Chapter 2

First examples

Keynote Questions

(a) Let $F_0, F_1, F_2 \in \Bbbk[X, Y, Z]$ be three general homogeneous cubic polynomials in three variables. Up to scalars, how many linear combinations $t_0 F_0 + t_1 F_1 + t_2 F_2$ factor as a product of a linear and a quadratic polynomial? (Answer on page 65.)

(b) Let $F_0, F_1, F_2, F_3 \in \Bbbk[X, Y, Z]$ be four general homogeneous cubic polynomials in three variables. How many linear combinations $t_0 F_0 + t_1 F_1 + t_2 F_2 + t_3 F_3$ factor as a product of three linear polynomials? (Answer on page 65.)

(c) Let A, B, C be general homogeneous polynomials of degree d in three variables. Up to scalars, for how many triples $t = (t_0, t_1, t_2) \neq (0, 0, 0)$ is $(A(t), B(t), C(t))$ a scalar multiple of (t_0, t_1, t_2)? (Answer on page 55.)

(d) Let S_d denote the space of homogeneous polynomials of degree d in two variables. If $V \subset S_d$ and $W \subset S_e$ are general linear spaces of dimensions a and b with $a + b = d + 2$, how many pairs $(f, g) \in V \times W$ are there (up to multiplication of each of f and g by scalars) such that $f \mid g$? (Answer on page 56.)

(e) Let $S \subset \mathbb{P}^3$ be a smooth cubic surface and $L \subset \mathbb{P}^3$ a general line. How many planes containing L are tangent to S? (Answer on page 50.)

(f) Let $L \subset \mathbb{P}^3$ be a line, and let S and $T \subset \mathbb{P}^3$ be surfaces of degrees s and t containing L. Suppose that the intersection $S \cap T$ is the union of L and a smooth curve C. What are the degree and genus of C? (Answer on page 71.)

In this chapter we illustrate the general theory introduced in the preceding chapter with a series of examples and applications.

The first section is a series of progressively more interesting computations of Chow rings of familiar varieties, with easy applications. Following this, in Section 2.2 we see an example of a different kind: We use facts about the Chow ring to describe some geometrically interesting loci in the projective space of cubic plane curves.

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Finally, in Section 2.4 we briefly describe intersection theory on surfaces, a setting in which the theory takes a particularly simple and useful form. As one application, we describe in Section 2.4.3 the notion of *linkage*, a tool used classically to understand the geometry of curves in $\mathbb{P}^3$.

2.1 The Chow rings of $\mathbb{P}^n$ and some related varieties

So far we have not seen any concrete examples of the intersection product or pullback. The first interesting case where this occurs is projective space.

Theorem 2.1. *The Chow ring of* $\mathbb{P}^n$ *is*

$$A(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1}),$$

where $\zeta \in A^1(\mathbb{P}^n)$ *is the rational equivalence class of a hyperplane; more generally, the class of a variety of codimension* k *and degree* d *is* $d\zeta^k$.

In particular, the theorem implies that $A^m(\mathbb{P}^n) \cong \mathbb{Z}$ for $0 \le m \le n$, generated by the class of an $(n-m)$-plane. The natural proof, given below, uses the intersection product.

Proof: Let $\{p\} \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ be a complete flag of subspaces. Applying Proposition 1.17 to the affine stratification with strata $U_i = \mathbb{P}^i \setminus \mathbb{P}^{i-1}$, we see that $A^k(\mathbb{P}^n)$ is generated by the class of $\mathbb{P}^{n-k}$, and thus by the class of any $(n-k)$-plane $L \subset \mathbb{P}^n$. Using Proposition 1.21, we get $A^n(\mathbb{P}^n) = \mathbb{Z}$. Since a general $(n-k)$-plane L intersects a general k-plane M transversely in one point, multiplication by $[M]$ induces a surjective map $A^k(\mathbb{P}^n) \to A^n(\mathbb{P}^n) = \mathbb{Z}$, so $A^k(\mathbb{P}^n) = \mathbb{Z}$ for all k.

An $(n-k)$-plane $L \subset \mathbb{P}^n$ is the transverse intersection of k hyperplanes, so

$$[L] = \zeta^k,$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the class of a hyperplane. Finally, since a subvariety $X \subset \mathbb{P}^n$ of dimension $n-k$ and degree d intersects a general k-plane transversely in d points, we have $\deg([X]\zeta^{n-k}) = d$. Since $\deg(\zeta^n) = 1$, we conclude that $[X] = d\zeta^k$. □

Here are two interesting qualitative results that follow from Theorem 2.1:

Corollary 2.2. *A morphism from* $\mathbb{P}^n$ *to a quasi-projective variety of dimension strictly less than* n *is constant.*

Proof: Let $\varphi : \mathbb{P}^n \to X \subset \mathbb{P}^m$ be the map, which we may assume is surjective onto X. The preimage of a general hyperplane section of X is disjoint from the preimage of a general point of X. But if $0 < \dim X < n$ then the preimage of a hyperplane section of X has dimension $n-1$ and the preimage of a point has dimension > 0. Since any two such subvarieties of $\mathbb{P}^n$ must meet, this is a contradiction. □

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Corollary 2.3. *If $X \subset \mathbb{P}^n$ is a variety of dimension m and degree d then $A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d)$, while if $m < m' \leq n$ then $A_{m'}(\mathbb{P}^n \setminus X) = \mathbb{Z}$. In particular, m and d are determined by the isomorphism class of $\mathbb{P}^n \setminus X$.*

Proof: Part (b) of Proposition 1.14 shows that there are exact sequences $A_i(X) \to A_i(\mathbb{P}^n) \to A_i(\mathbb{P}^n \setminus X) \to 0$. Furthermore $A_m(X) = \mathbb{Z}$ by part (b) of Proposition 1.8, while $A_{m'}(X) = 0$ for $m < m' \leq n$. By Theorem 2.1, we have $A_i(\mathbb{P}^n) = \mathbb{Z}$ for $0 \leq i \leq n$, and the image of the generator of $A_m(X)$ in $A_m(\mathbb{P}^n)$ is d times the generator of $A_i(\mathbb{P}^n)$. The results of the corollary follow. □

Theorem 2.1 implies the analog of Poincaré duality for $A(\mathbb{P}^n)$: $A_k(\mathbb{P}^n)$ is dual to $A^k(\mathbb{P}^n)$ via the intersection product. The reader should be aware that in cases where the Chow groups and the homology groups are different, Poincaré duality generally does *not* hold for the Chow ring; for example, when X is a variety, $A_{\dim X}(X) \cong \mathbb{Z}$, but $A_0(X)$ need not even be finitely generated.

One aspect of Theorem 2.1 may, upon reflection, seem strange: why is it that only the dimension and degree of a variety $X \subset \mathbb{P}^n$ are preserved under rational equivalence, and not other quantities such as (in the case of X a curve) the arithmetic genus?

First of all, to understand why this may appear curious, we recall from Eisenbud and Harris [2000, Proposition III-56] (see also Corollary B.12) that, if B is reduced and connected, then a closed subscheme $\mathcal{Y} \subset B \times \mathbb{P}^n$ is flat over B if and only if the fibers all have the same Hilbert polynomial. Thus, for example, if $Z \subset \mathbb{P}^1 \times \mathbb{P}^n$ is an irreducible surface dominating $\mathbb{P}^1$, then the fibers Z_0 and Z_∞ will be one-dimensional subschemes of $\mathbb{P}^n$ having not only the same degree, but also the same arithmetic genus. Why does this not contradict the assertion of Theorem 2.1 that curves C and $C' \subset \mathbb{P}^3$ of the same degree d but different genera *are* rationally equivalent?

The explanation is that both can be deformed, in families parametrized by $\mathbb{P}^1$, to schemes C_0, C'_0 supported on a line $L \subset \mathbb{P}^3$ and having multiplicity d, so that $\langle C \rangle \sim \langle C_0 \rangle = d\langle L \rangle$ as cycles, and likewise for C'. The difference in the genera of C and C' will be reflected in two things: the scheme structure along the line in the flat limits C_0 and C'_0, and the presence and multiplicity of embedded points in these limits.

For an example of the former, note that the schemes $C_0 = V((x, y)^2)$ and $C'_0 = V(x, y^3)$ are both supported on the line $L = V(x, y)$, and both have multiplicity 3, but the arithmetic genus of C_0 is 0, while that of C'_0 is 1 (after all, it is a plane cubic!). But the mechanism by which we associate a cycle to a scheme does not see the difference in the scheme structure; we have $\langle C_0 \rangle = \langle C'_0 \rangle = 3\langle L \rangle$. Similarly, a twisted cubic curve $C \subset \mathbb{P}^3$ can be deformed to a scheme generically isomorphic to either C_0 or C'_0; the difference in the arithmetic genus is accounted for by the fact that in the latter case the limiting scheme will necessarily have an embedded point. But again, rational equivalence does not "see" the embedded point; we have $[C] = 3[L]$ regardless.

2.1.1 Bézout's theorem

As an immediate consequence of Theorem 2.1, we get a general form of Bézout's theorem:

Corollary 2.4 (Bézout's theorem). *If $X_1, \dots, X_k \subset \mathbb{P}^n$ are subvarieties of codimensions $c_1, \dots, c_k$, with $\sum c_i \leq n$, and the X_i intersect generically transversely, then*

$$\deg(X_1 \cap \dots \cap X_k) = \prod \deg(X_i).$$

In particular, two subvarieties $X, Y \subset \mathbb{P}^n$ having complementary dimension and intersecting transversely will intersect in exactly $\deg(X) \cdot \deg(Y)$ *points.*

Using multiplicities we can extend this formula to the more general case where we assume only that the varieties intersect dimensionally transversely (that is, all components of the intersection $Z = \bigcap X_i$ have codimension equal to $\sum c_i$), as long as the X_i are generically Cohen–Macaulay along each component of their intersection. In this case, the intersection multiplicity $m_{Z_\alpha}(X_1, \dots, X_k)$ of the X_i along a component Z_α of their intersection, as described in Section 1.3.7, is equal to the multiplicity of the scheme Z at a general point of Z_α.

Corollary 2.5. *Suppose $X_1, \dots, X_k \subset \mathbb{P}^n$ are subvarieties of codimensions $c_1, \dots, c_k$ whose intersection is a scheme Z of pure dimension $n - \sum c_i$, with irreducible components $Z_1, \dots, Z_t$. If the X_i are Cohen–Macaulay at a general point of each Z_α, then*

$$[Z] = \sum [Z_j] = \prod [X_i];$$

equivalently,

$$\deg Z = \sum \deg Z_j = \prod \deg X_i.$$

Note that by the degree of a subscheme $Z \subset \mathbb{P}^n$ of dimension m we mean $m!$ times the leading coefficient of the Hilbert polynomial; in case Z is irreducible this will be equal to the degree of the reduced scheme Z_{red} times the multiplicity of the scheme, and more generally it will be given by

$$\deg(Z) = \sum \mathrm{mult}_{Z_i}(Z) \deg(Z_{\mathrm{red}}),$$

where the Z_i are the irreducible components of Z of maximal dimension m.

The Cohen–Macaulay hypothesis is satisfied if, for example, the X_i are all hypersurfaces; thus the classical case of two curves intersecting in $\mathbb{P}^2$ is covered.

There is a standard example that shows that the Cohen–Macaulay hypothesis is necessary:

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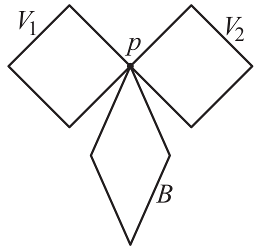

Figure 2.1 Let $A = V_1 \cup V_2 \subset \mathbb{P}^4$, where the V_i are general 2-planes, and let B be a 2-plane passing through the point $V_1 \cap V_2$. The degree of the product $[B][A]$ in $A(\mathbb{P}^4)$ is 2, as one sees by moving B to a plane B' transverse to A, but the length of the local ring of $B \cap A$ is 3.

Example 2.6. Let $X = \mathbb{P}^4$, let $V_1, V_2 \subset \mathbb{P}^4$ be general 2-planes and let $A = V_1 \cup V_2$.

Since V_1 and V_2 are general, they meet in a single point p. Let B be a 2-plane that passes through p and does not meet A anywhere else, and let B' be a 2-plane that does not pass through p and meets each of V_1, V_2 in a single (necessarily reduced) point. The cycles $\langle B \rangle$ and $\langle B' \rangle$ are rationally equivalent in $\mathbb{P}^4$. The intersection $B' \cap A$ consists of two reduced points, so $\deg(B' \cap A) = 2$ (see Figure 2.1).

However, the degree of the scheme $B \cap A$ is strictly greater than 2: Since the Zariski tangent space to the scheme $A = V_1 \cup V_2$ at the point p is all of $T_p(\mathbb{P}^4)$, the tangent space to the intersection $B \cap A$ at p must be all of $T_p(B)$. In other words, $B \cap A$ must contain the "fat point" at p in the plane B (that is, the scheme defined by the square of the ideal of p in B), and so must have degree at least 3.

In fact, we can see that the degree of the scheme $B \cap A$ is equal to 3 by a local calculation, as follows. Since B meets A only at the point p, we have to show that the length of the Artinian ring $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(B) + \mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ is 3. Let $S = \Bbbk[x_0, \dots, x_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4$. We may choose V_1, V_2 and B to have homogeneous ideals

$$I(A) = (x_0, x_1) \cap (x_2, x_3) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3),$$
$$I(B) = (x_0 - x_2, x_1 - x_3).$$

Modulo $I(B)$, we can eliminate the variables x_2 and x_3 and the ideal $I(A)$ becomes (x_0^2, x_0x_1, x_1^2). Passing to the affine open subset where $x_4 \neq 0$, this is the square of the maximal ideal corresponding to the origin in B. Therefore $\mathcal{O}_{\mathbb{P}^4,p}/(\mathcal{I}(B) + \mathcal{I}(A))\mathcal{O}_{\mathbb{P}^4,p}$ has basis $\{1, x_0/x_4, x_1/x_4\}$, and hence its length is 3.

