6 Divisors

Exercise 6.1. *Let X be a scheme satisfying (*). Then $X \times \mathbb{P}^n$ also satisfies (*) and* $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{P}^n$.

Solution. As in the proof of Proposition II.6.6 we see immediately that $X \times \mathbb{P}^1$ is noetherian, integral, and separated. To see that it is regular in codimension one, note that it can be covered by (two) open affines of the form $X \times \mathbb{A}^1$. Each of these is shown to be regular in codimension one in the proof of II.6.6 and so $X \times \mathbb{P}^1$ is regular in codimension one.

After Proposition II.6.5 and II.6.5 we have an exact sequence

$$\mathbb{Z} \xrightarrow{i} \mathrm{Cl}(X \times \mathbb{P}^1) \xrightarrow{j} \mathrm{Cl}\, X \to 0$$

The first map sends n to nZ where Z is the closed subscheme $\pi_2^{-1}\infty \subset X \times \mathbb{P}^1$ (where $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection), and the second is the composition of $\mathrm{Cl}(X \times \mathbb{P}^1) \to \mathrm{Cl}(X \times \mathbb{A}^1) \xleftarrow{\sim} \mathrm{Cl}\, X$. Consider the map $\mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^1)$ that sends $\sum n_i Z_i$ to $\sum n_i \pi_1^{-1} Z_i$. The composition $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^1) \to \mathrm{Cl}(X \times \mathbb{A}^1) \xleftarrow{\sim} \mathrm{Cl}(X)$ sends a prime divisor Z to $\pi_1^{-1}Z$, then $(X \times \mathbb{A}^1) \cap \pi_1^{-1} Z$, and then back to Z since $(X \times \mathbb{A}^1) \cap \pi_1^{-1} Z$ is the preimage of Z under the projection $X \times \mathbb{A}^1 \to X$. Hence, the epimorphism in the exact sequence above is split.

We now show that the morphism $\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^1)$ is split as well, by defining a morphism $\mathrm{Cl}(X \times \mathbb{P}^1) \to \mathbb{Z}$ which splits i. Let $k : \mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^1)$ denote the morphism we used to split j. Then we send a divisor ξ to $\xi - kj\xi$. This is in the kernel of j (since $jk = id$) and therefore in the image of i. So it remains only to see that i is injective.

Suppose that $nZ \sim 0$ for some integer n. Taking the "other" $X \times \mathbb{A}^1$ we have Z as $\pi_2^{-1}0$ under the projection $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$. In the open subset $X \times \mathbb{A}^1$ we have Z as X embedded at the origin. So the local ring of Z in the function field $K(t)$ (where K is the function field of X) is $K[t]_{(t)}$. Since $nZ \sim 0$ there is a function $f \in K(t)$ such that $v_Z(f) = n$ and $v_Y(f) = 0$ for every other prime divisor Y. So f is of the form $t^n \frac{g(t)}{h(t)}$ where $g, h \in K[t]$ and $t \nmid g(t), h(t)$. If the degree of g and h is 0 then changing coordinates back $t \mapsto t^{-1}$ we see that $v_Y(f) = -n$ where Y is another copy of X embedded at the origin, or infinity, depending on which coordinates we are using; the one opposite to Z at any rate. If one of g or h has degree higher than zero then, it will have an irreducible factor in $K[t]$, which will correspond to a prime divisor of the form $\pi_2^{-1}x$ for some $x \in \mathbb{P}^1$, and the value of f will not be zero at this prime divisor. Hence, there is no rational function with $(f) = nZ$ and so i is injective. Hence $\mathrm{Cl}(X \times \mathbb{P}^1) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4. *Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \dots, x_n]$ be a square free nonconstant polynomial, i.e., in the unique factorization of f into ir-*

reducible polynomials, there are no repeated factors. Let $A = k[x_1, \dots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring.

Solution. Let $B = k[x_1, \dots, x_n], L = \operatorname{Frac} B$ and consider the quotient field K of A. In this field we have $\frac{1}{g+zh}\frac{g-zh}{g-zh} = \frac{g-zh}{g^2-fh^2}$ since $z^2 = f$ in A, and so every element can be written in the form $g' + zh'$ where $g', h' \in L$. Hence, $K = L[z]/(z^2 - f)$. This is a degree 2 extension of L with automorphism $\sigma : z \mapsto -z$ and is therefore Galois. So we have the situation of Problem 5.14 from Atiyah-Macdonald (with badly chosen notation). Let A^c be the integral closure of A in K. We will show that $A = A^c$ by showing that for $\alpha = f + zg \in K$ (with $g, h \in L$) we have $\alpha \in A^c$ if and only if $f, g \in B$.

The minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. So if $g, h \in B$ then $\alpha \in A^c$. Conversely, suppose that $\alpha \in A^c$. Then $\alpha + \sigma\alpha = 2f$ and $\alpha - \sigma\alpha = 2g$ are both σ invariant and in A^c and are therefore in B, by the Atiyah-Macdonald exercise.

