Appendix A

The moving lemma

For many years, the development of intersection theory was based on a result known as the "moving lemma." This came in two flavors, the *basic moving lemma* and the *strong moving lemma*.

Lemma A.1 (Basic moving lemma)**.** *Let X be a smooth, quasi-projective variety.*

(a) Given cycles A, B on X, there exists a cycle A' rationally equivalent to A and generically transverse to B.
(b) The resulting class $[A' \cap B] \in A(X)$ is independent of the choice of such an A'.

Given the validity of these assertions, the intersection product may be defined by the formula $[A][B] = [A' \cap B]$.

Lemma A.2 (Strong moving lemma)**.** *Let $f : Y \to X$ be a morphism of smooth, quasi-projective varieties.*

(a) Given a cycle A on X, there exists a cycle $A' = \sum n_i A_i$ rationally equivalent to A and generically transverse to f; that is, such that the preimage $f^{-1}(A_i) \subset Y$ is generically reduced of the same codimension as A_i.
(b) The class $\left[\sum n_i f^{-1}(A')\right] \in A(Y)$ is independent of the choice of such an A'.

Given this, we can define a pullback map $f^* : A(X) \to A(Y)$, making the Chow ring into a contravariant functor. Note that the strong moving lemma is not just a generalization of the basic one — the basic moving lemma is just the strong one in case the map f is an inclusion — but also a strengthening: even in case f is an inclusion $Y \hookrightarrow X$, it produces a class in $A(Y)$ whose pushforward to X is the product $[A][Y]$, rather than simply a class in $A(X)$. This is called a *semi-refined* intersection product (the prefix "semi-" is because this is not the full degree of refinement possible; in Fulton's theory, subject to mild hypotheses it is possible to associate to a pair of subvarieties $A, B \subset X$ a class supported on the actual intersection $A \cap B$).

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

A method of proving part (a) of the basic moving lemma was put forward in Chow [1956] and Samuel [1956], following ideas of Severi [1933], and we will give the details in the first section of this appendix. (For other treatments see Samuel [1971], Roberts [1972a] and Hoyt [1971], as well as the discussion in Fulton [1984, Chapter 11].) In addition, part (a) of Lemma A.2 may be deduced from part (a) of Lemma A.1; we will do this in Section A.2 below.

It is also possible to use the same argument to prove part (b) of the basic version, by moving the second cycle B to a cycle B' generically transverse not only to all the components of A' but also to all subvarieties appearing in the rational equivalence between A and A'; this is carried out in the Stacks Project [2015, Tag 0AZ6] of de Jong and others. None of these approaches, however, suffice to prove part (b) of the strong moving lemma.

The Fulton–MacPherson approach to the definition of the intersection product, extended and detailed in Fulton [1984], has made the moving lemma unnecessary, and gives a technically superior and more general approach to intersection products. Though the direct proofs of part (b) (in either version) have remained controversial, the Fulton–MacPherson theory implies that the statements are correct. (If one is willing to work with rational coefficients, there is an alternative approach via K-theory as well.)

In our view, part (a) of the moving lemma, even though superseded (and rendered unnecessary) by the Fulton–MacPherson approach, still has heuristic importance, hence this appendix.

A.1 Generic transversality to a cycle

All existing proofs of part (a) of Lemma A.1 are based on an approach proposed by Severi, called the *cone construction*. The idea is this: We are given cycles A and B in a smooth variety $X \subset \mathbb{P}^N$, and want to find a cycle A', rationally equivalent to A and generically transverse to B. We will do this by expressing the cycle A as a difference of two cycles $A = E - A^1$, where E is the generically transverse intersection of X with another subvariety $\Phi \subset \mathbb{P}^N$ (so that E can be moved, by applying a linear transformation g of $\mathbb{P}^N$, to a cycle $E^1 = g\Phi \cap X$ generically transverse to B), and A^1 is better situated with respect to B than A. "Better situated" here means two things: if the intersection $A \cap B$ was not dimensionally transverse, then $A^1 \cap B$ will have strictly smaller dimension than $A \cap B$, and if $A \cap B$ is dimensionally transverse then $A^1 \cap B$ will actually be generically transverse. (It is called the cone construction because the variety Φ used is a cone over A, with vertex a general linear space $\Gamma \subset \mathbb{P}^N$.) If we carry out this process repeatedly, we will arrive at the desired cycle A'.