Given that we sometimes have $[A \cap B] \neq [A][B]$, it is natural to look for a correction term. In the example above, the set-theoretic intersection is a point, so this comes down to looking for a formula that will predict the difference in multiplicities $3-2 = 1$. Of course the correction term should reflect nontransversality, and one measure of nontransversality is the quotient $I(A) \cap I(B)/(I(A) \cdot I(B))$. In the case above one can compute this, finding that the quotient is a finite-dimensional vector space of length 1 — just the correction term we need. Now for any pair of ideals I, J in any ring R, the quotient $(I \cap J)/(I \cdot J)$ is isomorphic to $\mathrm{Tor}_1^R(R/I, R/J)$ (see Eisenbud [1995, Exercise A3.17]). With this information, knowing a special case proven earlier by Auslander and Buchsbaum, Serre [2000] produced a general formula (originally published in 1957):

Theorem 2.7 (Serre's formula)**.** *Suppose that $A, B \subset X$ are dimensionally transverse subschemes of a smooth scheme X and Z is an irreducible component of $A \cap B$. The intersection multiplicity of A and B along Z is*

$$m_Z(A, B) = \sum_{i=0}^{\dim X} (-1)^i \operatorname{length}_{\mathcal{O}_{A\cap B,Z}}\left(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z})\right).$$

The first term of the alternating sum in Serre's formula is

$$\operatorname{length}_{\mathcal{O}_{A\cap B,Z}} \mathrm{Tor}_0^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z}) = \operatorname{length}_{\mathcal{O}_{A\cap B,Z}} \mathcal{O}_{X,Z}/(\mathcal{I}_A + \mathcal{I}_B),$$

which is precisely the multiplicity of Z in the subscheme $A \cap B$; the remaining terms, involving higher Tors, are zero in the Cohen–Macaulay case and may be viewed as correction terms. We note that this formula is used relatively rarely in practice, since there are many alternatives, such as the one given by Fulton [1984, Chapter 7].

2.1.2 Degrees of Veronese varieties

Let

$$\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N, \quad \text{with } N = \binom{n+d}{n} - 1,$$

be the *Veronese map*

$$[Z_0, \dots, Z_n] \mapsto [\dots, Z^I, \dots],$$

where Z^I ranges over all monomials of degree d in $n+1$ variables. The image $\Phi = \Phi_{n,d} \subset \mathbb{P}^N$ of the Veronese map $\nu = \nu_{n,d}$ is called the *d-th Veronese variety of $\mathbb{P}^n$*, as is any subvariety of $\mathbb{P}^N$ projectively equivalent to it. This variety may be characterized, up to automorphisms of the target $\mathbb{P}^N$, as the image of the map associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$; in other words, by the property that the preimages $\nu^{-1}(H) \subset \mathbb{P}^n$ of hyperplanes $H \subset \mathbb{P}^N$ comprise all hypersurfaces of degree d in $\mathbb{P}^n$.

There is another attractive description, at least in characteristic 0: writing $\mathbb{P}^n = \mathbb{P}V$, where V is an $(n+1)$-dimensional vector space, $\nu_{n,d}$ is projectively equivalent to the map taking $\mathbb{P}V \to \mathbb{P}\operatorname{Sym}^d V$ by $[v] \mapsto [v^d]$; for if the coordinates of v are $v_0, \dots, v_n$ then the coordinates of v^d are

$$\frac{d!}{\prod_i d_i!}(v_0^{d_0} \cdots v_n^{d_n}).$$

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If the characteristic is 0 then the coefficients are nonzero, so we may rescale by an automorphism of $\mathbb{P}^N$ to get the standard Veronese map above.

We can use Corollary 2.4 to compute the degrees of Veronese varieties:

Proposition 2.8. *The degree of* $\Phi_{n,d}$ *is* d^n.

Proof: The degree of Φ is the cardinality of its intersection with n general hyperplanes $H_1, \dots, H_n \subset \mathbb{P}^N$; since the map ν is one-to-one, this is in turn the cardinality of the intersection $f^{-1}(H_1) \cap \dots \cap f^{-1}(H_n) \subset \mathbb{P}^n$. The preimages of the hyperplanes H_i are n general hypersurfaces of degree d in $\mathbb{P}^n$. By Bézout's theorem, the cardinality of their intersection is d^n. $\square$

2.1.3 Degree of the dual of a hypersurface

The same idea allows us to compute the degree of the *dual variety* of a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d, that is, the set of points $X^* \subset \mathbb{P}^{n*}$ corresponding to hyperplanes of $\mathbb{P}^n$ that are tangent to X. (In Chapter 10 we will generalize this notion substantially, discussing the duals of varieties of higher codimension and singular varieties as well.)

The set X^* is a variety because it is the image of X under the *Gauss map* $\mathcal{G}_X : X \to \mathbb{P}^{n*}$, a morphism that sends a point $p \in X$ to its tangent hyperplane $\mathbb{T}_p X$; in coordinates, if X is the zero locus of the homogeneous polynomial $F(Z_0, \dots, Z_n)$, then $\mathcal{G}_X$ is given by the formula

$$\mathcal{G}_X : p \mapsto \left[\frac{\partial F}{\partial Z_0}(p), \dots, \frac{\partial F}{\partial Z_n}(p)\right].$$

To see that this map is well-defined, note first that, since X is smooth, the partials of F have no common zeros on X (and this implies, by Euler's relation, that they do not have any common zeros in $\mathbb{P}^n$). Thus $\mathcal{G}_X$ defines a morphism $\mathbb{P}^n \to \mathbb{P}^{n*}$. When $p \in X$, Euler's relation shows that the vector $\mathcal{G}_X(p)$ is orthogonal to the vector $\tilde{p}$ representing the point p; thus the linear functional represented by $\mathcal{G}_X(p)$ induces a functional on the tangent space to $\mathbb{P}^n$, and the zero locus of this functional is the tangent space to X at p.

If $d = 1$, the map $\mathcal{G}_X$ is constant and X^* is a point. But if $d > 1$, then the fact that the partials of F have no common zeros says that the map $\mathcal{G}_X$ is finite: If $\mathcal{G}_X$ were constant on a complete curve $C \subset X$, the restrictions to C of the partials of F would be scalar multiples of each other, and so would have a common zero.

In particular, if $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d \geq 2$, the dual variety $X^* \subset \mathbb{P}^{n*}$ is again a hypersurface, though not usually smooth. The smoothness hypothesis is necessary here; for example, the dual Q^* of the quadric cone $Q = V(XZ - Y^2) \subset \mathbb{P}^3$ is a conic curve in $\mathbb{P}^{3*}$.

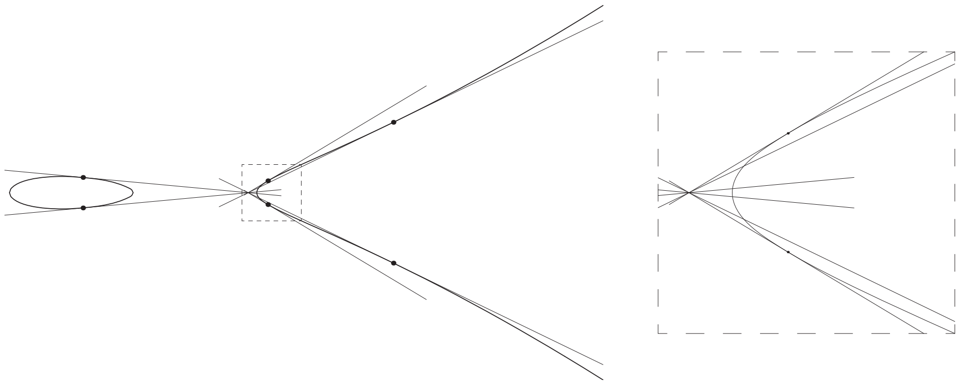

Figure 2.2 Six of the lines through a general point are tangent to a smooth plane cubic (but often not all the lines are defined over $\mathbb{R}$).

We will see in Corollary 10.21 that when X is a smooth hypersurface the map $\mathcal{G}_X$ is birational onto its image as well as finite. (This requires the hypothesis of characteristic 0; strangely enough, it may be false in characteristic p, where for example a general tangent line to a smooth plane curve may be bitangent!) We will use this now to deduce the degree of the dual hypersurface:

Proposition 2.9. *If $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d > 1$, then the dual of X is a hypersurface of degree $d(d-1)^{n-1}$.*

Proof: The degree of the dual variety $X^* \subset \mathbb{P}^{n*}$ is the number of points of intersection of X^* and $n-1$ general hyperplanes $H_i \subset \mathbb{P}^{n*}$. Since by Corollary 10.21 the map $\mathcal{G}_X : X \to X^* \subset \mathbb{P}^{n*}$ is birational, this is the same as the number of points of intersection of the preimages $\mathcal{G}_X^{-1}(H_i)$. Since $\mathcal{G}_X$ is given by the partial derivatives of the defining equation F of X, the preimages of these hyperplanes are the intersections of X with the hypersurfaces $Z_i \subset \mathbb{P}^n$ of degree $d-1$ in $\mathbb{P}^n$ given by general linear combinations of these partial derivatives. Inasmuch as the partials of F have no common zeros, Bertini's theorem (Theorem 0.5) tells us that the hypersurfaces given by $n-1$ general linear combinations will intersect transversely with X. By Bézout's theorem the number of these points of intersection is the product of the degrees of the hypersurfaces, that is, $d(d-1)^{n-1}$. $\square$

For example, suppose that X is a smooth cubic curve in $\mathbb{P}^2$. By the above formula, the degree of X^* is 6. Since a general line in $\mathbb{P}^{2*}$ corresponds to the set of lines through a general point $p \in \mathbb{P}^2$, there will be exactly six lines in $\mathbb{P}^2$ through p tangent to X, as shown in Figure 2.2.

Proposition 2.9 gives us the answer to Keynote Question (e): Since the planes containing the line L form a general line in the dual projective space $\mathbb{P}^{3*}$, the number of such planes tangent to a smooth cubic surface $S \subset \mathbb{P}^3$ is $3 \cdot 2^2 = 12$.

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2.1.4 Products of projective spaces

Though the Chow ring of a smooth variety behaves like cohomology in many ways, there are important differences. For example the cohomology ring of the product of two spaces is given modulo torsion by the Künneth formula $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$, but in general there is no analogous Künneth formula for the Chow rings of products of varieties. Even for a product of two smooth curves C and D of genera $g, h \geq 1$ we have no algorithm for calculating $A^1(C \times D)$, and no idea at all what $A^2(C \times D)$ looks like, beyond the fact that it cannot be in any sense finite-dimensional (Mumford [1962]).

However the Chow ring of the product of a variety with a projective space does obey the Künneth formula, as we will prove in a more general context in Theorem 9.6 (Totaro [2014] proved it for products of any two varieties with affine stratifications). For the moment we will content ourselves with the special case where both factors are projective spaces:

Theorem 2.10. *The Chow ring of* $\mathbb{P}^r \times \mathbb{P}^s$ *is given by the formula*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s).$$

Equivalently, if $\alpha, \beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ *denote the pullbacks, via the projection maps, of the hyperplane classes on* $\mathbb{P}^r$ *and* $\mathbb{P}^s$ *respectively, then*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}).$$

Moreover, the class of the hypersurface defined by a bihomogeneous form of bidegree (d, e) *on* $\mathbb{P}^r \times \mathbb{P}^s$ *is* $d\alpha + e\beta$.

Proof: We proceed exactly as in Theorem 2.1. We may construct an affine stratification of $\mathbb{P}^r \times \mathbb{P}^s$ by choosing flags of subspaces

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \mathbb{P}^r \quad \text{and} \quad \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{s-1} \subset \Gamma_s = \mathbb{P}^s,$$

with $\dim \Lambda_i = i = \dim \Gamma_i$, and taking the closed strata to be

$$\Xi_{a,b} = \Lambda_{r-a} \times \Gamma_{s-b} \subset \mathbb{P}^r \times \mathbb{P}^s.$$

The open strata

$$\tilde{\Xi}_{a,b} := \Xi_{a,b} \setminus (\Xi_{a-1,b} \cup \Xi_{a,b-1})$$

of this stratification are affine spaces. Invoking Proposition 1.17, we conclude that the Chow groups of $\mathbb{P}^r \times \mathbb{P}^s$ are generated by the classes $\varphi_{a,b} = [\Xi_{a,b}] \in A^{a+b}(\mathbb{P}^r \times \mathbb{P}^s)$. Since $\Xi_{a,b}$ is the transverse intersection of the pullbacks of a hyperplanes in $\mathbb{P}^r$ and b hyperplanes in $\mathbb{P}^s$, we have

$$\varphi_{a,b} = \alpha^a \beta^b,$$

and in particular $\alpha^{r+1} = \beta^{s+1} = 0$. This shows that $A(\mathbb{P}^r \times \mathbb{P}^s)$ is a homomorphic image of

$$\mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1}) = \mathbb{Z}[\alpha]/(\alpha^{r+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^{s+1}).$$

On the other hand, $\Xi_{r,s}$ is a single point, so $\deg \varphi_{r,s} = 1$. The pairing

$$A^{p+q}(\mathbb{P}^r \times \mathbb{P}^s) \times A^{r+s-p-q}(\mathbb{P}^r \times \mathbb{P}^s) \to \mathbb{Z}, \quad ([X],[Y]) \to \deg([X][Y])$$

sends $(\alpha^p\beta^q, \alpha^m\beta^n)$ to 1 if $p + m = r$ and $q + n = s$, because in this case the intersection is transverse and consists of one point, and to 0 otherwise, since then the intersection is empty. This shows that the monomials of bidegree (p, q), for $0 \le p \le r$ and $0 \le q \le s$, are linearly independent over $\mathbb{Z}$, proving the first statement.

If $F(X, Y)$ is a bihomogeneous polynomial with bidegree (d, e), then, because $F(X, Y)/X_0^d Y_0^e$ is a rational function on $\mathbb{P}^r \times \mathbb{P}^s$, the class of the hypersurface X defined by $F = 0$ is d times the class of the hypersurface $X_0 = 0$ plus e times the class of the hypersurface $Y_0 = 0$; that is, $[X] = d\alpha + e\beta$. □

2.1.5 Degrees of Segre varieties

The Segre variety $\Sigma_{r,s}$ is by definition the image of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^{(r+1)(s+1)-1}$ under the map

$$\sigma_{r,s} : \big([X_0, \ldots, X_r], [Y_0, \ldots, Y_s]\big) \mapsto [\ldots, X_iY_j, \ldots].$$

The map $\sigma_{r,s}$ is an embedding because on each open set where one of the X_i and one of the Y_j are nonzero the rest of the coordinates can be recovered from the products.

If V and W are vector spaces of dimensions $r + 1$ and $s + 1$, we may express $\sigma_{r,s}$ without choosing bases by the formula

$$\sigma_{r,s} : \mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W),$$
$$(v, w) \mapsto v \otimes w.$$

For example, the map $\sigma_{1,1}$ is defined by the four forms $a = X_0Y_0$, $b = X_0Y_1$, $c = X_1Y_0$, $d = X_1Y_1$, and these satisfy the equation $ac - bd = 0$; thus the Segre variety $\Sigma_{1,1}$ is the nonsingular quadric in $\mathbb{P}^3$.