Exercise 6.5. Quadric Hypersurfaces. *Let char $k \neq 2$, and let X be the affine quadric hypersurface* $\operatorname{Spec} k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \cdots + x_r^2)$.

a *Show that X is normal if $r \geq 2$.*

b *Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \cdots + x_r^2$. Now imitate the method of (6.5.2) to show that:*

 (a) If $r = 2$ then $\operatorname{Cl} X \cong \mathbb{Z}/2\mathbb{Z}$*;*

 (b) If $r = 3$ then $\operatorname{Cl} X \cong \mathbb{Z}$*;*

 (c) If $r \geq 4$ then $\operatorname{Cl} X = 0$.

c *Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equation. Show that:*

 (a) If $r = 2$, $\operatorname{Cl} Q \cong \mathbb{Z}$*, and the class of a hyperplane section $Q.H$ is twice the generator;*

 (b) If $r = 3$, $\operatorname{Cl} Q \cong \mathbb{Z} \oplus \mathbb{Z}$*;*

 (c) If $r \geq 4$, $\operatorname{Cl} Q \cong \mathbb{Z}$*, generated by $Q.H$.*

d *Prove Klein's theorem, which says that if $r \geq 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbb{P}^n$ such that $Y \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection.*

Solution. a Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \cdots + x_r^2)$. By taking $f = x_1^2 + \cdots + x_r^2$, if we can show that f is square free, then we will have the situation of Exericise II.6.4 and so A will be integrally closed, implying that X is normal. But the polynomial f has degree 2 and so it is a product of at most 2 other nonconstant polynomials, which by degree, must be linear. Suppose $\sum a_i x_i$ is a linear polynomials such that $(\sum a_i x_i)^2 = f$.

Then $a_i^2 = 1$ for all $i = 0, \dots, r$, and $2a_ia_j = 0$ for $i \neq j \in \{0, \dots, r\}$. But this implies that $2 = 2a_i^2a_j^2 = 0$ and we have assumed that k doesn't have characteristic 2. Hence f is square free.

b We assume -1 has a square root i in k, otherwise there isn't a suitable change of coordinates. Take the change of coordinates $x_0 \mapsto \frac{y_0+y_1}{2}$ and $x_1 \mapsto \frac{y_0-y_1}{i2}$. Then $x_0^2 + x_1^2 = y_0y_1$.

Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0x_1+x_2^2+\cdots+x_r^2)$. Now we imitate Example II.6.5.2. We take the closed subscheme of $\mathbb{A}^{n+1}$ with ideal $\langle x_1, x_2^2+\cdots+x_r^2\rangle$. This is a subscheme of X and is in fact $V(x_1)$ considering $x_1 \in A$. We have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - Z) \to 0$$

Now since $V(x_1) \cap X = X - Z$ the coordinate ring of $X - Z$ is

$$k[x_0, x_1, x_1^{-1}, x_2, \dots, x_n]/(x_0x_1 + x_2^2 + \cdots + x_r^2)$$

As in Example II.6.5.2 since $x_0 = -x_1^{-1}(x_2^2 + \cdots + x_r^2)$ in this ring we can eliminate x_0 and since every element of the ideal $(x_0x_1 + x_2^2 + \cdots + x_r^2)$ has an x_0 term, we have an isomorphism between the coordinate ring of $X - Z$ and $k[x_1, x_1^{-1}, x_2, \dots, x_n]$. This is a unique factorization domain so by Proposition II.6.2 $\mathrm{Cl}(X - Z) = 0$. So we have a surjection $\mathbb{Z} \to \mathrm{Cl}(X)$ which sends n to $n \cdot Z$.

$r = 2$ In this case the same reasoning as in Example II.6.5.2 works. Let $\mathfrak{p} \subset A$ be the prime associated to the generic point of Z. Then $\mathfrak{m}_\mathfrak{p}$ is generated by x_2 and $x_1 = x_0^{-1}x_2^2$ so $v_Z(x_1) = 2$. Since Z is cut out by x_1 there can be no other prime divisors Y with $v_Y(x_1) \neq 0$. It remains to see that Z is not a principle divisor. If it were then $\mathrm{Cl}(X)$ would be zero and by Proposition II.6.2 this would imply that A is a unique factorization domain (since A is normal by the first part of this exercise) which would imply that every hieght one prime ideal is principle. Consider the prime ideal $\langle x_1, x_2\rangle$ of A which defines Z. Let $\mathfrak{m} = (x_0, x_1, \dots, x_n)$. we have $\mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension n over k with basis $\{\overline{x}_i\}$. The ideal $\mathfrak{m}$ contains $\mathfrak{p}$ and its image in $\mathfrak{m}/\mathfrak{m}^2$ is a subspace of dimension at least 2. Hence, $\mathfrak{p}$ cannot be principle.

$r = 3$ We use Example II.6.6.1 and Exercise II.6.3(b). Using a similar change of coordinates as the beginning of this part of this exercise, we see that X is the affine cone of the projective quadric of Example II.6.6.1. This, by Exercise II.6.3(b) we have an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Cl}(X) \to 0$. We already know that $\mathrm{Cl}(X)$ is $\mathbb{Z}, \mathbb{Z}/n$ or 0. Tensoring with $\mathbb{Q}$ gives an exact sequence $\mathbb{Q} \to \mathbb{Q}^2 \to \mathrm{Cl}(X) \otimes \mathbb{Q} \to 0$ of $\mathbb{Q}$ vector spaces. Hence, $\mathrm{Cl}(X) = \mathbb{Z}$, as the other two cases contradict the exactness of the sequence of $\mathbb{Q}$-vector spaces.

