Chapter 2

Natural Constructions on Manifolds

The goal of this chapter is to introduce the basic terminology used in differential geometry. The key concept is that of tangent space at a point which is a first order approximation of the manifold near that point. We will be able to transport many notions in linear analysis to manifolds via the tangent space.

2.1 The tangent bundle

2.1.1 *Tangent spaces*

We begin with a simple example which will serve as a motivation for the abstract definitions.

Example 2.1.1. Consider the sphere

$$(S^2) : x^2 + y^2 + z^2 = 1 \quad \text{in } \mathbb{R}^3.$$

We want to find the plane passing through the North pole $N(0,0,1)$ that is "closest" to the sphere. The classics would refer to such a plane as an *osculator plane*.

The natural candidate for this osculator plane would be a plane given by a linear equation that best approximates the defining equation $x^2 + y^2 + z^2 = 1$ in a neighborhood of the North pole. The linear approximation of $x^2 + y^2 + z^2$ near N seems like the best candidate. We have

$$x^2 + y^2 + z^2 - 1 = 2(z-1) + O(2),$$

where $O(2)$ denotes a quadratic error. Hence, the osculator plane is $z = 1$, Geometrically, it is the horizontal *affine* plane through the North pole. The *linear* subspace $\{z = 0\} \subset \mathbb{R}^3$ is called the *tangent space* to S^2 at N.

The above construction has one deficiency: it is *not intrinsic*, i.e., it relies on objects "outside" the manifold S^2. There is one natural way to fix this problem. Look at a smooth path $\gamma(t)$ on S^2 passing through N at $t = 0$. Hence, $t \mapsto \gamma(t) \in \mathbb{R}^3$, and

$$|\gamma(t)|^2 = 1. \tag{2.1.1}$$

If we differentiate (2.1.1) at $t = 0$ we get $(\dot\gamma(0), \gamma(0)) = 0$, i.e., $\dot\gamma(0) \perp \gamma(0)$, so that $\dot\gamma(0)$ lies in the linear subspace $z = 0$. We deduce that the tangent space consists of the tangents to the curves on S^2 passing through N.

This is apparently no major conceptual gain since we still regard the tangent space as a subspace of $\mathbb{R}^3$, and this is still an extrinsic description. However, if we use the stereographic projection from the South pole we get local coordinates (u, v) near N, and any curve $\gamma(t)$ as above can be viewed as a curve $t \mapsto (u(t), v(t))$ in the (u, v) plane. If $\phi(t)$ is another curve through N given in local coordinates by $t \mapsto (\underline{u}(t), \underline{v}(t))$, then

$$\dot\gamma(0) = \dot\phi(0) \iff \big(\, \dot u(0), \dot v(0)\,\big) \;=\; \big(\, \dot{\underline{u}}(0), \dot{\underline{v}}(0)\,\big).$$

The right hand side of the above equality defines an equivalence relation $\sim$ on the set of smooth curves passing trough $(0, 0)$. Thus, there is a bijective correspondence between the tangents to the curves through N, and the equivalence classes of "$\sim$". This equivalence relation is now intrinsic modulo one problem: "$\sim$" may depend on the choice of the local coordinates. Fortunately, as we are going to see, this is a non-issue. □

Definition 2.1.2. Let M^m be a smooth manifold and p_0 a point in M. Two smooth paths $\alpha,\ \beta : (-\varepsilon, \varepsilon) \to M$ such that $\alpha(0) = \beta(0) = p_0$ are said to have a *first order contact* at p_0 if there exist local coordinates $(x) = (x^1, \dots, x^m)$ near p_0 such that

$$\dot x_\alpha(0) = \dot x_\beta(0),$$

where $\alpha(t) = (x_\alpha(t)) = (x^1_\alpha(t), \dots, x^m_\alpha(t))$, and $\beta(t) = (x_\beta(t)) = (x^1_\beta(t), \dots, x^m_\beta(t))$. We write this $\alpha \sim_1 \beta$. □

Lemma 2.1.3. $\sim_1$ *is an equivalence relation.*

Sketch of proof. The binary relation $\sim_1$ is obviously reflexive and symmetric, so we only have to check the transitivity. Let $\alpha \sim_1 \beta$ and $\beta \sim_1 \gamma$. Thus there exist local coordinates $(x) = (x^1, \dots, x^m)$ and $(y) = (y^1, \dots, y^m)$ near p_0 such that $(\dot x_\alpha(0)) = (\dot x_\beta(0))$ and $(\dot y_\beta(0)) = (\dot y_\gamma(0))$. The transitivity follows from the equality

$$\dot y^i_\gamma(0) = \dot y^i_\beta(0) = \sum_j \frac{\partial y^i}{\partial x^j} \dot x^j_\beta(0) = \sum_j \frac{\partial y^i}{\partial x^j} \dot x^j_\alpha(0) = \dot y^j_\alpha(0). \qquad \square$$

Definition 2.1.4. A *tangent vector* to M at p is a first-order-contact equivalence class of curves through p. The equivalence class of a curve $\alpha(t)$ such that $\alpha(0) = p$ will be temporarily denoted by $[\dot\alpha(0)]$. The set of these equivalence classes is denoted by T_pM, and is called the *tangent space* to M at p. □

Lemma 2.1.5. T_pM *has a natural structure of vector space.*

Proof. Choose local coordinates $(x^1, \dots, x^m)$ near p such that $x^i(p) = 0$, $\forall i$, and let α and β be two smooth curves through p. In the above local coordinates the curves α, β become $\big(x^i_\alpha(t)\big)$, $\big(x^i_\beta(t)\big)$. Construct a new curve γ through p given by

$$\big(x^i_\gamma(t)\big) = \big(x^i_\alpha(t) + x^i_\beta(t)\big).$$

Set $[\dot\alpha(0)] + [\dot\beta(0)] := [\dot\gamma(0)]$. For this operation to be well defined one has to check two things.

(a) The equivalence class $[\dot\gamma(0)]$ is independent of coordinates.
(b) If $[\dot{\alpha_1}(0)] = [\dot{\alpha_2}(0)]$ and $[\dot\beta_1(0)] = [\dot\beta_2(0)]$ then

$$[\dot{\alpha_1}(0)] + [\dot\beta_1(0)] = [\dot{\alpha_2}(0)] + [\dot\beta_2(0)].$$

We let the reader supply the routine details. □

Exercise 2.1.6. Finish the proof of the Lemma 2.1.5. □

From this point on we will omit the brackets $[\,-\,]$ in the notation of a tangent vector. Thus, $[\dot\alpha(0)]$ will be written simply as $\dot\alpha(0)$.

As one expects, all the above notions admit a nice description using local coordinates. Let $(x^1, \dots, x^m)$ be coordinates near $p \in M$ such that $x^i(p) = 0$, $\forall i$. Consider the curves

$$e_k(t) = (t\delta^1_k, \dots, t\delta^m_k), \quad k = 1, \dots, m,$$

where δ^i_j denotes Kronecker's delta symbol. We set

$$\frac{\partial}{\partial x^k}(p) := \dot e_k(0). \tag{2.1.2}$$

Note that these vectors depend on the local coordinates $(x^1, \dots, x^m)$. Often, when the point p is clear from the context, we will omit it in the above notation.

Lemma 2.1.7. $\big(\frac{\partial}{\partial x^k}(p)\big)_{1\leq k\leq m}$ *is a basis of* T_pM.

Proof. It follows from the obvious fact that any path through the origin in $\mathbb{R}^m$ has first order contact with a linear one $t \mapsto (a_1t, \dots, a_mt)$. □

Exercise 2.1.8. Let $F : \mathbb{R}^N \to \mathbb{R}^k$ be a smooth map. Assume that
(a) $M = F^{-1}(0) \neq \emptyset$;
(b) $\operatorname{rank} D_xF = k$, for all $x \in M$.
Then M is a smooth manifold of dimension $N - k$ and

$$T_xM = \ker D_xF, \quad \forall x \in M.$$ □

Example 2.1.9. We want to describe $T_{\mathbb{1}}G$, where G is one of the Lie groups discussed in Section 1.2.2.
(a) $G = O(n)$. Let $(-\varepsilon, \varepsilon) \ni s \mapsto T(s)$ be a smooth path of orthogonal matrices such that $T(0) = \mathbb{1}$. Then $T^t(s) \cdot T(s) = \mathbb{1}$. Differentiating this equality at $s = 0$ we get

$$\dot T^t(0) + \dot T(0) = 0.$$

The matrix $\dot{T}(0)$ defines a vector in $T_{\mathbb{1}}O(n)$, so the above equality states that this tangent space lies inside the space of skew-symmetric matrices, i.e., $T_{\mathbb{1}}O(n) \subset \underline{\mathrm{o}}(n)$. On the other hand, we proved in Section 1.2.2 that $\dim G = \dim \underline{\mathrm{o}}(n)$ so that

$$T_{\mathbb{1}}O(n) = \underline{\mathrm{o}}(n).$$

(b) $G = SL(n;\mathbb{R})$. Let $(-\varepsilon, \varepsilon) \ni s \mapsto T(s)$ be a smooth path in $SL(n;\mathbb{R})$ such that $T(0) = \mathbb{1}$. Then $\det T(s) = 1$ and differentiating this equality at $s = 0$ we get (see Exercise 1.1.3)

$$\operatorname{tr} \dot{T}(0) = 0.$$

Thus, the tangent space at $\mathbb{1}$ lies inside the space of traceless matrices, i.e. $T_{\mathbb{1}}SL(n;\mathbb{R}) \subset \underline{\mathrm{sl}}(n;\mathbb{R})$. Since (according to Exercise 1.2.25) $\dim SL(n;\mathbb{R}) = \dim \underline{\mathrm{sl}}(n;\mathbb{R})$ we deduce

$$T_{\mathbb{1}}SL(n;\mathbb{R}) = \underline{\mathrm{sl}}(n;\mathbb{R}). \qquad \square$$

Exercise 2.1.10. Show that $T_{\mathbb{1}}U(n) = \underline{\mathrm{u}}(n)$ and $T_{\mathbb{1}}SU(n) = \underline{\mathrm{su}}(n)$. $\square$

2.1.2 *The tangent bundle*

In the previous subsection we have naturally associated to an arbitrary point p on a manifold M a vector space T_pM. It is the goal of the present subsection to coherently organize the family of tangent spaces $(T_pM)_{p\in M}$. In particular, we want to give a rigorous meaning to the intuitive fact that T_pM depends smoothly upon p.

We will organize the disjoint union of all tangent spaces as a smooth manifold TM. There is a natural surjection

$$\pi : TM = \bigsqcup_{p\in M} T_pM \to M, \quad \pi(v) = p \Longleftrightarrow v \in T_pM.$$

Any local coordinate system $x = (x^i)$ defined over an open set $U \subset M$ produces a natural basis $\big(\frac{\partial}{\partial x^i}(p)\big)$ of T_pM, for any $p \in U$. Thus, an element $v \in TU = \bigsqcup_{p\in U} T_pM$ is completely determined if we know

- which tangent space does it belong to, i.e., we know $p = \pi(v)$,
- the coordinates of v in the basis $\big(\frac{\partial}{\partial x^i}(p)\big)$,

$$v = \sum_i X^i(v)\left(\frac{\partial}{\partial x^i}(p)\right).$$

We thus have a bijection

$$\Psi_x : TU \to U^x \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m,$$

where U^x is the image of U in $\mathbb{R}^m$ via the coordinates (x^i). We can now use the map Ψ_x to transfer the topology on $\mathbb{R}^m \times \mathbb{R}^m$ to TU. Again, we have to make sure this topology is independent of local coordinates.

To see this, pick a different coordinate system $y = (y^i)$ on U. The coordinate independence referred to above is equivalent to the statement that the transition map

$$\Psi_y \circ \Psi_x^{-1} : U^x \times \mathbb{R}^m \longrightarrow TU \longrightarrow U^y \times \mathbb{R}^m$$

is a homeomorphism.

Let $A := (\overline{x}, X) \in U^x \times \mathbb{R}^m$. Then $\Psi_x^{-1}(A) = (p, \dot{\alpha}(0))$, where $x(p) = \overline{x}$, and $\alpha(t) \subset U$ is a path through p given in the x coordinates as

$$\alpha(t) = \overline{x} + tX.$$

Denote by $F : U^x \to U^y$ the transition map $x \mapsto y$. Then

$$\Psi_y \circ \Psi_x^{-1}(A) = \big(y(\overline{x}); Y^1, \ldots, Y^m\big),$$

where $\dot{\alpha}(0) = (\dot{y}^j_\alpha(0)) = \sum Y^j \frac{\partial}{\partial y_j}(p)$, and $\big(y_\alpha(t)\big)$ is the description of the path $\alpha(t)$ in the coordinates y^j. Applying the chain rule we deduce

$$Y^j = \dot{y}^j_\alpha(0) = \sum_i \frac{\partial y^j}{\partial x^i} \dot{x}^i(0) = \sum_i \frac{\partial y^j}{\partial x^i} X^i. \tag{2.1.3}$$

This proves that $\Psi_y \circ \Psi_x^{-1}$ is actually smooth.

The natural topology of TM is obtained by patching together the topologies of TU_γ, where $(U_\gamma, \phi_\gamma)_\gamma$ is an atlas of M. A set $D \subset TM$ is open if its intersection with any TU_γ is open in TU_γ. The above argument shows that TM is a smooth manifold with (TU_γ, Ψ_γ) a defining atlas. Moreover, the natural projection $\pi : TM \to M$ is a smooth map.

Definition 2.1.11. The smooth manifold TM described above is called the *tangent bundle* of M. □

Proposition 2.1.12. *A smooth map $f : M \to N$ induces a smooth map $Df : TM \to TN$ such that*
(a) $Df(T_pM) \subset T_{f(p)}N$, $\forall p \in M$
(b) The restriction to each tangent space $D_pF : T_pM \to T_{f(p)}N$ is linear. The map Df is called the differential *of f, and one often uses the alternate notation $f_* = Df$.*

Proof. Recall that T_pM is the space of tangent vectors to curves through p. If $\alpha(t)$ is such a curve ($\alpha(0) = p$), then $\beta(t) = f(\alpha(t))$ is a smooth curve through $q = f(p)$, and we define

$$Df(\dot{\alpha}(0)) := \dot{\beta}(0).$$

One checks easily that if $\alpha_1 \sim_1 \alpha_2$, then $f(\alpha_1) \sim_1 f(\alpha_2)$, so that Df is well defined. To prove that the map $Df : T_pM \to T_qN$ is linear it suffices to verify this in any particular local coordinates $(x^1, \ldots, x^m)$ near p, and $(y^1, \ldots, y^n)$ near q, such that

$x^i(p) = 0$, $y^j(q) = 0$, $\forall i, j$, since any two choices differ (infinitesimally) by a linear substitution. Hence, we can regard f as a collection of maps

$$(x^1, \dots, x^m) \mapsto (y^1(x^1, \dots, x^m), \dots, y^n(x^1, \dots, x^m)).$$

A basis in T_pM is given by $\left\{ \frac{\partial}{\partial x_i} \right\}$, while a basis of T_qN is given by $\left\{ \frac{\partial}{\partial y_j} \right\}$.

If $\alpha, \beta : (-\varepsilon, \varepsilon) \to M$ are two smooth paths such that $\alpha(0) = \beta(0) = p$, then in local coordinates they have the description

$$\alpha(t) = (x^1_\alpha(t), \dots, x^m_\alpha(t)), \;\; \beta(t) = (x^1_\beta(t), \dots, x^m_\beta(t)),$$

$$\dot\alpha(0) = (\dot x^1_\alpha(0), \dots, \dot x^m_\alpha(0)), \;\; \dot\beta(0) = (\dot x^1_\beta(0), \dots, \dot x^m_\beta(0))$$

Then

$$F(\alpha(t)) = \big(y^1(x^i_\alpha(t)), \dots, y^n(x^i_\alpha(t))\big), \;\; F(\beta(t)) = \big(y^1(x^i_\beta(t)), \dots, y^n(x^i_\beta(t))\big),$$

$$F(\alpha(t) + \beta(t)) = \big(y^1(x^i_\alpha(t) + x^i_\beta(t)), \dots, y^n(x^i_\alpha(t) + x^i_\beta(t))\big),$$

$$DF\big(\dot\alpha(0)\big) = \sum_{j=1}^n \Big(\sum_{i=1}^m \frac{\partial y^j}{\partial x^i}\dot x^i_\alpha\Big)\frac{\partial}{\partial y_j}, \;\; DF\big(\dot\beta(0)\big) = \sum_{j=1}^n \Big(\sum_{i=1}^m \frac{\partial y^j}{\partial x^i}\dot x^i_\beta\Big)\frac{\partial}{\partial y_j},$$

$$DF\big(\dot\alpha(0) + \dot\beta(0)\big) = \frac{d}{dt}|_{t=0} F\big(\alpha(t) + \beta(t)\big) = \sum_{j=1}^n \Big(\sum_{i=1}^m \frac{\partial y^j}{\partial x^i}(\dot x^i_\alpha + \dot x^i_\beta)\Big)\frac{\partial}{\partial y_j}$$

$$= DF\big(\dot\alpha(0)\big) + DF\big(\dot\beta(0)\big).$$

This shows that $Df : T_pM \to T_qN$ is the linear operator given in these bases by the matrix $\left(\frac{\partial y^j}{\partial x^i}\right)_{1 \le j \le n,\, 1 \le i \le m}$. In particular, this implies that Df is also smooth. □

2.1.3 *Sard's Theorem*

In this subsection we want to explain rigorously a phenomenon with which the reader may already be intuitively acquainted. We describe it first in a special case.

Suppose M is a submanifold of dimension 2 in $\mathbb{R}^3$. Then, a simple thought experiment suggests that *most* horizontal planes will not be tangent to M. Equivalently, Iif we denote by f the restriction of the function z to M, then for most real numbers h the level set $f^{-1}(h)$ does not contain a point where the differential of f is zero, so that most level sets $f^{-1}(h)$ are smooth submanifolds of M of codimension 1, i.e., smooth curves on M.

We can ask a more general question. Given two smooth manifolds X, Y, a smooth map $f : X \to Y$, is it true that for "most" $y \in Y$ the level set $f^{-1}(y)$ is a *smooth* submanifold of X of codimension $\dim Y$? This question has a positive answer, known as Sard's theorem.

Definition 2.1.13. Suppose that Y is a smooth, connected manifold of dimension m.

(a) We say that a subset $S \subset Y$ is *negligible* if, for any coordinate chart of Y, $\Psi : U \to \mathbb{R}^m$, the set $\Psi(S \cap U) \subset \mathbb{R}^m$ has Lebesgue measure zero in $\mathbb{R}^m$.
(b) Suppose $F : X \to Y$ is a smooth map, where X is a smooth manifold. A point $x \in X$ is called a *critical* point of F, if the differential $D_x F : T_x X \to T_{F(x)} Y$ is not surjective.

We denote by $\boldsymbol{Cr}_F$ the set of critical points of F, and by $\Delta_F \subset Y$ its image via F. We will refer to Δ_F as the discriminant set of F. The points in Δ_F are called the *critical values* of F. □

Exercise 2.1.14. Define

$$\mathcal{Z} := \bigl\{ (x, a, b, c) \in \mathbb{R}^4;\ \ ax^2 + bx + c = 0;\ \ a \neq 0 \bigr\}.$$

(a) Prove that $\mathcal{Z}$ is a smooth submanifold of $\mathbb{R}^4$.
(b) Define $\pi : \mathcal{Z} \to \mathbb{R}^3$ by $(x, a, b, c) \stackrel{\pi}{\longmapsto} (a, b, c)$. Compute the discriminant set of π. □

Suppose U, V are finite dimensional real Euclidean vector spaces, $\mathcal{O} \subset U$ is an open subset, and $F : \mathcal{O} \to V$ is a smooth map. Then a point $u \in \mathcal{O}$ is a *critical point* of F if and only if

$$\operatorname{rank}(D_u F : U \to V) < \dim V.$$

Exercise 2.1.15. Show that for every $q \in N \setminus \Delta_F$ the fiber $f^{-1}(q)$ is either empty, or a submanifold of M of codimension $\dim N$. □

Theorem 2.1.16 (Sard). *Let $f : \mathcal{O} \to V$ be a smooth map as above. Then the discriminant set Δ_F is negligible.*

Proof. We follow the elegant approach of J. Milnor [74] and L. Pontryagin [82]. Set $n = \dim U$, and $m = \dim V$. We will argue inductively on the dimension n.