Proposition 2.11. *The degree of the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ is*

$$\deg \Sigma_{r,s} = \binom{r+s}{r}.$$

Proof: The degree of $\Sigma_{r,s}$ is the number of points in which it meets the intersection of $r+s$ hypersurfaces in $\mathbb{P}^{(r+1)(s+1)-1}$. Since $\sigma_{r,s}$ is an embedding, we may compute this number by pulling back these hypersurfaces to $\mathbb{P}^r \times \mathbb{P}^s$ and computing in the Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$. Thus $\deg \Sigma_{r,s} = \deg(\alpha + \beta)^{r+s}$, which gives the desired formula because $(\alpha + \beta)^{r+s} = \binom{r+s}{r}\alpha^r\beta^s$. □

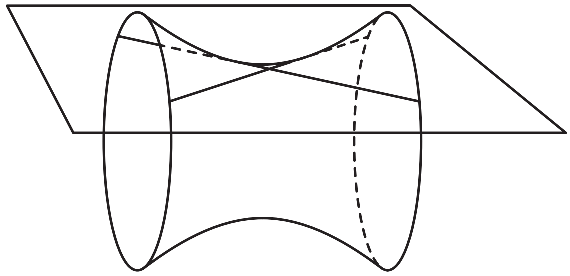

Figure 2.3 A tangent plane to a quadric in $\mathbb{P}^3$ meets the quadric in two lines, one from each ruling.

For instance, the Segre variety $\mathbb{P}^1 \times \mathbb{P}^r \subset \mathbb{P}^{2r+1}$ has degree $r+1$. These varieties are among those called *rational normal scrolls* (see Section 9.1.1). The simplest of these is the smooth quadric surface $Q \subset \mathbb{P}^3$, which is the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$; the pullbacks α and β of the point classes via the two projections are the classes of the lines of the two rulings of Q, and we have $\zeta = \alpha + \beta$, where ζ is the hyperplane class on $\mathbb{P}^3$ restricted to Q — a fact that is apparent if we look at the intersection of Q with any tangent plane, as in Figure 2.3.

This discussion can be generalized to arbitrary products of projective spaces (see Exercise 2.30).

2.1.6 The class of the diagonal

Next we will find the class δ of the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$ in the Chow group $A^r(\mathbb{P}^r \times \mathbb{P}^r)$, and more generally the class γ_f of the graph of a map $f : \mathbb{P}^r \to \mathbb{P}^s$. Apart from the applications of such a formula, this will introduce the *method of undetermined coefficients*, which we will use many times in the course of this book. (Another approach to this problem, via specialization, is given in Exercise 2.31.)

By Theorem 2.10, we have

$$A(\mathbb{P}^r \times \mathbb{P}^r) = \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{r+1}),$$

where $\alpha, \beta \in A^1(\mathbb{P}^r \times \mathbb{P}^r)$ are the pullbacks, via the two projection maps, of the hyperplane class in $A^1(\mathbb{P}^r)$. The class $\delta = [\Delta]$ of the diagonal is expressible as a linear combination

$$\delta = c_0\alpha^r + c_1\alpha^{r-1}\beta + c_2\alpha^{r-2}\beta^2 + \cdots + c_r\beta^r$$

for some $c_0, \ldots, c_r \in \mathbb{Z}$. We can determine the coefficients c_i by taking the product of both sides of this expression with various classes of complementary codimension: Specifically, if we intersect both sides with the class $\alpha^i\beta^{r-i}$ and take degrees, we have

$$c_i = \deg(\delta \cdot \alpha^i \beta^{r-i}).$$

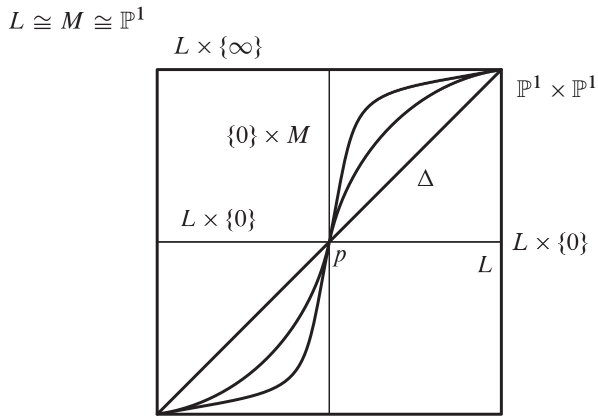

Figure 2.4 $[\Delta][L \times \{0\}] = 1 = [\Delta][\{0\} \times M]$, so $[\Delta] = [\{0\} \times M] + [L \times \{0\}]$, as one also sees from the degeneration in the figure.

We can evaluate the product $\delta \cdot \alpha^i \beta^{r-i}$ directly: If Λ and Γ are general linear subspaces of codimension i and $r - i$, respectively, then $[\Lambda \times \Gamma] = \alpha^i \beta^{r-i}$. Moreover,

$$(\Lambda \times \Gamma) \cap \Delta \cong \Lambda \cap \Gamma$$

is a reduced point, so

$$c_i = \deg(\delta \cdot \alpha^i \beta^{r-i}) = \#(\Delta \cap (\Lambda \times \Gamma)) = \#(\Lambda \cap \Gamma) = 1.$$

Thus

$$\delta = \alpha^r + \alpha^{r-1}\beta + \cdots + \alpha\beta^{r-1} + \beta^r.$$

See Figure 2.4. (This formula and its derivation will be familiar to anyone who has had a course in algebraic topology. As partisans we cannot resist pointing out that algebraic geometry had it first!)

2.1.7 The class of a graph

Let $f : \mathbb{P}^r \to \mathbb{P}^s$ be the morphism given by $(s + 1)$ homogeneous polynomials F_i of degree d that have no common zeros:

$$f : [X_0, \ldots, X_r] \mapsto [F_0(X), F_1(X), \ldots, F_s(X)].$$

By Corollary 2.2, we must have $s \geq r$. Let $\Gamma_f \subset \mathbb{P}^r \times \mathbb{P}^s$ be the graph of f. What is its class $\gamma_f = [\Gamma_f] \in A^s(\mathbb{P}^r \times \mathbb{P}^s)$?

As before, we can write

$$\gamma_f = c_0\alpha^r \beta^{s-r} + c_1\alpha^{r-1}\beta^{s-r+1} + c_2\alpha^{r-2}\beta^{s-r+2} + \cdots + c_r\beta^s$$

for some $c_0, \ldots, c_r \in \mathbb{Z}$, and as before we can determine the coefficients c_i in this expression by intersecting both sides with a cycle of complementary dimension:

$$c_i = \deg(\gamma_f \cdot \alpha^i \beta^{r-i}) = \#(\Gamma_f \cap (\Lambda \times \Phi))$$

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for general linear subspaces $\Lambda \cong \mathbb{P}^{r-i}$ and $\Phi \cong \mathbb{P}^{s-r+i} \subset \mathbb{P}^s$. By Theorem 1.7 the intersection $\Gamma_f \cap (\Lambda \times \Phi)$ is generically transverse.

Finally, $\Gamma_f \cap (\Lambda \times \Phi)$ is the zero locus in Λ of $r - i$ general linear combinations of the polynomials $F_0, \dots, F_s$. By Bertini's theorem, the corresponding hypersurfaces will intersect transversely, and by Bézout's theorem the intersection will consist of d^{r-i} points. Thus we arrive at the formula:

Proposition 2.12. *If $f : \mathbb{P}^r \to \mathbb{P}^s$ is a regular map given by polynomials of degree d on $\mathbb{P}^r$, the class γ_f of the graph of f is given by*

$$\gamma_f = \sum_{i=0}^{r} d^i \alpha^i \beta^{s-i} \in A^s(\mathbb{P}^r \times \mathbb{P}^s).$$

Using this formula, we can answer a general form of Keynote Question (c). A sequence $F_0, \dots, F_r$ of general homogeneous polynomials of degree d in $r+1$ variables defines a map $f : \mathbb{P}^r \to \mathbb{P}^r$, and we can count the fixed points

$$\{t = [t_0, \dots, t_r] \in \mathbb{P}^r \mid f(t) = t\}.$$

Since the F_i are general, we can take them to be general translates under $\mathrm{GL}_{r+1} \times \mathrm{GL}_{r+1}$ of arbitrary polynomials, so the cardinality of this set is the degree of the intersection of the graph γ_f of f with the diagonal $\Delta \subset \mathbb{P}^r \times \mathbb{P}^r$. This is

$$\begin{aligned} \deg(\delta \cdot \gamma_f) &= \deg\big((\alpha^r + \alpha^{r-1}\beta + \cdots + \beta^r) \cdot (d^r \alpha^r + d^{r-1}\alpha^{r-1}\beta + \cdots + \beta^r)\big) \\ &= d^r + d^{r-1} + \cdots + d + 1; \end{aligned}$$

in particular, if A, B, C are general forms of degree d in three variables then there are exactly $d^2 + d + 1$ points $t = [t_0, t_1, t_2] \in \mathbb{P}^2$ such that $[A(t), B(t), C(t)] = [t_0, t_1, t_2]$, and this is the answer to Keynote Question (c).

Note that in the case $d = 1$ and $s = r$, Proposition 2.12 implies that a general $(r+1) \times (r+1)$ matrix has $r + 1$ eigenvalues. It also follows that an arbitrary matrix has at least one eigenvalue.

2.1.8 Nested pairs of divisors on $\mathbb{P}^1$

We consider here one more example of an intersection theory problem involving products of projective spaces; this one will allow us to answer Keynote Question (d). To set this up, let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the projectivization of the space of homogeneous polynomials of degree d on $\mathbb{P}^1$ (equivalently, the space of effective divisors of degree d on $\mathbb{P}^1$). For any pair of natural numbers d and e with $e \geq d$, we consider the locus

$$\Phi = \{(f, g) \in \mathbb{P}^d \times \mathbb{P}^e \mid f \mid g\}.$$

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Alternatively, if we think of $\mathbb{P}^d$ as parametrizing divisors of degree d on $\mathbb{P}^1$, we can write this as

$$\Phi = \{(D, E) \in \mathbb{P}^d \times \mathbb{P}^e \mid E \geq D\}.$$

Since the projection map $\pi : \Phi \to \mathbb{P}^d$ has fibers isomorphic to $\mathbb{P}^{e-d}$, we see that Φ is irreducible of dimension e, or codimension d, in $\mathbb{P}^d \times \mathbb{P}^e$. We ask: What is the class of Φ in $A^d(\mathbb{P}^d \times \mathbb{P}^e)$?

Let $\sigma, \tau \in A^1(\mathbb{P}^d \times \mathbb{P}^e)$ be the pullbacks of the hyperplane classes in $\mathbb{P}^d$ and $\mathbb{P}^e$, respectively. A priori, we can write

$$[\Phi] = \sum c_i \sigma^i \tau^{d-i},$$

where each coefficient c_i is given by the degree of the product $[\Phi] \cdot \sigma^{d-i} \tau^{e-d+i}$; that is, the number of points of intersection of Φ with the product $\Lambda \times \Gamma$ of general linear spaces $\Lambda \cong \mathbb{P}^i \subset \mathbb{P}^d$ and $\Gamma \cong \mathbb{P}^{d-i} \subset \mathbb{P}^e$. This is exactly the number asked for in Keynote Question (d), but it may not be clear at first glance how to evaluate it.

The key to doing this is the observation that, abstractly, the variety Φ is isomorphic to a product $\mathbb{P}^d \times \mathbb{P}^{e-d}$: Specifically, it is the image of $\mathbb{P}^d \times \mathbb{P}^{e-d}$ under the map

$$\begin{aligned} \alpha : \mathbb{P}^d \times \mathbb{P}^{e-d} &\to \mathbb{P}^d \times \mathbb{P}^e, \\ (D, D') &\mapsto (D, D + D'). \end{aligned}$$

Furthermore, the pullback map $\alpha^* : A(\mathbb{P}^d \times \mathbb{P}^e) \to A(\mathbb{P}^d \times \mathbb{P}^{e-d})$ is readily described. Let $\sigma, \mu \in A^1(\mathbb{P}^d \times \mathbb{P}^{e-d})$ be the pullbacks of the hyperplane classes from $\mathbb{P}^d$ and $\mathbb{P}^{e-d}$, respectively. Since α commutes with the projection on the first factor, we see that $\alpha^*(\sigma) = \sigma$; since the composition $\mathbb{P}^d \times \mathbb{P}^{e-d} \to \mathbb{P}^d \times \mathbb{P}^e \to \mathbb{P}^e$ is given by bilinear forms on $\mathbb{P}^d \times \mathbb{P}^{e-d}$, we have $\alpha^*(\tau) = \sigma + \mu$. To evaluate the coefficient c_i, we write

$$\begin{aligned} \deg([\Phi] \cdot \sigma^{d-i} \tau^{e-d+i}) &= \deg(\alpha^*(\sigma^{d-i} \tau^{e-d+i})) \\ &= \deg(\sigma^{d-i} (\sigma + \mu)^{e-d+i}) \\ &= \binom{e-d+i}{i}; \end{aligned}$$

thus

$$[\Phi] = \sum \binom{e-d+i}{i} \sigma^i \tau^{d-i},$$

and correspondingly the answer to Keynote Question (d) is $\binom{e-d+a-1}{a-1}$.

2.1.9 The blow-up of $\mathbb{P}^n$ at a point

We will see in Chapter 13 how to describe the Chow ring of a blow-up in general. In this chapter, both to illustrate some of the techniques introduced so far and because the formulas derived will be useful in the interim, we will discuss two special cases: here the

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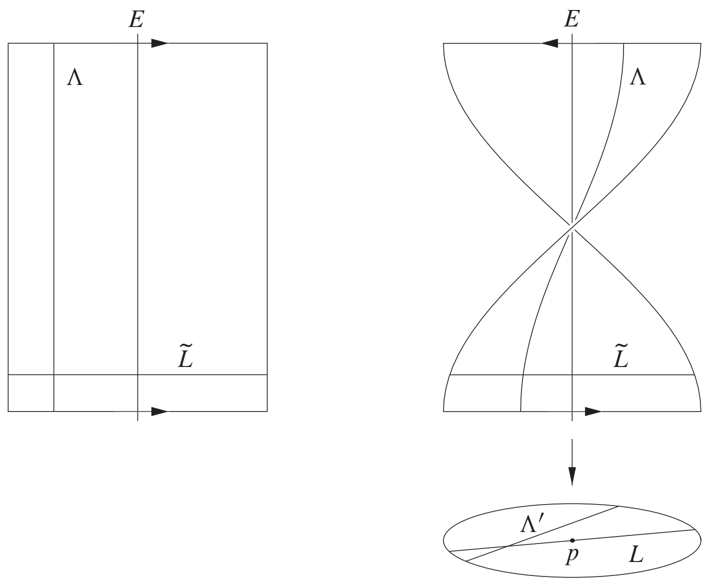

Figure 2.5 Blow-up of $\mathbb{P}^2$.

blow-up of $\mathbb{P}^n$ at a point for any $n \geq 2$ and in Section 2.4.4 the blow-up of any smooth surface at a point.

