Appendix B

Direct images, cohomology and base change

B.1 Can you define a bundle by its fibers?

To study the lines on a cubic surface back in Section 5.1, we needed to construct "the" vector bundle on the Grassmannian of lines in $\mathbb{P}^3$ whose fiber at the point representing a line L is the space of cubic forms on L. This specification is at best incomplete: the condition determines only the rank of the bundle. In the example we needed an additional property: we wanted the restriction map from the space of cubic forms on $\mathbb{P}^3$ to the line L to be induced by a map of bundles on the Grassmannian; that is, we needed the construction to be functorial in some reasonable sense.

To make things precise in this and many similar cases, we constructed the desired sheaves as *direct images*, and used the *theorem on cohomology and base change* to justify their properties. In this appendix we will give a gentle treatment of these important ideas. Much of the material is derived from Mumford [2008]; see also Arbarello et al. [1985, Chapter 4].

To state the problem more generally, suppose that we are given a family of varieties X_b with sheaves $\mathcal{F}_b$ on them, parametrized by the points b of a base variety B. As usual, by a family of varieties we mean a map $\pi : X \to B$, the "members" of the family being the fibers $X_b := \pi^{-1}(b)$. Similarly, by a family of sheaves we mean a sheaf $\mathcal{F}$ on X, with the members of the family being the sheaves $\mathcal{F}|_{X_b}$. We can expect nice results only if the members of the family "belong" together in some reasonable sense, which we generally take to be the condition that $\mathcal{F}$ is flat over B. Given such data, we ask whether there is a functorial construction of a sheaf $\mathcal{G}$ on B whose fiber $\mathcal{G}_b$ at a point b is the space of global sections of $\mathcal{F}_b$.

Such a sheaf $\mathcal{G}$ may or may not exist, as we shall soon see. Nevertheless, under very general circumstances we can define a sheaf $\pi_*\mathcal{F}$ on B, called the *direct image of $\mathcal{F}$ under π*, that is functorial in $\mathcal{F}$ and comes equipped with canonical maps $\varphi_b : (\pi_*\mathcal{F})_b \to H^0(\mathcal{F}|_{X_b})$ for $b \in B$. The theorem on cohomology and base change gives conditions under which all the φ_b are isomorphisms, in which case $\mathcal{G} := \pi_*\mathcal{F}$ will have the property we wish.

We present three versions of the theorem on cohomology and base change. The first — and the most often applied! — is Theorem B.5. A useful extension is the version given in Theorem B.9. The most general version, Theorem B.11, is paradoxically also the simplest, and easily implies the others. We prove these results, after various preliminaries, in Section B.5.

Before describing the results, we pause to explain an example that we will follow throughout this appendix:

Example B.1 (Two and three points in $\mathbb{P}^2$)**.** Let $\{p, q\} \subset \mathbb{P}^2$ be a set of two distinct points. Since $h^0(\mathcal{O}_{\{p,q\}}(d)) = 2$ and $h^1(\mathcal{O}_{\{p,q\}}(d)) = 0$ for all $d \in \mathbb{Z}$, the long exact sequence in cohomology coming from the short exact sequence $0 \to \mathcal{I}_{\{p,q\}} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\{p,q\}} \to 0$ immediately yields

$$h^0(\mathcal{I}_{\{p,q\}}(d)) = \begin{cases} \binom{2+d}{2} - 2 & \text{if } d \geq 1, \\ 0 & \text{if } d \leq 0, \end{cases}$$

$$h^1(\mathcal{I}_{\{p,q\}}(d)) = \begin{cases} 0 & \text{if } d \geq 1, \\ 1 & \text{if } d = 0, \\ 2 & \text{if } d < 0. \end{cases}$$

Ideal sheaves of sets of three distinct points $\{p, q, r\}$ can be analyzed similarly, using the sequence $0 \to \mathcal{I}_{\{p,q,r\}} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\{p,q,r\}} \to 0$, but there is a difference: When $d = 1$ there are two cases, depending on whether some linear form vanishes on all three points; we have

$$h^0(\mathcal{I}_{\{p,q,r\}}(1)) = h^1(\mathcal{I}_{\{p,q,r\}}(1)) = \begin{cases} 1 & \text{if } r \text{ lies on the line } \overline{p,q}, \\ 0 & \text{otherwise.} \end{cases}$$

It is thus interesting to consider a family of ideal sheaves of triples $\{p, q, r\}$ of distinct points as one of the points crosses the line joining the other two.

We prefer to work with a projective family, so we fix points p and q in $\mathbb{P}^2$ and let r move along a line $B \subset \mathbb{P}^2$ containing neither p nor q. To set this up, we consider in $B \times \mathbb{P}^2$ three families of points: the constant families

$$\Gamma_p = B \times \{p\} \quad \text{and} \quad \Gamma_q = B \times \{q\},$$

contained in $B \times \mathbb{P}^2$, and a family of points moving along B, that is, the diagonal

$$\Delta = \{(r, r) \in B \times \mathbb{P}^2 \mid r \in B\} \subset B \times B.$$

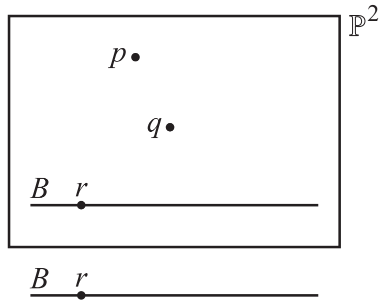

Figure B.1 The fibers of $\Gamma' \subset \Gamma$ over $r \in B$ are the sets $\{p, q\} \subset \{p, q, r\} \subset \mathbb{P}^2$.

We take

$$\Gamma = \Gamma_p \cup \Gamma_q \cup \Delta \subset B \times \mathbb{P}^2,$$

which we regard (via the projection map $\pi : \Gamma \to B$) as a family over B, with fiber over $r \in B$ the triple $\{p, q, r\} \subset \mathbb{P}^2$. Let Γ' be the trivial subfamily

$$\Gamma' = \Gamma_p \cup \Gamma_q \subset B \times \mathbb{P}^2$$

whose fiber over each point of B is the pair of fixed points $\{p, q\}$. See Figure B.1.

We now ask: Given an integer d, are there sheaves $\mathcal{G}, \mathcal{G}'$ on B whose fibers at a point $r \in B$ are the spaces of forms of degree d vanishing on Γ and Γ', respectively, and a map $\mathcal{G} \to \mathcal{G}'$ inducing the obvious inclusion of spaces of forms?

To see that this question fits into the former context, let $\mathcal{F} \subset \mathcal{F}'$ be the ideal sheaves of Γ and Γ' in $B \times \mathbb{P}^2$. Abusing notation slightly, we write $\mathcal{O}_{\mathbb{P}^2}(d)$ for the pullback to $B \times \mathbb{P}^2$ of $\mathcal{O}_{\mathbb{P}^2}(d)$ on $\mathbb{P}^2$ and $\mathcal{F}(d)$ for $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^2}(d)$. Thus

$$\mathcal{F}(d)|_{\pi^{-1}(r)} = \mathcal{I}_{\{p,q,r\}}(d),$$

the d-th twist of the ideal sheaf of $\{p, q, r\}$ in $\mathbb{P}^2$, and similarly for $\mathcal{F}'(d)$.

The answer is that such sheaves exist when $d \neq 1$ (and can, by Theorem B.5, be taken to be $\pi_*\mathcal{F}$ and $\pi_*\mathcal{F}'$, defined below). But no such $\mathcal{G} \to \mathcal{G}'$ exists when $d = 1$!

By our computation of $H^0(\mathcal{I}_{\{p,q,r\}}(1))$ above, the sheaf $\mathcal{G}$ would be a skyscraper sheaf concentrated at the unique point r_0 on the intersection of the line B with the line $\overline{p, q}$. Furthermore, the map $H^0(\mathcal{I}_{\{p,q,r_0\}}(1)) \to H^0(\mathcal{I}_{\{p,q\}}(1))$ is an isomorphism, so functoriality would imply that the map on fibers $\mathcal{G}_{r_0} \to \mathcal{G}'_{r_0}$ would be an isomorphism.

On the other hand, the fibers of $\mathcal{G}'$ would all be equal to the one-dimensional vector space $H^0(\mathcal{I}_{\{p,q\}}(1))$, so $\mathcal{G}'$ would be a line bundle. Since the only map from a skyscraper sheaf to a line bundle is 0, this is a contradiction, showing that the desired functorial construction is impossible! As we shall see in Example B.10, $\pi_*\mathcal{F} = 0$, and thus the fiber $(\pi_*\mathcal{F})_{r_0}$ is not equal to $H^0(\mathcal{F}|_{\pi^{-1}(r_0)})$.

B.2 Direct images

We remind the reader of our global convention that “sheaf” means coherent sheaf unless otherwise stated.

It becomes clear how to define $\pi_*\mathcal{F}$ if we add another natural condition to our desiderata: in cases where the fibers of $\pi_*\mathcal{F}$ are the spaces $H^0(\mathcal{F}|_{X_b})$, we would like algebraic families of elements of $H^0(\mathcal{F}|_{X_b})$ to give rise to sections of $\mathcal{F}$ in a way that is compatible with the identification of $\mathcal{G}_b$ with $H^0(\mathcal{F}|_{X_b})$. Here we interpret the phrase “algebraic family of elements” to mean “section of $\mathcal{G}$ defined over the preimage of an open set of B.”