For every positive integer k we denote by $\boldsymbol{Cr}_F^k \subset \boldsymbol{Cr}_F$ the set of points $u \in \mathcal{O}$ such that all the partial derivatives of F up to order k vanish at u. We obtain a decreasing filtration of closed sets

$$\boldsymbol{Cr}_F \supset \boldsymbol{Cr}_F^1 \supset \boldsymbol{Cr}_F^2 \supset \cdots .$$

The case $n = 0$ is trivially true so we may assume $n > 0$, and the statement is true for any $n' < n$, and any m. The inductive step is divided into three intermediary steps.

Step 1. The set $F(\boldsymbol{Cr}_F \setminus \boldsymbol{Cr}_F^1)$ is negligible.
Step 2. The set $F(\boldsymbol{Cr}_F^k \setminus \boldsymbol{Cr}_F^{k+1})$ is negligible for all $k \geq 1$.
Step 3. The set $F(\boldsymbol{Cr}_F^k)$ is negligible for some sufficiently large k.

Step 1. Set $\boldsymbol{Cr}_F' := \boldsymbol{Cr}_F \setminus \boldsymbol{Cr}_F^1$. We will show that there exists a countable open cover $\bigl\{\mathcal{O}_j\bigr\}_{j \geq 1}$ of $\boldsymbol{Cr}_F'$ such that $F(\mathcal{O}_j \cap \boldsymbol{Cr}_F')$ is negligible for all $j \geq 1$. Since

$\boldsymbol{Cr}'_F$ is separable, it suffices to prove that every point $u \in \boldsymbol{Cr}'_F$ admits an open neighborhood $\mathcal{N}$ such that $F\big(\mathcal{N} \cap \boldsymbol{Cr}'_F\big)$ is negligible.

Suppose $u_0 \in \boldsymbol{Cr}'_F$. Assume first that there exist local coordinates $(x^1, \dots, , x^n)$ defined in a neighborhood $\mathcal{N}$ of u_0, and local coordinates $(y^1, \dots, y^m)$ near $v_0 = F(u_0)$ such that,

$$x^i(u_0) = 0, \quad \forall i = 1, \dots n, \;\; y^j(v_0) = 0, \;\; \forall j = 1, \dots m,$$

and the restriction of F to $\mathcal{N}$ is described by functions $y^j = y^j(x^1, \dots, x^m)$ such that $y^1 = x^1$.

For every $t \in \mathbb{R}$ we set

$$\mathcal{N}_t := \big\{ (x^1, \dots, x^n) \in \mathcal{N}; \;\; x^1 = t \,\big\},$$

and we define

$$G_t : \mathcal{N}_t \to \mathbb{R}^{m-1}, \;\; (t, x^2, \dots, x^n) \mapsto \big(\, y^2(t, x^2, \dots, x^n), \dots, y^m(t, x^2, \dots, x^n) \,\big).$$

Observe that

$$\mathcal{N} \cap \boldsymbol{Cr}'_F = \bigcup_t \boldsymbol{Cr}_{G_t}\,.$$

The inductive assumption implies that the sets $\boldsymbol{Cr}_{G_t}$ have trivial $(n-1)$-dimensional Lebesgue measure. Using Fubini's theorem we deduce that $\mathcal{N} \cap \boldsymbol{Cr}'_F$ has trivial n-dimensional Lebesgue measure.

To conclude Step 1 is suffices to prove that the above simplifying assumption concerning the existence of nice coordinates is always fulfilled. To see this, choose local coordinates $(s^1, \dots, s^n)$ near u_0 and coordinates $(y^1, \dots, y^m)$ near v_0 such that

$$s^i(u_0) = 0, \quad \forall i = 1, \dots, n, \;\; y^j(v_0) = 0, \;\; \forall j = 1, \dots, m,$$

The map F is then locally described by a collection of functions $y^j(s^1, \dots, s^n)$, $j = 1, \dots, n$. Since $u \in \boldsymbol{Cr}'_F$, we can assume, after an eventual re-labelling of coordinates, that $\frac{\partial y^1}{\partial s^1}(u_0) \neq 0$. Now define

$$x^1 = y^1(s^1, \dots, s^n), \;\; x^i = s^i, \;\; \forall i = 2, \dots, n.$$

The implicit function theorem shows that the collection of functions $(x^1, \dots, x^n)$ defines a coordinate system in a neighborhood of u_0. We regard y^j as functions of x^i. From the definition we deduce $y^1 = x^1$.

Step 2. Set $\boldsymbol{Cr}_F^{(k)} := \boldsymbol{Cr}_F^k \setminus \boldsymbol{Cr}_F^{k+1}$. Since $u_0 \in \boldsymbol{Cr}_F^{(k)}$, we can find local coordinates $(s^1, \dots, s^n)$ near u_0 and coordinates $(y^1, \dots, y^m)$ near v_0 such that

$$s^i(u_0) = 0, \quad \forall i = 1, \dots, n, \;\; y^j(v_0) = 0, \;\; \forall j = 1, \dots, m,$$

$$\frac{\partial^j y^1}{\partial (s^1)^j}(u_0) = 0, \;\; \forall j = 1, \dots, k,$$

and

$$\frac{\partial^{k+1} y^1}{\partial (s^1)^{k+1}}(u_0) \neq 0.$$

Define

$$x^1(s) = \frac{\partial^k y^1}{\partial(s^1)^k},$$

and set $x^i := s^i, \ \forall i = 2, \dots, n$.

Then the collection (x^i) defines smooth local coordinates on an open neighborhood $\mathcal{N}$ of u_0, and $\boldsymbol{Cr}_F^k \cap \mathcal{N}$ is contained in the hyperplane $\{x^1 = 0\}$. Define

$$G : \mathcal{N} \cap \{x^1 = 0\} \to V; \ \ G(x^2, \dots, x^m) = F(0, x^2, \dots, x^n).$$

Then

$$\boldsymbol{Cr}_G^k \cap \mathcal{N} = \boldsymbol{Cr}_G^k, \ \ F(\boldsymbol{Cr}_F^k \cap \mathcal{N}) = G(\boldsymbol{Cr}_G^k),$$

and the induction assumption implies that $G(\boldsymbol{Cr}_G^k)$ is negligible. By covering $\boldsymbol{Cr}_F^{(k)}$ with a countably many open neighborhood $\{\mathcal{N}_\ell\}_{\ell > 1}$ such that $F(\boldsymbol{Cr}_F^k \cap \mathcal{N}_\ell)$ is negligible we conclude that $F(\boldsymbol{Cr}_F^{(k)})$ is negligible.

Step 3. Suppose $k > \frac{n}{m}$. We will prove that $F(\boldsymbol{Cr}_F^k)$ is negligible. More precisely, we will show that, for every compact subset $S \subset \mathcal{O}$, the set $F(S \cap \boldsymbol{Cr}_F^k)$ is negligible.

From the Taylor expansion around points in $\boldsymbol{Cr}_F^k \cap S$ we deduce that there exist numbers $r_0 \in (0,1)$ and $\lambda_0 > 0$, depending only on S, such that, if C is a cube with edge $r < r_0$ which intersects $\boldsymbol{Cr}_F^k \cap S$, then

$$\operatorname{diam} F(C) < \lambda_0 r^k,$$

where for every set $A \subset V$ we define

$$\operatorname{diam}(A) := \sup\big\{ \, |a_1 - a_2|; \ \ a_1, a_2 \in A \, \big\}.$$

In particular, if μ_m denotes the m-dimensional Lebesgue measure on V, and μ_n denotes the n-dimensional Lebesgue measure on U, we deduce that there exists a constant $\lambda_1 > 0$ such that

$$\mu_m\big(F(C)\big) \le \lambda_1 r^{mk} = \lambda_1 \mu_n(C)^{mk/n}.$$

Cover $\boldsymbol{Cr}_F^k \cap S$ by finitely many cubes $\{C_\ell\}_{1 \le \ell \le L}$, of edges $< r_0$, such that their interiors are disjoint. For every positive integer P, subdivide each of the cubes C_ℓ into P^n sub-cubes C_ℓ^i of equal sizes. For every sub-cube C_ℓ^σ which intersects $\boldsymbol{Cr}_F^k$ we have

$$\mu_m(C_\ell^\sigma) \le \lambda_1 \mu_n(C_\ell^\sigma)^{mk/n} = \frac{\lambda_1}{P^{mk}} \mu_n(C_\ell).$$

We deduce that

$$\mu_m(C_\ell \cap \boldsymbol{Cr}_F^k) \le \sum_\sigma \mu_m(C_\ell^\sigma \cap \boldsymbol{Cr}_F^k) \le P^{n-mk} \mu_n(C).$$

If we let $P \to \infty$ in the above inequality, we deduce that when $k > \frac{n}{m}$ we have

$$\mu_m(C_\ell \cap \boldsymbol{Cr}_F^k) = 0, \ \ \forall \ell = 1, \dots, L. \qquad \square$$

Theorem 2.1.16 admits the following immediate generalization.

Corollary 2.1.17 (Sard). *Suppose $F : X \to Y$ is a smooth map between two smooth manifolds. Then its discriminant set is negligible.* □

Definition 2.1.18. A smooth map $f : M \to N$ is called *immersion* (resp. *submersion*) if for every $p \in M$ the differential $D_p f : T_p M \to T_{f(p)} N$ is injective (resp. surjective). A smooth map $f : M \to N$ is called an *embedding* if it is an injective immersion. □

Suppose $F : M \to N$ is a smooth map, and $\dim M \geq \dim N$. Then F is a submersion if and only if the discriminant set Δ_F is empty.

Exercise 2.1.19. Suppose $F : M \to N$ is a smooth map, and $S \subset N$ is a smooth submanifold of N. We say that F is *transversal* to S if for every $x \in F^{-1}(S)$ we have

$$T_{F(x)} N = T_{F(x)} S + D_x F(T_x M).$$

Prove that if F is transversal to S, then $F^{-1}(S)$ is a submanifold of M whose codimension is equal to the codimension of S in N. □

Exercise 2.1.20. Suppose Λ, X, Y are smooth, connected manifolds, and $F : \Lambda \times X \to Y$ is a smooth map

$$\Lambda \times X \ni (\lambda, x) \mapsto F_\lambda(x) \in Y.$$

Suppose S is a submanifold of Y such that F is transversal to S. Define

$$\mathcal{Z} = F^{-1}(S) \subset \Lambda \times X,$$

$$\Lambda_0 = \big\{ \lambda \in \Lambda;\;\; F_\lambda : X \to Y \text{ is not transversal to } S \big\}.$$

Prove that Λ_0 is contained in the discriminant set of the natural projection $\mathcal{Z} \to \Lambda$. In particular, Λ_0 must be negligible. □

2.1.4 *Vector bundles*

The tangent bundle TM of a manifold M has some special features which makes it a very particular type of manifold. We list now the special ingredients which enter into this special structure of TM since they will occur in many instances. Set for brevity $E := TM$, and $F := \mathbb{R}^m$ ($m = \dim M$). We denote by $\mathrm{Aut}(F)$ the Lie group $\mathrm{GL}(n, \mathbb{R})$ of linear automorphisms of F. Then

(a) E is a smooth manifold, and there exists a surjective submersion $\pi : E \to M$. For every $U \subset M$ we set $E\,|_U := \pi^{-1}(U)$.

(b) From (2.1.3) we deduce that there exists a *trivializing cover*, i.e., an open cover $\mathcal{U}$ of M, and for every $U \in \mathcal{U}$ a diffeomorphism

$$\Psi_U : E\,|_U \to U \times F, \quad v \mapsto (p = \pi(v), \Phi^U_p(v))$$

(b1) Φ_p is a diffeomorphism $E_p \to F$ for any $p \in U$.
(b2) If $U,\ V \in \mathfrak{U}$ are two trivializing neighborhoods with non empty overlap $U \cap V$ then, for any $p \in U \cap V$, the map $\Phi_{VU}(p) = \Phi_p^V \circ (\Phi_p^U)^{-1} : F \to F$ is a linear isomorphism, and moreover, the map

$$p \mapsto \Phi_{VU}(p) \in \mathrm{Aut}(F)$$

is smooth.

In our special case, the map $\Phi_{VU}(p)$ is explicitly defined by the matrix (2.1.3)

$$A(p) = \left(\frac{\partial y^j}{\partial x^i}(p)\right)_{1 \leq i,j \leq m}.$$

In the above formula, the functions (x^i) are local coordinates on U, and the functions (y^j) are local coordinates on V.

The properties (a) and (b) make no mention of the special relationship between $E = TM$ and M. There are many other quadruples (E, π, M, F) with these properties and they deserve a special name.

Definition 2.1.21. A *vector bundle* is a quadruple (E, π, M, F) such that

- E, M are smooth manifolds,
- $\pi : E \to M$ is a surjective submersion,
- F is a vector space over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and
- the conditions (a) and (b) above are satisfied.

The manifold E is called the *total space*, and M is called the *base space*. The vector space F is called the *standard fiber*, and its dimension (over the field of scalars $\mathbb{K}$) is called the *rank* of the bundle. A *line bundle* is a vector bundle of rank one.□

Roughly speaking, a vector bundle is a smooth family of vector spaces. Note that the properties (b1) and (b2) imply that the fibers $\pi^{-1}(p)$ of a vector bundle have a natural structure of linear space. In particular, one can add elements in the same fiber. Moreover, the addition and scalar multiplication operations on $\pi^{-1}(p)$ depend smoothly on p. The smoothness of the addition operation this means that the addition is a smooth map

$$+ : E \times_M E = \Big\{ (u, v) \in E \times E;\ \ \pi(u) = \pi(v) \Big\} \to E.$$

The smoothness of the scalar multiplication means that it is smooth map

$$\mathbb{R} \times E \to E.$$

There is an equivalent way of defining vector bundles. To describe it, let us introduce a notation. For any vector space F over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we denote by $\mathrm{GL}_{\mathbb{K}}(F)$, (or simply $\mathrm{GL}(F)$ if there is no ambiguity concerning the field of scalars $\mathbb{K}$) the Lie group of linear automorphisms $F \to F$.

According to Definition 2.1.21, we can find an open cover (U_α) of M such that each of the restrictions $E_\alpha = E|_{U_\alpha}$ is isomorphic to a product $\Psi_\alpha : E_\alpha \cong F \times U_\alpha$. Moreover, on the overlaps $U_\alpha \cap U_\beta$, *the transition maps* $g_{\alpha\beta} = \Psi_\alpha \Psi_\beta^{-1}$ can be viewed as smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(F).$$

They satisfy the *cocycle condition*

(a) $g_{\alpha\alpha} = \mathbb{1}_F$
(b) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = \mathbb{1}_F$ over $U_\alpha \cap U_\beta \cap U_\gamma$.

Conversely, given an open cover (U_α) of M, and a collection of smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(F)$$

satisfying the cocycle condition, we can reconstruct a vector bundle by gluing the product bundles $E_\alpha = F \times U_\alpha$ on the overlaps $U_\alpha \cap U_\beta$ according to the gluing rules

$$\text{the point } (v, x) \in E_\alpha \text{ is identified with the point } \big(g_{\beta\alpha}(x)v, x \big) \in E_\beta \ \ \forall x \in U_\alpha \cap U_\beta.$$

The details are carried out in the exercise below.

We will say that the map $g_{\beta\alpha}$ *is the transition from* the α-trivialization to the β-trivialization, and we will refer to the collection of maps $(g_{\beta\alpha})$ satisfying the cocycle condition as a *gluing cocycle*. We will refer to the cover (U_α) as above as a *trivializing cover*.

Exercise 2.1.22. Consider a smooth manifold M, a vector space V, an open cover (U_α), and smooth maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$$

satisfying the cocycle condition. Set

$$X := \bigcup_\alpha V \times U_\alpha \times \{\alpha\}.$$

We topologize X as the disjoint union of the topological spaces $U_\alpha \times V$, and we define a relation $\sim \subset X \times X$ by

$$V \times U_\alpha \times \{\alpha\} \ni (u, x, \alpha) \sim (v, x, \beta) \in V \times U_\beta \times \{\beta\} \overset{def}{\Longleftrightarrow} x = y, \ \ v = g_{\beta\alpha}(x)u.$$

(a) Show that $\sim$ is an equivalence relation, and $E = X/\sim$ equipped with the quotient topology has a natural structure of smooth manifold.
(b) Show that the projection $\pi : X \to M$, $(u, x, \alpha) \mapsto x$ descends to a submersion $E \to M$.
(c) Prove that (E, π, M, V) is naturally a smooth vector bundle. □

Definition 2.1.23. A description of a vector bundle in terms of a trivializing cover, and a gluing cocycle is called a *gluing cocycle description* of that vector bundle. □

Exercise 2.1.24. Find a gluing cocycle description of the tangent bundle of the 2-sphere. □

In the sequel, we will prefer to think of vector bundles in terms of gluing cocycles.

Definition 2.1.25. (a) A *section* in a vector bundle $E \xrightarrow{\pi} M$ defined over the open subset $u \in M$ is a smooth map $s : U \to E$ such that

$$s(p) \in E_p = \pi^{-1}(p), \ \ \forall p \in U \Longleftrightarrow \pi \circ s = \mathbb{1}_U.$$

The space of smooth sections of E over U will be denoted by $\Gamma(U, E)$ or $C^\infty(U, E)$. Note that $\Gamma(U, E)$ is naturally a vector space.
(b) A section of the tangent bundle of a smooth manifold is called a *vector field* on that manifold. The space of vector fields over on open subset U of a smooth manifold is denoted by $\mathrm{Vect}\,(U)$. □

Proposition 2.1.26. *Suppose $E \to M$ is a smooth vector bundle with standard fiber F, defined by an open cover $(U_\alpha)\alpha \in \mathcal{A}$, and gluing cocycle*

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(F).$$

Then there exists a natural bijection between the vector space of smooth sections of E, and the set of families of smooth maps $\{\, s_\alpha : U_\alpha \to F;\ \ \alpha \in \mathcal{A} \,\}$, satisfying the following gluing condition on the overlaps

$$s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x), \ \ \forall x \in U_\alpha \cap U_\beta. \qquad \square$$

Exercise 2.1.27. Prove the above proposition. □

Definition 2.1.28. (a) Let $E^i \xrightarrow{\pi_i} M_i$ be two smooth vector bundles. A *vector bundle map* consists of a pair of smooth maps $f : M_1 \to M_2$ and $F : E^1 \to E^2$ satisfying the following properties.

- The map F covers f, i.e., $F(E^1_p) \subset E^2_{f(p)}$, $\forall p \in M_1$. Equivalently, this means that the diagram below is commutative

$$\begin{array}{ccc} E^1 & \xrightarrow{F} & E^2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

- The induced map $F : E^1_p \to E^2_{f(p)}$ is linear.

The composition of bundle maps is defined in the obvious manner and so is the identity morphism so that one can define the notion of bundle isomorphism in the standard way.
(b) If E and F are two vector bundles over the same manifold, then we denote by $\mathrm{Hom}(E, F)$ the space of bundle maps $E \to F$ which *cover the identity* $\mathbb{1}_M$. Such bundle maps are called *bundle morphisms.* □

For example, the differential Df of a smooth map $f : M \to N$ is a bundle map $Df : TM \to TN$ covering f.

Definition 2.1.29. Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. A *bundle endomorphism* of E is a bundle morphism $F : E \to E$. An automorphism (or *gauge transformation*) is an invertible endomorphism. □

Example 2.1.30. Consider the trivial vector bundle $\underline{\mathbb{R}}^n_M \to M$ over the smooth manifold M. A section of this vector bundle is a smooth map $u : M \to \mathbb{R}^n$. We can think of u as a smooth family of vectors $(u(x) \in \mathbb{R}^n)_{x \in M}$.

An endomorphism of this vector bundle is a smooth map $A : M \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$. We can think of A as a smooth family of $n \times n$ matrices

$$A_x = \begin{bmatrix} a^1_1(x) & a^1_2(x) & \cdots & a^1_n(x) \\ a^2_1(x) & a^2_2(x) & \cdots & a^2_n(x) \\ \vdots & \vdots & \vdots & \vdots \\ a^n_1(x) & a^n_2(x) & \cdots & a^n_n(x) \end{bmatrix}, \quad a^i_j(x) \in C^\infty(M).$$

The map A is a gauge transformation if and only if $\det A_x \neq 0$, $\forall x \in M$. □

Exercise 2.1.31. Suppose $E_1, E_2 \to M$ are two smooth vector bundles over the smooth manifold with standard fibers F_1, and respectively F_2. Assume that both bundle are defined by a common trivializing cover $(U_\alpha)_{\alpha \in \mathcal{A}}$ and gluing cocycles

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(F_1), \quad h_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(F_2).$$

Prove that there exists a bijection between the vector space of bundle morphisms $\mathrm{Hom}(E, F)$, and the set of families of smooth maps

$$\big\{ T_\alpha : U_\alpha \to \mathrm{Hom}(F_1, F_2); \;\; \alpha \in \mathcal{A} \big\},$$

satisfying the gluing conditions

$$T_\beta(x) = h_{\beta\alpha}(x) T_\alpha(x)(x) g^{-1}_{\beta\alpha}, \;\; \forall x \in U_{\alpha\beta}.$$ □

Exercise 2.1.32. Let V be a vector space, M a smooth manifold, $\{U_\alpha\}$ an open cover of M, and $g_{\alpha\beta}, \; h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ two collections of smooth maps satisfying the cocycle conditions. Prove the two collections define isomorphic vector bundles if and only they are *cohomologous*, i.e., there exist smooth maps $\phi_\alpha : U_\alpha \to \mathrm{GL}(V)$ such that

$$h_{\alpha\beta} = \phi_\alpha g_{\alpha\beta} \phi^{-1}_\beta.$$ □

2.1.5 *Some examples of vector bundles*

In this section we would like to present some important examples of vector bundles and then formulate some questions concerning the global structure of a bundle.

