M}^* := \operatorname{Hom}_R(\overline{M}, R)$.

Proof: Apply $\operatorname{Hom}_S(M, \_)$ to $0 \longrightarrow S \xrightarrow{\cdot t} S \longrightarrow S/tS \longrightarrow 0$ giving

$$0 \longrightarrow \operatorname{Hom}_S(M, S) \xrightarrow{\cdot t} \operatorname{Hom}_S(M, S) \longrightarrow \operatorname{Hom}_S(M, S/tS)$$

$$\longrightarrow \operatorname{Ext}^1_S(M, S) \xrightarrow{\cdot t} \operatorname{Ext}^1_S(M, S) \longrightarrow \operatorname{Ext}^1_S(M, S/tS) \longrightarrow \cdots .$$

Note that by the Ext-shifting lemma

$$\operatorname{Hom}_S(M, R) \simeq \operatorname{Hom}_R(\overline{M}, R) \quad \text{and} \quad \operatorname{Ext}^1_S(M, R) \simeq \operatorname{Ext}^1_R(\overline{M}, R).$$

Hence it follows immediately from the above long exact sequence that (1) implies (2).
Since $t \in \operatorname{Jac}(S)$, by NAK it is also clear from the same long exact sequence that to prove (1), it suffices to prove $\operatorname{Ext}^1_S(M, R)$ $(\simeq \operatorname{Ext}^1_R(\overline{M}, R)) = 0$.

Let K and C denote the kernel and cokernel respectively of the natural map $\overline{M} \longrightarrow \overline{M}^{**}$. The assumptions give that $K_{\mathfrak{p}} = 0$ and $C_{\mathfrak{p}} = 0$ for every $\mathfrak{p}$ in R such that $\operatorname{depth}(R_{\mathfrak{p}}) \leq 2$. Hence $\operatorname{ann}_R(K)$ and $\operatorname{ann}_R(C)$ are not contained in any such $\mathfrak{p}$ i.e. $\operatorname{depth}_{\operatorname{ann}_R(K)}(R) \geq 3$ and $\operatorname{depth}_{\operatorname{ann}_R(C)}(R) \geq 3$.

Let I be the cokernel $\overline{M}/K$. We have the two short exact sequences: $0 \longrightarrow K \longrightarrow \overline{M} \longrightarrow I \longrightarrow 0$ and $0 \longrightarrow I \longrightarrow \overline{M}^{**} \longrightarrow C \longrightarrow 0$.

By applying $\operatorname{Hom}(\_, R)$ to both, we get

$$0 \longrightarrow I^* \longrightarrow \overline{M}^* \longrightarrow K^* \longrightarrow \operatorname{Ext}^1_R(I,R) \longrightarrow \operatorname{Ext}^1_R(\overline{M},R) \longrightarrow \operatorname{Ext}^1_R(K,R) \text{ and}$$

$$\begin{aligned} 0 \longrightarrow C^* \longrightarrow \overline{M}^{**} \longrightarrow I^* \longrightarrow \operatorname{Ext}^1_R(C,R) \longrightarrow \\ \longrightarrow \operatorname{Ext}^1_R(\overline{M}^{**},R) \longrightarrow \operatorname{Ext}^1_R(I,R) \longrightarrow \operatorname{Ext}^2_R(C,R). \end{aligned}$$

Recall that by lemma 35, $\operatorname{Ext}^i_R(K,R) = \operatorname{Ext}^i_R(C,R) = 0$ for $i < 3$. So from the first exact sequence above it is clear that in order to prove $\operatorname{Ext}^1_R(\overline{M},R) = 0$, it is enough to show that $\operatorname{Ext}^1_R(I,R) = 0$.

From the second exact sequence above, we get that $\operatorname{Ext}^1_R(I,R) \simeq \operatorname{Ext}^1_R(\overline{M}^{**},R)$ which is zero since $\overline{M}^{**}$ is free over R. This proves the lemma. $\square$

Aside: In general, if N is a finitely generated R-module, such that $N_{\mathfrak{p}} = 0$ for every $\mathfrak{p}$ such that $\operatorname{depth}(\mathfrak{p}) \leq k$, then $l = \operatorname{depth}_{\operatorname{ann}_R(N)}(R) \geq k+1$. This can be seen as follows:

Let $a_1, \ldots, a_l$ be a maximal regular sequence in $\operatorname{ann}_R(N)$. Then $\operatorname{ann}_R(N) \subseteq Q$ for every Q in $Ass_R(R/(a_1, \ldots, a_l))$. This means that $\operatorname{depth}(R_Q) = l$. Since $N_Q \neq 0$, $l \geq k+1$.

Lemma 39 *Let T be a Noetherian ring, M a finitely generated T-module. Suppose that there is an element $x \in \operatorname{Jac}(T)$ such that x is a non-zerodivisor on M. If M/xM is free over T/xT, then M is a free T-module.*

Proof: Let $z_1, \ldots, z_r$ be elements of M such their images $\overline{z_1}, \ldots, \overline{z_r}$ in M/xM form a basis for M/xM over T/xT. Then $M \subseteq Tz_1 + \cdots + Tz_r + xM$ which implies that $M = Tz_1 + \cdots + Tz_r$ by NAK.

Let e_i, $1 \leq i \leq r$ be the standard basis for T^r. Then the map $e_i \mapsto z_i$ maps T^r onto M. Let N be its kernel. Let $n = (t_1, \ldots, t_r) \in N$. Then $\sum_{i=1}^r t_i z_i = 0$ in M. Hence $\sum_{i=1}^r t_i \overline{z_i} = 0$ in M/xM. Since $\overline{z_i}$ is a basis for M/xM, $t_i \in xT$ for each i. Write $t_i = xt_i'$ for some t_i' in T. Then we have $\sum_{i=1}^r xt_i' z_i = 0$ which implies that $\sum_{i=1}^r t_i' z_i = 0$ since x is a non-zerodivisor on M. Therefore, as above, $t_i' \in xT$ for each i. Continuing thus, we see that for each i, $t_i \in \bigcap_{n=1}^\infty x^n = 0$ by Krull's Intersection

Theorem. Thus $t_i = 0$ for each i, which proves that $N = 0$ i.e. $M \simeq T^r$. Thus M is a free T-module. $\square$

Proof of Lipman's Theorem: Suppose $j([M]) = 1$ for a rank 1, reflexive S-module M. Then $(M/tM)^{**} \simeq R$ is free. Moreover since $j_{\mathfrak{p}}$ is injective for every $\mathfrak{p}$ such that $\operatorname{depth}(R_{\mathfrak{p}}) \leq 2$, $M_{\mathfrak{p}} \simeq S_{\mathfrak{p}}$ is a free $S_{\mathfrak{p}}$-module. In particular, $\overline{M_{\mathfrak{p}}}$ is reflexive (since it is free). By Lemma 38, $\operatorname{Ext}^1_S(M, S) = 0$ and $\overline{M}^* = \overline{M^*}$. But R is integrally closed. Hence by remark 8, $\overline{M}^* \simeq \overline{M}^{***}$. Since $\overline{M}^{**}$ is free, so is $\overline{M}^{***}$ which implies that $\overline{M^*}$ is a free R-module of rank 1 i.e. $\overline{M^*} \simeq R$. Therefore by lemma 39, M^* is a free S-module and hence $M^* \simeq S$. Then $M \simeq M^{**} \simeq S^* \simeq S$ which proves that j is injective. $\square$

Corollary 40 *Let R be an integrally closed Noetherian domain, $S := R[[T]]$. If R satisfies S_3 and R_2, then the natural map $Cl(R) \xrightarrow{i} Cl(S)$ is an isomorphism. In particular, if R is a UFD, then so is S.*

Proof: Recall that $i([M]) = [M \otimes_R S]$ for any rank 1 reflexive R-module M. The map $Cl(R) \xrightarrow{i} Cl(S)$ is injective since $R \longrightarrow S$ is faithfully flat. In Lipman's Theorem, let $t = T$. If $j_{\mathfrak{p}} : Cl(S_{\mathfrak{p}}) \longrightarrow Cl(R_{\mathfrak{p}})$ is injective for every $\mathfrak{p}$ in the set $\{\mathfrak{p} \in \operatorname{Spec}(S) : T \in \mathfrak{p}, \operatorname{depth}(R_{\mathfrak{p}}) \leq 2\}$, then by Lipman's Theorem, $j : Cl(S) \longrightarrow Cl(R)$ is injective.

Now since R is S_3, $\operatorname{depth}(R_{\mathfrak{p}}) \leq 2$ implies that $\dim(R_{\mathfrak{p}}) \leq 2$. Hence by R_2, $R_{\mathfrak{p}}$ is a regular local ring. This gives us $R_{\mathfrak{p}}[[T]]$ is regular and hence a UFD. This means that $j_{\mathfrak{p}}$ is trivially injective. (Note that since $\mathfrak{p} = (\mathfrak{p} \cap R, T)$, $R_{\mathfrak{p}}[[T]] \simeq S_{\mathfrak{p}}$). Hence $j : Cl(S) \longrightarrow Cl(R)$ is injective.

So, we have $Cl(R) \overset{i}{\hookrightarrow} Cl(S) \overset{j}{\hookrightarrow} Cl(R)$. If M is a rank 1, reflexive R-module,

$$(j \circ i)([M]) = j([M \otimes_R S]) = [(M \otimes_R S \otimes_S R)^{**}] = [M^{**}] = [M]$$

i.e. $j \circ i : Cl(R) \longrightarrow Cl(R)$ is the identity map on $Cl(R)$. Hence i is an isomorphism. $\square$

§ 1.11 *Auslander's Theorems*

In this section we are aiming for the following two theorems of Auslander:

Theorem 41 (Auslander) *Let*

$$\mathfrak{S} := \{(R, M) : (R, \mathfrak{m}, \mathsf{k}) \text{ is a Noetherian local ring and } M \text{ is a finitely generated reflexive } R\text{-module (i.e. } M \simeq M^{**})\}.$$

Suppose that $\mathfrak{S}$ satisfies the following three conditions:
(i) $(R, M) \in \mathfrak{S} \implies (R_{\mathfrak{p}}, M_{\mathfrak{p}}) \in \mathfrak{S}$ *for every* $\mathfrak{p} \in \operatorname{Spec}(R)$,

(ii) $(R, M) \in \mathfrak{S}$ *and* $\operatorname{depth}(R) \leq 3 \implies M$ *is free, and*
(iii) $\operatorname{depth}(R) > 3$, $(R, M) \in \mathfrak{S}$ *and* $M_{\mathfrak{p}}$ *is free for every* $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \{\mathfrak{m}\} \implies$ *there is an element* $x \in \mathfrak{m}_R$*, a non-zerodivisor on* R *such that* $(R/xR, (M/xM)^{**}) \in \mathfrak{S}$.

Then, for every $(R, M) \in \mathfrak{S}$, M *is free.*

Theorem 42 (Auslander) *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a regular local ring,* M *a reflexive* R*-module such that* $\operatorname{Hom}(M, M) \simeq M^{\oplus t}$*. Then* M *is free.*

A lemma we use in this section is the following:

Lemma 43 *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a Noetherian local ring. Consider an exact sequence* $0 \to K \to A \to B \to C \to 0$ *of* R*-modules, where* $\lambda_R(K) < \infty$ *and* $\lambda_R(C) < \infty$*. Then* $\operatorname{Ext}^i_R(A, R) \simeq \operatorname{Ext}^i_R(B, R)$ *for* $i < \operatorname{depth}(R) - 1$.

Proof: Let $I = \operatorname{Coker}(K \to A) = \operatorname{Ker}(B \to C)$. Thus we get two short exact sequences
$$0 \to K \to A \to I \to 0 \text{ and } 0 \to I \to B \to C \to 0.$$

Apply $\operatorname{Hom}_R(\_, R)$ to the two short exact sequences to get

$$0 \to K^* \to A^* \to I^* \to \operatorname{Ext}^1_R(K, R) \to \operatorname{Ext}^1_R(A, R) \to \operatorname{Ext}^1_R(I, R) \to \cdots$$

and

$$0 \to C^* \to B^* \to I^* \to \operatorname{Ext}^1_R(C, R) \to \operatorname{Ext}^1_R(B, R) \to \operatorname{Ext}^1_R(I, R) \to \cdots .$$

Since K and C are modules of finite length, $\operatorname{Ext}^i_R(K, R) = 0 = \operatorname{Ext}^i_R(C, R)$ for $i < \operatorname{depth}(R)$.

Thus from the above long exact sequences, we get $\operatorname{Ext}^i_R(A, R) \simeq \operatorname{Ext}^i_R(I, R)$ for $i < \operatorname{depth}(R)$ and $\operatorname{Ext}^i_R(B, R) \simeq \operatorname{Ext}^i_R(I, R)$ for $i < \operatorname{depth}(R) - 1$ which proves the lemma. □

Proof of Theorem 41: Induct on $\dim(R)$. If $\dim(R) \leq 3$, then $\operatorname{depth}(R) \leq 3$ and hence M is free whenever $(R, M) \in \mathfrak{S}$ by (ii).

Suppose $(R, M) \in \mathfrak{S}$. Assume that M' is free whenever $(R', M') \in \mathfrak{S}$ with $\dim(R') < \dim(R)$. If $\operatorname{depth}(R) \leq 3$, M is free by (ii). Hence we may assume that $\operatorname{depth}(R) > 3$. By (i), we have $(R_{\mathfrak{p}}, M_{\mathfrak{p}}) \in \mathfrak{S}$ for every prime ideal $\mathfrak{p}$ in R. Since $\dim(R_{\mathfrak{p}}) < \dim(R)$ for $\mathfrak{p} \neq \mathfrak{m}$, by assumption, $M_{\mathfrak{p}}$ is free. Choose an $x \in \mathfrak{m}$ as in (iii). Since $\dim(\overline{R}) < \dim(R)$, by induction $\overline{M}^{**}$ is free, where $^-$ denotes going modulo x.

Apply $\operatorname{Hom}_R(M, \_)$ to the short exact sequence $0 \to R \xrightarrow{\cdot x} R \to \overline{R} \to 0$ to get the long exact sequence

$$0 \to M^* \xrightarrow{\cdot x} M^* \to \overline{M}^* \to \operatorname{Ext}^1_R(M, R) \xrightarrow{\cdot x} \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(M, \overline{R}) \to \cdots . \quad (*)$$

Claim: $\mathrm{Ext}^1_R(M,R)=0$.
Since $M \simeq M^{**}$, x is a non-zerodivisor on M. By NAK, to prove the claim it is enough to prove that $\mathrm{Ext}^1_R(M,\overline{R}) \simeq \mathrm{Ext}^1_{\overline{R}}(\overline{M},\overline{R})=0$.

Let K and C be defined by the exact sequence $0 \to K \to \overline{M} \to \overline{M}^{**} \to C \to 0$. Both K and C have finite length, since for every $\mathfrak{p} \in \mathrm{Spec}(R)\setminus\{\mathfrak{m}\}$, $M_{\mathfrak{p}}$ (and therefore $\overline{M}_{\mathfrak{p}}$) is free (and hence reflexive). Since $\mathrm{depth}(\overline{R}) \geq 3$, by Lemma 43, we have

$$\mathrm{Ext}^i_{\overline{R}}(\overline{M}^{**},\overline{R}) \simeq \mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{R}) \quad i=0,1. \qquad (**)$$

Thus in order to prove the claim, we now need to prove that $\mathrm{Ext}^1_{\overline{R}}(\overline{M}^{**},\overline{R})=0$. But by (iii), $(\overline{R},\overline{M}^{**}) \in \mathfrak{S}$. Hence by induction, $\overline{M}^{**}$ is $\overline{R}$-free. This implies that $\mathrm{Ext}^1_{\overline{R}}(\overline{M}^{**},\overline{R})=0$ and hence proves the claim that $\mathrm{Ext}^1_R(M,R)=0$.

Therefore, by $(*)$, we have $\overline{M}^{*} \simeq \overline{M^{*}}$. But by $(**)$ we have $\overline{M}^{*} \simeq \overline{M}^{***}$. Hence $\overline{M}^{**}$ being $\overline{R}$-free forces $\overline{M}^{***}$ to be $\overline{R}$-free, which in turn implies that $\overline{M}^{*}$ is $\overline{R}$-free.

Thus $\overline{M^{*}}$ is $\overline{R}$-free which implies that M^* is R-free. Therefore M^{**} is R-free. But M is reflexive, which proves the result. $\square$

Proof of Theorem 42: Let $\mathfrak{S}$ be the class of all pairs (R,M), where $(R,\mathfrak{m},\mathsf{k})$ is a regular local ring and M is a finitely generated reflexive R-module such that $\mathrm{Hom}_R(M,M) \simeq M^{\oplus t}$ for some t. We will prove that $\mathfrak{S}$ satisies the conditions (i) - (iii) of Theorem 41.

Clearly (i) is true. In order to prove (ii), if $\dim(R) \leq 2$, we claim that since M is a finitely generated reflexive R-module, it is free. To see this consider a presentation $F \xrightarrow{\phi} G \to M^* \to 0$ of M^*. Applying *, we get $0 \to M^{**} \to G^* \xrightarrow{\phi^*} F^* \to C \to 0$, where $C=\mathrm{Coker}(\phi^*)$. Since $\dim(R) \leq 2$ and R is regular, $\mathrm{pd}_R(C) \leq 2$ forcing M^{**} to be free. Since M is reflexive, this implies that M is free.
(To sum it all up, M is free since it is a second syzygy.)

Thus without loss of generality we may assume that $\dim(R)=3$. Since M is reflexive, as seen above, it is a second syzygy and hence $\mathrm{depth}(M) \geq 2$. If $\mathrm{depth}(M)=3$, then by the Auslander-Buchsbaum formula, $\mathrm{pd}_R(M)=0$, i.e. M is free. So we may assume that $\mathrm{depth}(M)=2$ and hence $\mathrm{pd}_R(M)=1$.

Note that by our reductions, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for any non-maximal prime ideal $\mathfrak{p}$ in R. Moreover, by tensoring $\mathrm{Hom}_R(M,M) \simeq M^{\oplus t}$ with K, the fraction field of R, we see that t is the rank of M. Hence (ii) follows by the following proposition.

Proposition 44 *Let* $(R,\mathfrak{m},\mathsf{k})$ *be a Noetherian local ring such that* $\mathrm{depth}(R) \geq 3$. *Suppose that M is a finitely generated reflexive R-module such that:*
a) $M_{\mathfrak{p}}$ *is* $R_{\mathfrak{p}}$*-free for every* $\mathfrak{p} \in \mathrm{Spec}(R)\setminus\{\mathfrak{m}\}$.
b) $\mathrm{pd}_R(M) \leq 1$.
c) $\lambda_R(\mathrm{Ext}^1_R(\mathrm{Hom}_R(M,M),R)) \leq \lambda_R(\mathrm{Ext}^1_R(M^{\oplus t},R))$, *where t is the rank of M.*
Then M is free.

In order to prove (iii) in Theorem 42, let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then x is a non-zerodivisor on $M \simeq M^{**}$. Applying $\mathrm{Hom}_R(M, \_)$ to the short exact sequence $0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0$ and denoting going modulo x by $^-$, we get the long exact sequence

$$0 \to \mathrm{Hom}_R(M,M) \xrightarrow{x} \mathrm{Hom}_R(M,M) \to \mathrm{Hom}_R(M,\overline{M}) \to \mathrm{Ext}^1_R(M,M).$$

Note that $\mathrm{Hom}_R(M,M) \simeq M^{\oplus t}$, $\mathrm{Hom}_R(M,\overline{M}) \simeq \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M})$ and $\mathrm{Ext}^1_R(M,M)$ has finite length (since $M_\mathfrak{p}$ is free for $\mathfrak{p} \neq \mathfrak{m}$). Hence, from the long exact sequence, we get a short exact sequence $0 \to \overline{M}^{\oplus t} \to \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M}) \to L \to 0$, where the cokernel L has finite length.

Since $\mathrm{depth}(\overline{R}) > 2$ and L has finite length, we have $\mathrm{Ext}^i_{\overline{R}}(L,\overline{R}) = 0$, $i = 0,1,2$. Applying $\mathrm{Hom}_{\overline{R}}(\_,\overline{R})$ to the above short exact sequence, we get $\mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M})^{**} \simeq (\overline{M}^{**})^{\oplus t}$. It suffices to show that $\mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M})^{**} \simeq \mathrm{Hom}_{\overline{R}}(\overline{M}^{**},\overline{M}^{**})$ to finish the proof, since this will imply that $(\overline{R},\overline{M}^{**}) \in \mathfrak{S}$. By applying the following lemma twice, the proof is complete. □

Lemma 45 *Let $(R,\mathfrak{m},\mathsf{k})$ be a Cohen-Macaulay local ring, M a finitely generated R-module, such that $M_\mathfrak{p}$ is free for every $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{ht}(\mathfrak{p}) \leq 1$. Then, $\mathrm{Hom}_R(M,M)^* \simeq \mathrm{Hom}_R(M^*,M^*)$.*

Note: The key point in the proof of the lemma as well as Prop. 44 is the existence of a natural map $M^* \otimes_R M \to \mathrm{Hom}_R(M,M)$, given by $f \otimes m \mapsto [x \mapsto f(x)m]$, which is an isomorphism if and only if M is free.

Let K and C be the kernel and the cokernel respectively of this map. Then we get an exact sequence

$$0 \to K \to M^* \otimes_R M \to \mathrm{Hom}_R(M,M) \to C \to 0. \qquad (*)$$

Proof: Let K and C be as in $(*)$. Since $M_\mathfrak{p}$ is free whenever $\mathrm{ht}\mathfrak{p} \leq 1$, we have $K_\mathfrak{p} = 0 = C_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathrm{ht}(\mathfrak{p}) \leq 1$. Hence $\mathrm{ht}(\mathrm{ann}(K)) \geq 2$ and $\mathrm{ht}(\mathrm{ann}(C)) \geq 2$. This gives us $\mathrm{Ext}^1_R(K,R) = 0 = \mathrm{Ext}^1_R(C,R)$, which forces $\mathrm{Hom}_R(M,M)^* \simeq (M^* \otimes_R M)^*$. But by the $\mathrm{Hom} - \otimes$ adjointness, $(M^* \otimes_R M)^* \simeq \mathrm{Hom}_R(M^*,M^*)$ proving the lemma. □

Let us now prove Prop. 44, to complete the proof of Auslander's Theorems.

Proof of Proposition 44: Let K and C be as in $(*)$. By (i), K and C are $\mathfrak{m}$-primary. This forces $\mathrm{Ext}^1_R(M^* \otimes_R M, R) \simeq \mathrm{Ext}^1_R(\mathrm{Hom}_R(M,M),R)$ by Lemma 43 since $1 < \mathrm{depth}(R) - 1$.

Thus (c) can be replaced by (c′)

$$\lambda_R(\mathrm{Ext}^1_R(M^* \otimes_R M, R)) \leq \lambda_R(\mathrm{Ext}^1_R(M^{\oplus t}, R)), \qquad t = \mathrm{rank}(M).$$

We have reduced the problem to proving M is free, assuming (a), (b) and (c$'$). Suppose M is not free. Without loss of generality , we may assume that $\mathrm{pd}_R(M) = 1$. Let

$$(\#) \qquad 0 \to R^{n-t} \xrightarrow{\phi} R^n \to M \to 0$$

be a minimal resolution of M over R.

Step 1: We claim that $\lambda_R(\mathrm{Ext}^1_R(M, M)) > t\lambda_R(\mathrm{Ext}^1_R(M, R))$.
Note that since $M_\mathfrak{p}$ is free for every $\mathfrak{p} \neq \mathfrak{m}$, $\mathrm{Ext}^i_R(M, \_)_\mathfrak{p} = 0$ for every $\mathfrak{p} \neq \mathfrak{m}$ and $\mathrm{Ext}^i_R(M, \_)$ are all of finite length.
Set $L = \mathrm{Ext}^1_R(M, R)$. Tensor $(\#)$ with L to get

$$0 \to \mathrm{Tor}^R_1(M, L) \to L^{n-t} \xrightarrow{\phi \otimes 1} L^n \to M \otimes_R L \to 0.$$

Note that $\mathrm{Ker}(\phi \otimes 1) \neq 0$ since $\mathrm{soc}(L^{n-t}) \mapsto 0$. Hence

$$\lambda_R(M \otimes_R N) + (n-t)\lambda_R(L) > n\lambda_R(L), \text{ i.e. } \lambda_R(M \otimes_R L) > t\lambda_R(L).$$

In order to prove the claim, we show that $M \otimes_R L \simeq \mathrm{Ext}^1_R(M, M)$.
Apply * to $(\#)$ to get

$$0 \to M^* \to R^{*n} \xrightarrow{\phi^*} R^{*(n-t)} \to L \to 0.$$

Tensoring with M gives an exact sequence

$$M \otimes_R R^{*n} \to M \otimes_R R^{*(n-t)} \to M \otimes_R L \to 0. \qquad (1)$$

Apply $\mathrm{Hom}_R(\_, M)$ to $(\#)$ to get

$$\mathrm{Hom}_R(R^n, M) \to \mathrm{Hom}_R(R^{n-t}, M) \to \mathrm{Ext}^1_R(M, M) \to 0. \qquad (2)$$

(1) and (2) give $M \otimes_R L \simeq \mathrm{Ext}^1_R(M, M)$, which proves the claim.

Step 2: We claim that $\mathrm{Ext}^1_R(M, M) \hookrightarrow \mathrm{Ext}^1_R(M^* \otimes M, R)$.
Applying $\otimes_R M^*$ to $(\#)$, we get the exact sequence

$$0 \to \mathrm{Tor}^R_1(M, N) \to M^* \otimes_R R^{n-t} \to M^* \otimes_R R^n \to M^* \otimes_R M \to 0.$$

Since $M_\mathfrak{p}$ is free for all $\mathfrak{p} \neq \mathfrak{m}$, $\mathrm{Tor}^R_1(M, N)$ has finite length. But $M^* \otimes_R R^{n-t}$ is reflexive. Hence $\mathrm{Tor}^R_1(M, N) = 0$. Thus we get a short exact sequence

$$0 \to M^* \otimes_R R^{n-t} \to M^* \otimes_R R^n \to M^* \otimes_R M \to 0.$$

Applying $\mathrm{Hom}_R(\_, R)$ to the above sequence, we get

$$0 \to (M^* \otimes_R M)^* \to (M^* \otimes_R R^n)^* \to (M^* \otimes R^{n-t})^* \to \mathrm{Ext}^1_R(M^* \otimes_R M, R).$$

By the $\mathrm{Hom} - \otimes$ adjointness we know that, for any N, $(N \otimes M^*)^* \simeq \mathrm{Hom}_R(N, M^{**})$. But in this case, $M^{**} \simeq M$. Hence the above sequence becomes

$$\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(R^n, M) \to \mathrm{Hom}_R(R^{n-t}, M) \to \mathrm{Ext}^1_R(M^* \otimes_R M, R).$$

This proves step 2 since $\mathrm{Ext}^1_R(M, M) \simeq \mathrm{Coker}(\mathrm{Hom}_R(R^n, M) \to \mathrm{Hom}_R(R^{n-t}, M))$.

Combining Step 1 and Step 2, we get a contradiction to (c′), proving that $\mathrm{pd}_R(M) \neq 1$. Thus M is free. $\square$

Exercises

(1) Let R be a Noetherian domain and let $\mathfrak{q}$ be a height one prime. If $0 \neq x \in \mathfrak{q}$, give a description of $\mathfrak{q}^*(x)$.

(2) Let R be a Noetherian integrally closed domain and let $\mathfrak{q}$ be a prime of height at least two. If $0 \neq x \in \mathfrak{q}$, give a description of $\mathfrak{q}^*(x)$.

(3) Let R be a Noetherian ring. Prove that R satisfies Serre's condition S_i iff whenever $x_k \in R$, $1 \leq j \leq i-1$ and $(x_1, ..., x_j)$ has height j, then $(x_1, ..., x_j)$ has no associated primes of height greater than j.

(4) Let R be a Noetherian domain and let $x \in R$ be a prime element. If R_x is integrally closed, prove that R is integrally closed. Extend this to localizing at a multiplicatively closed set generated by prime elements.

(5) Prove that the ring $R = \mathsf{k}[x, y, z]_{(x,y,z)}/(f)$ is a UFD, where k is a field of characteristic 0 or characteristic at least 11, and $f = x^2 + y^3 + z^7$.

(6) Let R be an integrally closed domain, and suppose that $x, y, z \in R$ are elements which satisfy the following conditions:

(a) x is a prime element.

(b) x, y form a regular sequence.

(c) There exists positive integers i, j, k such that $z^{i-1} \notin (x, y)R$, but $z^i \in (x^j, y^k)$ and such that $ijk - ij - jk - ik \geq 0$.

Prove that $R[[T]]$ is not a UFD.

(7) Let M be a finitely generated module over a ring R. Let A be a r by s matrix presenting M, i.e. there is an exact sequence

$$R^s \xrightarrow{A} R^r \to M \to 0.$$

Prove that the r by r minors of A annihilate M.

(8) Let X be a generic 2 by 3 matrix over a field k, and let R be the quotient ring of $\mathsf{k}[X]$ modulo the ideal of 2 by 2 minors of X. Show that R is integrally closed, and find the class group.

(9) Let X be a generic symmetric 3 by 3 matrix over a field k, and let R be the quotient ring of $\mathsf{k}[X]$ modulo the ideal of 2 by 2 minors of X. Show that R is integrally closed, and find the class group.

(10) Let R be an integrally closed Noetherian domain, and let M be a finitely generated R module. Fix an arbitrary prime filtration $0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_n = M$, where $M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1}$. Define a map from M to the class group of R by sending M to the sum of the classes of all $\mathfrak{p}_i$ which are height one (counted as many times as they appear in the filtration). Prove that this map is well-defined, and that it agrees with the map from M to the class group of R which we defined in class.

(11) Let R be a Noetherian ring and let M be a finitely generated R-module. Set $(\quad)^* = \mathrm{Hom}_R(\quad, R)$. Prove that $M \cong M^{**}$ if and only if the canonical map from M to M^{**} is an isomorphism.

(12) Let R, S be Noetherian integrally closed domains, and assume that $\phi : R \longrightarrow S$ is a faithfully flat map. Assume that rank one projective R-modules are free. Prove that the induced map on class groups $Cl(R) \longrightarrow Cl(S)$ is injective.

(13) Let $R = \oplus_{i \geq 0} R_i$ be a Noetherian graded integrally closed domain over a field $\mathsf{k} = R_0$. Let $\mathfrak{l}$ be an extension field of k, and set $S = R \otimes_{\mathsf{k}} \mathfrak{l}$. Prove that the map from $R \longrightarrow S$ is faithfully flat. Assuming that S is an integrally closed domain, prove that the embedding of R into S induces an injection on the class groups of R and S.