Definition B.2. Given a morphism of schemes $\pi : X \to B$ and a sheaf $\mathcal{F}$ on X, we define the *direct image* $\pi_*\mathcal{F}$ of $\mathcal{F}$ to be the quasi-coherent sheaf on B that assigns to each open subset $U \subset B$ the space of sections of $\mathcal{F}$ on the open set $\pi^{-1}(U)$, that is,

$$(\pi_*\mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U)).$$

It is immediate that $\pi_*(\mathcal{F})$ is a sheaf if $\mathcal{F}$ is, and that the construction is functorial in $\mathcal{F}$.

This definition is particularly natural if we think of a presheaf $\mathcal{F}$ as a contravariant functor

$$\mathcal{U} \to (\text{Sets}), \quad U \mapsto \mathcal{F}(U),$$

from the category $\mathcal{U}$ whose objects are open sets of X and whose morphisms are inclusions to the category of sets. Since π induces a covariant functor $\pi^{-1} : V \mapsto \pi^{-1}(V)$ from the category of open sets of B to that of X, we may simply compose to get the presheaf

$$\pi_*\mathcal{F} = \mathcal{F} \circ \pi^{-1}.$$

Note that by definition $H^0(\pi_*\mathcal{F}) = (\pi_*\mathcal{F})(B) = \mathcal{F}(X) = H^0(\mathcal{F})$.

When X and B are affine varieties, or more generally when the morphism π is affine (for example when π is a finite map), the sheaf $\pi_*\mathcal{F}$ is easy to understand: Giving $\mathcal{F}$ is equivalent to giving an $\mathcal{O}_X$-module, and it follows at once from the definitions that $\pi_*\mathcal{F}$ corresponds to the *same* module, viewed as a $\mathcal{O}_B$-module via the map of (sheaves of) rings $\pi^* : \mathcal{O}_B \to \mathcal{O}_X$. In particular we see that even when $\mathcal{F}$ is coherent $\pi_*\mathcal{F}$ may be only quasi-coherent.

The situation is very different when π is a projective morphism. A fundamental result of Serre (Theorem B.8) shows that when $\mathcal{F}$ is a coherent sheaf and π is a projective morphism $\pi_*\mathcal{F}$ is coherent. (Recall that a projective morphism is one that factors as the inclusion of X as a closed subset of some $B \times \mathbb{P}^n$ and the projection to B; since any morphism can be factored through its graph, any morphism of projective varieties is a projective morphism.) With a little more effort, the results can all be extended to proper morphisms; see Grothendieck [1963, Theorem 3.2.2.1].

The theorems on cohomology and base change give information not only about fibers of the direct image sheaf, but about more general pullbacks ("base changes") as well; we pause to put things into this more general context.

We may think about the fiber X_b of π over a point b as coming from a pullback, or *base change* diagram

$$\begin{tikzcd} X_b \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ \{b\} \arrow[r, "\rho"] & B \end{tikzcd}$$

The restriction of $\mathcal{F}$ to X_b can also be thought of as $\rho'^*(\mathcal{F})$, and from the definition of π'_* we see that $\pi'_*(\mathcal{F}|_{X_b}) = H^0(\mathcal{F}|_{X_b})$. Thus the theorem on cohomology and base change is about the comparison of $\pi'_*(\rho'^*\mathcal{F})$ and $\rho^*(\pi_*\mathcal{F})$.

More generally, for any map $\rho : B' \to B$ we can consider the pullback of the family $X \to B$ to B':

$$\begin{tikzcd} X' = X \times_B B' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

We say that the map $\pi' : X' \to B'$ and sheaf $\mathcal{F}' = \rho'^*\mathcal{F}$ are obtained from the map $\pi : X \to B$ and sheaf $\mathcal{F}$ by *base change*. In this situation, there is a natural map

$$\varphi_{B'} : \rho^*(\pi_*\mathcal{F}) \to \pi'_*(\rho'^*\mathcal{F}),$$

constructed as follows:

Applying the definitions, we see that the module of sections of $\rho^*(\pi_*\mathcal{F})$ over an open set $U' \subset B'$ is the direct limit over all open sets $U \subset B$ such that $\rho^{-1}(U) \supset U'$ of the $\mathcal{O}_{B'}$-modules

$$A_U := \mathcal{O}_{U'} \otimes \mathcal{O}_B\big(\mathcal{F}(\pi^{-1}(U))\big).$$

On the other hand, the module of sections of $\pi'_*(\rho'^*\mathcal{F})$ over U' is

$$\rho'^*(\mathcal{F})(\pi'^{-1}(U')),$$

which is the limit over all open subsets $V \subset X$ such that $\rho^{-1}(V) \supset \pi'^{-1}(U')$ of the $\mathcal{O}_{X'}$-modules

$$B_V := \mathcal{O}_{\pi^{-1}(U')} \otimes_{\mathcal{O}_X} \mathcal{F}(V).$$

Each open set U entering into the former limit gives rise to a $V = \pi^{-1}(U)$ that enters into the latter limit, and since $\mathcal{O}_{\pi^{-1}(U')}$ is a module over $\mathcal{O}_{U'}$ there is an induced map $A_U \to B_{\pi^{-1}(U)}$. The map

$$\varphi_{B'} : \rho^*(\pi_*\mathcal{F}) \to \pi'_*(\rho'^*\mathcal{F})$$

is the natural map induced between the limits.

When $B' = \{b\}$ is a closed point of B we write φ_b instead of $\varphi_{\{b\}}$. In this case, $\rho^*(\pi_*\mathcal{F})$ is the fiber of $\pi_*\mathcal{F}$ at b and $\pi'_*(\rho'^*\mathcal{F})$ is the set of global sections of $\mathcal{F}$ on the fiber X_b, and we see that *the image of φ_b is simply the set of global sections of $\mathcal{F}|_{X_b}$ that extend to an open neighborhood of X_b in X*. On the other hand, the kernel consists of sections defined in some small neighborhood V of $\pi^{-1}(b)$ that vanish on $\pi^{-1}(b)$, but cannot be expressed in terms of functions pulled back from any small neighborhood of b and vanishing at b.

In general, we ask when the natural maps $\varphi_{B'}$ are isomorphisms; in other words, *when does the formation of the direct image of $\mathcal{F}$ commute with base change?*

There are two cases that are easy:

(1) If π is affine, then the maps $\varphi_{B'}$ are isomorphisms for any quasi-coherent sheaf $\mathcal{F}$. (This follows at once from the description of π_* for affine morphisms given above.)
(2) If $\rho : B' \to B$ is a flat map, then the maps $\varphi_{B'}$ are isomorphisms. (This is an immediate consequence of Theorem B.11. See Hartshorne [1977, Proposition III.9.3] for a direct proof.) Since the inclusion of a point is not generally flat, this case is not, in practice, very useful.

The simplest case of a projective morphism is that of a finite morphism. The following proposition summarizes the situation in that case:

Proposition B.3. *If $\pi : X \to B$ is a finite morphism of quasi-projective varieties and $\mathcal{F}$ is a coherent sheaf on X, then $\pi_*\mathcal{F}$ is a coherent sheaf on B and the maps $\varphi : (\pi_*\mathcal{F})_b \to H^0(\mathcal{F}|_{X_b})$ are isomorphisms for all closed points $b \in B$. Moreover, the following are equivalent:*

(a) $\pi_*\mathcal{F}$ is a vector bundle on B.
(b) $\mathcal{F}$ is flat over B.
(c) The dimension of $H^0(\mathcal{F}|_{X_b})$ as a vector space over the residue class field $\kappa(b)$ is independent of the closed point $b \in B$.

Proof: Since sheaves are defined locally and the preimage of an affine subset under a finite map is again affine, we may suppose from the outset that both X and B are affine. Let $X = \operatorname{Spec} R$ and $B = \operatorname{Spec} S$, and let $M = H^0(\mathcal{F})$ be the R-module corresponding to $\mathcal{F}$. Because the varieties are affine, $\pi_*\mathcal{F}$ is represented by M regarded as a module over S via the map $\pi^* : S \to R$. The ring R is by hypothesis a finitely generated S-module, so M is a finitely generated S-module as well. The maps φ_b are isomorphisms because, writing $\mathfrak{m}_b \subset S$ for the maximal ideal corresponding to b, both $(\pi_*\mathcal{F})|_b$ and $H^0(\mathcal{F}|_{X_b})$ may be identified canonically with $M/\mathfrak{m}_b M$. (The same proof shows that the map $\varphi_{B'}$ is an isomorphism for any closed set B'.) The equivalence of parts (a) and (c) is proven in Proposition B.15 below. The equivalence of (a) and (b) is Eisenbud [1995, Exercise 6.2]. □

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Example B.4 (Direct images of $\mathcal{O}_{\Gamma'}(d)$ and $\mathcal{O}_{\Gamma}(d)$)**.** Returning to the situation of Example B.1, note that the families Γ' and Γ are finite over B. For every p, q, r and every d, the space of global sections of $\mathcal{O}_{\{p,q\}}(d)$ is two-dimensional and that of $\mathcal{O}_{\{p,q,r\}}(d)$ is three-dimensional. From Proposition B.3 we see that $\pi_*\mathcal{O}_{\Gamma'}(d)$ and $\pi_*\mathcal{O}_{\Gamma}(d)$ are vector bundles of ranks 2 and 3, respectively. The inclusion $\Gamma' \subset \Gamma$ induces an inclusion of these bundles. Since we can choose a fixed basis of functions on the fibers of Γ', and the pullback of $\mathcal{O}_B(d)$ to Γ' is the trivial bundle, the bundle $\pi_*\mathcal{O}_{\Gamma'}(d)$ is the trivial bundle $\mathcal{O}_B^2$.