$r \geq 4$ In this case we claim that Z is principle. Consider the ideal (x_1) in A. Its corresponding closed subset is Z and so if we can show that (x_1) is prime, then Z will be the principle divisor associated to the rational function x_1. Showing that (x_1) is prime is the same as showing that $A/(x_1)$ is integral, which is the same as showing that $\frac{k[x_0,\dots,x_n]}{(x_1,x_2^2+\dots+x_r^2)}$ is integral since $(x_1, x_0x_1 + x_2^2 + \dots + x_r^2) = (x_1, x_2^2 + \dots + x_r^2)$. This is the same as showing that $\frac{k[x_0,x_2,\dots,x_n]}{(x_2^2+\dots+x_r^2)}$ is integral (where the variable x_1 is missing on the top) which is the same as showing that $f = x_2^2 + \dots + x_r^2$ is irreducible. Suppose f is a product of more than one nonconstant polynomial. Since it has degree two, it is the product of at most two linear polynomials, say $a_0x_0+a_2x_2+\dots+a_nx_n$ and $b_0x_0 + b_2x_2 + \dots + b_nx_n$. Expanding the product of these two linear polynomials and comparing coefficients with f we find that (I) $a_ib_i = 1$ for $2 \leq i \leq r$, and (II) $a_ib_j + a_jb_i = 0$ for $2 \leq i, j \leq r$ and $i \neq j$. Without loss of generality we can assume that $a_2 = 1$. The relation (I) implies that $b_2 = 1$, and in general, $a_i = b_i^{-1}$ for $2 \leq i \leq r$. Putting this in the second relation gives (III) $a_i^2 + a_j^2 = 0$ for $2 \leq i \neq j \leq r$ and this together with the assumption that $a_2 = 1$ implies that (IV) $a_j^2 = -1$ for each $2 < j \leq r$. But if $r \geq 4$ then we have from (III) that $a_3^2 + a_4^2 = 0$ which contradicts (IV). Hence $x_2^2 + \dots + x_r^2$ is irreducible, so $\frac{k[x_0,x_2,\dots,x_n]}{(x_2^2+\dots+x_r^2)}$ is integral, so $A/(x_1)$ is integral, so (x_1) is prime and hence Z is the principle divisor corresponding to x_1. So $\mathrm{Cl}(X) = 0$.

c For each of these we use the exact sequence of Exercise II.6.3(b).

$r = 2$ We have an exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(Q) \to \mathbb{Z}/2 \to 0$ where the first mophism sends 1 to the class of $H \cdot Q$ a hyperplane section. Tensoring with $\mathbb{Q}$ we get an exact sequence $\mathbb{Q} \xrightarrow{2} \mathrm{Cl}(Q) \otimes \mathbb{Q} \to 0 \to 0$ and so since $\mathrm{Cl}(Q)$ is an abelian group we see that it is $\mathbb{Z} \oplus T$ where T is some torsion group. Tensoring with $\mathbb{Z}/p$ for a prime p we get either $\mathbb{Z}/2 \xrightarrow{0} Cl(Q) \otimes (\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$ if $p = 2$ or $\mathbb{Z}/p \xrightarrow{2} \mathrm{Cl}(Q) \otimes (\mathbb{Z}/p) \to 0 \to 0$ if $p \neq 2$. Hence, $T = 0$, and so $\mathrm{Cl}(Q) \cong \mathbb{Z}$ and the class of a hyperplane section is twice the generator.

$r = 3$ This is Example II.6.6.1.

$r \geq 4$ We have an exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(Q) \to 0 \to 0$, hence, $\mathrm{Cl}(Q) = \mathbb{Z}$ and it is generated by $Q \cdot H$.

Exercise 6.6. *Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$ of (6.10.2).*

a *Show that three points P, Q, R of X are collinear if and only if $P+Q+R = 0$ in the group law on X. (Note that the point $P_0 = (0, 1, 0)$ is the zero element in the group structure on X).*

b *A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.*

c *A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point (an inflection point of a plane curve is a nonsingular point P of the curve, whose tangent line (Exercise I.7.3) has intersection multiplicity ≥ 3 with the curve at P.)*

d *Let $k = \mathbb{C}$. Show that the points of X with coordinates in $\mathbb{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?*

Solution. a Suppose that P, Q, R are collinear. Then there is a line L on which they all lie and since every line meets X in exactly three points (counting multiplicities) P, Q, R are the only points where L meets X. In $\mathbb{P}^2$ any line is equivlant to z and so $P + Q + R \sim 3P_0$ as divisors, hence $(P - P_0) + (Q - P_0) + (R - P_0) \sim (P_0 - P_0)$ as divisors, and therefore $P + Q + R = 0$ in the group law on X.

Conversely, suppose that $P+Q+R = 0$ in the group law on X. If P, Q, R are not all distinct, then they are collinear in $\mathbb{P}^2$ since any two points are collinear in $\mathbb{P}^2$. Suppose they are distinct and consider the unique line L on which P and Q lie. This line intersects X in a unique third point T and we have $P + Q + T \sim 3P_0$. Hence, $P + Q + T = 0$ in the group law on X and therefore $R = -P - Q = T$. So P, Q, R are collinear.

b Recall that the tangent line to P is the unique line $T_P(X)$ whose intersection multiplicity with X at P is > 1 (Exercise I.7.3).

If $P = P_0$ then certainly the tangent line passes through P_0. Suppose that $P \neq P_0$ has order 2 and consider the tangent line $T_P(X)$ to P. This line intersects X in three points (counting multiplicities) and since it hits P with multiplicity greater than one, these three points are P, P and R for some other point R (which is possibly also P). Now P, P and R being collinear means that $P + P + R + 0$ in the group law on X. But P has order 2 and so we see that $R = 0 = P_0$. Hence, the tangent line $T_P(X)$ passes through P_0.

Conversely, suppose that the tangent line $T_P(X)$ passes through P_0. Since P_0 is the identity, it has order 2 so suppose that $P \neq P_0$. Again, $T_P(X)$ hits X in three points (counting multiplicities) of which at least two are P, and since we have assumed that $P_0 \neq P$ these three points are P, P and P_0. Hence, $P + P + P_0 = 0$ and since $P_0 = 0$ we see that P has order 2.

c If P is an inflection point then the intersection multiplicity of $T_P(X)$ and X at P is ≥ 3. Since X has degree three it can't be more than three and so we see that it is exactly three. So the three points of X that $T_P(X)$ hits, counting multiplicites, are all P, and so $P + P + P = 0$ in the group law. Hence, P has order three.