We remark that most of the salient points of the proof are already present in the very simplest case, where $X \subset \mathbb{P}^3$ is a smooth surface and $A = B \subset X$ a (possibly) singular curve; see Figure A.1.

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

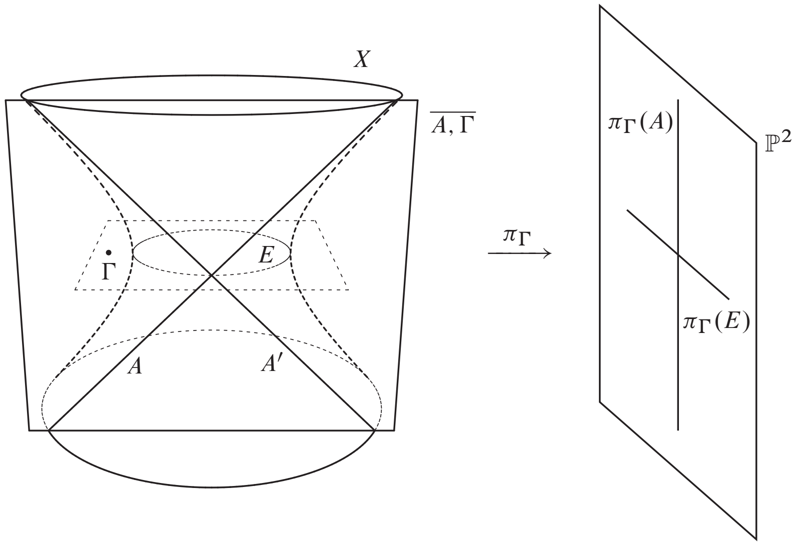

Figure A.1 $\overline{A, \Gamma} \cap X = A + A'$ and $\langle A \rangle \sim \langle E \rangle - \langle A' \rangle$.

Proof of part (a) of Lemma A.1: Set $n = \dim X$ and $a = \operatorname{codim}_X A$. We may assume that B is equidimensional, and we set $b = \operatorname{codim}_X B$.

We will construct sequences of equidimensional cycles of codimension a on X

$$A = A^0,\ A^1,\ A^2,\ \ldots \quad \text{and} \quad E^1,\ E^2,\ \ldots$$

satisfying the following conditions:

(a) A^i is rationally equivalent to $E^{i+1} - A^{i+1}$ on X.
(b) E^i is generically transverse to B.
(c) If C is a component of $A^{i+1} \cap B$ and $C \subset A^i \cap B$, then C is properly contained in a component of $A^i \cap B$; in particular, $\operatorname{codim} C > \operatorname{codim} A^i \cap B$.
(d) A^{i+1} is generically transverse to $B \setminus (A^i \cap B)$.

In what follows, the word "component" refers to an irreducible, nonembedded component. By Theorem 0.2, every component of $A \cap B$ has codimension $\leq a + b$. Thus part (c) shows that for $m > a + b - \operatorname{codim}(A \cap B)$ there are no components of $A^m \cap B$ that are contained in A^{m-1}, and part (d) then shows that A^m is generically transverse to B. By part (a) we have

$$A \sim \sum_{i=1}^{m} (-1)^{i+1} E^i + (-1)^m A^m,$$

and by part (b) all the E^i are generically transverse to B, so this will establish the theorem.

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

By induction it suffices to show that, given an equidimensional cycle A, we can produce E^1 and A^1 satisfying the given conditions. Without loss of generality we may assume that A is the cycle associated to a subvariety, which we also call A.

We begin by embedding X in a projective space $\mathbb{P}^N$ in such a way that no three points of X are collinear and no two tangent planes to X at distinct points meet. The following lemma shows that it suffices to replace whatever embedding $X \subset \mathbb{P}^{N'}$ we are originally given by its composition with the third Veronese map $\nu_3 : \mathbb{P}^{N'} \to \mathbb{P}^N$ (the embedding by the linear system of cubics):

Lemma A.3. *Let $\nu_3 : \mathbb{P}^N \to \mathbb{P}^M$ be the third Veronese map. No three points of $\nu_3(\mathbb{P}^N)$ are collinear, and the tangent planes at distinct points of $\nu_3(\mathbb{P}^N)$ are disjoint.*