(14) Prove an integrally closed Noetherian domain R is a UFD if and only if $R_{\mathfrak{m}}$ is a UFD for all maximal ideals $\mathfrak{m}$ of R and rank one projective R-modules are free.

(15) Find the class group of $R = \mathbb{C}[x, y, z]/(xy - z^n)$.

(16) Let R be an integrally closed Noetherian domain and let M be a finitely generated torsion-free R-module. Prove that there are at most finitely many primes $\mathfrak{p}_1, ..., \mathfrak{p}_n$ of height two such that $M_{\mathfrak{p}_i}$ is not $R_{\mathfrak{p}_i}$-free.

(17) Let $R = \mathbb{C}[[x, y, z]]/(f)$ be integrally closed. Give an algorithm to decide whether or not $Cl(R)$ is torsion-free.

(18) Let R be a Noetherian ring and let $I \subseteq J$ be ideals. Prove that $I = J$ iff $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for every associated prime $\mathfrak{p}$ of I.

(19) Let R be an integrally closed Noetherian domain and let M be a finitely generated R-module. Prove that M is reflexive if and only if every regular sequence x, y in R is a regular sequence on M.

(20) A polynomial $f \in R[X_1, ..., X_n]$ is said to be *primitive* if the coefficients of f generated the unit ideal. Prove that the set of primitive polynomials forms a multiplicatively closed set W. Set $R(X_1, ..., X_n) = R[X_1, ..., X_n]_W$. Prove that the natural map from $Cl(R) \longrightarrow Cl(R(X_1, ..., X_n))$ is an isomorphism.

(21) Prove that the ring $R = \mathsf{k}[x, y, z]_{(x,y,z)}/(f)$ is a UFD, if k is the real numbers, and $f = x^2 + y^2 + z^m$.

(22) Let k be a field of characteristic 0. Prove that the ring $\mathsf{k}[x^n, x^{n-1}y, ..., xy^{n-1}, y^n]$ generated by all monomials in x, y of degree n is integrally closed and has a cyclic class group of order n.

(23) (For those who know tight closure.) Let R be a complete local integrally closed domain of positive (possibly large) characteristic with algebraically closed residue field. Let x, y be a regular sequence of test elements. Prove there is a map from $(x, y)^*/(x, y)$ to $Cl(R[[T]])$ which is bijective (and make sense out of this statement!). In particular, given an element $z \in (x, y)^*$, $z \notin (x, y)$, show there is a nonzero element of the class group of $Cl(R[[T]])$ which can be constructed from z.

(24) Let R be a Noetherian ring, and $t \in R$ a non-zerodivisor such that R is complete in the t-adic topology. Assume that R/tR is an integrally closed domain. Prove that R is an integrally closed domain.

(25) Let R be a Noetherian ring, and $t \in R$ a non-zerodivisor such that R is complete in the t-adic topology. Assume that R/tR is a seminormal domain. Prove that R is a seminormal domain.

2. Coefficient Fields and Cohen's Structure Theorem(s)

For the results proved in this section, the ring need not necessarily be Noetherian unless otherwise mentioned.

Lemma 1 (Hensel's Lemma) *Let $(R, \mathfrak{m}, \mathsf{k})$ be a complete and separated local ring (separated means $\bigcap_{n\geq 0} \mathfrak{m}^n = 0$). Let $F(X)$ be a monic polynomial in $R[X]$; write $^-$ to denote going modulo $\mathfrak{m}$. Assume $\overline{F} = g \cdot h$ in $\mathsf{k}[X]$, where g and h are monic polynomials such that $(g,h) = 1$. Then there are liftings G and H of g and h respectively, both monic such that $F = G \cdot H$ in $R[X]$.*

Proof: Inductively we claim that there are monic polynomials $G_n, H_n \in R[X]$ such that:
a) $\overline{G_n} = g$ and $\overline{H_n} = h$,
b) $F - G_n \cdot H_n \in \mathfrak{m}^n R[X]$ and
c) $G_{n-1} \equiv G_n (mod\ \mathfrak{m}^{n-1}R[X])$ and $H_{n-1} \equiv H_n (mod\ \mathfrak{m}^{n-1}R[X])$.

If we prove that G_n and H_n exist for all n with the above properties, then $G := \lim G_n$ and $H := \lim H_n$ exist. Note that $F - G \cdot H \in \bigcap \mathfrak{m}^n R[X] = 0$ by assumption. Let G_1 and H_1 be any monic liftings of g and h respectively. Then (a) holds by construction and (b) holds since $\overline{F - G_1 H_1} = \overline{F} - gh = 0$.

Assume that G_n and H_n have been chosen satisfying (a) - (c). By (b), $F - G_n H_n \in \mathfrak{m}^n R[X]$. Choose $y_1, \ldots, y_l \in \mathfrak{m}^n, L_i(X) \in R[X]$ such that $F - G_n H_n = \Sigma_{i=1}^l y_i L_i$.

Since $(g,h) = 1$, there are $a_i(X)$ and $b_i(X)$ in $\mathsf{k}[X]$ such that $\overline{L_i} = a_i g + b_i h$. We can also assume that $\deg b_i < \deg g$ by the division algorithm. We claim that $\deg a_i < \deg h$.

Note that $\deg g + \deg h = \deg F$. On the other hand, $\deg(F - G_n H_n) < \deg F$ since the leading term cancels. Hence $\deg L_i < \deg F = \deg g + \deg h$. This forces $\deg a_i g < \deg g + \deg h$ which proves the claim.

Lift $a_i(X)$ and $b_i(X)$ to $A_i(X)$ and $B_i(X)$ respectively without changing the respective degrees. Then $F - G_n H_n = \Sigma_{i=1}^l y_i L_i$ and $L_i - A_i G - B_i H \in \mathfrak{m}R[X]$ since $\overline{L_i - A_i G_n - B_i H_n} = \overline{L_i} - a_i g - b_i h = 0$. Hence

$$F - G_n H_n - \Sigma y_i (A_i G_n + B_i H_n) \in \mathfrak{m}^{n+1} R[X] \qquad (*)$$

Define $G_{n+1} := G_n + \Sigma y_i B_i$ and $H_{n+1} := H_n + \Sigma y_i A_i$. Then
(a) $\overline{G_{n+1}} = \overline{G_n} = g$ and $\overline{H_{n+1}} = \overline{H_n} = h$ since $y_i \in \mathfrak{m}$.
(b) By $(*)$, $F - G_{n+1} H_{n+1} \in \mathfrak{m}^{n+1} R[X]$.
(c) $G_n \equiv G_{n+1} (mod\ \mathfrak{m}^{n+1} R[X])$ and $H_n \equiv H_{n+1} (mod\ \mathfrak{m}^{n+1} R[X])$ since $y_i \in \mathfrak{m}^n$.
Thus G_n and H_n exist for each n satisfying (a) - (c) which completes the proof. $\square$

Corollary 2 *Let R be a ring satisfying the hypothesis of Hensel's Lemma. With notations as above, let $F(X)$ be a monic polynomial in $R[X]$ be such that $\overline{F(X)}$ has a simple root α over k. Then F has a root $a \in R$ such that $\overline{a} = \alpha$.*

Example 1 Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring complete in the I-adic topology for an ideal I in R. Suppose that $\overline{e^2} \equiv \overline{e}(mod\ I)$. Then there is an element $e \in R$ such that $e \mapsto \overline{e}(modI)$ and $e^2 = e$.

Definition 1 *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a local ring. If there is a field* $L \subseteq R$ *such that the restriction* $L \subseteq R \xrightarrow{\pi} \mathsf{k}$ *is an isomorphsm, i.e.* $\pi|_L : L \xrightarrow{\sim} \mathsf{k}$*, then* L *is called a coefficient field for* R*.*

Example 2 1. Let $R = \mathsf{k}[X, Y]_{(X)}$ where $\mathsf{k}$is a field. The residue field is $\mathsf{k}(Y)$ and there is a copy of $\mathsf{k}(Y)$ in R, i.e. there is a coefficient field for R.
2. Let $V := \mathbb{R}[X]_{(X^2+1)}$. Then $V/\mathfrak{m}_V \simeq \mathbb{C}$. But $\mathbb{C}$ is not a subfield of V, i.e. V does not have a coefficient field.

The obvious question one can ask at this point is: When does a ring $(R, \mathfrak{m}, \mathsf{k})$ have a coefficient field? The following two theorems due to Cohen give some conditions under which the ring has a coefficient field.

Theorem 3 (I.S.Cohen) *Suppose* $(R, \mathfrak{m}, \mathsf{k})$ *is a complete and separated local ring such that* $\mathbb{Q} \subseteq R$ *(i.e.* R *has equal characteristic 0). Then* R *has a coefficient field.*

Proof: Consider the set of all subfields in R, this set is non-empty as $\mathbb{Q} \subseteq R$. Use Zorn's Lemma to choose a maximal such subfield, say L. We claim that L is a coefficient field for R. Note that $L \simeq \pi(L)$ where $\pi : R \twoheadrightarrow \mathsf{k}$ is the canonical projection. The content of the theorem is that $\pi(L) = \mathsf{k}$.

Let $\alpha \in \mathsf{k}$. Suppose α is transcendental over $\pi(L)$. Let $a \in R$ be a lift of α. Then $a \notin L$. Now for any $f(X) \in L[X]$, $\pi(f(a)) = \overline{f}(\alpha) \neq 0$. Hence $f(a) \notin \mathfrak{m}$. This implies that $L(a)$ is a subfield of R contradicting the maximality of L. Thus α must be algebraic over $\pi(L)$.

Let $f(X)$ be the minimal monic polynomial for α over $\pi(L)$. Passing to $\mathsf{k}[X]$, we have $f(X) = (X - \alpha)h(X)$ and $(X - \alpha, h(X)) = 1$ (α is a simple root of $f(X)$ since we are in characteristic 0). Let $F(X) = \pi^{-1}(f(X))$ in $L[X]$ (this makes sense since $L \simeq \pi(L)$). Thus F is a monic polynomial over R which factors modulo $\mathfrak{m}$. By Hensel's lemma, $F(X) = (X - a)H(X)$ where $a \in R$ is a lift of α. Hence a is algebraic over L, which implies that $L[a]$ is a subfield of R. By maximality of L, $a \in L$. Hence $\pi(a) = \alpha \in \pi(L)$. Thus L is a coefficient field for R. $\square$

Theorem 4 (I.S.Cohen) *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a complete and separated local ring such that* $\mathbb{Z}_p \subseteq R$*, where* $p \in \mathbb{N}$ *is a prime (i.e.* R *has characteristic* p*). Then* R *has a coefficient field.*

Proof: Let us first consider the case when $\mathfrak{m}^2 = 0$. We claim that $R^p := \{r^p : r \in R\}$ is a field.

Note that since $\mathrm{char}(R) = p$, R^p is a ring. If r^p is a non-zero element in R^p, then $r \notin \mathfrak{m}$. Thus r is a unit and hence the inverse of r^p is $(r^{-1})^p \in R^p$. Now choose any maximal subfield L in R containing R^p. We claim that L is a coefficient field for R.

Consider $\pi(L) \subseteq \mathsf{k}$ where π is the natural projection of R onto k. We want to show that $\pi(L) = \mathsf{k}$. By way of contradiction suppose $\alpha \in \mathsf{k} \setminus \pi(L)$. Lift α to $a \in R$. Note that $a^p \in R^p \subseteq L$. Since $\alpha^p = \pi(a^p)$, the minimal monic polynomial for α over $\pi(L)$ must be $X^p - \pi(a^p)$. This polynomial is irreducible over $\pi(L)$. We use the fact that if F is a field of characteristic p, α algebraic over F such that $\alpha^p \in F$, then either $\alpha \in F$ or the minimal monic polynomial of α over F is $X^p - \alpha^p$.

Thus it follows that $X^p - a^p$ is irreducible over L. Hence a is algebraic over L and clearly $L[a] \subseteq R$ is a larger field than L contradicting the maximality of L. This proves that $\pi(L) = \mathsf{k}$ when $\mathfrak{m}^2 = 0$. (In fact we only used $\mathfrak{m}^p = 0$).

Let us now consider the general case. Inductively we claim that there exist coefficient fields $L_n \subseteq R/\mathfrak{m}^n$, with maps $L_{n+1} \twoheadrightarrow L_n$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L_n \arrow[r] & R/\mathfrak{m}^n \\ 0 \arrow[r] & L_{n+1} \arrow[r] \arrow[u, two heads] & R/\mathfrak{m}^{n+1} \arrow[u, two heads, "\pi_n"'] \end{tikzcd}$$

commutes for all $n \in \mathbb{N}$.

In such a case, $\varprojlim_n L_n \subseteq \varprojlim_n R/\mathfrak{m}^n = \widehat{R} = R$ since R is complete. Clearly $\varprojlim_n L_n$ will be a coefficient field for R.

Choose $L_1 = \mathsf{k}$. L_2 exists by the first case. Suppose we have constructed $L_n \subseteq R/\mathfrak{m}_n$. If π_n is the natural surjection from $\mathfrak{m}^{n+1}$ to $\mathfrak{m}^n$, define $A := \pi_n^{-1}(L_n) \subseteq R/\mathfrak{m}^{n+1}$. A is a local subring of $R/\mathfrak{m}^{n+1}$ such that $A/\mathfrak{m}_A \simeq L_n \simeq \mathsf{k}$. Moreover, $\mathfrak{m}_A^2 = 0$. Hence by the first case we can find a coefficient field for $R/\mathfrak{m}^{n+1}$ say L_{n+1} which proves the result.

Corollary 5 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a complete and separated local ring. Suppose that $\mathbb{Z}_p \subseteq R$ and k is perfect (i.e. $\mathsf{k} = \mathsf{k}^p$). Then there is a unique coefficient field L for R given by $L := \bigcap_{n \in \mathbb{N}} R^{p^n}$.*

Proof: First of all, we claim that $L = \bigcap_{n \in \mathbb{N}} R^{p^n}$ is a field. Consider $a \in L$. This means that for each $n \in \mathbb{N}$, there is a $b_n \in R$, such that $b_n^{p^n} = a$. If $a \in \mathfrak{m}$, then $b_n \in \mathfrak{m}$ for each n. Since the ring R is separated, this implies that $a = 0$. If a is not in $\mathfrak{m}$, then a is a unit in R. Hence so are each of the b_n's. This forces $a^{-1} \in R^{p^n}$ for each n. Thus $a \in L$, $a \neq 0$, implies that $a^{-1} \in L$. Therefore L is a subfield of R.

Secondly, we will show that any arbitrary coefficient field $\mathbb{F}$ for R (such a field exists by Theorem 4) is contained in L. Since the coefficient field must be a maximal subfield, this will prove that L is indeed a coefficient field for R.

Let $\mathbb{F}$ be a coefficient field for R. Then $\mathbb{F} \simeq \mathsf{k}$. Since k is perfect, $\mathbb{F} = \mathbb{F}^{p^n} \subseteq R^{p^n}$ for each $n \in \mathbb{N}$. Hence $\mathbb{F} \subseteq \bigcap_{n\in\mathbb{N}} R^{p^n} = L$. $\square$

We will now prove a lemma that will be used in the following two theorems. This is a version of NAK for complete rings.

Lemma 6 (Complete NAK) *Let R be a ring, $I \subseteq R$ be an ideal such that R is complete in the I-adic topology. Let M be an R-module such that $\bigcap_{n\in\mathbb{N}} I^n M = 0$. If $x_1, \ldots, x_r \in M$ is such that the elements $x_i + IM$, $1 \le i \le r$, generate M/IM, then $M = Rx_1 + \cdots + Rx_r$.*

Proof: Set $N := Rx_1 + \cdots + Rx_r$. Then $M = N + IM$. Let $u \in M$. Then

$$u = \Sigma a_{i_0} x_i + m_1 \text{ where } a_{i_0} \in R \text{ and } m_1 \in IM.$$

Since $IM = IN + I^2M$, we get

$$u = \Sigma a_{i_0} x_i + \Sigma a_{i_1} x_i + m_2 \text{ where } a_{i_1} \in I \text{ and } m_2 \in I^2 M.$$

Inductively, there we get $a_{i_n} \in I^n$, $m_{n+1} \in I^{n+1}M$ such that

$$u = \Sigma[(a_{i_0} + a_{i_1} + \cdots + a_{i_n})x_i] + m_{n+1}.$$

The partial sums $\Sigma(a_{i_0} + a_{i_1} + \cdots + a_{i_n})$ form a Cauchy sequences in R in the I-adic topology, so there are $\tilde{r}_i$ such that $u = \Sigma \tilde{r}_i x_i$ such that $u - \Sigma \tilde{r}_i x_i \in \bigcap_{n\in\mathbb{N}} I^n M = 0$. Thus $u \in N$, proving the lemma. $\square$

Theorem 7 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a complete and separated local ring. Assume that R contains a field (and hence a coefficient field by Cohen's Theorems). Let k be a coefficient field for R. If $\mathfrak{m} = (x_1, \ldots, x_r)$ then there is a map $\phi : \mathsf{k}[[T_1, \ldots, T_r]] \twoheadrightarrow R$ defined by $T_i \mapsto x_i$. In particular, R is Noetherian.*

Proof: Note that ϕ makes sense since R is complete i.e.

$$\sum_{\substack{\alpha_v \in \mathsf{k} \\ v=(v_1,\ldots,v_r)\in\mathbb{N}^r}} \alpha_v \underline{T}^v \mapsto \sum_{\substack{\alpha_v \in \mathsf{k} \\ v=(v_1,\ldots,v_r)\in\mathbb{N}^r}} \alpha_v \underline{x}^v \in R.$$

The content of the theorem is that ϕ is onto. Set $S := \mathsf{k}[[T_1, \ldots, T_r]]$, $I = (T_1, \ldots, T_r)$ and $M = R$ as an S-module via the map ϕ. Note that $R/IR = R/\mathfrak{m} \simeq \mathsf{k}$ which is generated by one element as an S-module. Hence by the complete NAK, R is generated by a single element, namely the image of 1, i.e. $R \simeq S/\mathrm{ann}_S R$. $\square$

Theorem 8 (Complete Version of Noether Normalization) *Let $(R, \mathfrak{m}, \mathsf{k})$ be a complete Noetherian ring containing a field. Set $d := \dim(R)$, $x_1, \ldots, x_d$ be a system of parameters . Fix a coefficient field $\mathsf{k}$and let $S := \mathsf{k}[[T_1, \ldots, T_d]] \xrightarrow{\phi} R$ be defined by $T_i \mapsto x_i$. Then ϕ is injective and R becomes a finitely generated S-module via the map ϕ.*

Proof: Let $I = (T_1, \ldots, T_d) \subseteq S$. Then $R/IR \simeq R/(x_1, \ldots, x_d)$ is Artinian and hence finitely generated over $S/I \simeq \mathsf{k}$. Fix a generating set $\bar{y}_1, \ldots, \bar{y}_n$ of R/IR over $k \simeq S/I$. By the complete NAK, $R = Sy_1 + \cdots + Sy_n$. This implies that R is integral over $\phi(S)$. Hence

$$d = \dim(R) = \dim(\phi(S)) \leq \dim(S) - \mathrm{ht}(\mathrm{Ker}(\phi)) = d - \mathrm{ht}(\mathrm{Ker}(\phi)).$$

Therefore $\mathrm{ht}(\mathrm{Ker}(\phi)) = 0$ and hence $\mathrm{Ker}(\phi) = 0$ completing the proof. $\square$

3. Matlis Duality and Gorenstein Rings

§ 3.1 *Review of Injective Modules*

Theorem 1 *Let R be a (commutative) ring, E an R-module. Then the following are equivalent:*
(a) *Let $0 \to M \xrightarrow{i} N$ be an injection of R-modules. Then every homomorphism $f : M \to E$ induces a map $\tilde{f} : N \to E$ extending f.*
(b) *(Baer's Criterion) Let I be an ideal in R. Then every homomorphism $f : I \to E$ induces a map $\tilde{f} : R \to E$ extending f.*
(c) $\mathrm{Hom}_R(\_, E)$ *preserves short exact sequences (contravariantly).*
(d) *Whenever $E \subseteq M$, $E|M$ (i.e. E splits off M).*

Proof: It is clear that (a) implies (b). In order to prove the converse, let M, N, E and $f : M \to E$ be as in (a). We can think of M as a submodule of N. Let

$$\Lambda := \{(K, f_K) : M \subseteq K \subseteq N \text{ and } f_K|_M = f\}.$$

Partially order Λ by $(K, f_K) \leq (L, f_L)$ if $K \subseteq L \subseteq N$ and $f_L|_K = f_K$. By Zorn's Lemma, there is a maximal element (K, f_K) in Λ.

Let x be a non-zero element in N. Set $I := (K :_R x)$. Consider the map $g : I \to E$ given by $g(i) = f_K(ix)$. By (b), g extends to $\tilde{g} : R \to E$. Hence we get a map $h : K + Rx \to E$ defined by $h|_K = f_K$ and $h(x) = \tilde{g}(1)$. Since $(K + Rx, h) \in \Lambda$, by maximality of (K, f_K), x is in K, i.e. $K = N$.

In order to prove the equivalence of (a) and (c), let K be the cokernel of the map $i : M \hookrightarrow N$. We have the short exact sequence $0 \to M \xrightarrow{i} N \to K \to 0$. Applying $\mathrm{Hom}_R(\_, E)$ we get

$$0 \to \mathrm{Hom}_R(K, E) \to \mathrm{Hom}_R(N, E) \xrightarrow{\mathrm{Hom}_R(i,E)} \mathrm{Hom}_R(M, E) \to \mathrm{Ext}^1_R(K, E) \to \cdots$$

Now (a) is true if and only if $\mathrm{Hom}_R(i, E)$ is surjective i.e $\mathrm{Hom}_R(\_, E)$ preserves short exact sequences.

If $E \subseteq M$, by applying (a) to $0 \to E \to M$, we see that $E|M$. We will prove the converse after a short discussion.

Definition 1 *Any module E satisfying any of the properties (a) - (c) in theorem 1 is said to be an injective R-module.*

Some ways to construct injective modules:
1. If E_is are injective R-modules, then so is $\prod_i E_i$.

Warning: $\bigoplus E_i$ is injective if and only if R is Noetherian.

2. If E is an injective R-module and $I|E$, then I is also injective.

3. If $R \to S$ is a ring homomorphism and E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.

Proof: $\mathrm{Hom}_R(S, E)$ is an injective S-module if and only if $\mathrm{Hom}_S(\_, \mathrm{Hom}_R(S, E))$ preserves short exact sequences of S-modules. By the $\mathrm{Hom} - \otimes$ adjointness this happens if and only if $\mathrm{Hom}_R(\_ \otimes_S S, E)$ preserves short exact sequences of R-modules. But this is true since E is an injective R-module.

Definition 2 *An R-module M is said to be divisible if whenever t is a non-zerodivisor in R and u is an element of M, one can find a v in M (not necessarily unique) such that $u = tv$.*

Some examples of divisible modules:
1. Let R be a domain. Then its fraction field K is a divisible R-module.

2. If N is a submodule of a divisible module M, then M/N is divisible.

3. Direct sums of divisible modules are divisible.

4. Let E be an injective R-module. Then E is divisible.
Proof: Let t be a non-zerodivisor in R and $u \in E$. Since E is injective, we have

$$\begin{tikzcd} & E \arrow[r, equal] & E \\ 0 \arrow[r] & R \arrow[u, "1\mapsto u"] \arrow[r, "\cdot t"] & R \arrow[u, "f"'] \end{tikzcd}$$

where $f(t) = u$, i.e. $tf(1) = u$. This proves that E is divisible.

The question is: When does the converse of 4 hold? The following proposition answers this for us.

Proposition 2 *If R is a PID, then every divisible R-module is injective.*

Proof: Let E be divisible over R. We will use Baer's criterion to prove that E is injective. Let $I = (t)$ be a nonzero ideal in R and $f : I \to E$ be a homomorphism. We want to show that f extends to $\tilde{f} : R \to E$. Since E is divisible, there is an element v such that $f(t) = tv$. Define $\tilde{f}(1) := v$. This extends f to R. $\square$

Exercise: Let R be a domain. Then a torsion-free and divisible R-module is injective.

Remark 1
1. From the above proposition, with $R = \mathbb{Z}$, we see that $\mathsf{K} = \mathbb{Q}$ is an injective $\mathbb{Z}$-module and hence so is $\mathbb{Q}/\mathbb{Z}$.

2. Let M be an arbitrary $\mathbb{Z}$-module. Then for some N we have the short exact sequence $0 \to N \to \oplus\mathbb{Z} \to M \to 0$. Hence $M \simeq (\oplus\mathbb{Z})/N \hookrightarrow (\oplus\mathbb{Q})/N$ which is divisible and hence injective.
Thus every $\mathbb{Z}$-module injects into an injective $\mathbb{Z}$-module.

3. Let R be any ring, and let E be an injective $\mathbb{Z}$-module. Hence $\mathrm{Hom}_{\mathbb{Z}}(R, E)$ is an injective R-module.

Let M be an arbitrary R-module. By the previous remark, there is an injective $\mathbb{Z}$-module I, such that $M \overset{\phi}{\hookrightarrow} I$ as a $\mathbb{Z}$-module. Consider the induced map $\Phi : M \to \mathrm{Hom}_{\mathbb{Z}}(R, I)$ defined by $u \mapsto [r \mapsto \phi(ru)]$. If u is a nonzero element of M, then $\phi(u) \neq 0$ and hence $\Phi(u)(1) = \phi(u) \neq 0$, i.e. the image of u in $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is nonzero. Therefore, we have $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I)$ which is an injective R-module.

Thus we have proved:

Proposition 3 *Every R-module injects into an injective R-module.*

Proof of (d) implies (a) in Theorem 1: Let E be an R-module such that whenever $E \subseteq M$, $E|M$ for any R-module M. Now by Prop. 3, there is an injective R-module I such that we can embed $E \hookrightarrow I$. Therefore $E|I$ and hence is injective. $\square$

Injective Resolutions

A consequence of Prop.3 is the existence of an injective resolution for an R-module M. An injective resolution of M is an exact sequence

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

where each I^i is an injective R-module. The existence of an injective resolution can be shown as follows: If M is an R-module, there is an injective R-module I^0 such that $M \hookrightarrow I^0$. Let C^0 be the cokernel of this map. Then there is an injective R-module I^1 such that $C^0 \hookrightarrow I^1$. Looking at the cokernel C^1 of this inclusion we get another injective I^2 such that $C^1 \hookrightarrow I^2$. Putting all these together we get an injective resolution for M: $0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$.

We say that M has a finite injective dimension over R if there is an exact sequence

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

where each I^i is injective.

We define $\mathrm{id}_R(M) = \min\{n : M$ has an injective resolution of length $n\}$ and set $\mathrm{id}_R(M) = \infty$ if M has no finite injective resolutions.

§ 3.2 *More on Injectives: Essential Extensions and the Injective Hull*

We have seen that every R-module can be embedded into an injective R-module. The goal in this section is to find a "minimal" such module.

Definition 3 *Given R-modules M and N, $M \subseteq N$, we say that N is an essential extension over M (or N is essential over M) if every nonzero submodule K of N has a nontrivial intersection with M i.e. if $0 \neq K \subseteq N$, then $K \cap M \neq 0$.*

Remark 2
1. Let R be a domain with fraction field K. Then $R \subseteq \mathsf{K}$ is essential since given any nonzero R-submodule L of K, by clearing denominators $L \cap R \neq 0$.

Now K is also torsion-free and divisible as an R-module. Hence K is an injective R-module. Let I be any injective R-module such that $R \subseteq I$. We will show that $\mathsf{K} \subseteq I$.

The inclusion $R \overset{i}{\hookrightarrow} I$ induces a map $\mathsf{K} \overset{\tilde{i}}{\to} I$ since I is injective and $R \subseteq \mathsf{K}$. We have $\mathrm{Ker}(\tilde{i}) \cap R = \mathrm{Ker}(i)$. Therefore as $\mathrm{Ker}(i) = 0$ and K is essential over R, $\mathrm{Ker}(\tilde{i}) = 0$, i.e. $\mathsf{K} \subseteq I$.

Thus K is a minimal injective R-module containing R in the sense that whenever I is any injective R-module containing R, $\mathsf{K} \hookrightarrow I$.

2. The above proof shows that if E is essential over M and I is an injective R-module containing M, $E \hookrightarrow I$.

3. Essentialness is transitive, i.e. if M_1 is essential over M_2 and M_2 is essential over M_3, then M_1 is essential over M_3.

4. If M_1 is essential over M_3 and $M_3 \subseteq M_2 \subseteq M_1$, then M_1 is essential over M_2 and M_2 is essential over M_3.

5. If $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$ are essential, then so is $N_1 \oplus N_2 \subseteq M_1 \oplus M_2$.

Proposition 4 *An R-module E is injective if and only if there is no proper essential extension of E.*

Proof: Let us assume that E is injective. Suppose L is an essential extension of E. Since E is injective by Theorem 1, $E|L$, i.e. there is a submodule N of L such that

$L = E \oplus N$. But $N \cap E = 0$ and $E \subseteq L$ is essential. Hence $N = 0$ which implies that $E = L$.

Conversely suppose that E has no proper essential extensions. Let I be an injective R-module such that $E \hookrightarrow I$. If $E \neq I$, since E has no proper essential extensions, there is a nonzero submodule N of I such that $E \cap N = 0$. By Zorn's Lemma, we can find a maximal such N. Then $E \hookrightarrow I/N$ and by maximality of N, this is an essential extension. Therefore $E \simeq I/N$. Now $E \cap N = 0$ and $I = E + N$ implies that $I \simeq E \oplus N$, i.e. $E|I$ and hence is injective. □

Theorem 5 *Let R be a ring, M and E R-modules such that $M \subseteq E$. Then the following are equivalent:*
(1) *E is a maximal essential extension of M.*
(2) *E is a minimal injective R-module containing M.*
(3) *E is injective and essential over M.*

Proof:
(1) $\Rightarrow$ (3): E is essential over M by assumption and is injective by Prop 4 and by using transitivity of essentialness.
(3) $\Rightarrow$ (2): Suppose there is an injective R-module I such that $M \subseteq I \subseteq E$. Then since E is essential over M, E is essential over I. Since I is injective, by Prop. 4, $E = I$.
(2) $\Rightarrow$ (1): Let $\Lambda := \{N : M \subseteq N \subseteq E, N \text{ is an essential extension of } M\}$. By Zorn's Lemma, using transitivity of essentialness, Λ has a maximal element N. We will show that N has no proper essential extensions.

If L is essential over N, then L is essential over M. Hence since E is an injective module containing M, by remark 2.2, $L \hookrightarrow E$. Thus $N = L$. This implies that N is injective by Prop. 4. But by hypothesis, E is a minimal injective module containing M, hence $E = N$. Thus E is a maximal essential extension of M. □

Definition 4 *Let M be a fixed R-module. Any module satisfying E containing M satisfying any one (and hence all) of the above equivalent conditions is said to be an injective hull (or injective envelope) of M.*

Existence and Uniqueness of the Injective Hull:

Let M be a fixed R module. Let I be an injective R-module containing M. By the proof of (2) implies (1) in Theorem 5, there is a maximal essential extension E of M contained in I. Moreover there is no proper essential extension of E. Hence E is an injective hull of M.