Recall that the *blow-up* of $\mathbb{P}^n$ at a point p is the morphism $\pi : B \to \mathbb{P}^n$, where $B \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ is the closure of the graph of the projection $\pi_p : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1}$ from p, and π is the projection on the first factor:

$$\begin{tikzcd} & B \arrow[dl, "\pi"'] \arrow[dr, "\alpha"] & \\ \mathbb{P}^n \arrow[rr, dotted, "\pi_p"] & & \mathbb{P}^{n-1} \end{tikzcd}$$

Since the graph of the projection is isomorphic to the source $\mathbb{P}^n \setminus \{p\}$, B is irreducible. It is not hard to write explicit equations for B and to show that it is smooth as well; see, for example, Section IV.2 of Eisenbud and Harris [2000].

The *exceptional divisor* $E \subset B$ is defined to be $\pi^{-1}(p)$, the preimage of p in B, which, as a subset of $\mathbb{P}^n \times \mathbb{P}^{n-1}$, is $\{p\} \times \mathbb{P}^{n-1}$. Some other obvious divisors on B are the preimages of the hyperplanes of $\mathbb{P}^n$. If the hyperplane $H \subset \mathbb{P}^n$ contains p, then its preimage is the sum of two irreducible divisors, E and $\widetilde{H}$; the latter is called the *strict transform*, or *proper transform*, of H. More generally, if $Z \subset \mathbb{P}^n$ is any subvariety, we define the strict transform of Z to be the closure in B of the preimage $\pi^{-1}(Z \setminus \{p\})$. See Figure 2.5.

To compute the Chow ring of B, we start from a stratification of B, using the geometry of the projection map $\alpha : B \to \mathbb{P}^{n-1}$ to the second factor. We do this by first choosing a stratification of the target $\mathbb{P}^{n-1}$, and taking the preimages in B of these strata.

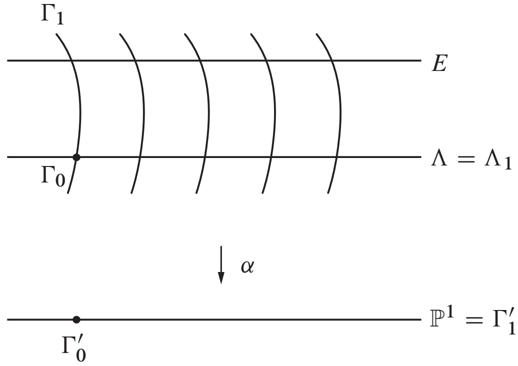

Figure 2.6 Blow-up of $\mathbb{P}^2$ as $\mathbb{P}^1$-bundle.

Then we choose a divisor $\Lambda \subset B$ that maps isomorphically by α to $\mathbb{P}^{n-1}$ — a *section* of α — and take, as additional strata, the intersections of these preimages with Λ.

We will choose as our section the preimage $\Lambda = \pi^{-1}(\Lambda')$ of a hyperplane $\Lambda' \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ not containing the point p. (There are other possible choices of a section, such as the exceptional divisor $E \subset B$; see Exercise 2.37.)

To carry this out, let

$$\Gamma_0' \subset \Gamma_1' \subset \cdots \subset \Gamma_{n-2}' \subset \Gamma_{n-1}' = \mathbb{P}^{n-1}$$

be a flag of linear subspaces and, for $k = 1, 2, \ldots, n$, let

$$\Gamma_k = \alpha^{-1}(\Gamma_{k-1}') \subset B.$$

Since the fibers of $\alpha : B \to \mathbb{P}^{n-1}$ are projective lines, the dimension of Γ_k is k. Next, for $k = 0, 1, \ldots, n-1$, we set

$$\Lambda_k = \Gamma_{k+1} \cap \Lambda,$$

so that Λ_k is the preimage of Γ_k' under the isomorphism $\alpha|_\Lambda : \Lambda \to \mathbb{P}^{n-1}$.

The subvarieties $\Gamma_1, \ldots, \Gamma_n, \Lambda_0, \ldots, \Lambda_{n-1}$ are the closed strata of a stratification of B, with inclusion relations

$$\begin{tikzcd}
\Lambda_0 \arrow[r, hook] \arrow[dr, hook] & \Lambda_1 \arrow[r, hook] \arrow[dr, hook] & \cdots \arrow[r, hook] & \Lambda_{n-2} \arrow[r, hook] \arrow[dr, hook] & \Lambda_{n-1} \arrow[dr, hook] & \\
& \Gamma_1 \arrow[r, hook] & \Gamma_2 \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_{n-1} \arrow[r, hook] & \Gamma_n = B
\end{tikzcd}$$

As we will soon see, this is an affine stratification, so that the classes of the closed strata generate the Chow group $A(B)$. (In fact, the open strata are isomorphic to affine spaces, and it follows from Totaro [2014] that they generate $A(B)$ freely; we will verify this independently when we determine the intersection products.)

To visualize this, we think of the blow-up B as the total space of a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$ via the projection map α; for example, this is the picture that arises if we take the standard picture of the blow-up of $\mathbb{P}^2$ at a point (shown in Figure 2.5) and "unwind" it as in Figure 2.6.

Proposition 2.13. *Let B be the blow-up of $\mathbb{P}^n$ at a point, with $n \geq 2$. With notation as above, the Chow ring $A(B)$ is the free abelian group on the generators $[\Lambda_k] = [\Lambda_{n-1}]^{n-k}$ for $k = 0, \dots, n-1$ and $[\Gamma_k] = [\Gamma_{n-1}]^{n-k}$ for $k = 1, \dots, n$. The class of the exceptional divisor E is $[\Lambda_{n-1}] - [\Gamma_{n-1}]$. If we set $\lambda = [\Lambda_{n-1}]$ and $e = [E]$, then*

$$A(B) \cong \frac{\mathbb{Z}[\lambda, e]}{(\lambda e, \lambda^n + (-1)^n e^n)}$$

as rings.

Proof: We start by verifying that the open strata $\Gamma_1^\circ, \dots, \Gamma_n^\circ, \Lambda_0^\circ, \dots, \Lambda_{n-1}^\circ$ of the stratification of B with closed strata Γ_k, Λ_k are isomorphic to affine spaces. This is immediate for the strata Λ_k°. For the strata Γ_k°, we choose coordinates $(x_0, \dots, x_n)$ on $\mathbb{P}^n$ so that $p = (1, 0, \dots, 0)$ and $\Lambda' \subset \mathbb{P}^n$ is the hyperplane $x_0 = 0$. By definition,

$$B = \{((x_0, \dots, x_n), (y_1, \dots, y_n)) \in \mathbb{P}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \text{ for all } i, j \geq 1\}.$$

Say the $(k-1)$-plane $\Gamma'_{k-1} \subset \mathbb{P}^{n-1}$ is given by $y_1 = \cdots = y_{n-k} = 0$. We can write the open stratum $\Gamma_k^\circ = \alpha^{-1}(\Gamma'_{k-1} \setminus \Gamma'_{k-2}) \cap (B \setminus \Lambda)$ as

$$\Gamma_k^\circ = \{((1, 0, \dots, 0, \lambda, \lambda y_{n-k+2}, \dots, \lambda y_n), (0, \dots, 0, 1, y_{n-k+2}, \dots, y_n))\}.$$

The functions $\lambda, y_{n-k+2}, \dots, y_n$ give an isomorphism of Γ_k° with $\mathbb{A}^k$.

It follows that the classes

$$\lambda_k = [\Lambda_k] \quad \text{and} \quad \gamma_k = [\Gamma_k] \quad \text{in } A_k(B)$$

generate the Chow groups of B.

We next compute the intersection products. Since Λ_k is the preimage of a k-plane in $\mathbb{P}^n$ not containing p, and any two such planes are linearly equivalent in $\mathbb{P}^n$, the classes of their pullbacks are all equal to λ_k. Similarly, the class of the proper transform of any k-plane in $\mathbb{P}^n$ containing p is γ_k. Having these representative cycles for the classes λ_k and γ_k makes it easy to determine their intersection products.

For example, a general k-plane in $\mathbb{P}^n$ intersects a general l-plane transversely in a general $(k+l-n)$-plane; thus

$$\lambda_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n.$$

Similarly, the intersection of a general k-plane in $\mathbb{P}^n$ containing p with a general l-plane not containing p is a general $(k+l-n)$-plane not containing p, so that

$$\gamma_k \lambda_l = \lambda_{k+l-n} \quad \text{for all } k + l \geq n,$$

and likewise

$$\gamma_k \gamma_l = \gamma_{k+l-n} \quad \text{for all } k + l \geq n + 1.$$

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Note the restriction $k+l \geq n+1$ on the last set of products: In the case $k+l=n$, the proper transforms of a general k-plane through p and a general l-plane through p are disjoint.

This determines the Chow ring of B. The pairing $A_k(B) \times A_{n-k}(B) \to A_0(B) \cong \mathbb{Z}$ is given by

$$\lambda_k \lambda_{n-k} = \lambda_k \gamma_{n-k} = \gamma_k \lambda_{n-k} = 1 \quad \text{and} \quad \gamma_k \gamma_{n-k} = 0.$$

This is nondegenerate, so the classes $\lambda_0, \ldots, \lambda_{n-1}$ and $\gamma_1, \ldots, \gamma_n$ freely generate $A(B)$.

It follows that we can express the class of the exceptional divisor E in terms of the generators Λ_{n-1} and Γ_{n-1} of $A_{n-1}(B)$. The most geometric way to do this is to observe that Λ'_{n-1} is linearly equivalent in $\mathbb{P}^n$ to a hyperplane $\Sigma \subset \mathbb{P}^n$ containing p, so the pullback of Σ is linearly equivalent to the union of the exceptional divisor E and a divisor D. Since D projects to a hyperplane of $\mathbb{P}^{n-1}$, it is contained in the preimage Γ of such a hyperplane. Since Γ is a $\mathbb{P}^1$-bundle over its image, it is irreducible. We see upon comparing dimensions that $D = \Gamma$. Since any two hyperplanes in $\mathbb{P}^{n-1}$ are rationally equivalent, so are their preimages in B; thus $\Lambda_{n-1} \sim D + E \sim \Gamma_{n-1} + E$, or $[E] = \lambda_{n-1} - \gamma_{n-1}$.

We now turn to the ring structure of $A(B)$. Let $\lambda = [\Lambda_{n-1}]$ and $e = [E] = \lambda - \gamma_{n-1}$. Since $\Lambda_{n-1} \cap E = \varnothing$, we have

$$\lambda e = 0.$$

Also,

$$\lambda_k = \lambda^{n-k} \quad \text{for } k = 0, \ldots, n-1,$$

and, since $\gamma_{n-1} = \lambda - e$,

$$\gamma_k = \gamma_{n-1}^{n-k} = (\lambda - e)^{n-k} = \lambda^{n-k} + (-1)^{n-k} e^{n-k} \quad \text{for } k = 1, \ldots, n.$$

It follows that λ and e generate $A(B)$ as a ring. In addition to the relation $\lambda e = 0$, they satisfy the relation

$$0 = \gamma_{n-1}^n = (\lambda - e)^n = \lambda^n + (-1)^n e^n.$$

Thus the Chow ring is a homomorphic image of the ring

$$A' := \mathbb{Z}[\lambda, e]/(\lambda e, \lambda^n + (-1)^n e^n).$$

For $m = 1, \ldots, n-1$, it is clear that every homogeneous element of degree m in A' is a $\mathbb{Z}$-linear combination of e^m and λ^m. Since for $0 < m < n$ the group $A^m(B)$ is a free $\mathbb{Z}$-module of rank 2, this implies that the map $A' \twoheadrightarrow A$ is an isomorphism. $\square$

We have computed the intersection products of the Λ_k and Γ_k by taking representatives that meet transversely (indeed, the possibility of doing so motivated our choice of Λ as a cross section of α above). Since E is the only irreducible variety in the class $[E]$ we cannot give a representative for e^2 quite as easily. But as we have seen,

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$[E] = [\Lambda_{n-1}] - [\Gamma_{n-1}]$ and both Λ_{n-1} and Γ_{n-1} are transverse to E (this illustrates the conclusion of the moving lemma!). It follows that

$$e^2 = [E \cap (\Lambda - \Gamma)] = -[E \cap \Gamma_{n-1}].$$

Since E projects isomorphically to $\mathbb{P}^{n-1}$ and Γ projects to a hyperplane in $\mathbb{P}^{n-1}$, we see that $E \cap \Gamma_{n-1}$ is a hyperplane in E; that is, $[E]^2$ is the *negative* of the class of a hyperplane in E.

The Chow ring of the blow-up of $\mathbb{P}^3$ along a line is worked out in Exercises 2.38–2.40. More generally, we will see how to describe the Chow ring of a general projective bundle in Chapter 9, and the Chow ring of a more general blow-up in Chapter 13.

2.1.10 Intersection multiplicities via blow-ups

We can use the description of the Chow ring of the blow-up B of $\mathbb{P}^n$ at a point to prove Proposition 1.29, relating the intersection multiplicity of two subvarieties $X, Y \subset \mathbb{P}^n$ of complementary dimension at a point to the multiplicities of X and Y at p. (The same argument will apply to subvarieties of an arbitrary smooth variety once we have described the Chow ring of a general blow-up in Section 13.6.) The idea is to compare the intersection $X \cap Y \subset \mathbb{P}^n$ of X and Y in $\mathbb{P}^n$ with the intersection $\widetilde{X} \cap \widetilde{Y} \subset B$ of their proper transforms in the blow-up.

We start by finding the class of the proper transforms:

Proposition 2.14. *Let $X \subset \mathbb{P}^n$ be a k-dimensional variety and $\widetilde{X} \subset B$ its proper transform in the blow-up B of $\mathbb{P}^n$ at a point p. If X has degree d and multiplicity $m = \operatorname{mult}_p(X)$ at p, then the class of the proper transform is*

$$[\widetilde{X}] = (d - m)\lambda_k + m\gamma_k \in A(B).$$

Proof: This follows from two things: the definition of the multiplicity of X at p as the degree of the projectivized tangent cone $\mathbb{T}C_pX$ (Section 1.3.8), and the identification of the projectivized tangent cone $\mathbb{T}C_pX$ to X at p with the intersection of the proper transform $\widetilde{X}$ with the exceptional divisor $E \cong \mathbb{P}^{n-1} \subset B$ (on page 36).

