Lecture 24

· Thm 20.7: $\Psi_n : \mathrm{Gal}(K_{\pi,n}/K) \to \left(\mathcal{O}_K/\pi^n\mathcal{O}_K\right)^\times$

s.t. $\Psi_n(\sigma) \cdot_{F_f} x = \sigma(x) \quad \forall x \in \mu_{f,n} \quad \sigma \in \mathrm{Gal}(K_{\pi,n}/K)$

Remark: Can show:

Ψ_n does not depend on Lubin-Tate series f.

Set $K_{\pi,\infty} := \bigcup_{n=1}^{\infty} K_{\pi,n}$

Ψ: $\mathrm{Gal}(K_{\pi,\infty}/K) \cong \varprojlim_n \left(\mathcal{O}_K/\pi^n\mathcal{O}_K\right)^\times \cong \mathcal{O}_K^\times$

Theorem 20.8: (Generalized local Kronecker-Weber)

$K^{ab} = K_{\pi,\infty} K^{ur}$

Art_K defined by

$$K^\times \cong \mathbb{Z} \times \mathcal{O}_K^\times \longrightarrow \mathrm{Gal}(K^{ur}/K) \times \mathrm{Gal}(K_{\pi,\infty}/K)$$

$$\pi^n u \leftrightarrow (n, \quad u \quad) \mapsto (\mathrm{Fr}_{K^{ur}/K}^n, \Psi^{-1}(u))$$

Remark: Independent of choice of π.

§ Local Kronecker-Weber ** Non-examinable **

Let L/K finite Galois. Define function

$\phi := \phi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}$

$$\phi(s) = \int_0^s \frac{1}{[G_0 : G_t]} dt$$

Convention: $t \in [-1, 0)$, $\frac{1}{[G_0 : G_t]} = [G_t : G_0]$.

For $m \leq s < m+1$ $(m \in \mathbb{Z}_{\geq -1})$.

$$\phi(s) = \begin{cases} s & m=-1 \\ \frac{1}{|G_0|}\left(|G_1| + \dots + |G_m| + (s-m)|G_{m+1}|\right) & m \geq 0 \end{cases}$$

Thus: • ϕ is continuous + piecewise linear

• ϕ is strictly increasing.

$\Rightarrow$ Can define $\psi_{L/K} := \phi_{L/K}^{-1}$

Definition 21.1: (Upper numbering) The higher ramification groups in upper numbering is defined by

$$G^s(L/K) := G_{\psi_{L/K}(s)}(L/K)$$

Key point: $G_s(L/K)$ behaves well w.r.t. subgroups

$G^s(L/K)$ " " " quotients

$L/F/K$, $L/F/K$ Galois. Then

• $G_s(L/F) = G_s(L/K) \cap \mathrm{Gal}(L/F)$

If also F/K Galois,

$$G^t(L/K)\,\mathrm{Gal}(L/F)/\mathrm{Gal}(L/F) = G^t(F/K).$$

(Herbrand's theorem)

Eg. $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\zeta_{p^n})$

$k \in \mathbb{Z}$, $1 \leq k \leq n-1$.

For $p^{k-1} - 1 < s \leq p^k - 1$,

$G_s \cong \{m \in (\mathbb{Z}/p^n\mathbb{Z})^\times \mid m \equiv 1 \mod p^k\}$

Since G_s jumps at p^{k-1}, $\phi_{L/H}$ is linear.
on $[p^{k-1}-1, p^k-1]$. Thus to compute $\phi_{L/K}$, suffices to compute $\phi_{L/K}(p^k-1)$.

Compute: $\phi_{L/K}(p^k-1)=k \quad (1 \le k \le n-1)$

$$\Rightarrow G^s \cong \begin{cases} (\mathbb{Z}/p^n\mathbb{Z})^\times & \text{if } s \le 0 \\ (1+p^k\mathbb{Z})/p^n\mathbb{Z} & \text{if } k-1 < s \le k \ (1 \le k \le n \ldots \\ \{1\} & \text{if } s > n-1. \end{cases}$$

Note G^s jumps at $1, \ldots, n-1$ integers — a priori not clear.

Definition 21.2: We say i is a jump in the filtration $\{G^s\}_{s \in \mathbb{R}_{\ge -1}}$ if $G^i \ne G^j \ \forall j > i$.

4 Theorem 21.3: (Hasse–Arf)
If $Gal(L/K)$ is abelian, then the jumps of the filtration $\{G^s\}_{s \in \mathbb{R}_{\ge -1}}$ can only be integers.

Proof: Omit. [Serre: Local fields, Chapter V, §7].

Lemma 21.4: L/K tot. ram. abelian extension. $G := Gal(L/K)$. $G^n = \{1\} \Rightarrow [L:K] \mid q^{n-1}(q-1)$
$(q = |k|)$.

Proof: (Sketch) Hasse–Arf $\Rightarrow$ at most n jumps

$G^0/G^1 \hookrightarrow k^*$ $\Rightarrow |G^0/G^1| \mid q-1$

$G^i/G^{i+1} \hookrightarrow (k,+)$ $\Rightarrow |G^i/G^{i+1}| \mid q$ $i = 1, \dots n-$

$\square$

Lemma 21.5: $L_1, L_2 \subseteq K^{ab}$ s.t.

$G^s(L_1/K)^n = 1$ $\quad G^s(L_2/K)^n = 1$

Then $G^s(L_1L_2/K)^n = 1$.

Proof of Theorem 20.8: (Sketch.)

Let $\bar{\sigma} \in Gal(K^{ur}K_{\pi,\infty}/K)$ correspond to

$(Fr_{K^{ur}/K}, id) \in Gal(K^{ur}/K) \times Gal(K_{\pi,\infty}/K)$

Let $\sigma \in Gal(K^{ab}/K)$ s.t. $\sigma|_{K_{\pi,\infty}K^{ur}} = \bar{\sigma}$

Set $K_\sigma = (K^{ab})^\sigma$. Galois theory $\Rightarrow$

- $K^{ab} = K_\sigma K^{ur}$
- $K_{\pi,\infty} \subseteq K_\sigma$
- K_σ tot. ram. abelian.

Let $F \subseteq K_\sigma$ finite. Assume $G^n(F/K) = \{1\}$

Set $L = F K_{\pi,n}$

$G^n(L/K) = \{1\}$ by Lemma 21.5.

Lemma 21.4 $\Rightarrow |Gal(L/K)| \leq q^{n-1}(q-1) = |Gal(K_{\pi,n}/K)|$

$\Rightarrow L = K_{\pi,n} \Rightarrow F \subseteq K_{\pi,n}$ $\square$