Proof: By Proposition 7.10, any subscheme of degree ≤ 4 in $\mathbb{P}^N$ imposes independent conditions on cubics. Thus any subscheme of $\nu_3(\mathbb{P}^N)$ of degree $d \leq 4$ spans a plane of dimension $\geq d - 1$. In particular, no three points of $\nu_3(\mathbb{P}^N)$ lie on a line (this also follows from the fact that $\nu_3(\mathbb{P}^N)$ is cut out by quadrics and contains no lines). If the tangent planes to $\nu_3(\mathbb{P}^N)$ at points p, q met in some point r, then the lines $L_1 = \overline{p, r}$ and $L_2 = \overline{q, r}$ would be contained in the 2-plane $\overline{p, q, r}$, and this plane would contain the subscheme $(L_1 \cap X) \cup (L_2 \cap X)$, which has length at least 4, a contradiction. $\square$

Thus we may assume from the outset that no three points of X in $\mathbb{P}^N$ are collinear, and that any two tangent planes to X at distinct points are disjoint.

Lemma A.4. *Let X and A be as above and $\pi_\Gamma : X \to \mathbb{P}^n$ be the linear projection from a general $N - n - 1$ plane $\Gamma \subset \mathbb{P}^N$. We may write*

$$\pi_\Gamma^{-1}(\pi_\Gamma(A)) = A \cup A'_\Gamma$$

as schemes, where A'_Γ is a generically reduced scheme of pure codimension a that does not contain A.

Write $\mathbb{G} := \mathbb{G}(N - n - 1, N)$ for the Grassmannian of $(N - n - 1)$-dimensional planes in $\mathbb{P}^N$; the statement above means that the conclusion holds for all planes in an open dense subset of $\mathbb{G}$.

Proof: To simplify the notation, we set $\widetilde{A}_\Gamma := \pi_\Gamma^{-1}(\pi(A)) = X \cap \overline{A, \Gamma}$ (see Figure A.1).

By Theorem 0.2, the components of $\widetilde{A}_\Gamma$ have codimension in X at most the codimension of the irreducible variety $\pi_\Gamma(A)$ in $\mathbb{P}^n$. Since $\pi_\Gamma : X \to \mathbb{P}^n$ is finite, $\pi_\Gamma(A)$ has dimension $n - a$. Thus every component of $\pi_\Gamma^{-1}(\pi_\Gamma(A))$ is of pure codimension a and maps surjectively to $\pi_\Gamma(A)$. In particular, A itself is a component of $\widetilde{A}_\Gamma$.

Thus it suffices to prove that $\widetilde{A}_\Gamma$ is generically reduced. Since every component of $\widetilde{A}_\Gamma$ surjects onto $\pi_\Gamma(A)$, it even suffices to show that the general fiber of π_Γ is reduced. The fibers of π_Γ are the intersections of X with the $(N - n)$-dimensional

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

planes containing Γ, and a general such plane Σ is a general $(N-n)$-plane containing a general $(N-n-1)$-plane Γ. Thus Σ is general in the space of all $(N-n)$-planes, so reducedness follows from Bertini's theorem. □

With notation as in Lemma A.4, we set $A^1_\Gamma := A'_\Gamma$. The situation is illustrated in Figure A.1. We can now establish conditions (a) and (b) of the proof of Lemma A.1:

Lemma A.5. *With notation as above, if* $\Gamma \in \mathbb{G}$ *is general then*

$$[A] = [E^1] - [A^1_\Gamma] \in A(X),$$

where E^1 *is generically transverse to* B.

Proof: We have

$$A = \widetilde{A}_\Gamma - A^1_\Gamma$$

where $\widetilde{A}_\Gamma$ is the generically transverse intersection of X with the cone $\overline{A,\Gamma}$. Let $g \in \mathrm{PGL}_{N+1}$ be a general automorphism of $\mathbb{P}^N$. By the argument of part (c) of the Kleiman transversality theorem (Theorem 1.7),

$$\widetilde{A}_\Gamma \sim E' := X \cap g(\overline{A,\Gamma}),$$

and by part (a) of the same theorem $g(\overline{A,\Gamma})$, and hence E', will be generically transverse to B. □

Completion of the proof of Lemma A.1: With notation and hypotheses as above, it suffices to show that, for general Γ, the cycle corresponding to the scheme A^1_Γ satisfies (c) and (d). Since an intersection of open dense subsets of $\mathbb{G}$ is again open and dense, it suffices to do this for one component of B at a time, so we may assume that B is (the cycle associated to) a subvariety, which we also call B.

Condition (c): Consider a component C of $A^1_\Gamma \cap B$ that is contained in A, so that in fact

$$C \subset A \cap A^1_\Gamma \cap B \subset A \cap B.$$

We must show that C is not a component of $A \cap B$.

