Algebraic Topo logy Homework 2

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§ *Problems from 1.2*

EXERCISE 1.2.1. Show that the free product $G * H$ of nontrivial groups G and H has trivial center, and that the only elements of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Proof: Recall that two elements of $G * H$ are equal if and only if their reductions are identical. We use this fact without comment.

Suppose that $g \in G$ and $h \in H$ are both nontrivial elements. Then both ghg^{-1} and h are reduced in $G * H$, and hence are not equal as they are of different lengths. This means $gh \neq hg$ for all nontrivial elements $g \in G$ and $h \in H$.

Now suppose we have some reduced word $w_1 w_2 ... w_n \in G * H$ where $w_i \in G \cup H$ for $1 \leq i \leq n$ and $n \geq 2$. Again let $g \in G$ and $h \in H$ be reduced words. We have four cases to consider.

(1) If $w_1, w_k \in G$, then hw and wh are both reduced and are hence not equal.

(2) If $w_1, w_k \in H$, then gw and wg are both reduced and are hence not equal.

(3) If $w_1 \in G$ and $w_k \in H$, then $w_2 \in H$ by the assumption that w is reduced. Hence both $gw_2...w_k$ and wg are reduced, and since $k \geq 2$, we have that $gw \neq wg$.

(4) If $w_1 \in H$ and $w_k \in G$, then $w_2 \in G$ and we get $hw \neq wh$ by the same argument as above.

Thus, every nontrivial element of $G * H$ fails to commute with some other element, meaning the center of $G * H$ is trivial.

We now show that the only elements of $G * H$ are the conjugates of finite-order elements of G and H. Let $w \in G * H$ be finite order, i.e. assume $w^k = 1$ where 1 is the empty word for some $k \in \mathbb{N}$.

First, notice that w must have an odd number of letters. If $w = w_1...w_{2n}$ is reduced, then w_1 and w_{2n} belong to different groups, and therefore $w^2 = w_1...w2nw_1...w2n$ is also reduced. Successive multiplication of w with itself will only make the word longer. w must therefore have an odd number of elements in order to reduce upon successive multiplication. Thus the reduced form of w is $w_1...w_{2n+1}$.

As previously noted, we need w to shrink upon successive products. This means that w_1 and w_{2k+1} must multiply to 1 in either H or G, i.e. $w_1 = w_{2n+1}^{-1}$. Similarly, $w_2 = w_{2n}^{-1}$, $w_3 = w_{2n-1}^{-1}$, and $w_n = w_{n+2}^{-1}$. This observation means that

$$(w_1...w_n)^{-1} = w_n^{-1}...w_1{-}1 = w_{n+2}...w_{2n+1}.$$

Therefore

$$w = (w_1...w_n)w_{n+1}(w_{n+2}...w_{2n+1})$$

and finally,

$$w^k = (w_1...w_n)w_{n+1}^k(w_{n+2}...w_{2n+1}) = 1 \implies w_{n+1}^k = 1$$

And since w_{n+1} must be an element in either H or G, we conclude that w is the conjugate of some finite order element in G or H.

□

EXERCISE 1.2.2. Let $X \subseteq \mathbb{R}^m$ be the union of convex open sets $X_1, ..., X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.

Proof: We proceed by induction on n. If $n = 1$, then X itself is a convex open set and is hence homeomorphic to an open ball in $\mathbb{R}^m$, so there is nothing to prove. We do the $n = 2$ case too as a warm up, since it features precisely the setup required for Van Kampen's theorem. We can cover X with X_1 and X_2, each of which is a convex set, such that $X_1 \cap X_2 \neq \emptyset$. The intersection of convex sets is convex (this is a fact I believe I may use) and hence $X_1 \cap X_2$ is path-connected. Hence, by Van Kampen, for any $x_0 \in X_1 \cap X_2$ we have

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0)/N \cong 1 * 1/N \cong 1$$

since both X_1 and X_2 are simply connected.

Now suppose the statement of the problem holds for $1, ..., n$ and that X is a union of open convex sets $X_1, ..., X_{n+1}$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for distinct i, j, k. By the inductive hypothesis the set $Y = X_1 \cup ... \cup X_n \subset X$ is simply connected. To apply Van Kampen we simply need that $Y \cap X_{n+1}$ is path connected. Choose $x, y \in Y \cap X_{n+1}$. First, notice that we may write

$$Y \cap X_{n+1} = (X_1 \cap X_{n+1}) \cup ... \cup (X_n \cap X_{n+1}),$$

so there is some i and j such that $x \in X_i \cap X_{n+1}$ and $y \in X_j \cap X_{n+1}$. By assumption, the intersection

$$X_i \cap X_j \cap X_{n+1}$$

is nonempty, so we may choose some $z \in X_i \cap X_j \cap X_{n+1}$. Because the intersection of convex sets is convex, we may connect x with z in $X_i \cap X_{n+1}$ via a line segment α and z with y in $X_j \cap X_{n+1}$ via a line segment β. The path $\alpha \cdot \beta$ obtained by concatenating these two line segments is then a path from x to y, hence $Y \cap X_{n+1}$ is path connected. Applying Van Kampen as before, we get that

