Toric Geometry: Theorems and Definitions

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Contents

1 Notation and Preliminaries

Definition 1.1.

- $\mathfrak{h} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is the upper half plane in $\mathbb{C}$
- $\mathrm{GL}_2(\mathbb{R})^+ = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) > 0\}$ is the set of all automorphisms of $\mathbb{R}^2$ with positive determinant
- $\mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det(g) = 1\}$ is the standard special linear subgroup of $\mathrm{GL}_2(\mathbb{R})$
- $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \,\middle|\, a,b,c,d \in \mathbb{Z} \right\}$ is the special linear subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

Lemma 1.2. The group $GL_2(\mathbb{R})^+$ acts transitively on $\mathfrak{h}$ by Möbius transformations (fractional linear transformations). That is, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and any $\tau = x + iy \in \mathfrak{h}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d} \in \mathfrak{h}$$

and

$$\tau = x + iy = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i.$$

Lemma 1.3. Suppose $k \in \mathbb{Z}$, $f : \mathfrak{h} \longrightarrow \mathbb{C}$ is a meromorphic function, and $g \in \mathrm{GL}_2(\mathbb{R})^+$. We define the **weight k action of** g on f to be the right action $(f \mapsto f|_k[g])$ where $f|_k[g] : \mathfrak{h} \longrightarrow \mathbb{C}$ is a meromorphic function defined

$$f|_k[g](\tau) = \det(g) f(g\tau) j(g,\tau)^k,$$

where if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $j(g,\tau) = (c\tau + d)$.

Definition 1.4. Let $k \in \mathbb{Z}$ and let $\Gamma \leq SL_2(\mathbb{Z})$ be a subgroup of finite index. A **weakly modular function** of weight k and level Γ is a meromorphic function f on $\mathfrak{h}$ such that for all $\gamma \in \Gamma$, $f|_k[\gamma] = f$. A **modular function** (of weight k and level Γ) is a weakly modular function which is meromorphic at ∞. A **modular form** is a weakly modular function which is holomorphic in $\mathfrak{h}$ and at ∞. A **cuspoidal modular form** is a modular form which vanishes at ∞.

2 Modular Forms on $SL_2(\mathbb{Z})$