# Part III Commutative Algebra

Example Sheet IV, 2021, Solutions

1. Since localization preserves exact sequences, and $S^{-1}(A^n) = (S^{-1}A)^n$, we obtain an exact sequence

$$(S^{-1}A)^{n_1} \to (S^{-1}A)^{n_2} \to S^{-1}M \to 0.$$

Applying $\mathrm{Hom}_{S^{-1}A}(\cdot, S^{-1}N)$ to this gives an exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to (S^{-1}N)^{n_2} \to (S^{-1}N)^{n_1},$$

using the fact that for any ring $A$, $\mathrm{Hom}_A(A^n, N) = \prod_{i=1}^n N = N^n$. Note this is just the universal property for direct sum. [Note: this point fails for an infinite direct sum, as the infinite product and sum don't agree.] On the other hand, applying $\mathrm{Hom}_A(\cdot, N)$ to the exact sequence stated in the problem gives similarly

$$0 \to \mathrm{Hom}_A(M, N) \to N^{n_2} \to N^{n_1},$$

and we may again localize, giving

$$0 \to S^{-1}\mathrm{Hom}_A(M, N) \to (S^{-1}N)^{n_2} \to (S^{-1}N)^{n_1}.$$

By inspection, the two maps $(S^{-1}N)^{n_2} \to (S^{-1}N)^{n_1}$ agree (both given by the transpose of the matrix defining the map $A^{n_1} \to A^{n_2}$), and hence the two kernels are isomorphic.

2. (a) This is an argument from IA Numbers&Sets. Let $A$ be a UFD with field of fractions $K$, and suppose $p/q \in K$ with $p, q$ relatively prime. If $p/q$ is integral over $A$, then we have an equation

$$(p/q)^n + a_1(p/q)^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$. Multiplying by $q^n$ gives an equation

$$p^n = -(a_1 p^{n-1} q + \cdots + a_n q^n),$$

so that $p^n$ is divisible by $q$. But this is impossible if $p$ and $q$ are relatively prime unless $q$ is a unit in $A$. Thus $p/q \in A$.

(b) We first note $A$ is an integral domain, as $y^2 - x^3$ is easily seen to be prime. $A$ is not normal: note $y/x \in K$, the field of fractions of $A$, but the equation

$$u^2 - x = 0$$

is monic in $u$ and satisfied by $u = y/x$, hence an equation of integrality for $y/x$. Thus $A$ is (unsurprisingly, given the way the question was phrased) not normal.

Now define a map $\varphi : k[x, y] \to k[u]$ given by $x \mapsto u^2$, $y \mapsto u^3$. I claim $\ker\varphi = (x^2 - y^3)$, and hence $\varphi$ induces an inclusion of rings $A \subseteq k[u]$. Clearly $(x^2 - y^3) \subseteq \ker\varphi$, and $\ker\varphi$ is prime as $k[x, y]/\ker\varphi$ is a subring of $k[u]$, hence an integral domain. There are many ways to show then that $\ker\varphi = (y^2 - x^3)$, e.g., by explicit calculation. Here is a rather stupid way. Note that $\dim k[x, y] = 2$, so if $\ker\varphi \neq (y^2 - x^3)$, then $\ker\varphi$ is maximal and by the Hilbert Nullstellensatz, $k[x, y]/\ker\varphi$ is a finite field extension of $k$, of course contained in $k[u]$. However the image contains $u^2$, which is transcendental over $k$, which is impossible.

Next note that under this inclusion we obtain an isomorphism of fields of fractions, since $u$ may be written as the image of $y/x$ under the induced map $A_{(0)} \to k[u]_{(0)}$. Thus we have a chain of inclusions

$$A \subseteq k[u] \subseteq A_{(0)} = k(u).$$

Since $k[u]$ is integrally closed in $k(u)$ by (a), it follows that the integral closure of $A$ is contained in $k[u]$. On the other hand, $u$ is integral over $A$ by the earlier discussion, so the integral closure of $A$ agrees with $k[u]$.

3. Let $x \in B$ be integral over $A$, say satisfying

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

with $a_i \in A$, assumed to be a relation of minimal degree. If $x \notin A$, then $n \geq 2$ and

$$x(x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}) = -a_n \in A,$$

and under the assumption that $B \setminus A$ is multiplicatively closed, we see necessarily that $a' = x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1} \in A$. But then

$$x^{n-1} + a_1 x^{n-2} + \cdots + (a_{n-1} - a') = 0$$

is an equation of integrality for $x$ of one smaller degree, contradicting minimality of $n$.