Example 2.1.33. (The tautological line bundle over $\mathbb{RP}^n$ and $\mathbb{CP}^n$). First, let us recall that a rank one vector bundle is usually called a *line bundle*. We consider only the complex case. The total space of the *tautological* or *universal* line bundle over $\mathbb{CP}^n$ is the space

$$\mathcal{U}_n = \mathcal{U}_n^{\mathbb{C}} = \left\{ (z, L) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n; \;\; z \text{ belongs to the line } L \subset \mathbb{C}^{n+1} \right\}.$$

Let $\pi : \mathcal{U}_n^{\mathbb{C}} \to \mathbb{CP}^n$ denote the projection onto the second component. Note that for every $L \in \mathbb{CP}^n$, the fiber through $\pi^{-1}(L) = \mathcal{U}_{n,L}^{\mathbb{C}}$ coincides with the one-dimensional subspace in $\mathbb{C}^{n+1}$ defined by L. □

Example 2.1.34. (The tautological vector bundle over a Grassmannian). We consider here for brevity only complex Grassmannian $\mathbf{Gr}_k(\mathbb{C}^n)$. The real case is completely similar. The total space of this bundle is

$$\mathcal{U}_{k,n} = \mathcal{U}_{k,n}^{\mathbb{C}} = \left\{ (z, L) \in \mathbb{C}^n \times \mathbf{Gr}_k(\mathbb{C}^n) \; ; \; z \text{ belongs to the subspace } L \subset \mathbb{C}^n \right\}.$$

If π denotes the natural projection $\pi : \mathcal{U}_{k,n} \to \mathbf{Gr}_k(\mathbb{C}^n)$, then for each $L \in \mathbf{Gr}_k(\mathbb{C}^n)$ the fiber over L coincides with the subspace in $\mathbb{C}^n$ defined by L. Note that $\mathcal{U}_{n-1}^{\mathbb{C}} = \mathcal{U}_{1,n}^{\mathbb{C}}$. □

Exercise 2.1.35. Prove that $\mathcal{U}_n^{\mathbb{C}}$ and $\mathcal{U}_{k,n}^{\mathbb{C}}$ are indeed smooth vector bundles. Describe a gluing cocycle for $\mathcal{U}_n^{\mathbb{C}}$. □

Example 2.1.36. A complex line bundle over a smooth manifold M is described by an open cover $(U_\alpha)_{\alpha \in \mathcal{A}}$, and smooth maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*,$$

satisfying the cocycle condition

$$g_{\gamma\alpha}(x) = g_{\gamma\beta}(x) \cdot g_{\beta\alpha}(x), \;\; \forall x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Consider for example the manifold $M = S^2 \subset \mathbb{R}^3$. Denote as usual by N and S the North and respectively South pole. We have an open cover

$$S^2 = U_0 \cup U_\infty, \;\; U_0 = S^2 \setminus \{S\}, \;\; U_1 = S^2 \setminus \{N\}.$$

In this case, we have only a single nontrivial overlap, $U_N \cap U_S$. Identify U_0 with the complex line $\mathbb{C}$, so that the North pole becomes the origin $z = 0$.

For every $n \in \mathbb{Z}$ we obtain a complex line bundle $L_n \to S^2$, defined by the open cover $\{U_0, U_1\}$ and gluing cocycle

$$g_{10}(z) = z^{-n}, \;\; \forall z \in \mathbb{C}^* = U_0 \setminus \{0\}.$$

A smooth section of this line bundle is described by a pair of smooth functions

$$u_0 : U_0 \to \mathbb{C}, \ \ u_1 : U_1 \to \mathbb{C},$$

which along the overlap $U_0 \cap U_1$ satisfy the equality $u_1(z) = z^{-n}u_0(z)$. For example, if $n \geq 0$, the pair of functions

$$u_0(z) = z^n, \ \ u_1(p) = 1, \ \ \forall p \in U_1$$

defines a smooth section of L_n. □

Exercise 2.1.37. We know that $\mathbb{CP}^1$ is diffeomorphic to S^2. Prove that the universal line bundle $\mathcal{U}_n \to \mathbb{CP}^1$ is isomorphic with the line bundle L_{-1} constructed in the above example. □

Exercise 2.1.38. Consider the incidence set

$$\mathcal{I} := \big\{ \,(x, L) \in (\,\mathbb{C}^{n+1} \setminus \{0\}\,) \times \mathbb{CP}^n; \ \ z \in L \,\big\}.$$

Prove that the closure of $\mathcal{I}$ in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ is a smooth manifold diffeomorphic to the total space of the universal line bundle $\mathcal{U}_n \to \mathbb{CP}^n$. This manifold is called the *complex blowup* of $\mathbb{C}^{n+1}$ at the origin. □

The family of vector bundles is very large. The following construction provides a very powerful method of producing vector bundles.

Definition 2.1.39. Let $f : X \to M$ be a smooth map, and E a vector bundle over M defined by an open cover (U_α) and gluing cocycle $(g_{\alpha\beta})$. The *pullback of E by f* is the vector bundle f^*E over X defined by the open cover $f^{-1}(U_\alpha)$, and the gluing cocycle $(\,g_{\alpha\beta} \circ f\,)$. □

One can check easily that the isomorphism class of the pullback of a vector bundle E is independent of the choice of gluing cocycle describing E. The pullback operation defines a linear map between the space of sections of E and the space of sections of f^*E.

More precisely, if $s \in \Gamma(E)$ is defined by the open cover (U_α), and the collection of smooth maps $(\,s_\alpha\,)$, then the pullback f^*s is defined by the open cover $f^{-1}(U_\alpha)$, and the smooth maps $(\,s_\alpha \circ f\,)$. Again, there is no difficulty to check the above definition is independent of the various choices.

Exercise 2.1.40. For every positive integer k consider the map

$$p_k : \mathbb{CP}^1 \to \mathbb{CP}^1, \ \ p_k([z_0, z_1]) = [z_0^k, z_1^k].$$

Show that $p_k^*L_n \cong L_{kn}$, where L_n is the complex line bundle $L_n \to \mathbb{CP}^1$ defined in Example 2.1.36. □

Exercise 2.1.41. Let $E \to X$ be a rank k (complex) smooth vector bundle over the manifold X. Assume E is *ample*, i.e. there exists a finite family $s_1, \dots, s_N$ of smooth sections of E such that, for any $x \in X$, the collection $\{s_1(x), \dots, s_N(x)\}$ spans E_x. For each $x \in X$ we set

$$S_x := \Big\{ v \in \mathbb{C}^N \ ; \ \sum_i v^i s_i(x) = 0 \Big\}.$$

Note that $\dim S_x = N - k$. We have a map $F : X \to \mathbf{Gr}_k(\mathbb{C}^N)$ defined by $x \mapsto S_x^{\perp}$.
(a) Prove that F is smooth.
(b) Prove that E is isomorphic with the pullback $F^*\mathcal{U}_{k,N}$. □

Exercise 2.1.42. Show that any vector bundle over a smooth *compact* manifold is ample. Thus any vector bundle over a compact manifold is a pullback of some tautological bundle! □

The notion of vector bundle is trickier than it may look. Its definition may suggest that a vector bundle looks like a direct product $manifold \times$ vector space since this happens at least locally. We will denote by $\underline{\mathbb{K}}^n_M$ the bundle $\mathbb{K}^r \times M \to M$.

Definition 2.1.43. A rank r vector bundle $E \xrightarrow{\pi} M$ (over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is called *trivial* or *trivializable* if there exists a bundle isomorphism $E \cong \underline{\mathbb{K}}^r_M$. A bundle isomorphism $E \to \underline{\mathbb{K}}^r_M$ is called a *trivialization* of E, while an isomorphism $\underline{\mathbb{K}}^r \to E$ is called a *framing* of E.

A pair (trivial vector bundle, trivialization) is called a *trivialized*, or *framed* bundle. □

Remark 2.1.44. Let us explain why we refer to a bundle isomorphism $\varphi : \underline{\mathbb{K}}^r_M \to E$ as a framing.

Denote by $(e_1, \dots, e_r)$ the canonical basis of $\mathbb{K}^r$. We can also regard the vectors e_i as constant maps $M \to \mathbb{K}^r$, i.e., as (special) sections of $\underline{\mathbb{K}}^r_M$. The isomorphism φ determines sections $f_i = \varphi(e_i)$ of E with the property that for every $x \in M$ the collection $(f_1(x), \dots, f_r(x))$ is a *frame* of the fiber E_x.

This observation shows that we can regard any framing of a bundle $E \to M$ of rank r as a collection of r sections $u_1, \dots, u_r$ which are *pointwise* linearly independent. □

One can naively ask the following question. Is every vector bundle trivial? We can even limit our search to tangent bundles. Thus we ask the following question.

Is it true that for every smooth manifold M the tangent bundle TM is trivial (as a vector bundle)?

Let us look at some positive examples.

Example 2.1.45. $TS^1 \cong \underline{\mathbb{R}}_{S^1}$ Let θ denote the angular coordinate on the circle. Then $\frac{\partial}{\partial\theta}$ is a globally defined, nowhere vanishing vector field on S^1. We thus get a map

$$\underline{\mathbb{R}}_{S^1} \to TS^1, \ \ (s,\theta) \mapsto (s\frac{\partial}{\partial\theta}, \theta) \in T_\theta S^1$$

which is easily seen to be a bundle isomorphism.

Let us carefully analyze this example. Think of S^1 as a Lie group (the group of complex numbers of norm 1). The tangent space at $z = 1$, i.e., $\theta = 0$, coincides with the subspace $\boldsymbol{Re}\, z = 0$, and $\frac{\partial}{\partial\theta}\,|_1$ is the unit vertical vector $\boldsymbol{j}$.

Denote by R_θ the counterclockwise rotation by an angle θ. Clearly R_θ is a diffeomorphism, and for each θ we have a linear isomorphism

$$D_\theta\,|_{\theta=0}\, R_\theta : T_1S^1 \to T_\theta S^1.$$

Moreover,

$$\frac{\partial}{\partial\theta} = D_\theta\,|_{\theta=0}\, R_\theta \boldsymbol{j}.$$

The existence of the trivializing vector field $\frac{\partial}{\partial\theta}$ is due to to our ability to "move freely and coherently" inside S^1. One has a similar freedom inside a Lie group as we are going to see in the next example. □

Example 2.1.46. For any Lie group G the tangent bundle TG is trivial.

To see this let $n = \dim G$, and consider $e_1, \dots, e_n$ a basis of the tangent space at the origin, $T_{\mathbb{1}}G$. We denote by R_g the right translation (by g) in the group defined by

$$R_g : \ x \mapsto x \cdot g, \ \ \forall x \in G.$$

R_g is a diffeomorphism with inverse $R_{g^{-1}}$ so that the differential DR_g defines a linear isomorphism $DR_g : T_{\mathbb{1}}G \to T_gG$. Set

$$E_i(g) = DR_g(e_i) \in T_gG, \ \ i = 1, \cdots, n.$$

Since the multiplication $G \times G \to G$, $(g,h) \mapsto g \cdot h$ is a smooth map we deduce that the vectors $E_i(g)$ define *smooth* vector fields over G. Moreover, for every $g \in G$, the collection $\{E_1(g), \dots, E_n(g)\}$ is a basis of T_gG so we can define without ambiguity a map

$$\Phi : \underline{\mathbb{R}}^n_G \to TG, \ \ (g; X^1, \dots X^n) \mapsto (g; \sum X^i E_i(g)).$$

One checks immediately that Φ is a vector bundle isomorphism and this proves the claim. In particular TS^3 is trivial since the sphere S^3 is a Lie group (unit quaternions). (Using the Cayley numbers one can show that TS^7 is also trivial; see [83] for details.) □

We see that the tangent bundle TM of a manifold M is trivial if and only if there exist vector fields $X_1, \dots, X_m$ ($m = \dim M$) such that for each $p \in M$, $X_1(p), \dots, X_m(p)$ span T_pM. This suggests the following more refined question.

Problem *Given a manifold M, compute $v(M)$, the maximum number of pointwise linearly independent vector fields over M. Obviously $0 \leq v(M) \leq \dim M$ and TM is trivial if and only if $v(M) = \dim M$. A special instance of this problem is the celebrated vector field problem: compute $v(S^n)$ for any $n \geq 1$.*

We have seen that $v(S^n) = n$ for n =1,3 and 7. Amazingly, these are the only cases when the above equality holds. This is a highly nontrivial result, first proved by J.F.Adams in [2] using very sophisticated algebraic tools. This fact is related to many other natural questions in algebra. For a nice presentation we refer to [69].

The methods we will develop in this book will not suffice to compute $v(S^n)$ for any n, but we will be able to solve "half" of this problem. More precisely we will show that

$$v(S^n) = 0 \text{ if and only if } n \text{ is an even number.}$$

In particular, this shows that TS^{2n} is not trivial. In odd dimensions the situation is far more elaborate (a complete answer can be found in [2]).

Exercise 2.1.47. $v(S^{2k-1}) \geq 1$ for any $k \geq 1$. □

The quantity $v(M)$ can be viewed as a measure of nontriviality of a tangent bundle. Unfortunately, its computation is highly nontrivial. In the second part of this book we will describe more efficient ways of measuring the extent of nontriviality of a vector bundle.

2.2 A linear algebra interlude

We collect in this section some classical notions of linear algebra. Most of them might be familiar to the reader, but we will present them in a form suitable for applications in differential geometry. This is perhaps the least glamorous part of geometry, and unfortunately cannot be avoided.

☞ **Convention.** *All the vector spaces in this section will tacitly be assumed finite dimensional, unless otherwise stated.*

2.2.1 *Tensor products*

Let E, F be two vector spaces over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Consider the (infinite) direct sum

$$\mathcal{T}(E,F) = \bigoplus\nolimits_{(e,f)\in E\times F} \mathbb{K}.$$

Equivalently, the vector space $\mathcal{T}(E,F)$ can be identified with the space of functions $c : E \times F \to \mathbb{K}$ with finite support. The space $\mathcal{T}(E,F)$ has a natural basis consisting of "Dirac functions"

$$\delta_{e,f} : E \times F \to \mathbb{K},\ \ (x,y) \mapsto \begin{cases} 1 \text{ if } (x,y) = (e,f) \\ 0 \text{ if } (x,y) \neq (e,f). \end{cases}$$

In particular, we have an injection[1]

$$\delta : E \times F \to \mathfrak{T}(E, F), \ \ (e, f) \mapsto \delta_{e,f}.$$

Inside $\mathfrak{T}(E, F)$ sits the linear subspace $\mathcal{R}(E, F)$ spanned by

$$\lambda\delta_{e,f} - \delta_{\lambda e,f},\ \lambda\delta_{e,f} - \delta_{e,\lambda f},\ \delta_{e+e',f} - \delta_{e,f} - \delta_{e',f},\ \delta_{e,f+f'} - \delta_{e,f} - \delta_{e,f'},$$

where $e, e' \in E,\ f, f' \in F$, and $\lambda \in \mathbb{K}$. Now define

$$E \otimes_{\mathbb{K}} F := \mathfrak{T}(E, F)/\mathcal{R}(E, F),$$

and denote by π the canonical projection $\pi : \mathfrak{T}(E, F) \to E \otimes F$. Set

$$e \otimes f := \pi(\delta_{e,f}).$$

We get a natural map

$$\iota : E \times F \to E \otimes F, \ \ e \times f \mapsto e \otimes f.$$

Obviously ι is bilinear. The vector space $E \otimes_{\mathbb{K}} F$ is called the *tensor product* of E and F over $\mathbb{K}$. Often, when the field of scalars is clear from the context, we will use the simpler notation $E \otimes F$. The tensor product has the following universality property.

Proposition 2.2.1. *For any bilinear map $\phi : E \times F \to G$ there exists a* unique *linear map $\Phi : E \otimes F \to G$ such that the diagram below is commutative.*

$$\begin{tikzcd} E \times F \arrow[r, "\iota"] \arrow[dr, "\phi"'] & E \otimes F \arrow[d, dashed, "\Phi"] \\ & G \end{tikzcd} .$$

□

The proof of this result is left to the reader as an exercise. Note that if (e_i) is a basis of E, and (f_j) is a basis of F, then $(e_i \otimes f_j)$ is a basis of $E \otimes F$, and therefore

$$\dim_{\mathbb{K}} E \otimes_{\mathbb{K}} F = (\dim_{\mathbb{K}} E) \cdot (\dim_{\mathbb{K}} F).$$

Exercise 2.2.2. Using the universality property of the tensor product prove that there exists a natural isomorphism $E \otimes F \cong F \otimes E$ uniquely defined by $e \otimes f \mapsto f \otimes e$. □

The above construction can be iterated. Given three vector spaces $E_1,\ E_2,\ E_3$ over the same field of scalars $\mathbb{K}$ we can construct two triple tensor products:

$$(E_1 \otimes E_2) \otimes E_3 \ \text{ and } \ E_1 \otimes (E_2 \otimes E_3).$$

Exercise 2.2.3. Prove there exists a natural isomorphism of $\mathbb{K}$-vector spaces

$$(E_1 \otimes E_2) \otimes E_3 \cong E_1 \otimes (E_2 \otimes E_3).$$

□

[1] *A word of caution: δ is not linear!*

The above exercise implies that there exists a unique (up to isomorphism) triple tensor product which we denote by $E_1 \otimes E_2 \otimes E_3$. Clearly, we can now define multiple tensor products: $E_1 \otimes \cdots \otimes E_n$.

Definition 2.2.4. (a) For any two vector spaces U, V over the field $\mathbb{K}$ we denote by $\mathrm{Hom}(U, V)$, or $\mathrm{Hom}_{\mathbb{K}}(U, V)$ the space of $\mathbb{K}$-linear maps $U \to V$.
(b) The dual of a $\mathbb{K}$-linear space E is the linear space E^* defined as the space $\mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$ of $\mathbb{K}$-linear maps $E \to \mathbb{K}$. For any $e^* \in E^*$ and $e \in E$ we set

$$\langle e^*, e\rangle := e^*(e). \qquad \square$$

The above constructions are functorial. More precisely, we have the following result.

Proposition 2.2.5. *Suppose E_i, F_i, G_i, $i = 1, 2$ are $\mathbb{K}$-vector spaces. Let $T_i \in \mathrm{Hom}(E_i, F_i)$, $S_i \in \mathrm{Hom}(F_i, G_i)$, $i = 1, 2$, be two linear operators. Then they naturally induce a linear operator*

$$T = T_1 \otimes T_2 : E_1 \otimes E_2 \to F_1 \otimes F_2, \;\; S_1 \otimes S_2 : F_1 \otimes F_2$$

uniquely defined by

$$T_1 \otimes T_2(e_1 \otimes e_2) = (T_1 e_1) \otimes (T_2 e_2), \;\; \forall e_i \in E_i,$$

and satisfying

$$(S_1 \otimes S_2) \circ (T_1 \otimes T_2) = (S_1 \circ T_1) \otimes (S_2 \circ T_2).$$

(b) Any linear operator $A : E \to F$ induces a linear operator $A^\dagger : F^* \to E^*$ uniquely defined by

$$\langle A^\dagger f^*, e\rangle = \langle f^*, Ae\rangle, \;\; \forall e \in E, \;\; f^* \in F^*.$$

The operator $A^\dagger$ is called the transpose *or* adjoint *of A. Moreover,*

$$(A \circ B)^\dagger = B^\dagger \circ A^\dagger, \;\; \forall A \in \mathrm{Hom}(F, G), \;\; B \in \mathrm{Hom}(E, F). \qquad \square$$

Exercise 2.2.6. Prove the above proposition. $\square$

Remark 2.2.7. Any basis $(\boldsymbol{e}_i)_{1 \le i \le n}$ of the n-dimensional $\mathbb{K}$-vector space determines a basis $(\boldsymbol{e}^i)_{1 \le i \le n}$ of the dual vector space V^* uniquely defined by the conditions

$$\langle \boldsymbol{e}^i, \boldsymbol{e}_j\rangle = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

We say that the basis $(\boldsymbol{e}^i)$ is *dual* to the basis $(\boldsymbol{e}_j)$. The quantity (δ^i_j) is called the *Kronecker symbol.*

A vector $\boldsymbol{v} \in V$ admits a decomposition

$$\boldsymbol{v} = \sum_{i=1}^{n} v^i \boldsymbol{e}_i,$$

while a vector $\boldsymbol{v}^* \in V^*$ admits a decomposition

$$\boldsymbol{v}^* = \sum_{i=1}^{n} v_i^* \boldsymbol{e}^i.$$

Moreover,

$$\langle \boldsymbol{v}^*, \boldsymbol{v} \rangle = \sum_{i=1}^{n} v_i^* v^i.$$

Classically, a vector $\boldsymbol{v}$ in V is represented by a one-column matrix

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix},$$

while a vector $\boldsymbol{v}^*$ is represented by a one-row matrix

$$\boldsymbol{v}^* = [v_1^* \quad \dots \quad v_n^*].$$

Then

$$\langle \boldsymbol{v}^*, \boldsymbol{v} \rangle = [v_1^* \quad \dots \quad v_n^*] \cdot \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix},$$

where the $\cdot$ denotes the multiplication of matrices. □

Using the functoriality of the tensor product and of the dualization construction one proves easily the following result.