Conversely, if P has order three then $P+P+P=0$ then the three points P,P,P are collinear. That is, there is a line L such that L intersects X in the unique point P with intersection multiplicity three. Since there is a unique line of $\mathbb{P}^2$ that intersects X at P with multiplicity greater than one–the tangent line–we see that the tangent line intersects X at P with multiplicity three, and therefore P is an inflection point.

d If the base field is $\mathbb{C}$ then the elliptic curve is isomorphic as an abelian variety to the quotient of the complex plane by a lattice $\mathbb{Z}^2$.

Exercise 6.7. *Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0,* $\operatorname{CaCl}^0 X$*, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.*

Exercise 6.8. *a Let $f : X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphim of Picard groups, $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.*

b If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphism of (6.16).

c If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f : X \to \mathbb{P}^n$ is the inclusion map, then f^* on Pic *agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).*

Exercise 6.9. Singular curves.

Exercise 6.10. The Grothendieck Group $K(X)$. *Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group genertaed byt all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.*

a If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

b If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the rank of $\mathscr{F}$ to be $\dim_k \mathscr{F}_\xi$ where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism $\operatorname{rank} : K(X) \to \mathbb{Z}$.

c If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0$$

where the first map is extension by zero, and the second map is restriction.

Solution. a Let $\mathscr{F}$ be a coherent sheaf on X. Then $\mathscr{F}$ corresponds to a finitely generated $k[t]$-module M. We take a presentation $k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0$ of M and since $k[t]$ is a principle ideal domain, we can choose the first morphism to be injective.[1] Hence, we arrive at an exact sequence $0 \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathscr{F} \to 0$ so in the Grothendieck group we have $\gamma(\mathscr{F}) = (m-n)\gamma(\mathcal{O}_X)$. So the morphism $\mathbb{Z} \to K(X)$ sending n to $n\gamma(\mathcal{O}_X)$ is surjective. To see that this morphism is injective, we use the rank homomorphism from the next part of this exercise to split it.

b First we show that it defines a homomorphism. Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherent sheaves on X. Since this sequence is exact, it is exact at every stalk. In particular, it is exact at the stalk at the generic point ξ. So we have an exact sequence of finitely generated $\mathcal{O}_\xi$-modules $0 \to \mathscr{F}'_\xi \to \mathscr{F}_\xi \to \mathscr{F}''_\xi \to 0$. Hence, $\dim_K \mathscr{F}_\xi = \dim_K \mathscr{F}''_\xi + \dim_K \mathscr{F}'_\xi$. So rank is a well-defined homomorphism.

To see that it is surjective, notice that $\gamma(\mathcal{O}_X) \mapsto 1$, and so $n \cdot \gamma(\mathcal{O}_X) \mapsto n$.

c *Surjectivity on the right.* Every coherent sheaf $\mathscr{F}$ on $X - Y$ can be extended to a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_{X-Y} = \mathscr{F}$ by Exercise II.5.15. So the morphism on the right is surjective.

Exactness in the middle. Suppose that $\mathscr{F}$ is a coherent sheaf on X with support in Y. We will show (below) that there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_n = 0$ such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y. Assuming we have such a finite filtration, we have $\gamma(\mathscr{F}_i) = \gamma(\mathscr{F}_{i+1}) + \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$ in $K(X)$ and so $\gamma(\mathscr{F}) = \sum_{i=0}^{n-1} \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$. Hence, the class represented by $\mathscr{F}$ is in the image of $K(Y) \to K(X)$. Now if $\sum n_i \gamma(\mathscr{F}_i)$ is in the kernel of $K(X) \to K(X-Y)$ the *Proof of claim.* Let $i : Y \to X$ be the closed embedding of Y into X and consider the two functors $i_* : Coh(X) \to Coh(Y)$ (Exercise II.5.5) and $i^* : Coh(Y) \to Coh(X)$. These functors are adjoint (page 110) and so we have a natural morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ for any coherent sheaf $\mathscr{F}$ on X. Let $\operatorname{Spec} A$ be an open affine subscheme of X on which $\mathscr{F}$ has the form $\widetilde{M}$. Closed subschemes of affine schemes correspond to ideals bijectively and so $\operatorname{Spec} A \cap Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$ and the morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ restricted to $\operatorname{Spec} A$ has the form $M \to M/IM$. Thus we see that η is surjective. Let $\mathscr{F}_0 = \mathscr{F}$ and define $\mathscr{F}_j$ inductively as $\mathscr{F}_j = \ker(\mathscr{F}_{j-1} \to i_* i^* \mathscr{F}_j)$. It follows from our

[1] *If N is a submodule of a free A-module M of rank n where A is an integral PID then N is free.* Induction on n. If $n = 1$ then a submodule is an ideal and since A is a PID the ideal is of the form (a) for some $a \in A$. Since A is integral the map $b \mapsto ab$ is an isomorphism of modules. Now suppose $M = A^n$. Consider the submodule A^{n-1} of elements whose last component is zero. Then by the inductive hypothesis $N' = A^{n-1} \cap N$ is free; let $m_1, \dots, m_r$ be a basis for N' as a free A-module. If $\pi : A^n \to A$ is projection onto the last component then its image is an ideal I of A. If $I = 0$ then $N' = N$ and we are done. If not, choose an elemen $n \in N$ such that $\pi n = a$ where $(a) = I$. Then we claim that $N = N' \oplus An$. Certainly, $N' + An \subseteq N$. If $m \in N$ then $m = (m - (\pi m)n) + (\pi m)n$ is a decomposition into an element of N' and of An so $N' + An \supseteq N$ and therefore $N' + An = N$, so it remains to see that $N' \oplus An \to N' + An$ is injective. Suppose (x, bn) is in the kernel. Then $x + bn = 0$ and so $\pi(x + bn) = 0$. But $\pi(x + bn) = ba$ and since A is integral this implies that $b = 0$. Hence, $x + 0n = 0$ and so $x = 0$. So $N' \oplus An \to N' + An = N$ is an isomorphism.

definition that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y so we just need to show that the filtration $\mathscr{F} \supseteq \mathscr{F}_1 \supseteq \dots$ is finite.