Given these, the proposition follows from the observation that if $i : E \hookrightarrow B$ is the inclusion, then $i^*(\lambda_k) = 0$ (λ_k is represented by the cycle Λ_k, which is disjoint from E) and $i^*(\gamma_k)$ is the class of a $(k-1)$-plane in $E \cong \mathbb{P}^{n-1}$ (γ_k is represented by the cycle Γ_k, which intersects E transversely in a $(k-1)$-plane). This says that the coefficient of γ_k in the expression above for $[\widetilde{X}]$ must be the multiplicity $m = \operatorname{mult}_p(X)$; the coefficient of λ_k similarly follows by restricting to a hyperplane not containing p. $\square$

Now suppose we are in the setting of Proposition 1.29: $X, Y \subset \mathbb{P}^n$ are dimensionally transverse subvarieties of complementary dimensions k and $n - k$, having multiplicities m and m' respectively at p. If, as we supposed in the statement of the proposition, the projectivized tangent cones to X and Y at p are disjoint (that is, $\widetilde{X} \cap \widetilde{Y} \cap E = \varnothing$), then the intersection multiplicity $m_p(X, Y)$ of X and Y at p is simply the difference between the intersection number $\deg([X][Y])$ of X and Y in $\mathbb{P}^n$ and the intersection number $\deg([\widetilde{X}][\widetilde{Y}])$ of their proper transforms in B; by Proposition 2.14 and our description of the Chow ring $A(B)$, this is just mm'.

2.2 Loci of singular plane cubics

This section represents an important shift in viewpoint, from studying the Chow rings of common and useful algebraic varieties to studying Chow rings of *parameter spaces*. It is a hallmark of algebraic geometry that the set of varieties (and more generally, schemes, morphisms, bundles and other geometric objects) with specified numerical invariants may often be given the structure of a scheme itself, sometimes called a parameter space. Applying intersection theory to the study of such a parameter space, we learn something about the geometry of the objects parametrized, and about geometrically characterized classes of these objects. This gets us into the subject of *enumerative geometry*, and was one of the principal motivations for the development of intersection theory in the 19th century.

By way of illustration, we will focus on the family of curves of degree 3 in $\mathbb{P}^2$: plane cubics. Plane cubics are parametrized by the set of homogeneous cubic polynomials $F(X, Y, Z)$ in three variables, modulo scalars, that is, by $\mathbb{P}^9$.

There is a continuous family of isomorphism classes of smooth plane cubics, parametrized naturally by the affine line (see Hartshorne [1977]), but there are only a finite number of isomorphism classes of singular plane cubics:

- irreducible plane cubics with a node;
- irreducible plane cubics with a cusp;
- plane cubics consisting of a smooth conic and a line meeting it transversely;
- plane cubics consisting of a smooth conic and a line tangent to it;
- plane cubics consisting of three nonconcurrent lines (“triangles”);
- plane cubics consisting of three concurrent lines (“asterisks”);
- cubics consisting of a double line and a line; and finally
- cubics consisting of a triple line.

These are illustrated in Figures 2.7–2.9, where the arrows represent specialization, as explained below.

The locus in $\mathbb{P}^9$ of points corresponding to singular curves of each type is an orbit of PGL_3 and a locally closed subset of $\mathbb{P}^9$. These loci, together with the open subset $U \subset \mathbb{P}^9$ of smooth cubics, give a stratification of $\mathbb{P}^9$. We may ask: What are the closed strata of this stratification like? What are their dimensions? What containment relations hold among them? Where is each one smooth and singular? What are their tangent spaces and tangent cones? What are their degrees as subvarieties of $\mathbb{P}^9$?

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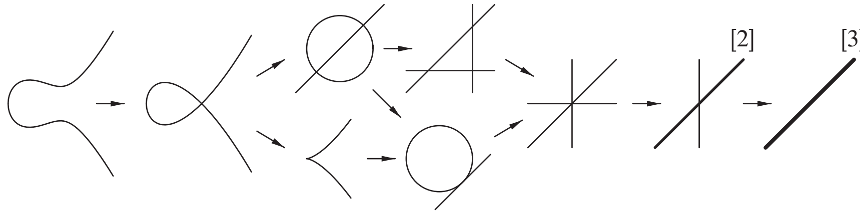

Figure 2.7 Hierarchy of singular plane cubic curves.

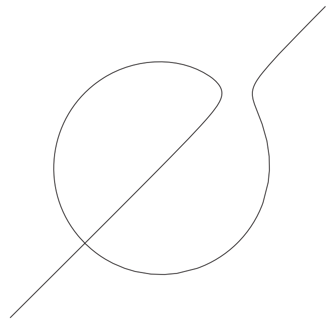

Figure 2.8 Nodal cubic about to become the union of a conic and a transverse line: $y^2 - x^2(x+1) + 100(x-y)\big((x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 - \frac{1}{2}\big)$.

Some of these questions are easy to answer. For example, the dimensions are given in Figure 2.7, and the reader can verify them as an exercise. The specialization relationships (when one orbit is contained in the closure of another, as indicated by arrows in the chart) are also easy, because to establish that one orbit lies in the closure of another it suffices to exhibit a one-parameter family $\{C_t \subset \mathbb{P}^2\}$ of plane cubics with an open set of parameter values t corresponding to one type and a point corresponding to the other. The noninclusion relations are subtler — why, for example, is a triangle not a specialization of a cuspidal cubic? — but can also be proven by focusing on the singularities of the curves.

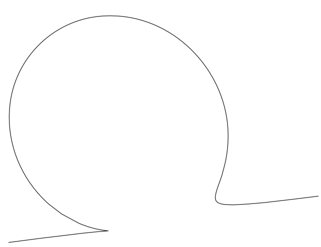

Figure 2.9 Cuspidal cubic about to become the union of a conic and a tangent line: $y^2 - x^3 + 7y(x^2 + (y-1)^2 - 1)$.

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The tangent spaces require more work; we will give some examples in Exercises 2.42–2.43, in the context of establishing a transversality statement, and we will see more of these, as well as some tangent cones, in Section 7.7.3.

In the rest of this section we will focus on the question of the degrees of these loci; we will find the answer in the case of the loci of reducible cubics, triangles and asterisks. In the exercises we indicate how to compute the degrees of the other loci of plane cubics, except for the loci of irreducible cubics with a node and of irreducible cubics with a cusp; these will be computed in Section 7.3.2 and Section 11.4 respectively.

The calculations here barely scratch the surface of the subject; see for example Aluffi [1990; 1991] for a beautiful and extensive treatment of the enumerative geometry of plane cubics. Moreover, the answers to analogous questions for higher-degree curves or hypersurfaces of higher dimension — for example, about the stratification by singularity type — remain mysterious. Even questions about the dimension and irreducibility of these loci are mostly open; they are a topic of active research. See Greuel et al. [2007] for an introduction to this area.

For example, it is known that for $0 \leq \delta \leq \binom{d}{2}$ the locus of plane curves of degree d having exactly δ nodes is irreducible of codimension δ in the projective space $\mathbb{P}^N$ of all plane curves of degree d (see, for example, Harris and Morrison [1998]), and its degree has also been determined (Caporaso and Harris [1998]). But we do not know the answers to the analogous questions for plane curves with δ nodes and κ cusps, and when it comes to more complicated singularities even existence questions are open. For example, for $d > 6$ it is not known whether there exists a rational plane curve $C \subset \mathbb{P}^2$ of degree d whose singularities consist of just one double point.

2.2.1 Reducible cubics

Let $\Gamma \subset \mathbb{P}^9$ be the closure of the locus of cubics consisting of a conic and a transverse line (equivalently, the locus of reducible and/or nonreduced cubics). We can describe Γ as the image of the map

$$\tau : \mathbb{P}^2 \times \mathbb{P}^5 \to \mathbb{P}^9$$

from the product of the space $\mathbb{P}^2$ of homogeneous linear forms and the space $\mathbb{P}^5$ of homogeneous quadratic polynomials to $\mathbb{P}^9$, given simply by multiplication: $(F, G) \mapsto FG$. Inasmuch as the coefficients of the product FG are bilinear in the coefficients of F and G, the pullback $\tau^*(\zeta)$ of the hyperplane class $\zeta \in A^1(\mathbb{P}^9)$ is the sum

$$\tau^*(\zeta) = \alpha + \beta,$$

where α and β are the pullbacks to $\mathbb{P}^2 \times \mathbb{P}^5$ of the hyperplane classes on $\mathbb{P}^2$ and $\mathbb{P}^5$.

By unique factorization of polynomials, the map τ is birational onto its image; it follows that the degree of Γ is given by

$$\deg(\Gamma) = \deg(\tau^*(\zeta)^7) = \deg((\alpha + \beta)^7) = 21,$$

and this is the answer to Keynote Question (a).

Another way to calculate the degree of Γ is described in Exercises 2.42–2.44.

2.2.2 Triangles

A similar analysis gives the answer to Keynote Question (b) — how many cubics in a three-dimensional linear system factor completely, as a product of three linear forms. Here, the key object is the closure $\Sigma \subset \mathbb{P}^9$ of the locus of such totally reducible cubics, which we may call *triangles*; the keynote question asks us for the number of points of intersection of Σ with a general 3-plane. By Bertini's theorem this is the degree of Σ.

Since Σ is the image of the map

$$\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9,$$
$$([L_1], [L_2], [L_3]) \mapsto [L_1 L_2 L_3],$$

we can proceed as before, with the one difference that the map is now no longer birational, but rather is generically six-to-one. Thus if $\alpha_1, \alpha_2, \alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ are the pullbacks of the hyperplane classes in the factors $\mathbb{P}^2$ via the three projections, so that

$$\mu^*(\zeta) = \alpha_1 + \alpha_2 + \alpha_3,$$

we get

$$\deg(\Sigma) = \frac{1}{6} \deg(\alpha_1 + \alpha_2 + \alpha_3)^6 = \frac{1}{6}\binom{6}{2, 2, 2} = 15.$$

This is the answer to Keynote Question (b): In a general three-dimensional linear system of cubics, there will be exactly 15 triangles.

2.2.3 Asterisks

By an *asterisk*, we mean a cubic consisting of the sum of three concurrent lines. To see that the closure of this locus is indeed a subvariety of $\mathbb{P}^9$ and to calculate its degree, let

$$\mu : \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^9$$

be as in Section 2.2.2, and consider the subset

$$\Phi = \{(L_1, L_2, L_3) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \mid L_1 \cap L_2 \cap L_3 \neq \varnothing\};$$

the locus $A \subset \mathbb{P}^9$ of asterisks is then the image $\mu(\Phi)$ of Φ under the map μ. If we write the line L_i as the zero locus of the linear form

$$a_{i,1}X + a_{i,2}Y + a_{i,3}Z,$$

then the condition that $L_1 \cap L_2 \cap L_3 \neq \varnothing$ is equivalent to the equality

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = 0.$$

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The left-hand side of this equation is a homogeneous trilinear form on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, from which we see that Φ is a closed subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and A is a closed subset of $\mathbb{P}^9$. Moreover, we see that the class of Φ is

$$[\Phi] = \alpha_1 + \alpha_2 + \alpha_3 \in A^1(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2),$$

so that the pullback via μ of five general hyperplanes in $\mathbb{P}^9$ will intersect Φ in

$$\deg([\Phi](\alpha_1 + \alpha_2 + \alpha_3)^5) = \deg(\alpha_1 + \alpha_2 + \alpha_3)^6 = \binom{6}{2,2,2} = 90$$

points. Since the map $\mu|_\Phi : \Phi \to A$ has degree 6, it follows that the degree of the locus $A \subset \mathbb{P}^9$ of asterisks is 15.

2.3 The circles of Apollonius

Apollonius posed the problem of constructing the circles tangent to three given circles. Using Bézout's theorem we can count them.

Theorem 2.15. *If D_1, D_2 and D_3 are three general circles, there are exactly eight circles tangent to all three.*

2.3.1 What is a circle?

We first need to say what we mean by a circle in complex projective space. While circles are usually characterized in terms of a metric, in fact they have a purely algebro-geometric definition. Starting from the affine equation

$$(x-a)^2 + (y-b)^2 = r^2 \tag{2.1}$$

of a circle of radius r centered at a point (a, b) in $\mathbb{A}^2$, and homogenizing with respect to a new variable z, we get $(x-az)^2 + (y-bz)^2 = r^2z^2$. We think of the line $z = 0$ as the "line at infinity," and we see that the circle passes through the two points

$$\circ_+ = (1, i, 0) \quad \text{and} \quad \circ_- := (1, -i, 0)$$

on the line at infinity; these are called the *circular points*. Conversely, it is an easy exercise to see that the equation of any smooth conic passing through the two circular points can be put into put into the form (2.1).

We thus define a *circle* to be a conic in $\mathbb{P}^2$ with coordinates x, y, z passing through the two circular points on the line at infinity $z = 0$; equivalently, a circle is a conic $C = V(f) \subset \mathbb{P}^2$ whose defining equation f lies in the ideal $(z, x^2 + y^2)$. (This formulation makes sense over any field of characteristic $\neq 2$.) We see from this that the set of circles is a three-dimensional linear subspace in the space $\mathbb{P}^5$ of all conics in $\mathbb{P}^2$.

Much geometry can be done in this context. For example, a direct calculation shows that the *center* of the circle is the point of intersection of the tangent lines to the circle at the circular points; in particular, the coordinates of the center are rational functions of the coefficients of its defining equation.

Note that when we characterize circles as conics containing the circular points p, q at infinity, we are including singular conics that pass through these points, and we see that there are two kinds of singular circles: unions of the line at infinity $\overline{\circ_+, \circ_-}$ with another line in $\mathbb{P}^2$, and unions $L \cup M$ of lines with $\circ_+ \in L$ and $\circ_- \in M$. It is easy to see from the equations that these are the limits of smooth circles of radius r as $r \to \infty$ and $r \to 0$, respectively. (When the radius of a circle goes to 0, we may think the circle shrinks to a point, but that is because we are seeing only points in $\mathbb{R}^2$: over $\mathbb{C}$, the conic $x^2 + y^2 = 0$ consists of the two lines $x = \pm iy$.)

2.3.2 Circles tangent to a given circle

Next, we have to define what we mean when we say two circles are tangent. Let $D \subset \mathbb{P}^2$ be a smooth circle. If C is any other circle, we can write the intersection $C \cap D$, viewed as a divisor on D, as the sum

$$C \cap D = \circ_+ + \circ_- + p + q.$$

In these terms, we make the following definition:

Definition 2.16. We say that the circle C is tangent to the circle D if $p = q$.

In other words, C and D are tangent if they have two coincident intersections in addition to their intersection at the circular points; this includes the case where C, D have intersection multiplicity 3 at p or q. Let Z_D be the variety of circles tangent to a given smooth circle D. We will show that Z_D is a quadric cone in the $\mathbb{P}^3$ of circles.