Proposition 2.2.8. *(a) There exists a natural isomorphism*

$$E^* \otimes F^* \cong (E \otimes F)^*,$$

uniquely defined by

$$E^* \otimes F^* \ni e^* \otimes f^* \longmapsto L_{e^* \otimes f^*} \in (E \otimes F)^*,$$

where

$$\langle L_{e^* \otimes f^*}, x \otimes y \rangle = \langle e^*, x \rangle \langle f^*, y \rangle, \ \ \forall x \in E, \ \ y \in F.$$

In particular, this shows $E^* \otimes F^*$ can be naturally identified with the space of bilinear maps $E \times F \to \mathbb{K}$.

(b) The adjunction morphism $E^* \otimes F \to \operatorname{Hom}(E, F)$, *given by*

$$E^* \otimes F \ni e^* \otimes f \longmapsto T_{e^* \otimes f} \in \operatorname{Hom}(E, F),$$

where

$$T_{e^* \otimes f}(x) := \langle e^*, x \rangle f, \ \ \forall x \in E,$$

is an isomorphism.[2] □

[2]The finite dimensionality of E is absolutely necessary. This adjunction formula is known in Bourbaki circles as "*formule d'adjonction chêr à Cartan*".

Exercise 2.2.9. Prove the above proposition. □

Let V be a vector space. For $r, s \geq 0$ set

$$\mathcal{T}_s^r(V) := V^{\otimes r} \otimes (V^*)^{\otimes s},$$

where by definition $V^{\otimes 0} = (V^*)^{\otimes 0} = \mathbb{K}$. An element of $\mathcal{T}_s^r$ is called *tensor of type (r,s)*.

Example 2.2.10. According to Proposition 2.2.8 a tensor of type $(1,1)$ can be identified with a linear endomorphism of V, i.e.,

$$\mathcal{T}_1^1(V) \cong \operatorname{End}(V),$$

while a tensor of type $(0,k)$ can be identified with a k-linear map

$$\underbrace{V \times \cdots \times V}_{k} \to \mathbb{K}.$$

□

A tensor of type $(r,0)$ is called *contravariant*, while a tensor of type $(0,s)$ is called *covariant*. The *tensor algebra* of V is defined to be

$$\mathcal{T}(V) := \bigoplus_{r,s} \mathcal{T}_s^r(V).$$

We use the term algebra since the tensor product induces bilinear maps

$$\otimes : \mathcal{T}_s^r \times \mathcal{T}_{s'}^{r'} \to \mathcal{T}_{s+s'}^{r+r'}.$$

The elements of $\mathcal{T}(V)$ are called *tensors*.

Exercise 2.2.11. Show that $\big(\mathcal{T}(V), +, \otimes\big)$ is an associative algebra. □

Example 2.2.12. It is often useful to represent tensors using coordinates. To achieve this pick a basis (e_i) of V, and let (e^i) denote the dual basis in V^* uniquely defined by

$$\langle e^i, e_j \rangle = \delta_j^i = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

We then obtain a basis of $\mathcal{T}_s^r(V)$

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s} /\ 1 \leq i_\alpha,\, j_\beta \leq \dim V\}.$$

Any element $T \in \mathcal{T}_s^r(V)$ has a decomposition

$$T = T_{j_1 \ldots j_s}^{i_1 \ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots, \otimes e^{j_s},$$

where we use Einstein convention to sum over indices which appear twice, once as a *superscript*, and the second time as *subscript*.

Using the adjunction morphism in Proposition 2.2.8, we can identify the space $\mathcal{T}_1^1(V)$ with the space $\operatorname{End}(V)$ a linear isomorphisms. Using the bases (e_i) and (e^j), and Einstein's convention, the adjunction identification can be described as the correspondence which associates to the tensor $A = a_j^i e_i \otimes e^j \in \mathcal{T}_1^1(V)$, the linear operator $L_A : V \to V$ which maps the vector $v = v^j e_j$ to the vector $L_A v = a_j^i v^j e_i$. □

On the tensor algebra there is a natural *contraction* (or *trace*) operation

$$\mathrm{tr}\,:\mathcal{T}^r_s \to \mathcal{T}^{r-1}_{s-1}$$

uniquely defined by

$$\mathrm{tr}\,(v_1\otimes\cdots\otimes v_r\otimes u^1\otimes\cdots\otimes u^s) := \langle u^1, v_1\rangle v_2\otimes\cdots v_r\otimes u^2\otimes\cdots\otimes u^s,$$

$\forall v_i\in V,\ u^j\in V^*$.

In the coordinates determined by a basis (e_i) of V, the contraction can be described as

$$(\mathrm{tr}\,T)^{i_2\dots i_r}_{j_2\dots j_s} = \left(T^{ii_2\dots i_r}_{ij_2\dots j_s}\right),$$

where again we use Einstein's convention. In particular, we see that the contraction coincides with the usual trace on $\mathcal{T}^1_1(V)\cong \mathrm{End}\,(V)$.

2.2.2 *Symmetric and skew-symmetric tensors*

Let V be a vector space over $\mathbb{K}=\mathbb{R},\mathbb{C}$. We set $\mathcal{T}^r(V) := \mathcal{T}^r_0(V)$, and we denote by $\mathcal{S}_r$ the group of permutations of r objects. When $r=0$ we set $\mathcal{S}_0 := \{1\}$.

Every permutation $\sigma\in\mathcal{S}_r$ determines a linear map $\mathcal{T}^r(V)\to\mathcal{T}^r(V)$, uniquely determined by the correspondences

$$v_1\otimes\cdots\otimes v_r \longmapsto v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r)},\ \ \forall v_1,\dots,v_r\in V.$$

We denote this action of $\sigma\in\mathcal{S}_r$ on an arbitrary element $t\in\mathcal{T}^r(V)$ by σt.

In this subsection we will describe two subspaces invariant under this action. These are special instances of the so called *Schur functors.* (We refer to [34] for more general constructions.) Define

$$\boldsymbol{S}_r : \mathcal{T}^r(V)\to\mathcal{T}^r(V),\ \ \boldsymbol{S}_r(t) := \frac{1}{r!}\sum_{\sigma\in\mathcal{S}_r}\sigma t,$$

and

$$\boldsymbol{A}_r : \mathcal{T}^r(V)\to\mathcal{T}^r(V),\ \ \boldsymbol{A}_r(t) := \begin{cases} \frac{1}{r!}\sum_{\sigma\in\mathcal{S}_r}\epsilon(\sigma)\sigma t & \text{if } r\le\dim V\\ 0 & \text{if } r>\dim V\end{cases}.$$

Above, we denoted by $\epsilon(\sigma)$ the signature of the permutation σ. Note that

$$\boldsymbol{A}_0 = \boldsymbol{S}_0 = \mathbb{1}_{\mathbb{K}}.$$

The following results are immediate. Their proofs are left to the reader as exercises.

Lemma 2.2.13. *The operators $\boldsymbol{A}_r$ and $\boldsymbol{S}_r$ are projectors of $\mathcal{T}^r(V)$, i.e.,*

$$\boldsymbol{S}_r^2 = \boldsymbol{S}_r,\ \ \boldsymbol{A}_r^2=\boldsymbol{A}_r.$$

Moreover,

$$\sigma\boldsymbol{S}_r(t)=\boldsymbol{S}_r(\sigma t)=\boldsymbol{S}_r(t),\ \sigma\boldsymbol{A}_r(t)=\boldsymbol{A}_r(\sigma t)=\epsilon(\sigma)\boldsymbol{A}_r(t),\ \ \forall t\in\mathcal{T}^r(V). \qquad \square$$

Definition 2.2.14. A tensor $T \in \mathcal{T}^r(V)$ is called *symmetric* (respectively *skew-symmetric*) if

$$\boldsymbol{S}_r(T) = T \quad \text{(respectively } \boldsymbol{A}_r(T) = T\text{)}.$$

The nonnegative integer r is called the *degree* of the (skew-)symmetric tensor.

The space of symmetric tensors (respectively skew-symmetric ones) of degree r will be denoted by $\boldsymbol{S}^r V$ (and respectively $\Lambda^r V$). □

Set

$$\boldsymbol{S}^\bullet V := \bigoplus_{r \geq 0} \boldsymbol{S}^r V, \text{ and } \Lambda^\bullet V := \bigoplus_{r \geq 0} \Lambda^r V.$$

Definition 2.2.15. The *exterior product* is the bilinear map

$$\wedge : \Lambda^r V \times \Lambda^s V \to \Lambda^{r+s} V,$$

defined by

$$\omega^r \wedge \eta^s := \frac{(r+s)!}{r!s!} \boldsymbol{A}_{r+s}(\omega \otimes \eta), \ \ \forall \omega^r \in \Lambda^r V, \ \eta^s \in \Lambda^s V. \qquad \square$$

Proposition 2.2.16. *The exterior product has the following properties.*
(a) (Associativity)

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma), \ \ \forall \alpha, \beta\, \gamma \in \Lambda^\bullet V.$$

In particular,

$$v_1 \wedge \cdots \wedge v_k = k! \boldsymbol{A}_k(v_1 \otimes \ldots \otimes v_k) = \sum_{\sigma \in \mathcal{S}_k} \epsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}, \ \ \forall v_i \in V.$$

(b) (Super-commutativity)

$$\omega^r \wedge \eta^s = (-1)^{rs} \eta^s \wedge \omega^r, \ \ \forall \omega^r \in \Lambda^r V, \ \ \omega^s \in \Lambda^s V.$$

Proof. We first define a new product "$\wedge_1$" by

$$\omega^r \wedge_1 \eta^s := \boldsymbol{A}_{r+s}(\omega \otimes \eta),$$

which will turn out to be associative and will force $\wedge$ to be associative as well.

To prove the associativity of $\wedge_1$ consider the quotient algebra $\mathcal{Q}^* = \mathcal{T}^*/\mathcal{J}^*$, where $\mathcal{T}^*$ is the *associative* algebra $(\bigoplus_{r \geq 0} \mathcal{T}^r(V), +, \otimes)$, and $\mathcal{J}^*$ is the bilateral ideal generated by the set of squares $\{v \otimes v / \ v \in V\}$. Denote the (obviously associative) multiplication in $\mathcal{Q}^*$ by $\cup$. The natural projection $\pi : \mathcal{T}^* \to \mathcal{Q}^*$ induces a linear map $\pi : \Lambda^\bullet V \to \mathcal{Q}^*$. We will complete the proof of the proposition in two steps.

Step 1. We will prove that the map $\pi : \Lambda^\bullet V \to \mathcal{Q}^*$ is a linear isomorphism, and moreover

$$\pi(\alpha \wedge_1 \beta) = \pi(\alpha) \cup \pi(\beta). \tag{2.2.1}$$

In particular, $\wedge_1$ is an associative product.

The crucial observation is

$$\pi(T) = \pi(\boldsymbol{A}_r(T)), \ \ \forall t \in \mathcal{T}^r(V). \tag{2.2.2}$$

It suffices to check (2.2.2) on monomials $T = e_1 \otimes \cdots \otimes e_r$, $e_i \in V$. Since

$$(u+v)^{\otimes 2} \in \mathcal{I}^*, \ \ \forall u,\, v \in V$$

we deduce $u \otimes v = -v \otimes u (\operatorname{mod} \mathcal{I}^*)$. Hence, for any $\sigma \in \mathcal{S}_r$

$$\pi(e_1 \otimes \cdots \otimes e_r) = \epsilon(\sigma)\pi(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(r)}). \tag{2.2.3}$$

When we sum over $\sigma \in \mathcal{S}_r$ in (2.2.3) we obtain (2.2.2).

To prove the injectivity of π note first that $\boldsymbol{A}_*(\mathcal{I}^*) = 0$. If $\pi(\omega) = 0$ for some $\omega \in \Lambda^\bullet V$, then $\omega \in \ker \pi = \mathcal{I}^* \cap \Lambda^\bullet V$ so that

$$\omega = \boldsymbol{A}_*(\omega) = 0.$$

The surjectivity of π follows immediately from (2.2.2). Indeed, any $\pi(T)$ can be alternatively described as $\pi(\omega)$ for some $\omega \in \Lambda^\bullet V$. It suffices to take $\omega = \boldsymbol{A}_*(T)$.

To prove (2.2.1) it suffices to consider only the special cases when α and β are monomials:

$$\alpha = \boldsymbol{A}_r(e_1 \otimes \cdots \otimes e_r), \ \ \beta = \boldsymbol{A}_s(f_1 \otimes \cdots \otimes f_s).$$

We have

$$\pi(\alpha \wedge_1 \beta) = \pi\left(\boldsymbol{A}_{r+s}(\boldsymbol{A}_r(e_1 \otimes \cdots \otimes e_r) \otimes \boldsymbol{A}_s(f_1 \otimes \cdots \otimes f_s))\right)$$

$$\overset{(2.2.2)}{=} \pi\left(\boldsymbol{A}_r(e_1 \otimes \cdots \otimes e_r) \otimes \boldsymbol{A}_s(f_1 \otimes \cdots \otimes f_s)\right)$$

$$\overset{def}{=} \pi(\boldsymbol{A}_r(e_1 \otimes \cdots \otimes e_r)) \cup \pi(\boldsymbol{A}_s(f_1 \otimes \cdots \otimes f_s)) = \pi(\alpha) \cup \pi(\beta).$$

Thus $\wedge_1$ is associative.

Step 2. The product $\wedge$ is associative. Consider $\alpha \in \Lambda^r V$, $\beta \in \Lambda^s V$ and $\gamma \in \Lambda^t V$. We have

$$(\alpha \wedge \beta) \wedge \gamma = \left(\frac{(r+s)!}{r!s!}\alpha \wedge_1 \beta\right) \wedge \gamma = \frac{(r+s)!}{r!s!}\frac{(r+s+t)!}{(r+s)!t!}(\alpha \wedge_1 \beta) \wedge_1 \gamma$$

$$= \frac{(r+s+t)!}{r!s!t!}(\alpha \wedge_1 \beta)\wedge_1 \gamma = \frac{(r+s+t)!}{r!s!t!}\alpha \wedge_1 (\beta \wedge_1 \gamma) = \alpha \wedge (\beta \wedge \gamma).$$

The associativity of $\wedge$ is proved. The computation above shows that

$$e_1 \wedge \cdots \wedge e_k = k! \boldsymbol{A}_k(e_1 \otimes \cdots \otimes e_k).$$

(b) The supercommutativity of $\wedge$ follows from the supercommutativity of $\wedge_1$ (or $\cup$). To prove the latter one uses (2.2.2). The details are left to the reader. $\square$

Exercise 2.2.17. Finish the proof of part (b) in the above proposition. $\square$

The space $\Lambda^\bullet V$ is called the *exterior algebra* of V. $\wedge$ is called the *exterior product* . The exterior algebra is a $\mathbb{Z}$-graded algebra, i.e.,

$$(\Lambda^r V) \wedge (\Lambda^s V) \subset \Lambda^{r+s} V, \ \ \forall r, s.$$

Note that $\Lambda^r V = 0$ for $r > \dim V$ (pigeonhole principle).

Definition 2.2.18. Let V be an n-dimensional $\mathbb{K}$-vector space. The one dimensional vector space $\Lambda^n V$ is called the *determinant line* of V, and it is denoted by $\det V$. □

There exists a natural injection $\iota_V : V \hookrightarrow \Lambda^\bullet V$, $\iota_V(v) = v$, such that

$$\iota_V(v) \wedge \iota_V(v) = 0, \ \ \forall v \in V.$$

This map enters crucially into the formulation of the following universality property.

Proposition 2.2.19. *Let V be a vector space over $\mathbb{K}$. For any $\mathbb{K}$-algebra A, and any linear map $\phi : V \to A$ such that $(\phi(x)^2 = 0$, there exists an unique morphism of $\mathbb{K}$-algebras $\Phi : \Lambda^\bullet V \to A$ such that the diagram below is commutative*

$$\begin{tikzcd} V \arrow[r, hook, "\iota_V"] \arrow[dr, "\phi"'] & \Lambda^\bullet V \arrow[d, dashed, "\Phi"] \\ & A \end{tikzcd} ,$$

i.e., $\Phi \circ \iota_V = \phi$.

Exercise 2.2.20. Prove Proposition 2.2.19. □

The space of symmetric tensors $\boldsymbol{S}^\bullet V$ can be similarly given a structure of associative algebra with respect to the product

$$\alpha \cdot \beta := \boldsymbol{S}_{r+s}(\alpha \otimes \beta), \ \ \forall \alpha \in \boldsymbol{S}^r V, \ \beta \in \boldsymbol{S}^s V.$$

The symmetric product "$\cdot$" is also commutative.

Exercise 2.2.21. Formulate and prove the analogue of Proposition 2.2.19 for the algebra $\boldsymbol{S}^\bullet V$. □

It is often convenient to represent (skew-)symmetric tensors in coordinates. If $e_1, \dots, e_n$ is a basis of the vector space V then, for any $1 \leq r \leq n$, the family

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_r} / \ 1 \leq i_1 < \cdots < i_r \leq n \}$$

is a basis for $\Lambda^r V$ so that any degree r skew-symmetric tensor ω can be uniquely represented as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \omega^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r}.$$

Symmetric tensors can be represented in a similar way.

The $\Lambda^\bullet$ and $\boldsymbol{S}^\bullet$ constructions are functorial in the following sense.

Proposition 2.2.22. *Any linear map $L : V \to W$ induces a natural morphisms of algebras*

$$\Lambda^\bullet L : \Lambda^\bullet V \to \Lambda^\bullet W, \;\; \boldsymbol{S}^\bullet L : \boldsymbol{S}^\bullet V \to \boldsymbol{S}^\bullet W$$

uniquely defined by their actions on monomials

$$\Lambda^\bullet L(v_1 \wedge \cdots \wedge v_r) = (Lv_1) \wedge \cdots \wedge (Lv_r),$$

and

$$\boldsymbol{S}^\bullet L(v_1, \dots, v_r) = (Lv_1) \cdots (Lv_r).$$

Moreover, if $U \xrightarrow{A} V \xrightarrow{B} W$ are two linear maps, then

$$\Lambda^r(BA) = (\Lambda^r B)(\Lambda^r A), \;\; \boldsymbol{S}^r(BA) = (\boldsymbol{S}^r B)(\boldsymbol{S}^r A). \qquad \square$$

Exercise 2.2.23. Prove the above proposition. $\square$

In particular, if $n = \dim_{\mathbb{K}} V$, then any linear endomorphism $L : V \to V$ defines an endomorphism

$$\Lambda^n L : \det V = \Lambda^n V \to \det V.$$

Since the vector space $\det V$ is 1-dimensional, the endomorphism $\Lambda^n L$ can be identified with a scalar $\det L$, the determinant of the endomorphism L.

Definition 2.2.24. Suppose V is a finite dimensional vector space and $A : V \to V$ is an endomorphism of V. For every positive integer r we denote by $\sigma_r(A)$ the trace of the induced endomorphism

$$\Lambda^r A : \Lambda^r V \to \Lambda^r V,$$

and by $\psi_r(A)$ the trace of the endomorphism

$$\boldsymbol{S}^r(A) : \boldsymbol{S}^r V \to \boldsymbol{S}^r V.$$

We define

$$\sigma_0(A) = 1, \;\; \psi_0(A) = \dim V. \qquad \square$$

Exercise 2.2.25. Suppose V is a complex n-dimensional vector space, and A is an endomorphism of V.