We have $\mathcal{O}_{\Gamma}(d) = \mathcal{O}_{\Gamma'}(d) \oplus \mathcal{O}_{\Delta}(d)$, where $\Delta = \{(r, r) \in B \times \mathbb{P}^2\}$, as before, and projects isomorphically to each factor. Thus $\mathcal{O}_{\Delta}(d)$, the pullback of $\mathcal{O}_B(d)$ from the first factor, pushes forward to $\mathcal{O}_B(d)$ on the second factor for every d, and $\pi_*\mathcal{O}_{\Gamma}(d) \cong \mathcal{O}_B^2 \oplus \mathcal{O}_B(d)$ is a nontrivial bundle.

For more general projective morphisms neither condition (b) nor condition (c) of Proposition B.3 alone will imply that $\pi_*\mathcal{F}$ is a vector bundle or that the maps φ_b are isomorphisms. But conditions (b) and (c) together do imply both of these conclusions. This result is the most often used special case of the theorem on cohomology and base change (Theorem B.9):

Theorem B.5 (Cohomology and base change, version 1)**.** *Let $\pi : X \to B$ be a projective morphism of varieties and let $\mathcal{F}$ be a coherent sheaf on X that is flat over B. If the dimension of $H^0(\mathcal{F}|_{\pi^{-1}(b)})$ is independent of the closed point $b \in B$, then $\pi_*\mathcal{F}$ is a vector bundle of rank equal to $h^0(\mathcal{F}|_{\pi^{-1}(b)})$, and the comparison map*

$$\varphi_b : (\pi_*\mathcal{F})_b \to H^0(\mathcal{F}|_{\pi^{-1}(b)})$$

is an isomorphism for every closed point $b \in B$. More generally, if B' is any scheme, $\rho : B' \to B$ is a morphism, and

$$\begin{tikzcd} X' = X \times_B B' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

is a pullback diagram, then the natural map

$$\varphi_{B'} : \rho^*\pi_*\mathcal{F} \to \pi'_*\rho'^*\mathcal{F}$$

is an isomorphism.

Theorem B.5 is subsumed by Theorem B.9, which will be proven in Section B.5.

We note that although we allow B' to be an arbitrary scheme it is necessary for the formulation above to assume that B is a variety (being reduced would be enough).

We will make use of the following natural "adjunction" maps, defined for any morphism $\pi : X \to B$ of schemes as follows:

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(a) If $\mathcal{F}$ is a quasi-coherent sheaf on X, there is a natural map

$$\epsilon_{\mathcal{F}} : \pi^*\pi_*\mathcal{F} \to \mathcal{F},$$

defined by the condition that, on the preimage of an open set $U \subset B$, $\epsilon_{\mathcal{F}}$ is the map

$$\mathcal{O}_X(\pi^{-1}(U)) \otimes_{\mathcal{O}_Y(U)} \mathcal{F}(\pi^{-1}(U)) \to \mathcal{F}(\pi^{-1}(U))$$

sending each section $1 \otimes \sigma$ to σ. This is well-defined because $\mathcal{F}(\pi^{-1}(U))$ is an $\mathcal{O}_X(\pi^{-1}(U))$-module.

(b) Given a quasi-coherent sheaf $\mathcal{G}$ on B, there is a natural map

$$\eta_{\mathcal{G}} : \mathcal{G} \to \pi_*\pi^*\mathcal{G},$$

defined by the condition that, on any open set $U \subset B$, $\eta_{\mathcal{G}}$ is the map

$$\begin{aligned} \mathcal{G}(U) = \mathcal{O}_B(U) \otimes_{\mathcal{O}_B(U)} \mathcal{G}(U) \xrightarrow{\pi^*\otimes 1} &\mathcal{O}_X(\pi^{-1}(U)) \otimes_{\mathcal{O}_B(U)} \mathcal{G}(U) \\ &= \pi^*(\mathcal{G})(\pi^{-1}(U)) \\ &= (\pi_*\pi^*\mathcal{G})(U). \end{aligned}$$

For example, $\pi^*\mathcal{O}_B = \mathcal{O}_X$, so $\eta_{\mathcal{O}_B}$ is a map $\mathcal{O}_B \to \pi_*\mathcal{O}_X$.

As an application that does not mention base change, we derive a result about line bundles that we used in studying projective bundles (Chapter 9):

Corollary B.6. *Suppose $\pi : X \to B$ is a flat, projective morphism and that all the fibers of π are reduced and connected.*

(a) $\eta_{\mathcal{O}_B} : \mathcal{O}_B \to \pi_*\mathcal{O}_X$ is an isomorphism.
(b) If $\mathcal{L}, \mathcal{L}'$ are line bundles on X, then $\mathcal{L}|_{\pi^{-1}(b)} \cong \mathcal{L}'|_{\pi^{-1}(b)}$ for all $b \in B$ if and only if $\mathcal{L} \cong (\pi^*\mathcal{M}) \otimes \mathcal{L}'$ for some line bundle $\mathcal{M}$ on B, that is, if $\mathcal{L}$ and $\mathcal{L}'$ differ by tensoring with a line bundle pulled back from B.

Remark. The result fails without flatness, for example in the case when π is the embedding of a proper closed subscheme of B.

Proof: To say that X is flat over B means that $\mathcal{O}_X$ is flat over B. Since flatness is a local property, this implies that any line bundle on X is flat over B, so we may apply Theorem B.9 to line bundles on X.

(a) The map $\eta_{\mathcal{O}_B}$ takes the global section 1 of $\mathcal{O}_B$ to the global section 1 in $\pi_*\mathcal{O}_X$. Because X is flat, $1 \in \pi_*\mathcal{O}_X$ is not annihilated by any nonzero (local) section of $\mathcal{O}_B$, so $\eta_{\mathcal{O}_B}$ is injective.

Since $\pi^{-1}(b)$ is a reduced, connected projective variety for every $b \in B$, the vector space $H^0(\mathcal{O}_X|_{\pi^{-1}(b)}) = H^0(\mathcal{O}_{\pi^{-1}(b)})$ is one-dimensional. With this, Theorem B.5 shows that $\pi_*\mathcal{O}_X$ is a line bundle with fiber $H^0(\mathcal{O}_{\pi^{-1}(b)})$. It follows that the restriction of $\eta_{\mathcal{O}_B}$ to any fiber is surjective. By Nakayama's lemma, $\eta_{\mathcal{O}_B}$ itself is surjective.

(b) First, suppose that $\mathcal{L}' = \mathcal{L} \otimes \pi^*\mathcal{M}$ for some line bundle $\mathcal{M}$ on B. We have

$$\mathcal{L}'|_{\pi^{-1}(b)} = (\mathcal{L} \otimes \pi^*\mathcal{M})|_{\pi^{-1}(b)} = \mathcal{M}_b \otimes \mathcal{L}|_{\pi^{-1}(b)} \cong \mathcal{L}|_{\pi^{-1}(b)}.$$

For the converse, given $\mathcal{L}$ and $\mathcal{L}'$, we may multiply both by $\mathcal{L}'^{-1}$, and thus reduce to the case where $\mathcal{L}' = \mathcal{O}_X$, so that $\mathcal{L}|_{\pi^{-1}(b)}$ is trivial for each $b \in B$. Our hypothesis then implies that $H^0(\mathcal{L}|_{\pi^{-1}(b)})$ is one-dimensional for every $b \in B$, so by Theorem B.9 $\pi_*\mathcal{L}$ is a line bundle.

We will complete the proof by showing that $\epsilon_{\mathcal{L}} : \pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Since both the source and target are line bundles, it suffices to show that $\epsilon_{\mathcal{L}}$ is surjective, and for this we may by Nakayama's lemma restrict to a fiber. By Theorem B.5, the fiber of $\pi_*\mathcal{L}$ at a point b is $H^0(\mathcal{L}|_{\pi^{-1}(b)})$, so $(\pi^*\pi_*\mathcal{L})|_{\pi^{-1}(b)}$ is the trivial line bundle of rank 1, generated by $1 \otimes \sigma$ for any nonzero global section σ of $\mathcal{L}|_{\pi^{-1}(b)}$. Since $\epsilon_{\mathcal{L}}$ sends $1 \otimes \sigma$ to σ, and $\mathcal{L}|_{\pi^{-1}(b)}$ is a trivial line bundle, we see that $\epsilon_{\mathcal{L}}$ restricts to an isomorphism on each fiber, as required. □

B.3 Higher direct images

Let $\pi : X \to B$ be a morphism and let $\mathcal{F}$ be a sheaf on X. The direct image functor $\mathcal{F} \mapsto \pi_*\mathcal{F}$ is a generalization of the functor sending a sheaf to its vector space of global sections. The *higher direct image functors* $\mathcal{F} \mapsto R^i\pi_*\mathcal{F}$ have the same relation to the higher cohomology, and may be defined as right derived functors of π_* or as the sheafification of the presheaf $U \mapsto H^i(\mathcal{F}|_{\pi^{-1}(U)})$ (in the case $i = 0$ the sheafification is unnecessary). For this see Hartshorne [1977, Section III.8]. Here we will take a more concrete approach, defining the right derived functors via the Čech complex and for simplicity sticking to the case of a projective morphism.