If E' is any injective hull of M, then using injectivity of E and essentialness of E' over M, $E' \hookrightarrow E$, by remark 2.2. But since E' is injective, by Prop. 4, E cannot be a proper essential extension of E' i.e. $E \simeq E'$. Thus the injective hull is unique up to isomorphism and this module is denoted by $E_R(M)$.

Example 1 If R is a domain with fraction field K, then by remark 2.1, $E_R(R) = \mathsf{K}$.

Warning: We will see that for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $E_R(R/\mathfrak{p})$ is not the same as $E_{R/\mathfrak{p}}(R/\mathfrak{p}) = \kappa(\mathfrak{p})$, the fraction field of $R/\mathfrak{p}$. In general, the base ring makes a big difference.

Theorem 6 (Matlis) *Let R be Noetherian. Then*
(1) *E is an indecomposable injective R-module if and only if $E \simeq E_R(R/\mathfrak{p})$ for some $\mathfrak{p}$ in* Spec*(R).*
(2) *Every injective R-module can be written as a direct sum of indecomposable injective R-modules.*

Proof: (1) Let E be an indecomposable injective R-module. Choose $\mathfrak{p} \in \operatorname{Ass}_R(E)$ (defined to be the set $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} = \operatorname{ann}_R(x)$ for some nonzero $x \in E\}$. Then $R/\mathfrak{p} \hookrightarrow E$ and hence $E_R(R/\mathfrak{p})|E$. This gives us $E \simeq E_R(R/\mathfrak{p})$ since E is indecomposable.

We now need to prove that for every $\mathfrak{p} \in \operatorname{Spec}(R)$, $E_R(R/\mathfrak{p})$ is indecomposable. Suppose $E_R(R/\mathfrak{p}) = E_1 \oplus E_2$. We have $R/\mathfrak{p} \overset{i}{\hookrightarrow} E_R(R/\mathfrak{p}) = E_1 \oplus E_2$. Let $i(\bar{1}) = y = (y_1, y_2)$ in $E_1 \oplus E_2$. Hence we have $\mathfrak{p} = \operatorname{ann}_R(y) = \operatorname{ann}_R(y_1) \cap \operatorname{ann}_R(y_2)$. This forces (without loss of generality) $\mathfrak{p} = \operatorname{ann}_R(y_1)$ and $\mathfrak{p} \subseteq \operatorname{ann}_R(y_2)$. But then $R/\mathfrak{p}$ embeds into E_1 via the projection map, and since E_1 is injective and $E_R(R/\mathfrak{p})$ is essential over $R/\mathfrak{p}$, the projection map from $E_R(R/\mathfrak{p})$ onto E_1 must be an embedding, i.e. $E_R(R/\mathfrak{p}) = E_1$, proving $E_R(R/\mathfrak{p})$ is indecomposable.
(2) Let E be an injective R-module. Consider the set Λ; an element of Λ is a collection of indecomposable injective submodules of E say $\{E_i\}$ such that $\Sigma E_i = \oplus E_i$. Order by inclusion. We will first show that Λ is not empty.

Choose $\mathfrak{p}$ in $\operatorname{Ass}_R(E)$. As in the proof of (1), $E_R(R/\mathfrak{p}) \hookrightarrow E$ and is indecomposable by 1. Hence $\{E_R(R/\mathfrak{p})\} \in \Lambda$, i.e. Λ is not empty. We can apply Zorn's Lemma to get a maximal such $\{E_i\}$ in Λ. We need the following lemma:

Lemma 7 *Let R be a Noetherian ring and E_i be injective R-modules. Then $\bigoplus\limits_i E_i$ is injective.*

By the lemma, for every $\{E_i\} \in \Lambda$, $\Sigma E_i = \oplus E_i$ is injective. Let $\{E_i\}$ be maximal in Λ and let $I := \oplus E_i$. Then $I|E$, i.e. there is a submodule N of E such that $E = I \oplus N$. Note that $N|I$ and hence is injective. If $N \neq 0$, there is a $\mathfrak{p}$ in $\operatorname{Spec}(R)$ such that $E_R(R/\mathfrak{p}) \hookrightarrow N$. This implies that $\{E_i\} \cup \{E_R(R/\mathfrak{p})\} \in \Lambda$, which contradicts the maximality of $\{E_i\}$. Hence $N = 0$ and $E = \Sigma E_i = \oplus E_i$. $\square$

Proof of Lemma 7: We use Baer's criterion to prove that $\oplus E_i$ is injective. Let I be an ideal in R and $f : I \to \oplus E_i$ be given. Since I is finitely generated, there are $i_1, \ldots, i_k$ such that $f(I) \subseteq \oplus_{j=1}^k E_{i_j}$. Since each E_i is injective, f extends to a map

$\tilde{f}: R \to \oplus_{j=1}^k E_{i_j}$, proving the lemma. $\square$

Remark 3
1. The proof of (1) in Theorem 5 shows that $\mathrm{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$. In fact something much more general is true. For a finitely generated R-module M, $\mathrm{Ass}(E_R(M)) = \mathrm{Ass}(M)$.

To prove this, firstly observe that $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(E_R(M))$. For the other inclusion, consider a prime ideal $\mathfrak{p} \in \mathrm{Ass}(E_R(M))$. Then $R/\mathfrak{p} \hookrightarrow E_R(M)$. Since $E_R(M)$ is an essential extension of M, $N := M \cap R/\mathfrak{p} \neq 0$. Hence $\emptyset \neq \mathrm{Ass}(N) \subseteq \mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Since $\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$, this proves the other inclusion.

2. The converse of lemma 7 is true but harder to prove.

Example 2 We know that $E_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Q}$ and that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. In addition

$$\mathbb{Q}/\mathbb{Z} \quad \simeq \bigoplus_{p\in\mathbb{Z},\ \text{p a prime}} E_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}).$$

See exercise 16.

Theorem 8 *Let R be a Noetherian ring. Then*

$$E_R(R/\mathfrak{p}) = E_{R_\mathfrak{p}}(\kappa(\mathfrak{p})).$$

Proof: First of all, we claim that $E_R(R/\mathfrak{p})$ is an $R_\mathfrak{p}$-module. Let $s \in R \setminus \mathfrak{p}$. Since $\mathrm{Ass}_R(E_R(R/\mathfrak{p})) = \{\mathfrak{p}\}$, s is a non-zerodivisor on $E_R(R/\mathfrak{p})$. This means that $sE \simeq E$. But $sE \subseteq E$ is injective and hence $sE|E$. The indecomposablility of E forces $E = sE$. Therefore s is a unit on E i.e. E has a $R_\mathfrak{p}$-module structure.

The map $R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p})$ factors through $\kappa(\mathfrak{p})$ since $\kappa(\mathfrak{p})$ is essential over $R/\mathfrak{p}$. Now $R/\mathfrak{p} \hookrightarrow E_R(R/\mathfrak{p})$ is an essential extension as R-modules, hence so is $\kappa(\mathfrak{p}) \hookrightarrow E_R(R/\mathfrak{p})$. This implies that $\kappa(\mathfrak{p}) \hookrightarrow E_R(R/\mathfrak{p})$ is *a fortiori* an essential extension as $R_\mathfrak{p}$-modules. We need to prove that $E_R(R/\mathfrak{p})$ is an injective $R_\mathfrak{p}$-module to complete the proof.

This can be seen as follows:

$$E_R(R/\mathfrak{p}) = (E_R(R/\mathfrak{p}))_\mathfrak{p} = (\mathrm{Hom}_R(R, , E_R(R/\mathfrak{p})))_\mathfrak{p}$$

$$= \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E_R(R/\mathfrak{p})_\mathfrak{p}) \quad \text{since } R_\mathfrak{p} \text{ is flat over} R$$

$$= \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E_R(R/\mathfrak{p})) = \mathrm{Hom}_R(R_\mathfrak{p}, E_R(R/\mathfrak{p}))$$

which is an injective $R_\mathfrak{p}$-module. $\square$

Example 3
Let $\mathfrak{p} = \mathrm{Ker}(\mathbb{C}[X,Y,Z] \twoheadrightarrow \mathbb{C}[t^3,t^4,t^5])$. Then $\mathfrak{p} = (X^3 - YZ, Y^2 - XZ, Z^2 - X^2Y)$. Write

$$E_R(R/\mathfrak{p}^n) = E_R(R/\mathfrak{p})^{\oplus a(n)} \bigoplus E_R(\mathbb{C})^{\oplus b(n)}$$

where $R = \mathbb{C}[X,Y,Z]$. Find $a(n)$ and $b(n)$ as functions of n. (See exercise 24.)

Theorem 9 *Let R be a Noetherian ring, $I \subseteq \mathfrak{p} \subseteq R$ be ideals, $\mathfrak{p} \in \mathrm{Spec}(R)$. Then*

$$E_{R/I}(R/\mathfrak{p}) = \mathrm{Hom}_R(R/I, E_R(R/\mathfrak{p})).$$

Proof: We already know that $\mathrm{Hom}_R(R/I, E_R(R/\mathfrak{p}))$ is an injective R/I-module. Therefore it is enough to show that it is essential over $R/\mathfrak{p}$.

Now $\mathrm{Hom}_R(R/I, E_R(R/\mathfrak{p})) \simeq 0 :_{E_R(R/\mathfrak{p})} I \subseteq E_R(R/\mathfrak{p})$. Since $R/\mathfrak{p} \subseteq E_R(R/\mathfrak{p})$, and $I \subseteq \mathfrak{p}$, we obtain that $R/\mathfrak{p} \subseteq 0 :_{E_R(R/\mathfrak{p})} I$. Moreover, since $E_R(R/\mathfrak{p})$ is essential over $R/\mathfrak{p}$, so is $0 :_{E_R(R/\mathfrak{p})} I$, by remark 2.4. This proves that $\mathrm{Hom}_R(R/I, E_R(R/\mathfrak{p}))$ is the injective hull of $R/\mathfrak{p}$ over R/I. $\square$

Corollary 10 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and $E = E_R(\mathsf{k})$ be the injective hull over R of k. Then for any ideal I in R, $E_{R/I}(\mathsf{k}) = 0 :_E I$.*

Proof: By Theorem 9, $E_{R/I}(\mathsf{k}) \simeq \mathrm{Hom}_R(R/I, E) \simeq 0 :_E I$. $\square$

§ 3.3 *Matlis Duality : Study of $E_R(\mathsf{k}) = E$*

Notations and Remarks:

1. By $(R, \mathfrak{m}, \mathsf{k}, E)$, we mean that R is a Noetherian local ring with unique maximal ideal $\mathfrak{m}$, residue field $\mathsf{k} = R/\mathfrak{m}$ and $E := E_R(\mathsf{k})$, the injective hull of the residue field k over R.

2. By $(\_)^\vee$ we mean $\mathrm{Hom}_R(\_, E)$. Recall that $^\vee$ preserves short exact sequences (contravariantly).

3. Cor. 10 can be rephrased as: For any ideal I in $(R, \mathfrak{m}, \mathsf{k}, E)$,

$$E_{R/I}(\mathsf{k}) \simeq (R/I)^\vee \simeq 0 :_E I.$$

Let us first study the case when $\dim(R) = 0$.

Theorem 11 *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be zero-dimensional, M a finitely generated R-module. Then*
(1) $\mathsf{k}^\vee \simeq \mathsf{k}$.
(2) $\lambda_R(M) = \lambda_R(M^\vee)$. *In particular, $M^\vee$ is finitely generated .*
(3) $\mathrm{Hom}_R(E, E) \simeq R$ *(i.e. $R \simeq R^{\vee\vee}$).*
(4) *More generally, the natural map $M \to M^{\vee\vee}$ is an isomorphism.*

Comment: $^\vee$ basically flips the structure of the module, $M^\vee$ is looking at M upside down. We will make this more precise as we go along.

Proof: (1) We have $\mathsf{k}^\vee = \mathrm{Hom}_R(\mathsf{k}, E) \simeq E_{\mathsf{k}}(\mathsf{k})$ by Cor. 10. But $E_{\mathsf{k}}(\mathsf{k}) \simeq \mathsf{k}$ since k is both essential and divisible (hence injective) as a module over itself. Thus $\mathsf{k}^\vee \simeq \mathsf{k}$.
(2) Since $\mathfrak{m} \in \mathrm{Ass}_R(M)$, $\mathsf{k} \hookrightarrow M$. Let N be the cokernel of this inclusion. By applying $^\vee$ to the short exact sequence $0 \to \mathsf{k} \to M \to N \to 0$, we get the short exact sequence $0 \to N^\vee \to M^\vee \to \mathsf{k}^\vee \to 0$. Note that $\lambda_R(N) = \lambda_R(M) - 1$. Hence an induction on $\lambda_R(M)$ together with (1) gives (2).
(3) Consider the map $R \to \mathrm{Hom}_R(E, E) = R^{\vee\vee}$ given by $r \mapsto \mu_r$ (multiplication by r). In order to prove that this map is an isomorphism, it is enough to show that it is a monomorphism (or epimorphism), since by (2) $\lambda_R(R) = \lambda_R(R^\vee) = \lambda_R(R^{\vee\vee})$. We will show that it is injective.

Suppose for some $r \in R$, μ_r is the zero map i.e. $rE = 0$. Hence $\mathrm{Hom}_R(R/rR, E) \simeq 0 :_E r = E$. Therefore by Cor. 10, $E \simeq E_{R/rR}(\mathsf{k})$ which gives us $\lambda(E) = \lambda(E_{R/rR}(\mathsf{k}))$. But by (2), $\lambda(R) = \lambda(E)$ and $\lambda(E_{R/rR}(\mathsf{k})) = \lambda(R/rR)$. This forces $\lambda(rR) = 0$ i.e. $r = 0$.
(4) Take a presentation $G \to F \to M \to 0$ of M, where F and G are free modules. From this, we get

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\simeq"] & F \arrow[r] \arrow[d, "\simeq"] & M \arrow[r] \arrow[d] & 0 \\
G^{\vee\vee} \arrow[r] & F^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

where the second row is exact by the exactness of $^{\vee\vee}$. Since $G \simeq G^{\vee\vee}$ and $F \simeq F^{\vee\vee}$, by the five lemma, $M \simeq M^{\vee\vee}$. (One needs to check that the squares commute). $\square$

Remark 4
1. Hidden in the proof of (3) above is a fact worth noting separately: E is a faithful R-module.

2. Note that we did not need $\dim(R) = 0$ in the proof of (1).

Theorem 12 *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Noetherian local ring. Then*
(1) *Every element of E is killed by $\mathfrak{m}^n$ for some n.*
(2) $\mathsf{k}^\vee \simeq \mathsf{k}$.
(3) $M^\vee \neq 0$ *for an arbitrary nonzero module M.*
(4) $E \simeq E_{\widehat{R}}(\mathsf{k})$.
(5) $\mathrm{Hom}_R(E, E)$ $(= R^{\vee\vee}) \simeq \widehat{R}$.
(6) *E is Artinian.*

Proof: (1) Let z be a nonzero element in E. Then $\emptyset \neq \mathrm{Ass}_R(Rz) \subseteq \mathrm{Ass}_R(E) = \{\mathfrak{m}\}$. Thus Rz is a finite length R-module. So $\mathfrak{m}^n z = 0$ for some n.
(2) The proof is the same as in (1) of Theorem 11 as noted in remark 4.2.
(3) If M is a nonzero finitely generated R-module, then $M/\mathfrak{m}M \neq 0$ by NAK. This gives a surjection $M \twoheadrightarrow M/\mathfrak{m}M \twoheadrightarrow \mathsf{k}$. We get an inclusion $\mathsf{k}^\vee \hookrightarrow M^\vee$ by applying $^\vee$. Therefore, by using (2), $M^\vee \neq 0$.

Suppose M is not finitely generated . Let m be a nonzero element in M. Then $(Rm)^\vee \neq 0$. Since we have a surjection $M^\vee \twoheadrightarrow (Rm)^\vee$, $M^\vee \neq 0$.
(4) Firstly notice that E is an $\widehat{R}$-module since by (1) $E = \bigcup_{n \geq 0}(0 :_E \mathfrak{m}^n)$. This is due to the fact that each $0 :_E \mathfrak{m}^n$ is a $\widehat{R}$-module since $\widehat{R}/\widehat{\mathfrak{m}^n} \simeq R/\mathfrak{m}^n$.

We want to prove that $E \simeq E_{\widehat{R}}(\mathsf{k})$. Note that since $\mathsf{k} \subseteq E$ is an essential extension of R-modules, it is also an essential extension of $\widehat{R}$-modules. This implies that $\mathsf{k} \subseteq E \subseteq E_{\widehat{R}}(\mathsf{k})$. It is enough to prove that $\mathsf{k} \subseteq E_{\widehat{R}}(\mathsf{k})$ is an essential extension as R-modules to prove the required isomorphism, since E is a maximal essential extension of k as R-modules.

Note that the R-submodules of $E_{\widehat{R}}(\mathsf{k})$ are precisely the $\widehat{R}$-submodules of $E_{\widehat{R}}(\mathsf{k})$ by (1). Therefore $E \simeq E_{\widehat{R}}(\mathsf{k})$.
(5) Let $E_n = 0 :_E \mathfrak{m}^n = \mathrm{Hom}_R(R/\mathfrak{m}^n, E)$. Then $E_n \simeq E_{R/\mathfrak{m}^n}(\mathsf{k})$ by Cor. 10.

Claim: $\mathrm{Hom}_R(E,E) \simeq \varprojlim_n \mathrm{Hom}_R(E_n, E_n)$ under the map $f \leftrightarrow (f_n)$ where $f_n = f|_{E_n}$.

The crucial point is that $f|_{E_n}(E_n) \subseteq E_n$. This is true since if $z \in E_n$, then $\mathfrak{m}^n f(z) = f(\mathfrak{m}^n z) = 0$, i.e. $f(z) \in E_n$. Then using the fact that $\bigcup_n E_n = E$, the above claim is true.

But $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) \simeq R/\mathfrak{m}^n$ by the zero-dimensional case (Theorem 11.3) since $E_n = 0 :_E \mathfrak{m}^n = E_{R/\mathfrak{m}^n}(\mathsf{k})$ by Cor. 10. Thus we have

$$\mathrm{Hom}_R(E,E) \simeq \varprojlim_n \mathrm{Hom}_R(E_n, E_n) \simeq \varprojlim_n R/\mathfrak{m}^n = \widehat{R}.$$

(6) Finally we want to prove that E is Artinian. Let $\{N_n\}$ be a decreasing chain of submodules of E. By applying $^\vee$, we get $E^\vee \twoheadrightarrow N_1^\vee \twoheadrightarrow N_2^\vee \twoheadrightarrow \cdots$. By (5), $E^\vee = \widehat{R}$. Set $J_i = \mathrm{Ker}(\widehat{R} \twoheadrightarrow N_i)$. We have $J_1 \subseteq J_2 \subseteq \cdots$. Since $\widehat{R}$ is Noetherian, there is an n such that $J_i = J_n$ whenever $i \geq n$. This implies that $N_i^\vee = N_n^\vee$ for all $i \geq n$ as $N_i^\vee \simeq \widehat{R}/J_i$.

Let C_i be the cokernel of the inclusion $N_{i+1} \hookrightarrow N_i$. By applying $^\vee$, we get $0 \to C_i^\vee \to N_i^\vee \to N_{i+1}^\vee \to 0$. This forces $C_i^\vee = 0$ for $i \geq n$. Therefore by (3), $C_i = 0$ for $i \geq n$ which proves that E is Artinian. $\square$

Theorem 13 (Matlis Duality) *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Noetherian local ring. Then there is a one-one arrow reversing correspondence between finitely generated $\widehat{R}$-modules and Artinian R-modules, i.e.*

$$\{\textit{finitely generated } \widehat{R}\textit{-modules}\} \underset{\vee}{\overset{\vee}{\rightleftharpoons}} \{\textit{Artinian } R\textit{-modules}\}.$$

In particular, if M is either an Artinian R-module, or a finitely generated $\widehat{R}$-module, $M^{\vee\vee} \simeq M$.

Some remarks about Artinian R-modules: Let M be an Artinian R-module.
1. M is an $\widehat{R}$-module. To see this, consider $x \in M$. Then there is an n such that $\mathfrak{m}^n x = \mathfrak{m}^{n-1}x$. Therefore by NAK, $\mathfrak{m}^n x = 0$. If $\hat{r} \in \widehat{R}$, choose $r \in R$ such that $\hat{r} - r \in \mathfrak{m}^n \widehat{R}$. Then we can define $\hat{r}x = rx$ which gives a $\widehat{R}$- module structure on M.

2. M is essential over $\mathrm{soc}_R(M) = 0 :_M \mathfrak{m}$. If N is a nonzero submodule of M, for a nonzero $x \in N$, there is a smallest $n \in \mathbb{N}$ such that $\mathfrak{m}^n x = 0$. Then $0 \neq \mathfrak{m}^{n-1}x \subseteq \mathrm{soc}_R(M) \cap N$.

Proof of Theorem 13: Let M be a finitely generated $\widehat{R}$-module. Then we have $\widehat{R}^{\oplus n} \to M \to 0$ for some n. This gives us $0 \to M^\vee \to (\widehat{R}^\vee)^{\oplus n}$, so $M^\vee \hookrightarrow E^{\oplus n}$ and hence is Artinian.

Suppose M is an Artinian R-module. Then $\mathrm{soc}_R(M)$ is a finite dimensional vector space over k. (If not, then we can find an infinite descending chain of subspaces of $\mathrm{soc}_R(M)$ which are necessarily R-submodules of M). Hence $\mathrm{soc}_R(M) \simeq \mathsf{k}^{\oplus t}$ for some $t \in \mathbb{N}$. Therefore $E_R(\mathrm{soc}_R(M)) \simeq E^{\oplus t}$. Since $\mathrm{soc}_R(M) \subseteq M$ is essential, $M \hookrightarrow E^{\oplus t}$. This gives us $\widehat{R}^{\oplus t} \simeq (E^\vee)^{\oplus t} \twoheadrightarrow M^\vee$, so $M^\vee$ is a finitely generated $\widehat{R}$-module.

Now we need to show that if M is either an Artinian R-module, or a finitely generated $\widehat{R}$-module, $M^{\vee\vee} \simeq M$. First, suppose that M is a finitely generated $\widehat{R}$-module. Note that $\widehat{R}^{\vee\vee} \simeq E^\vee \simeq \widehat{R}$. As in Theorem 11.4, taking a presentation of M, applying ${}^{\vee\vee}$ and using the five lemma, we see that $M \simeq M^{\vee\vee}$.

If M is Artinian, then as we have seen above, there is a $t_0 \in \mathbb{N}$ such that $M \hookrightarrow E^{\oplus t_0}$. Let C be the cokernel of this inclusion. Then C is Artinian. Hence there is a $t_1 \in \mathbb{N}$ such that $0 \to M \to E^{\oplus t_0} \to E^{\oplus t_1}$ is exact. Since $E^{\vee\vee} \simeq \widehat{R}^\vee \simeq E$, we can apply ${}^{\vee\vee}$ to the above "injective presentation" of M and use five lemma to conclude that $M \simeq M^{\vee\vee}$. $\square$

Example 4 Let M, N be finitely generated modules over a complete Noetherian local ring $(R, \mathfrak{m}, \mathsf{k}, E)$. Then

$$\text{(a) } \mathrm{Tor}_i^R(M,N)^\vee \simeq \mathrm{Ext}_R^i(M, N^\vee) \quad \text{and} \quad \text{(b) } \mathrm{Tor}_i^R(M, N^\vee) \simeq \mathrm{Ext}_R^i(M,N)^\vee.$$

In order to prove (a), consider a free resolution $\mathbf{F}_\bullet$ of N. Then $\mathrm{Tor}_i^R(M,N) = H_i(M \otimes_R \mathbf{F}_\bullet)$. By the exactness of $^\vee$, we have

$$\mathrm{Tor}_i^R(M,N)^\vee \simeq H^i((M \otimes_R \mathbf{F}_\bullet)^\vee)$$

$$= H^i(\mathrm{Hom}_R(M \otimes_R \mathbf{F}_\bullet, E)) \simeq H^i(\mathrm{Hom}_R(M, Hom(\mathbf{F}_\bullet, E)))$$

by the Hom-$\otimes$ adjointness. But $\mathrm{Hom}(\mathbf{F}_\bullet, E)$ is an injective resolution of $N^\vee$. Hence $H^i(\mathrm{Hom}_R(M, Hom(\mathbf{F}_\bullet, E))) = \mathrm{Ext}_R^i(M, N^\vee)$. This proves part (a).
Part (b) can be done as an exercise. See exercise 6.

Note: The above example can be generalised, but one has to be careful!!!

Proposition 14 *Let R be a Noetherian ring, $\mathfrak{p}$ and $\mathfrak{q}$ prime ideals in R. Then*
(a)

$$[E_R(R/\mathfrak{p})]_\mathfrak{q} = \begin{cases} 0 \text{ for } \mathfrak{p} \not\subseteq \mathfrak{q} \\ E_R(R/\mathfrak{p}) \text{ for } \mathfrak{p} \subseteq \mathfrak{q} \end{cases}$$

(b)

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_R(R/\mathfrak{q})_\mathfrak{p}) = \begin{cases} \kappa(\mathfrak{p}) \text{ for } \mathfrak{p} = \mathfrak{q} \\ 0 \text{ for } \mathfrak{p} \neq \mathfrak{q} \end{cases}$$

Proof: (a) We have proved in Theorem 8 that $E_R(R/\mathfrak{p})$ is an $R_\mathfrak{p}$-module, even more an $R_\mathfrak{q}$-module if $\mathfrak{p} \subseteq \mathfrak{q}$.

If $\mathfrak{p} \not\subseteq \mathfrak{q}$, choose $x \in \mathfrak{p}$, $x \notin \mathfrak{q}$. For every $z \in E_R(R/\mathfrak{p})$, there is an $n \in \mathbb{N}$ such that $x^n z = 0$. This implies that $\frac{z}{1} = 0$ in $[E_R(R/\mathfrak{p})]_\mathfrak{q}$ since $x \notin \mathfrak{q}$.
(b) If $\mathfrak{q} \not\subseteq \mathfrak{p}$, $E_R(R/\mathfrak{q})_\mathfrak{p} = 0$ by (a).

If $\mathfrak{q} = \mathfrak{p}$, then $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_R(R/\mathfrak{p})_\mathfrak{p}) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_{R_\mathfrak{p}}(\kappa(\mathfrak{p})))$ since $E_R(R/\mathfrak{p})_\mathfrak{p} = E_R(R/\mathfrak{p}) \simeq E_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ by Theorem 8. But $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))) = \kappa(\mathfrak{p})^\vee \simeq \kappa(\mathfrak{p})$ by Theorem 11.1. Therefore $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_R(R/\mathfrak{p})_\mathfrak{p}) \simeq \kappa(\mathfrak{p})$.

Now suppose $\mathfrak{q}$ is strictly contained in $\mathfrak{p}$. We have $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_R(R/\mathfrak{q})_\mathfrak{p}) \simeq (\mathrm{Hom}_R(R/\mathfrak{p}, E_R(R/\mathfrak{q})))_\mathfrak{p}$ by flatness of $R_\mathfrak{p}$ over R. Consider $f : R/\mathfrak{p} \to E_R(R/\mathfrak{q})$. Then $\mathfrak{p} f(\bar{1}) = 0$. Hence $\mathfrak{p} \subseteq \mathrm{ann}_R(f(\bar{1}))$. But $\mathrm{Ass}_R(E_R(R/\mathfrak{q})) = \{\mathfrak{q}\}$ and this forces $\mathrm{ann}_R(x) \subseteq \mathfrak{q}$ for every nonzero x in $E_R(R/\mathfrak{q})$. Therefore $f(\bar{1}) = 0$ since $\mathfrak{p}$ is not contained in $\mathfrak{q}$. $\square$

Corollary 15 *Let R be a Noetherian ring and I be an injective R-module. Write $I \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} E_R(R/\mathfrak{p})^{t_\mathfrak{p}}$. Then $t_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})}(\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p}))$. In particular, $t_\mathfrak{p}$ does not depend on the decomposition chosen.*

Proof: Localize $I \simeq \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} E_R(R/\mathfrak{p})^{t_\mathfrak{p}}$ at $\mathfrak{p}$ and use Prop. 14(b).

Remark 5 Combining Theorem 6 and Cor. 15, we see that any injective module over a Noetherian ring R has a unique direct sum decomposition (up to isomorphism) into indecomposable (injective) R-modules.

Corollary 16 *Let I be an injective module over a Noetherian ring R. For every prime ideal $\mathfrak{p}$ in R, $I_{\mathfrak{p}}$ is an injective $R_{\mathfrak{p}}$-module.*

Proof: Write $I \simeq \bigoplus_{\mathfrak{q} \in \operatorname{Spec}(R)} E_R(R/\mathfrak{q})^{t_{\mathfrak{q}}}$. Then $I_{\mathfrak{p}} \simeq \bigoplus_{(\mathfrak{q} \in \operatorname{Spec}(R),\ \mathfrak{q} \subseteq \mathfrak{p})} E_R(R/\mathfrak{q})^{t_{\mathfrak{q}}}$ by Prop. 14(a). Hence $I_{\mathfrak{p}}$ is injective. □

Strange but true: From the above proof, we see that $I \twoheadrightarrow I_{\mathfrak{p}}$.

§ 3.4 *Zero-dimensional Gorenstein Rings*

Theorem 17 *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a zero-dimensional Noetherian local ring. The following are equivalent:*
(1) $\operatorname{id}_R(R) < \infty$.
(2) *R is injective (as a module over itself).*
(3) $R \simeq E$.
(4) *$\operatorname{soc}_R(R)$ is a 1-dimensional k-vector space.*
(5) *The ideal (0) in R is irreducible.*
(6) *For every ideal I in R, $0 :_R (0 :_R I) = I$.*

Definition 5 *When any one (and hence all) of the above conditions are satisfied, we say that (the zero-dimensional ring) R is Gorenstein.*

Recall that:
1. For any R-module M,
$\operatorname{id}_R(M) := \inf\{n\colon 0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0$, where each E^i is injective$\}$.
2. An ideal I in R is *irreducible* if $I = J \cap K$ for ideals J and K in R implies that $I = J$ or $I = K$.
3. For any R-module M, $\operatorname{soc}_R(M) := 0 :_M \mathfrak{m}$.