(a) Prove that if A is diagonalizable and its eigenvalues are $a_1, \dots, a_n$, then

$$\sigma_r(A) = \sum_{1 \le i_1 < \cdots < i_r \le n} a_{i_1} \cdots a_{i_r}, \;\; \psi_r(A) = a_1^r + \cdots + a_n^r.$$

(b) Prove that for every sufficiently small $z \in \mathbb{C}$ we have the equalities

$$\det(\mathbb{1}_V + zA) = \sum_{j \ge 0} \sigma_r(A) z^r, \;\; -\frac{d}{dz} \log \det(\mathbb{1} - zA) = \sum_{r \ge 1} \psi_r(A) z^{r-1}. \qquad \square$$

The functors $\Lambda^\bullet$ and $\boldsymbol{S}^\bullet$ have an *exponential like* behavior, i.e., there exists a *natural* isomorphism

$$\Lambda^\bullet(V \oplus W) \cong \Lambda^\bullet V \otimes \Lambda^\bullet W. \tag{2.2.4}$$

$$\boldsymbol{S}^\bullet(V \oplus W) \cong \boldsymbol{S}^\bullet V \otimes \boldsymbol{S}^\bullet W. \tag{2.2.5}$$

To define the isomorphism in (2.2.4) consider the bilinear map

$$\phi : \Lambda^\bullet V \times \Lambda^\bullet W \to \Lambda^\bullet(V \oplus W),$$

uniquely determined by

$$\phi(v_1 \wedge \cdots \wedge v_r\,,\, w_1 \wedge \cdots \wedge w_s) = v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s.$$

The universality property of the tensor product implies the existence of a linear map

$$\Phi : \Lambda^\bullet V \otimes \Lambda^\bullet W \to \Lambda^\bullet(V \oplus W),$$

such that $\Phi \circ \iota = \phi$, where ι is the inclusion of $\Lambda^\bullet V \times \Lambda^\bullet W$ in $\Lambda^\bullet V \otimes \Lambda^\bullet W$. To construct the inverse of Φ, note that $\Lambda^\bullet V \otimes \Lambda^\bullet W$ is naturally a $\mathbb{K}$-algebra by

$$(\omega \otimes \eta) * (\omega' \otimes \eta') = (-1)^{\deg \eta \cdot \deg \omega'}(\omega \wedge \omega') \otimes (\eta \wedge \eta').$$

The vector space $V \oplus W$ is naturally embedded in $\Lambda^\bullet V \otimes \Lambda^\bullet W$ via the map given by

$$(v, w) \mapsto \psi(v, w) = v \otimes 1 + 1 \otimes w \in \Lambda^\bullet V \otimes \Lambda^\bullet W.$$

Moreover, for any $x \in V \oplus W$ we have $\psi(x) * \psi(x) = 0$. The universality property of the exterior algebra implies the existence of a unique morphism of $\mathbb{K}$-algebras

$$\Psi : \Lambda^\bullet(V \oplus W) \to \Lambda^\bullet V \otimes \Lambda^\bullet W,$$

such that $\Psi \circ \iota_{V \oplus W} = \psi$. Note that Φ is also a morphism of $\mathbb{K}$-algebras, and one verifies easily that

$$(\Phi \circ \Psi) \circ \iota_{V \oplus W} = \iota_{V \oplus W}.$$

The uniqueness part in the universality property of the exterior algebra implies $\Phi \circ \Psi = \textit{identity}$. One proves similarly that $\Psi \circ \Phi = \textit{identity}$, and this concludes the proof of (2.2.4).

We want to mention a few general facts about $\mathbb{Z}$-graded vector spaces, i.e., vector spaces equipped with a direct sum decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

(We will always assume that each V_n is finite dimensional.) The vectors in V_n are said to be *homogeneous, of degree* n. For example, the ring of polynomials $\mathbb{K}[x]$ is a $\mathbb{K}$-graded vector space. The spaces $\Lambda^\bullet V$ and $\boldsymbol{S}^\bullet V$ are $\mathbb{Z}$-graded vector spaces.

The direct sum of two $\mathbb{Z}$-graded vector spaces V and W is a $\mathbb{Z}$-graded vector space with

$$(V \oplus W)_n := V_n \oplus W_n.$$

The tensor product of two $\mathbb{Z}$-graded vector spaces V and W is a $\mathbb{Z}$-graded vector space with

$$(V \otimes W)_n := \bigoplus_{r+s=n} V_r \otimes W_s.$$

To any $\mathbb{Z}$-graded vector space V one can naturally associate a formal series $P_V(t) \in \mathbb{Z}[[t, t^{-1}]]$ by

$$P_V(t) := \sum_{n \in \mathbb{Z}} (\dim_{\mathbb{K}} V_n) t^n.$$

The series $P_V(t)$ is called the *Poincaré series* of V.

Example 2.2.26. The Poincaré series of $\mathbb{K}[x]$ is

$$P_{\mathbb{K}[x]}(t) = 1 + t + t^2 + \cdots + t^{n-1} + \cdots = \frac{1}{1-t}. \qquad \square$$

Exercise 2.2.27. Let V and W be two $\mathbb{Z}$-graded vector spaces. Prove the following statements are true (whenever they make sense).
(a) $P_{V \oplus W}(t) = P_V(t) + P_W(t)$.
(b) $P_{V \otimes W}(t) = P_V(t) \cdot P_W(t)$.
(c) $\dim V = P_V(1)$. $\square$

Definition 2.2.28. Let V be a $\mathbb{Z}$-graded vector space. The *Euler characteristic* of V, denoted by $\chi(V)$, is defined by

$$\chi(V) := P_V(-1) = \sum_{n \in \mathbb{Z}} (-1)^n \dim V_n,$$

whenever the sum on the right-hand side makes sense. $\square$

Remark 2.2.29. If we try to compute $\chi(\mathbb{K}[x])$ using the first formula in Definition 2.2.28 we get $\chi(\mathbb{K}[x]) = 1/2$, while the second formula makes no sense (divergent series). $\square$

Proposition 2.2.30. *Let V be a $\mathbb{K}$-vector space of dimension n. Then*

$$P_{\Lambda^\bullet V}(t) = (1+t)^n \text{ and } P_{\boldsymbol{S}^\bullet V}(t) = \left(\frac{1}{1-t}\right)^n = \frac{1}{(n-1)!} \left(\frac{d}{dt}\right)^{n-1} \left(\frac{1}{1-t}\right).$$

In particular, $\dim \Lambda^\bullet V = 2^n$ *and* $\chi(\Lambda^\bullet V) = 0$.

Proof. From (2.2.4) and (2.2.5) we deduce using Exercise 2.2.27 that for any vector spaces V and W we have

$$P_{\Lambda^\bullet(V\oplus W)}(t) = P_{\Lambda^\bullet V}(t)\cdot P_{\Lambda^\bullet W}(t) \ \text{ and } \ P_{\boldsymbol{S}^\bullet(V\oplus W)}(t) = P_{\boldsymbol{S}^\bullet V}(t)\cdot P_{\boldsymbol{S}^\bullet W}(t).$$

In particular, if V has dimension n, then $V\cong\mathbb{K}^n$ so that

$$P_{\Lambda^\bullet V}(t) = (P_{\Lambda^\bullet \mathbb{K}}(t))^n \ \text{ and } \ P_{\boldsymbol{S}^\bullet V}(t) = (P_{\boldsymbol{S}^\bullet \mathbb{K}}(t))^n.$$

The proposition follows using the equalities

$$P_{\Lambda^\bullet \mathbb{K}}(t) = 1+t, \ \text{ and } \ P_{\boldsymbol{S}^\bullet \mathbb{K}}(t) = P_{\mathbb{K}[x]}(t) = \frac{1}{1-t}. \qquad \square$$

2.2.3 *The "super" slang*

The aim of this very brief section is to introduce the reader to the "super" terminology. We owe the "super" slang to the physicists. In the quantum world many objects have a special feature not present in the Newtonian world. They have parity (or chirality), and objects with different chiralities had to be treated differently.

The "super" terminology provides an algebraic formalism which allows one to deal with the different parities on an equal basis. From a strictly syntactic point of view, the "super" slang adds the attribute super to most of the commonly used algebraic objects. In this book, the prefix "s-" will abbreviate the word "super".

Definition 2.2.31. (a) A s-space is a $\mathbb{Z}_2$-graded vector space, i.e., a vector space V equipped with a direct sum decomposition $V = V_0\oplus V_1$.

(b) A s-algebra over $\mathbb{K}$ is a $\mathbb{Z}_2$-graded $\mathbb{K}$-algebra, i.e., a $\mathbb{K}$-algebra $\mathcal{A}$ together with a direct sum decomposition $\mathcal{A} = \mathcal{A}^0\oplus\mathcal{A}^1$ such that $\mathcal{A}^i\cdot\mathcal{A}^j\subset\mathcal{A}^{i+j\,(mod\,2)}$. The elements in $\mathcal{A}^i$ are called homogeneous of degree i. For any $a\in\mathcal{A}^i$ we denote its degree (mod 2) by $|a|$. The elements in $\mathcal{A}^0$ are said to be *even* while the elements in $\mathcal{A}^1$ are said to be *odd*.

(c) The *supercommutator* in a s-algebra $\mathcal{A} = \mathcal{A}^0\oplus\mathcal{A}^1$ is the bilinear map

$$[\bullet,\bullet]_s : \mathcal{A}\times\mathcal{A}\to\mathcal{A},$$

defined on homogeneous elements $\omega^i\in\mathcal{A}^i$, $\eta^j\in\mathcal{A}^j$ by

$$[\omega^i,\eta^j]_s := \omega^i\eta^j - (-1)^{ij}\eta^j\omega^j.$$

An s-algebra is called *s-commutative*, if the suppercommutator is trivial, $[\bullet,\bullet]_s\equiv 0$.
$\square$

Example 2.2.32. Let $E = E^0\oplus E^1$ be a s-space. Any linear endomorphism $T\in\operatorname{End}(E)$ has a block decomposition

$$T = \begin{bmatrix} T_{00} & T_{01}\\ T_{10} & T_{11}\end{bmatrix},$$

where $T_{ji} \in \mathrm{End}\,(E^i, E^j)$. We can use this block decomposition to describe a structure of s-algebra on $\mathrm{End}\,(E)$. The even endomorphisms have the form

$$\begin{bmatrix} T_{00} & 0 \\ 0 & T_{11} \end{bmatrix},$$

while the odd endomorphisms have the form

$$\begin{bmatrix} 0 & T_{01} \\ T_{10} & 0 \end{bmatrix}.$$

□

Example 2.2.33. Let V be a finite dimensional space. The exterior algebra $\Lambda^\bullet V$ is naturally a s-algebra. The even elements are gathered in

$$\Lambda^{even} V = \bigoplus_{r \text{ even}} \Lambda^r V,$$

while the odd elements are gathered in

$$\Lambda^{odd} V = \bigoplus_{r \text{ odd}} \Lambda^r V.$$

The s-algebra $\Lambda^\bullet V$ is s-commutative. □

Definition 2.2.34. Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be a s-algebra. An *s-derivation* on $\mathcal{A}$ is a linear operator on $D \in \mathrm{End}\,(\mathcal{A})$ such that, for any $x \in \mathcal{A}$,

$$[D, L_x]_s^{\mathrm{End}(\mathcal{A})} = L_{Dx}, \tag{2.2.6}$$

where $[\ ,\ ]_s^{\mathrm{End}(\mathcal{A})}$ denotes the supercommutator in $\mathrm{End}\,(\mathcal{A})$ (with the s-structure defined in Example 2.2.32), while for any $z \in \mathcal{A}$ we denoted by L_z the left multiplication operator $a \mapsto z \cdot a$.

An s-derivation is called *even* (respectively *odd*), if it is even (respectively *odd*) as an element of the s-algebra $\mathrm{End}\,(\mathcal{A})$. □

Remark 2.2.35. The relation (2.2.6) is a super version of the usual Leibniz formula. Indeed, assuming D is homogeneous (as an element of the s-algebra $\mathrm{End}\,(\mathcal{A})$) then equality (2.2.6) becomes

$$D(xy) = (Dx)y + (-1)^{|x||D|} x(Dy),$$

for any homogeneous elements $x, y \in \mathcal{A}$. □

Example 2.2.36. Let V be a vector space. Any $u^* \in V^*$ defines an odd s-derivation of $\Lambda^\bullet V$ denoted by i_{u^*} uniquely determined by its action on monomials.

$$i_{u^*}(v_0 \wedge v_1 \wedge \cdots \wedge v_r) = \sum_{i=0}^{r} (-1)^i \langle u^*, v_i \rangle v_0 \wedge v_1 \wedge \ \dots\ \wedge \hat{v}_i \wedge \cdots \wedge v_r.$$

As usual, a hat indicates a missing entry. The derivation i_{u^*} is called the *interior derivation* by u^* or the *contraction* by u^*. Often, one uses the alternate notation $u^* \lrcorner$ to denote this derivation. □

Exercise 2.2.37. Prove the statement in the above example. □

Definition 2.2.38. Let $\mathcal{A} = (\mathcal{A}^0 \oplus \mathcal{A}^1, +, [\ ,\ ])$ be an s-algebra over $\mathbb{K}$, not necessarily associative. For any $x \in \mathcal{A}$ we denote by R_x the right multiplication operator, $a \mapsto [a, x]$. $\mathcal{A}$ is called an *s-Lie algebra* if it is s-anticommutative, i.e.,

$$[x, y] + (-1)^{|x||y|}[y, x] = 0, \text{ for all homogeneous elements } x, y \in \mathcal{A},$$

and $\forall x \in \mathcal{A}$, R_x is a s-derivation.

When $\mathcal{A}$ is purely even, i.e., $\mathcal{A}^1 = \{0\}$, then $\mathcal{A}$ is called simply a *Lie algebra.* The multiplication in a (s-) Lie algebra is called the (s-)bracket. □

The above definition is highly condensed. In down-to-earth terms, the fact that R_x is a s-derivation for all $x \in \mathcal{A}$ is equivalent with the *super Jacobi identity*

$$[[y, z], x] = [[y, x], z] + (-1)^{|x||y|}[y, [z, x]], \tag{2.2.7}$$

for all homogeneous elements $x, y, z \in \mathcal{A}$. When $\mathcal{A}$ is a purely even $\mathbb{K}$-algebra, then $\mathcal{A}$ is a *Lie algebra* over $\mathbb{K}$ if $[\ ,\ ]$ is anticommutative and satisfies (2.2.7), which in this case is equivalent with the classical *Jacobi identity*,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \ \forall x, y, z \in \mathcal{A}. \tag{2.2.8}$$

Example 2.2.39. Let E be a vector space (purely even). Then $\mathcal{A} = \mathrm{End}\,(E)$ is a Lie algebra with bracket given by the usual commutator: $[a, b] = ab - ba$. □

Proposition 2.2.40. *Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be a s-algebra, and denote by $\mathrm{Der}_s(\mathcal{A})$ the vector space of s-derivations of $\mathcal{A}$.*
(a) For any $D \in \mathrm{Der}_s(\mathcal{A})$, its homogeneous components $D^0,\ D^1 \in \mathrm{End}\,(\mathcal{A})$ are also s-derivations.
(b) For any $D,\ D' \in \mathrm{Der}_s(\mathcal{A})$, the s-commutator $[D, D']_s^{\mathrm{End}(A)}$ is again an s-derivation.
(c) $\forall x \in \mathcal{A}$ the bracket $B^x : a \mapsto [a, x]_s$ is a s-derivation called the bracket derivation *determined by x. Moreover*

$$[B^x, B^y]_s^{\mathrm{End}\,(\mathcal{A})} = B^{[x,y]_s}, \ \forall x, y \in \mathcal{A}.$$ □

Exercise 2.2.41. Prove Proposition 2.2.40. □

Definition 2.2.42. Let $E = E^0 \oplus E^1$ and $F = F^0 \oplus F^1$ be two s-spaces. Their s-tensor product is the s-space $E \otimes F$ with the $\mathbb{Z}_2$-grading,

$$(E \otimes F)^\epsilon := \bigoplus_{i+j \equiv \epsilon\,(2)} E^i \otimes F^j, \ \epsilon = 0, 1.$$

To emphasize the super-nature of the tensor product we will use the symbol "$\hat{\otimes}$" instead of the usual "$\otimes$". □

Exercise 2.2.43. Show that there exists a natural isomorphism of s-spaces

$$V^* \widehat{\otimes} \Lambda^\bullet V \cong \mathrm{Der}_s(\Lambda^\bullet V),$$

uniquely determined by $v^* \times \omega \mapsto D^{v^* \otimes \omega}$, where $D^{v^* \otimes \omega}$ is s-derivation defined by

$$D^{v^* \otimes \omega}(v) = \langle v^*, v \rangle \omega, \ \ \forall v \in V.$$

Notice in particular that any s-derivation of $\Lambda^\bullet V$ is uniquely determined by its action on $\Lambda^1 V$. (When $\omega = 1$, $D^{v^* \otimes 1}$ coincides with the internal derivation discussed in Example 2.2.36.) □

Let $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ be an s-algebra over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$. A *supertrace* on $\mathcal{A}$ is a $\mathbb{K}$-linear map $\tau : \mathcal{A} \to \mathbb{K}$ such that,

$$\tau([x, y]_s) = 0 \ \ \forall x, y \in \mathcal{A}.$$

If we denote by $[\mathcal{A}, \mathcal{A}]_s$ the linear subspace of $\mathcal{A}$ spanned by the supercommutators

$$\big\{ [x, y]_s \ ; \ x, y \in \mathcal{A} \big\},$$

then we see that the space of s-traces is isomorphic with the dual of the quotient space $\mathcal{A}/[\mathcal{A}, \mathcal{A}]_s$.

Proposition 2.2.44. *Let $E = E_0 \oplus E_1$ be a finite dimensional s-space, and denote by $\mathcal{A}$ the s-algebra of endomorphisms of E. Then there exists a canonical s-trace* tr_s *on $\mathcal{A}$ uniquely defined by*

$$\mathrm{tr}_s \mathbb{1}_E = \dim E_0 - \dim E_1.$$

In fact, if $T \in \mathcal{A}$ has the block decomposition

$$T = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix},$$

then

$$\mathrm{tr}_s T = \mathrm{tr}\, T_{00} - \mathrm{tr}\, T_{11}.$$ □

Exercise 2.2.45. Prove the above proposition. □

2.2.4 *Duality*

Duality is a subtle and fundamental concept which permeates all branches of mathematics. This section is devoted to those aspects of the atmosphere called duality which are relevant to differential geometry. In the sequel, *all vector spaces will be tacitly assumed finite dimensional*, and we will use Einstein's convention without mentioning it. $\mathbb{K}$ will denote one of the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2.46. A *pairing* between two $\mathbb{K}$-vector spaces V and W is a bilinear map $B : V \times W \to \mathbb{K}$. □

Any pairing $B : V \times W \to \mathbb{K}$ defines a linear map

$$\mathbb{I}_B : V \to W^*, \;\; v \mapsto B(v, \bullet) \in W^*,$$

called the *adjunction morphism* associated to the pairing.

Conversely, any linear map $L : V \to W^*$ defines a pairing

$$B_L : V \times W \to \mathbb{K}, \;\; B(v,w) = (Lv)(w), \;\; \forall v \in V, \;\; w \in W.$$

Observe that $\mathbb{I}_{B_L} = L$. A pairing B is called a *duality* if the adjunction map $\mathbb{I}_B$ is an isomorphisms.

Example 2.2.47. The natural pairing $\langle \bullet, \bullet \rangle : V^* \times V \to \mathbb{K}$ is a duality. One sees that $\mathbb{I}_{\langle \bullet, \bullet \rangle} = \mathbb{1}_{V^*} : V^* \to V^*$. This pairing is called the *natural duality* between a vector space and its dual. □

Example 2.2.48. Let V be a finite dimensional real vector space. Any symmetric nondegenerate quadratic form $(\bullet, \bullet) : V \times V \to \mathbb{R}$ defines a (self)duality, and in particular a natural isomorphism

$$\mathfrak{L} := \mathbb{I}_{(\bullet,\bullet)} : V \to V^*.$$

When $(\bullet, \bullet)$ is positive definite, then the operator $\mathfrak{L}$ is called *metric duality*. This operator can be nicely described in coordinates as follows. Pick a basis (e_i) of V, and set

$$g_{ij} := (e_i, e_j).$$

Let (e^j) denote the dual basis of V^* defined by

$$\langle e^j, e_i \rangle = \delta_i^j, \;\; \forall i, j.$$

The action of $\mathfrak{L}$ is then

$$\mathfrak{L} e_i = g_{ij} e^j.$$

□

Example 2.2.49. Consider V a real vector space and $\omega : V \times V \to \mathbb{R}$ a skew-symmetric bilinear form on V. The form ω is said to be *symplectic* if this pairing is a duality. In this case, the induced operator $\mathbb{I}_\omega : V \to V^*$ is called *symplectic duality*. □

Exercise 2.2.50. Suppose that V is a real vector space, and $\omega : V \times V \to \mathbb{R}$ is a symplectic duality. Prove the following.