It is visually obvious that the family of circles in $\mathbb{R}^2$ tangent to a given circle is two-dimensional. To prove this algebraically we consider the incidence correspondence

$$\Phi = \{(r, C) \in D \times \mathbb{P}^3 \mid C \text{ is tangent to } D \text{ at } r\},$$

where when r is a circular point the condition should be interpreted as saying the intersection multiplicity $m_r(C, D)$ is ≥ 3. The condition that a curve $f = 0$ meet a curve D with multiplicity m at a smooth point $r \in D$ means that the function $f|_D$ vanishes to order m at r; it is thus m linear conditions on the coefficients of the equation f. This shows that, for each point $r \in D$, the fiber of Φ over r is a $\mathbb{P}^1$, cut out by two linear equations in the space of circles. It follows that Φ is irreducible of dimension 2. Since almost all circles tangent to D are tangent at a single point, the map $\Phi \to \mathbb{P}^3$ sending (r, C) to C is birational. Thus the image Z_D of Φ in $\mathbb{P}^3$ also two-dimensional.

To show that $Z_D \subset \mathbb{P}^3$ is a quadric, let $L \subset \mathbb{P}^3$ be a general line, corresponding to a pencil of circles $\{C_t\}_{t \in \mathbb{P}^1}$. If f and g are the defining equations of C_0 and C_∞, the

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rational function f/g has two zeros (where C_0 meets D, aside from the circular points) and two poles (where C_∞ meets D, aside from the circular points), so f/g gives a map $D \to \mathbb{P}^1$ of degree 2.

The circles C_t tangent to D correspond to the branch points of this map; by the classical Riemann–Hurwitz formula, there will be two such points. Thus the degree of $L \cap Z_D$ is 2, and we see that Z_D is a quadric surface. On the other hand, if $C \neq D$ is tangent to D at $r \in D$, then every member of the linear space of circles jointing C to D satisfies the linear condition for tangency at r, so $Z_D \subset \mathbb{P}^3$ is a cone with vertex corresponding to D, as claimed.

2.3.3 Conclusion of the argument

Now let D_1, D_2, D_3 be three circles. If the intersection $A := Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$ is finite, then Bézout's theorem implies that $\deg A = 2^3 = 8$. To prove that the intersection is finite for nearly all triples of circles D_i, we consider the incidence correspondence

$$\Psi := \{(D_1, D_2, D_3, C) \in (\mathbb{P}^3)^4 \mid C \text{ is tangent to each of the } D_i\}.$$

If we project onto the last factor, the fiber is Z_C^3, and thus has dimension 6, so $\dim \Psi = 9$. Thus the projection to the nine-dimensional space consisting of all triples (D_1, D_2, D_3) cannot have generic fiber of positive dimension.

We have now shown that, counting with multiplicity, there are eight circles tangent to three general circles D_1, D_2, D_3. To prove that there are really eight distinct circles, we would need to prove that the intersection $Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$ is transverse. In Section 8.2.3 we will see how to do this directly, by identifying explicitly the tangent spaces to the loci Z_D. For now we will be content to give an example of the situation where the eight circles are distinct: it is shown on the cover of this book!

Another approach to the circles of Apollonius, via the notion of *theta-characteristics*, is given in Harris [1982]. There is also an analogous notion of a *sphere* in $\mathbb{P}^3$; see for example Exercise 13.32.

2.4 Curves on surfaces

Aside from enumerative problems, intersection products appeared in algebraic geometry as a central tool in the theory of surfaces, developed mostly by the Italians in the late 19th and early 20th centuries. In this section we describe some of the basic ideas. This will serve to illustrate the use of intersection products in a simple setting, and also provide us with formulas that will be useful throughout the book. A different treatment of some of this material is in the last chapter of Hartshorne [1977]; and much more can be found, for example, in Beauville's beautiful book on algebraic surfaces [1996].

Throughout this section we will use some classical notation: If S is a smooth projective surface and $\alpha, \beta \in A^1(S)$, we will write $\alpha \cdot \beta$ for the degree $\deg(\alpha\beta)$ of their product $\alpha\beta \in A^2(S)$, and we refer to this as the *intersection number* of the two classes. Further, if $C \subset S$ is a curve we will abuse notation and write C for the class $[C] \in A^1(S)$. Thus, for example, if $C, D \subset S$ are two curves, we will write $C \cdot D$ in place of $\deg([C] \cdot [D])$ and we will write C^2 for $\deg([C]^2)$. The reader should not be misled by this notation into thinking that $A^2(S) = \mathbb{Z}$ — as we have already remarked, the group $A^2(S)$ need not even be finite-dimensional in any reasonable sense.

2.4.1 The genus formula

One of the first formulas in which intersection products appeared was the *genus formula*, a straightforward rearrangement of the adjunction formula that describes the genus of a smooth curve on a smooth projective surface (we will generalize it to some singular curves in Section 2.4.6). If $C \subset S$ is a smooth curve of genus g on a smooth surface, then

$$K_C = (K_S + C)|_C;$$

since the degree of the canonical class of C is $2g - 2$, this yields

$$g = \frac{C^2 + K_S \cdot C}{2} + 1. \tag{2.2}$$

Example 2.17 (Plane curves)**.** By way of examples, consider first a smooth curve $C \subset \mathbb{P}^2$ of degree d. If we let $\zeta \in A^1(\mathbb{P}^2)$ be the class of a line, we have $[C] = d\zeta$ and $K_{\mathbb{P}^2} = -3\zeta$, so the genus of C is

$$g = \frac{-3d + d^2}{2} + 1 = \frac{(d-1)(d-2)}{2}.$$

Thus we recover, for example, the well-known fact that lines and smooth conics have genus 0 while smooth cubics have genus 1.

Example 2.18 (Curves on a quadric)**.** Now suppose that $Q \subset \mathbb{P}^3$ is a smooth quadric surface, and that $C \subset Q$ is a smooth curve of bidegree (d, e) — that is, a curve linearly equivalent to d times a line of one ruling plus e times a line of the other (equivalently, in terms of the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, the zero locus of a bihomogeneous polynomial of bidegree (d, e)). Let α and $\beta \in A^1(Q)$ be the classes of the lines of the two rulings of Q, as in the discussion in Section 2.1.5 above, and let $\zeta = \alpha + \beta$ be the class of a plane section of Q. Applying adjunction to $Q \subset \mathbb{P}^3$, we have

$$K_Q = (K_{\mathbb{P}^3} + Q)|_Q = -2\zeta = -2\alpha - 2\beta.$$

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Thus, by the genus formula,

$$\begin{aligned} g &= \frac{(d\alpha + e\beta)^2 - 2(\alpha + \beta)(d\alpha + e\beta)}{2} + 1 \\ &= \frac{2de - 2d - 2e}{2} + 1 \\ &= (d-1)(e-1). \end{aligned}$$

2.4.2 The self-intersection of a curve on a surface

We can sometimes use the genus formula to determine the self-intersection of a curve on a surface. For example, suppose that $S \subset \mathbb{P}^3$ is a smooth surface of degree d and $L \subset S$ is a line. Letting $\zeta \in A^1(S)$ denote the plane class and applying adjunction to $S \subset \mathbb{P}^3$, we have $K_S = (d-4)\zeta$, so that $L \cdot K_S = d - 4$; since the genus of L is 0, the genus formula yields

$$0 = \frac{L^2 + d - 4}{2} + 1,$$

or

$$L^2 = 2 - d.$$

The cases $d = 1$ (a line on a plane) and $d = 2$ are probably familiar already; in the case $d \geq 3$, the formula implies the qualitative statement that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines*. (See Exercise 2.60 below for a sketch of a proof, and Exercise 2.59 for an alternative derivation of $L^2 = 2 - d$.)

We note in passing that we could similarly ask for the degree of the self-intersection of a 2-plane $\Lambda \cong \mathbb{P}^2 \subset X$ on a smooth hypersurface $X \subset \mathbb{P}^5$. This is far harder (as the reader may verify, neither of the techniques suggested in this chapter for calculating the self-intersection of a line on a smooth surface $S \subset \mathbb{P}^3$ will work); the answer is given in Exercise 13.22.

2.4.3 Linked curves in $\mathbb{P}^3$

Another application of the genus formula yields a classical relation between what are called *linked curves* in $\mathbb{P}^3$.

Let $S, T \subset \mathbb{P}^3$ be smooth surfaces of degrees s and t, and suppose that the scheme-theoretic intersection $S \cap T$ consists of the union of two smooth curves C and D with no common components. Let the degrees of C and D be c and d, and let their genera be g and h respectively. By Bézout's theorem, we have

$$c + d = st,$$

so that the degree of C determines the degree of D. What is much less obvious is that the degree and genus of C determine the degree and genus of D. Here is one way to derive the formula.

To start, we use the genus formula (2.2) to determine the self-intersection of C on S: Since $K_S = (s-4)\zeta$, we have

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 = \frac{C^2 + (s-4)c}{2} + 1,$$

and hence

$$C^2 = 2g - 2 - (s-4)c$$

(generalizing our formula in Section 2.4.2 for the self-intersection of a line). Next, since $[C] + [D] = t\zeta \in A^1(S)$, we can write the intersection number of C and D on S as

$$C \cdot D = C(t\zeta - C) = tc - (2g - 2 - (s-4)c) = (s+t-4)c - (2g-2).$$

This in turn allows us to determine the self-intersection of D on S:

$$D^2 = D(t\zeta - C) = td - ((s+t-4)c - (2g-2)).$$

Applying the genus formula to D, we obtain

$$\begin{aligned} h &= \frac{D^2 + K_S \cdot D}{2} + 1 \\ &= \frac{td - ((s+t-4)c - (2g-2)) + (s-4)d}{2} + 1. \end{aligned}$$

Simplifying, we get

$$h - g = \frac{s+t-4}{2}(d-c); \tag{2.3}$$

in English, *the difference in the genera of C and D is proportional to the difference in their degrees, with ratio* $(s+t-4)/2$.

The answer to Keynote Question (f) is a special case of this: If $L \subset \mathbb{P}^3$ is a line, and S and T general surfaces of degrees s and t containing L, then, writing $S \cap T = L \cup C$, we see that C is a curve of degree $st - 1$ and genus

$$h = \frac{(s+t-4)(st-2)}{2}.$$

As is often the case with enumerative formulas, this is just the beginning of a much larger picture. The theory of *liaison* describes the relationship between the geometry of linked curves such as C and D above. The theory in general is far more broadly applicable (the curves C and D need only be Cohen–Macaulay, and we need no hypotheses at all on the surfaces S and T), and ultimately provides a complete answer to the question of when two given curves $C, D \subset \mathbb{P}^3$ can be connected by a series of curves $C = C_0, C_1, \ldots, C_{n-1}, C_n = D$ with C_i and C_{i+1} linked as above. We will see a typical application of the notion of linkage in Exercise 2.62 below; for the general theory, see Peskine and Szpiro [1974].

2.4.4 The blow-up of a surface

The blow-up of a point on a surface plays an important role in the theory of surfaces, and we will now explain a little of this theory. Locally (in the analytic or étale topology), such blow-ups look like the blow-up of $\mathbb{P}^2$ at a point, which was treated in Section 2.1.9.

To fix notation, we let $p \in S$ be a point in a smooth projective surface and write $\pi : \widetilde{S} \to S$ for the blow-up of S at p. We write $E = \pi^{-1}(p) \subset \widetilde{S}$ for the preimage of p, called the *exceptional divisor*, and $e \in A^1(\widetilde{S})$ for its class. We will use the following definitions and facts:

- $\pi : \widetilde{S} \to S$ is birational, and if $q \in E \subset \widetilde{S}$ is any point of the exceptional divisor, then there are generators z, w for the maximal ideal of $\mathcal{O}_{\widetilde{S},q}$ and generators x, y for the maximal ideal of $\mathcal{O}_{S,p}$ such that $\pi^* x = zw$, $\pi^* y = w$, and E is defined locally by the equation $w = 0$. In particular, $\widetilde{S}$ is smooth and E is a Cartier divisor.
- If C is a smooth curve through p, then the *proper transform* $\widetilde{C}$ of C, which is by definition the closure in $\widetilde{S}$ of $\pi^{-1}(C \setminus \{p\})$, meets E transversely in one point.
- More generally, if C has an *ordinary m-fold point at p*, then $\widetilde{C}$ meets E transversely in m distinct points. Here we say that C has an ordinary m-fold point at p if the completion of the local ring of C at p has the form

$$\hat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x, y]\!] \Big/ \left(\prod_{i=1}^{m} (x - \lambda_i y) \right)$$

  for some distinct $\lambda_1, \ldots, \lambda_m \in \Bbbk$; geometrically, this says that, near p, C consists of the union of m smooth branches meeting pairwise transversely at p.

We can completely describe $A(\widetilde{S})$ in terms of $A(S)$:

Proposition 2.19. *Let S be a smooth projective surface and $\pi : \widetilde{S} \to S$ the blow-up of S at a point p; let $e \in A^1(\widetilde{S})$ be the class of the exceptional divisor.*

(a) $A(\widetilde{S}) = A(S) \oplus \mathbb{Z}e$ as abelian groups.
(b) $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$ for any $\alpha, \beta \in A^1(S)$.
(c) $e \cdot \pi^*\alpha = 0$ for any $\alpha \in A^1(S)$.
(d) $e^2 = -[q]$ for any point $q \in E$ (in particular, $\deg(e^2) = -1$).

Proof: We first show that π_* and π^* are inverse isomorphisms between $A^2(S)$ and $A^2(\widetilde{S})$. By the moving lemma, if $\alpha \in A_0(S)$ is any class, we can write $\alpha = [A]$ for some $A \in Z_0(S)$ with support disjoint from p; thus $\pi_*\pi^*\alpha = \alpha$. Likewise, if $\alpha \in A_0(\widetilde{S})$ is any class, we can write $\alpha = [A]$ for some $A \in Z_0(S)$ with support disjoint from E; thus $\pi^*\pi_*\alpha = \alpha$.

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We next turn to A^1. If $\alpha \in A^1(S)$ is any class, we can write $\alpha = [A]$ for some $A \in Z_1(S)$ with support disjoint from p; thus $\pi_*\pi^*\alpha = \alpha$. On the other hand, the kernel of the pushforward map $\pi_* : Z_1(\widetilde{S}) \to Z_1(S)$ is just the subgroup generated by e, the class of E. Thus we have an exact sequence

$$0 \longrightarrow \langle e \rangle \longrightarrow A^1(\widetilde{S}) \longrightarrow A^1(S) \longrightarrow 0,$$

with $\pi^* : A^1(S) \to A^1(\widetilde{S})$ splitting the sequence.

It remains to show that the class e is not torsion in $A^1(\widetilde{S})$. This follows from the formula $\deg e^2 = -1$, which we will prove independently below.