Just as the higher cohomology of a sheaf can be used to derive information about global sections, the higher direct image sheaves shed light on the direct image itself. There are other applications as well. For example, in Chapter 14 we used higher direct images to study jumping lines.

We will deal with both the cohomology of sheaves and the homology of complexes. To limit confusion, we will use H^i to denote the i-th Čech cohomology functor applied to a sheaf, while H^i will denote the i-th homology of a complex.

Suppose that $\pi : X \to B$ is a projective morphism; that is, π factors as a closed immersion $X \subset B \times \mathbb{P}^n$ and the projection $B \times \mathbb{P}^n \to B$. If $\mathcal{F}$ is a sheaf on X, we may regard $\mathcal{F}$ as a sheaf on $\mathbb{P} := B \times \mathbb{P}^n$. We write $\mathbb{P} = \mathrm{Proj}(S)$, where S is the sheaf of graded algebras $\mathcal{O}_B[x_0, \dots, x_n]$, and thus $\mathcal{F}$ is the sheafification of a sheaf of graded S-modules.

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Let $U_i \subset \mathbb{P}$ be the open subscheme $x_i \neq 0$, and let

$$\mathcal{C}^\bullet : \quad \bigoplus_i \mathcal{O}_{\mathbb{P}}|_{U_i} \longrightarrow \bigoplus_{i,j} \mathcal{O}_{\mathbb{P}}|_{U_i \cap U_j} \longrightarrow \cdots$$

be the Čech complex on $\mathbb{P}$. Note that each term $\mathcal{C}^i$ is $\mathbb{Z}$-graded. If $\mathcal{F}$ is a quasi-coherent sheaf on X we define $R^i\pi_*\mathcal{F}$ to be the degree-0 part of the homology of the complex $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet$ at the i-th term, that is,

$$R^i\pi_*\mathcal{F} := (\mathrm{H}^i(\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{C}^\bullet))_0.$$

What makes this somewhat technical definition useful is that

$$(\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}|_{U_i})_0$$

is the sheaf of modules over

$$\mathcal{O}_B[x_0, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n]$$

corresponding to the restriction of the sheaf $\mathcal{F}$ to the open set $U_i = B \times \mathbb{A}^n$. This and the assumption that $\mathcal{F}$ is a sheaf show that $R^0\pi_*\mathcal{F} = \pi_*\mathcal{F}$. Also, when $U \subset B$ is affine, so that each $U_i \cap \pi^{-1}(U)$ is affine, we get

$$R^i\pi_*\mathcal{F}|_{\pi^{-1}(U)} = H^i(\mathcal{F}|_{\pi^{-1}(U)}).$$

Since any sheaf is determined by its restriction to affine open subsets, this property (together with the restriction morphisms) characterizes $R^i\pi_*\mathcal{F}$ and shows that the definition is independent of the embedding $X \subset B \times \mathbb{P}^n$ that we chose. If $b \in B$ is a point or a subvariety, then

$$H^i(\mathcal{F}_b) = \mathrm{H}^i(\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet).$$

This is generally not equal to the fiber

$$(R^i\pi_*\mathcal{F})_b = \kappa(b) \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet)$$

of the higher direct image. However, if z is a cycle or boundary in $\mathcal{F} \otimes \mathcal{C}^\bullet$, then $1 \otimes z$ is a cycle or boundary in $\kappa(b) \otimes \mathcal{F} \otimes \mathcal{C}^\bullet$, so we get maps $R^i\pi_*\mathcal{F} \to H^i(\mathcal{F}|_{X_b})$ that in turn induce comparison maps

$$\varphi_b^i : (R^i\pi_*\mathcal{F})_b \to H^i(\mathcal{F}|_{X_b}).$$

In the previous section we asked when the groups $H^i(\mathcal{F}|_{X_b})$ are the fibers of a sheaf, and when this sheaf is a vector bundle. Again, we will give sufficient conditions for these things to be the case by giving conditions for the maps φ_b^i to be isomorphisms and for $R^i\pi_*\mathcal{F}$ to be a vector bundle.

We start with some properties of the sheaves $R^i\pi_*\mathcal{F}$:

Proposition B.7. *Let $\pi : X \to B$ be a projective morphism.*

(a) (Restriction to open sets) Let $U \subset B$ be an open subset, and let $\pi' : \pi^{-1}(U) \to U$ be the restriction of π. If $\mathcal{F}$ is any quasi-coherent sheaf on X, then $(R^i\pi_*\mathcal{F})|_U = R^i\pi'_*(\mathcal{F}|_{\pi^{-1}(U)})$.

(b) (Long exact sequence) The functor π_* is left exact, and the functors $R^i\pi_*$ are the right derived functors of π_*. In particular, if

$$\epsilon : 0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of quasi-coherent sheaves on X, then there are natural "connecting homomorphisms" η_i making the sequence

$$\cdots \longrightarrow R^i\pi_*\mathcal{F} \xrightarrow{R^i\pi_*\alpha} R^i\pi_*\mathcal{G} \xrightarrow{R^i\pi_*\beta} R^i\pi_*\mathcal{H} \xrightarrow{\eta_i} R^{i+1}\pi_*\mathcal{F} \longrightarrow \cdots$$

exact.

(c) (Push-pull formula) If $\mathcal{E}$ is a vector bundle on B and $\mathcal{F}$ is a quasi-coherent sheaf on X, then

$$R^i\pi_*(\pi^*\mathcal{E} \otimes \mathcal{F}) \cong \mathcal{E} \otimes R^i\pi_*\mathcal{F}.$$

Proof: (a) Since $\mathcal{O}_U$ is flat over $\mathcal{O}_B$, the restriction to U commutes with taking homology.

(b) The terms of the complex $\mathcal{C}^\bullet$ are flat over $\mathcal{O}_X$, so when we tensor $\mathcal{C}^\bullet$ with the short exact sequence ϵ we get a short exact sequence of complexes, and thus a long exact sequence of homology sheaves. Taking the degree-0 part preserves exactness. Since the long exact sequence begins with

$$0 \longrightarrow \pi_*\mathcal{F} \longrightarrow \pi_*\mathcal{G} \longrightarrow \cdots,$$

we see that π_* is left exact.

To show that $R^i\pi_*$ is the i-th right derived functor of π_*, it now suffices to show that $R^i\pi_*\mathcal{F} = 0$ when $\mathcal{F}$ is injective (Eisenbud [1995, Appendix A3.9]), or more generally flasque. It suffices to prove this after restricting to an affine subset $U \subset B$. Since the restriction of a flasque sheaf to an open subset is flasque, the result follows from the corresponding result for cohomology.

(c) The sheaf $\pi^*\mathcal{E}$ is also a vector bundle, and thus flat, so tensoring with $\pi^*\mathcal{E}$ commutes with taking homology:

$$\mathrm{H}^i(\pi^*\mathcal{E} \otimes \mathcal{F} \otimes \mathcal{C}^\bullet) = \pi^*\mathcal{E} \otimes \mathrm{H}^i(\mathcal{F} \otimes \mathcal{C}^\bullet).$$

Taking the degree-0 part yields the desired formula. □

Theorem B.8 (Serre's coherence theorem)**.** *If $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X, then $R^i\pi_*\mathcal{F}$ is coherent for each i.*

The proof involves some useful ideas from homological commutative algebra.

Proof: Since the formation of $R^i\pi_*\mathcal{F}$ commutes with the restriction to an open set in the base, it suffices to treat the case where $B = \operatorname{Spec} A$ is affine. The Čech complex $\mathcal{C}^\bullet$ is the direct limit of the duals of the Koszul complexes that are S-free resolutions of the ideals $(x_0^m, \ldots, x_n^m) \subset S$, and direct limits commute with taking homology. Thus if M is any finitely generated graded $S = A[x_0, \ldots, x_n]$-module representing the sheaf $\mathcal{F}$, the homology of $\mathcal{F} \otimes \mathcal{C}^\bullet$ is

$$R^i\pi_*\mathcal{F} = \lim_m \operatorname{Ext}^i_S((x_0^m, \ldots, x_n^m), M)_0.$$

Write $\mathfrak{m}$ for the "irrelevant" ideal $(x_0, \ldots, x_n) \subset S$. For each m there is an integer $N(m)$ such that $\mathfrak{m}^{N(m)} \subset (x_0^m, \ldots, x_n^m) \subset \mathfrak{m}^m$. It follows that

$$\lim_m \operatorname{Ext}^i_S((x_0^m, \ldots, x_n^m), M) = \lim_m \operatorname{Ext}^i_S(\mathfrak{m}^m, M).$$

Each term $\operatorname{Ext}^i_S(\mathfrak{m}^m, M)$ of this limit is a finitely generated S-module, so its degree-0 part is a finitely generated A-module, and it suffices to show that the natural map

$$\operatorname{Ext}^i_S(\mathfrak{m}^m, M)_0 \to \operatorname{Ext}^i_S(\mathfrak{m}^{m+1}, M)_0$$

is an isomorphism for large m. From the long exact sequence in Ext_S, and the fact that $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ is a direct sum of copies of $A(-m)$ (the free A-module of rank 1 with generator in degree m), we see that it is enough to prove that $\operatorname{Ext}^i_S(A(-m), M)_0 = 0$ when m is large. Disentangling the degree shifts, we see that

$$\operatorname{Ext}^i_S(A(-m), M)_0 = \operatorname{Ext}^i_S(A, M)(m)_0 = \operatorname{Ext}^i_S(A, M)_m.$$

However, A is annihilated (as an S-module) by $\mathfrak{m}$, so $\operatorname{Ext}^i_S(A, M)$ is annihilated by $\mathfrak{m}$. Since it is a finitely generated S-module, it can only be nonzero in finitely many degrees, whence, indeed, $\operatorname{Ext}^i_S(A, M)_m = 0$ when m is large. $\square$

We remark that it is possible, using the notion of Castelnuovo–Mumford regularity, to bound the degree m for which $\operatorname{Ext}^i_S(A, M)_m = 0$ in terms of the data in a free resolution of M (similar to Smith [2000]), so the proof just given allows effective computation of the functors $R^i\pi_*\mathcal{F}$. For a different proof see Hartshorne [1977, Theorem III.8.8].