(a) The V has even dimension.

(b) If (e_i) is a basis of V, and $\omega_{ij} := \omega(e_i, e_j)$, then $\det(\omega_{ij})_{1 \le i,j \le \dim V} \neq 0$. □

The notion of duality is compatible with the functorial constructions introduced so far.

Proposition 2.2.51. *Let $B_i : V_i \times W_i \to \mathbb{R}$ $(i = 1, 2)$ be two pairs of spaces in duality. Then there exists a natural duality*

$$B = B_1 \otimes B_2 : (V_1 \otimes V_2) \times (W_1 \otimes W_2) \to \mathbb{R},$$

uniquely determined by

$$\mathbb{I}_{B_1 \otimes B_2} = \mathbb{I}_{B_1} \otimes \mathbb{I}_{B_2} \Longleftrightarrow B(v_1 \otimes v_2, w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2). \qquad \square$$

Exercise 2.2.52. Prove Proposition 2.2.51. □

Proposition 2.2.51 implies that given two spaces in duality $B : V \times W \to \mathbb{K}$ there is a naturally induced duality

$$B^{\otimes n} : V^{\otimes r} \times W^{\otimes r} \to \mathbb{K}.$$

This defines by restriction a pairing

$$\Lambda^r B : \Lambda^r V \times \Lambda^r W \to \mathbb{K}$$

uniquely determined by

$$\Lambda^r B \left(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r\right) := \det \left(B\left(v_i, w_j\right)\right)_{1 \leq i,j \leq r}.$$

Exercise 2.2.53. Prove the above pairing is a duality. □

In particular, the natural duality $\langle \bullet, \bullet \rangle : V^* \times V \to \mathbb{K}$ induces a duality

$$\langle \bullet, \bullet \rangle : \Lambda^r V^* \times \Lambda^r V \to \mathbb{R},$$

and thus defines a natural isomorphism

$$\Lambda^r V^* \cong \left(\Lambda^r V\right)^*.$$

This shows that we can regard the elements of $\Lambda^r V^*$ as skew-symmetric r-linear forms $V^r \to \mathbb{K}$.

A duality $B : V \times W \to \mathbb{K}$ naturally induces a duality $B^\dagger : V^* \times W^* \to \mathbb{K}$ by

$$B^\dagger(v^*, w^*) :=\langle v^*, \mathbb{I}_B^{-1} w^* \rangle,$$

where $\mathbb{I}_B : V \to W^*$ is the adjunction isomorphism induced by the duality B.

Now consider a (*real*) Euclidean vector space V. Denote its inner product by $(\bullet, \bullet)$. The self-duality defined by $(\bullet, \bullet)$ induces a self-duality

$$(\bullet, \bullet) : \Lambda^r V \times \Lambda^r V \to \mathbb{R},$$

determined by

$$(v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r) := \det\left(\,(v_i, w_j)\,\right)_{1 \leq i,j \leq r}. \tag{2.2.9}$$

The right hand side of (2.2.9) is a Gramm determinant, and in particular, the bilinear form in (2.2.9) is symmetric and positive definite. Thus, we have proved the following result.

Corollary 2.2.54. *An inner product on a real vector space V naturally induces an inner product on the tensor algebra $\mathcal{T}(V)$, and in the exterior algebra $\Lambda^\bullet V$.* □

In a Euclidean vector space V the inner product induces the metric duality $\mathcal{L} : V \to V^*$. This induces an operator $\mathcal{L} : T^r_s(V) \to T^{r-1}_{s+1}(V)$ defined by

$$\mathcal{L}(v_1 \otimes \ldots \otimes v_r \otimes u^1 \otimes \cdots \otimes u^s) = (v_2 \otimes \cdots \otimes v_r) \otimes ((\mathcal{L}v_1 \otimes u^1 \otimes \cdots \otimes u^s). \quad (2.2.10)$$

The operation defined in (2.2.10) is classically referred to as *lowering the indices.*

The reason for this nomenclature comes from the coordinate description of this operation. If $T \in \mathcal{T}^r_s(V)$ is given by

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$

then

$$(\mathcal{L}T)^{i_2 \ldots i_r}_{jj_1 \ldots j_r} = g_{ij} T^{ii_2 \ldots i_r}_{j_1 \ldots j_s},$$

where $g_{ij} = (e_i, e_j)$. The inverse of the metric duality $\mathcal{L}^{-1} : V^* \to V$ induces a linear operation $\mathcal{T}^r_s(V) \to \mathcal{T}^{r+1}_{s-1}(V)$ called *raising the indices.*

Exercise 2.2.55 (Cartan). *Let V be an Euclidean vector space. For any $v \in V$ denote by e_v (resp. i_v) the linear endomorphism of $\Lambda^* V$ defined by $e_v \omega = v \wedge \omega$ (resp. $i_v = \imath_{v^*}$ where $\imath_{v^*}$ denotes the interior derivation defined by $v^* \in V^*$-the metric dual of v; see Example 2.2.36). Show that for any $u, v \in V$*

$$[e_v, i_u]_s = e_v i_u + i_u e_v = (u, v) \mathbb{1}_{\Lambda^\bullet V}. \qquad \square$$

Definition 2.2.56. Let V be a *real* vector space. A *volume form* on V is a non-trivial linear form on the determinant line of V, $\mu : \det V \to \mathbb{R}$. $\square$

Equivalently, a volume form on V is a nontrivial element of $\det V^*$ ($n = \dim V$). Since $\det V$ is 1-dimensional, a choice of a volume form corresponds to a choice of a basis of $\det V$.

Definition 2.2.57. (a) An *orientation* on a vector space V is a *continuous, surjective* map

$$\boldsymbol{or} : \det V \setminus \{0\} \to \{\pm 1\}.$$

We denote by $\boldsymbol{Or}(V)$ the set of orientations of V. Observe that $\boldsymbol{Or}(V)$ consists of precisely two elements.

(b) A pair $(V, \boldsymbol{or})$, where V is a vector space, and $\boldsymbol{or}$ is an orientation on V is called an *oriented vector space.*

(c) Suppose $\boldsymbol{or} \in \boldsymbol{Or}(V)$. A basis ω of $\det V$ is said to be *positively oriented* if $\boldsymbol{or}(\omega) > 0$. Otherwise, the basis is said to be *negatively oriented.* $\square$

There is an equivalent way of looking at orientations. To describe it, note that any nontrivial volume form μ on V uniquely specifies an orientation $\boldsymbol{or}_\mu$ given by

$$\boldsymbol{or}_\mu(\omega) := \operatorname{sign} \mu(\omega), \ \ \forall \omega \in \det V \setminus \{0\}.$$

We define an equivalence relation on the space of nontrivial volume forms by declaring

$$\mu_1 \sim \mu_2 \iff \mu_1(\omega)\mu_2(\omega) > 0, \ \ \forall \omega \in \det V \setminus \{0\}.$$

Then

$$\mu_1 \sim \mu_2 \iff \boldsymbol{or}_{\mu_1} = \boldsymbol{or}_{\mu_2}.$$

To every orientation $\boldsymbol{or}$ we can associate an equivalence class $[\mu]_{\boldsymbol{or}}$ of volume forms such that

$$\mu(\omega)\boldsymbol{or}(\omega) > 0, \ \ \forall \omega \in \det V \setminus \{0\}.$$

Thus, we can identify the set of orientations with the set of equivalence classes of nontrivial volume forms.

Equivalently, to specify an orientation on V it suffices to specify a basis ω of $\det V$. The associated orientation $\boldsymbol{or}_\omega$ is uniquely characterized by the condition

$$\boldsymbol{or}_\omega(\omega) = 1.$$

To any basis $\{e_1, ..., e_n\}$ of V one can associate a basis $e_1 \wedge \cdots \wedge e_n$ of $\det V$. Note that a permutation of the indices $1, \dots, n$ changes the associated basis of $\det V$ by a factor equal to the signature of the permutation. Thus, to define an orientation on a vector space, it suffices to specify a total ordering of a given basis of the space.

An *ordered* basis of an *oriented* vector space $(V, \boldsymbol{or})$ is said to be *positively oriented* if so is the associated basis of $\det V$.

Definition 2.2.58. Given two orientations $\boldsymbol{or}_1, \boldsymbol{or}_2$ on the vector space V we define

$$\boldsymbol{or}_1/\boldsymbol{or}_2 \in \{\pm 1\}$$

to be

$$\boldsymbol{or}_1/\boldsymbol{or}_2 := \boldsymbol{or}_1(\omega)\boldsymbol{or}_2(\omega), \ \ \forall \omega \in \det V \setminus \{0\}.$$

We will say that $\boldsymbol{or}_1/\boldsymbol{or}_2$ is the *relative signature* of the pair of orientations $\boldsymbol{or}_1, \boldsymbol{or}_2$. $\square$

Assume now that V is an Euclidean space. Denote the Euclidean inner product by $g(\bullet, \bullet)$. The vector space $\det V$ has an induced Euclidean structure, and in particular, there exist *exactly* two length-one-vectors in $\det V$. If we fix one of them, call it ω, and we think of it as a basis of $\det V$, then we achieve two things.

- First, it determines a volume form μ_g defined by

$$\mu_g(\lambda\omega) = \lambda.$$

- Second, it determines an orientation on V.

Conversely, an orientation $\boldsymbol{or} \in \boldsymbol{Or}(V)$ uniquely selects a length-one-vector $\omega = \omega_{\boldsymbol{or}}$ in $\det V$, which determines a volume form $\mu_g = \mu_g^{\boldsymbol{or}}$. Thus, we have proved the following result.

Proposition 2.2.59. *An orientation* $\boldsymbol{or}$ *on an Euclidean vector space* (V, g) *canonically selects a volume form on* V*, henceforth denoted by* $\mathrm{Det}_g = \mathrm{Det}_g^{\boldsymbol{or}}$. □

Exercise 2.2.60. Let (V, g) be an n-dimensional Euclidean vector space, and $\boldsymbol{or}$ an orientation on V. Show that, for any basis $v_1, \dots v_n$ of V, we have

$$\mathrm{Det}_g^{\boldsymbol{or}}(v_1 \wedge \cdots \wedge v_n) = \boldsymbol{or}(v_1 \wedge \cdots \wedge v_n)\sqrt{(\det g(v_i, v_j))}.$$

If $V = \mathbb{R}^2$ with its standard metric, and the orientation given by $e_1 \wedge e_2$, prove that

$$|\,\mathrm{Det}_g^{\boldsymbol{or}}(v_1 \wedge v_2)|$$

is the area of the parallelogram spanned by v_1 and v_2. □

Definition 2.2.61. Let $(V, g, \boldsymbol{or})$ be an oriented, Euclidean space and denote by $\mathrm{Det}_g^{\boldsymbol{or}}$ the associated volume form. The *Berezin integral* or (*berezinian*) is the linear form

$$\vec{\int_g} : \Lambda^\bullet V \to \mathbb{R},$$

defined on homogeneous elements by

$$\vec{\int_g} \omega = \begin{cases} 0 & \text{if } \deg \omega < \dim V \\ \mathrm{Det}_g^{\boldsymbol{or}} \omega & \text{if } \deg \omega = \dim V \end{cases}.$$ □

Definition 2.2.62. Let $\omega \in \Lambda^2 V$, where $(V, g, \boldsymbol{or})$ is an oriented, Euclidean space. We define its *pfaffian* as

$$\boldsymbol{Pf}\,(\omega) = \boldsymbol{Pf}_g^{\boldsymbol{or}}(\omega) := \vec{\int_g} \exp \omega = \begin{cases} 0 & \text{if } \dim V \text{ is odd} \\ \frac{1}{n!}\mathrm{Det}_g^{\boldsymbol{or}}(\omega^{\wedge n}) & \text{if } \dim V = 2n \end{cases},$$

where $\exp \omega$ denotes the exponential in the (nilpotent) algebra $\Lambda^\bullet V$,

$$\exp \omega := \sum_{k \geq 0} \frac{\omega^k}{k!}.$$ □

If (V, g) is as in the above definition, $\dim V = N$, and $A : V \to V$ is a skew-symmetric endomorphism of V, then we can define $\omega_A \in \Lambda^2 V$ by

$$\omega_A = \sum_{i<j} g(Ae_i, e_j) e_i \wedge e_j = \frac{1}{2} \sum_{i,j} g(Ae_i, e_j) e_i \wedge e_j,$$

where $(e_1, \dots, e_N)$ is a *positively oriented orthonormal basis* of V. The reader can check that ω_A is independent of the choice of basis as above. Notice that

$$\omega_A(u, v) = g(Au, v), \ \ \forall u, v \in V.$$

We define the *pfaffian* of A by

$$\boldsymbol{Pf}(A) := \boldsymbol{Pf}(\omega_A).$$

Example 2.2.63. Let $V = \mathbb{R}^2$ denote the standard Euclidean space oriented by $e_1 \wedge e_2$, where $e_1 e_2$ denotes the standard basis. If

$$A = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix},$$

then $\omega_A = \theta e_1 \wedge e_2$ so that $\boldsymbol{Pf}(A) = \theta$. □

Exercise 2.2.64. Let $A : V \to V$ be a skew-symmetric endomorphism of an oriented Euclidean space V. Prove that $\boldsymbol{Pf}(A)^2 = \det A$. □

Exercise 2.2.65. Let $(V, g, \boldsymbol{or})$ be an oriented Euclidean space of dimension $2n$. Consider $A : V \to V$ a skewsymmetric endomorphism and a positively oriented orthonormal frame $e_1, \dots, e_{2n}$. Prove that

$$\boldsymbol{Pf}_g^{\boldsymbol{or}}(A) = \frac{(-1)^n}{2^n n!} \sum_{\sigma \in \mathcal{S}_{2n}} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)}$$

$$= (-1)^n \sum_{\sigma \in \mathcal{S}'_{2n}} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where $a_{ij} = g(e_i, Ae_j)$ is the (i,j)-th entry in the matrix representing A in the basis (e_i), and $\mathcal{S}'_{2n}$ denotes the set of permutations $\sigma \in \mathcal{S}_{2n}$ satisfying

$$\sigma(2k-1) < \min\{\sigma(2k), \sigma(2k+1)\}, \ \ \forall k. \qquad □$$

Let $(V, g, \boldsymbol{or})$ be an n-dimensional, oriented, real Euclidean vector space. The metric duality $\mathcal{L}_g : V \to V^*$ induces both a metric, and an orientation on V^*. In the sequel we will continue to use the same notation $\mathcal{L}_g$ to denote the metric duality $\mathcal{T}_s^r(V) \to \mathcal{T}_s^r(V^*) \cong \mathcal{T}_s^r(V)$.

Definition 2.2.66. Suppose $(V, g, \boldsymbol{or})$ oriented Euclidean space, and r is a non-negative integer, $r \leq \dim V$. The r-th *Hodge pairing* is the pairing

$$\Xi = \Xi_{g,\boldsymbol{or}}^r : \Lambda^r V^* \times \Lambda^{n-r} V \to \mathbb{R},$$

defined by

$$\Xi(\omega^r, \eta^{n-r}) := \mathrm{Det}_g^{\boldsymbol{or}}\big(\mathcal{L}_g^{-1}\omega^r \wedge \eta^{n-r}\big), \ \ \omega^r \in \Lambda^r V^*, \ \ \eta^{n-r} \in \Lambda^{n-r} V. \qquad □$$

Exercise 2.2.67. Prove that the Hodge pairing is a duality. □

Definition 2.2.68. The Hodge $*$-operator is the adjunction isomorphism

$$* = \mathbb{I}_\Xi : \Lambda^r V^* \to \Lambda^{n-r} V^*$$

induced by the Hodge duality. □

The above definition obscures the meaning of the $*$-operator. We want to spend some time clarifying its significance.

Let $\alpha \in \Lambda^r V^*$ so that $*\alpha \in \Lambda^{n-r} V^*$. Denote by $\langle \bullet, \bullet \rangle$ the standard pairing

$$\Lambda^{n-r} V^* \times \Lambda^{n-r} V \to \mathbb{R},$$

and by $(\bullet, \bullet)$ the induced metric on $\Lambda^{n-r} V^*$. Then, by definition, for every $\beta \in \Lambda^{n-s} V^*$ the operator $*$ satisfies

$$\operatorname{Det}_g(\mathcal{L}_g^{-1}\alpha \wedge \mathcal{L}_g^{-1}\beta) = \left\langle *\alpha, \mathcal{L}_g^{-1}\beta \right\rangle = (*\alpha, \beta), \ \ \forall \beta \in \Lambda^{n-r} V^*. \tag{2.2.11}$$

Let ω denote the unit vector in $\det V$ defining the orientation. Then (2.2.11) can be rewritten as

$$\langle \alpha \wedge \beta, \omega \rangle = (*\alpha, \beta), \ \ \forall \beta \in \Lambda^{n-r} V^*.$$

Thus

$$\alpha \wedge \beta = (*\alpha, \beta) \operatorname{Det}_g^{\boldsymbol{or}}, \ \ \forall \alpha \in \Lambda^r V^*, \ \ \forall \beta \in \Lambda^{n-r} V^*. \tag{2.2.12}$$

Equality (2.2.12) uniquely determines the action of $*$.

Example 2.2.69. Let V be the standard Euclidean space $\mathbb{R}^3$ with standard basis $e_1, \ e_2, \ e_3$, and orientation determined by $e_1 \wedge e_2 \wedge e_3$. Then

$$*e_1 = e_2 \wedge e_3, \ \ *e_2 = e_3 \wedge e_1, \ \ *e_3 = e_1 \wedge e_2,$$

$$*1 = e_1 \wedge e_2 \wedge e_3, \ \ *(e_1 \wedge e_2 \wedge e_3) = 1,$$

$$*(e_2 \wedge e_3) = e_1, \ \ *(e_3 \wedge e_1) = e_2, \ \ *(e_1 \wedge e_2) = e_3. \qquad \square$$

The following result is left to the reader as an exercise.

Proposition 2.2.70. *Suppose* $(V, g, \boldsymbol{or})$ *is an oriented, real Euclidean space of dimension* n. *Then the associated Hodge* $*$*-operator satisfies*

$$*(*\omega) = (-1)^{p(n-p)}\omega \ \ \ \forall \omega \in \Lambda^p V^*,$$

$$\operatorname{Det}_g(*1) = 1,$$

and

$$\alpha \wedge *\beta = (\alpha, \beta) * 1, \ \ \forall \alpha \in \Lambda^k V^*, \ \forall \beta \in \Lambda^{n-k} V^*. \qquad \square$$

Exercise 2.2.71. Let (V, g, ε) be an n-dimensional, oriented, Euclidean space. For every $t > 0$ Denote by g_t the rescaled metric $g_t = t^2 g$. If $*$ is the Hodge operator corresponding to the metric g and orientation $\boldsymbol{or}$, and $*_t$ is the Hodge operator corresponding to the metric g_t and the same orientation, show that

$$\operatorname{Det}_{tg}^{\boldsymbol{or}} = t^n \operatorname{Det}_g^{\boldsymbol{or}},$$

and

$$*_t \omega = t^{n-2p} * \omega \ \ \ \forall \omega \in \Lambda^p V^*. \qquad \square$$

We conclude this subsection with a brief discussion of densities.

Definition 2.2.72. Let V be a real vector space. For any $r \geq 0$ we define an *r-density* to be a function $f : \det V \to \mathbb{R}$ such that

$$f(\lambda u) = |\lambda|^r f(u), \ \ \forall u \in \det V \setminus \{0\}, \ \forall \lambda \neq 0. \qquad \square$$

The linear space of r-densities on V will be denoted by $|\Lambda|_V^r$. When $r = 1$ we set $|\Lambda|_V := |\Lambda|_V^1$, and we will refer to 1-densities simply as densities.

Example 2.2.73. Any Euclidean metric g on V defines a canonical 1-density $|\operatorname{Det}_g| \in |\Lambda|_V^1$ which associated to each $\omega \in \det V$ its length, $|\omega|_g$. $\square$

Observe that an orientation $\boldsymbol{or} \in \boldsymbol{Or}(V)$ defines a natural *linear* isomorphism

$$\imath_{\boldsymbol{or}} : \det V^* \to |\Lambda|_V \quad \det V^* \ni \mu \mapsto \imath_{\boldsymbol{or}}\mu \in |\Lambda|_V, \tag{2.2.13}$$

where

$$\imath_{\boldsymbol{or}}\mu(\omega) = \boldsymbol{or}(\omega)\mu(\omega), \ \ \forall \omega \in \det V \setminus \{0\}.$$

In particular, an orientation in a Euclidean vector space canonically identifies $|\Lambda|_V$ with $\mathbb{R}$.

