D-Modules, Unit F-Crystals, and Hodge Theory

Definitions, Theorems, Remarks, and Notable Examples

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1 Basics

Here we cover basic definitions and theorems in the theory of D-modules with a heavy emphasis on examples.

1.1 Weyl Algebra

Let K be a field of characteristic 0. We construct the Weyl algebra in two ways and prove that these constructions produce isomorphic rings.

Definition 1.1. Let K be a field of characteristic 0 and let $K[X] = K[x_1, ..., x_n] = \Gamma(X, \mathcal{O}_X)$ be the polynomial ring over K in n variables, and let $X = \mathbb{A}^n_K = \mathbb{A}^n$. Consider the algebra of K-linear operators $\mathrm{End}_K(K[X])$ and more specifically the operators $\hat{x}_i, \partial_j \in \mathrm{End}_K(K[X])$ for $1 \leq, i, j \leq n$. These are defined

$$\hat{x}_i : K[X] \longrightarrow K[X],\ f \mapsto x_i \cdot f$$

and

$$\partial_j : K[X] \longrightarrow K[X],\ f \mapsto \frac{\partial f}{\partial x_j}.$$

These are both linear operators, and they satisfy the relation

$$[\partial_j, \hat{x}_i] = \partial_j \hat{x}_i - \hat{x}_i \partial_j = \delta_{ij}.$$