Part (b) of the proposition simply recalls the fact that π^* is a ring homomorphism. For part (c) we use the push-pull formula:

$$\pi_*(e \cdot \pi^*\alpha) = \pi_* e \cdot \alpha = 0.$$

For part (d), let $C \subset S$ be any curve smooth at p, so that the proper transform $\widetilde{C} \subset \widetilde{S}$ of C will intersect E transversely at one point q. We have then

$$\pi^*[C] = [\widetilde{C}] + e,$$

and intersecting both sides with the class e yields

$$0 = [q] + e^2,$$

so the self-intersection number of e is $\deg e^2 = -1$. □

2.4.5 Canonical class of a blow-up

We can express the canonical class of $\widetilde{S}$ in terms of the canonical class of S as follows:

Proposition 2.20. *With notation as above,*

$$K_{\widetilde{S}} = \pi^* K_S + e.$$

Proof: We must show that if ω is a rational 2-form on S, regular and nonzero at p, then the pullback $\pi^*\omega$ vanishes simply along E. Let $q \in E \subset \widetilde{S}$, and let (z, w) be generators of the maximal ideal of $\mathcal{O}_{\widetilde{S},q}$ such that there are generators (x, y) for the maximal ideal of $\mathcal{O}_{S,p}$ with

$$\pi^* x = zw \quad \text{and} \quad \pi^* y = w.$$

It follows that

$$\pi^* dx = z\,dw + w\,dz \quad \text{and} \quad \pi^* dy = dw.$$

Thus

$$\pi^*(dx \wedge dy) = w(dz \wedge dw).$$

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Since the local equation of E at q is $w = 0$, this shows that $\pi^* dx$ vanishes simply along E, as required. □

2.4.6 The genus formula with singularities

It will be useful in a number of situations to have a version of the genus formula (2.2) that gives the geometric genus of a possibly singular curve $C \subset S$. (The *geometric genus* of a reduced curve is the genus of its normalization.) To start with the simplest case, suppose that $C \subset S$ is a curve smooth away from a point $p \in C$ of multiplicity m. Assume moreover that p is an ordinary m-fold point, so that in particular the proper transform $\widetilde{C}$ is smooth. We can invoke the genus formula on $\widetilde{S}$ to give a formula for the genus g of $\widetilde{C}$ in terms of intersection numbers on S.

As divisors,

$$\pi^* C = \widetilde{C} + mE,$$

so that

$$[\widetilde{C}] = \pi^*[C] - me.$$

From Proposition 2.20, we have

$$K_{\widetilde{S}} = \pi^* K_S + e,$$

and, putting this together with the genus formula for $\widetilde{C} \subset \widetilde{S}$ and Proposition 2.19, we have

$$\begin{aligned} g &= \frac{\widetilde{C}^2 + K_{\widetilde{S}} \cdot \widetilde{C}}{2} + 1 \\ &= \frac{(\pi^* C - me)^2 + (\pi^* K_S + e)(\pi^* C - me)}{2} + 1 \\ &= \frac{C^2 + K_S \cdot C}{2} + 1 - \binom{m}{2}. \end{aligned}$$

More generally, if $C \subset S$ has singular points $p_1, \dots, p_\delta$ of multiplicity $m_1, \dots, m_\delta$, and the proper transform $\widetilde{C}$ of C in the blow-up $\mathrm{Bl}_{\{p_1,\dots,p_\delta\}}$ of S at the points p_i is smooth (as will, in particular, be the case if the p_i are ordinary m_i-fold points of C), we have

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 - \sum \binom{m_i}{2}.$$

One can extend this further, to general singular curves $C \subset S$, by using iterated blow-ups, or by generalizing the adjunction formula, using the fact that any curve on a smooth surface has a canonical bundle (see for example Hartshorne [1977, Theorem III.7.11]).

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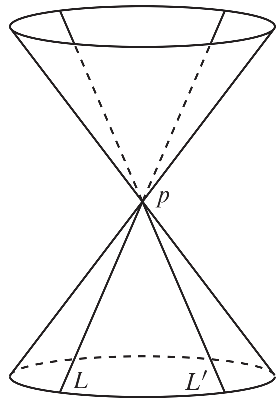

Figure 2.10 The degree of intersection of two lines on a quadric cone is $\frac{1}{2}$.

2.5 Intersections on singular varieties

In this section we discuss the problems of defining intersection products on singular varieties. To begin with, the moving lemma may fail if X is even mildly singular:

Example 2.21 (Figure 2.10)**.** Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be a smooth conic and $X = \overline{p, C} \subset \mathbb{P}^3$ the cone with vertex $p \notin \mathbb{P}^2$. Let $L \subset X$ be a line (which necessarily contains p). We claim that every cycle on X that is rationally equivalent to L has support containing p, and thus the conclusion of part (a) of the moving lemma does not hold for X.

To show this, we first remark that the degrees of any two rationally equivalent curves on X are the same; that is, there is a function $\deg : A_1(X) \to \mathbb{Z}$ taking each irreducible curve to its degree. For, if $i : X \to \mathbb{P}^3$ is the inclusion, then for any curve D on X we have

$$\deg D = \deg(\zeta \cdot i_*([D])),$$

where ζ is the class of a hyperplane in $\mathbb{P}^3$. In particular $\deg L = 1$ is odd.

Now let $D \subset X$ be any curve not containing p. We claim that the degree of D must be even. To see this, observe that the projection map $\pi_p : D \to C$ is a finite map whose fibers are the intersections of D with the lines of X; it follows that a general line in X will intersect D transversely in $\deg(\pi_p)$ points. Now let $H \subset \mathbb{P}^3$ be a general plane through p. H intersects X in the union of two general lines $L, L' \subset X$, and so meets D transversely in $2\deg(\pi_p)$ points, so $\deg D$ is even. It follows that any cycle of dimension 1 on X, effective or not, whose support does not contain p has even degree, and hence cannot be rationally equivalent to L.

Retaining the notation of Example 2.21, one might hope to define an intersection product on $A(X)$ even without the moving lemma. It seems natural to think that since two distinct lines $L, L' \subset X$ through p meet in the reduced point p, we would have $[L][L'] = [p]$. However, if ζ is the class of a general plane section $H \cap X$ of X through p,

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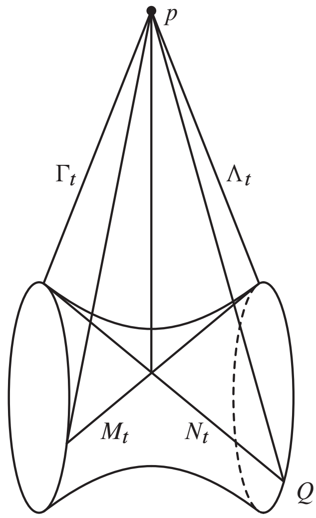

Figure 2.11 The intersection product of $[\Lambda_t]$ and the class of a line cannot be defined.

then (since such a hyperplane meets each L transversely in one point) we might also expect $\zeta[L] = [p]$. But ζ is rationally equivalent to the union of two lines through p. Thus, if both expectations were satisfied, we would have

$$[p] = \zeta[L'] = 2[L][L'] = 2[p].$$

Applying the degree map, we would get the contradiction $1 = 2$.

There is a way around the difficulty, if we work in the ring $A(X) \otimes \mathbb{Q}$: We can take the product of the classes of two lines to be one-half the class of the point p, and our contradiction is resolved. As Mumford has pointed out, something similar can be done for all normal surfaces (see Example 8.3.11 of Fulton [1984]). But in higher dimensions there are more difficult problems, as the following example shows:

Example 2.22. Let $Q \subset \mathbb{P}^3 \subset \mathbb{P}^4$ be a smooth quadric surface, and let $X = \overline{p, Q}$ be the cone in $\mathbb{P}^4$ with vertex $p \notin \mathbb{P}^3$. The quadric Q contains two families of lines $\{M_t\}$ and $\{N_t\}$, and the cone X is correspondingly swept out by the two families of 2-planes $\{\Lambda_t = \overline{p, M_t}\}$ and $\{\Gamma_t = \overline{p, N_t}\}$; see Figure 2.11.

Now, any line $L \subset X$ not passing through the vertex p maps, under projection from p, to a line of Q; that is, it must lie either in a plane Λ_t or in a plane Γ_t; lines on X that do pass through p lie on one plane of each type. Note that since lines $M_t, M_{t'} \subset Q$ of the same ruling are disjoint for $t \neq t'$, while lines M_t and $N_{t'}$ of opposite rulings meet in a point, a general line $M \subset X$ lying in a plane Λ_t is disjoint from $\Lambda_{t'}$ for $t \neq t'$ and meets each plane Γ_s transversely in a point. Thus, if there were any intersection product on $A(X)$ satisfying the fundamental condition $(*)$ of Theorem 1.5, we would have

$$[M][\Lambda_t] = 0 \quad \text{and} \quad [M][\Gamma_t] = [q]$$

for some point $q \in X$. Likewise, for a general line $N \subset X$ lying in a plane Γ_t, the opposite would be true; that is, we would have

$$[N][\Lambda_t] = [r] \quad \text{and} \quad [N][\Gamma_t] = 0.$$

But the lines M and N — indeed, any two lines on X — are rationally equivalent! Since any two lines in Λ_t are rationally equivalent, the line M is rationally equivalent to the line of intersection $\Lambda_t \cap \Gamma_s$. Since any two lines in Γ_s are rationally equivalent, the line of intersection (and thus also M) is rationally equivalent to an arbitrary line in Γ_s. Since a point cannot be rationally equivalent to 0 on X, we have a contradiction. Thus products such as $[M][\Lambda_t]$ cannot be defined in $A(X)$.

Despite this trouble, one can still define $f_M^*[\Lambda_t]$ and $f_N^*[\Lambda_t]$ using methods of Fulton [1984]. In fact, one can define the pullback f^* for an inclusion morphism $f : B \hookrightarrow X$ that is a "regular embedding" (which means that B is locally a complete intersection in X), or for the composition of such a morphism with a flat map.

Example 2.22 also shows that, even though f_{M*} is well-defined, pullbacks cannot be defined, at least in a way that makes the push-pull formula valid. If X were smooth, then by the push-pull formula $[M][\Lambda_t]$ would be equal to $f_{M*}([M]f_M^*[\Lambda_t])$, where the product $[M]f_M^*[\Lambda_t]$ should be interpreted as being in $A(M)$. This product is well-defined, as are the pullback and pushforward. But they do not allow us to compute the product $[M][\Lambda_t]$; since $[M] = [N]$ in $A(X)$, we would arrive at the contradiction

$$0 = f_{M*}(f_M^*[\Lambda_t]) = [M][\Lambda_t] = [N][\Lambda_t] = f_{N*}(f_N^*[\Lambda_t]) = [r].$$

There are, however, certain cycles (such as those represented by Chern classes of bundles) with which one can intersect, and this leads to a notion of "Chow cohomology" groups $A^*(X)$, which play a role relative to the Chow groups analogous to the role of cohomology relative to homology in the topological context: we have intersection products

$$A^c(X) \otimes A^d(X) \to A^{c+d}(X)$$

and

$$A^c(X) \otimes A_k(X) \to A_{k-c}(X)$$

analogous to cup and cap products in topology. In the present volume we will avoid all of this by sticking for the most part to the case of intersections on smooth varieties, where we can simply equate $A^c(X) = A_{\dim X - c}(X)$; for the full treatment, see Fulton [1984, Chapters 6, 8 and 17], and, for a visionary account of what might be possible, Srinivas [2010].

2.6 Exercises

Exercise 2.23. Let $\nu = \nu_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ be the quadratic Veronese map. If $C \subset \mathbb{P}^2$ is a plane curve of degree d, show that the image $\nu(C)$ has degree $2d$. (In particular, this means that the Veronese surface $S \subset \mathbb{P}^5$ contains only curves of even degree!) More generally, if $\nu = \nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ is the degree-d Veronese map and $X \subset \mathbb{P}^n$ is a variety of dimension k and degree e, show that the image $\nu(X)$ has degree $d^k e$.

Exercise 2.24. Let $\sigma = \sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre map, and let $X \subset \mathbb{P}^r \times \mathbb{P}^s$ be a subvariety of codimension k. Let the class $[X] \in A^k(\mathbb{P}^r \times \mathbb{P}^s)$ be given by

$$[X] = c_0\alpha^k + c_1\alpha^{k-1}\beta + \cdots + c_k\beta^k$$

(where $\alpha, \beta \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ are the pullbacks of the hyperplane classes, and we take $c_i = 0$ if $i > s$ or $k - i > r$).

(a) Show that all $c_i \geq 0$.
(b) Calculate the degree of the image $\sigma(X) \subset \mathbb{P}^{(r+1)(s+1)-1}$.
(c) Using (a) and (b), show that any linear space $\Lambda \subset \Sigma_{r,s} \subset \mathbb{P}^{(r+1)(s+1)-1}$ contained in the Segre variety lies in a fiber of either the map $\Sigma_{r,s} \cong \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^r$ or the corresponding map to $\mathbb{P}^s$.

Exercise 2.25. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi : (x_0, x_1, x_2) \dashrightarrow \left(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right),$$

or, equivalently,

$$\varphi : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1),$$

and let $\Gamma_\varphi \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of φ. Find the class

$$[\Gamma_\varphi] \in A^2(\mathbb{P}^2 \times \mathbb{P}^2).$$

Exercise 2.26. Let $\sigma : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ be the Segre map. Find the class of the graph of σ in $A(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^8)$.

Exercise 2.27. Let $s : \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^{2*}$ be the rational map sending $(p, q) \in \mathbb{P}^2 \times \mathbb{P}^2$ to the line $\overline{p, q}$. Find the class of the graph of s in $A(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*})$.

Exercise 2.28. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Suppose that X has an ordinary double point (that is, a point $p \in X$ such that the projective tangent cone $\mathbb{T}C_pX$ is a smooth quadric), and is otherwise smooth. What is the degree of the dual hypersurface $X^* \subset \mathbb{P}^{n*}$?

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Exercise 2.29. Let $p \in X \subset \mathbb{P}^n$ be a variety of degree d and dimension k, and suppose that $p \in X$ is a point of multiplicity m (see Section 1.3.8 for the definition). Assuming that the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ is birational onto its image, what is the degree of $\pi_p(X)$?
Hint: Use Proposition 2.14.

Exercise 2.30. Show that the Chow ring of a product of projective spaces $\mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_k}$ is

$$\begin{aligned} A(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}) &= \bigotimes A(\mathbb{P}^{r_i}) \\ &= \mathbb{Z}[\alpha_1, \dots, \alpha_k]/(\alpha_1^{r_1+1}, \dots, \alpha_k^{r_k+1}), \end{aligned}$$

where $\alpha_1, \dots, \alpha_k$ are the pullbacks of the hyperplane classes from the factors. Use this to calculate the degree of the image of the Segre embedding

$$\sigma : \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k} \hookrightarrow \mathbb{P}^{(r_1+1)\cdots(r_k+1)-1}$$

corresponding to the multilinear map $V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$.