Proof: (3) ⇒ (2) ⇒ (1) is immediate.
(1) ⇒ (3): Consider an injective resolution

$$0 \to R \to I^0 \to I^1 \to \cdots \to I^n \to 0$$

of R. The only indecomposable injective R-module is E since $\operatorname{Spec}(R) = \{\mathfrak{m}\}$. Therefore each $I^i \simeq E^{\oplus b_i}$; we can choose $b_i < \infty$ since R and E are Artinian. Applying $^\vee$ and rewriting $E^\vee$ as R and $R^\vee$ as E, we get a long exact sequence

$$0 \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to E \to 0.$$

which implies that $\mathrm{pd}_R(E) < \infty$. By the Auslander-Buchsbaum formula, $\mathrm{pd}_R(E) + \mathrm{depth}(E) = \mathrm{depth}(R)$. Since $\dim(R) = 0$, $\mathrm{depth}(E) = \mathrm{depth}(R) = 0$ and hence $\mathrm{pd}_R(E) = 0$ i.e. E is free. (An alternate argument can be given more simply–refine the resolution to a minimal free resolution of E. If E is not free, then there is an injective map of two free modules at the end of the resolution; but this is impossible since any socle element goes to zero.) Therefore $E \simeq R^{\oplus m}$. But $\lambda_R(R) = \lambda_R(E)$ by Theorem 11.2 which forces $m = 1$. Thus $E \simeq R$.
(3) $\Rightarrow$ (6): We want to prove that $0 :_R (0 :_R I) = I$. Note that $I \subseteq 0 :_R (0 :_R I)$.

Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Applying $^\vee$ we get the short exact sequence $0 \to (R/I)^\vee \to E \to I^\vee \to 0$. Since $R \simeq E$ (by hypothesis), $(R/I)^\vee \simeq 0 :_R I$ and hence $I^\vee \simeq \mathrm{Hom}_R(I, R) \simeq R/(0 :_R I)$.

Applying $^\vee$ again, we get $I^{\vee\vee} \simeq \mathrm{Hom}_R(R/(0 :_R I), R) \simeq 0 :_R (0 :_R I)$. Since $I^{\vee\vee} \simeq I$ by Theorem 11.4, we get $I = 0 :_R (0 :_R I)$.
(6) $\Rightarrow$ (4): Let x be a nonzero element in $\mathrm{soc}_R(R)$. Then $0 :_R x = \mathfrak{m}$. Therefore $\mathsf{k} \simeq (x) = 0 :_R (0 :_R x) = 0 :_R \mathfrak{m}$.
(4) $\Rightarrow$ (3): We know that $\mathsf{k} \simeq \mathrm{soc}_R(R) \subseteq R$ is an essential extension. Since E is the maximal essential extension of k, $R \hookrightarrow E$. Therefore $R \simeq E$ since $\lambda_R(R) = \lambda_R(E)$.
(4) $\Rightarrow$ (5): Suppose $J \neq (0)$, $K \neq (0)$ are ideals in R. Since $\mathrm{soc}_R(R) \subseteq R$ is an essential extension, $J \cap \mathrm{soc}_R(R) \neq (0)$ and $K \cap \mathrm{soc}_R(R) \neq (0)$. But $\mathrm{soc}_R(R)$ has length 1 which forces $\mathrm{soc}_R(R) \subseteq J \cap K$. Hence $J \cap K \neq (0)$, i.e. (0) is irreducible.
(5) $\Rightarrow$ (4): If $\dim(\mathrm{soc}_R(R)) \geq 2$, choose u, v linearly independent in $\mathrm{soc}_R(R)$. Then $(u) \cap (v) = (0)$ contradicts the irreducibility of (0). $\square$

Remark 6 Note that if R is Gorenstein then $R \simeq E$ means that $M^* \simeq M^\vee$ for every R-module M. Therefore $M^{\vee\vee} \simeq M$ implies that M is reflexive.

Theorem 18 *Let $(S, \mathfrak{m}, \mathsf{k}, E)$ be a zero-dimensional Noetherian local ring. Let I be an ideal in S and $R := S/I$. The following are equivalent:*
(1) *R is Gorenstein.*
(2) *$0 :_E I \simeq R$.*
(3) *$0 :_E I$ is a cyclic R-module.*
(4) *There is a nonzero element u in E such that $0 :_S u = I$.*

Proof: (1) $\Rightarrow$ (2): $0 :_E I \simeq E_{S/I}(\mathsf{k})$ by Cor. 10. Hence $0 :_E I \simeq R$ since $R = S/I$ and $R \simeq E_R(\mathsf{k})$ by assumption.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (4): Let u be a generator for $0 :_E I$ and let $J = 0 :_S u$. Then $I \subseteq J$. Since u generates $0 :_E I \simeq R^\vee$, $JR^\vee = 0$. This implies that $JR \simeq JR^{\vee\vee} = 0$ Therefore $J \subseteq I$, which proves $I = 0 :_S u$.

(4) $\Rightarrow$ (1): Let $I = 0 :_S u$. Therefore $Ru \simeq Su$ and hence $Ru \hookrightarrow E$.

Consider $0 \to Ru \to E \to E/Ru \to 0$. Apply $^\vee$ to get $0 \to (E/Ru)^\vee \to S \to \operatorname{Hom}_S(R, E) \to 0$. Clearly $I \cdot \operatorname{Hom}_S(R, E) = 0$. Therefore $\operatorname{Hom}_S(R, E) \simeq S/J$ where $I \subseteq J$. Counting lengths we see that

$$\lambda_S(S/J) \leq \lambda_S(S/I) = \lambda_S(R) = \lambda_S(\operatorname{Hom}_S(R, E)) = \lambda_S(S/J).$$

This forces $J = I$ and $R \simeq \operatorname{Hom}_S(R, E) \simeq E_R(\mathsf{k})$. (The last isomorphism is by Cor. 10). $\square$

Exercise: Let R be a Noetherian local ring, I be any unmixed ideal in R of height 0. Prove that $0 :_R (0 :_R I) = I$ if and only if $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ in R of height 0.

A way to get Gorenstein rings:
Let $R := \mathsf{k}[X_1, \ldots, X_n]$, $\operatorname{char}(\mathsf{k}) = 0$. Set $E := \mathsf{k}[Y_1, \ldots, Y_n]$ where Y_i acts on R as $\frac{\partial}{\partial X_i}$, i.e. $Y_i X_j = \delta_{ij}$. Take $f(Y_1, \ldots, Y_n) \in E$; then $R/(0 :_R f)$ is Gorenstein.

Example 5 The ring $S := \mathsf{k}[X_1, \ldots, X_n]/(X_iX_j, X_i^2 - X_j^2)_{i \neq j}$ is Gorenstein.

If we take $f = Y_1^2 + \cdots + Y_n^2$ ($= \frac{\partial^2}{\partial X_1^2} + \cdots + \frac{\partial^2}{\partial X_n^2}$); then $\operatorname{ann}_R(f) = (X_iX_j, X_i^2 - X_j^2)_{i \neq j}$ which shows that S is Gorenstein.

Remark 7 We can show that $\mathsf{k}[X_1, \ldots, X_n]/(X_iX_j, X_i^2 - X_j^2)_{i \neq j}$ is Gorenstein by showing that the socle is one-dimensional over k. See Example 10.

Discussion: Let V be an n-dimensional vector space over k. Recall that a *symmetric bilinear form* on V is a pairing $V \times V \longrightarrow \mathsf{k}$ given by $(v, w) \mapsto < v, w > \in \mathsf{k}$ such that $< \, , \, >$ is linear in each variable and is symmetric, i.e. $< v, w > = < w, v >$ for every $v, w \in V$.

For a subspace, $W \subseteq V$, we define

$$W^\perp = \{v \in V :< w, v >= 0 \text{ for every } w \in W\}.$$

We say that the form $< \, , \, >$ is *non-degenerate* if $V^\perp = 0$.

Let $\{v_1, \ldots, v_n\}$ be a k-basis for V. Given a symmetric bilinear form $< \, , \, >$ on V, we can associate to it a symmetric matrix as follows:

$$< \, , \, > \quad \longleftrightarrow \quad (< v_i, v_j >).$$

One can check that $< \, , \, >$ is non-generate if and only if the matrix $(< v_i, v_j >)$ is invertible.

If $< \, , \, >$ is a symmetric bilinear form on V, then construct a ring $R := \mathsf{k} \oplus V \oplus R_2$ as a k-vector space, where $R_2 = \mathsf{k}$. The multiplicative structure on R is as follows: In degree 0, k acts as scalars, $R_1 \cdot R_2 = 0 = R_2 \cdot R_2$ and for $x, y \in R_1,\ x \cdot y =< x, y >$. Associativity follows trivially since $R_1^3 = 0$.

Conversely, let $R = \mathsf{k} \oplus R_1 \oplus R_2$, with Hilbert function $1, n, 1$, i.e. $\dim_{\mathsf{k}}(R_1) = n$ and $R_2 \simeq \mathsf{k}$. Fix a generator Δ of R_2. Associated to R, there is a symmetric bilinear form on R_1 given by $< x, y >= \alpha$ for $x,\ y \in R_1$, where $x \cdot y = \alpha\Delta$.

Question: The ring R is an Artinian graded ring. When is it Gorenstein?

Note that $R \simeq \mathsf{k}[X_1, \dots, X_n]/I$, where I is a homogeneous ideal such that $\mathfrak{m}^3 \subseteq I$. Now R is Gorenstein if and only if $0 :_R \mathfrak{m} = \operatorname{soc}(R)$ is a one-dimensional k-vector space. Since $\mathfrak{m}^3 = 0$, we see that $\mathfrak{m}^2 \subseteq \operatorname{soc}(R)$. Hence R is Gorenstein if and only if $\mathfrak{m}^2 = \operatorname{soc}(R)$.

In other words, R is Gorenstein if and only if $R_1 \cap \operatorname{soc}(R) = 0$, i.e. $\operatorname{soc}(R)$ does not contain any linear forms. Note that

$$x \in 0 :_R \mathfrak{m} \cap R_1 \Leftrightarrow x \cdot y = 0 \text{ for all } y \in R_1 \ \Leftrightarrow < x, R_1 >= 0.$$

Thus R is Gorenstein if and only if $< \, , \, >$ is a non-degenerate symmetric bilinear form on R_1.

Example 6 Let V be a 3-dimensional vector space over a field k. Then the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ corresponds to a non-degenerate symmetric bilinear form on V. The corresponding ring $R = \mathsf{k} \oplus \mathbb{R}_1 \oplus \mathbb{R}_2$ is Gorenstein, where $R_1 = \mathsf{k}x \oplus \mathsf{k}y \oplus \mathsf{k}z$, $R_2 = \mathsf{k}\Delta$ satisfying $x^2 = y^2 = z^2 = \Delta$, $xy = 0$, $yz = 0$ and $xz = 0$,
i.e. $R \simeq \mathsf{k}[X, Y, Z]/(X^2 - Y^2, X^2 - Z^2, XY, YZ, XZ)$ is Gorenstein.

Remark 8 Gorenstein rings with Hilbert function $1, n, n, 1$ have not been classified.

Question: If R is a graded zero-dimensional Gorenstein ring with Hilbert function $1, n, m, n, 1$, is it necessary that $m \geq 2n - 2$?

§ 3.5 *Free Resolutions of Gorenstein Quotients of Regular Local Rings*

Set-up: Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension n. Let $R := T/I$ be a zero-dimensional quotient of T. Since $\operatorname{depth}(R) = 0$, by the Auslander-Buchsbaum formula, $\operatorname{pd}_T(R) = \operatorname{depth}(T) = n$. Consider a minimal free resolution of R:

$$\mathbf{F}_\bullet : \quad 0 \to T^{b_n} \xrightarrow{\phi_n} \cdots \to T^{b_1} \xrightarrow{\phi_1} T \to R. \quad (*)$$

Theorem 19 *With notation as above, $b_n = \dim_{\mathsf{k}}(\mathrm{soc}(R))$. In particular, R is Gorenstein if and only if $b_n = 1$.*

Proof: Compute $\mathrm{Tor}_n^T(\mathsf{k}, R)$ in two different ways. Since $(*)$ is a minimal resolution of R over T, $\mathrm{Tor}_n^T(\mathsf{k}, R) \simeq \mathsf{k}^{b_n}$.

Now, a resolution of $\mathsf{k}$is given by the Koszul complex of a minimal set of generators of $\mathfrak{m}_T = (x_1, \dots, x_n)$. We have

$$0 \longrightarrow T \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} T^n \longrightarrow \cdots \longrightarrow T^n \longrightarrow T \longrightarrow \mathsf{k} \to 0.$$

Therefore $\mathrm{Tor}_n^T(\mathsf{k}, R)$ is isomorphic to the homology of $0 \longrightarrow R \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} R^n$ which is $\mathrm{soc}(R)$. □

Corollary 20 *If $f_1, \dots, f_n$ is a regular sequence in $\mathfrak{m}_T$, then $R := T/(f_1, \dots, f_n)$ is Gorenstein.*

Proof: A resolution of R over T is given by the Koszul complex $K(f_1, \dots, f_n; T)$ and hence $b_n = 1$. □

Corollary 21 (Serre) *If $\dim(T) = 2$ and $R := T/I$ is Gorenstein, then $\mu(I) = 2$, i.e. $I = (f, g)$ where f, g is a regular sequence in T.*

Proof: Let $\mu(I) = b$. Since $\dim(T) = 2$ and R is Gorenstein, there is a resolution of R over T that looks like

$$0 \to T \to T^b \to T \to R \to 0.$$

Tensoring with $Q(T)$, the fraction field of T, gives $b = 2$. □

Corollary 22 *If T is a regular local ring of dimension n and $f_1, \dots, f_n$ is a regular sequence, then $(f_1, \dots, f_n)$ is an irreducible ideal in T.*

Proof: Since $R := T/(f_1, \dots, f_n)$ is Gorenstein, $(0) \subseteq R$ is irreducible which is equivalent to $(f_1, \dots, f_n) \subseteq T$ being irreducible. □

A natural question that arises from Cor. 20 is the following:
We know that $\mathrm{soc}_T(T/(f_1, \dots, f_n))$ is 1-dimensional. Is there any way to compute it?
Answer: Let $\mathfrak{m}_T = (x_1, \dots, x_n)$. Write $f_i = \sum_{j=1}^n a_{ij} x_j$, $i = 1, \dots, n$. Let $A := (a_{ij})_{n \times n}$. Then $\det(A)$ is a generator for $\mathrm{soc}_T(T/(f_1, \dots, f_n))$.

Example 7 Let $x_1^{l_1} = f_1, \dots, x_n^{l_n} = f_n$. Then $\det(A) = x_1^{l_1-1} \cdots x_n^{l_n-1}$ represents the socle.

In the polynomial case there is another answer.
Let $T = \mathsf{k}[X_1, \dots, X_n]$, $\mathrm{rad}(f_1, \dots, f_n) = (X_1, \dots, X_n)$ and $I = (\partial f_i/\partial X_j)_{1 \le i,j \le n}$. Then $\det(I)$ represents the socle of $T/(f_1, \dots, f_n)$.

The two answers overlap in the graded case when the f_i's are homogeneous of degree m_i such that $\mathrm{char}(\mathsf{k}) \not| \, m_i$.
By Euler's formula,

$$m_i f i = \sum X_j \frac{\partial f_i}{\partial X_j}.$$

Since $(m_1 \cdots m_n)^{-1} \in \mathsf{k}$, we can write

$$f_i = \sum X_j \frac{1}{m_i} \frac{\partial f_i}{\partial X_j}.$$

By the first case $\det(\frac{1}{m_i}\frac{\partial f_i}{\partial X_j})$ represents the socle. This agrees with the second answer since $\det(\frac{1}{m_i}\frac{\partial f_i}{\partial X_j}) = \frac{1}{m_1 \cdots m_n}\det(\frac{\partial f_i}{\partial X_j})$.

Example 8 Let us compute $\mathrm{soc}_T(T/(f,g))$ where $T = \mathsf{k}[X,Y]$, $f = X^3 + Y^7$ and $g = X^2Y^3$. With notations as before,

$$A = \begin{bmatrix} X^2 & Y^6 \\ XY^3 & 0 \end{bmatrix}.$$

Therefore $\det(A) = -XY^9$. Hence $\mathrm{soc}_T(R) \simeq (XY^9)$.
Every ideal containing (f,g) properly must contain XY^9.

Theorem 23 *With the same notations as in Theorem 19, $E_R(\mathsf{k}) \simeq \mathrm{Coker}(\phi_n^*)$ and has a free resolution*

$$0 \to T^* \overset{\phi_1^*}{\to} (T^*)^{b_1} \to \cdots \to (T^*)^{b_n} \to E_R(\mathsf{k}) \to 0 \qquad (**)$$

over T. In particular, $\mathrm{Ext}_T^n(R,T) \simeq E_R(\mathsf{k})$.

Proof: Apply $\mathrm{Hom}_T(\_, T)$ to $(*)$. The homology is $\mathrm{Ext}_T^i(R,T)$. But $\mathrm{depth}_{\mathrm{ann}(R)}(T) = n$. Hence $\mathrm{Ext}_T^i(R,T) = 0$ for $i < n$ and

$$0 \to T^* \overset{\phi_1^*}{\to} \cdots \overset{\phi_n^*}{\to} (T^*)^{b_n} \to \mathrm{Ext}_T^n(R,T) \to 0$$

is exact. By the Ext-shifting lemma, $\mathrm{Ext}_T^n(R,T) \simeq \mathrm{Hom}_{T/(f_1,\dots,f_n)}(R, T/(f_1,\dots,f_n))$ for a regular sequence $\underline{f} := f_1, \dots, f_n$ in I. By Cor. 20, $T/(\underline{f})$ is Gorenstein and hence $E_{T/(\underline{f})}(\mathsf{k}) \simeq T/(\underline{f})$. Then $\mathrm{Hom}_{T/(\underline{f})}(R, T/(\underline{f})) \simeq \mathrm{Hom}_{T/(\underline{f})}(R, E_{T/(\underline{f})}(\mathsf{k})) \simeq E_R(\mathsf{k})$ by Theorem 9. $\square$

Corollary 24 $\mu(E_R(\mathsf{k})) = b_n$.

Proof: Since ϕ_i^* has entries in $\mathfrak{m}_T$, $(**)$ is a minimal resolution for $E_R(\mathsf{k})$. This implies that $\mu(E_R(\mathsf{k})) = b_n$. $\square$

Definition 6 *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a zero-dimensional Noetherian local ring. The number $\mu(E_R(\mathsf{k})) = \dim_{\mathsf{k}}(\mathrm{soc}_T(R)) = b_n$ is called the type of R.*

Remark 9 Observe that in the above proof we have $E_R(\mathsf{k}) \simeq \mathrm{Hom}(T/I, T/\underline{f}) \simeq ((\underline{f}) :_T I)/(\underline{f})$. From this we get:
R is Gorenstein if and only if $E_R(\mathsf{k})$ is cyclic ($b_n = 1$) if and only if $(\underline{f}) :_T I = (\underline{f}, g)$. But since $T/\underline{f}$ is Gorenstein, $I = (\underline{f}) :_T ((\underline{f}) :_T I) = (\underline{f} :_T g)$.

Conversely if $g \notin (\underline{f})$, reversing the above arguments shows that $T/((\underline{f}) :_T g)$ is Gorenstein.

Example 9 Let $I = (X^3, Y^3, Z^3)$, $g = X^2 + Y^2 + Z^2$. Then

$$I :_T g = (X^3, Y^3, Z^3, XYZ, X(Y^2 - Z^2), Y(X^2 - Z^2), Z(X^2 - Y^2))$$

is a 7-generated Gorenstein ideal.

Theorem 25 (J. Watanabe) *Let $I \subseteq T := \mathsf{k}[X, Y, Z]$ be such that T/I is a zero-dimensional Gorenstein ring. Then I has an odd number of generators.*

§ 3.6 *Teter's Theorem*

In this section, we are aiming for the following theorem of Teter.

Theorem 26 (Teter,1974) *Let $(R, \mathfrak{m}_R, \mathsf{k}, E)$ be an Artinian ring*[1]*. Then the following are equivalent:*
1 *$R \simeq S/(\Delta)$, where $(S, \mathfrak{m}_S, \mathsf{k})$ is a (zero-dimensional) Gorenstein* [2] *ring and $(\Delta) = \mathrm{soc}(S)$.*
2 *There is an isomorphism $\mathfrak{m}_R \xrightarrow{\phi} \mathfrak{m}_R^\vee$ such that for every x, y in $\mathfrak{m}_R$, $\phi(x)(y) = \phi(y)(x)$.*

General Discussion

Let $(R, \mathfrak{m}_R, \mathsf{k})$ be an Artinian ring. Define

$$g(R) = \min\{\lambda(S) - \lambda(R) : S \text{ is a Gorenstein ring mapping onto } R\}.$$

[1]All rings in this section are local

[2]In this section, a Gorenstein ring is a self-injective ring i.e. is zero dimensional.

The number $g(R)$ gives a numerical value for how close can one get to an Artinian ring R by a Gorenstein ring.
The question is: How does one intrinsically compute $g(R)$?
Comments :
1. Note that $g(R)$ is zero if and only if R is Gorenstein.
2. When is $g(R) = 1$? This occurs if and only if $R \simeq S/\mathrm{soc}(S)$ for a Gorenstein ring S and is answered completely by Teter.
3. Observe that $g(R)$ is always finite. This can be seen as follows:

By Cohen's Structure Theorem, we can write R as the quotient of a regular local ring T by an $\mathfrak{m}_T$-primary ideal. Let $R := T/K$ where K is $\mathfrak{m}_T$-primary. If $\dim(T) = d$, choose a regular sequence $x_1, \ldots, x_d$ in K and write $S := T/(x_1, \ldots, x_d)$. Then S is a complete intersection ring and hence Gorenstein. If we set $J := K/(x_1, \ldots, x_d)$, we see that $R \simeq S/J$ and hence $g(R) < \infty$.
4. One can say something even better. We will show that $g(R) \leq \lambda(R)$. Let $E = E_R(\mathsf{k})$ and define $S := R \oplus E$. Then S is a local ring under the following operations:

$$(r, u) + (s, v) = (r + s, u + v) \text{ and } (r, u) \cdot (s, v) = (rs, rv + su).$$

Note that E is an ideal in S and that $E^2 = 0$. Hence $\mathfrak{m}_S := \mathfrak{m} \oplus E$, the unique maximal ideal in S, is nilpotent, showing that S is Artinian.

Now we know that $\mathrm{Hom}_R(S, E)$ is an injective S-module. But

$$\mathrm{Hom}_R(S, E) = \mathrm{Hom}_R(R \oplus E, E) \simeq \mathrm{Hom}_R(R, E) \oplus \mathrm{Hom}_R(E, E) \simeq E \oplus R$$

by Theorem 11. Thus $\mathrm{Hom}_R(S, E) \simeq S$, i.e. S is self-injective and hence Gorenstein.

Since $\lambda_R(S) = 2\lambda(R)$, and S maps onto R via the natural projection, $g(R) \leq \lambda_R(S) - \lambda(R) = \lambda(R)$.

Example 10 The ring $S = \mathsf{k}[X_1, \ldots, X_n]/(X_iX_j, X_i^2 - X_j^2)_{i \neq j}$ is Gorenstein. Let $\mathfrak{m} = (X_1, \ldots, X_n)$, $I = (X_iX_j, X_i^2 - X_j^2)_{i \neq j}$. Then $\mathfrak{m}^3 \subseteq I$.

There is a one dimensional space of quadrics in $\mathfrak{m}/I$, i.e. $\dim(\mathfrak{m}/I)_2 = 1$, for example, X^2 generates this space.

Let $l = \Sigma \alpha_i X_i$ be any linear form. If $\alpha_j \neq 0$, then $X_j(l) = \alpha_j X_i^2 \notin I$. Hence $l\mathfrak{m} \not\subseteq I$. In other words, there are no linear forms in the socle of S. Therefore $\mathrm{soc}(S) = (X_1^2)$ has dimension 1 over k. Hence S is Gorenstein.
The Hilbert series of S is $H_S(t) = 1 + nt + t^2$.

Since $S/\mathrm{soc}(S) \simeq \mathsf{k}[X_1, \ldots, X_n]/\mathfrak{m}^2 =: R$, $g(R) = 1$.

Question: Suppose that $I = I_r(X_{r \times s})$ $(r < s)$ defines a Cohen-Macaulay quotient. How close can we get to I by a Gorenstein ideal J?

Lemma 27 *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian ring. Then the following are equivalent:*
1. *There is an isomorphism $\phi : \mathfrak{m} \to \mathfrak{m}^\vee$ such that*

$$\forall x, y \in \mathfrak{m}, \quad \phi(x)(y) = \phi(y)(x).$$

2. *There is an isomorphism $\theta : \mathfrak{m}^\vee \to \mathfrak{m}$ such that*

$$\forall f, g \in \mathfrak{m}^\vee, \quad g(\theta(f)) = f(\theta(g)).$$

3. *There is a surjective homomorphism $\psi : E \to \mathfrak{m}$ such that*

$$\forall u, v \in E, \quad \psi(u)(v) = \psi(v)(u).$$

Remark 10 Let $\mathfrak{m} \overset{i}{\hookrightarrow} R$. This induces a surjective map $E \overset{i^\vee}{\twoheadrightarrow} \mathfrak{m}^\vee$, such that

$$\text{for every } x \in \mathfrak{m} \text{ and } u \in E, \quad i^\vee(u)(x) = xu.$$

Proof: The map $\mathrm{Hom}_R(R, E) \xrightarrow{i^\vee} \mathrm{Hom}_R(\mathfrak{m}, E)$ is given by $u \mapsto u \circ i$. Hence $i^\vee(u)(x) = xu$.

Proof of the Lemma: (1) $\Rightarrow$ (2): Set $\theta := \phi^{-1}$. We need to prove $\forall f, g \in \mathfrak{m}^\vee$, $g(\phi^{-1}(f)) = f(\phi^{-1}(g))$.

Write $x = \phi^{-1}(f)$ and $y = \phi^{-1}(g)$. Then we have $\phi(x)(y) = \phi(y)(x)$. Rewriting this in terms of f, g and ϕ^{-1} gives us the required property.
(2)$\Rightarrow$(3): We have an isomorphism $\mathfrak{m}^\vee \xrightarrow{\theta} \mathfrak{m}$. Define $E \xrightarrow{\psi} \mathfrak{m}$ by $\psi = \theta \circ i^\vee$. Let u, v be in E. Write $i^\vee(u) = f$ and $i^\vee(v) = g$. We have $g(\theta(f)) = f(\theta(g))$ rewriting which we get $i^\vee(v)(\psi(u)) = i^\vee(u)(\psi(v))$. By the above remark, this is the same as $\psi(u)v = \psi(v)u$.
(3) $\Rightarrow$ (1): We have the two surjective maps $E \overset{\psi}{\twoheadrightarrow} \mathfrak{m}$ and $E \overset{i^\vee}{\twoheadrightarrow} \mathfrak{m}^\vee$. Note that since E is Artinian, it is essential over its socle. Moreover, $\mathrm{soc}(E)$ is one-dimensional. Hence $\lambda(\mathrm{Ker}(\psi)) = 1 = \lambda(\mathrm{Ker}(i^\vee))$ forces $\mathrm{Ker}(\psi) = \mathrm{soc}(E) = \mathrm{Ker}(i^\vee)$.

Define $\phi : \mathfrak{m} \to \mathfrak{m}^\vee$ as $x \mapsto i^\vee(\psi(u))$ for any preimage u of x under ψ. We want to show that this map is well-defined. Suppose $\psi(u) = \psi(v)$ for $u, v \in E$. Then $u - v \in \mathrm{Ker}(\psi) = \mathrm{Ker}(i^\vee)$, i.e. $i^\vee(u) = i^\vee(v)$. Thus ϕ is well-defined and hence by the Snake Lemma, is an isomorphism.

Let $x, y \in \mathfrak{m}$. Choose $u, v \in E$ such that $\psi(u) = x$ and $\psi(v) = y$. We have $\psi(u)v = \psi(v)u$, i.e. $i^\vee(u)(\psi(v)) = i^\vee(v)(\psi(u))$ by the above remark. By definition $\phi(x) = i^\vee(u)$ and $\phi(y) = i^\vee(v)$. Hence we get $\phi(x)(y) = \phi(y)(x)$. $\square$

Proof of Teter's theorem: (1) $\Rightarrow$ (2): Let $R \simeq S/\mathrm{soc}(S)$, where S is Gorenstein. Since S is Gorenstein, it is self-injective and hence $\mathrm{Hom}_S(\_, S)$ is exact. Since $\mathrm{soc}(S) \simeq \mathsf{k}$, we have the short exact sequence $0 \to \mathsf{k} \to S \to R \to 0$. By applying

$\operatorname{Hom}_S(\_, S)$, we get $0 \to \operatorname{Hom}(R, S) \to S \to \mathsf{k} \to 0$ since $\operatorname{Hom}_S(\mathsf{k}, S) \simeq \operatorname{soc}(S) \simeq \mathsf{k}$. This gives us $\operatorname{Hom}_S(R, S) \simeq \mathfrak{m}_S$. But $\operatorname{Hom}_S(R, S) = \operatorname{Hom}_S(R, E_S(\mathsf{k})) \simeq E_R(\mathsf{k}) = E$. Thus

$$R \simeq S/\operatorname{soc}(S) \implies E \simeq \mathfrak{m}_S.$$

(Observe that $\mathfrak{m}_S \cdot \operatorname{soc}(S) = 0$, hence $\mathfrak{m}_S$ is an R-module.)

Define $\psi : E \twoheadrightarrow \mathfrak{m}_R$ by the canonical surjection $\mathfrak{m}_S \twoheadrightarrow \mathfrak{m}_R$. Let $u, v \in \mathfrak{m}_S$. Then $\psi(u)(v) = u \cdot v = \psi(v)(u)$. The statement of (2) follows by the lemma.

The idea of (2) $\Rightarrow$ (1): We use the fact that R is a quotient a regular local ring $(T, \mathfrak{m}_T, \mathsf{k})$ by an ideal I contained in $\mathfrak{m}_T^2$. This follows from Cohen's Structure theorem.

We are looking for a Gorenstein ring $S := T/J$, $J \subseteq I$ such that $I = J :_T \mathfrak{m}_T$. Teter looks in the vector space $I/\mathfrak{m}_T I$ for a subspace V of codimension 1, that lifts back to an ideal J in T such that $S = T/J$ is actually Gorenstein.
Note that if we can find such a J, then $\mathfrak{m}_T I \subseteq J$.