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0)/N \cong 1 * 1/N \cong 1$$

since both Y and X_{n+1} are simply connected, and we are done. $\square$

EXERCISE 1.2.4 Let $X \subseteq \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

Proof: Set $Y = \mathbb{R}^3 - X$ and consider the unit sphere $S^2 \subset \mathbb{R}^3$ and let $\{p_1, ..., p_{2n}\}$. Each line through the origin intersects S^2 at exactly two points (antipodal points in fact) and hence $X \cap S^2 = \{p_1, ..., p_{2n}\}$.

I first claim that Y deformation retracts onto $S^2 - \{p_1, ..., p^2\}$. This is actually not difficult, simply retract each point $y \in Y$ to the surface of S^2 along the ray connecting y to the origin in $\mathbb{R}^3$. This means that

$$\pi_1(Y, y_0) \cong \pi_1(S^2 - \{p_1, ..., p_{2n}\}).$$

Next, I claim that $S^2 - \{p_1, ..., p_{2n}\}$ deformation retracts onto a wedge of $2n - 1$ circles. This again isn't too hard, but does require more description. Without loss of generality, assume that $p_1, ..., p_{2n-1}$ all lie on a great circle containing $p_{2n} = N$, the north pole. Around each point p_i with $1 \leq i < 2n$ we may place a copy of S^1 such that it intersects the loops around its nearest neighbors at exactly one point, giving us a wedge of $2n - 1$ copies of S^1. The two nearest neighbors to N will intersect only one of these loops each. From the interior of each of these loops in $S^2 - \{p_1, ..., p_{2n}\}$ we deformation retract to the outer boundary, and for any point $y \in S^2 - \{p_1, ..., p_2 n\}$ not contained in one of these loops we deformation retract along the great arc connecting y to N away from N. This gives us a deformation retract of $S^2 - \{p_1, ..., p_{2n}\}$ to $\bigvee_{i=1}^{2n-1} S^1$, and thus by Proposition 1.17,

$$\pi_1(Y, y_0) \cong \pi_1(S^2 - \{p_1, ..., p_{2n}\}) \cong \pi_1\left(\bigvee_{i=1}^{2n-1} S^1, y_0\right) \cong \overbrace{\mathbb{Z} * ... * \mathbb{Z}}^{2n-1 \text{ times}} \cong F_{2n-1}.$$

Hence $\pi_1(Y, y_0)$ is isomorphic to the free group in F_{2n-1} generators. □

EXERCISE 1.2.7 Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

Proof: Take $X^0 = \{\text{pt}\}$, and let the one skeleton X^1 consist of a single interval with both endpoints attached to pt. That is, X^1 is a circle.

We obtain X by attaching a single disk to X^1 in the following way. Regard S^1, the boundary of D^2, as a square with sides labeled a, b, c and d starting from the top and moving anticlockwise. Our attaching map $\varphi : S^1 \longrightarrow X^1$ is defined as follows.