2.2.5 *Some complex linear algebra*

In this subsection we want to briefly discuss some aspects specific to linear algebra over the field of complex numbers.

Let V be a complex vector space. Its *conjugate* is the complex vector space $\overline{V}$ which coincides with V as a *real* vector space, but in which the multiplication by a scalar $\lambda \in \mathbb{C}$ is defined by

$$\lambda \cdot v := \overline{\lambda} v, \ \ \forall v \in V.$$

The vector space V has a complex dual V_c^* that can be identified with the space of complex linear maps $V \to \mathbb{C}$. If we forget the complex structure we obtain a *real* dual V_r^*consisting of all real-linear maps $V \to \mathbb{R}$.

Definition 2.2.74. A *Hermitian metric* is a complex bilinear map

$$(\bullet, \bullet) : V \times \overline{V} \to \mathbb{C}$$

satisfying the following properties.

- The bilinear from $(\bullet, \bullet)$ is positive definite, i.e.

$$(v, v) > 0, \ \ \forall v \in V \setminus \{0\}.$$

- For any $u, v \in V$ we have $(u, v) = \overline{(v, u)}$. $\square$

A Hermitian metric defines a duality $V \times \overline{V} \to \mathbb{C}$, and hence it induces a *complex linear* isomorphism

$$\mathfrak{L} : \overline{V} \to V_c^*, \ \ v \mapsto (\cdot, v) \in V_c^*.$$

If V and W are complex Hermitian vector spaces, then any complex linear map $A : V \to W$ induces a complex linear map

$$A^* : \overline{W} \to V_c^* \quad A^* w := \Big(v \mapsto \langle Av, w \rangle \Big) \in V_c^*,$$

where $\langle \bullet, \bullet \rangle$ denotes the natural duality between a vector space and its dual. We can rewrite the above fact as

$$\langle Av, w \rangle = \langle v, A^* w \rangle.$$

A complex linear map $\overline{W} \to V_c^*$ is the same as a complex linear map $W \to \overline{V_c}^*$. The metric duality defines a complex linear isomorphism $\overline{V_c}^* \cong V$ so we can view the *adjoint* A^* as a complex linear map $W \to V$.

Let $h = (\bullet, \bullet)$ be a Hermitian metric on the complex vector space V. If we view $(\bullet, \bullet)$ as an object over $\mathbb{R}$, i.e., as an $\mathbb{R}$-bilinear map $V \times V \to \mathbb{C}$, then the Hermitian metric decomposes as

$$h = \boldsymbol{Re}\, h - \boldsymbol{i}\, \omega, \quad \boldsymbol{i} := \sqrt{-1}, \ \ \omega = -\, \boldsymbol{Im}\, h.$$

The real part is an inner product on the *real* space V, while ω is a *real*, skew-symmetric bilinear form on V, and thus can be identified with an element of $\Lambda^2_{\mathbb{R}} V^*$. ω is called the *real 2-form associated to the Hermitian metric*.

It is convenient to have a coordinate description of the abstract objects introduced above. Let V be an n-dimensional complex vector space and h a Hermitian metric on it. Pick an unitary basis $e_1, ..., e_n$ of V, i.e., $n = \dim_{\mathbb{C}} V$, and $h(e_i, e_j) = \delta_{ij}$. For each j, we denote by f_j the vector $\boldsymbol{i} e_j$. Then the collection $e_1, f_1, \dots, e_n, f_n$ is an $\mathbb{R}$-basis of V. Denote by $e^1, f^1, \dots, e^n, f^n$ the dual $\mathbb{R}$-basis in V^*. Then

$$\boldsymbol{Re}\, h\,(e_i, e_j) = \delta_{ij} = \boldsymbol{Re}\, h(f_i, f_j) \ \text{ and } \ \boldsymbol{Re}\, h\,(e_i, f_j) = 0,$$

i.e.,

$$\boldsymbol{Re}\, h = \sum_i (e^i \otimes e^i + f^i \otimes f^i).$$

Also

$$\omega(e_i, f_j) = -\boldsymbol{Im}\, h(e_i, \boldsymbol{i} e_j) = \delta_{ij}, \ \ \omega(e_i, e_j) = \omega(f_i, f_j) = 0 \ \ \forall i, j,$$

which shows that

$$\omega = -\boldsymbol{Im}\, h = \sum_i e^i \wedge f^i.$$

Any complex space V can be also thought of as a real vector space. The multiplication by $\boldsymbol{i} = \sqrt{-1}$ defines a *real* linear operator which we denote by J. Obviously J satisfies $J^2 = -\mathbb{1}_V$.

Conversely, if V is a real vector space then any real operator $J : V \to V$ as above defines a complex structure on V by

$$(a + b\boldsymbol{i})v = av + bJv, \ \ \forall v \in V, \ a + b\boldsymbol{i} \in \mathbb{C}.$$

We will call an operator J as above a *complex structure*.

Let V be a real vector space with a complex structure J on it. The operator J has no eigenvectors on V. The natural extension of J to the complexification of V, $V_{\mathbb{C}} = V \otimes \mathbb{C}$, has two eigenvalues $\pm\boldsymbol{i}$, and we have a splitting of $V_{\mathbb{C}}$ as a direct sum of *complex* vector spaces (eigenspaces)

$$V_{\mathbb{C}} = \ker(J - \boldsymbol{i}) \oplus (\ker J + \boldsymbol{i}).$$

Exercise 2.2.75. Prove that we have the following isomorphisms of *complex* vector spaces

$$V \cong \ker(J - \boldsymbol{i}) \qquad \overline{V} \cong (\ker J + \boldsymbol{i}). \qquad \square$$

Set

$$V^{1,0} := \ker(J - \boldsymbol{i}) \cong_{\mathbb{C}} V \qquad V^{0,1} := \ker(J + \boldsymbol{i}) \cong_{\mathbb{C}} \overline{V}.$$

Thus $V_{\mathbb{C}} \cong V^{1,0} \oplus V^{0,1} \cong V \oplus \overline{V}$. We obtain an isomorphism of **$\mathbb{Z}$-graded complex** vector spaces

$$\Lambda^{\bullet} V_{\mathbb{C}} \cong \Lambda^{\bullet} V^{1,0} \otimes \Lambda^{\bullet} V^{0,1}.$$

If we set

$$\Lambda^{p,q} V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1},$$

then the above isomorphism can be reformulated as

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^{p,q} V. \tag{2.2.14}$$

Note that the complex structure J on V induces by duality a complex structure J^* on V_r^*, and we have an isomorphism of *complex vector spaces*

$$V_c^* = (V, J)_c^* \cong (V_r^*, J^*).$$

We can define similarly $\Lambda^{p,q} V^*$ as the $\Lambda^{p,q}$-construction applied to the real vector space V_r^* equipped with the complex structure J^*. Note that

$$\Lambda^{1,0} V^* \cong (\Lambda^{1,0} V)_c^*,$$

$$\Lambda^{0,1} V^* \cong (\Lambda^{0,1} V)_c^*,$$

and, more generally

$$\Lambda^{p,q}V^* \cong (\Lambda^{p,q}V)^*_c.$$

If h is a Hermitian metric on the complex vector space (V, J), then we have a natural isomorphism of complex vector spaces

$$V^*_c \cong (V^*_r, J^*) \cong_{\mathbb{C}} (V, -J) \cong_{\mathbb{C}} \overline{V},$$

so that

$$\Lambda^{p,q}V^* \cong_{\mathbb{C}} \Lambda^{q,p}V.$$

The Euclidean metric $g = \boldsymbol{Re}\, h$, and the associated 2-form $\omega = -\boldsymbol{Im}\, h$ are related by

$$g(u, v) = \omega(u, Jv), \;\; \omega(u, v) = g(Ju, v), \;\; \forall u, v \in V. \tag{2.2.15}$$

Moreover, ω is a $(1, 1)$-form. To see this it suffices to pick a unitary basis (e_i) of V, and construct as usual the associated real orthonormal basis $\{e_1, f_1, \cdots, e_n, f_n\}$ $(f_i = Je_i)$. Denote by $\{e^i, f^i \; ; \; i = 1, \dots, n\}$ the dual orthonormal basis in V^*_r. Then $J^*e^i = -f^i$, and if we set

$$\varepsilon^i := \frac{1}{\sqrt{2}}(e^i + \boldsymbol{i}f^i), \;\; \bar{\varepsilon}^j := \frac{1}{\sqrt{2}}(e^j - \boldsymbol{i}f^j),$$

then

$$\Lambda^{1,0}V^* = \mathrm{span}_{\mathbb{C}}\{\varepsilon^i\} \;\; \Lambda^{0,1} = \mathrm{span}_{\mathbb{C}}\{\bar{\varepsilon}^j\},$$

and

$$\omega = \boldsymbol{i} \sum \varepsilon^i \wedge \bar{\varepsilon}^i.$$

Let V be a complex vector space, and $e_1, \dots, e_n$ be a basis of V over $\mathbb{C}$. This is not a *real* basis of V since $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$. We can however complete this to a real basis. More precisely, the vectors $e_1, \boldsymbol{i}e_1, \dots, e_n, \boldsymbol{i}e_n$ form a real basis of V.

Proposition 2.2.76. *Suppose* $(e_1, \dots, e_n)$ *and* $(f_1, \dots, f_n)$ *are two* complex bases *of* V, *and* $Z = (z^j_k)_{1 \leq j,k}$ *is the complex matrix describing the transition from the basis* e *to the basis* f, *i.e.*,

$$f_k = \sum_j z^j_k e_j, \;\; \forall 1 \leq k \leq n.$$

Then

$$f_1 \wedge \boldsymbol{i}f_1 \wedge \cdots \wedge f_n \wedge \boldsymbol{i}f_n = |\det Z|^2 e_1 \wedge \boldsymbol{i}e_1 \wedge \cdots \wedge e_n \wedge \boldsymbol{i}e_n.$$

Proof. We write

$$z_k^j = x_k^j + \boldsymbol{i} y_k^j, \;\; x_k^j, y_k^j \in \mathbb{R}, \;\; \hat{e}_j = \boldsymbol{i} e_j, \;\; \hat{f}_k = \boldsymbol{i} f_k.$$

Then

$$f_k = \sum_j (x_k^j + \boldsymbol{i} y_k^j) e_j = \sum_j x_k^j e_j + \sum_j y_k^j \hat{e}_j,$$

and

$$\hat{f}_k = -\sum_j y_k^j e_j + \sum_j x_k^j \hat{e}_j.$$

Then, if we set $\epsilon_n = (-1)^{n(n-1)/2}$, we deduce

$$f_1 \wedge \boldsymbol{i} f_1 \wedge \cdots \wedge f_n \wedge \boldsymbol{i} f_n = \epsilon_n f_1 \wedge \cdots \wedge f_n \wedge \hat{f}_1 \wedge \cdots \wedge \hat{f}_n$$

$$= \epsilon_n (\det \hat{Z}) e_1 \wedge \cdots \wedge e_n \wedge \hat{e}_1 \wedge \cdots \wedge \hat{e}_n = (\det \widehat{Z}) e_1 \wedge \boldsymbol{i} e_1 \wedge \ldots \wedge e_n \wedge \boldsymbol{i} e_n,$$

where $\widehat{Z}$ is the $2n \times 2n$ real matrix

$$\widehat{Z} = \begin{bmatrix} \boldsymbol{Re}\, Z & -\boldsymbol{Im}\, Z \\ \boldsymbol{Im}\, Z & \boldsymbol{Re}\, Z \end{bmatrix} = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix},$$

and X, Y denote the $n \times n$ real matrices with entries (x_k^j) and respectively (y_k^j).

We want to show that

$$\det \widehat{Z} = |\det Z|^2, \;\; \forall Z \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n). \tag{2.2.16}$$

Let

$$\mathcal{A} := \big\{ Z \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n); \;\; |\det Z|^2 = \det \widehat{Z} \,\big\}.$$

We will prove that $\mathcal{A} = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$.

The set $\mathcal{A}$ is nonempty since it contains all the diagonal matrices. Clearly $\mathcal{A}$ is a closed subset of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$, so it suffices to show that $\mathcal{A}$ is dense in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$.

Observe that the correspondence

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \ni Z \to \widehat{Z} \in \mathrm{End}_{\mathbb{R}}(\mathbb{R})$$

is an endomorphism of $\mathbb{R}$-algebras. Then, for every $Z \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$, and any complex linear automorphism $T \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n)$, we have

$$\widehat{TZT^{-1}} = \widehat{T}\widehat{Z}\widehat{T}^{-1}.$$

Hence

$$Z \in \mathcal{A} \Longleftrightarrow TZT^{-1} \in \mathcal{A}, \;\; \forall T \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n).$$

In other words, if a complex matrix Z satisfies (2.2.16), then so will any of its conjugates. In particular, $\mathcal{A}$ contains *all the diagonalizable $n \times n$ complex matrices*, and these form a dense subset of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ (see Exercise 2.2.77). □

Exercise 2.2.77. Prove that the set of diagonalizable $n \times n$ complex matrices form a dense subset of the vector space $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$. □

Definition 2.2.78. The *canonical orientation* of a complex vector space V, $\dim_{\mathbb{C}} V = n$, is the orientation defined by $e_1 \wedge \boldsymbol{i}e_1 \wedge ... \wedge e_n \wedge \boldsymbol{i}e_n \in \Lambda^{2n}_{\mathbb{R}} V$, where $\{e_1, \dots, e_n\}$ is any complex basis of V. □

Suppose h is a Hermitian metric on the complex vector space V. Then $g = \boldsymbol{Re}\, h$ defines is real, Euclidean metric on V regarded as a real vector space. The canonical orientation $\boldsymbol{or}_c$ on V, and the metric g define a volume form $\mathrm{Det}_g^{\boldsymbol{or}_c} \in \Lambda^{2n}_{\mathbb{R}} V_r^*$, $n = \dim_{\mathbb{C}} V$, and a pffafian

$$\boldsymbol{Pf}_h = \boldsymbol{Pf}_h^{\boldsymbol{or}_c} : \Lambda^2_{\mathbb{R}} V^* \to \mathbb{R}, \;\; \Lambda^2 V_r^* \ni \eta \mapsto \frac{1}{n!} g(\eta^n, \mathrm{Det}_g).$$

If $\omega = -\boldsymbol{Im}\, h$ is real 2-form associated with the Hermitian metric h, then

$$\mathrm{Det}_g = \mathrm{Det}_g^{\boldsymbol{or}_c} = \frac{1}{n!}\omega^n,$$

and we conclude

$$\boldsymbol{Pf}_h\, \omega = 1.$$

2.3 Tensor fields

2.3.1 *Operations with vector bundles*

We now return to geometry, and more specifically, to vector bundles.

Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let $E \to M$ be a rank r $\mathbb{K}$-vector bundle over the smooth manifold M. According to the definition of a vector bundle, we can find an open cover (U_α) of M such that each restriction $E\,|_{U_\alpha}$ is trivial: $E\,|_{U_\alpha} \cong V \times U_\alpha$, where V is an r-dimensional vector space over the field $\mathbb{K}$. The bundle E is obtained by gluing these trivial pieces on the overlaps $U_\alpha \cap U_\beta$ using a collection of transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ satisfying the cocycle condition.

Conversely, a collection of gluing maps as above satisfying the cocycle condition uniquely defines a vector bundle. In the sequel, we will exclusively think of vector bundles in terms of gluing cocycles.

Let E, F be two vector bundles over the smooth manifold M with standard fibers V_E and respectively V_F, given by a (common) open cover (U_α), and gluing cocycles

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V_E), \;\; \text{and respectively} \;\; h_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V_F).$$

Then the collections

$$g_{\alpha\beta} \oplus h_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V_E \oplus V_F), \;\; g_{\alpha\beta} \otimes h_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V_E \otimes V_F),$$

$$(g^\dagger_{\alpha\beta})^{-1} : U_{\alpha\beta} \to \mathrm{GL}(\,V_E^*\,), \;\; \Lambda^r g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(\Lambda^r V_E),$$

where † denotes the transpose of a linear map, satisfy the cocycle condition, and therefore define vector bundles which we denote by $E \oplus F$, $E \otimes F$, E^*, and respectively $\Lambda^r E$. In particular, if $r = \operatorname{rank}_{\mathbb{K}} E$, the bundle $\Lambda^r E$ has rank 1. It is called the *determinant line bundle* of E, and it is denoted by $\det E$.

The reader can check easily that these vector bundles are independent of the choices of transition maps used to characterize E and F (use Exercise 2.1.32). The bundle E^* is called the *dual* of the vector bundle E. The direct sum $E \oplus F$ is also called the *Whitney sum* of vector bundles. All the functorial constructions on vector spaces discussed in the previous section have a vector bundle correspondent. (Observe that a vector space can be thought of as a vector bundle over a point.)

These above constructions are natural in the following sense. Let E' and F' be vector bundles over the same smooth manifold M'. Any bundle maps $S : E \to E'$ and $T : F \to F'$, both covering the *same diffeomorphism* $\phi : M \to M'$, induce bundle morphisms

$$S \oplus T : E \oplus F \to E' \oplus T', \;\; S \otimes T : E \otimes F \to E' \otimes F',$$

covering ϕ, a morphism

$$S^\dagger : (E')^* \to E^*,$$

covering ϕ^{-1} etc.

Exercise 2.3.1. Prove the assertion above. □

Example 2.3.2. Let E, F, E' and F' be vector bundles over a smooth manifold M. Consider bundle isomorphisms $S : E \to E'$ and $T : F \to F'$ covering the same diffeomorphism of the base, $\phi : M \to M$. Then $(S^{-1})^\dagger : E^* \to (E')^*$ is a bundle isomorphism covering ϕ, so that we get an induced map $(S^{-1})^\dagger \otimes T : E^* \otimes F \to (E')^* \otimes F'$. Note that we have a natural identification

$$E^* \otimes F \cong \operatorname{Hom}(E, F). \qquad \square$$

Definition 2.3.3. Let $E \to M$ be a $\mathbb{K}$-vector bundle over M. A *metric* on E is a section h of $E^* \otimes_{\mathbb{K}} \overline{E^*}$ ($\overline{E} = E$ if $\mathbb{K} = \mathbb{R}$) such that, for any $m \in M$, $h(m)$ defines a metric on E_m (Euclidean if $\mathbb{K} = \mathbb{R}$ or Hermitian if $\mathbb{K} = \mathbb{C}$). □

2.3.2 *Tensor fields*

We now specialize the previous considerations to the special situation when E is the tangent bundle of M, $E \cong TM$. The *cotangent bundle* is then

$$T^*M := (TM)^*.$$

We define the tensor bundles of M

$$\mathcal{T}^r_s(M) := \mathcal{T}^r_s(TM) = (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

Definition 2.3.4. (a) A *tensor field* of type (r, s) over the open set $U \subset M$ is a section of $\mathcal{T}^r_s(M)$ over U.
(b) A degree r *differential form* (r-form for brevity) is a section of $\Lambda^r(T^*M)$. The space of (smooth) r-forms over M is denoted by $\Omega^r(M)$. We set

$$\Omega^\bullet(M) := \bigoplus_{r \geq 0} \Omega^r(M).$$

(c) A *Riemannian metric* on a manifold M is a metric on the tangent bundle. More precisely, it is a symmetric $(0, 2)$-tensor g, such that for every $x \in M$, the bilinear map

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

defines a Euclidean metric on T_xM. □

If we view the tangent bundle as a smooth family of vector spaces, then a tensor field can be viewed as a smooth selection of a tensor in each of the tangent spaces. In particular, a Riemann metric defines a smoothly varying procedure of measuring lengths of vectors in tangent spaces.

Example 2.3.5. It is often very useful to have a local description of these objects. If $(x^1, \dots, x^n)$ are local coordinates on an open set $U \subset M$, then the vector fields $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ trivialize $TM\,|_U$, i.e., they define a framing of the restriction of TM to U. We can form a dual framing of $T^*M\,|_U$, using the 1-forms dx^i, $i = 1, \dots, n$. They satisfy the duality conditions

$$\langle dx^i, \frac{\partial}{\partial x_j} \rangle = \delta^i_j, \;\; \forall i, j.$$

A basis in $\mathcal{T}^r_s(T_xM)$ is given by

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s};\; 1 \leq i_1, \dots, i_r \leq n,\; 1 \leq j_1, \dots, j_s \leq n \right\}.$$

Hence, any tensor $T \in \mathcal{T}^r_s(M)$ has a local description

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}.$$

In the above equality we have used Einstein's convention. In particular, an r-form ω has the local description

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \omega_{i_1 \dots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

while a Riemann metric g has the local description

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j, \;\; g_{ij} = g_{ji}. \quad □$$

Remark 2.3.6. (a) A covariant tensor field, i.e., a $(0, s)$-tensor field S, naturally defines a $C^\infty(M)$-multilinear map

$$S : \bigoplus_1^s \mathrm{Vect}\,(M) \to C^\infty(M),$$

$$(X_1, \ldots, X_s) \mapsto \big(p \mapsto S_p\big(X_1(p), \ldots, X_s(p)\big) \in C^\infty(M).$$

Conversely, any such map uniquely defines a $(0, s)$-tensor field. In particular, an r-form η can be identified with a skew-symmetric $C^\infty(M)$-multilinear map

$$\eta : \bigoplus_1^r \mathrm{Vect}\,(M) \to C^\infty(M).$$

Notice that the wedge product in the exterior algebras induces an associative product in $\Omega^\bullet(M)$ which we continue to denote by $\wedge$.