Theorem B.9 (Cohomology and base change, version 2)**.** *Let $\pi : X \to B$ be a projective morphism of schemes, with B connected and quasi-projective, and let $\mathcal{F}$ be a coherent sheaf on X.*

(a) If B is reduced then there is a dense open set $U \subset B$ such that $R^i\pi_*\mathcal{F}|_U$ is a vector bundle, and such that for all closed points $b \in U$ the fiber $(R^i\pi_*\mathcal{F}|_U)_b$ is equal to $H^i(\mathcal{F}|_{X_b})$.

(b) Suppose that $\mathcal{F}$ is flat over B. Let i be an integer. If $H^j(\mathcal{F}|_{X_b}) = 0$ for all $j > i$ and all closed points $b \in B$, then for every closed point $b \in B$ the comparison map

$$\varphi^i_b : (R^i\pi_*\mathcal{F})_b \to H^i(\mathcal{F}|_{X_b})$$

is an isomorphism.

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(c) *Suppose that $\mathcal{F}$ is flat over B and B is reduced. If for some i the function $b \mapsto \dim_{\kappa(b)} H^i(\mathcal{F}|_{X_b})$ is constant, then $R^i\pi_*\mathcal{F}$ is a vector bundle of rank equal to $\dim_{\kappa(b)} H^i(\mathcal{F}|_{X_b})$, and for every closed point $b \in B$ the comparison maps*

$$\varphi_b^i : (R^i\pi_*\mathcal{F})_b \to H^i(\mathcal{F}|_{X_b}),$$
$$\varphi_b^{i-1} : (R^{i-1}\pi_*\mathcal{F})_b \to H^{i-1}(\mathcal{F}|_{X_b})$$

are isomorphisms. More generally, if B' is any scheme, $\rho : B' \to B$ is a morphism, and

$$\begin{tikzcd} X' = X \times_B B' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

is a pullback diagram, then the natural map

$$\varphi_{B'}^i : \rho^* R^i\pi_*\mathcal{F} \to R^i\pi'_*\rho'^*\mathcal{F}$$

is an isomorphism.

Theorem B.9 will be proven in Section B.5.

Example B.10 (Continuation of Example B.1)**.** Using Theorem B.9 we can easily compute the sheaves $R^i\pi_*\mathcal{I}_\Gamma(d)$ and $R^i\pi_*\mathcal{I}_{\Gamma'}(d)$ introduced in Example B.1. The sheaves $\mathcal{I}_\Gamma$ and $\mathcal{I}_{\Gamma'}$ are flat over $B \cong \mathbb{P}^1$ because they have no torsion (see for example Eisenbud and Harris [2000, Theorem II-29]). The functions $b \mapsto h^i(\mathcal{I}_{\Gamma'_b}(d))$ are constant for all d, since Γ'_b is itself constant. The computation discussed in Example B.1 shows that the same is true for $\mathcal{I}_{\Gamma_b}(d)$ as long as $d \geq 2$. It follows that for all d

$$R^0\mathcal{I}_{\Gamma'}(d) = \mathcal{O}_B^{\binom{d+2}{2}-2},$$
$$R^1\mathcal{I}_{\Gamma'}(d) = 0,$$

and for $d \geq 2$

$$R^0\mathcal{I}_{\Gamma}(d) = \mathcal{O}_B^{\binom{d+2}{2}-3},$$
$$R^1\mathcal{I}_{\Gamma}(d) = 0.$$

On the other hand, if $d = 1$, we will prove that

$$R^0\mathcal{I}_\Gamma(d) = 0,$$
$$R^1\mathcal{I}_\Gamma(d) = \mathcal{O}_{r_0}.$$

To this end we first apply Theorem B.9, which shows that $R^i\pi_*\mathcal{O}_{B\times\mathbb{P}^2}(1)$ is a vector bundle for each i and its fiber over $r \in B$ is $H^i\mathcal{O}_{\mathbb{P}^2}(1)$, while $R^0\pi_*\mathcal{O}_\Gamma(1)$ is a vector bundle of rank 3 whose fiber over r is the set of functions on $\{p, q, r\}$. We now use Proposition B.7(b) to obtain the exact sequence

$$0 \longrightarrow R^0\mathcal{I}_\Gamma(1) \longrightarrow R^0\pi_*\mathcal{O}_{B\times\mathbb{P}^2}(1) \xrightarrow{e} R^0\pi_*\mathcal{O}_\Gamma(1) \longrightarrow R^1\mathcal{I}_\Gamma(1) \longrightarrow 0.$$

The map labeled e, restricted to the fiber over r, is (after choosing coordinates) the map $\Bbbk^3 \to \Bbbk^3$ sending each linear form on $\mathbb{P}^2$ to the vector of its values at the (homogeneous coordinates of the) three points p, q, r. When the points are non-collinear, this map is an isomorphism. A map of vector bundles that is generically an isomorphism is a monomorphism of sheaves, so $R^0\mathcal{I}_\Gamma(1) = \operatorname{Ker} e = 0$. The unique fiber where the rank of e drops is r_0, and there the rank of e is 2. At the fiber over r_0 the image of the map is two-dimensional. It follows that $R^1\mathcal{I}_\Gamma(1) = \operatorname{coker} e$ is the skyscraper sheaf of length 1 concentrated at the point r_0, as claimed.

B.4 The direct image complex

We now turn to the most general and simplest version of the theorem on base change and cohomology. To simplify the notation, we will identify quasi-coherent sheaves over an affine scheme $B = \operatorname{Spec} A$ with their modules of global sections.

Theorem B.11 (Cohomology and base change, version 3)**.** *Let $\pi : X \to B$ be a projective morphism to an affine scheme $B = \operatorname{Spec} A$, and let $\mathcal{F}$ be a sheaf on X that is flat over B. Suppose that the maximum dimension of a fiber of π is n. There is a complex*

$$\mathcal{P}^\bullet : \cdots \longrightarrow P^0 \longrightarrow \cdots \longrightarrow P^n \longrightarrow 0$$

of finitely generated projective A-modules such that:

(a) $R^i\pi_*(\mathcal{F}) \cong \mathrm{H}^i(\mathcal{P}^\bullet)$ for all i.
(b) For every $b \in B$ and $i \in \mathbb{Z}$ there is an isomorphism

$$H^i(\mathcal{F}|_{X_b}) \cong \mathrm{H}^i(\kappa(b) \otimes_A \mathcal{P}^\bullet).$$

More generally, for every pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\rho'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ B' \arrow[r, "\rho"] & B \end{tikzcd}$$

with $B' = \operatorname{Spec} A'$,

$$R^i\pi'_*\rho'^*(\mathcal{P}^\bullet) \cong \mathrm{H}^i(A' \otimes_A \mathcal{P}^\bullet).$$

Moreover, if B is reduced, then we may choose $\mathcal{P}^\bullet$ so that $P^i = 0$ for $i < 0$.

Theorem B.11 is proven at the end of this section. A version where the base B is any quasi-projective scheme is given in Exercise B.18.

The second statement of (b) could be improved to say that $\rho^* R\pi_*\mathcal{F}$ is quasi-isomorphic (or, equivalently, isomorphic in the derived category) to $R\pi_*\rho'^*\mathcal{F}$.

https://doi.org/10.1017/CBO9781139062046.018 Published online by Cambridge University Press

The complex $\mathcal{P}^\bullet$ in Theorem B.11 is not unique up to isomorphism, but it is unique up to the equivalence relation, called *quasi-isomorphism*, generated by maps of complexes that induce isomorphisms on homology (for right-bounded projective complexes, as in our case, this comes down to homotopy equivalence). The class of $\mathcal{P}^\bullet$ modulo quasi-isomorphism is called the *direct image complex of* $\mathcal{F}$, written $R\pi_*\mathcal{F}$, which is usually treated as an element of the derived category of (right-bounded) complexes of coherent sheaves on B. We say that $\mathcal{P}^\bullet$ *represents* $R\pi_*\mathcal{F}$.

Abstract as Theorem B.11 may seem, the construction of $R\pi_*\mathcal{F}$ can be performed explicitly in examples of modest size, for instance by the computer algebra package *Macaulay2*; see Exercise B.20.

Theorem B.11 makes the proof of most other statements about base change easy, as we shall see in the next section. Here is a taste:

Corollary B.12. *Let $\pi : X \to B$ be a projective morphism of schemes and let $\mathcal{F}$ be a sheaf on X that is flat over B. For each i, the dimension function*

$$B \ni b \mapsto \dim_{\kappa(b)} H^i(\mathcal{F}|_{X_b})$$

is an upper-semicontinuous function (in particular, it takes its smallest value on an open set). Moreover, the Euler characteristic

$$\chi(\mathcal{F}|_{X_b}) := \sum (-1)^i \dim_{\kappa(b)} H^i(\mathcal{F}|_{X_b})$$

is constant on connected components of B.