Since Γ is general, every component of $A \cap B$ contains points p such that Γ does not meet the tangent plane to X at p. The map $\pi_\Gamma : X \to \mathbb{P}^n$ is nonsingular at such points. Since $\pi_\Gamma^{-1}(\pi_\Gamma(A)) = A' \cup A$, such points cannot lie in $A \cap A'$. Consequently C must be properly contained in some component of $A \cap B$, as required.

Condition (d): Finally, we wish to show that for general Γ the intersection of A^1_Γ with $B^* = B \setminus A \cap B$ is generically transverse, or equivalently that $\widetilde{A}_\Gamma$ and B^* are generically transverse.

We first prove the weaker statement that $\widetilde{A}_\Gamma$ and B^* are dimensionally transverse. Consider the incidence correspondence Ψ defined by

$$\Psi := \{(\Gamma, p, q) \in \mathbb{G} \times A \times B^* \mid \Gamma \cap \overline{p,q} \neq \varnothing\}.$$

The fiber of Ψ over any point $(p, q) \in A \times B^*$ is isomorphic to the set $\Sigma(\overline{p, q})$ of $(N-n-1)$-planes $\Gamma \in \mathbb{G}^*$ meeting the line $\overline{p, q}$. By Theorem 4.1 this is an irreducible variety of codimension n in $\mathbb{G}^*$. Since the projection $\Psi \to A \times B^*$ is proper, it follows that Ψ is irreducible of dimension

$$\dim \Psi = \dim \mathbb{G}^* + \dim A + \dim B - n.$$

This implies that for general Γ the fiber of Ψ over Γ has dimension $\dim A + \dim B - n$. Since this fiber surjects onto $A_\Gamma^1 \cap B^*$, we see that $\dim A_\Gamma^1 \cap B^* \leq \dim A + \dim B - n$, so by Theorem 0.2 we have equality. That is, for general Γ the sets A_Γ and B^* are dimensionally transverse; every component of their intersection has dimension $\dim A + \dim B - n$.

Let $\mathbb{G}^* \subset \mathbb{G}$ be the open dense set consisting of the planes Γ disjoint from X such that $\pi_\Gamma : A \to \pi_\Gamma(A)$ is birational and $\widetilde{A}_\Gamma$ is generically reduced.

To prove the generic transversality of $\widetilde{A}_\Gamma$ and B^* for general Γ, we next consider $\Psi_0 \subset \Psi$, where

$$\Psi_0 := \{(\Gamma, p, q) \in \mathbb{G}^* \times A \times B^* \mid \pi_\Gamma(p) = \pi_\Gamma(q),\ \widetilde{A}_\Gamma \text{ is not transverse to } B^* \text{ at } q\}.$$

If $\widetilde{A}_\Gamma$ and B^* were not generically transverse for generic Γ, then for an open set of Γ in $\mathbb{G}$ the fiber of Ψ_0 over Γ would surject to at least one component of $A_\Gamma^1 \cap B^*$, and thus would have dimension $\geq \dim A + \dim B - n$. This would imply that $\dim \Psi_0 \geq \dim \mathbb{G} + \dim A + \dim B - n = \dim \Psi$. Thus it suffices to show that $\dim \Psi_0 < \dim \Psi$.

To do this, we will write Ψ_0 as the union of five subsets

$$\Psi_0 = \Psi_1 \cup \Psi_2 \cup \Psi_3 \cup \Psi_4 \cup \Psi_5,$$

defined in terms of the reasons why A_Γ might not be transverse to B^* at q. To start, the intersection of A_Γ^1 and B at q will be nontransverse if:

(1) q is a singular point of B.

The intersection will also be nontransverse if q is a singular point of $\widetilde{A}_\Gamma$. Since $q \in X$, we have $q \notin \Gamma$, so q is singular on $\widetilde{A}_\Gamma$ if and only if $\pi_\Gamma(q) = \pi_\Gamma(p)$ is singular on $\pi_\Gamma(A)$. This can occur only if one of the following occurs:

(2) $q \in \overline{\Gamma, p}$ with p a singular point of A.
(3) $q \in \overline{\Gamma, p}$ and $q \in \overline{\Gamma, p'}$ for two distinct points $p, p' \in A$.
(4) $q \in \overline{\Gamma, p}$ and $\Gamma \cap \mathbb{T}_p A \neq \varnothing$.