4. (a) If $I \neq A$, then there exists a maximal ideal $\mathfrak{m} \supseteq I$. Since $k$ is algebraically closed, we know from lecture that $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. But then $Z(\mathfrak{m}) = \{(a_1, \ldots, a_n)\}$, and $I \subseteq \mathfrak{m}$ implies $Z(\mathfrak{m}) \subseteq Z(I)$, so $Z(I)$ is non-empty.

   (b) If $f^n \in I(Z(I))$, then $f \in I(Z(I))$ is immediate, and hence $\sqrt{I} \subseteq I(Z(I))$ without any hypotheses on $k$. Conversely, suppose that $f \notin \sqrt{I}$. Then there exists a prime ideal $\mathfrak{p} \subseteq A$ with $I \subseteq \mathfrak{p}$, $f \notin \mathfrak{p}$. Let $\bar{f}$ be the image of $f$ in $B = A/\mathfrak{p}$, and set $C = B_{\bar{f}} = B[\bar{f}^{-1}]$, the localization of $B$ at $\bar{f}$. Note this is a subring of the field of fractions of $A/\mathfrak{p}$, hence is non-zero. Let $\mathfrak{m}$ be a maximal ideal of $C$. Since $C$ is a finitely generated $k$-algebra, $C/\mathfrak{m}$ is a finite field extension of $k$, i.e., $C/\mathfrak{m} \cong k$. Let $a_1, \ldots, a_n$ be the images of $x_1, \ldots, x_n$ under the composition

$$A \to A/\mathfrak{p} \to C \to C/\mathfrak{m} \cong k.$$

   Tracing through these maps, the image of $f$ is $f(a_1, \ldots, a_n)$ and is necessarily non-zero since $\bar{f}$ is invertible in $C$. On the other hand, the kernel of this map is $\mathfrak{m}' = (x_1 - a_1, \ldots, x_n - a_n)$ as usual, with $\mathfrak{m}' \supseteq \mathfrak{p} \supseteq I$, so $\{(a_1, \ldots, a_n)\} = Z(\mathfrak{m}') \subseteq Z(I)$. Thus we see $f$ does not vanish at some point of $Z(I)$, i.e., $f \notin I(Z(I))$.

5. Let $\Sigma$ be the set of all local subrings of $K$: note this is non-empty since $K$ is local. Order $\Sigma$ by domination, and suppose $A_1 \leq A_2 \leq A_3 \cdots$ is a chain. Set $A = \bigcup_{i \geq 1} A_i$, $\mathfrak{m} = \bigcup_{i \geq 1} \mathfrak{m}_i$, where $\mathfrak{m}_i \subseteq A_i$ is the maximal ideal of $A_i$. Then $A$ is a ring and $\mathfrak{m}$ is an ideal. If $a \in A \setminus \mathfrak{m}$, then $a \in A_i$ for some $i$, but $a \notin \mathfrak{m}_i$. Thus $a$ is invertible in $A_i$, hence is invertible in $A$. Further, $\mathfrak{m} \neq A$, as it is immediate from the definition of domination that $\mathfrak{m} \cap A_i = \mathfrak{m}_i$. This shows $\mathfrak{m}$ is the unique maximal ideal of $A$. Thus the hypotheses of Zorn's lemma holds, and maximal elements exist.

   Now let $(A, \mathfrak{m}) \in \Sigma$ be maximal. Let $\Omega$ be the algebraic closure of $A/\mathfrak{m}$, and $g : A \to \Omega$ the obvious homomorphism with kernel $\mathfrak{m}$. Now consider the set $\Sigma'$ of pairs $(B, h)$ with $B$ a subring of $K$ and $h : B \to \Omega$, as in lecture, with ordering as in lecture. Then $\Sigma'$ is non-empty (containing $(A, g)$), and hence we may find a maximal element $(B, h)$ with $A \subseteq B$ and $h|_A = g$. Then $B$ is a valuation ring by the lectures, and $\ker h = \mathfrak{n}$ is the maximal ideal of $B$. Then $\mathfrak{m} = \ker g = \ker h|_A = (\ker h) \cap A = \mathfrak{n} \cap A$, so $(B, \mathfrak{n})$ dominates $(A, \mathfrak{m})$. By maximality of the latter in $\Sigma$, $A = B$ is a valuation ring.

