Foundations of Data Science and Machine Learning – *Homework 3*

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Last compiled February 20, 2023

EXERCISE 2. Let $\mathbf{X} \in \mathbb{R}^{n \times d}$ be a matrix whose n rows are the data points $\mathbf{x}_1, ..., \mathbf{x}_n \in \mathbb{R}^d$, and let $\chi = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$. Consider the k-means optimization problem: find a partition $C_1, ..., C_k$ which minimizes, among all partitions of $[n]$ into k subsets,

EXERCISE 3. Find the mapping $\varphi(\mathbf{x})$ that gives rise to the polynomial kernel

$$K(\mathbf{x}, \mathbf{y}) = (x_1 x_2 + y_1 y_2)^2.$$

Proof: Consider the map $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined $\varphi(x_1, x_2) = (x_1^2, x_2^2, \sqrt{2} x_1 x_2)$. Interestingly, this is similar to the map one considers from a polynomial ring $R[x_1, x_2]$ to its 2^{nd} Veronese subring $R[x_1^2, x_1 x_2, x_2^2]$. We then have that

$$\begin{aligned} \varphi(\mathbf{x}) \cdot \varphi(\mathbf{y})^T &= (x_1^2, x_2^2, \sqrt{2} x_1 x_2) \cdot (y_1^2, y_2^2, \sqrt{2} y_1 y_2) \\ &= x_1^2 y_1^2 + x_2^2 y_2^2 + 2 x_1 y_1 x_2 y_2 \\ &= (x_1 y_1 + x_2 y_2)^2, \end{aligned}$$

hence φ gives rise to the desired kernel. $\square$