Accordingly, we set

$$\Psi_1 := \{(\Gamma, p, q) \in \Psi_0 \mid q \in B_{\mathrm{sing}}\} \subset \Psi,$$
$$\Psi_2 := \{(\Gamma, p, q) \in \Psi_0 \mid p \in A_{\mathrm{sing}}\} \subset \Psi,$$
$$\Psi_3 := \{(\Gamma, p, q) \in \Psi_0 \mid \text{there exists } p' \neq p \in A \text{ with } \Gamma \cap \overline{p', q} \neq \varnothing\} \subset \Psi,$$
$$\Psi_4 := \{(\Gamma, p, q) \in \Psi_0 \mid p \in A_{\mathrm{sm}} \text{ and } \Gamma \cap \mathbb{T}_p A \neq \varnothing\} \subset \Psi,$$

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

where $A_{\text{sing}} \subset A$ denotes the singular locus, $A_{\text{sm}} = A \setminus A_{\text{sing}}$, and similarly for B.

Now suppose that $(p,q) \in \Psi_0$ is not in $\bigcup_1^4 \Psi_i$, so that $\widetilde{A}_\Gamma$ and B^* are both smooth at q. The intersection of $\widetilde{A}_\Gamma$ and B^* will be nontransverse if and only if the tangent spaces of these two varieties at q fail to be transverse. The tangent plane to the cone $\overline{A,\Gamma}$ at q is the span of Γ and the tangent space $\mathbb{T}_p A$. This span fails to intersect B transversely at q only if the three linear spaces Γ, $\mathbb{T}_p A$ and $\mathbb{T}_q B$ fail to span all of $\mathbb{P}^N$. From our hypothesis on the embedding of X, it follows that $\mathbb{T}_p A$ and $\mathbb{T}_q B$ are disjoint. Thus a necessary condition for nontransversality at q in this case is that:

(5) Γ is not transverse to $\overline{\mathbb{T}_p A, \mathbb{T}_q B}$.

Since $\dim \overline{\mathbb{T}_p A, \mathbb{T}_q B} = 2n - a - b + 1$ and $\dim \Gamma = N - n$, the relevant set is

$$\Psi_5 := \{(\Gamma, p, q) \in \Psi_0 \mid p \in A_{\text{sm}},\ q \in B_{\text{sm}},\ \dim(\Gamma \cap \overline{\mathbb{T}_p A, \mathbb{T}_q B}) > n - a - b\}.$$

We can compute the dimensions of Ψ_1 and Ψ_2 just as we computed the dimension of Ψ itself. Since A and B are reduced, the sets A_{sing} and B^*_{sing} have strictly smaller dimension than A and B^*. Noting again that the fibers of the projection $\Psi \to A \times B^*$ all have codimension n in $\mathbb{G}$, we see that Ψ_1 and Ψ_2 have strictly smaller dimensions than Ψ.

The set Ψ_4 dominates $A_{\text{sm}} \times B^*$, but with strictly smaller fibers than Ψ: By our hypothesis on the embedding of X in $\mathbb{P}^N$, we have $q \notin \mathbb{T}_p A$. Also $p \notin \Gamma$. If $\Gamma \cap \mathbb{T}_p A \neq \varnothing$ then, in addition to meeting the line $\overline{p,q}$ in at least a point, Γ must intersect the $(a+1)$-plane $\overline{q, \mathbb{T}_p A}$ in at least a line. This is a proper subvariety of the Schubert cycle $\Sigma_n(\overline{p,q})$, so Ψ_4 has smaller dimension than Ψ.

Similarly, the fiber of Ψ_5 over any point $(p,q) \in A_{\text{sm}} \times B^*_{\text{sm}}$ is a proper subvariety of $\Sigma_n(\overline{p,q})$, and again we conclude that $\dim \Psi_5 < \dim \Psi$.

To compute the dimension of Ψ_3 we introduce one more incidence correspondence. Set

$$\widetilde{\Psi}_3 := \{(\Gamma, p, p', q) \in \mathbb{G}^* \times A \times A \times B^* \mid p \neq p',\ \Gamma \cap \overline{p,q} \neq \varnothing \text{ and } \Gamma \cap \overline{p',q} \neq \varnothing\}.$$

Since Ψ_3 is the image of $\widetilde{\Psi}_3$ under a projection to $\mathbb{G}^* \times A \times B^*$, it suffices to show that $\dim \widetilde{\Psi}_3 < \dim \Psi$.