Proof of (2) $\Rightarrow$ (1): We have $\phi : \mathfrak{m}_R \xrightarrow{\simeq} \mathfrak{m}_R^\vee$. By the Hom $-\otimes$ adjointness, $\phi \in \operatorname{Hom}_R(\mathfrak{m}_R, \operatorname{Hom}_R(\mathfrak{m}_R, E))$ gives a map $\tilde{\phi} \in \operatorname{Hom}_R(\mathfrak{m}_R \otimes_R \mathfrak{m}_R, E)$ defined by $\tilde{\phi}(x \otimes y) = \phi(x)(y)$ for any $x, y \in \mathfrak{m}_R$. Note that the condition of (2) implies that

$$\tilde{\phi}(x \otimes y) = \tilde{\phi}(y \otimes x).$$

We have

$$\begin{tikzcd}
\mathfrak{m}_T/I \otimes \mathfrak{m}_T/I \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"] & E \\
\mathfrak{m}_T^2/\mathfrak{m}_T I \arrow[r, "="] & \mathfrak{m}_T^2/\mathfrak{m}_T I \arrow[u, dashed, "\widehat{\phi}"']
\end{tikzcd}$$

where $(x + I) \otimes (y + I) \overset{\tilde{\phi}}{\mapsto} \phi(x + I)(y + I)$ and $(x + I) \otimes (y + I) \overset{\pi}{\mapsto} (xy + \mathfrak{m}_T I)$.

We claim that there is a map $\widehat{\phi} : \mathfrak{m}_T^2/\mathfrak{m}_T I \to E$ such that:
(a) the above diagram commutes.
(b) $\widehat{\phi}|_{(I/\mathfrak{m}_T I)} \neq 0$.
(c) $\operatorname{Ker}(\widehat{\phi}|_{(I/\mathfrak{m}_T I)}) =: J/\mathfrak{m}_T I$ is a subspace of $I/\mathfrak{m}_T I$ of codimension 1 and
(d) $J :_T \mathfrak{m}_T = I$.

In order to prove (a), it is enough to prove that $\operatorname{Ker}(\pi)$ is generated by elements in $\mathfrak{m}_T/I \otimes \mathfrak{m}_T/I$ of the form $(x + I) \otimes (y + I) - (y + I) \otimes (x + I)$. In such a case $\tilde{\phi}$ restricts to $\widehat{\phi}$ making the diagram commute.

Let $\mathfrak{m}_T$ be minimally generated by $x_1, \ldots, x_n$. Let $\Sigma(\bar{a}_i \otimes \bar{b}_i)$ be an element of $\operatorname{Ker}(\pi)$, where $\bar{x} = x + I$. Since $a_i, b_i \in \mathfrak{m}_T$, without loss of generality we may assume that $\Sigma(\bar{a_i} \otimes \bar{b_i}) = \Sigma_{i=1}^n(\bar{c_i} \otimes \bar{x_i}) \in \mathrm{Ker}(\pi)$. Hence $\Sigma_{i=1}^n c_i x_i \in \mathfrak{m}_T I$. This implies that there are elements $u_i \in I$ such that $\Sigma_{i=1}^n c_i x_i = \Sigma_{i=1}^n u_i x_i$ in T. Hence $\Sigma_{i=1}^n (c_i - u_i)x_i = 0$ in T. But $x_1, \ldots, x_n$ is a regular sequence on T, so $(c_1, \ldots, c_n) - (u_1, \ldots, u_n)$ can be written in terms of the Koszul syzygies i.e.

$$(c_1 - u_1, \ldots, c_n - u_n) = \Sigma_{i<j} u_{ij}(x_j e_i - x_i e_j)$$

where $\{e_i\}_{i=1}^n$ is the standard basis of T^n.

Going modulo I, we see that

$$(\bar{c}_1, \ldots, \bar{c}_n) = \Sigma_{i<j} \bar{u_{ij}}(0, \ldots, \bar{x_j}, \ldots, -\bar{x_i}, \ldots, 0).$$

Let $(a_1, \ldots, a_n) \overset{\bullet}{\otimes} (b_1, \ldots, b_n)$ denote $\Sigma(a_i \otimes b_i)$. Then

$$\Sigma(\bar{c_i} \otimes \bar{x_i}) = (\bar{c}_1, \ldots, \bar{c}_n) \overset{\bullet}{\otimes} (\bar{x}_1, \ldots, \bar{x}_n)^T$$

$$= \Sigma_{i<j} \bar{u_{ij}}(0, \ldots, \bar{x_j}, \ldots, -\bar{x_i}, \ldots, 0) \overset{\bullet}{\otimes} (\bar{x}_1, \ldots, \bar{x}_n)^T = \Sigma \bar{u_{ij}}(\bar{x_i} \otimes \bar{x_j} - \bar{x_j} \otimes \bar{x_i})$$

verifying (a).

We now have a map $\mathfrak{m}_T^2/\mathfrak{m}_T I \xrightarrow{\widehat{\phi}} E$ where $\widehat{\phi}(\overline{\Sigma a_i b_i}) = \Sigma\phi(a_i)(b_i)$. Restrict $\widehat{\phi}$ to $I/\mathfrak{m}_T I$, call it g. Since $\mathfrak{m}_T \cdot I/\mathfrak{m}_T I = 0$, $g : I/\mathfrak{m}_T I \to \mathrm{soc}(E) \simeq \mathsf{k}$.

Let $J \subseteq T$ be defined by $J/\mathfrak{m}_T I = \mathrm{Ker}(g)$. Then $J :_T \mathfrak{m}_T = I$ as can be seen as follows:
Suppose u is an element of T such that $u \cdot \mathfrak{m}_T \subseteq J$, i.e. for each j, $u \cdot x_j \in J$. Then $g(\overline{ux_j}) = 0$ since $J/\mathfrak{m}_T I = \mathrm{Ker}(g)$. So $\phi(\bar{u})(\bar{x_j}) = 0$ for each j. But $\phi(\bar{u}) \in \mathfrak{m}_R^\vee$ and this says $\phi(\bar{u})(\mathfrak{m}_R) = 0$. Hence $\phi(\bar{u}) = 0$. But ϕ is an isomorphism. Hence $u \in I$ which proves (d).

Note that the exact sequence $0 \to J/\mathfrak{m}_T I \to I/\mathfrak{m}_T I \xrightarrow{g} \mathsf{k}$ shows that $\lambda(I/J) \leq 1$. But $\lambda((J :_T \mathfrak{m}_T)/J) \geq 1$. Hence $\lambda(I/J) = 1$ and therefore $g : I/\mathfrak{m}_T I \twoheadrightarrow \mathsf{k}$ which proves (b) and (c).
Thus $S := T/J$ is a Gorenstein ring (since $\mathrm{soc}(S) \simeq (J :_T \mathfrak{m}_T)/J$ is one dimensional) such that $R \simeq S/\mathrm{soc}(S)$. $\square$

The following theorem is an improvement of Teter's Theorem.

Theorem 28 (Huneke,Vraciu) *Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian ring such that $1/2 \in R$, $\mathrm{soc}(R) \subseteq \mathfrak{m}_R^2$. Then the following are equivalent.* 1. *There is a Gorenstein ring S such that $R \simeq S/\mathrm{soc}(S)$.*
2. *There is a surjective map $E \twoheadrightarrow \mathfrak{m}_R$.*

Obvious question: If $E \twoheadrightarrow I$ for some ideal I in R, does there exist a Gorenstein ring S mapping onto R such that $\lambda(S) - \lambda(R) \leq \lambda(R/I)$?

General question on socles: Let $I \subseteq \mathsf{k}[X_1, \ldots, X_n] =: S$ such that $R := S/I$ is Cohen-Macaulay (maybe Gorenstein). Does there exist an ideal $\Delta \not\subseteq I$ such that for every system of parameters $f_1, \ldots, f_d$, the image of Δ in $R/(f_1, \ldots, f_d)$ generates the socle?

Remark 11 Condition (2) of Teter's theorem is equivalent by Lemma 27 to (2′): there exists $E \xrightarrow{f} \!\!\!\!\to \mathfrak{m}_R$ such that for every u and v in E, $f(u)(v) = f(v)(u)$.

Proof of Theorem: (1) $\Rightarrow$ (2) follows from Teter's theorem and (2′).

To prove (2) $\Rightarrow$ (1), given $f : E \twoheadrightarrow \mathfrak{m}_R$, we construct another $g : E \twoheadrightarrow \mathfrak{m}_R$ such that $g(u)(v) = g(v)(u)$ and then invoke Teter's theorem to conclude the proof.

Let us postpone the proof of (2) $\Rightarrow$ (1) until the end of the following
Discussion: We will construct an involution $\sim$ on $E^* = \mathrm{Hom}_R(E, R)$. Let $f \in \mathrm{Hom}_R(E, R)$. Fix $u \in E$. Consider $\phi_{f,u} : E \to E$ defined by $\phi_{f,u}(v) = f(v) \cdot u$. Since $\mathrm{Hom}_R(E, E) \simeq R$, there is an element $r_{f,u} \in R$ such that $\phi_{f,u}(v) = r_{f,u} \cdot v$.

Define $\tilde{f} : E \to R$ by $\tilde{f}(u) = r_{f,u}$. Verify that $\tilde{f} \in E^*$. Note that for every u and v in E, we have

$$\tilde{f}(u)v = r_{f,u}(v) = \phi_{f,u}(v) = f(v)u \qquad\qquad \text{Property P.}$$

Also note that $\tilde{\tilde{f}} = f$. This happens due to Property P. In particular, $E^* \xrightarrow{\sim} E^*$ is an isomorpism and $(\sim)^2 = Id$. So $\sim$ is an involution.
Proof of (2) $\Rightarrow$ (1): Let $g = f + \tilde{f}$. Note that for every u, v in E, $g(u)v = f(u)v + \tilde{f}(u)v = \tilde{f}(v)u + f(v)u$ by Property P. Hence

$$g(u)v = g(v)u \text{ for every } u, v \in E.$$

To prove g is onto, it suffices to prove that $\lambda(\mathrm{Ker}(g)) = 1$, since $\lambda(E) = \lambda(R) = \lambda(\mathfrak{m}_R) + 1$. So we need to prove that $\mathrm{Ker}(g) \subseteq \mathrm{soc}(E)$.
Quick Remark: Note that

$$\mathfrak{m}_R(f(u)v - f(v)u) = 0 \text{ for every } u, v \in E.$$

This follows since $f(u)v - f(v)u \in \mathrm{Ker}(f) = \mathrm{soc}(E)$ (by counting lengths).

Let $u \in \mathrm{Ker}(g)$. We will show that $u \in \mathrm{soc}(E)$ which will finish the proof that g is surjective.
Case 1: $u \in \mathfrak{m}_R E$.
Write $u = \Sigma_i r_i u_i$, $r_i \in \mathfrak{m}_R$ and $u_i \in E$. Then for every $v \in E$,

$$f(u)v = (\Sigma r_i f(u_i))v = \Sigma(r_i f(u_i)v) = \Sigma(r_i u_i f(v)) = uf(v)$$

by the above remark. Hence $u \in \operatorname{Ker}(g)$ implies that for every $v \in E$,

$$0 = g(u)v = f(u)v + \tilde{f}(u)v = f(u)v + f(v)u = 2f(u)v.$$

Since $1/2 \in R$, $f(u)v = 0$ for every $v \in E$. But E is faithful, hence $f(u) = 0$, i.e. $u \in \operatorname{Ker}(f) = \operatorname{soc}(E)$.
Case 2: $u \notin \mathfrak{m}_R E$.
We will show that this contradicts $\operatorname{soc}(R) \subseteq \mathfrak{m}_R^2$.
Claim: $\mathfrak{m}_R^2 u = 0$.
Let $r \in \mathfrak{m}_R$. Then $ru \in \operatorname{Ker}(g)$ and $ru \in \mathfrak{m}_R E$. By case 1, this gives us $ru \in \operatorname{soc}(E)$ which implies that $\mathfrak{m}_R^2 u = 0$. The proof is complete using the following lemma.

Lemma 29 *A minimal generator $u \in E$ such that $\mathfrak{m}_R^2 u = 0$ corresponds to a socle element $x \in \operatorname{soc}(R)$ such that $x \notin \mathfrak{m}_R^2$.*

Proof: We have $u \in 0 :_E \mathfrak{m}_R^2 \simeq \operatorname{Hom}_R(R/\mathfrak{m}_R^2, E) = E_{R/\mathfrak{m}_R^2}(\mathsf{k})$.

Consider the following where $J = 0 :_E \mathfrak{m}_R^2$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{m}_R J \arrow[r] \arrow[d] & J \arrow[r, "u\mapsto(1,0,\ldots,0)"] \arrow[d] & \mathsf{k}^s \arrow[r] \arrow[d, "\text{not zero}"] & 0 \\
0 \arrow[r] & \mathfrak{m}_R E \arrow[r] & E \arrow[r, "u\mapsto(1,0,\ldots,0))"] & \mathsf{k}^t \arrow[r] & 0
\end{tikzcd}$$

Applying $\operatorname{Hom}_R(\_, E)$, we get

$$\begin{tikzcd}
 & \operatorname{soc}(R) \arrow[d, equal] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{k}^t \arrow[r] \arrow[d, "\text{not zero}"] & R \arrow[r] \arrow[d] & (\mathfrak{m}_R E)^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathsf{k}^s \arrow[r] & R/\mathfrak{m}_R^2 \arrow[r] & (\mathfrak{m}_R J)^\vee \arrow[r] & 0
\end{tikzcd}$$

Hence there is an element $x \in \operatorname{soc}(R)$ such that $x \in \mathfrak{m}_R^2$. $\square$

§ 3.7 *Gorenstein Rings in Arbitrary Dimensions*

Definition 7 *A Noetherian local ring $(R, \mathfrak{m}, \mathsf{k}, E)$ is Gorenstein if* $\operatorname{id}_R(R) < \infty$.

Remark 12 If R is Gorenstein, then so is $R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ in R. This is true due to the fact that if $\mathbf{I}^\bullet$ is an injective resolution of R over R, then $\mathbf{I}^\bullet{}_\mathfrak{p}$ is an injective resolution of $R_\mathfrak{p}$ over $R_\mathfrak{p}$.

Thus we can define a Noetherian ring R to be Gorenstein if and only if $R_\mathfrak{p}$ is Gorenstein for every prime ideal $\mathfrak{p}$ in R (or equivalently $R_\mathfrak{m}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ in R).

Minimal Injective Resolutions

Let R be a Noetherian ring and M be an R-module.

Definition 8 *We say that an injective resolution* $0 \to M \to I^0 \xrightarrow{\phi_0} I^1 \xrightarrow{\phi_1} \dots$ *of* M *over* R *is minimal if*

$$E_R(I^i/\phi_{i-1}(I^{i-1})) \simeq I^{i+1}.$$

The following theorem gives a homological criterion for an injective resolution to be minimal.

Theorem 30 *Let* R *be a Noetherian ring and* $M \subseteq I$ R*-modules where* I *is injective. Then* $I \simeq E_R(M)$ *if and only if for every* $\mathfrak{p}$ *in* $\mathrm{Spec}(R)$ *the map* $\theta_\mathfrak{p}$: $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ *is an isomorphism.*

Remark 13 Note that $\theta_\mathfrak{p}$ is always injective since localization is exact and Hom is left exact. The advantage of the theorem is that it is a local property. The disadvantage, however, is that we need to check the condition for every prime ideal $\mathfrak{p}$ in R.

Proof of Theorem 30: Suppose $I = E_R(M)$ so that $M \subseteq I$ is essential. By remark 13 we need to show that $\theta_\mathfrak{p} : \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ is surjective for every $\mathfrak{p}$ in $\mathrm{Spec}(R)$.

Fix $\mathfrak{p}$ in $\mathrm{Spec}(R)$ and a homomorphism $\phi \in \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p}) \simeq (\mathrm{Hom}_R(R/\mathfrak{p}, I))_\mathfrak{p}$. Suppose $\phi \neq 0$. Choose $\psi : R/\mathfrak{p} \to I$ and $s \notin \mathfrak{p}$ such that $\psi/s = \phi$. Then $\psi \neq 0$. Let $\psi(\bar{1}) = z$. Since $M \subseteq I$ is essential, $Rz \cap M \neq 0$. Let r be an element in R such that $rz \in M$, $rz \neq 0$. This forces $r \notin \mathfrak{p}$. Let $\chi : R/\mathfrak{p} \to M$ be defined by $\bar{1} \mapsto rz$. Then $s\chi/sr \in \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$. We claim that $s\chi/sr = \phi$. This is true since $\phi(\bar{1}) = z/1$ and $s\chi/sr(\bar{1}) = z/1$.

Conversely suppose that $\theta_\mathfrak{p}$ is an isomorphism for each $\mathfrak{p}$ in $\mathrm{Spec}(R)$. Let N be a non-zero submodule of I. We want to prove that $N \cap M \neq 0$. Let $\mathfrak{p} \in Ass_R(N)$. Then we have $R/\mathfrak{p} \hookrightarrow N \hookrightarrow I$. Let $\bar{1} \mapsto z$ in N under the map. This extends to a map $\phi : \kappa(\mathfrak{p}) \to I_\mathfrak{p}$ defined by $1 \mapsto z/1$. Choose $\psi : \kappa(\mathfrak{p}) \to M_\mathfrak{p}$ such that $\theta_\mathfrak{p}(\psi) = \phi$.

It follows that $z/1 \in M_\mathfrak{p}$ which implies that there is an $s \notin \mathfrak{p}$ such that $sz \in M$. Since $\mathfrak{p} = \mathrm{ann}_R(z)$, $sz \neq 0$ i.e. $sz \in M \cap N$ is a non-zero element. Thus I is essential over M. $\square$

Corollary 31 *Let R be a Noetherian ring, M an R-module and*

$$\mathbf{I}^\bullet \ : \quad 0 \to M \to I^0 \to I^1 \to \dots$$

be an injective resolution of M over R. Then $\mathbf{I}^\bullet$ is minimal if and only if the maps $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{n+1}_\mathfrak{p})$ are zero for each n and $\mathfrak{p}$ in $\mathrm{Spec}(R)$.

Proof: Let $N_n = \mathrm{Ker}(I^n \to I^{n+1})$. Then by definition, $\mathbf{I}^\bullet$ is minimal if and only if $I^n \simeq E_R(N_n)$. By Theorem 30, this happens if and only if $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (N_n)_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p})$ for each n and $\mathfrak{p}$ in $\mathrm{Spec}(R)$.
Localize the left exact sequence $0 \to N_n \to I^n \to I^{n+1}$ at $\mathfrak{p}$ and apply $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \_)$. Using the facts that localization is flat and Hom is left-exact, the sequence $0 \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (N_n)_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{n+1}_\mathfrak{p})$ is exact.
Hence

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (N_n)_\mathfrak{p}) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p}) \Leftrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (I^n)_\mathfrak{p}) \xrightarrow{0} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{n+1}_\mathfrak{p})$$

proving the result. □

Definition 9 *Let R be a Noetherian ring, $\mathfrak{p}$ a prime ideal in R and M an R-module. Then the Bass numbers $\mu_j(\mathfrak{p}; M)$ are defined to be $\dim_{\kappa(\mathfrak{p})}(\mathrm{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}))$ for each $j \geq 0$.*

Corollary 32 *Let R be Noetherian, M be an R-module and $0 \to M \to I^0 \to I^1 \to \dots$ be a minimal resolution of M. Write $I^j = \bigoplus E_R(R_\mathfrak{p})^{a_j(\mathfrak{p})}$. Then $a_j(\mathfrak{p}) = \mu_j(\mathfrak{p}; M)$.*

Proof: We have $\mathrm{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) = H^j(\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathbf{I}^\bullet{}_\mathfrak{p}))$. By Cor. 31, the differentials are 0. Hence $\mathrm{Ext}^j_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^j_\mathfrak{p})$ where $I^j_\mathfrak{p} = \bigoplus (E_R(R_\mathfrak{p})_\mathfrak{p})^{a_j(\mathfrak{p})}$. But by Prop. 14,

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E_R(R/\mathfrak{q})_\mathfrak{p}) = \begin{cases} \kappa(\mathfrak{p}) \text{ for } \mathfrak{p} = \mathfrak{q} \\ 0 \text{ for } \mathfrak{p} \neq \mathfrak{q} \end{cases}$$

which proves the corollary. □

Corollary 33 *If M is finitely generated, then $\mu_j(\mathfrak{p}; M)$ is finite for every $j \geq 0$ and $\mathfrak{p}$ in* $\mathrm{Spec}(R)$.

Proof: The proof follows since $\mathrm{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ can be computed from a minimal projective resolution of $\kappa(\mathfrak{p})$ and the fact that these modules are finitely generated over $R_\mathfrak{p}$. □

To Paraphrase: For an R-module M, given any $\mathfrak{p}$ in $\mathrm{Spec}(R)$, $\mu_i(\mathfrak{p} : M)$ is the number of copies of $E_R(R/\mathfrak{p})$ at the ith spot in a minimal resolution of M over R. Moreover, if M is finitely generated , then $\mu_i(\mathfrak{p} : M)$ is finite.

Lemma 34 (Bass) *Let R be a Noetherian ring and M a finitely generated R-module. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R such that* $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$. *If $\mu_i(\mathfrak{p}, M) \neq 0$, then $\mu_{i+1}(\mathfrak{q}, M) \neq 0$.*

Proof: Localizing at $\mathfrak{q}$, we may assume that R is a local ring with maximal ideal $\mathfrak{q} = \mathfrak{m}$. Choose $x \in \mathfrak{m} \setminus \mathfrak{p}$. Consider the short exact sequence $0 \to R/\mathfrak{p} \xrightarrow{\cdot x} R/\mathfrak{p} \to R/(\mathfrak{p}, x) \to 0$. Apply $\mathrm{Hom}_R(\_, M)$ to get

$$\ldots \to \mathrm{Ext}^i_R(R/\mathfrak{p}, M) \xrightarrow{\cdot x} \mathrm{Ext}^i_R(R/\mathfrak{p}, M) \to \mathrm{Ext}^{i+1}_R(R/(\mathfrak{p}, x), M) \to \ldots.$$

By NAK, $\mathrm{Ext}^{i+1}_R(R/(\mathfrak{p}, x), M) \neq 0$.

Since $\mathrm{ht}(\mathfrak{m}/\mathfrak{p}) = 1$, $\sqrt{(\mathfrak{p}, x)} = \mathfrak{m}$, so we filter $R/(\mathfrak{p}, x)$ with copies of $\mathsf{k} = R/\mathfrak{m}$. Hence if $\mathrm{Ext}^{i+1}_R(\mathsf{k}, M) = 0$, then so is $\mathrm{Ext}^{i+1}_R(R/(\mathfrak{p}, x), M)$. □

Corollary 35 *Let M be a finitely generated module over a Noetherian local ring R. Then*

$$\mathrm{id}_R(M) = \sup\{I : \mathrm{Ext}^i_R(\mathsf{k}, M) \neq 0\}.$$

Thus at the last spot in a minimal injective resolution, $E_R(\mathsf{k})$ appears.

Proof: Let $0 \to M \to I^0 \to I^1 \to \ldots$ be a minimal injective resolution of M over R. If $I^i \neq 0$, then there is a $\mathfrak{p}$ in $\mathrm{Spec}(R)$ such that $\mu_i(\mathfrak{p}; M) \neq 0$. Then $\mu_{i+\mathrm{ht}(\mathfrak{m}/\mathfrak{p})}(\mathfrak{m}; M) \neq 0$ by Bass' Lemma. Hence $\mathrm{Ext}^{i+\mathrm{ht}(\mathfrak{m}/\mathfrak{p})}_R(\mathsf{k}, M) \neq 0$. □

Some Applications

Theorem 36 *A regular local ring R is Gorenstein.*

Proof: Let $\dim(R) = d$. Let $\mathbf{I}^\bullet : 0 \to M \to I^0 \to I^1 \to \ldots$ be a minimal injective resolution of M. Then we claim that $I^n = 0$ for $n > d$.

For every $\mathfrak{p}$ in $\mathrm{Spec}(R)$, $\mathrm{pd}_{R_\mathfrak{p}}(\kappa(\mathfrak{p})) = \dim(R_\mathfrak{p}) \leq d$. Hence $\mathrm{Ext}^n_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), R_\mathfrak{p}) = 0$ for $n > d$. Thus $\mu_n(\mathfrak{p}; R) = 0$ for $n > d$ which implies that $\mathrm{id}_R(R) \leq d$, i.e. R is Gorenstein. □

Observation: If R is a regular local ring, then $\mathrm{id}_R(R) = \dim(R)$. From the proof of theorem 36, $\mathrm{id}_R(R) \leq \dim(R)$. The other inequality follows from the fact that $\mathrm{Ext}^d_R(\mathsf{k}, R) \neq 0$ (which follows from Bass' Lemma).

Theorem 37 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, M a finitely generated R-module and x be an element in R which is a non-zerodivisor on both M and R. Then $\mathrm{id}_R(M) < \infty$ if and only if $\mathrm{id}_{R/xR}(M/xM) < \infty$.*

Proof: Let $\mathbf{I}^\bullet$ be a minimal injective resolution of M. Recall that this means $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p}) \xrightarrow{0} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I^{n+1}_\mathfrak{p})$ for every $\mathfrak{p}$ in $\mathrm{Spec}(R)$. Apply $\mathrm{Hom}_R(R/xR, \_)$ to $\mathbf{I}^\bullet$. Since $\mathrm{pd}_R(R/xR) = 1$, $\mathrm{Ext}^i_R(R/xR, M) = 0$ for $i \geq 2$ i.e. $\mathrm{Hom}_R(R/xR, I^{\geq 2})$ is an injective resolution of N over R/xR, where $N = \mathrm{Ker}(\mathrm{Hom}_R(R/xR, I^2) \to \mathrm{Hom}_R(R/xR, I^3))$. In fact, it is a minimal injective resolution.

To prove this we need to show that for every $\mathfrak{p}$ in $\mathrm{Spec}(R/xR)$, the map

$$\mathrm{Hom}_{(R/xR)_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Hom}_R(R/xR, I^n)_\mathfrak{p}) \to \mathrm{Hom}_{(R/xR)_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Hom}_R(R/xR, I^{n+1})_\mathfrak{p})$$

is zero for every $n \geq 2$. This is true since $\mathrm{Hom}_{(R/xR)_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Hom}_R(R/xR, I^n)_\mathfrak{p}) \simeq \mathrm{Hom}_{R-\mathfrak{p}}(\kappa(\mathfrak{p}), I^n_\mathfrak{p})$ for all n by $\mathrm{Hom} - \otimes$ adjointness and the corresponding maps are 0 as noted before. The minimality proves that $\mathrm{id}_{R/xR}(N) < \infty$ if and only if $\mathrm{id}_R(M) < \infty$.

Note that $N = \Im(\mathrm{Hom}_R(R/xR, I^1) \to \mathrm{Hom}_R(R/xR, I^2))$ as $\mathrm{Ext}^2_R(R/xR, M) = 0$. Since $\mathrm{Ext}^0_R(R/xR, M) = 0$, we have the sequence $0 \to \mathrm{Hom}_R(R/xR, I^0) \xrightarrow{i} \mathrm{Hom}_R(R/xR, I^1) \xrightarrow{\pi} N \to 0$ which is exact on both ends. The middle homology is $\mathrm{Ext}^1_R(R/xR, M)$ (i.e. the only non-vanishing Ext is $\mathrm{Ext}^1_R(R/xR, M)$).

Let $Z = \mathrm{Ker}(\pi)$ and $W = \mathrm{Coker}(\pi)$(i.e.

$$0 \to \mathrm{Hom}_R(R/xR, I^0) \xrightarrow{i} \mathrm{Hom}_R(R/xR, I^1) \to W \to 0$$

is exact). Let $E^i = \mathrm{Hom}_R(R/xR, I^i)$. Then $E^1/Z \simeq N$ and $E^1/E^0 \simeq W$. By Snake Lemma, we get a short exact sequence $0 \to Z/E^0 \to W \to N \to 0$ i.e. there is a short exact sequence $0 \to \mathrm{Ext}^1_R(R/xR, M) \to W \to N \to 0$.

Note that since $\mathrm{id}_{R/xR}(W) = \sup\{i : \mathrm{Ext}^i_{(R/xR)}(\mathsf{k}, W) \neq 0\}$, W has finite injective dimension over R/xR. Thus we get the following equivalences:

$$\mathrm{id}_R(M) < \infty \Leftrightarrow \mathrm{id}_{R/xR}(N) < \infty \Leftrightarrow \mathrm{id}_{R/xR}(\mathrm{Ext}^1_R(R/xR, M)) < \infty.$$

Finally compute $\mathrm{Ext}^1_R(R/xR, M)$ from the projective resolution $0 \to R \xrightarrow{\cdot x} R \to R/xR \to 0$. Applying $\mathrm{Hom}_R(\_, M)$ we get

$$0 \to \mathrm{Hom}_R(R/xR, M) \to M \xrightarrow{\cdot x} M \to \mathrm{Ext}^1_R(R/xR, M) \to \mathrm{Ext}^1_R(R, M) \to \dots$$

Since R is free, $\mathrm{Ext}^1_R(R, M) = 0$. Moreover, $\mathrm{Hom}_R(R/xR, M) = 0$ since x is a non-zerodivisor on M. Thus $\mathrm{Ext}^1_R(R/xR, M) \simeq M/xM$ proving the result. □

Corollary 38 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring, $x_1, \dots, x_n$ a regular sequence in R. Then R is a Gorenstein ring if and only if $R/(x_1, \dots, x_n)$ is Gorenstein.*

Proof: The proof is immediate from theorem 37 by induction. □

Corollary 39 *If R is a regular local ring and $f_1, \dots, f_n$ is a regular sequence, then $R/(f_1, \dots, f_n)$ is Gorenstein i.e. a complete intersection ring is Gorenstein.*

Proof: Using the fact that a regular local ring is Gorenstein, the statement follows from Cor. 38. $\square$

Theorem 40 *Let R be a Gorenstein ring. Then R is Cohen-Macaulay.*

Proof: Without loss of generality we may assume that $(R, \mathfrak{m}, \mathsf{k})$ is local. If R is zero-dimensional, then there is nothing to prove. Hence we may assume that $\dim(R) > 0$. Induct on $\dim(R)$. It is enough to prove that $\operatorname{depth}(R) > 0$, i.e. there is a non-zerodivisor x in R. This will imply that R/xR is Gorenstein and hence Cohen-Macaulay, by induction which in turn forces R to be Cohen-Macaulay.

Suppose $\operatorname{depth}(R) = 0$. There exists a short exact sequence $0 \to \mathsf{k} \to R \to N \to 0$. Applying $\operatorname{Hom}(\_, R)$ (and noting that $\operatorname{Ext}^i_R(R, R) = 0$ for $i > 0$) we get $\operatorname{Ext}^i_R(\mathsf{k}, R) \simeq \operatorname{Ext}^{i+1}_R(N, R)$ for $i > 0$. If $t = \operatorname{id}_R(R)$, $\operatorname{Ext}^t_R(\mathsf{k}, R) \neq 0$ by Bass' Lemma and hence $\operatorname{Ext}^{t+1}_R(N, R) \neq 0$, which is impossible since $\operatorname{id}_R(R) = t$.