Exercise 2.31. For $t \neq 0$, let $A_t : \mathbb{P}^r \to \mathbb{P}^r$ be the automorphism

$$[X_0, X_1, X_2, \dots, X_r] \mapsto [X_0, tX_1, t^2X_2, \dots, t^rX_r].$$

Let $\Phi \subset \mathbb{A}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the locus

$$\Phi^\circ = \{(t, p, q) \mid t \neq 0 \text{ and } q = A_t(p)\}.$$

Describe the fiber of Φ over the point $t = 0$, and deduce once again the formula of Section 2.1.6 for the class of the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$.

In the simplest case, this construction is a rational equivalence between a smooth plane section of a quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (the diagonal, in terms of suitable identifications of the factors with $\mathbb{P}^1$) and a singular one (the sum of a line from each ruling), as in Figure 2.12.

Exercise 2.32. Let

$$\Psi = \{(p, q, r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n \mid p, q \text{ and } r \text{ are collinear in } \mathbb{P}^n\}.$$

(Note that this includes all diagonals.)

(a) Show that this is a closed subvariety of codimension $n-1$ in $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$.
(b) Use the method of undetermined coefficients to find the class

$$\psi = [\Psi] \in A^{n-1}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n).$$

(We will see a way to calculate the class $[\psi]$ using Porteous' formula in Exercise 12.9.)

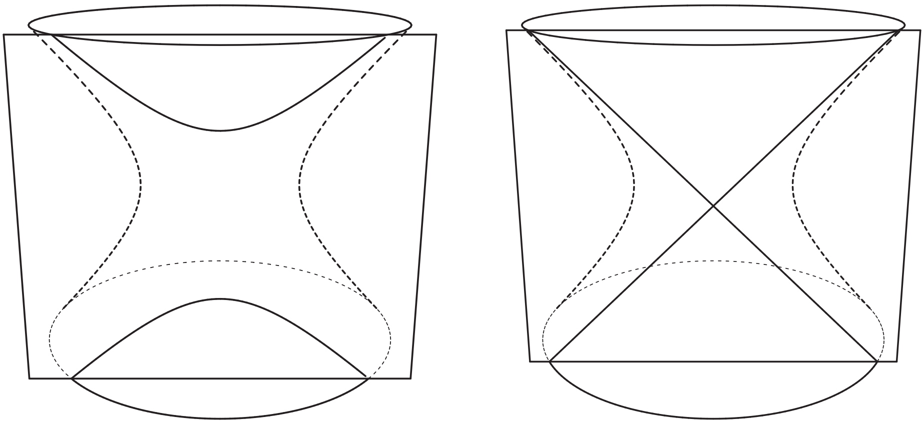

Figure 2.12 The diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to a sum of fibers.

Exercise 2.33. Suppose that $(F_0, \dots, F_r)$ and $(G_0, \dots, G_r)$ are general $(r+1)$-tuples of homogeneous polynomials in $r+1$ variables, of degrees d and e respectively, so that in particular the maps $f : \mathbb{P}^r \to \mathbb{P}^r$ and $g : \mathbb{P}^r \to \mathbb{P}^r$ sending x to $(F_0(x), \dots, F_r(x))$ and $(G_0(x), \dots, G_r(x))$ are regular. For how many points $x = (x_0, \dots, x_r) \in \mathbb{P}^r$ do we have $f(x) = g(x)$?

The next two exercises set up Exercise 2.36, which considers when a point $p \in \mathbb{P}^2$ will be collinear with its images under several maps:

Exercise 2.34. Consider the locus $\Phi \subset (\mathbb{P}^2)^4$ of 4-tuples of collinear points. Find the class $\varphi = [\Phi] \in A^2((\mathbb{P}^2)^4)$ of Φ by the method of undetermined coefficients, that is, by intersecting with cycles of complementary dimension.

Exercise 2.35. With $\Phi \subset (\mathbb{P}^2)^4$ as in the preceding problem, calculate the class $\varphi = [\Phi]$ by using the result of Exercise 2.32 on the locus $\Psi \subset (\mathbb{P}^2)^3$ of triples of collinear points and considering the intersection of the loci $\Psi_{1,2,3}$ and $\Psi_{1,2,4}$ of 4-tuples (p_1, p_2, p_3, p_4) with (p_1, p_2, p_3) and (p_1, p_2, p_4) each collinear.

Exercise 2.36. Let A, B and $C : \mathbb{P}^2 \to \mathbb{P}^2$ be three general automorphisms. For how many points $p \in \mathbb{P}^2$ are the points p, $A(p)$, $B(p)$ and $C(p)$ collinear?

Exercise 2.37. Let B be the blow-up of $\mathbb{P}^n$ at a point p, with exceptional divisor E as in Section 2.1.9. With notation as in that section, show that there is an affine stratification with closed strata Γ_k for $k = 1, \dots, n$ and $E_k := \Gamma_k \cap E$ for $k = 0, \dots, n-1$. Let e_k be the class of E_k. Show that $e_{n-1} = \lambda_{n-1} - \gamma_{n-1}$ to describe the classes γ_k in terms of λ_k and e_k and vice versa. Conclude that the classes $\gamma_k = [\Gamma_k]$ and e_k form a basis for the Chow group $A(B)$.

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Exercises 2.38–2.40 deal with the blow-up of $\mathbb{P}^3$ along a line. To fix notation, let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a line $L \subset \mathbb{P}^3$, that is, the graph $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ of the rational map $\pi_L : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ given by projection from L. Let $\alpha : X \to \mathbb{P}^1$ be projection on the second factor.

Exercise 2.38. Let $H \subset \mathbb{P}^3$ be a plane containing L and $\widetilde{H} \subset X$ its proper transform. Let $J \subset \mathbb{P}^3$ be a plane transverse to L, $\widetilde{J} \subset X$ its proper transform (which is equal to its preimage in X) and $M \subset J$ a line not meeting L. Show that the subvarieties

$$X, \ \widetilde{H}, \ \widetilde{J}, \ \widetilde{J} \cap \widetilde{H}, \ M, \ M \cap \widetilde{H}$$

are the closed strata of an affine stratification of X, with open strata isomorphic to affine spaces. In particular, since only one (the subvariety $M \cap \widetilde{H}$) is a point, deduce that $A^3(X) \cong \mathbb{Z}$.

Exercise 2.39. Let $h = [\widetilde{H}]$, $j = [\widetilde{J}] \in A^1(X)$ and $m = [M] \in A^2(X)$ be the classes of the corresponding strata. Show that

$$h^2 = 0, \quad j^2 = m \ \text{ and } \ \deg(jm) = \deg(hm) = 1.$$

Conclude that

$$A(X) = \mathbb{Z}[h, j]/(h^2, j^3 - hj^2).$$

Exercise 2.40. Now let $E \subset X$ be the exceptional divisor of the blow-up, and $e = [E] \in A^1(X)$ its class. What is the class e^2?

Exercise 2.41. Let $\mathbb{P}^5$ be the space of conic curves in $\mathbb{P}^2$.

(a) Find the dimension and degree of the locus of double lines (in characteristic $\neq 2$).
(b) Find the dimension and degree of the locus $\Delta \subset \mathbb{P}^5$ of singular conics (that is, line pairs and double lines).

Exercises 2.42–2.54 deal with some of the loci in the space $\mathbb{P}^9$ of plane cubics described in Section 2.2.

Exercise 2.42. Let $\mathbb{P}^9$ be the space of plane cubics and $\Gamma \subset \mathbb{P}^9$ the locus of reducible cubics. Let $L, C \subset \mathbb{P}^2$ be a line and a smooth conic intersecting transversely at two points $p, q \in \mathbb{P}^2$; let $L + C$ be the corresponding point of Γ. Show that Γ is smooth at $L + C$, with tangent space

$$\mathbb{T}_{L+C}\Gamma = \mathbb{P}\{\text{homogeneous cubic polynomials } F \mid F(p) = F(q) = 0\}.$$

Exercise 2.43. Using the preceding exercise, show that, if $p_1, \dots, p_7 \in \mathbb{P}^2$ are general points and $H_i \subset \mathbb{P}^9$ is the hyperplane of cubics containing p_i, then the hyperplanes $H_1, \dots, H_7$ intersect Γ transversely — that is, the degree of Γ is the number of reducible cubics through $p_1, \dots, p_7$.

https://doi.org/10.1017/CBO9781139062046.004 Published online by Cambridge University Press

Exercise 2.44. Calculate the number of reducible plane cubics passing through seven general points $p_1, \dots, p_7 \in \mathbb{P}^2$, and hence, by the preceding exercise, the degree of Γ.

Exercise 2.45. We can also calculate the degree of the locus $\Sigma \subset \mathbb{P}^9$ of triangles (that is, totally reducible cubics) directly, as in Exercises 2.42–2.44. To start, show that if $C = L_1 + L_2 + L_3$ is a triangle with three distinct vertices — that is, points $p_{i,j} = L_i \cap L_j$ of pairwise intersection — then Σ is smooth at C with tangent space

$$\mathbb{T}_{L+C}\Sigma = \mathbb{P}\{\text{homogeneous cubic polynomials } F \mid F(p_{i,j}) = 0 \text{ for all } i, j\}.$$

Exercise 2.46. Using the preceding exercise,

(a) show that if $p_1, \dots, p_6 \in \mathbb{P}^2$ are general points, then the degree of Σ is the number of triangles containing $p_1, \dots, p_6$; and
(b) calculate this number directly.

Exercise 2.47. Consider a general asterisk — that is, the sum $C = L_1 + L_2 + L_3$ of three distinct lines all passing through a point p. Show that the variety $\Sigma \subset \mathbb{P}^9$ of triangles is smooth at C, with tangent space the space of cubics double at p. Deduce that the space $A \subset \mathbb{P}^9$ of asterisks is also smooth at C.

Exercise 2.48. Let $p_1, \dots, p_5 \in \mathbb{P}^2$ be general points. Show that any asterisk containing $\{p_1, \dots, p_5\}$ consists, possibly after relabeling the points, of the sum of the line $L_1 = \overline{p_1, p_2}$, the line $L_2 = \overline{p_3, p_4}$ and the line $L_3 = \overline{p_5, (L_1 \cap L_2)}$.

Exercise 2.49. Using the preceding two exercises, show that, if $p_1, \dots, p_5 \in \mathbb{P}^2$ are general points, then the hyperplanes H_{p_i} intersect the locus $A \subset \mathbb{P}^9$ of asterisks transversely, and calculate the degree of A accordingly.

Exercise 2.50. Show that (in characteristic $\neq 3$) the locus $Z \subset \mathbb{P}^9$ of triple lines is a cubic Veronese surface, and deduce that its degree is 9.

Exercise 2.51. Let $X \subset \mathbb{P}^9$ be the locus of cubics of the form $2L + M$ for L and M lines in $\mathbb{P}^2$.

(a) Show that X is the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under a regular map such that the pullback of a general hyperplane in $\mathbb{P}^9$ is a hypersurface of bidegree $(2, 1)$.
(b) Use this to find the degree of X.

Exercise 2.52. If you try to find the degree of the locus X of the preceding problem by intersecting X with hyperplanes $H_{p_1}, \dots, H_{p_4}$, where

$$H_p = \{C \in \mathbb{P}^9 \mid p \in C\},$$

you get the wrong answer (according to the preceding problem). Why? Can you account for the discrepancy?

Exercise 2.53. Let $\mathbb{P}^2$ denote the space of lines in the plane and $\mathbb{P}^5$ the space of plane conics. Let $\Phi \subset \mathbb{P}^2 \times \mathbb{P}^5$ be the closure of the locus of pairs

$$\{(L, C) \mid C \text{ is smooth and } L \text{ is tangent to } C\}.$$

Show that Φ is a hypersurface, and, assuming characteristic 0, find its class $[\Phi] \in A^1(\mathbb{P}^2 \times \mathbb{P}^5)$.

Exercise 2.54. Let $Y \subset \mathbb{P}^9$ be the closure of the locus of reducible cubics consisting of a smooth conic and a tangent line. Use the result of Exercise 2.53 to determine the degree of Y.

Exercise 2.55. Let $\mathbb{P}^{14}$ be the space of quartic curves in $\mathbb{P}^2$, and let $\Sigma \subset \mathbb{P}^{14}$ be the closure of the space of reducible quartics. What are the irreducible components of Σ, and what are their dimensions and degrees?

Exercise 2.56. Find the dimension and degree of the locus $\Omega \subset \mathbb{P}^{14}$ of totally reducible quartics (that is, quartic polynomials that factor as a product of four linear forms).

Exercise 2.57. Again let $\mathbb{P}^{14}$ be the space of plane quartic curves, and let $\Theta \subset \mathbb{P}^{14}$ be the locus of sums of four concurrent lines. Using the result of Exercise 2.34, find the degree of Θ.

Exercise 2.58. Find the degree of the locus $A \subset \mathbb{P}^{14}$ of the preceding problem, this time by calculating the number of sums of four concurrent lines containing six general points $p_1, \dots, p_6 \in \mathbb{P}^2$, assuming transversality.

A natural generalization of the locus of asterisks, or of sums of four concurrent lines, would be the locus, in the space $\mathbb{P}^N$ of hypersurfaces of degree d in $\mathbb{P}^n$, of *cones*. We will indeed be able to calculate the degree of this locus in general, but it will require more advanced techniques than we have at our disposal here; see Section 7.3.4 for the answer.

Exercise 2.59. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d and $L \subset S$ a line. Calculate the degree of the self-intersection of the class $\lambda = [L] \in A^1(S)$ by considering the intersection of S with a general plane $H \subset \mathbb{P}^3$ containing L.

Exercise 2.60. Let S be a smooth surface. Show that if $C \subset S$ is any irreducible curve such that the corresponding point in the Hilbert scheme $\mathcal{H}$ of curves on S (see Section 6.3) lies on a positive-dimensional irreducible component of $\mathcal{H}$, then the degree $\deg(\gamma^2)$ of the self-intersection of the class $\gamma = [C] \in A^1(S)$ is nonnegative. Using this and the preceding exercise, prove the statement made in Section 2.4.2 that *a smooth surface $S \subset \mathbb{P}^3$ of degree 3 or more can contain only finitely many lines*.

Exercise 2.61. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve. Show that

(a) if C has genus 2, it must lie on a quadric surface;
(b) if C has genus 1, it cannot lie on a quadric surface; and

(c) if C has genus 0, it may or may not lie on a quadric surface (that is, some rational quintic curves do lie on quadrics and some do not).

Exercise 2.62. Let $C \subset \mathbb{P}^3$ be a smooth quintic curve of genus 2. Show that C lies on a quadric surface Q and a cubic surface S with intersection $Q \cap S$ consisting of the union of C and a line.

Exercise 2.63. Use the result of Exercise 2.62 — showing that a smooth quintic curve of genus 2 is linked to a line in the complete intersection of a quadric and a cubic — to find the dimension of the subset of the Hilbert scheme corresponding to smooth curves of degree 5 and genus 2.