To estimate the dimension of $\widetilde{\Psi}_3$, consider the projection to $A \times A \times B^*$. By our hypothesis on the embedding of X in $\mathbb{P}^N$, any three points of X span a 2-plane. Also $q \in X$, so $q \notin \Gamma$. Thus the conditions $\Gamma \cap \overline{p,q} \neq \varnothing$ and $\Gamma \cap \overline{p',q} \neq \varnothing$ amount to saying that $\dim(\Gamma \cap \overline{p, p', q}) \geq 1$. This describes the Schubert cycle $\sigma_{n,n}(\overline{p, p', q})$, which has codimension $2n$ in $\mathbb{G}^*$ by Theorem 4.1. We thus have

$$\dim \widetilde{\Psi}_3 = \dim \mathbb{G} + 2a + b - 2n < \dim \Psi,$$

as required. Putting this all together, we get that $\dim \Psi_0 < \dim \Psi$, completing the proof of Lemma A.1. $\square$

https://doi.org/10.1017/CBO9781139062046.017 Published online by Cambridge University Press

A.2 Generic transversality to a morphism

Let $f : Y \to X$ be a morphism of smooth varieties. Recall that a subvariety $A \subset X$ is said to be *dimensionally transverse to* f if the codimension of $f^{-1}(A)$ in Y is the same as the codimension of A in X, and *generically transverse to* f if in addition $f^{-1}(A)$ is generically reduced (Definition 1.22). In this section we will show that there is a finite collection of subvarieties of X, depending on f, such that if A is generically transverse to each of these subvarieties then A is generically transverse to f. (This would not be true without our standing hypothesis of characteristic 0, or at least a hypothesis that f is generically separable; if f is not separable, then $f^{-1}(A)$ is necessarily nonreduced, so A is never generically transverse to f.) For each k set

$$\Phi_k^\circ(f) := \{x \in f(Y) \mid \dim f^{-1}(x) \geq k\},$$

and write

$$\begin{aligned}\Psi^\circ(f) := \{x \in X \mid \text{for some } y \in f^{-1}(x),\\ \operatorname{rank} df_y : T_yY \to T_xX \text{ is } < \min(\dim X, \dim Y)\}\end{aligned}$$

for the image of the singular locus of f; let $\Psi(f)$ and $\Phi_k(f)$ be the closures of $\Psi^\circ(f)$ and $\Phi_k^\circ(f)$.

Theorem A.6. *Suppose that $f : Y \to X$ is a morphism of varieties. If a subvariety $A \subset X$ is dimensionally transverse to each $\Phi_k(f)$ then A is dimensionally transverse to f. If in addition A is generically transverse to Ψ then A is generically transverse to f.*

If f is not separable, then $f^{-1}(A)$ is necessarily nonreduced, so A is never generically transverse to f.

Proof: First suppose that A is dimensionally transverse to each Φ_k. The dimension of Φ_k is $\leq \dim Y - k$, with strict inequality for $k > \dim Y - \dim f(Y)$ since Y is irreducible.

Let $k_0 = \dim Y - \dim f(Y)$, so that $\Psi_{k_0} = f(Y)$. For $k > k_0$, transversality to Ψ_k yields

$$\dim(A \cap \Phi_k) \leq \dim A - \operatorname{codim} f(Y) - k + k_0 - 1,$$

from which it follows that

$$\dim(f^{-1}(A \cap \Phi_k)) \leq \dim A - \operatorname{codim} f(Y) + k_0 - 1.$$

By Theorem 0.2 every component of $f^{-1}(A)$ has dimension $\geq \dim A - \operatorname{codim} f(Y) + k_0$. It follows that no component of $f^{-1}(A)$ is contained in $f^{-1}(\Phi_k)$ for $k > k_0$, and hence

$$\begin{aligned}\dim f^{-1}(A) &= k_0 + \dim(A \cap f(Y))\\ &= k_0 + \dim f(Y) - \operatorname{codim} A\\ &= \dim Y - \operatorname{codim} A,\end{aligned}$$

as required.

Because the characteristic of the ground field is 0, the branch locus Ψ is strictly contained in $f(Y)$ (this is the algebraic version of Sard's theorem; see Milnor [1965, Theorem 6.1]). Thus $A \cap f(Y)$ is not contained in Ψ. It follows that a general point of each component of $f^{-1}(A)$ is smooth, so that $f^{-1}(A)$ is generically reduced. $\square$