Note that we need $t > 0$ to use the fact that $\operatorname{Ext}^t_R(R, R) = 0$. We will show that $t = 0$ forces $\dim(R) = 0$ which will complete the proof. If $t = 0$, then R is injective as a module over itself. Hence $R \simeq \bigoplus E_R(R/\mathfrak{p})^{a(\mathfrak{p})}$. But by Cor. 35, $a(\mathfrak{m}) \neq 0$. Thus by counting lengths, we see that $a(\mathfrak{p}) = 0$ for $\mathfrak{p} \neq \mathfrak{m}$ and $a(\mathfrak{m}) = 1$, i.e. $R \simeq E_R(\mathsf{k})$ which implies that R is Artinian. $\square$

Let us prove an analogue of the Auslander-Buchsbaum formula which has plenty of applications. We will henceforth refer to it as **Formula 1**.

Theorem 41 (Formula 1) *Let M and N be finitely generated modules over a Noetherian local ring $(R, \mathfrak{m}, \mathsf{k})$ such that $\operatorname{id}_R(M) < \infty$. Then*

$$\operatorname{depth}(N) + \sup\{i : \operatorname{Ext}^i_R(N, M) \neq 0\} = \operatorname{id}_R(M).$$

Proof: Set $t := \operatorname{id}_R(M)$. Induct on $\operatorname{depth}(N)$. Let $\operatorname{depth}(N) = 0$.

Note that the formula is true for $N = \mathsf{k}$ by Cor. 35 since $\operatorname{depth}(\mathsf{k}) = 0$. Since $\operatorname{depth}(N) = 0$, we have a short exact sequence $0 \to \mathsf{k} \to N \to N' \to 0$. Apply $\operatorname{Hom}_R(\_, M)$ to get

$$\ldots \longrightarrow \operatorname{Ext}^t_R(N, M) \longrightarrow \operatorname{Ext}^t_R(\mathsf{k}, M) \longrightarrow \operatorname{Ext}^{t+1}_R(N', M) \longrightarrow \ldots$$

As observed before, $\operatorname{Ext}^t_R(\mathsf{k}, M) \neq 0$ by Cor. 35 and $\operatorname{Ext}^{t+1}_R(N', M) = 0$ since $\operatorname{id}_R(M) = t$. Hence $\operatorname{Ext}^t_R(N, M) \neq 0$.

Suppose $\operatorname{depth}(N) > 0$. Choose x in $\mathfrak{m}$, a non-zerodivisor on N. Applying $\operatorname{Hom}_R(\_, M)$ to the short exact sequence $0 \to N \xrightarrow{\cdot x} N \to N/xN \to 0$ we get

$$\ldots \longrightarrow \operatorname{Ext}^i_R(N, M) \xrightarrow{\cdot x} \operatorname{Ext}^i_R(N, M) \longrightarrow \operatorname{Ext}^{i+1}_R(N/xN, M) \longrightarrow \ldots$$

By NAK, $\sup\{i : \operatorname{Ext}^i_R(N, M) \neq 0\} = \sup\{i : \operatorname{Ext}^i_R(N/xN, M) \neq 0\} - 1$. Since $\operatorname{depth}(N/xN) = \operatorname{depth}(N) - 1$, the formula is true for N/xN by induction and hence for N. $\square$

Theorem 42 *Let $(S, \mathfrak{m}_S, \mathsf{k})$ be a regular local ring of dimension n. Let $R \simeq S/I$, where I is an ideal of height h in S. Then the following are equivalent:*
1. *R is Gorenstein.*
2. *R is Cohen-Macaulay and $b_h = 1$ where $b_i = \dim_{\mathsf{k}}(\mathrm{Tor}_i^S(\mathsf{k}, R))$ for all i.*
3. *$b_h = 1$.*

Proof: It is clear that (2) implies (3). For the converse, we need to show that R is Cohen-Macaulay. By the Auslander-Buchsbaum formula, $\mathrm{pd}_S(R) + \mathrm{depth}(R) = \mathrm{depth}(S) = n$. We know that $\dim(R) = n - h =: d$. In order to prove that R is Cohen-Macaulay, it is enough to show $\mathrm{pd}_S(R) = h$ i.e. $b_i = 0$ for every $i > h + 1$. Consider a minimal free resolution of R over S:

$$\mathbf{F}_\bullet : \quad \ldots S^{b_{h+1}} \to S \xrightarrow{\phi_h} S^{b_{h-1}} \xrightarrow{\phi_{h-1}} \ldots \to S^{b_1} \to S \to R \to 0.$$

Let $\phi(1) = (a_1, \ldots, a_{b_{h-1}})^T$. This vector is not zero since it spans $\mathrm{Ker}(\phi_{h-1})$. (Note that $\mathrm{Ker}(\phi_{h-1})$ is non-zero since $\mathbf{F}_\bullet$ is a minimal free resolution of R over S and by the Auslander-Buchsbaum formula, $\mathrm{pd}_S(R) \geq h$). Since S is a domain, this implies that $\mathrm{Ker}(\phi_h) = 0$. Hence $b_i = 0$ for $i > h$.
(1) $\Leftrightarrow$ (2): By theorem 40, if R is Gorenstein then it is Cohen-Macaulay. Hence, we may assume that R is Cohen-Macaulay and then prove R is Gorenstein if and only if $b_h = 1$. Since R is Cohen-Macaulay, there is a R-regular sequence $x_1, \ldots, x_d$ in S. Then $Tor_i^S(S/(x_1, \ldots, x_d)S), R) = 0$ for every $i > 0$. Therefore by tensoring a minimal resolution of R over S by $S/(x_1, \ldots, x_d)S$, we see that $b_i^S(R) = b_i^{S/(x_1,\ldots,x_d)S}(R/(x_1, \ldots, x_d)R)$. We may assume that (without loss of generality) $S/(x_1, \ldots, x_d)S$ is a regular local ring.

This reduces the problem to the case where $d = \dim(R) = 0$. In this case, by theorem 19, we know that R is Gorenstein if and only if $b_h = 1$.

Warning:
1. In (2) implies (3), we need S to be a domain. Let $S = \mathsf{k}[X, Y]/(XY)$. Then we have the linear resolution

$$\ldots \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} \ldots$$

2. Let $S = \mathsf{k}[X, Y, U, V]$, $I = (X, Y) \cap (U, V) = (XU, XV, YU, YV)$. Then $\mathrm{ht}(I) = 2$. Then $0 \to S \to S^4 \to S^4 \to S \to S/I \to 0$ is a minimal resolution of S/I over S. In this case $b_3 = 1$, but S/I is not Gorenstein, in fact not even Cohen-Macaulay. Observe that $b_2 = 4$, $b_2 \neq 1$.

§ 3.8 *Fibers of Flat Maps*

Theorem 43 *Let $(R, \mathfrak{m}_R, \mathsf{k}) \xrightarrow{\phi} (S, \mathfrak{m}_S, \mathfrak{l})$ be a flat local homomorphism of rings. Then*

1. $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$.
2. *If* $\mathfrak{m}_R S = \mathfrak{m}_S$*, then for an* R*-module* M *of finite length,* $\lambda_R(M) = \lambda_S(M \otimes_R S)$.
3 *If* $z_1, \dots, z_n$ *is a regular sequence on* $S/\mathfrak{m}_R S$*, then* $z_1, \dots, z_n$ *is* S*-regular and* $R \longrightarrow S/(z_1, \dots, z_n)$ *is flat.*
4. $\operatorname{depth}(R) + \operatorname{depth}(S/\mathfrak{m}_R S) = \operatorname{depth}(S)$.
5. S *is Cohen-Macaulay iff* R *and* $S/\mathfrak{m}_R S$ *are both Cohen-Macaulay.*
6. S *is Gorenstein iff* R *and* $S/\mathfrak{m}_R S$ *are both Gorenstein.*

Note that since ϕ is flat and local (i.e. $\phi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$), ϕ is injective and hence the extension $R \xrightarrow{\phi} S$ is faithfully flat.

Warning: The analogue of (5) (or (6)) by replacing Cohen-Macaulay (or Gorenstein) by "regular" is not true. as can be seen from the following example.

Example 11 The extension $R := \mathsf{k}[[X^2]] \longrightarrow \mathsf{k}[[X]] =: S$ is flat (where X is an indeterminate). Both R and S are regular but $S/\mathfrak{m}_R S \simeq \mathsf{k}[[X]]/(X^2)$ is not regular.

However, if R and $S/\mathfrak{m}_R S$ are both regular, then so is S.

Proof of Theorem 43:
(1) Induct on $\dim(R)$. Suppose $\dim(R) = 0$. Then $\mathfrak{m}_R$ is nilpotent. Hence $\mathfrak{m}_R S$ is nilpotent which implies $\dim(S) = \dim(S/\mathfrak{m}_R S)$.

Suppose $\dim(R) > 0$. Without loss of generality , we may assume that R is reduced by replacing R by R/N and S by S/NS, where N is the nilradical of R. Note that the extension $R \longrightarrow S$ is still flat. This is due to the fact that if $R \longrightarrow S$ is flat, T is any R-algebra, then $T \longrightarrow T \otimes_R S$ is also flat. Also note that going modulo N leaves the respective dimensions unchanged.
Let x be a non-zerodivisor on R. By tensoring $0 \to R \xrightarrow{\cdot x} R$ with S, we see that $y := \phi(x)$ is a non-zerodivisor on S. This gives us $\dim(R/xR) = \dim(R) - 1$ and $\dim(S/yS) = \dim(S) - 1$. The proof is complete by using induction on the extension $R/xR \longrightarrow S/yS$.
(2) Without loss of generality suppose $n := \lambda_R(M) < \infty$. Fix a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \text{ where } M_{i+1}/M_i \simeq R/\mathfrak{m}_R.$$

Tensoring with S gives us

$$0 = M_0 \otimes_R S \subseteq M_1 \otimes_R S \subseteq \cdots \subseteq M_n \otimes_R S = M \otimes_R S$$

$$\text{where } M_{i+1} \otimes_R S/M_i \otimes_R S \simeq R/\mathfrak{m}_R \otimes_R S \simeq S/\mathfrak{m}_R S = S/\mathfrak{m}_S.$$

Hence $\lambda_S(M \otimes_R S) = n$.

(3) We just need to prove the case $n = 1$. The statement follows for $n > 1$ by applying induction to the extension $R \to S/z_1S$.

Let z be a non-zerodivisor on $S/\mathfrak{m}_RS$. By tensoring the short exact sequence $0 \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n \to 0$ with S, we get the short exact sequence

$$0 \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \otimes_R S \to R/\mathfrak{m}_R^{n+1} \otimes_R S \to R/\mathfrak{m}_R^n \otimes_R S \to 0.$$

If $\dim_{\mathsf{k}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = d_n$, then $\mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \otimes_R S \simeq (S/\mathfrak{m}_RS)^{\oplus d_n}$. Hence we have the short exact sequence

$$0 \to (S/\mathfrak{m}_RS)^{\oplus d_n} \to S/\mathfrak{m}_R^{n+1}S \to S/\mathfrak{m}_R^nS \to 0.$$

By induction on n, we see that z is a non-zerodivisor on $S/\mathfrak{m}_R^nS$ for each n. Now if $u \in S$ is such that $z \cdot u = 0$, then $u \in \bigcap_{n=0}^{\infty} \mathfrak{m}_R^nS = (0)$ by the Krull Intersection theorem. Hence z is a non-zerodivisor on S.

Now we need to prove that $R \longrightarrow S/zS$ is flat. It is enough to show that $\mathrm{Tor}_1^R(M, S/zS) = 0$ for every finitely generated module M. By taking a prime filtration of M, we may assume that $M = R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ in R.

Consider the short exact sequence $0 \to S \xrightarrow{\cdot z} S \to S/zS \to 0$. Since S is flat over R, $\mathrm{Tor}_1^R(R/\mathfrak{p}, S) = 0$. Hence the induced long exact sequence on homology gives

$$0 \to \mathrm{Tor}_1^R(R/\mathfrak{p}, S/zS) \to S/\mathfrak{p}S \xrightarrow{\cdot z} S/\mathfrak{p}S \to S/(z, \mathfrak{p})S \to 0.$$

Since the map $R/\mathfrak{p}R \to S/\mathfrak{p}S$ is flat and the closed fiber (i.e. the ideals over the maximal ideal) of $R \to S$ and $R/\mathfrak{p} \to S/\mathfrak{p}S$ are the same, by the first part of (3) z is a non-zerodivisor on $S/\mathfrak{p}S$. This forces $\mathrm{Tor}_1^R(R/\mathfrak{p}, S/zS) = 0$.

(4) Choose a maximal regular sequence $x_1, \dots, x_s$ in R. By flatness of S over R, their images in S form an S-regular sequence. Hence by passing to $R/(x_1, \dots, x_s)$ and $S/(x_1, \dots, x_s)S$, we may assume that $\mathrm{depth}(R) = 0$.

Choose a regular sequence $z_1, \dots, z_t$ in S, which is a maximal regular sequence on $S/\mathfrak{m}_RS$. This is possible by (3). We can replace S by $S/(z_1, \dots, z_t)$ and assume that $\mathrm{depth}(S/\mathfrak{m}_R)S = 0$. Thus (4) is reduced to proving $\mathrm{depth}(S) = 0$ assuming $\mathrm{depth}(R) = 0$ and $\mathrm{depth}(S/\mathfrak{m}_RS) = 0$.

Now $\mathrm{depth}(R) = 0$ implies that $\mathsf{k} \hookrightarrow R$. Tensoring with S, we see that $S/\mathfrak{m}_RS \hookrightarrow S$. But $\mathrm{depth}(S/\mathfrak{m}_RS) = 0$ implies $\mathfrak{l} \hookrightarrow S/\mathfrak{m}_RS$. Thus $\mathfrak{l} \hookrightarrow S$ which forces $\mathrm{depth}(S) = 0$.

(5) = (1) + (4) which is a very well-known fact.

(6) Whichever direction we want to prove, by (5) we can assume that R, S and $S/\mathfrak{m}_RS$ are Cohen-Macaulay, since we know that Gorenstein rings are Cohen-Macaulay.

By using the same reductions as in (4), without loss of generality we may assume that $\mathrm{depth}(R) = 0 = \mathrm{depth}(S)$ and hence $\dim(R) = 0 = \dim(S)$ since they are both Cohen-Macaulay. We need to prove that

$$\dim_{\mathfrak{l}}(\mathrm{soc}(S)) = 1 \Leftrightarrow \dim_{\mathsf{k}}(\mathrm{soc}(R)) = 1 \text{ and } \dim_{\mathfrak{l}}(\mathrm{soc}(S/\mathfrak{m}_RS)) = 1.$$

Let $r = \dim_{\mathsf{k}}(\mathrm{soc}(R))$, $f = \dim_{\mathfrak{l}}(\mathrm{soc}(S/\mathfrak{m}_R S))$ and $s = \dim_{\mathfrak{l}}(\mathrm{soc}(S))$. Then $\mathsf{k}^r \hookrightarrow R$. By tensoring with S, we get $(S/\mathfrak{m}_R S)^r \hookrightarrow S$. Since $\mathfrak{l}^f \hookrightarrow S/\mathfrak{m}_R S$, we have $\mathfrak{l}^{rf} \hookrightarrow S$. Hence $s \geq rf$. This proves that if S is Gorenstein, then so are R and $S/\mathfrak{m}_R S$.

We have $0 :_R \mathfrak{m}_R \simeq \mathsf{k}^r$. Hence by flatness, $0 :_S \mathfrak{m}_R S \simeq (S/\mathfrak{m}_R S)^r$. We also have $0 :_S \mathfrak{m}_S \simeq \mathfrak{l}^s$. Since $\mathfrak{m}_R S \subseteq \mathfrak{m}_S$, $0 :_S \mathfrak{m}_S \subseteq 0 :_S \mathfrak{m}_R S$.

Now $0 :_S \mathfrak{m}_S \simeq 0 :_{0:_S \mathfrak{m}_R S} \mathfrak{m}_S \simeq 0 :_{(S/\mathfrak{m}_R S)^r} \mathfrak{m}_S$. Since $0 :_{S/\mathfrak{m}_R S} \mathfrak{m}_S \simeq \mathfrak{l}^f$, we see that $\mathfrak{l}^s \simeq 0 :_S \mathfrak{m}_S \simeq 0 :_{(S/\mathfrak{m}_R S)^r} \mathfrak{m}_S \simeq \mathfrak{l}^{rf}$. Hence $s = rf$. $\square$

Corollary 44 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. Then R is Gorenstein if and only if $\widehat{R}$ is Gorenstein.*

Proof: The proof is immediate from (6) in Theorem 40 since $\widehat{R}/\mathfrak{m}\widehat{R} \simeq \widehat{R}/\widehat{\mathfrak{m}}$ is Gorenstein. $\square$

Corollary 45 *If R is Gorenstein, then so is $R[X_1, \ldots, X_n]$.*

Proof: By induction on n, we need to prove that $R[X]$ is Gorenstein if R is. Let $Q \in \mathrm{Spec}(R[X])$ and $\mathfrak{q} = Q \cap R$. Since $(R[X])_Q \simeq (R_{\mathfrak{q}}[X])_Q$, without loss of generality we may assume that $(R, \mathfrak{m})$ is a local Gorenstein ring and $Q \cap R \simeq \mathfrak{m}$. Since $(R, \mathfrak{m}) \to (R[X])_Q$ is local and flat, to prove that $(R[X])_Q$ is Gorenstein, it is necessary and sufficient to prove that $(R[X]/\mathfrak{m}R[X])_Q$ is Gorenstein. But $(R[X]/\mathfrak{m}R[X])_Q \simeq \mathsf{k}[X]_Q$ is Gorenstein (since $\mathsf{k}[X]$ is a P.I.D). $\square$

Exercises

(1) Let R be a commutative Noetherian ring. Prove that the injective hull $E(R)$ of R is a flat R-module iff the tensor product of any two injective R-modules is injective.

(2) Let R be a commutative Noetherian ring. Prove that the injective hull $E(R)$ of R is a flat R-module iff R_p is Gorenstein for all minimal primes p of R.

(3) Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian complete local ring. By $^\vee$ denote $\mathrm{Hom}_R(\ \ , E)$ where E is the injective hull of k, the residue field of R. Prove that an R-module M satisfies $M \simeq M^{\vee\vee}$ (under the natural map) iff M has a finitely generated submodule N such that M/N is Artinian.

(4) Let $(R, \mathfrak{m}, \mathsf{k})$ be a 1-dimensional regular local ring with quotient field K and $E = E_R(\mathsf{k})$. Let L be the fraction field of the completion of R. Prove that $L \simeq \mathrm{Hom}_R(\mathsf{K}, E)$.

(5) Let $(R, \mathfrak{m}, \mathsf{k}), \mathsf{K}$, and E be as in exercise 4. Prove that $\mathrm{Ext}^1_R(E, R) \simeq R$. Prove that the extension $0 \to R \to \mathsf{K} \to E \to 0$ generates $\mathrm{Ext}^1_R(E, R)$.

(6) Define a module M over a commutative Noetherian local ring $(R, \mathfrak{m}, \mathsf{k}, E)$ to be <u>Matlis</u> <u>reflexive</u> if $M \simeq M^{\vee\vee}$ under the natural map where $^\vee$ is $\mathrm{Hom}_R(\ \ , E)$. Let M, N be Matlis reflexive R-modules. Prove that for all $i \geq 0$, $\mathrm{Ext}^i_R(M, N)$ and $\mathrm{Tor}^R_i(M, N)$ are also Matlis reflexive and

$$\mathrm{Ext}^i_R(M, N)^\vee \simeq \mathrm{Tor}^R_i(M, N^\vee).$$

(7) Let R be a local Gorenstein ring, and let $I \subseteq J$ be two ideals of height 0. Assume that $\dim(R) = 1$ and R/I is Cohen-Macaulay. If $(0 : J) + J = (I : J) + J$ and this ideal has positive height, then $I = 0$.

(8) Let R be a local commutative Noetherian ring and let M be an Artinian R-module. Let $f \in \mathrm{Hom}_R(M, M)$. If $\mathrm{Ker} f$ has finite length, prove that $\mathrm{Coker} f$ has finite length and

$$\lambda(\mathrm{Ker} f) \geq \lambda(\mathrm{Coker} f).$$

(9) Let $(R, \mathfrak{m})$ be a Gorenstein ring, and suppose that I is an ideal in R such that R/I is Gorenstein. Let $g \in R$, $g \notin I$. Prove that $R/(I : g)$ is Gorenstein if and only if $R/(I : (I : g))$ is Cohen-Macaulay.

(10) Let R be a commutative Noetherian ring. If G is a flat R-module and I is an injective R-module, prove that $\mathrm{Hom}_R(G, I)$ is an injective R-module.

(11) Let $(R, \mathfrak{m}, \mathsf{k})$ be a complete Gorenstein local ring of dimension d. Set E equal to an injective hull of the residue field k. Prove that $\mathrm{Ext}^d_R(E, R) \simeq R$, and $\mathrm{Ext}^i_R(E, R) = 0$ for $i \neq d$.

(12) Let $(R, \mathfrak{m})$ be a 0-dimensional Noetherian local ring with E = an injective hull of $R/\mathfrak{m}$, and $^\vee = \mathrm{Hom}_R(\quad, E)$. Show that the following statements are equivalent for an ideal $I \subseteq R$.

  (a) $I \simeq I^\vee$

  (b) There is an exact sequence
$$0 \to T \to E \to I \to 0$$
  such that $IT = 0$.

(13) Let I be an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m}, \mathsf{k}, E)$, and write I an an intersection of t distinct irreducible ideal, irredundantly. Prove that $E_R(R/I) \cong E^t$.

(14) Let R be a Noetherian local ring. Prove that an R-module M is Artinian iff M is an essential extension of its socle and its socle is finite-dimensional.

(15) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian local ring. Let M be a finitely generated R-module. Prove that $\mu(M) = \dim_{\mathsf{k}}(\mathrm{soc}(M^\vee))$.

(16) Show that $\mathbb{Q}/\mathbb{Z} \cong \oplus E_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z})$, where the sum ranges over all prime integers p.

(17) Let R be a Noetherian ring. Prove that the following two conditions are equivalent:

  (a) R is a Gorenstein ring.

  (b) For every finitely generated R-module M, there exists an integer n, depending on M, such that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geq n$.

(18) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. The *type* of R is by definition the type of the Artinian ring $R/(x_1, ..., x_d)$, where $x_1, ..., x_d$ is a system of parameters. Prove this is well-defined.

(19) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian local ring. We know that both R and E are faithful R-modules, both having the same length. Either prove or give a counterexample to the following claim: If M is a finitely generated faithful R-module, then the length of M is at least the length of R.

(20) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian local ring. Make a ring out of $S = R \oplus E$ by component-wise addition and multiplication as follows:

$$(r, u)(s, v) = (rs, rv + su).$$

Prove that S is a Gorenstein local ring which maps onto R.

(21) Let $(R, \mathfrak{m})$ be a local Gorenstein ring, and let M be a finitely generated R-module. Prove that M has finite projective dimension if and only if M has finite injective dimension.

(22) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian local ring. Prove or give a counterexample to the following statement: for every finitely generated R-module M, either M^{*^n} is reflexive for large n or $\mathsf{k}|M^{*^n}$ for large n.

(23) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be an Artinian local ring. Classify all finitely generated R-modules M such that $\operatorname{Hom}_R(M, M) \cong R$.

(24) Let $(R, \mathfrak{m})$ be a three dimensional regular local ring and let $\mathfrak{p}$ be a height two prime of R. Decompose

$$E_R(R/\mathfrak{p}^2) \cong E_R(R/m)^a \oplus E_R(R/\mathfrak{p})^b.$$

Prove that $a = \binom{n-1}{2}$ and $b = 2$ where $n = \mu(\mathfrak{p})$.

(25) Find $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$.

(26) Let $(R, \mathfrak{m})$ be a Noetherian local Gorenstein ring. Prove that a finitely generated R-module M is reflexive iff M is a second syzygy, i.e. there exists an exact sequence,

$$0 \to M \to F \to G$$

where F and G are finitely generated free R-modules.

(27) Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring. Prove that R is regular iff $\operatorname{id}_R(\mathsf{k}) < \infty$.

(28) Let $(R, \mathfrak{m}, \mathsf{k})$ be a regular local ring of dimension d. Assume that R/I is 0-dimensional and Gorenstein. Prove that $\operatorname{Ext}^d_R(R/I, R) \cong R/I$.

(29) Let $(R, \mathfrak{m}, \mathsf{k})$ be a 0-dimensional Gorenstein local ring with socle Rx. If $\mu(m) \geq 2$, prove that R/Rx is never Gorenstein.

(30) Let R be a Noetherian ring and let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals of R. Prove that $E_R(R/\mathfrak{p})$ is not isomorphic to $E_R(R/\mathfrak{q})$.

(31) Prove or give a counterexample to the following claim: let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Noetherian local ring, and suppose that

$$0 \to M \to F \to N \to 0$$

is exact, where F is finitely generated and free. If $E \otimes_R M \to E \otimes_R F$ is injective, then N is flat.

(32) Let E be an injective R-module, and let $I_1, ..., I_n$ be ideals of R. Prove that

$$\mathrm{ann}_E(I_1 \cap ... \cap I_n) = \sum_i \mathrm{ann}_E(I_i).$$

(33) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a complete Noetherian local ring and let M be a faithful R-module which is an essential extension of k. Prove that $M \cong E$.

(34) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a complete Noetherian local ring. If E is flat over R, prove that R is a 0-dimensional Gorenstein ring.

(35) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a complete Noetherian local ring. Prove that $E \otimes_R E \neq 0$ iff $\mathrm{depth}(R) = 0$.

(36) Prove or give a counterexample: Let (R, m) be a d-dimensional local Gorenstein ring and let M be a finitely generated R-module such that $\dim(M) = \mathrm{depth}(M) = d$. Then for any ideal q, generated by a system of parameters,

$$\lambda(M \otimes R/q) = \lambda(\mathrm{Hom}_R(M, R/q)).$$

(37) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Noetherian local ring, M a finitely generated R-module, and

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^j \to \cdots$$

a minimal injective resolution of M. Prove that if $\mathfrak{p} \in \operatorname{supp} M$, then the injective hull $E_R(R/\mathfrak{p})$ is a direct summand of I^j if and only if depth ${}_{R_\mathfrak{p}} M_\mathfrak{p} \leq j \leq \mathrm{id}_{R_\mathfrak{p}} M_\mathfrak{p}$ (which may be infinite).

(38) Let R be a Noetherian local ring. A finitely generated R-module M is said to be *Gorenstein projective* if $\operatorname{Ext}^i_R(M, R) = 0$ and $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, R), R) = 0$ for $i > 0$. Over a complete Noetherian local ring $(R, \mathfrak{m}, \mathsf{k}, E)$, given an Artinian module M, prove that $\operatorname{Hom}_R(M, E(\mathsf{k}))$ is Gorenstein projective if and only if $\operatorname{Hom}_R(E(\mathsf{k}), M)$ is nonzero and Gorenstein projective.

(39) Let R be a Noetherian domain and let E be an R-module which is both torsion-free and divisible. Prove that E is injective.

(40) Suppose that R is a Noetherian ring and $\mathfrak{p}$ and $\mathfrak{q}$ are primes, M a finitely generated R-module, and Q a prime minimal over $\mathfrak{p}+\mathfrak{q}$. Assume that $\mu_i(\mathfrak{p}, M) \neq 0$ and $\mu_j(\mathfrak{q}, M) \neq 0$. Prove or disprove: $\mu_{i+j}(Q, M) \neq 0$.

(41) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Noetherian local ring, and suppose that M is a finitely generated R-module having finite injective dimension. Prove that R is Cohen-Macaulay.

(42) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Artinian local ring. Prove that E has a minimal generator killed by $\mathfrak{m}^n$ if and only if R has a nonzero socle element which is not in $\mathfrak{m}^n$.

(43) Let $(R, \mathfrak{m}, \mathsf{k}, E)$ be a Artinian local ring. Prove that $E \otimes_R E$ and $E^* = \operatorname{Hom}_R(E, R)$ are Matlis duals. In general, consider the operations $(\quad)^\vee$ and $(\quad)^*$. Is there a relationship between these two operations on the class of finitely generated R-modules?

4. Canonical Modules

§ 4.1 *Canonical Modules for Homomorphic Images of Regular Local Rings*

Let us first see what the canonical module is for a Cohen-Macaulay quotient of a regular local ring. Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring, $R \simeq T/I$ be Cohen-Macaulay. Set $\mathrm{ht}(I) = c$. By the Auslander-Buchsbaum formula, R is Cohen-Macaulay $\Leftrightarrow$ $\mathrm{depth}(R) = \dim(R) \Leftrightarrow \mathrm{pd}(R) = \dim(T) - \dim(R) = \mathrm{ht}(I) = c$.

Consider a minimal resolution $\mathbf{F}_\bullet$ of R over T. By the above argument $\mathbf{F}_\bullet$ has length c. Let

$$\mathbf{F}_\bullet : \qquad 0 \to T^{b_c} \xrightarrow{\phi_c} T^{b_{c-1}} \to \cdots \to T^{b_1} \to T \to R \to 0.$$

Apply $\mathrm{Hom}_T(\_, T)$ (i.e. apply *).
Define the canonical module of R denoted by ω_R (or K_R) to be $\mathrm{Coker}(\phi^*)$, i.e.

$$(T^{b_{c-1}})^* \xrightarrow{\phi^*} (T^{b_c})^* \to \omega_R \to 0.$$

Question: Is the canonical module of R independent of the choice of T? The answer is yes, but the proof is postponed.

Lemma 1 *Let R, T be as above. Then*
1. *A free resolution of ω_R over T is given by*

$$0 \to T^* \to (T^{b_1})^* \to \cdots \to (T^{b_{c-1}})^* \xrightarrow{\phi^*} (T^{b_c})^* \to \omega_R \to 0.$$

2. $\mathrm{ann}_T(\omega_R) = I$.
3. *ω_R is Cohen-Macaulay.*

Proof:
(1) The cohomology of $\mathbf{F}_\bullet{}^*$ are the $\mathrm{Ext}_T(R, T)$ modules. Since $R \simeq T/I$ and I has a regular sequence of length c on T, $\mathrm{Ext}^i_T(R, T) = 0$ for all $i < c$. For $i = c$, the homology is $0 :_T I$, which is zero since T is a domain. Moreover, by definition $\mathrm{Ext}^c_T(R, T) = \omega_R$. Hence $\mathbf{F}_\bullet{}^*$ is a free resolution of ω_R over T.
(2) Since $\omega_R = \mathrm{Ext}^c_T(R, T)$, $I = \mathrm{ann}_T(R) \subseteq \mathrm{ann}_T(\omega_R)$. To prove the other inclusion, apply $\mathrm{Hom}_T(\_, T)$ again to $\mathbf{F}_\bullet{}^*$. We see that $\mathrm{Ext}^i_T(\omega_R, T) = 0$ for $i < c$ and $\mathrm{Ext}^0_T(\omega_R, T) = R$. Hence $\mathrm{ann}_T(\omega_R) \subseteq \mathrm{ann}_T(R) = I$.
(3) Since $\mathrm{ann}_T(\omega_R) = I$, $\dim(\omega_R) = \dim(R)$. And since $\mathrm{pd}(\omega_R) = c = \mathrm{pd}(R)$, by the Auslander-Buchsbaum formula, $\mathrm{depth}(\omega_R) = \mathrm{depth}(R)$. Since R is Cohen-Macaulay, this proves that $\mathrm{depth}(\omega_R) = \dim(\omega_R) = \dim(R)$. Hence ω_R is Cohen-Macaulay. □

Definition 1 *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a local ring and* M *a finitely generated* R*-module. We say that* M *is maximal Cohen-Macaulay (or MCM) if* $\operatorname{depth}(M) = \dim(M) = \dim(R)$.