On our open affine we have $\mathscr{F}_j|_{\operatorname{Spec} A} = I^j M$. Now the support of $\widetilde{M}$ contained in the closed subscheme $\operatorname{Spec} A/I = V(I)$ so by Exercise II.5.6(b) we have $\sqrt{\operatorname{Ann} M} \supseteq \sqrt{I} \supseteq I$. Since A is noetherian, every ideal is finitely generated. In particular, I is finitely generated. So there exists some N such that $\operatorname{Ann} M \supseteq I^N$ (see the proof of Exercise II.5.6(d) for details). Hence, $0 = I^N M$ and so the filtration is finite when restricted to an open affine. Since X is noetherian, there is a cover by finitely many affine opens $\{U_i\}$ and so if n_i is the point at which $\mathscr{F}_i|_{U_i} = 0$ then $\mathscr{F}_{\max\{n_i\}} = 0$. So the filtration is finite.

Exercise 6.11. The Grothendieck Group of a Nonsingular Curve. *Let X be a nonsingular curve over an algebraically closed field k.*

a *For any divisor $D = \sum n_i P_i$, let $\psi(D) = \sum n_i [k(P_i)] \in K(X)$ where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the stucture sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = [\mathcal{O}_D]$. Then use (6.18) to show that for any $D, \psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.*

b *For any coherent sheaf $\mathscr{F}$ on X, show that there exists locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\wedge^{r_0} \mathscr{E}_0) \otimes (\wedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it goves a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divsor, then $\det(\psi(D)) = \mathscr{L}(D)$.*

c *If $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \o \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank f, then $[\mathscr{F}] - r[\mathcal{O}_X] \in \operatorname{im} \psi$.*

d *Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto [\mathcal{O}_X]$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.*

Solution. a We denote the associated subscheme of D also by D. So its sheaf of ideals is $\mathscr{I}_D$. For each closed point $P \in X$ let $\mathscr{F}_P$ be the skyscraper sheaf $\operatorname{coker}((\mathscr{I}_D)_P \to \mathcal{O}_P)$ at P and zero elsewhere. There are surjections $\mathcal{O}_X \to \mathscr{F}_P$ for each P and so we have an exact sequence

$$0 \to \mathscr{I}_D \to \mathcal{O}_X \to \bigoplus_{P \in X} \mathscr{F}_P \to 0$$

Hence, $\mathcal{O}_D \cong \oplus \mathscr{F}_P$ and so $\gamma(\mathcal{O}_D) = \sum \gamma(\mathscr{F}_P)$. Now consider $\mathscr{F}_P$ for some $P \in X$ with $\mathscr{F}_P$ nonzero (there are only finitely many as there are only finitely many points in D). Choose a representation $\{(U_i, f_i)\}$ of the Cartier divisor corresponding to the Weil divisor D. Since D is effective, this can be chosen so that $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ for each i, and in this case the sheaf of ideals $\mathscr{I}_D$ is locally generated by f_i (by the definition on page 145). If U_i is an open that contains P then $v_P(f_i) = n$, where n is the coefficient of P in the sum D. So in the local ring $\mathcal{O}_P$ we have $f_i = t^n$ where t is a generator of $\mathfrak{m}_P$. The stalk of $\mathscr{F}_P$ at P is by our definition above $\operatorname{coker}((\mathscr{I}_D)_P \to \mathcal{O}_P)$ which we can now see to be $\mathcal{O}_P/\mathfrak{m}_P^n$. For each i we have an exact sequence of $\mathcal{O}_P$ modules $0 \to \mathfrak{m}_P^i/\mathfrak{m}_P^{i+1} \to \mathcal{O}_P/\mathfrak{m}_P^{i+1} \to \mathcal{O}_P/\mathfrak{m}_P^i \to 0$ and we have isomorphisms of $\mathcal{O}_P$-modules $\mathfrak{m}_P^i/\mathfrak{m}_P^{i+1} \cong \mathfrak{m}_P/\mathfrak{m}_P^2 \cong k$ so it follows that $\gamma(\mathscr{F}_P) = n\gamma(k(P))$. Combining this with the equality $\gamma(\mathcal{O}_D) = \sum \gamma(\mathscr{F}_P)$ shows that $\psi(D) = \gamma(\mathcal{O}_D)$.