Proof: It suffices to prove the result in the case where B is affine and connected, say $B = \operatorname{Spec} A$. Let $\mathcal{P}^\bullet$ be a complex of finitely generated projective A-modules with the properties given in Theorem B.11. Restricting to some possibly smaller open set of B, we may assume that $\mathcal{P}^\bullet$ is a complex of finitely generated free modules. For each $b \in B$ we get a complex of vector spaces by taking the fiber of $\mathcal{P}^\bullet$ at b. The maps $\varphi^i : P^i \to P^{i-1}$ in $\mathcal{P}^\bullet$ are given by matrices with entries in A, and thus the rank of $\varphi^i_b := \kappa(b) \otimes_A \varphi^i$ is a semicontinuous function of b. It follows that

$$\dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b) = \dim_{\kappa(b)} P^i|_b - \operatorname{rank} \varphi_b^{i+1} - \operatorname{rank} \varphi_b^i$$

is a semicontinuous function of b. Further, the Euler characteristic

$$\begin{aligned}\chi(\mathcal{F}|_{X_b}) &= \sum (-1)^i \dim_{\kappa(b)} \mathrm{H}^i(\mathcal{P}^\bullet|_b)\\ &= \sum (-1)^i \dim_{\kappa(b)}(P^i|_b)\\ &= \sum (-1)^i \operatorname{rank} P^i\end{aligned}$$

is a constant function of b. □

Example B.13 (Further continuation of Example B.1)**.** The tools above can be converted into algorithms for computing the direct image complex of a coherent sheaf — see Exercise B.20 — but sometimes one can understand the result without computation.

The derived category $D^b(\mathbb{P}^1)$ of bounded complexes of coherent sheaves on $\mathbb{P}^1$ (or on any smooth curve) is *formal*, in the sense that every bounded complex of locally free sheaves is quasi-isomorphic to the direct sum of its homology sheaves, or equivalently to the direct sum of locally free resolutions of its homology sheaves (see Exercise B.17).

Thus, if $\mathcal{F}$ is a family of sheaves on the family of varieties $\pi : X \to C$, where C is a smooth curve and $\mathcal{F}$ is flat over C with support of relative dimension n, the direct image complex may be taken simply to be the direct sum of locally free resolutions of the coherent sheaves $R^0\pi_*\mathcal{F}, R^1\pi_*\mathcal{F}, \ldots, R^n\pi_*\mathcal{F}$, each in its appropriate homological degrees.

Returning to Example B.1, we see for example that the direct image complex $R\pi_*\mathcal{I}_{\Gamma'}(1)$ is the quasi-isomorphism class of

$$0 \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

with nonzero term in cohomological degree 0, while $R\pi_*\mathcal{I}_\Gamma(1)$ is the quasi-isomorphism class of

$$0 \longrightarrow \mathcal{O}_B(-1) \xrightarrow{\sigma} \mathcal{O}_B \longrightarrow 0,$$

where $\mathcal{O}_B$ is in cohomological degree 1 and the differential is multiplication by a section (unique up to scalars) that vanishes at the point r_0. Further, the map $\mathcal{I}_\Gamma(1) \subset \mathcal{I}_{\Gamma'}(1)$ induces

$$\begin{tikzcd}
R\pi_*(\mathcal{I}_{\Gamma'}(1)) : 0 \arrow[r] & \mathcal{O}_B \arrow[r] & 0 \arrow[r] & 0 \\
R\pi_*(\mathcal{I}_{\Gamma}(1)) : 0 \arrow[r] \arrow[u] & \mathcal{O}_B \arrow[r, "\sigma"] \arrow[u] & \mathcal{O}_B(1) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The following example shows that the hypothesis of flatness in Theorem B.11 is essential:

Example B.14. Let $B = \mathbb{A}^2$ and let $\pi : X \to B$ be the blow-up of B at the origin. Let $\mathbb{P}^1 \cong E \subset X$ be the exceptional divisor and let $\mathcal{F}$ be the line bundle $\mathcal{O}(E)$. Note that $\mathcal{F}$ is not flat over B. We have $\chi(\mathcal{F}|_b) = 1$ for $b \neq 0$, but $\mathcal{F}|_0 = \mathcal{O}_{\mathbb{P}^1}(-1)$, so the dimension of $H^0(\mathcal{F}|_b)$ is not upper-semicontinuous and the Euler characteristic $\chi(\mathcal{F}|_b)$ is not constant.

B.5 Proofs of the theorems on cohomology and base change

We require two tools from commutative algebra. First, a fundamental method for proving that a sheaf is a vector bundle (that is, is locally free):

Proposition B.15. *A coherent sheaf $\mathcal{G}$ on a connected reduced scheme B is a vector bundle if and only the dimension of the $\kappa(b)$-vector space $\mathcal{G}_b$ is the same for all points $b \in B$; if B is quasi-projective, then it even suffices to check this for closed points. These conditions are satisfied, in particular, if $\mathcal{G}$ has a resolution*

$$\mathcal{P}^\bullet : \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

by vector bundles of finite rank that remains a resolution when tensored with $\kappa(b)$ for every $b \in B$.

Proof: If $\mathcal{G}$ is a vector bundle, then it has constant rank because B is connected, and this rank is the common dimension of $\mathcal{G}_b$ over $\kappa(b)$.

To prove the converse, and also the last statement of the proposition, we note that the problem is local, so we may assume that $B = \operatorname{Spec} A$, where A is a local ring with maximal ideal $\mathfrak{m}$ corresponding to the closed point $b_0 \in B$, and that $\mathcal{G}$ is the sheaf associated to a finitely generated A-module G. By Nakayama's lemma, a minimal set of generators of G corresponds to a map $f : F \to G$ from a free A-module F such that the induced map $(A/\mathfrak{m}) \otimes F \to (A/\mathfrak{m}) \otimes G$ is an isomorphism. In particular, the rank of the free module F is $\dim_{A/\mathfrak{m}}(A/(\mathfrak{m}) \otimes G) = \dim_{\kappa(b)} \mathcal{G}|_b$.

Let $K = \operatorname{Ker} \varphi$, and let P be a minimal prime of A. Since A is reduced, A_P is a field. Localizing at P, we get an exact sequence of finite dimensional vector spaces

$$0 \longrightarrow K_P \longrightarrow F_P \longrightarrow G_P \longrightarrow 0,$$

from which we have $\operatorname{rank} F = \dim_{A_P} G_P + \dim_{A_P} K_P$. By hypothesis, $\dim_{A_P} G_P = \dim_{A/\mathfrak{m}} G/\mathfrak{m}G = \operatorname{rank} F$, so $K_P = 0$. Since A is reduced, the only associated primes of F are the minimal primes of A, and thus $K \subset F$ itself must be zero.

To prove that it suffices to assume the constancy of the dimension of $\mathcal{G}_b$ at closed points in the quasi-projective case, it suffices to show that

$$\dim_{\kappa(\eta)} \mathcal{G}_\eta = \min_b \dim_{\kappa(b)} \mathcal{G}_b,$$

where η is a generic point and the minimum is take over the closed points in the closure of η. The inequality $\leq$ follows at once from Nakayama's lemma, since $\mathcal{G}$ is generated locally at b — and thus at η — by $\dim_{\kappa(b)} \mathcal{G}_b$ elements.

For the opposite inequality, choose elements $\{x_i\} \subset \mathcal{G}$ that form a $\kappa(\eta)$-basis for $\mathcal{G}_\eta$ and a set of generators $\{y_i\} \subset \mathcal{G}$ for $\mathcal{G}$. We can express all the y_i as linear combinations of the x_i with rational coefficients, using finitely many denominators. By the Nullstellensatz, there is a closed point b in the closure of η such that the product of these denominators is invertible at b, and it follows that the elements x_i span $\mathcal{G}_b$, so $\dim_{\kappa(\eta)} \mathcal{G}_\eta \geq \dim_{\kappa(b)} \mathcal{G}_b$ as required.

For the last statement of the corollary, suppose that $\mathcal{G}$ has a resolution $\mathcal{P}^\bullet$ with the given property. Since B is the spectrum of the local ring A, we may identify $\mathcal{P}^\bullet$ with a free resolution of $\mathcal{G}$. Since a minimal free resolution is a summand of any resolution

(Eisenbud [1995, Theorem 20.2]) the minimal free resolution $\mathcal{P}'^\bullet$ of $\mathcal{G}$ has the same property. But after tensoring with the residue class field $\kappa(b_0)$ the differentials in $\mathcal{P}'^\bullet$ become zero. Since by hypothesis the resolution remains acyclic, we must have $P'^0 = \mathcal{G}$, so $\mathcal{G}$ is free. $\square$

The proof of Theorem B.11 requires one more tool, a way of approximating a complex by a complex of free modules with good properties.

Proposition B.16. *Let A be a Noetherian ring, and let*

$$\mathcal{K}^\bullet : \cdots \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$

be a complex of (not necessarily finitely generated) flat A-modules whose homology modules are finitely generated and such that $K^m = 0$ for $m \gg 0$. There is a complex of finitely generated free A-modules $\mathcal{P}^\bullet$ with $P^m = 0$ for $m \gg 0$ and a map of complexes $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ such that for every A-module M the map

$$r \otimes_A M : \mathcal{P}^\bullet \otimes_A M \longrightarrow \mathcal{K}^\bullet \otimes_A M$$

induces an isomorphism on homology.