Example 1 By (3) in lemma 1, ω_R is MCM as an R-module when R is Cohen-Macaulay.

Proposition 2 *Let* T *be a regular local ring,* $R = T/I$ *be zero-dimensional. Then* $\omega_R \simeq E_R(\mathsf{k})$. *In particular,* R *is a zero-dimensional Gorenstein local ring if and only if* $R \simeq \omega_R$.

Proof: We know that $\omega_R \simeq \operatorname{Ext}^c_T(R, T)$, where $c = \operatorname{ht}(I)$ $(= \dim(T))$. Choose $\underline{x}$, a maximal regular sequence in I. Then

$$\omega_R \simeq \operatorname{Hom}_{T/(\underline{x})}(T/I, T/(\underline{x})) \quad \left(\simeq \frac{(\underline{x}) :_T I}{(\underline{x})}\right).$$

Now $S := T/(\underline{x})$ is a complete intersection ring and hence Gorenstein. Therefore, $S \simeq E_S(\mathsf{k})$. This gives us

$$\omega_R \simeq \operatorname{Hom}_S(R, S) \simeq \operatorname{Hom}_S(R, E_S(\mathsf{k})) \simeq E_R(\mathsf{k}).$$

The rest of the statement follows from the fact that R is a zero-dimensional Gorenstein ring if and only if $R \simeq E_R(\mathsf{k})$. $\square$

Let us now define the canonical module for a ring that is not necessarily Cohen-Macaulay.

Definition 2 *Let* T *be a regular Noetherian ring,* $R \simeq T/I$ *for an ideal* $I \subsetneq T$. *Set* $c := \operatorname{ht}(I)$. *We define a canonical module for* R *(with respect to* T*) to be* $\omega_R := \operatorname{Ext}^c_T(R, T)$.

Good Note: We are not assuming that T is local or that R is Cohen-Macaulay.
Bad Note: This makes for some problems.

Proposition 3 *Let* T *be a regular local ring,* $R \simeq T/I$ *be a Cohen-Macaulay quotient. Then the following are equivalent:*
1. ω_R *is cyclic.*
2. $\omega_R \simeq R$.
3. R *is Gorenstein.*

Proof: Set $\mathrm{ht}(I) =: c$. (3) $\Leftrightarrow$ (1) follows from the fact that R is Gorenstein if and only if $b_c = 1$ and (2) $\Rightarrow$ (1) is clear. Hence we only need to prove (1) $\Rightarrow$ (2).

Since ω_R is cyclic, there is an ideal J in T containing I such that $\omega_R \simeq T/J$. But by lemma 1.2, $\mathrm{ann}_T(\omega_R) = I$. Hence $J = I$, i.e. $\omega_R \simeq R$.

Discussion: Assume further that R is a domain, i.e. $I = \mathfrak{p}$ is a prime ideal in T. Since $T_\mathfrak{p}$ is a regular local ring, we can choose a maximal regular sequence $\underline{x} := x_1, \ldots, x_c \in \mathfrak{p}$, such that $(x_1, \ldots, x_c)_\mathfrak{p} = \mathfrak{p}T_\mathfrak{p}$. Notice that this implies $\underline{x} :_T \mathfrak{p} \not\subseteq \mathfrak{p}$. In particular, $\mathrm{ht}((\underline{x} :_T \mathfrak{p}) + \mathfrak{p}) > c$. Since $(\underline{x})$ is an unmixed ideal of height c, it follows that $(\underline{x} :_T \mathfrak{p}) + \mathfrak{p}$ is not contained in any associated prime of $(\underline{x})$. We claim that this forces $\underline{x} :_T \mathfrak{p} \cap \mathfrak{p} \subseteq \underline{x}$.

By Prime Avoidance, there is an element $y \in ((\underline{x} :_T \mathfrak{p}) + \mathfrak{p}) \setminus \cup_{Q \in \mathrm{Ass}(\underline{x})} Q$. Then $(\underline{x}) :_T y = (\underline{x})$. Now, we have $((\underline{x} :_T \mathfrak{p}) + \mathfrak{p})((\underline{x} :_T \mathfrak{p}) \cap \mathfrak{p}) \subseteq (\underline{x})$. Hence

$$y((\underline{x} :_T \mathfrak{p}) \cap \mathfrak{p}) \subseteq (\underline{x}) \implies ((\underline{x} :_T \mathfrak{p}) \cap \mathfrak{p}) \subseteq (\underline{x}) :_T y = (\underline{x}).$$

Thus we have $((\underline{x} :_T \mathfrak{p}) \cap \mathfrak{p}) \subseteq (\underline{x})$ which means that $((\underline{x} :_T \mathfrak{p}) \cap \mathfrak{p}) = \underline{x}$.

Recall that by the Ext Shifting Lemma,

$$\omega_R = \mathrm{Ext}^c_T(R, T) \simeq \mathrm{Hom}_{T/(\underline{x})}(T/\mathfrak{p}, T/(\underline{x})).$$

Hence

$$\omega_R \simeq (\underline{x} :_T \mathfrak{p})/\underline{x} \simeq ((\underline{x} :_T \mathfrak{p}) + \mathfrak{p})/\mathfrak{p} \subseteq R = T/\mathfrak{p}.$$

Upshot: If R is a domain, then $\omega_R \hookrightarrow R$.

Theorem 4 *Let R be a domain and $\omega_R \subseteq R$ via the embedding in the above discussion. Then* $\mathrm{ht}(\omega_R) = 1$ *and R/ω_R is Gorenstein.*

Proof: Set $d := \dim(R)$. By Lemma 1, ω_R is a Cohen-Macaulay R-module with $\mathrm{depth}(\omega_R) = d$. Consider the short exact sequence $0 \to \omega_R \to R \to R/\omega_R \to 0$.

Since $\mathrm{depth}(\omega_R) = \mathrm{depth}(R) = d$, $\mathrm{depth}(R/\omega_R) \geq d - 1$. But then

$$d > \dim(R/\omega_R) \geq \mathrm{depth}(R/\omega_R) \geq d - 1$$

forcing R/ω_R to be a Cohen-Macaulay module of dimension $d-1$. Hence $\mathrm{ht}(\omega_R) = 1$.

Now since $\mathrm{ht}_R(\omega_R) = 1$, $\mathrm{ht}_T((\underline{x} :_T \mathfrak{p}) + \mathfrak{p}) = \mathrm{ht}_T(\mathfrak{p}) + 1 = c + 1$, where T, $\mathfrak{p}$ and $\underline{x}$ are as in above discussion. Hence in order to prove that R/ω_R is Gorenstein, it is enough to prove that $\mathrm{Ext}^{c+1}_T(T/(\underline{x} :_T \mathfrak{p} + \mathfrak{p}), T)$ is cyclic by lemma 1.
Consider the short exact sequence

$$0 \to T/(\underline{x}) \to T/(\underline{x} :_T \mathfrak{p}) \oplus T/\mathfrak{p} \to T/(\underline{x} :_T \mathfrak{p} + \mathfrak{p}) \to 0.$$

Apply $\mathrm{Hom}_T(\_, T)$ to get

$$\mathrm{Ext}^c_T(T/(\underline{x}), T) \to \mathrm{Ext}^{c+1}_T(T/(\underline{x} :_T \mathfrak{p}+\mathfrak{p}), T) \to \mathrm{Ext}^{c+1}_T(T/\mathfrak{p}, T) \oplus \mathrm{Ext}^{c+1}_T(T/(\underline{x} :_T \mathfrak{p}), T).$$

Since $\mathrm{ht}(\underline{x}) = c$, $\mathrm{Ext}^c_T(T/(\underline{x}), T) \simeq \omega_{T/(\underline{x})}$. But $T/(\underline{x})$ is Gorenstein. Therefore $\omega_{T/(\underline{x})} \simeq T/(\underline{x})$. Hence, by the above sequence in Ext's, in order to prove that $\mathrm{Ext}^{c+1}_T(T/(\underline{x} :_T \mathfrak{p} + \mathfrak{p}), T)$ is cyclic, it is enough to prove that both $\mathrm{Ext}^{c+1}_T(T/\mathfrak{p}, T)$ and $\mathrm{Ext}^{c+1}_T(T/(\underline{x} :_T \mathfrak{p}), T)$ are zero.

We know that $T/\mathfrak{p}$ is Cohen-Macaulay. By Lemma 6, $T/(\underline{x} :_T \mathfrak{p})$ is also Cohen-Macaulay. Hence both have projective dimension c. So $\mathrm{Ext}^i_T(T/(\underline{x} :_T \mathfrak{p}), T) = 0$ and $\mathrm{Ext}^i_T(T/\mathfrak{p}, T) = 0$ for all $i > c$. This completes the proof. □

Corollary 5 (M.P.Murthy) *Let R be a Cohen-Macaulay ring with a canonical module ω_R. Further assume that R is a UFD. Then R is Gorenstein.*

Proof: Since $\mathrm{ht}(\omega_R) = 1$ and R/ω_R is Cohen-Macaulay, ω_R is an unmixed ideal of height 1 in R. Recall that if R is a UFD, the class group of R, $Cl(R) = 0$. Therefore the class of ω_R in $Cl(R)$ is zero. Hence $\omega_R \simeq R$, i.e. R is Gorenstein. □

In proof of theorem 4, we have used the fact that $T/(\underline{x} :_T \mathfrak{p})$ is Cohen-Macaulay. This follows from the following lemma:

Lemma 6 *Let T be regular local ring, I an ideal such that T/I is Cohen-Macaulay and $\underline{x}$ a maximal regular sequence in I. Then $T/(\underline{x} :_T I)$ is Cohen-Macaulay.*

More generally we prove

Theorem 7 (Peskine-Szpiro) *Let $(S, \mathfrak{m}_S, \mathsf{k})$ be a Gorenstein local ring, I an ideal in S such that $R := S/I$ is Cohen-Macaulay and $\underline{x}$ a maximal regular sequence in I. Then $S/(\underline{x} :_S I)$ is Cohen-Macaulay.*

Remark 1 This theorem due to Peskine and Szpiro is the fundamental theorem for linkage.

Proof of Theorem 7: Without loss of generality , we can replace S by $S/(\underline{x})$, so we may assume that $\underline{x} = 0$, i.e. $\mathrm{ht}(I) = 0$. Consider the short exact sequence

$$0 \to I \to S \to S/I \to 0 \qquad (*).$$

Since $\mathrm{depth}(S) = \mathrm{depth}(S/I) = \dim(S)$, we see that (by the depth lemma), $\mathrm{depth}(I) = \dim(S)$, i.e. I is a MCM S-module.
Apply $\mathrm{Hom}_S(\_, S)$ to $(*)$. Since $\mathrm{Ext}^i_S(S, S) = 0$ for all $i > 0$, we get

$$0 \to 0 :_S I \to S \to \mathrm{Hom}_S(I, S) \to \mathrm{Ext}^1_S(S/I, S) \to 0 \qquad (\#)$$

Remark 2 Let M be a MCM S-module. Since $\mathrm{id}(S) < \infty$, by Formula 1,

$$\mathrm{depth}(M) + \sup\{i : \mathrm{Ext}^i_S(M, S) \neq 0\} = \mathrm{depth}(S).$$

Hence $\mathrm{Ext}^i_S(M, S) = 0$ for all $i > 0$.

In particular, $\mathrm{Ext}^i_S(S/I, S) = 0$ for all $i > 0$. By (#), this forces

$$S/(0 :_S I) \simeq I^* := \mathrm{Hom}_S(I, S).$$

Thus Theorem 7 is true if we prove Theorem 8.

Theorem 8 *Let* $(S, \mathfrak{m}_S, \mathsf{k})$ *be a local Gorenstein ring. If* M *is MCM, then*
(a) M^* *is MCM.*
(b) $M \simeq M^{**}$*.*

Proof: Consider the short exact sequence $0 \to N \to F \to M \to 0$ where F is a finitely generated free S-module. Note that $\mathrm{Ext}^i_S(M, S) = 0$ for all $i > 0$ by Formula 1. Hence the corresponding long exact sequence on Exts gives $0 \to M^* \to F^* \to N^* \to 0$. Since N is also MCM, repeating the process with N instead of M, and continuing the same way, we see that there is an exact sequence

$$0 \longrightarrow M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots .$$

If the above sequence is finite, then M has finite projective dimension over S. In such a case, since M is MCM, $\mathrm{pd}M = 0$ by the Auslander-Buchsbaum formula, i.e. M is free. Then both (a) and (b) hold.
Hence we may assume that the above sequence is infinite.
(a) Using the lemma

Lemma 9 *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a short exact sequence of* S*-modules, and* $s := \mathrm{depth}(M_2) > \mathrm{depth}(M_3) =: t$, *then* $\mathrm{depth}(M_1) = t + 1$.

repeatedly, we see that M^* is MCM, i.e. (a) holds.
(b) Since N is MCM, part (a) proves that N^* is MCM and then $\mathrm{Ext}^1_S(N^*, S) = 0$. Applying * to $0 \to M^* \to F^* \to N^* \to 0$, we get that $0 \to N^{**} \to F^{**} \to M^{**} \to 0$ is a short exact sequence. Consider

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N^{**} \arrow[r] & F^{**} \arrow[r] & M^{**} \arrow[r] & 0
\end{tikzcd}$$

This proves that whenever M is a MCM module, the map $M \to M^{**}$ is surjective by Snake lemma. Applying the same fact to N, $N \twoheadrightarrow N^{**}$ which forces $M \to M^{**}$

to be injective. Thus $M \simeq M^{**}$. $\square$

Note that Lemma 9 can be proved by the Ext characterization of depth.

§ 4.2 *Canonical Modules over Cohen-Macaulay Rings (in more generality, in more depth)*

Setup: Let $(S, \mathfrak{m}_S, \mathsf{k})$ be a Gorenstein local ring. We say that a finitely generated S-module $M \in \mathrm{CM_S(i)}$ (or simply CM(i)) if $\operatorname{depth}(M) = \dim(M) = i$.
For example, if $\lambda_S(M) < \infty$, then $M \in \mathrm{CM(0)}$.
If $d := \dim(S)$, then $M \in \mathrm{CM(d)}$ if and only if M is a MCM module.

Theorem 10 *Let $(S, \mathfrak{m}_S, \mathsf{k})$ be a Gorenstein local ring of dimension n. Suppose that $M \in \mathrm{CM(i)}$. Then*
1. $\operatorname{Ext}^j_S(M, S) = 0$ *for all* $j \neq n - i$.
2. $\operatorname{Ext}^{n-i}_S(M, S) \in \mathrm{CM(i)}$.
3. $\operatorname{Ext}^{n-i}_S(\operatorname{Ext}^{n-i}_S(M, S), S) \simeq M$.

Proof: Since $\dim(M) = i$, $\operatorname{ht}(\operatorname{ann}_S(M)) = n-i$. Choose a maximal regular sequence $x_1, \ldots, x_{n-i} \in \operatorname{ann}_S(M)$. Let $\overline{S} = S/(x_1, \ldots, x_{n-i})$. Then $\dim(\overline{S}) = i$. We know that $\operatorname{Ext}^j_S(M, S) = 0$ for $j < n - i$ and $\operatorname{Ext}^j_S(M, S) \simeq \operatorname{Ext}^{j-(n-i)}_{\overline{S}}(M, \overline{S})$ for $j \geq n - i$. By replacing S by $\overline{S}$, without loss of generality we may assume that $\dim(S) = \dim(M) = n$, i.e. M is a MCM S-module.
Thus, assuming that M is a MCM S-module, statements (1) - (3) reduce to proving
(i) $\operatorname{Ext}^j_S(M, S) = 0$ for all $j > 0$.
(ii) $M^* \in \mathrm{CM(n)}$, i.e. M^* is MCM.
(iii) $M^{**} \simeq M$.
which have already been proved. The statement (i) is precisely remark 2 and statements (ii) and (iii) are the conclusions of Theorem 8. $\square$

Let us now see what a canonical module of a local ring $(R, \mathfrak{m}_R, \mathsf{k})$ is in a much more general setting than discussed before.

Definition 3 *Let $(S, \mathfrak{m}_S, \mathsf{k}_S)$ be a Gorenstein local ring of dimension n and $(R, \mathfrak{m}_R, \mathsf{k}_R)$ be a Noetherian local ring such that $R \in \mathrm{CM_S(d)}$. We write $\omega_R := \operatorname{Ext}^{n-d}_S(R, S)$ and call ω_R a canonical module of R (with respect to S).*

Remark 3 We will ultimately prove that ω_R is independent of S.

Some properties of ω_R

Corollary 11 (of Theorem 10) *With notations as in definition 3, we have*
(a) $\omega_R \in \mathrm{CM_S(d)}$ *and*
(b) $\mathrm{Ext}_S^{n-d}(\omega_R, S) \simeq R$.

Remark 4 If $S \twoheadrightarrow R$ and $\mathfrak{p} \in \mathrm{Spec}(S)$, then $\omega_{R_\mathfrak{p}} \simeq (\omega_R)_\mathfrak{p}$ (where $\omega_{R_\mathfrak{p}}$ is a canonical module of $R_\mathfrak{p}$ with respect to $S_\mathfrak{p}$).

Proof: Since $\dim(S) - \dim(R) = \dim(S_\mathfrak{p}) - \dim(R_\mathfrak{p}) = i$(say), we have

$\omega_{R_\mathfrak{p}} \simeq \mathrm{Ext}^i_{S_\mathfrak{p}}(R_\mathfrak{p}, S_\mathfrak{p}) \simeq (\mathrm{Ext}^i_S(R, S))_\mathfrak{p} \simeq (\omega_R)_\mathfrak{p}$. □

Proposition 12 *Let the notation be as in definition 3 of* ω_R. *Suppose* x *is a non-zerodivisor on* R. *Then*
1. x *is a non-zerodivisor on* ω_R.
2. $\omega_R/x\omega_R \simeq \omega_{R/xR}$ *(with respect to either* S *or* S/xS*).*

Proof: As in the proof of Theorem 10, without loss of generality we may assume that $\dim(R) = \dim(S) = d$, by going modulo a maximal regular sequence in $\mathrm{ann}_S(R)$. Hence R is MCM and $\omega_R = \mathrm{Hom}_S(R, S)$.

By remark 2, $\mathrm{Ext}^j_S(R, S) = 0$ for all $j > 0$. Hence applying $\mathrm{Hom}_S(\_, S)$ to the short exact sequence $0 \to R \xrightarrow{\cdot x} R \to R/xR \to 0$, we get another short exact sequence $0 \to \omega_R \xrightarrow{\cdot x} \omega_R \to \mathrm{Ext}^1_S(R/xR, S) \to 0$.

This proves (1) immediately and we further have

$$\frac{\omega_R}{x\omega_R} \simeq \mathrm{Ext}^1_S(R/xR, S) \simeq \mathrm{Hom}_{S/xS}(R/xR, S/xS) \simeq \omega_{R/xR}$$

with respect to either S or S/xS, which proves (2). □

Corollary 13 ω_R *is a MCM* R*-module.*

Proof: By Prop. 12, $\mathrm{depth}(R) \leq \mathrm{depth}(\omega_R)$. But since R is Cohen-Macaulay, $\mathrm{depth}(R) = \dim(R) \geq \dim(\omega_R) \geq \mathrm{depth}(\omega_R)$. □

Theorem 14 (Duality) *Let* $M \in \mathrm{CM_R(k)}$. *Then*
1. $\mathrm{Ext}^j_R(M, \omega_R) = 0$ *for all* $j \neq d - k$.
2. $\mathrm{Ext}^{d-k}_R(M, \omega_R) \in \mathrm{CM_R(k)}$.
3. $\mathrm{Ext}^{d-k}_R(\mathrm{Ext}^{d-k}_R(M, \omega_R), \omega_R) \simeq M$.
4. $\mathrm{Hom}_R(\omega_R, \omega_R) \simeq R$.

Corollary 15 *Let* w_1 *and* w_2 *be two canonical modules of* R *(with respect to Gorenstein rings* S_1 *and* S_2 *respectively). Then* $\mathrm{Ext}^j_R(w_1, w_2) = 0$ *for* $j \neq 0$.

Proof of Theorem 14: Choose a maximal regular sequence $\underline{x} := x_1, \ldots, x_{d-k}$ in $\mathrm{ann}_R(M)$. Let $\overline{R} := R/\underline{x}R$. By Prop. 12, $x_1, \ldots, x_{d-k}$ is a regular sequence on ω_R. This gives us $\mathrm{Ext}^i_R(M, \omega_R) = 0$ for $i < d-k$ and

$$\mathrm{Ext}^i_R(M, \omega_R) \simeq \mathrm{Ext}^{i-(d-k)}_{\overline{R}}(M, \omega_R/\underline{x}\omega_R) \simeq \mathrm{Ext}^{i-(d-k)}_{\overline{R}}(M, \omega_{\overline{R}}) \text{ for } i \geq d-k.$$

Replacing R by $\overline{R}$, we may assume that $\dim(M) = \dim(R) = d$. Now, replacing S by S modulo a maximal regular sequence in $\mathrm{ann}_S(R)$, we may assume that $\dim(S) = \dim(R) = d$. Then $\omega_R \simeq \mathrm{Hom}_S(R, S)$.
Using the lemma

Lemma 16 *With notations as above,* $\mathrm{Ext}^j_R(M, \omega_R) \simeq \mathrm{Ext}^j_S(M, S)$.

statements (1) - (3) follow from Theorem 10. To prove (4), replace M by R in (3). Note that $d = k$. By (3),

$$R \simeq \mathrm{Ext}^{d-k}_R(\mathrm{Ext}^{d-k}_R(R, \omega_R), \omega_R) = \mathrm{Hom}_R(\mathrm{Hom}_R(R, \omega_R), \omega_R) \simeq \mathrm{Hom}_R(\omega_R, \omega_R)$$

proving the theorem. □

Proof of Lemma 16: We want to prove that $\mathrm{Ext}^j_R(M, \omega_R) \simeq \mathrm{Ext}^j_S(M, S)$. Let $\mathbf{I}^\bullet$ be an injective resolution of S over itself. Then

$$\mathrm{Ext}^j_S(M, S) = H^j(\mathrm{Hom}_S(M, \mathbf{I}^\bullet)) = H^j(\mathrm{Hom}_S(M \otimes_R R, \mathbf{I}^\bullet))$$

$$\simeq H^j(\mathrm{Hom}_R(M, \mathrm{Hom}_R(R, \mathbf{I}^\bullet))) \text{ by the Hom} - \otimes \text{ adjointness.}$$

However $\mathrm{Ext}^j_S(R, S) = 0$ for $j > 0$ by Theorem 10 since $R \in \mathrm{CM_S(d)}$, where $d = \dim(S)$. Thus $\mathrm{Hom}_S(R, \mathbf{I}^\bullet)$ is an acyclic complex of injective R-modules with $H^0(\mathrm{Hom}_S(R, \mathbf{I}^\bullet)) \simeq \mathrm{Hom}_S(R, S) = \omega_R$. Hence $\mathrm{Hom}_S(R, \mathbf{I}^\bullet)$ is an injective resolution of ω_R over R which implies that $H^j(\mathrm{Hom}_R(M, \mathrm{Hom}_S(R, \mathbf{I}^\bullet))) = \mathrm{Ext}^j_R(M, \omega_R)$. This proves the lemma. □

The following is really a corollary, but is important enough to be accorded the status of a theorem.

Theorem 17 *Let ω_1 and ω_2 be two canonical modules for a Cohen-Macaulay local ring R (with respect to two Gorenstein rings S_1 and S_2). Then $\omega_1 \simeq w_2$.*

Corollary 18 *If R is Gorenstein, then $\omega_R \simeq R$.*

Proof: Compute ω_R over R: $\omega_R = \mathrm{Hom}_R(R, R) \simeq R$. □

Proof of Theorem 17: Let $\underline{x} = x_1, \dots, x_d$ be a maximal regular sequence in R (where $d := \dim(R)$). Set $R_i := R/(x_1, \dots, x_i)R$. Then $\omega_j/(x_1, \dots, x_i)\omega_j$ is a canonical module of R_i with respect to S_j by Prop. 12. Denote this by $w_j^{(i)}$. By induction on $d - i$, we claim that $w_1^{(i)} \simeq \omega_2^{(i)}$.

When $i = d$, R_d is Artinian. Hence by Prop. 2, $w_1^{(d)} \simeq E_{R_d}(\mathsf{k}) \simeq \omega_2^{(d)}$. Observe that $\mathrm{Ext}^1_{R_i}(\omega_1^{(i)}, \omega_2^{(i)}) = 0$ for each $i < d$ by Cor. 15. Hence the proof follows by induction using the following lemma. $\square$

Lemma 19 *Let M, N be finitely generated R-modules, x a non-zerodivisor on N. Assume further that $\mathrm{Ext}^1_R(M, N) = 0$. If $M/xM \simeq N/xN$, then $M \simeq N$.*

Proof: Let $\bar{\ }$ denote going modulo x. Apply $\mathrm{Hom}_R(M, \_)$ to the short exact sequence $0 \to N \xrightarrow{\cdot x} N \to \overline{N} \to 0$. Since $x \cdot \mathrm{Hom}_R(M, \overline{N}) = 0$, the induced long exact sequence gives the short exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \xrightarrow{\cdot x} \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, \overline{N}) \to 0. \qquad (*)$$

Since $\mathrm{Hom}_R(M, \overline{N}) \simeq \mathrm{Hom}_R(\overline{M}, \overline{N})$, an isomorphism between $\overline{M}$ and $\overline{N}$ induces a surjective map $\overline{\phi} : M \xrightarrow{\sim} \overline{N}$. By $(*)$, $\overline{\phi}$ can be lifted to a homomorphism $\phi : M \to N$. Since $\overline{\phi}$ is surjective, $\phi(M) + xN = N$. By NAK, ϕ is surjective.

We want to now prove that ϕ is injective. Let $K := \mathrm{Ker}(\phi)$. Then we have the short exact sequence $0 \to K \to M \xrightarrow{\phi} N \to 0$. Tensoring with $\overline{R}$, the induced long exact sequence on homology is

$$\cdots \mathrm{Tor}_1^R(N, \overline{R}) \to \overline{K} \to \overline{M} \xrightarrow{\overline{\phi}} \overline{N} \to 0.$$

Since x is a non-zerodivisor on N, $\mathrm{Tor}_1^R(N, \overline{R}) = 0$. This forces $\overline{K} = 0$ and hence by NAK, $K = 0$. $\square$

Definition 4 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring. We say that R has a canonical module ω_R if ω_R is finitely generated and $\widehat{\omega_R} = \omega_{\widehat{R}}$.*

Note: $\omega_{\widehat{R}}$ exists by Cohen's Structure Theorem.

Remark: If ω_R exists, it is unique up to isomorphism. This can be proved by using exercise (1) and theorem 17 for $\widehat{R}$.

Theorem 20 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring. Then R has a canonical module if and only if there is a Gorenstein local ring $(S, \mathfrak{m}_S, \mathsf{k})$ mapping onto R.*

Proof: If there is a Gorenstein local ring $(S, \mathfrak{m}_S, \mathsf{k})$ mapping onto R, then the canonical module exists and is unique up to isomorphism by construction.

To prove the converse, let ω_R be a canonical module for R. Set $S = R \oplus \omega_R$ as a set. Define an additive structure on S componentwise and multiplication by

$$(r_1, u_1) \cdot (r_2, u_2) = (r_1 r_2, r_1 u_2 + r_2 u_1) \quad r_1, r_2 \in R; u_1, u_2 \in \omega_R.$$

An alternative way is to think of the elements of S as matrices;

$$S = \left\{ \begin{pmatrix} r & u \\ 0 & r \end{pmatrix} : r \in R, u \in \omega_R \right\}$$

with addition and multiplication being the usual operations on matrices. We denote S with these operations by $R \ltimes \omega_R$.

Under these operations, S is a commutative Noetherian ring and $\omega_R \subseteq S$ is an ideal such that $\omega_R^2 = 0$. Moreover $S/\omega_R \simeq R$. Since ω_R is nilpotent and S/ω_R is local, S is also local.

We will now prove that S is Gorenstein. To prove this, let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters in R. By abuse of notation, we think of $\underline{x}$ a sequence of elements in S. By Prop. 12. $\underline{x}$ forms a regular sequence on ω_R. This forces $\underline{x}$ to be an S-regular sequence. Hence it is enough to prove that $S/\underline{x}$ is Gorenstein. But

$$S/\underline{x}S \simeq (R/\underline{x}) \ltimes (\omega_R/\underline{x}\omega_R) \simeq (R/\underline{x}) \ltimes (\omega_{R/\underline{x}}) \simeq (R/\underline{x}) \ltimes (E_{R/\underline{x}}(\mathsf{k})).$$

Thus it is enough to show that if $(R, \mathfrak{m}, \mathsf{k}, E)$ is an Artinian local ring, then $S = R \ltimes E$, with operations defined as above, is Gorenstein. This can be proven as an exercise. $\square$

Theorem 21 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring with a canonical module ω_R. The following are equivalent:*
1. *$\omega_R \simeq I \subseteq R$.*
2. *$\widehat{R}_{\mathfrak{p}}$ is Gorenstein for every minimal prime $\mathfrak{p}$ in $\widehat{R}$.*

Furthermore if either of (and hence both) these conditions hold, then $I \not\subseteq \bigcup_{Q \in \mathrm{Min}(R)} Q$.

Proof: (1) $\Rightarrow$ (2): Consider the completions of both sides in (1). We have

$$\omega_{\widehat{R}} = \widehat{\omega_R} \simeq \widehat{I} \simeq I\widehat{R} \subseteq \widehat{R}.$$

Thus $\omega_{\widehat{R}}$ is an ideal in $\widehat{R}$.