   Conversely, suppose $A$ is a valuation ring of $K$. Then $A$ is local with maximal ideal $\mathfrak{m}$. Suppose $(B, \mathfrak{n}) \geq (A, \mathfrak{m})$. If $x \in B \setminus A$, then $x^{-1} \in A$ is a non-unit. Thus $x^{-1} \in \mathfrak{m} \subseteq \mathfrak{n}$, so $1 = x \cdot x^{-1} \in \mathfrak{n}$, contradiction. Thus $(A, \mathfrak{m})$ is a maximal element of $\Sigma$.

6. (a) We first note this notion is well-defined independently of representative $x, y$: changing the representative changes $xy^{-1}$ by an element of $U$. Thus in what follows, we denote elements of $\Gamma$ by a choice of representative. To check this is a total order we need to check: (i) $x \leq x$, which is clear since $xx^{-1} = 1 \in A$. (ii) $a \leq b$ and $b \leq c$ implies $a \leq c$. But if $ab^{-1} \in A$ and $bc^{-1} \in A$, then $(ab^{-1})(bc^{-1}) = ac^{-1} \in A$. (iii) If $a \leq b$ and $b \leq a$, then $a = b$. But this implies $ab^{-1}, a^{-1}b \in A$, hence $ab^{-1} \in U$, so $a$, $b$ represent the same element in $\Gamma$. (iv) $a \leq b$ or $b \leq a$. Indeed, since either $ab^{-1} \in A$ or $a^{-1}b \in A$ as $A$ is a valuation ring, we have one of the two inequalities holding.

   Finally it is immediate that the group structure is respected, as $(xz)(yz)^{-1} = xy^{-1}$.

   (b) Assume $v(x) \leq v(y)$, i.e., $yx^{-1} \in A$. Then $(x + y)x^{-1} = 1 + yx^{-1} \in A$, so $v(x + y) \geq v(x)$.

7. First, it is immediate that $A$ is closed under addition and multiplication from properties (b) and (a) resepctively. Similarly, $\mathfrak{m}$ is closed under addition by (b) and under multiplication by arbitrary elements of $A$ by (a). Thus $\mathfrak{m}$ is an ideal. If $a \in A \setminus \mathfrak{m}$, then $v(a) = 0$, and hence $v(a^{-1}) = 0$ so $a^{-1} \in A$ and $a$ is invertible. Thus $\mathfrak{m}$ is the unique maximal ideal. Finally, if $a \in K^*$, either $v(a) \geq 0$ and $a \in A$, or $v(a) < 0 = v(1)$ and $0 = v(1) = v(a)v(a^{-1}) < v(1)v(a^{-1}) = v(a^{-1})$, so $a^{-1} \in A$.

8. To see $A$ is an integral domain, suppose we have non-zero elements $a = \sum_{\gamma \in I} a_\gamma z^\gamma$, $b = \sum_{\gamma \in J} b_\gamma z^\gamma$ with $a_\gamma, b_\gamma$ all non-zero, and suppose $ab = 0$. Let $\gamma_a$ be the smallest element of $I$, $\gamma_b$ the smallest element of $J$. (These exist as the group $\Gamma$ is totally ordered and $I$, $J$ are finite sets). Then the coefficient of $z^{\gamma_a + \gamma_b}$ in $ab$ is $a_\gamma b_\gamma \neq 0$, contradicting $ab = 0$. Thus $A$ is an integral domain.

   Now we show that $v_0$ satisfies Conditions (a),(b). First, (a) is immediate from a similar analysis as above: taking $a, b$ as above, $v(a) = \gamma_a$, $v(b) = \gamma_b$, and $v(ab) = \gamma_a + \gamma_b$. For (b), again taking $a, b$ as above, we note that if wlog $\gamma_a < \gamma_b$, then the coefficient of $z^{\gamma_a}$ in $a + b$ is $a_{\gamma_a} \neq 0$, and this is the minimal exponent appearing. Thus $v(\gamma_a + \gamma_b) = \gamma_a$. If on the other hand $\gamma_a = \gamma_b$, the coefficient of $z^{\gamma_a}$ in $a + b$ is $a_{\gamma_a} + b_{\gamma_b}$, which may or may not be zero. In the former case $v(a + b) > v(a) = v(b)$ and in the latter case $v(a + b) = v(a) = v(b)$.