If D' is some other effective divisor in the same linear equivalence class as D then we have

$$\begin{aligned}
\psi(D) &= \gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_D) \\
&\overset{6.18}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D)) \overset{6.13}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D')) \\
&\overset{6.18}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_{D'})) = \gamma(\mathcal{O}_{D'}) = \psi(D')
\end{aligned}$$

So ψ defines a homomorphism (for an arbitrary divisor D, write it as a difference of two effective divisors $D = D_+ - D_-$ and then we have $\psi(D) = \gamma(\mathcal{O}_{D_+}) - \gamma(\mathcal{O}_{D_-})$).

b *Existence of the exact sequence.* By Corollary II.5.18 we can write $\mathscr{F}$ as the quotient of a finite direct sum $\mathscr{E}_0 = \oplus\mathcal{O}(n_i)$ of twisted structure sheaves $\mathcal{O}(n_i)$ for various n_i. Let $\mathscr{E}_1$ be the kernel of the map $\mathscr{E}_0 \to \mathscr{F}$. At each closed point we then have an exact sequence

$$0 \to (\mathscr{E}_1)_x \to \mathcal{O}_x^{\oplus n} \to \mathscr{F}_x \to 0$$

That is, $(\mathscr{E}_1)_x$ is a submodule of $\mathcal{O}_x^{\oplus n}$. But each $\mathcal{O}_x$ is a reduced regular local ring of dimension one, and therefore a principle ideal domain (the only two ideals are zero since it is reduced, and $\mathfrak{m}$ which is principle since $\mathcal{O}_x$ is regular) and every submodule of a free module over a principle ideal domain is free. Hence $(\mathscr{E}_1)_x$ is free for every closed point x. Then by Exercise II.5.7 $\mathscr{E}_1$ is locally free.

Independence of $\mathscr{E}_1$ and $\mathscr{E}_0$. Suppose that we choose another locally free resolution $0 \to \mathscr{E}_1' \to \mathscr{E}_0' \to \mathscr{F} \to 0$. Consider the sequence $0 \to \mathscr{G} \to$

$\mathscr{E}_0 \oplus \mathscr{E}'_0 \to \mathscr{F} \to 0$. We have a diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{E}_0 \oplus \mathscr{E}'_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}'_0 \arrow[r, equal] \arrow[d] & \mathscr{E}'_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

and we know that the rows and the two right columns are exact. Hence, the left column is exact as well by the nine lemma. We also get a similar diagram using $\mathscr{E}'_1, \mathscr{E}'_0$ in the top row which gives an exact sequence $0 \to \mathscr{E}'_1 \to \mathscr{G} \to \mathscr{E}_0 \to 0$ as the left column. We know that $\mathscr{G}$ is a locally free sheaf by the same argument we used to show that existence of the exact sequence and so using the isomorphism of exercise II.5.16(d) we see that

$$\begin{aligned}
(\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}_1)^{-1} &\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}_1)^{-1} \otimes (\wedge\mathscr{E}'_0)^{-1} \otimes (\wedge\mathscr{E}'_0) \\
&\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{G})^{-1} \otimes (\wedge\mathscr{E}'_0) \\
&\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}'_1)^{-1} \otimes (\wedge\mathscr{E}_0)^{-1} \otimes (\wedge\mathscr{E}'_0) \\
&\cong (\wedge\mathscr{E}'_0) \otimes (\wedge\mathscr{E}'_1)^{-1}
\end{aligned}$$

So the determinant is independent of the resolution chosen.

The map det *defines a homomorphism* $K(X) \to \mathrm{Pic}(X)$. We need to show that whenever we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves, it holds that $\det \mathscr{F} \cong (\det \mathscr{F}') \otimes (\det \mathscr{F}'')$. Let $0 \to \mathscr{E}'_1 \to \mathscr{E}'_0 \to \mathscr{F}' \to 0$ be an exact sequence, and $\mathscr{E}_0 \to \mathscr{F}$ a surjective morphism with $\mathscr{E}_0, \mathscr{E}'_0, \mathscr{E}'_1$ all locally free. We define $\mathscr{G} = \ker(\mathscr{E}_0 \oplus \mathscr{E}'_0 \to \mathscr{F})$

and $\mathscr{H} = \ker \mathscr{E}_0 \to \mathscr{F}''$ to obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{E}_1' \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}_0' \arrow[r] \arrow[d] & \mathscr{E}_0 \oplus \mathscr{E}_0' \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whose columns and the lower two rows are exact by construction. Hence, by the nine lemma the top row is exact. $\mathscr{G}$ and $\mathscr{H}$ are locally free sheaves by the same argument we used to show the existence of the exact sequence above. using the isomorphism of exercise II.5.16(d) we see that

$$\begin{aligned}
\det \mathscr{F} &\cong (\wedge \mathscr{E}_0) \otimes (\wedge \mathscr{E}_0') \otimes (\wedge \mathscr{G})^{-1} \\
&\cong (\wedge \mathscr{E}_0) \otimes (\wedge \mathscr{E}_0') \otimes (\wedge \mathscr{H})^{-1} \otimes (\wedge \mathscr{E}_1')^{-1} \\
&\cong \det \mathscr{F}' \otimes \det \mathscr{F}''
\end{aligned}$$

Hence, $\det : K(X) \to \mathrm{Pic}(X)$ is a well defined homomorphism.

For a divisor D, $\det(\psi(D)) = \mathscr{L}(D)$*.* Suppose D is an effective divisor. Then we have an exact sequence $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ where Y is the corresponding closed subscheme. Since both $\mathcal{O}_X$ and $\mathscr{I}_Y$ are locally free of one, and by definition $\psi(D) = \gamma(\mathcal{O}_D)$ we have $\det(\psi(D)) = \mathcal{O}_X \otimes \mathscr{I}_Y^{-1} = \mathscr{I}_Y^{-1}$. Then using Proposition II.6.18 this is equal to $\mathscr{L}(-D)^{-1}$ and then by Proposition II.6.13 this is isomorphic to $\mathscr{L}(D)$. If D is not effective, write it as a difference of effective divisors and use the fact that det and ψ are both group homomorphisms together with Proposition II.6.13.

c To construct the injective morphism, the idea is to take a basis for the $K(X)$-vector space $\mathscr{F}_\xi$, and find a suitable $\mathscr{L}(D)$ such that this basis gives global sections of $\mathscr{L}(D) \otimes \mathscr{F}$. This defines a morphism $\mathcal{O}_X^{\oplus n} \to \mathscr{L}(D) \otimes \mathscr{F}$ which we show to be injective, and then tensor everything with $\mathscr{L}(D)^{-1}$.