(b) Let $f \in C^\infty(M)$. Its Fréchet derivative $Df : TM \to T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ is naturally a 1-form. Indeed, we get a smooth $C^\infty(M)$-linear map $df : \mathrm{Vect}\,(M) \to C^\infty(M)$ defined by

$$df(X)_m := Df(X)_{f(m)} \in T_{f(m)}\mathbb{R} \cong \mathbb{R}.$$

In the sequel we will always regard the differential of a smooth function f as a 1-form and to indicate this we will use the notation df (instead of the usual Df). □

Any diffeomorphism $f : M \to N$ induces bundle isomorphisms $Df : TM \to TN$ and $(Df^{-1})^\dagger : T^*M \to T^*N$ covering f. Thus, a diffeomorphism f induces a linear map

$$f_* : \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(N), \tag{2.3.1}$$

called the *push-forward* map. In particular, the group of diffeomorphisms of M acts naturally (and linearly) on the space of tensor fields on M.

Example 2.3.7. Suppose $f : M \to N$ is a diffeomorphism, and S is a $(0, k)$-tensor field on M, which we regard as a $C^\infty(M)$-multilinear map

$$S : \underbrace{\mathrm{Vect}\,(M) \times \cdots \times \mathrm{Vect}\,(M)}_{k} \to C^\infty(M).$$

Then f_*S is a $(0, k)$ tensor field on N. Let $q \in N$, and set $p := f^{-1}(q)$. Then, for any $Y_1, \ldots, Y_k \in T_qN$, we have

$$(f_*S)_q(Y_1, \ldots, Y_k) = S_p\big(f_*^{-1}Y_1, \ldots, (f_*)^{-1}Y_k\big)$$

$$= S_p\big((D_pf)^{-1}Y_1, \ldots, (D_pf)^{-1}Y_k\big). \qquad □.$$

For covariant tensor fields a more general result is true. More precisely, any smooth map $f : M \to N$ defines a linear map

$$f^* : \mathcal{T}^0_s(N) \to \mathcal{T}^0_s(M),$$

called the *pullback* by f. Explicitly, if S is such a tensor defined by a $C^\infty(M)$-multilinear map

$$S : (\mathrm{Vect}\,(N))^s \to C^\infty(N),$$

then f^*S is the covariant tensor field defined by

$$(f^*S)_p(X_1(p), \dots, X_s(p)) := S_{f(p)}\big(\,D_pf(X_1), \dots, D_pf(X_s)\,\big),$$

$\forall X_1, \dots, X_s \in \mathrm{Vect}\,(M)$, $p \in M$. Note that when f is a *diffeomorphism* we have

$$f^* = (f_*^{-1})^\dagger,$$

where f_* is the push-forward map defined in (2.3.1).

Example 2.3.8. Consider the map

$$F_* : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2, \quad (r, \theta) \mapsto (x = r\cos\theta, y = r\sin\theta).$$

The map F defines the usual polar coordinates. It is a diffeomorphism onto the open subset

$$U := \mathbb{R}^2 \setminus \{(x, 0);\ \ x \geq 0\}.$$

For simplicity, we will write ∂_r, ∂_x instead of $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial x}$ etc. We have

$$F^*dx = d(r\cos\theta) = \cos\theta dr - r\mathrm{sin}\theta d\theta, \quad F^*dy = d(r\sin\theta) = \mathrm{sin}\theta dr + r\cos\theta d\theta,$$

$$F^*(dx \wedge dy) = d(r\cos\theta) \wedge d(r\mathrm{sin}\theta) = (\cos\theta dr - r\mathrm{sin}\theta d\theta) \wedge (\mathrm{sin}\theta dr + r\cos\theta d\theta)$$

$$= r(\cos^2\theta + \mathrm{sin}^2\theta) dr \wedge d\theta = r dr \wedge d\theta.$$

To compute $F_*\partial_r$ and $F_*\partial_\theta$ we use the chain rule which implies

$$F_*\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\theta\partial_x + \sin\theta\partial_y = \frac{1}{r}(x\partial_x + y\partial_y)$$

$$= \frac{1}{(x^2 + y^2)^{1/2}}(x\partial_x + y\partial_y).$$

The Euclidean metric is described over U by the symmetric $(0, 2)$–tensor $g = dx^2 + dy^2$. The pullback of g by F is the symmetric $(0, 2)$–tensor

$$F^*(dx^2 + dy^2) = \big(\,d(r\cos\theta)\,\big)^2 + \big(\,d(r\sin\theta)\,\big)^2$$

$$= (\cos\theta dr - r\mathrm{sin}\theta d\theta)^2 + (\mathrm{sin}\theta dr + r\cos\theta d\theta)^2 = dr^2 + r^2 d\theta^2.$$

To compute F_*dr we need to express r as a function of x and y, $r = (x^2 + y^2)^{1/2}$, and then we have

$$F_*dr = (F^{-1})^*dr = d(x^2 + y^2)^{1/2} = x(x^2 + y^2)^{-1/2}dx + y(x^2 + y^2)^{-1/2}dy. \quad \square$$

All the operations discussed in the previous section have natural extensions to tensor fields. There exists a tensor multiplication, a Riemann metric defines a duality $\mathcal{L} : \mathrm{Vect}\,(M) \to \Omega^1(M)$ etc. In particular, there exists a contraction operator

$$\mathrm{tr}\,:\ \mathcal{T}^{r+1}_{s+1}(M) \to \mathcal{T}^r_s(M)$$

defined by

$$\mathrm{tr}\,(X_0 \otimes \cdots \otimes X_r) \otimes (\omega_0 \otimes \cdots \otimes \omega_s) = \omega_0(X_0)(X_1 \otimes \cdots X_r \otimes \omega_1 \otimes \cdots \otimes \omega_s),$$

$\forall X_i \in \mathrm{Vect}\,(M)$, $\forall \omega_j \in \Omega^1(M)$. In local coordinates the contraction has the form

$$\left\{\mathrm{tr}\,\left(T^{i_0 \dots i_r}_{j_0 \dots j_s}\right)\right\} = \{T^{ii_1 \dots i_r}_{ij_1 \dots j_s}\}.$$

Let us observe that a Riemann metric g on a manifold M induces metrics in all the associated tensor bundles $\mathcal{T}^r_s(M)$. If we choose local coordinates (x^i) on an open set U then, as explained above, the metric g can be described as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j,$$

while a tensor field T of type (r, s) can be described as

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}.$$

If we denote by (g^{ij}) the inverse of the matrix (g_{ij}), then, for every point $p \in U$, the length of $T(p) \in \mathcal{T}^r_s(M)_p$ is the number $|T(p)|_g$ defined by

$$|T(p)|_g = g_{i_1 k_1} \dots g_{i_r k_r} g^{j_1 \ell_1} \cdots g^{j_s \ell_s} T^{i_1 \dots i_r}_{j_1 \dots j_s} T^{k_1 \dots k_r}_{\ell_1 \dots \ell_s},$$

where in the above equalities we have used Einstein's convention.

The exterior product defines an exterior product on the space of smooth differential forms

$$\wedge : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M).$$

The space $\Omega^\bullet(M)$ is then an associative algebra with respect to the operations $+$ and $\wedge$.

Proposition 2.3.9. *Let $f : M \to N$ be a smooth map. The pullback by f defines a morphism of associative algebras $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$.* □

Exercise 2.3.10. Prove the above proposition. □

2.3.3 *Fiber bundles*

We consider it useful at this point to bring up the notion of fiber bundle. There are several reasons to do this.

On one hand, they arise naturally in geometry, and they impose themselves as worth studying. On the other hand, they provide a very elegant and concise language to describe many phenomena in geometry.

We have already met examples of fiber bundles when we discussed vector bundles. These were "smooth families of vector spaces". A fiber bundle wants to be a smooth family of copies of the same manifold. This is a very loose description, but it offers a first glimpse at the notion about to be discussed.

The model situation is that of direct product $X = F \times B$, where B and F are smooth manifolds. It is convenient to regard this as a family of manifolds $(F_b)_{b \in B}$. The manifold B is called the base, F is called the standard (model) fiber, and X is called the total space. This is an example of *trivial* fiber bundle.

In general, a fiber bundle is obtained by gluing a bunch of trivial ones according to a prescribed rule. The gluing may encode a symmetry of the fiber, and we would like to spend some time explaining what do we mean by symmetry.

Definition 2.3.11. (a) Let M be a smooth manifold, and G a Lie group. We say the group G acts on M from the left (respectively right), if there exists a smooth map

$$\Phi : G \times M \to M, \ \ (g, m) \mapsto T_g m,$$

such that $T_1 \equiv \mathbb{1}_M$ and

$$T_g(T_h m) = T_{gh} m \ \ (\text{respectively } T_g(T_h m) = T_{hg} m) \ \ \forall g, h \in G, \ m \in M.$$

In particular, we deduce that $\forall g \in G$ the map T_g is a diffeomorphism of M. For any $m \in M$ the set

$$G \cdot m = \{T_g m; \ g \in G\}$$

is called the *orbit* of the action through m.
(b) Let G act on M. The action is called *free* if $\forall g \in G$, and $\forall m \in M$ $T_g m \neq m$. The action is called *effective* if, $\forall g \in G$, $T_g \neq \mathbb{1}_M$. □

It is useful to think of a Lie group action on a manifold as encoding a symmetry of that manifold.

Example 2.3.12. Consider the unit sphere

$$S^1 = \{(x, yz) \in \mathbb{R}^3; \ \ x^2 + y^2 + z^2 = 1\}.$$

Then the counterclockwise rotations about the z-axis define a smooth left action of S^1 on S^2. More formally, if we use cylindrical coordinates (r, θ, z),

$$x = r\cos\theta, \ \ y = r\sin\theta, \ \ z = 0,$$

then for every $\varphi \in \mathbb{R} \bmod 2\pi \cong S^1$ we define $T_\varphi : S^2 \to S^2$ by

$$T_\varphi(r, \theta, z) = (r, (\theta + \varphi) \bmod 2\pi, z).$$

The resulting map $T : S^1 \times S^2 \to S^2$, $(\varphi, p) \mapsto T_\varphi(p)$ defines a left action of S^1 on S^2 encoding the rotational symmetry of S^2 about the z-axis. □

Example 2.3.13. Let G be a Lie group. A *linear representation* of G on a vector space V is a left action of G on V such that each T_g is a linear map. One says V is a G-*module*. For example, the tautological linear action of $SO(n)$ on $\mathbb{R}^n$ defines a linear representation of $SO(n)$. □

Example 2.3.14. Let G be a Lie group. For any $g \in G$ denote by L_g (resp. R_g) the left (resp right) translation by g. In this way we get the tautological left (resp. right) action of G on itself. □

Definition 2.3.15. A smooth *fiber bundle* is an object composed of the following:

(a) a smooth manifold E called the *total space*;
(b) a smooth manifold F called the *standard fiber*;
(c) a smooth manifold B called the *base*;
(d) a surjective submersion $\pi : E \to B$ called the *natural projection*;
(e) a collection of *local trivializations*, i.e., an open cover (U_α) of the base B, and diffeomorphisms $\Psi_\alpha : F \times U_\alpha \to \pi^{-1}(U_\alpha)$ such that

$$\pi \circ \Psi_\alpha(f, b) = b, \ \ \forall (f, b) \in F \times U_\alpha,$$

i.e., the diagram below is commutative.

$$\begin{tikzcd} F \times U_\alpha \arrow[rr, "\Psi_\alpha"] \arrow[dr] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

We can form the transition (gluing) maps $\Psi_{\alpha\beta} : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta}$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$, defined by $\Psi_{\alpha\beta} = \psi_\alpha^{-1} \circ \psi_\beta$. According to (e), these maps can be written as

$$\psi_{\alpha\beta}(f, b) = (T_{\alpha\beta}(b)f, b),$$

where $T_{\alpha\beta}(b)$ is a diffeomorphism of F depending smoothly upon $b \in U_{\alpha\beta}$.

We will denote this fiber bundle by (E, π, F, B).

If G is a Lie group, then the bundle (E, π, F, B) is called a G-*fiber bundle* if it satisfies the following additional conditions.

(f) There exists an effective left action of the Lie group G on F,

$$G \times F \ni (g, x) \mapsto g \cdot x = T_g x \in F.$$

The group G is called the *symmetry group of the bundle*.

(g) There exist smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ satisfying the cocycle condition

$$g_{\alpha\alpha} = 1 \in G, \ \ g_{\gamma\alpha} = g_{\gamma\beta} \cdot g_{\beta\alpha}, \ \ \forall \alpha, \beta, \gamma,$$

and such that

$$T_{\alpha\beta}(b) = T_{g_{\alpha\beta}(b)}.$$

We will denote a G-fiber bundle by (E, π, F, B, G). □

The choice of an open cover (U_α) in the above definition is a source of arbitrariness since there is no natural prescription on how to perform this choice. We need to describe when two such choices are equivalent.

Two open covers (U_α) and (V_i), together with the collections of local trivializations

$$\Phi_\alpha : F \times U_\alpha \to \pi^{-1}(U_\alpha) \text{ and } \Psi_i : F \times V_i \to \pi^{-1}(V_i)$$

are said to be equivalent if, for all α, i, there exists a smooth map

$$T_{\alpha i} : U_\alpha \cap V_i \to G,$$

such that, for any $x \in U_\alpha \cap V_i$, and any $f \in F$, we have

$$\Phi_\alpha^{-1}\Psi_i(f, x) = (T_{\alpha i}(x)f, x).$$

A G-bundle structure is defined by an equivalence class of trivializing covers.

As in the case of vector bundles, a collection of gluing data determines a G-fiber bundle. Indeed, if we are given a cover $(U_\alpha)_{\alpha \in A}$ of the base B, and a collection of transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle condition, then we can get a bundle by gluing the trivial pieces $U_\alpha \times F$ along the overlaps.

More precisely, if $b \in U_\alpha \cap U_\beta$, then the element $(f, b) \in F \times U_\alpha$ is identified with the element $(g_{\beta\alpha}(b) \cdot f, b) \in F \times U_\beta$.

Definition 2.3.16. Let $E \xrightarrow{\pi} B$ be a G-fiber bundle. A G-automorphism of this bundle is a diffeomorphism $T : E \to E$ such that $\pi \circ T = \pi$, i.e., T maps fibers to fibers, and for any trivializing cover (U_α) (as in Definition 2.3.15) there exists a smooth map $g_\alpha : U_\alpha \to G$ such that

$$\Psi_\alpha^{-1}T\Psi_\alpha(f, b) = (\, g_\alpha(b)f, b\,), \quad \forall b, f.$$ □

Definition 2.3.17. (a) A *fiber bundle* is an object defined by conditions (a)-(d) and (f) in the above definition. (One can think the structure group is the group of diffeomorphisms of the standard fiber).
(b) A *section* of a fiber bundle $E \xrightarrow{\pi} B$ is a smooth map $s : B \to E$ such that $\pi \circ s = \mathbb{1}_B$, i.e., $s(b) \in \pi^{-1}(b)$, $\forall\, b \in B$. □

Example 2.3.18. A rank r vector bundle (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is a $\mathrm{GL}(r, \mathbb{K})$-fiber bundle with standard fiber $\mathbb{K}^r$, and where the group $\mathrm{GL}(r, \mathbb{K})$ acts on $\mathbb{K}^r$ in the natural way. □

Example 2.3.19. Let G be a Lie group. A *principal G-bundle* is a G-fiber bundle with fiber G, where G acts on itself by left translations. Equivalently, a principal G-bundle over a smooth manifold M can be described by an open cover $\mathcal{U}$ of M and a G-cocycle, i.e., a collection of smooth maps

$$g_{UV} : U \cap V \to G \quad U, V \in \mathcal{U},$$

such that $\forall x \in U \cap V \cap W$ $(U, V, W \in \mathcal{U})$

$$g_{UV}(x)g_{VW}(x)g_{WU}(x) = 1 \in G. \qquad \square$$

Exercise 2.3.20. (Alternative definition of a principal bundle). Let P be a fiber bundle with fiber a Lie group G. Prove the following are equivalent.
(a) P is a principal G-bundle.
(b) There exists a free, right action of G on G,

$$P \times G \to P, \ \ (p, g) \mapsto p \cdot g,$$

such that its orbits coincide with the fibers of the bundle P, and there exists a trivializing cover

$$\{ \Psi_\alpha : G \times U_\alpha \to \pi^{-1}(U_\alpha) \},$$

such that

$$\Psi_\alpha(hg, u) = \Psi_\alpha(h, u) \cdot g, \ \ \forall g, h \in G, \ u \in U_\alpha. \qquad \square$$

Exercise 2.3.21. (The frame bundle of a manifold). Let M^n be a smooth manifold. Denote by $F(M)$ the set of frames on M, i.e.,

$$F(M) = \{(m; X_1, \dots, X_n);\ m \in M,\ X_i \in T_mM \text{ and } \operatorname{span}(X_1, \dots, X_n) = T_mM\}.$$

(a) Prove that $F(M)$ can be naturally organized as a smooth manifold such that the natural projection $p : F(M) \to M$, $(m; X_1, \dots, X_n) \mapsto m$ is a submersion.
(b) Show $F(M)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. The bundle $F(M)$ is called the *frame bundle* of the manifold M.
Hint: A matrix $T = (T^i_j) \in \mathrm{GL}(n, \mathbb{K})$ acts on the right on $F(M)$ by

$$(m; X_1, ..., X_n) \mapsto (m; (T^{-1})^i_1 X_i, ..., (T^{-1})^i_n X_i). \qquad \square$$

Example 2.3.22. (Associated fiber bundles). Let $\pi : P \to G$ be a principal G-bundle. Consider a trivializing cover $(U_\alpha)_{\alpha \in A}$, and denote by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ a collection of gluing maps determined by this cover. Assume G acts (on the left) on a smooth manifold F

$$\tau : G \times F \to F, \ \ (g, f) \mapsto \tau(g)f.$$

The collection $\tau_{\alpha\beta} = \tau(g_{\alpha\beta}) : U_{\alpha\beta} \to \mathrm{Diffeo}\,(F)$ satisfies the cocycle condition and can be used (exactly as we did for vector bundles) to define a G-fiber bundle with fiber F. This new bundle is independent of the various choices made (cover (U_α) and transition maps $g_{\alpha\beta}$). (Prove this!) It is called the bundle associated to P via τ and is denoted by $P \times_\tau F$. $\square$

Exercise 2.3.23. Prove that the tangent bundle of a manifold M^n is associated to $F(M)$ via the natural action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. $\square$

Exercise 2.3.24. (The Hopf bundle) If we identify the unit odd dimensional sphere S^{2n-1} with the submanifold

$$\left\{ (z_1, \dots, z_n) \in \mathbb{C}^n;\ |z_0|^2 + \cdots + |z_n|^2 = 1 \right\}$$

then we detect an S^1-action on S^{2n-1} given by

$$e^{i\theta} \cdot (z_1, \dots, z_n) = (e^{i\theta} z_1, \dots, e^{i\theta} z_n).$$

The space of orbits of this action is naturally identified with the complex projective space $\mathbb{CP}^{n-1}$.
(a) Prove that $\boldsymbol{p} : S^{2n-1} \to \mathbb{CP}^{n-1}$ is a principal S^1 bundle called Hopf bundle. ($\boldsymbol{p}$ is the obvious projection). Describe one collection of transition maps.
(b) Prove that the tautological line bundle over $\mathbb{CP}^{n-1}$ is associated to the Hopf bundle via the natural action of S^1 on $\mathbb{C}^1$. □

Exercise 2.3.25. Let E be a vector bundle over the smooth manifold M. Any metric h on E (euclidian or Hermitian) defines a submanifold $S(E) \subset E$ by

$$S(E) = \{ v \in E;\ |v|_h = 1 \}.$$

Prove that $S(E)$ is a fibration over M with standard fiber a sphere of dimension $\operatorname{rank} E - 1$. The bundle $S(E)$ is usually called the *sphere bundle* of E. □

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