   We now define $v : K^* \to \Gamma$ by $v(a/b) = v_0(a)v_0(b)^{-1}$, for $a, b, \in A$, $b \neq 0$. This is well-defined as if $a_1/b_1 = a_2/b_2$, $a_1 b_2 = a_2 b_1$ and $v_0(a_1)v_0(b_2) = v_0(a_2)v_0(b_1)$ from which it follows that $v(a_1/b_1) = v(a_2/b_2)$. It is immediately a group homomorphism. Further

$$\begin{aligned} v\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) &= v\left(\frac{a_1 b_2 + a_2 b_1}{b_1 b_2}\right) = v_0(a_1 b_2 + a_2 b_1) v_0(b_1 b_2)^{-1} \\ \geq & \ (\min\{v_0(a_1 b_2), v_0(a_2 b_1)\}) v_0(b_1 b_2)^{-1} = \min\{v(a_1/b_1), v(a_2/b_2)\}, \end{aligned}$$

as desired

9. Let $a, b \in K^*$ with $1 + a + b = 0$. Now in order for this equality to hold, the lowest degree terms (with respect to the totally ordered group $\Gamma$) must cancel, i.e., write $a, b$ as in the solution to the previous question. Since $1 = 1z^0$, we have the following possibilities. (1) $\gamma_a = \gamma_b \leq 0$; (2) $\gamma_a = 0$, $\gamma_b \geq 0$; (3) $\gamma_b = 0$, $\gamma_a \geq 0$. Thus the image of the map is contained in

$$T = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha = \beta \leq 0 \text{ or } \alpha = 0, \beta \geq 0 \text{ or } \alpha \geq 0, \beta = 0\}.$$

If $k$ is not the field with two elements, then the map is surjective onto $T$, otherwise it misses the origin. For example, for any $\alpha = \beta \leq 0$, we may take $a = z^\alpha$, $b = -z^\alpha - 1$, so $1 + a + b = 0$ and $(v(a), v(b)) = (\alpha, \alpha)$. However, if $\alpha = 0$ and the characteristic is 2, then $b = 0$, which isn't allowed. If $\alpha = 0, \beta \geq 0$, we may take $a = -1 + z^\beta$, $b = -z^\beta$, and similarly reversing the role of $\alpha$ and $\beta$. So the image of the given map is $T$ if the characteristic isn't 2. If $\operatorname{char} k = 2$ and $k$ has at least four elements, then take $a \in k \setminus \{0, 1\}$, $b = 1 + a$. However, if $k$ is $\mathbb{F}_2$, then we can't find non-zero $a, b$ with $1 + a + b = 0$ and $v(a) = v(b) = 0$.

10. Suppose $A$ is not a field, and let $K \neq A$ be the field of fractions of $A$. With $U$ the group of invertible elements of $A$, we have $v : K^* \to \Gamma = K^*/U$ as in Q6. Now suppose that the set $V := \{\gamma \in \Gamma \mid \gamma > 0\}$ does not have a minimal element. Thus we have an infinite sequence $\gamma_1 > \gamma_2 > \cdots > 0$. Define ideals

$$I_n = \{a \in A \setminus \{0\} \mid v(a) \geq \gamma_n\} \cup \{0\}.$$

This is immediately seen to be an ideal, and clearly

$$I_1 \subsetneq I_2 \subsetneq \cdots$$

because the value map $v$ is surjective. Thus $A$ is not Noetherian.

So if $A$ is Noetherian, the set $V$ has a minimal element, say $\gamma$. Then in fact $\gamma$ is a generator of $\Gamma$: if $\gamma' > 0$, then $\gamma' - n\gamma$, $n = 0, 1, 2, \ldots$ forms a decreasing sequence, and thus eventually $\gamma' - n\gamma \leq 0$. But if $\gamma' - n\gamma < 0$ and $\gamma' - (n-1)\gamma > 0$, we see from the first inequality that $\gamma' - (n-1)\gamma < \gamma$, contradicting minimality of $\gamma$. Thus $\gamma' = n\gamma$ for some $n$, and hence $\gamma$ generates $\Gamma$. Thus $\Gamma \cong \mathbb{Z}$.