Let $\mathfrak{p}$ be a minimal prime in $\widehat{R}$. We want to prove that $\widehat{R}_{\mathfrak{p}}$ is Gorenstein. We have $(\omega_{\widehat{R}})_{\mathfrak{p}} \subseteq \widehat{R}_{\mathfrak{p}}$. But

$$(\omega_{\widehat{R}})_{\mathfrak{p}} \simeq \omega_{\widehat{R}_{\mathfrak{p}}} \simeq E_{\widehat{R}_{\mathfrak{p}}}(\widehat{R}_{\mathfrak{p}}/\mathfrak{p}\widehat{R}_{\mathfrak{p}}).$$

Hence by counting lengths, $E_{\widehat{R}_\mathfrak{p}}(\widehat{R}_\mathfrak{p}/\mathfrak{p}\widehat{R}_\mathfrak{p}) \simeq \widehat{R}_\mathfrak{p}$, i.e. $\widehat{R}_\mathfrak{p}$ is Gorenstein.
Moreover $I\widehat{R} \not\subseteq \mathfrak{p}$, proving the last statement of the theorem.

We use the following lemma to prove the converse.

Lemma 22 *If R is a zero-dimensional Noetherian ring, M a finitely generated R-module and $M_\mathfrak{p} \simeq R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ in R, then $M \simeq R$.*

(2) $\Rightarrow$ (1) Let $\mathfrak{q}$ be a minimal prime in R. Choose $\mathfrak{p} \in \mathrm{Min}(\widehat{R})$ such that $\mathfrak{p} \cap R = \mathfrak{q}$. Since $R \to \widehat{R}$ is flat, so is $R_\mathfrak{q} \to \widehat{R}_{R\setminus\mathfrak{q}}$. Hence $R_\mathfrak{q} \to \widehat{R}_\mathfrak{p}$ is a flat, local map.

Suppose $\widehat{R}_\mathfrak{p}$ is Gorenstein. Hence by *Fibers of Flatness*, $R_\mathfrak{q}$ is Gorenstein. We will show that if $R_\mathfrak{q}$ is Gorenstein for every minimal prime $\mathfrak{q}$ in R, then ω_R is an ideal in R.

Let $\mathfrak{q}$ be a minimal prime in R. Then $(\omega_R)_\mathfrak{q} \simeq \omega_{R_\mathfrak{q}} \simeq R_\mathfrak{q}$ since $R_\mathfrak{q}$ is Gorenstein. Let $W = R \setminus \bigcup_{\mathfrak{q}\in\mathrm{Min}(R)} \mathfrak{q}$. By Aplying the lemma to R_W and $(\omega_R)_W$, we see that $(\omega_R)_W \overset{\phi}{\simeq} R_W$. Thus

$$\phi \in \mathrm{Hom}_{R_W}(\omega_{R_W}, R_W) \simeq (\mathrm{Hom}_R(\omega_R, R))_W.$$

Choose $\psi : \omega_R \to R$ and $w \in W$ such that $\phi = \psi/w$.

We claim that ψ is injective. To see this, observe that $(\mathrm{Ker}(\psi))_W = 0$. But W consists of non-zerodivisors on R and hence on ω_R (by Prop. 12). This forces $\mathrm{Ker}(\psi) = 0$ proving that ω_R is an ideal in R. $\square$

Remark: If R is Cohen-Macaulay, then $\mathrm{Ass}(R) = \mathrm{Min}(R)$. Hence, in this case, $I \not\subseteq \bigcup_{\mathfrak{q}\in\mathrm{Min}(R)} \mathfrak{q}$ if and only if I contains a non-zerodivisor.

Exercise: Prove lemma 22.

Theorem 23 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a 1-dimensional Cohen-Macaulay Noetherian local ring. If $\widehat{R}_\mathfrak{p}$ is Gorenstein for every minimal prime $\mathfrak{p}$ in $\widehat{R}$, then*
1. *R has a canonical module ω_R and*
2. *ω_R is isomorphic to an $\mathfrak{m}$-primary ideal of R.*

Proof: Note that (2) follows from (1) at once by Theorem 21 once we show that ω_R exists.

By Theorem 21 applied to $\widehat{R}$, we see that $\omega_{\widehat{R}} \subseteq \widehat{R}$. Let $\omega_{\widehat{R}} \simeq J \subseteq \widehat{R}$. Since $J \not\subseteq \bigcup_{\mathfrak{p}\in\mathrm{Min}(\widehat{R})} \mathfrak{p}$ and $\widehat{R}$ is 1-dimensional, J is $\mathfrak{m}$-primary. Hence there is an $n \in \mathbb{N}$ such that $\widehat{\mathfrak{m}}^n \subseteq J$. But $R/\mathfrak{m}^n \simeq \widehat{R}/\widehat{\mathfrak{m}}^n$. Hence $J/\widehat{\mathfrak{m}}^n \simeq I/\mathfrak{m}^n$ for some ideal I in R. Then

$$\widehat{I} \simeq I\widehat{R} = J \simeq \omega_{\widehat{R}}.$$

Thus, by the definition I is a canonical module of R, i.e. ω_R exists and $\omega_R \simeq I$ proving (1). $\square$

§ 4.3 *Some Characterizations of Gorenstein Rings*

We are aiming for the following classical result which almost started the definition of Gorenstein rings.

Theorem 24 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a 1-dimensional Noetherian local ring with an infinite residue field such that $\widehat{R}$ is reduced (i.e. R is analytically unramified). Let K be the total ring of quotients of R and $\overline{R}$ be the integral closure of R in K. Let $\mathfrak{C}_R := \{\alpha \in \mathsf{K} : \alpha\overline{R} \subseteq R\}$ be the conductor of $\overline{R}$ into R. Then R is Gorenstein if and only if $2\lambda(R/\mathfrak{C}_R) = \lambda(\overline{R}/\mathfrak{C}_R)$.*

Setup: Let R be a reduced ring, K be it's total ring of quotients and $\overline{R}$ be the integral closure of R in K. We assume throughout that $\overline{R}$ is a finitely generated R-module.

Discussion: If $\overline{R} = R(a_1/b_1) + \cdots + R(a_n/b_n)$, $a_i, b_i \in R$, $b_i \notin \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Ass}(R)$, then $(b_1, \ldots, b_n)\overline{R} \subseteq R$. Note that since R is reduced, $\operatorname{Ass}(R) = \operatorname{Min}(R)$.

Definition: Define $\mathfrak{C}_{\overline{R}/R}$ (sometimes denoted simply by $\mathfrak{C}_R$) $:= \{\alpha \in \mathsf{K} : \alpha\overline{R} \subseteq R\}$ to be *the conductor of $\overline{R}$ to R.*

Remark 5
1. $\mathfrak{C}_R \subseteq R$ is an ideal. Moreover $\mathfrak{C}_R$ is an ideal in $\overline{R}$. Finally, $\mathfrak{C}_R$ is the largest common ideal of R and $\overline{R}$.
2. Note that under the above assumptions, $\mathfrak{C}_R$ contains a non-zerodivisor. In particular, if R is a 1-dimensional local ring, then $\mathfrak{C}_R$ is $\mathfrak{m}$-primary (assuming $R \neq \overline{R}$).

Illustrative Examples:

Example 2 Let $R = \mathsf{k}[[t^3, t^5]]$ where k is a field. Then $\overline{R} = \mathsf{k}[[t]]$. Note that the powers of t in R are $0, 3, 5, 6, 8, 9, 10, \ldots$. Then $\mathfrak{C}_R = (t^8, t^9, t^{10})R = (t^8)\overline{R}$.

Note that $R/\mathfrak{C}_R \simeq \mathsf{k}\cdot 1 + \mathsf{k}\cdot t^3 + \mathsf{k}\cdot t^5 + \mathsf{k}\cdot t^6$ and $\overline{R}/R \simeq \mathsf{k}\cdot t + \mathsf{k}\cdot t^2 + \mathsf{k}\cdot t^4 + \mathsf{k}\cdot t^7$. As k-vector spaces $\lambda(\overline{R}/R) = 4 = \lambda(R/\mathfrak{C}_R)$. This is not a coincidence. The equality $\lambda(\overline{R}/R) = \lambda(R/\mathfrak{C}_R)$ holds since R is Gorenstein.

Exercise: If $(a, b) = 1, a, b > 0$. Prove that

$$\mathfrak{C}_R =< t^j : j \geq (a-1)(b-1) > R \text{ where } R = \mathsf{k}[[t^a, t^b]].$$

Example 3 Let $R = \mathsf{k}[[t^3, t^4, t^5]]$. Then $\overline{R} = \mathsf{k}[[t]]$ and $\mathfrak{C}_R \simeq (t^3, t^4, t^5)R = \mathfrak{m}_R$. In this case $\lambda(\overline{R}/R) = 2 \neq 1 = \lambda(R/\mathfrak{C}_R)$. This implies that R is not Gorenstein.

Remark 6 Let M, N be finitely generated R-submodules of K which contain a non-zerodivisor apiece. Then
1. $M \otimes_R \mathsf{K} \simeq \mathsf{K}$, $N \otimes_R \mathsf{K} \simeq \mathsf{K}$ and
2. $\mathrm{Hom}_R(M, N) \otimes_R \mathsf{K} \simeq \mathrm{Hom}_{\mathsf{K}}(M \otimes_R \mathsf{K}, N \otimes_R \mathsf{K}) \simeq \mathrm{Hom}_{\mathsf{K}}(\mathsf{K}, \mathsf{K}) \simeq \mathsf{K}$. Therefore $\mathrm{Hom}_R(M, N) \overset{i}{\hookrightarrow} \mathsf{K}$ as the Hom is torsion-free.

Let $f \in \mathrm{Hom}_R(M, N)$. Choose $x \in M \cap R$, a non-zerodivisor. Then $i(f)$ can be identified with $f(x)/x$. This can be seen as follows: For any $y \in M \cap R$, $x \cdot f(y) = y \cdot f(x)$ and hence $f(y) = y \cdot (f(x)/x)$.
This extends to all $y \in M$. Thus we have,

$$\mathrm{Hom}_R(M, N) = \{\alpha \in \mathsf{K} : \alpha M \subseteq N\}.$$

In particular,

$$\mathrm{Hom}_R(\overline{R}, R) = \{\alpha \in \mathsf{K} : \alpha \overline{R} \subseteq R\} = \mathfrak{C}_R.$$

We use the following lemma to prove the main theorem in this section.

Lemma 25 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring of dimension d with a canonical module ω_R. Let L be a finitely generated R-module. If $\lambda_R(L) < \infty$, then $L^\vee \simeq \mathrm{Ext}^d_R(L, \omega_R)$.*
In particular, $\lambda_R(\mathrm{Ext}^d_R(L, \omega_R)) = \lambda_R(L)$.

Proof: Choose a maximal regular sequence $\underline{x} = x_1, \ldots, x_d \in \mathrm{ann}_R(L)$. Since R is Cohen-Macaulay, $\underline{x}$ is not only a regular sequence on R, but also on ω_R. Hence

$$\mathrm{Ext}^d_R(L, \omega_R) \simeq \mathrm{Ext}^0_R(L, \omega_R/\underline{x}\omega_R) \simeq \mathrm{Hom}_R(L, \omega_{R/\underline{x}R}) \simeq \mathrm{Hom}_{R/\underline{x}R}(L, E_{R/\underline{x}R}(\mathsf{k}))$$

$$\simeq \mathrm{Hom}_{R/\underline{x}R}(L, \mathrm{Hom}_R(R/\underline{x}R, E_R(\mathsf{k}))) \simeq \mathrm{Hom}_R(L \otimes_R R/\underline{x}R, E_R(\mathsf{k})) \simeq L^\vee. \square$$

Definition: Let $(R, \mathfrak{m}, \mathsf{k})$ be a Cohen-Macaulay local ring. Then we define

$$\mathrm{type}(R) := \mu(\omega_{\widehat{R}}).$$

With this background, we will now prove the main theorem of this section. The following statement is stronger than what we stated in Theorem 24.

Theorem 26 *Let $(R, \mathfrak{m}, \mathsf{k})$ be a 1-dimensional reduced Noetherian local ring with an infinte residue field such that $\overline{R}$ is a finite R-module. Assume $\widehat{R}_{\mathfrak{p}}$ is Gorenstein for every minimal prime $\mathfrak{p}$ in $\widehat{R}$. Then $\lambda(\overline{R}/R) \geq \lambda(R/\mathfrak{C}_R) + \mathrm{type}(R) - 1$. The equality holds if R is Gorenstein. Thus R is Gorenstein if and only if $\lambda(R/\mathfrak{C}_R) = \lambda(\overline{R}/R)$.*

Remark: The condition $\lambda(R/\mathfrak{C}_R) = \lambda(\overline{R}/R)$ is usually written as

$$2\lambda(R/\mathfrak{C}_R) = \lambda(\overline{R}/\mathfrak{C}_R).$$

Proof of Theorem 26: By theorem 23, ω_R exists. Moreover, we can assume it to be an $\mathfrak{m}$-primary ideal in R. Hence $\text{type}(R) = \mu(\omega_R)$.

Consider the short exact sequence $0 \to \omega_R \to \omega_R\overline{R} \to \omega_R\overline{R}/\omega_R \to 0$. Apply $\text{Hom}(\_, \omega_R)$ and observe that (1) $\text{Hom}_R(\omega_R\overline{R}/\omega_R, \omega_R) = 0$ since ω_R contains a non-zerodivisor and (2) $\text{Ext}^1_R(\omega_R\overline{R}, \omega_R) = 0$ by Theorem 14 since $\omega_R\overline{R} \in \text{CM}_\text{R}(1)$ to get

$$0 \to \text{Hom}_R(\omega_R\overline{R}, \omega_R) \to \text{Hom}_R(\omega_R, \omega_R) \to \text{Ext}^1_R(\omega_R\overline{R}/\omega_R, \omega_R) \to 0. \qquad (*)$$

Since $(\omega_R \otimes_R \overline{R})/(torsion) \simeq \omega_R\overline{R}$, $\text{Hom}_R(\omega_R\overline{R}, \omega_R) \simeq \text{Hom}_R(\omega_R \otimes_R \overline{R}, \omega_R)$ which, by the $\text{Hom} - \otimes$ adjointness, is isomorphic to $\text{Hom}_R(\overline{R}, \text{Hom}_R(\omega_R, \omega_R))$. Thus using $\text{Hom}_R(\omega_R, \omega_R) \simeq R$,

$$\text{Hom}_R(\omega_R\overline{R}, \omega_R) \simeq \text{Hom}_R(\overline{R}, R) \simeq \mathfrak{C}_R.$$

and therefore $(*)$ reduces to

$$0 \to \mathfrak{C}_R \to R \to \text{Ext}^1_R(\omega_R\overline{R}/\omega_R, \omega_R)) \to 0.$$

Moreover, by lemma 25,

$$\lambda(\text{Ext}^1_R(\omega_R\overline{R}/\omega_R, \omega_R)) = \lambda(\omega_R\overline{R}/\omega_R).$$

Hence

$$\lambda(R/\mathfrak{C}_R) = \lambda(\omega_R\overline{R}/\omega_R) \qquad (**).$$

Since k is infinite, we can choose $x \in \omega_R$, a minimal reduction of ω_R. Hence $\omega_R\overline{R} = x\overline{R}$ which implies that

$$\lambda(\omega_R\overline{R}/\omega_R) = \lambda(x\overline{R}/\omega_R) = \lambda(x\overline{R}/xR) - \lambda(\omega_R/xR) = \lambda(\overline{R}/R) - \lambda(\omega_R/xR).$$

Thus $(**)$ gives $\lambda(\overline{R}/R) = \lambda(R/\mathfrak{C}_R) + \lambda(\omega_R/xR)$. Hence we just need to show that $\lambda(\omega_R/xR) \geq \text{type}(R) - 1$.

But x is a minimal generator of ω_R, so that

$$\lambda(\omega_R/xR) \geq \mu(\omega_R/xR) = \mu(\omega_R) - 1 = \text{type}(R) - 1$$

proving the inequality in the theorem.

Recall that R is Gorenstein if and only if $\text{type}(R) = 1$. Thus if R is Gorenstein, $\omega_R \simeq R$ and hence $(**)$ gives $\lambda(R/\mathfrak{C}_R) = \lambda(\overline{R}/R)$. Since $\text{type}(R) = 1$, this proves the equality in the theorem. If R is not Gorenstein, then $\text{type}(R) \geq 2$, forcing $\lambda(\overline{R}/R) > \lambda(R/\mathfrak{C}_R)$, proving the second part of the theorem. $\square$

Remark 7 We will now show that the assumption k is infinite is not necessary to prove the above theorem. We used the fact that k was infinite to claim the existence of an element $x \in \omega_R$ such that $\omega_R\overline{R} = x\overline{R}$ using a minimal reduction. What we need to show is that such an element exists even if the field is not necessarily infinite.

Note that since $\overline{R}$ is reduced, semilocal and integrally closed in its total ring of fractions, it is a direct product of DVRs. This is due to the following reason: The total ring of quotients K of $\overline{R}$ (or R) is $\kappa_1 \times \cdots \times \kappa_t$, where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\} = \mathrm{Min}R$ and $\kappa_i = \kappa(\mathfrak{p}_i)$, the residue field of $R_{\mathfrak{p}_i}$. Consider the idempotent elements $e_i \in \mathsf{K}$ corresponding to κ_i. Then $e_i^2 - e_i = 0$, and hence $e_i \in \overline{R}$ for each i.

Thus $\overline{R}$ is a principal ideal ring, i.e. every ideal in $\overline{R}$ is principal. In particular, there is an element $t \in \overline{R}$ such that $\omega_R\overline{R} = t\overline{R}$.
Warning: t need not be in R as can be seen in example 4 below.
To finish the proof, choose a non-zerodivisor $c \in \mathfrak{C}_R$. Then $ct \in R$ and $c\omega_R \simeq \omega_R$. Hence replace ω_R by $c\omega_R$ and t by ct. Thus we have $c\omega_R\overline{R} = ct\overline{R}$, concluding the proof. $\square$

The following is an example to illustrate the fact that the element t, chosen as in the above remark, need not be in R.

Example 4 Let $R = \mathbb{Z}_2[[X, Y]]/(XY(X+Y))$. Then R is a reduced 1-dimensional ring. We have $\mathfrak{m}_R\overline{R} = (X, Y)\overline{R}$ and

$$\overline{R} \overset{\phi}{\simeq} \mathbb{Z}_2[[Y]]\times\mathbb{Z}_2[[X]]\times\mathbb{Z}_2[[X]], \text{ where } \phi(f) = (f(\mathrm{mod}X), f(\mathrm{mod}Y), f(\mathrm{mod}(X+Y))).$$

The elements X, Y and $X+Y$ in $\overline{R}$ correspond to $(0, X, X)$, $(Y, 0, Y)$ and $(Y, X, 0)$ respectively under ϕ. Then $\mathfrak{m}_R\overline{R} \not\simeq f\overline{R}$, for $f = X$, Y or $X+Y$. But

$$\mathfrak{m}_R\overline{R} \simeq (Y, X, X)\overline{R}.$$

Notation:
1. By $\mathbb{W}$, we denote the set $\mathbb{N} \cup \{0\}$.

2. For a subset $H \subseteq \mathbb{W}$, by t^H we denote the set $\{t^i : i \in H\}$.

The following is an example to illustrate the theorem.

Example 5 We will show that $R = \mathsf{k}[t^6, t^7, t^8]$ is Gorenstein.
Let $H := \{i \in \mathbb{W} : t^i \in R\}$ and $C := \{i \in \mathbb{W} : t^i \in \mathfrak{C}_R\}$. Then

$$H = \{0, 6, 7, 8, 12, 13, 14, 15, 16, 18, 19, \ldots\} \text{ and } C = \{18, 19, 20, \ldots\}.$$

Thus

$$\lambda(\overline{R}/R) = |\mathbb{W} \setminus H| = |\{1, 2, 3, 4, 5, 9, 10, 11, 17\}| = 9 \text{ and}$$

$$\lambda(R/\mathfrak{C}_R) = |H \setminus C| = |\{0, 6, 7, 8, 12, 13, 14, 15, 16\}| = 9.$$

Hence by the above theorem, R is Gorenstein. In fact, R is a complete intersection ring.

Definition 5 *Let H be a subset of $\mathbb{W}$ containing 0, closed under addition, such that $c + \mathbb{W} \subseteq H$ for some $c \in \mathbb{W}$. H is said to be symmetric if there is an $m \in \mathbb{W}$ such that for every $n \geq 0$, $n \in H \Leftrightarrow m - n \notin H$.*

Proposition 27 *The ring $R = \mathsf{k}[[t^H]]$ is Gorenstein if and only if H is symmetric.*

Proof: Let $c \in \mathbb{W}$ be given by $\mathfrak{C}_R = (t^i : i \geq c)$. Note that by definition of c, $t^{c-1} \notin R$. Suppose $\mathsf{k}[[t^H]]$ is Gorenstein. Set $m = c - 1$.

If $n \in H$, then $t^n \in R$. Then $t^{c-1-n} \notin R$ else $t^{c-1} = t^n \cdot t^{c-1-n} \in R$ which is not true.

Since $\lambda(\overline{R}/R) = \lambda(R/\mathfrak{C}_R)$, H must contain exactly a half of the elements in $(\mathbb{W} \setminus H) \cup (H \setminus C) = \{0, 1, \cdots, c-1\}$. Hence $t^n \in R \Leftrightarrow t^{c-1-n} \notin R$, i.e. $n \in H \Leftrightarrow m - n \notin H$.

Conversely, if H is symmetric, then $|\mathbb{W} \setminus H| = |H \setminus C|$, i.e. $\lambda(\overline{R}/R) = \lambda(R/\mathfrak{C}_R)$, which implies that R is Gorenstein. □

Example 6

Question: Is $R = \mathsf{k}[[t^3, t^5, t^7]]$ Gorenstein? If not, what is ω_R?

In this case, $H = \{0, 3, 5, 6, 7, 8, \ldots\}$, hence $c = 5$. Therefore R is not Gorenstein since $c - 1$ has to be odd for H to be symmetric.

Let us now find out what the canonical module ω_R of R is.

There are two basic ways to find ω_R:

Method I: Find a Gorenstein ring $S \subseteq R$ and compute $\omega_R = \mathrm{Hom}_S(R, S)$.

We can choose $\mathsf{k}[[t^3]] \subseteq \mathsf{k}[[t^3, t^5, t^7]] = R$. A better choice is $S := \mathsf{k}[[t^3, t^5]] \subseteq R$, since R and S are birational and $\dim(S) = \dim(R)$. Note that S is a hypersurface and hence Gorenstein. In this case,

$$\omega_R \simeq \mathrm{Hom}_S(R, S) \simeq \{\alpha \in \mathsf{k}[[t]] : \alpha R \subseteq S\},$$

i.e. ω_R is the conductor of R to S. It is enough to find the set $\{j \in \mathbb{N} \cup \{0\} : t^j R \subseteq S\}$.

We have $H_R = \{0, 3, 5, 6, 7, 8, \ldots\}$ and $H_S = \{0, 3, 5, 6, 8, 9, 10, \ldots\}$. Hence $\{j \in \mathbb{N} \cup \{0\} : t^j R \subseteq S\} = \{3, 5\}$, i.e. $\omega_R \simeq < t^3, t^5 >$. Note that ω_R is also an ideal in R. In this case, $\mathrm{type}(R) = \mu(\omega_R) = 2$.

Method II: Map a regular local ring $T \twoheadrightarrow R$, compute a resolution of R over T and take the cokernel of the dual of the last map.

A Variation of method II: (works better if R is a domain). Let T be as above. Write $R \simeq T/I$, for an ideal $I \in T$. Let $\underline{x}$ be a maximal regular sequence in I. Then if $c = \mathrm{ht}(I)$, we have

$$\omega_R \simeq \mathrm{Ext}^c_T(R,T) \simeq \mathrm{Hom}_{T/(\underline{x})}(R, T/(\underline{x})) \simeq (\underline{x} :_T I)/(\underline{x}).$$

Let $T := \mathsf{k}[[X,Y,Z]] \xrightarrow{\phi} \mathsf{k}[[t^3,t^5,t^7]] = R$. Then

$$\mathfrak{p} := \mathrm{Ker}(\phi) = I_2\left(\begin{pmatrix} X & Z & Y \\ Y & X^3 & Z \end{pmatrix}\right) = (X^4 - YZ, Y^2 - XZ, Z^2 - X^3Y).$$

Let Δ_1, Δ_2 and Δ_3 respectively denote the three generators of $\mathfrak{p}$. Then

$$X\Delta_1 + Z\Delta_2 + Y\Delta_3 = 0 = Y\Delta_1 + X^3\Delta_2 + Z\Delta_3 \qquad (*).$$

Since Δ_1, Δ_2 is a regular sequence in $\mathfrak{p}$, $(\Delta_1,\Delta_2) :_T \mathfrak{p} = (\Delta_1,\Delta_2) :_T \Delta_3 = (Y,Z)$ by $(*)$. Therefore the image of (Y,Z) in R is a canonical module, i.e. $\omega_R \simeq (t^5,t^7)$.

Note that these generators of ω_R correspond to the third column of the matrix $\begin{pmatrix} X & Z & Y \\ Y & X^3 & Z \end{pmatrix}$. We could have chosen the first column (giving $\omega_R \simeq (t^3,t^5)$) or the second column (giving $\omega_R \simeq (t^7,t^9)$).

Exercises

(1) Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian rings. Assume that $R/\mathfrak{m} \simeq S/\mathfrak{n}$ under the induced map. Let M, N be two finitely generated R-modules. If $M \otimes_R S \simeq N \otimes_R S$, prove that $M \simeq N$.

(2) Let $(R, \mathfrak{m})$ be a Noetherian local ring, and assume that $M \in \mathrm{CM_R(i)}$. Is $R/\mathrm{ann}M$ Cohen-Macaulay?

(3) Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $M \in \mathrm{CM_R(i)}$. Assume that $x \in \mathfrak{m}$ is a non-zerodivisor M. Is $\mathrm{ann}(M/xM) = \mathrm{ann}M + Rx$?

(4) Let $R = \mathsf{k}[[t^7, t^9, t^{10}]]$ where k is a field of characteristic $\neq 7$.

(a) Compute ω_R by mapping a regular local ring onto R.

(b) Compute ω_R by considering $\mathrm{Hom}_B(R, B)$ where $B = \mathsf{k}[[t^7, t^9]]$.

(5) Let $(S, \mathfrak{n})$ be a RLR, $R = S/I$, and assume that $R \in \mathrm{CM_S(n-2)}$ where $n = \dim(S)$. Let C be an $t \times (t+1)$ matrix giving a resolution:

$$0 \longrightarrow S^t \xrightarrow{C} S^{t+1} \longrightarrow S \longrightarrow R \longrightarrow 0.$$

Assume that any two maximal minors of C form a regular sequence in S (this is always possible), which generate $I_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Min}(I)$.

Let B be any $t \times (t-1)$ matrix obtained from C by deleting 2 columns. Prove that

$$\omega_R \simeq I_{t-1}(B)R.$$

(6) (Stanley) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module $\omega_R \subseteq R$. Let $\underline{x} = x_1, \ldots, x_d$ be a maximal regular sequence on R. Let $S = R/(\underline{x})$ and $I = (\omega_R + (\underline{x}))/(\underline{x}) \subseteq S$. Show that $I^\vee = I$, where ${}^\vee = \mathrm{Hom}_S(\quad, E)$ with E an injective hull of the residue field of S.

(7) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let M be a finitely generated R-module . Assume that $\mathrm{Ext}^i_R(M, M) = 0$ for all $1 \le i$ and assume that $\mathrm{Hom}_R(M, M)$ is free. Prove that if R is Gorenstein, then M is free.

(8) (Greco) Let R be a complete local reduced ring of dimension one. Assume the residue field of R is infinite and assume that R is not Gorenstein. Let $\overline{R}$ be the

integral closure of R in its total quotient ring. Let $\mathfrak{C} \simeq \mathrm{Hom}_R(\overline{R}, R) \subseteq R$, the conductor of $\overline{R}$ to R. Prove or give a counterexample to the inequality:

$$\min\{\lambda(R/K) | \omega_R \simeq K \subseteq R\} \geq$$
$$e(R) + \lambda(R/\mathfrak{C}) - \lambda(\overline{R}/R)$$

where $e(R)$ is the multiplicity of R.
(So $e(R) = \min\{\lambda(R/(x) | x$ is a non-zerodivisor in $R\})$.

(9) Prove that the inequality in exercise 8 can be strict by considering $\mathsf{k}[[t^5, t^6, t^8]] = R$.

(10) Let $(R, \mathfrak{m})$ be a complete local domain satisfying S_2. Let $\mathfrak{C}$ be the conductor of $\overline{R}$, the integral closure of R, to R. Prove that $\mathfrak{C}$ is a height one unmixed ideal.

(11) Let R be a complete 1-dimensional domain with integral closure $\overline{R}$. Let $\mathfrak{C} = \mathrm{Hom}_R(\overline{R}, R)$, the conductor of $\overline{R}$ to R. Suppose that I is an ideal of R such that $I \supseteq \mathfrak{C}$. Let J be any other ideal of R isomorphic to I. Prove that $J\overline{R} \subseteq I\overline{R}$.

(12) (R. Wiegand) Let R be a 1-dimensional complete local domain which is Gorenstein. Prove that for every finitely generated torsion-free R-module M without nontrivial free summands, there is a ring S, $R \underset{\neq}{\subset} S \subseteq \overline{R}$ (= integral closure of R) such that M is an S-module. (Hint: consider the inverse of the "trace ideal", the ideal generated by all $f(x)$ where $f \in M^*$ and $x \in M$. Use exercise 11.)

(13) Prove that the assumption that R is Gorenstein in exercise 12 is needed by considering the canonical module of a Cohen-Macaulay 1-dimensional complete domain.

(14) Let $(R, \mathfrak{m})$ be a 1-dimensional Noetherian local ring whose completion is reduced. Let K be the total ring of quotients of R. Prove that R is Gorenstein iff $\mathfrak{m}^{-1} = \{x \in \mathsf{K} : x\mathfrak{m} \subseteq R\}$ is generated by 2 elements as an R-module.

(15) Let S be a Gorenstein local ring, and assume that $R = S/I$ is CM. Let J be generated by a maximal regular sequence in I, and assume that x is an element of S which is a non-zerodivisor on S/J. Prove that $(J : I) + Sx = (J, x) : (I, x)$. (Hint: consider the canonical modules of R and R/Rx.)

(16) Let R be a 1-dimensional Noetherian local domain with finite integral closure. Prove that the conductor of R cannot be contained in a principal ideal of R unless R is integrally closed.

(17) Let R be a Noetherian semilocal ring, and let M be a finitely generated R-module such that $M_{\mathfrak{m}} \simeq R_{\mathfrak{m}}^r$ for all maximal ideals. Prove that $M \simeq R^r$.

(18) Let k be a field, and let a, b be two relatively prime positive integers. Set $R = \mathsf{k}[[t^a, t^b]]$. Prove that the conductor of R is generated by all t^j where $j \geq (a-1)(b-1)$.

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