Problems from Hartshorne Chapter 2.2

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EXERCISE 1. Let A be an abelian group and defined the *constant presheaf* associated to A on the topological space X to be the preshseaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.

Proof: Let $\mathcal{C}$ be the constant sheaf on X, i.e. the sheaf defined as follows: for any open $U \subseteq X$, $\mathcal{C}(U)$ is the group of all continuous maps of U into A (where A is endowed with the discrete topology). Let $\mathcal{G}$ be any other sheaf on X.

Define $\theta : \mathcal{F} \longrightarrow \mathcal{C}$ as follows. For an open set U, let $\theta(U) : \mathcal{F}(U) = A \longrightarrow \mathcal{C}(U)A$ send a point $a \in A$ to the constant map $(x \mapsto a) \in \mathcal{C}(U)$.

Now suppose we have some morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$. We would like to define $\beta : \mathcal{C} \longrightarrow \mathcal{G}$ such that $\beta \circ \theta = \alpha$.

Fix an open subset $U \subseteq X$ and a section $f : U \longrightarrow A$ of $\mathcal{C}(U)$. Notice that $\{f^{-1}(a)\}_{a \in A}$ is an open cover of U and $f|_{f^{-1}(a)} = (x \mapsto a) = \theta(U)(a)$ for all $a \in A$. Consider the collection $\{\alpha(U)(a)\}_{a \in A}$ of sections in $\mathcal{G}(U)$. These satisfy the gluing compatibility condition, namely

$$\alpha(U)(a)|_{f^{-1}(a) \cap f^{-1}(b)} = \alpha(U)(b)|_{f^{-1}(a) \cap f^{-1}(b)}$$

and hence there is some element $g_f \in \mathcal{G}(U)$ such that $g_f|_{f^{-1}(a)} = \alpha(U)(a)|_{f^{-1}(a)}$ for all $a \in A$. We simply define $\beta(U)(f) = g_f$ to obtain a map $\beta(U) : \mathcal{C}(U) \longrightarrow \mathcal{G}(U)$. This satisfies the restriction requirements and hence β is a map of schemes. Furthermore, if $f = \theta(U)(a)$ for some $a \in A$, then f is the constant map $x \mapsto a$ and hence $f^{-1}(a) = U$, so $\beta(f) = \alpha(U)(a)$. This shows that $\alpha = \beta \circ \theta$, meaning $\mathcal{C}$ satisfies the universal property of the sheaf associated to $\mathcal{F}$. $\square$

EXERCISE 2.

(a) For any morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence ... $\longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^i \xrightarrow{\varphi^i} \mathcal{F}^{i+1} \longrightarrow$ ... of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof:

(a) Recall that for any $V \subseteq X$ containing a point P we have the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{F}_P & \xrightarrow{\varphi_P} & \mathcal{G}_P \end{array}$$

Start with an element $(t, V) \in \ker(\varphi_P)$. Then t is a section of $\mathcal{F}(V)$ by definition and by commutativity of the diagram we have that $\pi(\varphi(V)(t)) = (\varphi(V)(t), V) = 0$ in $\mathcal{G}_P$. This means that there is some open neighborhood $W \subset V$ of P such that $\varphi(U)(t)|_W = 0$ by the equivalence relation on $\mathcal{G}_P$, and since $\varphi(U)(t)|_W = \varphi(W)(t)$ we have that $\varphi(W)(t|_W) = 0$. Hence $t|_W = 0$ and so $t \in \ker \varphi(W)$. Hence $(t|_W, W) \in (\ker \varphi)_P$, and because $(t|_W, W)$ and (t, V) represent the same element in $\ker(\varphi_P)$, this shows the inclusion $\ker(\varphi_P) \subseteq (\ker \varphi)_P$.

For the other inclusion, take an element $(t, V) \in (\ker \varphi)_P$. This means that $t \in (\ker \varphi)(V) = \ker(\varphi(V))$ and hence $\varphi(V)(t) = 0$ in $\mathcal{G}(V)$. Composing with π gives $\pi(\varphi(V)(t)) = (\varphi(V)(t), V) = 0$ in $\mathcal{G}_P$. By commutativity, $\pi((t, V)) = (t, V) \in \mathcal{F}_P$ maps to 0 under φ_P, so $(t, V) \in \ker(\varphi_P)$. This gives us the other inclusion.

Now let's consider $\operatorname{im}(\varphi)$.

$\square$

Exercise 3.

(a) Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U and there are elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ and an open set U such that $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is not surjective.