11. We show $M_1$ is projective. Note that if $N$ is an $A$-module, a homomorphism $h : M_1 \to N$ is completely determined by $h(1, 0)$, since $h(a_1, 0) = (a_1, 0)h(1, 0)$. Further, this map is well-defined. Thus given a surjective map $g : M \to N$ and $h : M_1 \to N$, we lift to $h' : M_1 \to M$ by choosing a lift $m \in M$ of $h(1, 0)$, and then defining $h'$ by $h'(1, 0) = m$.

12. A resolution is

$$\cdots \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{(x,y)} A \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

where $d_1$ is given by the matrix $\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$ and $d_2$ is given by the matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We just need to check exactness. It is obviously exact at the $A/\mathfrak{m}$ and $A$ terms. Next consider exactness at the right-most $A^2$ term. Since $xy = 0$ in $A$, $\operatorname{im} d_1 \subseteq \ker(x, y)$. Now suppose $(f, g) \in \ker(x, y)$. Then $xf + yg = 0$ in $A$, so any representative of this expression in $k[x, y]$ must be divisible by $xy$. From this it immediately follows that any representative of $f$ must be divisble by $y$ and any representative for $g$ must be divisible by $x$. Then $(f, g) \in \operatorname{im} d_1$.

The exactness at any other place holds since the kernel of multiplication by $x$ is the ideal $(y)$ and the kernel of multiplication by $y$ is the ideal $(x)$.

To compute $\operatorname{Ext}^i_A(A/\mathfrak{m}, A)$, we apply $\operatorname{Hom}_A(\cdot, A)$ to the above resolution, giving a complex

$$A \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} \cdots$$

(Note we simply transpose each map, but $d_1, d_2$ come from symmetric matrices). The kernel of $\begin{pmatrix} x \\ y \end{pmatrix}$ is $(x) \cap (y) = (xy) = 0$, so $\operatorname{Hom}_A(A/\mathfrak{m}, A) = \operatorname{Ext}^0_A(A/\mathfrak{m}, A) = 0$. On the other hand, $\ker d_1$ is $(x) \oplus (y)$, so

$$\operatorname{Ext}^1_A(A/\mathfrak{m}, A) = \frac{\{(xf, yg) \in A^2 \mid f, g \in A\}}{\{(xf, yf) \mid f \in A\}}.$$

It is not difficult to see that this is isomorphic to $k$ via the map which take $(xf, yg) \mapsto f(0) - g(0)$.

For the calculation of $\mathrm{Tor}_i^A(A/\mathfrak{m}, A/\mathfrak{m})$, we tensor the resolution with $A/\mathfrak{m} = k$, giving

$$\cdots \to k^2 \to k^2 \to k^2 \to k$$

with all maps being zero. Thus

$$\mathrm{Tor}_i^A(A/\mathfrak{m}, A/\mathfrak{m}) = \begin{cases} k & i = 0 \\ k^2 & i > 0 \end{cases}$$

13. As $S_\mathfrak{m}$ is an integral domain, $a_1 = x^3 \in \mathfrak{m}$ is a non-zero divisor. Take $a_2 = y^3 \in \mathfrak{m}$. We need to verify that $a_2$ is not a zero-divisor in $S' = S_\mathfrak{m}/(a_1)$. So suppose that $(a/b) \cdot y^3 = 0$ in $S'$. Then we can write

$$\frac{a}{b}y^3 = \frac{c}{d}x^3$$

in $S_\mathfrak{m}$, $b, d \notin \mathfrak{m}$, or equivalently,

$$(ad)y^3 = (bc)x^3$$

in $S$. Viewing $S$ as a subring of $k[x, y]$, note this implies that $x^3|(ad)$. Now if $x|d$, then $d \in S \cap (x, y) \subseteq k[x, y]$, and $\mathfrak{m} = S \cap (x, y)$, so $d \in \mathfrak{m}$. This contradicts $d \notin \mathfrak{m}$. Thus $x^3|a$, and thus $a/b = 0$ in $S'$ already. Thus $y^3$ is not a zero-divisor in $S'$ and $a_1, a_2$ is a regular sequence. Since $\dim S_\mathfrak{m} = 2$, we see $S_\mathfrak{m}$ is Cohen-Macaulay.

14. (a) $\Rightarrow$: We have an exact sequence

$$M \to N \to C \to 0$$

where $C$ is the cokernel of $f$. Tensoring with $k$ preserves exactness, so $C \otimes_A k = 0$ if $f$ is minimal. Thus by Nakayama's lemma, $C = 0$ so $f$ is surjective. Now let $x \in \ker f$. Then $x$ represents an element $\bar{x} \in M/\mathfrak{m}M = M \otimes_A k$, and thus $\bar{x}$ maps to zero under $f \otimes \mathrm{id}$. But again since $f$ is minimal, this implies $\bar{x} = 0$, so $x \in \mathfrak{m}M$.