Cover X with finitely many open affines $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$. On each of these, the restriction of $\mathscr{F}$ has the form $\widetilde{M_i}$ for some A_i-module M_i. Now consider the stalk $\mathscr{F}_\xi$ of $\mathscr{F}$ at the generic point. Since X is integral each A_i is integral and so the generic point appears as (0) in each U_i, so we have isomorphisms $\mathscr{F}_\xi \cong (\operatorname{Frac} A_i) \otimes_{A_i} M_i$ for each i. If $e_1, \dots, e_n$ is a basis for $\mathscr{F}_\xi$ as a $K(X)$-vector space, then these isomorphisms gives each e_j as $\frac{m_{ij}}{a_i}$ for some $m_{ij} \in M_i$ and $a_i \in A_i$ (if for some i the denominators of each $\frac{m_{ij}}{a_{ij}}$ were not the same, multiply by $\frac{\prod_{k\neq j} a_{ik}}{\prod_{k\neq j} a_{ik}}$ to get $\frac{m'_{ij}}{\prod a_{ij}}$). Now we want to use the a_i to define a Cartier divisor but $\frac{a_i}{a_j}$ might not be in $\mathcal{O}_X(U_i \cap U_j)$. We rectify this by shrinking the U_i as follows. First define $U'_i = U_i \backslash V(a_i)$ for each i. If $\cup U'_i \neq X$ then its complement is a finite set of points (since X is a curve), each one of which is contained in $V(a_i)$ for some i (since $\{U_i\}$ was a cover). For each of these points x, choose a $V(a_i)$ that it is in, and put it back in U_i. So if Z_i is the set of points in $V(a_i)$ that we have decided to leave in U_i, we have $U'_i = U_i \backslash (V(a_i)\backslash Z_i)$. The end result is that for $i \neq j$, if x is a point in $V(a_i) \cup V(a_j)$ then $x \notin U'_i \cap U'_j$. So $V(a_i) \cap (U'_i \cap U'_j)$ and $V(a_j) \cap (U'_i \cap U'_j)$ are both empty. It follows that a_i and a_j are both invertible in $\mathcal{O}_X(U'_i \cap U'_j)$. [2]

So we can define a Cartier divisor $D' = \{(U'_i, a_i\}$ whose associated sheaf is locally generated by $\frac{1}{a_i}$ on U'_i. The point is that our basis vectors e_j from $\mathscr{F}_\xi$ are now sections $\frac{1}{a_i} \otimes m_{ij}$ of $\Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$. Futhermore, these sections agree on the intersections and so we have global sections $e_i \in \Gamma(X, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$ and this we obtain a morphism $\mathcal{O}_X^{\oplus n} \to \mathscr{L}(D') \otimes \mathscr{F}$. We claim that this is injective. To see this, it will be enough to show that the $\frac{1}{a_i} \otimes m_{ij}$ generate a free submodule of $\Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$. To see this let $M = \Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$ and consider the morphism $M \to M \otimes K(X)$. Let $A = \mathcal{O}_X(U'_i)$ and let $A^n \to M$ be the morphism defined by sending $(a_1, \ldots, a_n)$ to $\sum_j a_j \frac{1}{a_i} \otimes m_{ij}$. If $A^n \to M$ were to have a kernel, say N, then we would have an exact sequence

$$N \otimes K \to A^n \otimes K \to M \otimes K$$

but the second morphism is an isomorphism and so $N \otimes K$ is zero. Hence the composition $N \to N \otimes K \to A^n \otimes K$ is zero. But this is the same as the composition $N \to A^n \to A^n \otimes K$, and both of these maps are injective. Hence, $N = 0$.

So we have an injective morphism of sheaves $\mathcal{O}_X \to \mathscr{L}(D') \otimes \mathscr{F}$. Now we need just tensor with $\mathscr{L}(D')^{-1} = \mathscr{L}(D)$ and we obtain an exact sequence $0 \to \mathscr{L}(D)^{\oplus n} \to \mathscr{F} \to \mathscr{T} \to 0$ where $\mathscr{T}$ is the cokernel of $\mathscr{L}(D)^{\oplus n} \to \mathscr{F}$. To see that $\mathscr{T}$ is torsion, consider the stalk of this exact sequence at the generic point. We get an exact sequence of $K(X)$-vector spaces $0 \to V' \to V \to V'' \to 0$ and the ranks of V' and V are the same. Hence $\mathscr{T}_\xi = 0$ and $\mathscr{T}$ is torsion.

To show that $[\mathscr{F}] - r[\mathcal{O}_X]$ is in the image of ψ we first use the exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$ to see that $[\mathscr{F}] - r[\mathcal{O}_X] = r[\mathscr{L}(D)] + [\mathscr{T}] - r[\mathcal{O}_X]$. So if $[\mathscr{T}]$ and $[\mathscr{L}(D)] - [\mathcal{O}_X]$ are in the image of ψ then we are done.

[2] For any affine scheme $\operatorname{Spec} A$, if a is not invertible, then (a) is a proper ideal of A, and therefore contained in some maximal idea (Zorn's Lemma) $\mathfrak{m}$ which implies that $a \in \mathfrak{m}$ and so $\mathfrak{m} \in V(a)$. Therefore, if $V(a) = \varnothing$ then a is invertible.