Proof: We will construct a complex of finitely generated free modules $\mathcal{P}^\bullet$ with a map r to $\mathcal{K}^\bullet$ inducing an isomorphism on homology without the assumption of flatness, and then we will use the flatness hypothesis to show that any such $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces an isomorphism on homology after tensoring with the arbitrary module M.

We will construct the complex $\mathcal{P}^\bullet$ by downward induction on i, using the hypothesis that a map of complexes

$$\begin{tikzcd} K^{i+1} \arrow[r, "d^{i+1}"] & K^{i+2} \arrow[r, "d^{i+2}"] & \cdots \\ P^{i+1} \arrow[u, "r_{i+1}"] \arrow[r, "e^{i+1}"] & P^{i+2} \arrow[u, "r_{i+2}"] \arrow[r, "e^{i+2}"] & \cdots \end{tikzcd}$$

inducing isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+2$ has been constructed, with the additional property that the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective.

If i is sufficiently large that $K^m = 0$ for all $m \geq i+1$, then the choice $P^m = 0$ and $r_m = 0$ for $m \geq i+1$ satisfies these conditions, giving a base for the induction.

To make the inductive step from $i+1$ to i, we choose P^i to be the direct sum of two projective modules, $P^i = P^i_1 \oplus P^i_2$, where P^i_1 is chosen to map onto the kernel of the composite $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ and P^i_2 is chosen to map onto $\mathrm{H}^i(\mathcal{K}^\bullet)$. We define the differential e^i to be the given map on P^i_1 and zero on P^i_2. Also, we define r_i on P^i_2 by lifting the chosen map $P^i_2 \to \mathrm{H}^i(\mathcal{K}^\bullet)$ to a map $P^i_2 \to \operatorname{Ker} d^i$ and composing with the inclusion $\operatorname{Ker} d^i \subset K^i$. On the other hand, since r_{i+1} carries the image of P^i_1 to the kernel of the map $\operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$, which is by definition

$\operatorname{Im} d^i$, we may define r_i on P_1^i to be the lifting of this map $P_1^i \to \operatorname{Im} d^i$ to a map $P_1^i \to K^i$.

This gives a map of complexes

$$\begin{tikzcd}
 & K^i \arrow[r, "d^i"] & K^{i+1} \arrow[r, "d^{i+1}"] & K^{i+2} \arrow[r, "d^{i+2}"] & \cdots \\
P^i = P_1^i \oplus P_2^i \arrow[r, "e^i"] \arrow[ur, phantom] & P_1^i \arrow[u, "r_i"] & P^{i+1} \arrow[u, "r_{i+1}"] \arrow[r, "e^{i+1}"] & P^{i+2} \arrow[u, "r_{i+2}"] \arrow[r, "e^{i+2}"] & \cdots
\end{tikzcd}$$

It is clear from the construction that the r_i induce isomorphisms $\mathrm{H}^j(\mathcal{P}^\bullet) \to \mathrm{H}^j(\mathcal{K}^\bullet)$ for all $j \geq i+1$ and the composite map $\operatorname{Ker} e^{i+1} \to \operatorname{Ker} d^{i+1} \to \mathrm{H}^{i+1}(\mathcal{K}^\bullet)$ is surjective, so the induction is complete.

We now use the hypothesis that the K^i are flat, and suppose that we have proven that r_j induces an isomorphism $\mathrm{H}^j(\mathcal{P}^\bullet \otimes M) \to \mathrm{H}^j(\mathcal{K}^\bullet \otimes M)$ for every $j > i$ and for every module M. This is trivial in the range where K^j and P^j are both zero, so again we can do a downward induction.

Choose a surjection $F \to M$ from a free A-module, and let L be the kernel, so that

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$$

is a short exact sequence. Since all the K^i and the P^i are flat, we get short exact sequences of complexes by tensoring with $\mathcal{P}^\bullet$ and $\mathcal{K}^\bullet$, from which we get two long exact sequences by applying the higher direct image functors, and the comparison map $r : \mathcal{P}^\bullet \to \mathcal{K}^\bullet$ induces a commutative diagram

$$\begin{tikzcd}
H^i(\mathcal{K}^\bullet \otimes L) \arrow[r] & H^i(\mathcal{K}^\bullet \otimes F) \arrow[r] & H^i(\mathcal{K}^\bullet \otimes M) \arrow[r] & H^{i+1}(\mathcal{K}^\bullet \otimes L) \arrow[r] & H^{i+1}(\mathcal{K}^\bullet \otimes F) \\
H^i(\mathcal{P}^\bullet \otimes L) \arrow[r] \arrow[u, "r_i \otimes L"] & H^i(\mathcal{P}^\bullet \otimes F) \arrow[r] \arrow[u, "r_i \otimes F", "\cong"'] & H^i(\mathcal{P}^\bullet \otimes M) \arrow[r] \arrow[u, "r_i \otimes M"] & H^{i+1}(\mathcal{P}^\bullet \otimes L) \arrow[r] \arrow[u, "r_{i+1} \otimes L", "\cong"'] & H^{i+1}(\mathcal{P}^\bullet \otimes F) \arrow[u, "r_{i+1} \otimes F", "\cong"']
\end{tikzcd}$$

where, for any module N, we write $r_i \otimes N$ to denote the map $H^i(\mathcal{P}^\bullet \otimes N) \to H^i(\mathcal{K}^\bullet \otimes N)$ induced by r_i. The maps marked "$\cong$" are isomorphisms: $r_i \otimes F$ and $r_{i+1} \otimes F$ are isomorphisms because F is free, while $r_{i+1} \otimes L$ is an isomorphism by induction. A diagram chase (sometimes called the "five-lemma") immediately shows that the map $r_i \otimes M$ is a surjection. Since the module M was arbitrary, $r_i \otimes L$ is a surjection as well. Using this information, a second diagram chase shows that $r_i \otimes M$ is injective, completing the induction. □

Proof of Theorem B.11: Since π is projective we may write $X \subset \mathbb{P} := \mathbb{P}_A^n$ for some n, and we let U_i, $i = 0, \ldots, n$, be the standard open covering of $\mathbb{P}$ as in Section B.3. Let $\mathcal{C}^\bullet$ be the Čech complex defined there. Since $\mathcal{F}$ is flat and $(\mathcal{F} \otimes \mathcal{O}_{U_i \cap U_j \cap \ldots})_0$ is the module corresponding to the restriction of $\mathcal{F}$ to the affine open set $U_i \cap U_j \cap \cdots$, the modules of the complex $(\mathcal{F} \otimes \mathcal{C}^\bullet)_0$ are flat. By Theorem B.8 the homology of this complex is finitely generated, so we may apply Proposition B.16 and obtain a complex $\mathcal{P}^\bullet$ whose

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i-th homology is $R^i\pi_*\mathcal{F}$. Taking $M = \kappa(b)$ in the proposition, we see that $\mathcal{P}^\bullet$ has the second required property as well.

Finally, to show that we may choose $\mathcal{P}^\bullet$ with $P^i = 0$ for $i < 0$, note that for any choice of $\mathcal{P}^\bullet$ satisfying the proposition the homology $\mathrm{H}^i(\mathcal{P}^\bullet)$ is zero for $i < 0$. The last statement of Proposition B.15 shows that $P'^0 := \operatorname{coker}(P^{-1} \to P^0)$ is projective. The map r_0 induces a map $P'^0 \to \operatorname{coker}(K^{-1} \to K^0)$, and since P'^0 is projective we may lift this to a new map $r_0' : P'^0 \to K^0$. It follows from the construction that

$$\begin{tikzcd}
\cdots \arrow[r] & K^{-1} \arrow[r] & K^{-1} \arrow[r] & K^0 \arrow[r] & K^1 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u, "0"] & 0 \arrow[r] \arrow[u, "0"] & P'^0 \arrow[r] \arrow[u, "r_0'"] & P^1 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

again induces an isomorphism on homology. □

Proof of Theorem B.9: The statements being local on B, we may assume from the outset that B is affine. In parts (b) and (c) we have assumed that $\mathcal{F}$ is flat, and by the generic flatness theorem (see Eisenbud [1995, Section 14.2]) there is in any case an open set $U_1 \subset B$ over which $\mathcal{F}$ is flat. Thus even for part (a) we may assume that $\mathcal{F}$ is flat over B. Let $\mathcal{P}^\bullet$ be a complex of projective modules representing $R\pi_*\mathcal{F}$, as in Theorem B.11.

(a) Removing the intersections of the components of X and then passing to a connected component, we may harmlessly assume that X is integral. Shrinking the open set U further, we may assume that the ranks of the maps in the restricted complex $(\mathcal{P}^\bullet)_b$ are constant for all $b \in U$. It follows from Proposition B.15 that all the homology modules of $\mathcal{P}^\bullet$ are vector bundles. Thus $\mathcal{P}^\bullet$ is locally split, and forming its homology commutes with any pullback.