$\Leftarrow$: Surjectivity of $f$ implies surjectivity of $f \otimes \mathrm{id}$. Next let $x \in M$ represent an element of $\ker f \otimes \mathrm{id}$. Then $f(x) \in \mathfrak{m}N$, i.e., $f(x) = \sum a_i x_i$ for some $a_i \in \mathfrak{m}$, $x_i \in N$. Using surjectivity of $f$, choose $y_i \in M$ with $f(y_i) = x_i$. Then $f(x - \sum a_i y_i) = 0$, and thus $x - \sum a_i y_i \in \mathfrak{m}M$ by assumption. Thus $x \in \mathfrak{m}M$, and so $x$ represents zero in $M/\mathfrak{m}M$.

(b) Let $\bar{x}_1, \ldots, \bar{x}_n$ be a basis for $M \otimes_A k$ as a $k$-vector space, and let $x_1, \ldots, x_n$ be lifts to $M$. Then by Nakayama's lemma, $x_1, \ldots, x_n$ generate $M$, and we obtain a surjective map $f : F = A^n \to M$ taking the $i^{th}$ standard basis vector to $x_i$. Further, if $f(a_1, \ldots, a_n) = 0$, we have $\sum a_i x_i = 0$, so $\sum a_i \bar{x}_i = 0$, so by linear independence of the $\bar{x}_i$, $a_i \in \mathfrak{m}$ for all $i$. Thus the conditions of (a) are satisfied, so $f$ is minimal.

(c) Note by minimality of $g$, the matrix $(c_{ij})$ determining $f$ must satisfy $c_{ij} \in \mathfrak{m}$, by (a). Then the map $\mathrm{Ext}_A^i(k, A) \to \mathrm{Ext}_A^i(k, A)$ given by multplication $c_{ij}$ is the map induced by multiplication by $c_{ij}$ on $k$, but this is zero. (We used a similar, admittedly hand-wavy, argument in lecture). Thus in particular $(c_{ij})$ defines the zero map $\mathrm{Ext}_A^i(k, K) \to \mathrm{Ext}^i(k, F)$.

15. By Q14, (b), we may find a surjective minimal map from a free module $\epsilon : F_0 \to M$. Because $A$ is Noetherian, $\ker \epsilon$ is finitely generated, and we can again find a surjective minimal map from a free module $F_1 \to \ker \epsilon$, and take $d_1 : F_1 \to F_0$ to be the composition $F_1 \to \ker \epsilon \hookrightarrow F_0$. We repeat, just as in the construction of free resolutions, but at each step choose a minimal surjective map. In particular, $d_i$ is a composition

$$F_i \to \ker(d_{i-1}) \hookrightarrow F_{i-1}.$$

Now I claim that $d_i \otimes \mathrm{id} = 0$. Indeed, by Q14, (a), $\ker(d_{i-1}) \subseteq \mathfrak{m}F_{i-1}$. Thus given $\sum_j f_j \otimes a_j \in F_i \otimes_A k$, with $f_j \in F_i$, $a_j \in k$, we have $d_i(f_j) = \sum a_{jk} f'_k$ with $a_{jk} \in \mathfrak{m}$ and $f'_k \in F_{i-1}$ as $d_i(f_j) \in \ker(d_{i-1}) \subseteq \mathfrak{m}F_{i-1}$. Then

$$(d_i \otimes \mathrm{id})(\sum_j f_j \otimes a_j) = \sum_{j,k} a_{jk} f'_k \otimes a_j = \sum_{j,k} f'_k \otimes a_{jk} a_j = 0$$

since $a_{jk} \in \mathfrak{m}$. Thus we have constructed a minimal resolution.

16. We may compute $\mathrm{Tor}_i^A(k, M)$ using any free resolution of $M$, so in particular if we calculate it using a resolution of length $n = \mathrm{pd}_A M$, we see $\mathrm{Tor}_i^A(k, M) = 0$ for $i > n$.