(i) $[\mathscr{L}(D)] - [\mathcal{O}_X]$ is in the image of ψ. As we saw in part (a) of this exercise, for effective divisors D there is an exact sequence $0 \to \mathscr{L}(D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ (c.f. Proposition II.6.18) so in $K(X)$ we have $[\mathcal{O}_D] = [\mathcal{O}_X] - [\mathscr{L}(D)]$. Now if D is not necassarily effective, then it can be written as a difference $D = D_+ - D_-$ of effective divisors. Then we have $\psi(D) = [\mathcal{O}_{D_+}] - [\mathcal{O}_{D_-}] = [\mathcal{O}_X] - [\mathscr{L}(D_+)] - [\mathcal{O}_X] + [\mathscr{L}(D_-)] = [\mathscr{L}(D_-)] - [\mathscr{L}(D_+)]$. Now since $\mathscr{L}(D_-)^{-1}$ is locally free, tensoring with it preserves exact sequences, so $\Phi : [\mathscr{F}] \mapsto [\mathscr{F} \otimes \mathscr{L}(D_-)^{-1}]$ is a well defined (set) function on $K(X)$. So $\Phi(\psi(D)) = \Phi\big([\mathscr{L}(D_-)] - [\mathscr{L}(D_+)]\big) = [\mathcal{O}_X] - [\mathscr{L}(D)]$. But $\psi(D) = \sum n_i[k(P_i)]$ where $D = \sum n_i P_i$. and so $\psi(D)$ is unchanged by Φ. Hence $[\mathcal{O}_X] - [\mathscr{L}(D)]$ is in the image of ψ.

(ii) $[\mathscr{T}]$ is in the image of ψ. By Exercise II.5.6 the support of $\mathscr{T}$ is a closed subset of X. Since X is a curve, this is a finite set of points, so $\mathscr{T} = \oplus \mathscr{T}_{P_i}$ is a finite sum of skyscraper sheaves. If we can show that $[\mathscr{T}_P]$ is in the image of ψ for every coherent skyscraper sheaf $\mathscr{T}_P$ then we are done. As we are not assuming X complete, it is enough to do this in the affine case. So suppose that $X = \operatorname{Spec} A$ and that $\widetilde{M}$ is a coherent skyscraper sheaf, concentrated at the maximal prime $\mathfrak{p} \in \operatorname{Spec} A$. For each i we have an exact sequence $0 \to \mathfrak{p}^{i+1}M \to \mathfrak{p}^i M \to \mathfrak{p}^i M/\mathfrak{p}^{i+1}M \to 0$. The A-module $\mathfrak{p}^i M/\mathfrak{p}^{i+1}M$ is a finite rank $A/\mathfrak{p}$-module; that is, a finite dimensional vector space. Hence, $\mathfrak{p}^i M/\mathfrak{p}^{i+1}M \cong (A/\mathfrak{p})^{\oplus n_i}$ for some n_i. The associated sheaf to $A/\mathfrak{p}$ is the skyscraper sheaf $k(P)$ and so by induction, we have $[\widetilde{M}] = \sum_{i \geq 0} n_i[k(P)]$, if this sum is finite. As the support of $\widetilde{M}$ is $\mathfrak{p}$, Exercise II.5.6(b) shows that $\sqrt{\operatorname{Ann} M} = \mathfrak{p}$. The ring A is noetherian and so $\mathfrak{p}^N \subseteq \operatorname{Ann} M$ for some N. [3] This means that $\mathfrak{p}^N M = 0$. Hence, $n_j = 0$ for each $j > N$ and so the sum $[\widetilde{M}] = \sum_{i \geq 0} n_i[k(P)]$ is finite. Therefore, $[\mathscr{T}]$ is in the image of ψ.

d The diagram is

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r, bend right, "\psi"'] & K(X) \arrow[l, bend right, "\det"'] \arrow[r, bend right, "\text{rank}"'] & \mathbb{Z} \arrow[l, bend right, "n\gamma(\mathcal{O}_X)"'] \end{tikzcd}$$

It is fairly evident that $\operatorname{rank}(n\gamma(\mathcal{O}_X)) = n$ and $\det(n\gamma(\mathcal{O}_X)) = \mathcal{O}_X^{\otimes n} = \mathcal{O}_X = 1 \in \operatorname{Pic}(X)$. Furthermore, since ψ takes a divisor to a sum of skyscraper sheaves, and the rank of a skyscraper sheaf is zero, we have $\operatorname{rank} \circ \psi = 0$. So we just need to show that $\det \circ \psi = id_{\operatorname{Pic} X}$.

Suppose that D is an effective divisor and $\mathscr{L}(D)$ the corresponding invertible sheaf. Then by part (a) ψ sends D to $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_D)$. By Proposition II.6.18 this is equal to $\gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D))$. The homomorphism det then takes this to $\mathcal{O}_X \otimes (\mathscr{L}(-D))^\vee \cong (\mathscr{L}(-D))^\vee \cong \mathscr{L}(D)$. Hence $\det \circ \psi = id_{\operatorname{Pic} X}$.

[3] Since A is noetherian, $\mathfrak{p}$ is finitely generated. Let $a_1, \ldots, a_n$ be generators. For each i there is some n_i such that $a_i^{n_i} \in \operatorname{Ann} M$. Taking N high enough, every monomial of degree N in the a_i will contain at least one term of the form a_i^m with $m > n_i$. Hence, $\mathfrak{p}^N \subseteq \operatorname{Ann} M$.

Exercise 6.12. *Let X be a complete nonsingular curve. Show that there is a unique way to define the degree of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbb{Z}$, such that:*

a If D is a divisor, $\deg \mathscr{L}(D) = \deg D$;

b If $\mathscr{F}$ is a torsion sheaf then $\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$; and

c If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.