(b) Since $\mathrm{H}^i(\mathcal{P}^\bullet|_b) = H^i(\mathcal{F}|_{X_b})$, these spaces are zero for all $j > i$. If m is the greatest integer for which $\mathcal{P}^m \neq 0$, and $m > i$, then Nakayama's lemma implies that the map $\mathcal{P}^{m-1} \to \mathcal{P}^m$ is surjective, and thus split. Consequently we can build a quasi-isomorphic complex $\mathcal{P}'^\bullet$ by replacing $\mathcal{P}^{m-1}$ by $\mathcal{P}'^{m-1} := \operatorname{Ker}(\mathcal{P}^{m-1} \to \mathcal{P}^m)$ and truncating the complex there:

$$\begin{tikzcd}
\mathcal{P}'^\bullet : & \cdots \arrow[r] & \mathcal{P}^{m-2} \arrow[r] \arrow[d] & \mathcal{P}'^{m-1} \arrow[r] \arrow[d] & 0 & \\
\mathcal{P}^\bullet : & \cdots \arrow[r] & \mathcal{P}^{m-2} \arrow[r] & \mathcal{P}^{m-1} \arrow[r] & \mathcal{P}^m \arrow[r] & 0
\end{tikzcd}$$

Continuing in this way, we may assume that $\mathcal{P}^j = 0$ for all $j > i$. Since forming cokernels commutes with pullback,

$$\begin{aligned}
H^i(\mathcal{F}|_{\pi^{-1}b}) &= \operatorname{coker}(\mathcal{P}^{i-1}_b \to \mathcal{P}^i_b) \\
&= (\operatorname{coker}(\mathcal{P}^{i-1} \to \mathcal{P}^i))_b \\
&= (R^i\pi_*\mathcal{F})_b.
\end{aligned}$$

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(c) Let

$$\cdots \longrightarrow \mathcal{P}^{i-1} \xrightarrow{d^{i-1}} \mathcal{P}^i \xrightarrow{d^i} \mathcal{P}^{i+1} \longrightarrow \cdots$$

be the differentials of $\mathcal{P}^\bullet$. Since the ranks of the differentials d_b^{i-1} and d_b^i are lower-semicontinuous, the constancy of $\dim_{\kappa(b)} H^i(\mathcal{F}_{\pi^{-1}(b)})$ implies the constancy of the ranks of d_b^{i-1} and d_b^i.

Focusing for a moment on d^{i-1}, we see that since taking fibers commutes with taking images the fibers of the module $\operatorname{Im} d^{i-1}$ have constant rank. By Proposition B.15, $\operatorname{Im} d^{i-1}$ is a projective module, so the short exact sequence

$$0 \longrightarrow \operatorname{Ker} d^{i-1} \longrightarrow \mathcal{P}^{i-1} \longrightarrow \operatorname{Im} d^{i-1} \longrightarrow 0$$

splits, and thus stays exact under pullback along $\rho : B' \to B$. It follows that

$$R^{i-1}\pi'_*(\rho'^*\mathcal{F}) = \mathrm{H}^{i-1}(\rho^*\mathcal{P}^\bullet) \xrightarrow{\varphi^{i-1}_{B'}} \rho^*\mathrm{H}^{i-1}(\mathcal{P}^\bullet) = \rho^* R^{i-1}\pi_*(\mathcal{F})$$

is an isomorphism.

Of course, the same considerations hold for $R^i\pi_*\mathcal{F}$. Furthermore, since the map $\operatorname{Im} d^{i-1} \to \operatorname{Ker} d^i$ has constant rank on restriction to each closed point b, so does the cokernel $R^i\pi_*\mathcal{F}$, so this sheaf is projective, proving part (c). □

Remark. Suppose that $\pi : X \to B$ is a projective morphism and $\mathcal{F}$ is a coherent sheaf on X, flat over B, as in Theorem B.9, and suppose that $b \in B$ is a point at which $\dim_{\kappa(b)} H^i(\mathcal{F}_{X_b})$ "jumps"—i.e., is larger than it is for some points in any open neighborhood of b. From the constancy of the Euler characteristic $\chi(\mathcal{F}|_{X_b})$, it follows that some $\dim_{\kappa(b)} H^j(\mathcal{F}|_{X_b})$ with $j \neq i \bmod 2$ must jump too. But the proof above gives a tiny bit more: Since the rank of d_b^i or of d_b^{i-1} must have gone down, either $\dim_{\kappa(b)} H^{i+1}(\mathcal{F}|_{X_b})$ or $\dim_{\kappa(b)} H^{i-1}(\mathcal{F}|_{X_b})$ must jump at b. Colloquially, "the jumps occur in adjacent pairs."

B.6 Exercises

Exercise B.17. (a) Suppose that R is a ring. Show that if

$$\mathcal{C} : \cdots \to C^{i-1} \to C^i \to C^{i+1} \to \cdots$$

is a complex of R-modules and $H := H^i\mathcal{C}$ has a projective resolution of length 1 $\mathcal{P}_i : 0 \to Q^{i-1} \to P^i \to H^i \to 0$ then there is a map $\mathcal{P} \to \mathcal{C}$ inducing the identity map on H. Conclude that if R is a Dedekind domain any complex is quasi-isomorphic to a direct sum of projective resolutions of its homology modules, and thus to a direct sum of its homology modules themselves. Note that there is generally no map from the

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direct sum of the homology modules to $\mathcal{C}$; rather, there is a "roof" diagram with two maps that are quasi-isomorphisms, pointing in opposite directions:

$$\begin{tikzcd} & \bigoplus \mathcal{P}_i \arrow[dl] \arrow[dr] & \\ \bigoplus H^i\mathcal{C} & & \mathcal{C} \end{tikzcd}$$

(b) Now suppose that C is a smooth curve. Given any coherent sheaves $\mathcal{H}, \mathcal{G}$ on C, show that there is a resolution $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{H} \to 0$ by coherent sheaves such that $\mathrm{Ext}^1(\mathcal{P} \oplus \mathcal{Q}, \mathcal{G}) = 0$. Use this to imitate the argument of part (a), proving that every complex of coherent sheaves on C is quasi-isomorphic to its homology.

Exercise B.18. In Theorem B.11 we made the hypothesis that the base was affine in order to make use of projective resolutions, which do not generally exist over quasi-projective bases. Show that if B is quasi-projective, and $\mathcal{G}$ is a coherent sheaf on B, then one can resolve $\mathcal{G}$ by sums of line bundles, and that the twists of these bundles may be taken to be arbitrarily negative. Use this to give a version of Proposition B.16 that works for quasi-projective schemes. Show that this suffices to prove a version of Theorem B.11 where the object $R\pi_*\mathcal{F}$ is represented by a complex whose terms are sums of line bundles.

Exercise B.19. Let B be a curve of genus 1 over k and let $p \in B$ be a point. Let $X = B \times B$, and let $\pi : X \to B$ the projection onto the first factor. Let Δ be the diagonal in X, and consider the line bundle $\mathcal{F} = \mathcal{O}_X(\Delta - B \times p)$, so that $\mathcal{F}|_{b \times B} = \mathcal{O}_B(b - p)$. Show that $\pi_*\mathcal{F} = 0$, but the natural map $H^0(\mathcal{F}|_{p \times B}) \to \kappa(p) \otimes R^0\pi_*\mathcal{F}$ is not an isomorphism.

Show that $R^1\pi_*\mathcal{F}$ is a torsion sheaf, supported at p with fiber isomorphic to $H^1(\mathcal{F}|_{p\times B})$, and that the complex

$$0 \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{L} \longrightarrow 0$$

represents $R\pi_*\mathcal{F}$.

Exercise B.20. Get a copy of *Macaulay2* from the website `http://www.math.uiuc.edu/Macaulay2/`, and compute some direct image complexes, using the following computation as a model. In the code below, we work over the field `QQ` of rational numbers (for a larger computation we would use a finite field such as $\mathbb{Z}/31003$ for efficiency). The projection map $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ is expressed by writing the homogeneous coordinate ring of $\mathbb{P}^2 \times \mathbb{P}^1$ as a polynomial ring P in three variables over the polynomial ring B in two variables representing the base. The ideals $I \subset I' \subset P$ are the bihomogeneous ideals of the families $\Gamma \supset \Gamma'$. Thus the computation is prepared as follows:

```
B = QQ[s,t];
P = B[x_0,x_1,x_2];
I' = intersect(ideal(x_0,x_1), ideal(x_0,x_2));
I = intersect(I', ideal(x_1-x_2, s*x_0-t*x_1));
```

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We now compute the complex $R\pi_*(\mathcal{I}_{\Gamma'}(d))$ for d from 0 to 3. The computation uses code in the *Macaulay2* package *BGG*. *Macaulay2* does not abuse notation as we have in this chapter; both components of the twist $P(d,0)$ must be made explicit: *Macaulay2* notation for this shifted module is $P^{\{\{d,0\}\}}$. Note that the ideal I' is made into a module explicitly. The ** represents the tensor product.

```
needsPackage "BGG";
for d from 0 to 3 do
<<directImageComplex(module I'**P^{{d,0}}) << endl<<endl
```

The output is something like the following (with the difference that in the actual output the complexes are indexed homologically instead of cohomologically):

$$\begin{array}{ccccccc}
2 & & 1 & & 0 & & -1 \\
0 & \longleftarrow & B & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & B & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & B^4 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & B^8 & \longleftarrow & 0
\end{array}$$

We can do the same with $R\pi_*(\mathcal{I}_{\Gamma}(d))$

```
for d from 0 to 3 do
<<directImageComplex(module I**P^{{d,0}}) << endl<<endl
```

and obtain

$$\begin{array}{ccccccc}
2 & & 1 & & 0 & & -1 \\
0 & \longleftarrow & B^2 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & B & \longleftarrow & B & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & B^3 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & B^8 & \longleftarrow & B
\end{array}$$

where the map on the right in the last complex is a split inclusion, so that the last complex is quasi-isomorphic to

$$0 \longleftarrow 0 \longleftarrow B^7 \longleftarrow 0.$$

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