Now suppose given

$$0 \to F_{n'} \to \cdots \to F_0 \to M \to 0$$

a free resolution of length $n'$. If $i$ is the smallest integer for which $\mathrm{Tor}_{i+1}^A(k, M) = 0$, then we have

$$n' \geq n = \mathrm{pd}_A M \geq i.$$

But if this resolution is minimal, we calculate $\mathrm{Tor}_i^A(k, M)$ as the homology of the complex

$$\cdots \to F_2 \otimes_A k \to F_1 \otimes_A k \to F_0 \otimes_A k.$$

However, because the resolution is minimal, all maps of this complex are zero and hence $\mathrm{Tor}^A_{j+1}(k, M) = F_{j+1} \otimes_A k$ for all $j$. This is zero if and only if $F_{j+1} = 0$, showing $i \geq n' \geq n$. Thus $i = n$, and $n = n'$ if the resolution is minimal.

17. If $\mathrm{pd}_A M = 0$, then $M$ is free, as there is a resolution $0 \to F_0 \to M \to 0$ with $F_0$ free and isomorphic to $M$. Now $\operatorname{depth} A^n = \operatorname{depth} A$ is immediate from the definition of depth, so we get the desired equality.

Now suppose $\mathrm{pd}_A M = 1$, so we have by Q16 a minimal resolution

$$0 \longrightarrow F_1 \xrightarrow{f} F_0 \xrightarrow{g} M \longrightarrow 0.$$

After tensoring with $k$, we see the map $g$ becomes an isomorphism as $f \otimes \mathrm{id} = 0$. Thus $g$ is minimal, and we obtain that the induced maps $f_* : \mathrm{Ext}^i_A(k, F_1) \to \mathrm{Ext}^i_A(k, F_0)$ vanish by Q14(c). Thus the long exact sequence

$$\cdots \to \mathrm{Ext}^{i-1}_A(k, M) \to \mathrm{Ext}^i_A(k, F_1) \to \mathrm{Ext}^i_A(k, F_0) \to \mathrm{Ext}^i_A(k, M) \to \mathrm{Ext}^{i+1}_A(k, F_0) \to \cdots$$

breaks up into a collection of short exact sequences

$$0 \to \mathrm{Ext}^{i-1}_A(k, F_0) \to \mathrm{Ext}^{i-1}(k, M) \to \mathrm{Ext}^i_A(k, F_1) \to 0.$$

Now we know from the lecture the characterization that $\operatorname{depth} M$ is the smallest $i$ such that $\mathrm{Ext}^i_A(k, M) \neq 0$. However from the above exact sequences, it then follows that $\operatorname{depth} M = \operatorname{depth} F_1 - 1 = \operatorname{depth} A - 1$ as before.

We now go by induction. Suppose $\mathrm{pd}_A M = n$, and consider a minimal free resolution $F_\bullet \to M$. Let $M' = \ker(F_0 \to M)$, so that we also have a minimal free resolution

$$0 \to F_n \to \cdots \to F_1 \to M' \to 0.$$

Then $\mathrm{pd}_A M' = n - 1$ by Q16, and hence by the induction assumption $\operatorname{depth} M' = \operatorname{depth} A - n + 1$. Note in particular that $\operatorname{depth} M' < \operatorname{depth} A$ as $n \geq 2$. On the other hand, we also a short exact sequence

$$0 \to M' \to F_0 \to M \to 0$$

giving a long exact sequence

$$\cdots \to \mathrm{Ext}^{i-1}_A(k, F_0) \to \mathrm{Ext}^{i-1}_A(k, M) \to \mathrm{Ext}^i_A(k, M') \to \mathrm{Ext}^i_A(k, F_0) \to \cdots$$

Again using the characterization of depth, we then know that $\mathrm{Ext}^i_A(k, M') = 0$ for $i < \operatorname{depth} M'$, and also $\mathrm{Ext}^i_A(k, A) = 0$ for $i$ in the same range, so $\mathrm{Ext}^{i-1}_A(k, M) = 0$ for $i < \operatorname{depth} M'$. On the other hand, if $i = \operatorname{depth} M'$, we see $\mathrm{Ext}^{i-1}_A(k, M) \neq 0$. Thus $\operatorname{depth} M = \operatorname{depth} M' - 1$. So

$$\operatorname{depth} A = \mathrm{pd}_A M' + \operatorname{depth} M' = \mathrm{pd}_A M - 1 + \operatorname{depth} M + 1 = \mathrm{pd}_A M + \operatorname{depth} M,$$

completing the induction step.