1. Attach a to X^1 by wrapping it once clockwise.

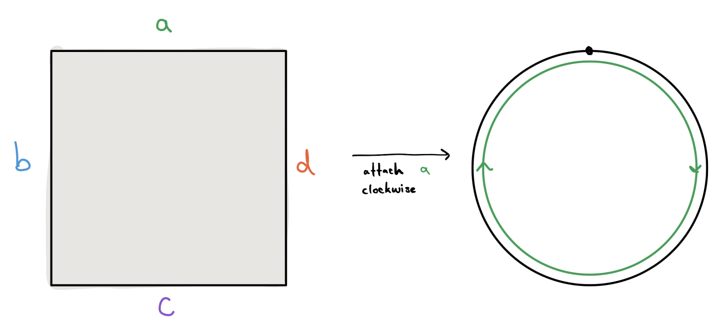

Figure 1: Wrap a clockwise around X^1

2. Collapse b and d to the basepoint pt.

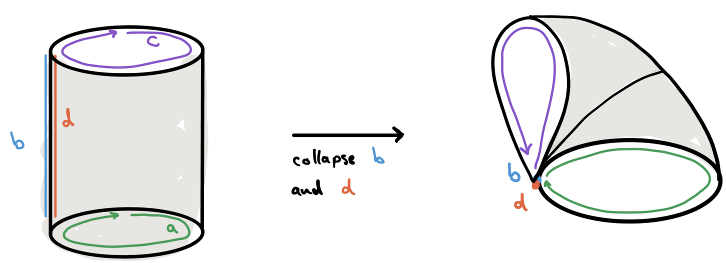

Figure 2: Collapse b and d to points

3. Attach c by wrapping it once around X^1 anticlockwise.

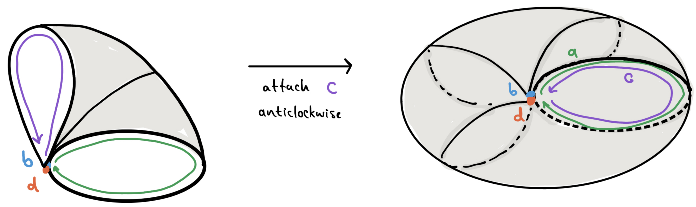

Figure 3: Wrap c anticlockwise around X^1

Ignore the fact that the inner X^1 circle appears to not be filled in Figure 3, it is. This process gives a CW-structure on X, and we can now compute $\pi_1(X, x_0)$ using Proposition 1.26. The inclusion $X^1 \longrightarrow X$ induces a surjection $\pi_1(X^1, x_0) \longrightarrow \pi_1(X, x_0)$ by part (a), whose kernel is generated by conjugations of the attaching map by change of basepoint maps. Choosing $x_0 = \text{pt}$, we then have that the kernel is generated by $[\varphi]$ itself. However, $[\varphi] = 0$ in $\pi_1(X^1, x_0)$ since it is the loop given by rotating once around X^1 in both directions. Hence we have an isomorphismk

$$\pi_1(X, x_0) \cong \pi_1(X^1, x_0) \cong \mathbb{Z}.$$

$\square$

Exercise 1.2.11. The **mapping torus** T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof: We consider first the case where $X = S^1 \vee S^1$. We can express X as a CW-complex with one 0-cell and two 1-cells through the following construction. Let x_0 be a 0-cell. Attach the ends of two 1-cells to x_0, and we have X.

Now, because f is basepoint preserving, if we take x_0 to be our basepoint, $x_0 \mapsto x_0$ which means that under the equivalence relation, $(x_0, 0) \mapsto (x_0, 1)$. As stated in Hatcher, we can regard T_f as the construction of $X \vee S^1$ with appropriate cells attached, i.e. as the space obtained by taking every k cell in X and attaching a $k + 1$ cell. This is visualized in the diagram below. By Proposition 1.26, we therefore have that $\pi_1(T_f) \cong \pi_1(X \vee S^1)/N$. However, this is precisely the fundamental group from question (8). Thus,

$$\pi_1(T_f) \approx (\mathbb{Z} * \mathbb{Z} * \mathbb{Z})/\langle aba^{-1}b^{-1}, cdc^{-1}d^{-1} \rangle$$

Where $a = f_*(a)$, etc.

We now consider the case where $X = S^1 \times S^1$. This is a torus. We once again regard T_f as the space obtained by attaching appropriate cells to $X \vee S^1$. This time we attach one 3-cell (for the 2-cell of the torus) and two two-cells (for the two 1-cells of the torus). One again, the wedge with S^1 is the result of attaching one 1-cell to the basepoint of X.

From part (b) of Proposition 1.26, we know that the 3-cell is simply connected and therefore doesn't affect $\pi_1(T_f)$. We therefore obtain almost exactly the same fundamental group as before, except that we have an extra 1-cell. This extra cell causes a and b to commute. Therefore,

$$\pi_1(T_f) \approx (\mathbb{Z} * \mathbb{Z} * \mathbb{Z}) / \langle aba^{-1}b^{-1}, cdc^{-1}d^{-1} \mid ab = ba \rangle$$

$\square$

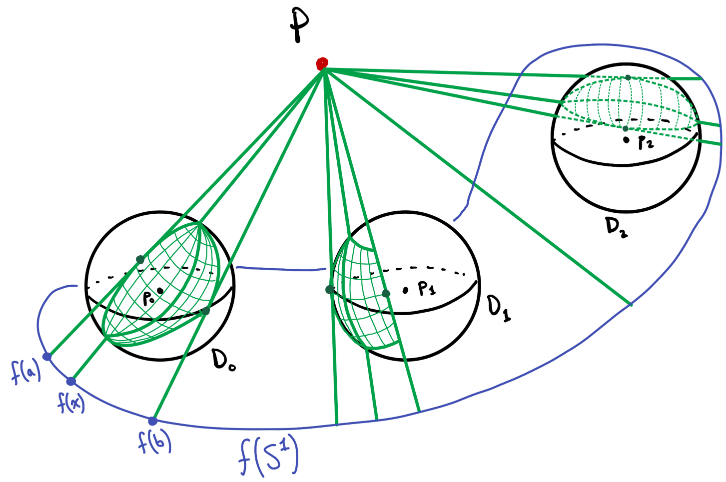

Figure 4: The homotopy in Case 1