Notes for Tropical Geometry

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2 Introduction/Motivation

Tropical geometry is the study of discrete structures appearing in limits of polynomial equations.
Course outline:

(1) Hypersurface amoebas, their skeleta, and tropical limits

(2)

3 Hypersurface amoebas, their skeleta, and tropical limits

3.1 Laurent polynomial ring

$\mathbb{C}[z_1^{\pm_1}, ..., z_n^{\pm}]$. Each such Laurent polynomial defines a holomorphic (algebraic) map $f : (\mathbb{C}^{\times})^n \longrightarrow \mathbb{C}$ whose zero locus $V(f) \subsetneq (\mathbb{C}^*)^n$ $f \neq 0$ is a **complex hypersurface.** The ring $\mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}]$ is a unique factorization domain which implies $f = f_1^{\alpha_1} \cdot ... \cdot f_m^{\alpha_m}$ where the f_i are ireducible, pairwise different, and hence $Z(f) = Z(f_1) \cup ... \cup Z(f_m)$. This locus is *always* a complex submanifold, even in the case of the nodal cubic for instance, of $\dim_{\mathbb{C}} = n - 1$ outside of a real codimension 2 subset $Z(f) \cap Z(\partial_1 f) \cap ... \cap Z(\partial_n f)$.

Example 3.1.

(a) $V(z + w) \subsetneq (\mathbb{C}^{\times})^2$ is isomorphic as a $\mathbb{C}$-manifold or as an algebraic variety to $\mathbb{C}^{\times}$. The map $\mathbb{C}^{\times} \mapsto V(z + w)$ given $u \mapsto (u, -u)$ parameterizes this curve.

(b) $V(z + w + 1) \subseteq (\mathbb{C}^{\times})^2$ is isomorphic to $\mathbb{C}^{\times} \setminus \{0, 1\}$ via the map $u \mapsto (u, 1 - u)$.

3.2 The Log Map

Forget phases and use logarithmic coordinates.

$$\text{Log} : (\mathbb{C}^{\times})^n \xrightarrow{1.1} \mathbb{R}^n_{>0} \xrightarrow{\log} \mathbb{R}^n$$

given by

$$(z_1, ..., z_n) \mapsto (|z_1|, ..., |z_n|) \mapsto (\log |z_1|, ..., \log |z_n|).$$

Definition 3.2. The **Hypersurface amoeba** of $f \in \mathbb{C}[z_1^{\pm}, ..., z_n^{\pm}] \setminus \{0\}$ is

$$\mathcal{A}_f = \text{Log}(V(f)) \subseteq \mathbb{R}^n$$

(Gelfand, Vapranov, Zelevabsky)

Example 3.3.

(a) $f = z + w$

(b) $f = z + w + 1$

(c) $f = 1 + 5zw + w^2 - z^2 + 3z^2w - z^2w^2$

(add pictures later) careful to draw these such that the complements of the amoeba are all convex.

Observations:

- connected cusps of $\mathbb{R}^n \setminus \mathbb{C}_f$ are convex in $\dim = 2$. $\mathcal{A}_f$ looks like a thickened graph. We'll sketch a proof of a more general result.

Recall: $\mathcal{U} \subseteq \mathbb{C}$, $f : \mathcal{U} \setminus \{p_1, ..., p_r\} \longrightarrow \mathbb{C}$ are meromorphic with mkr poles $(p_1, ..., p_r)$ and s zeros with multiplicity. This implies

$$s - r = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

This is the argument principle from complex analysis. Appears in the derivative of $\frac{1}{2\pi i} \int_{S^1} \log |f| dz$. This appears in the Jensen formula: $\mathcal{U} \subseteq \mathbb{C}$ an open subset and assume it contains a closed disk of radius r $\{z \mid |z| \leq r\} = D$. Important that it includes the boundary. Then if we have a holomorphic function $f : \mathcal{U} \longrightarrow \mathbb{C}$ with zeros of f in D $a_1, ..., a_k$ such that $0 < |a_1| \leq |a_2| \leq ... \leq |a_k|$ (with multiplicity) then we have

$$\frac{1}{2\pi i} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|}.$$

This is the Jensen formula.

Proof. (Rudin, "Real and complex analysis")

(1) Assume f has no zeros and hence that $\log |f|$ harmonic. Using the mean value property for harmonic functions (go review Analysis) yields the Jensen Formula.

(2) For the general case, suppose we have $|a_1|, ..., |a_n| < r$, and then that $|a_{m+1}|, ..., |a_k| = r$. Consider $g(z) = f(z) \cdot \prod_{j=1}^{m} \frac{r^2 - \overline{a}_j z}{r(a_j - z)} \prod_{j=m+1}^{k} \frac{a_j}{a_j - z}$ with no zeros in $|z| \leq r$. This implies

$$g(0) = f(0) \cdot \prod_{j=1}^{m} \frac{r}{a_j}$$

by our first case.

(3) $|z| = r$, so on the boundary, we have

$$|\frac{r^2 - a_j z}{r(a_j - z)}| = \frac{1}{r} |\frac{r^2 \overline{z} - a_j |z|^2}{r(a_j - z)}| = \frac{r}{r} = 1$$

$$\implies \log |g(re^{i\theta})| = \log |f(re^{i\theta})| - \sum_{j=m+1}^{k} \log \overbrace{|1 - e^{i(\theta - \theta_j)}|}^{a_j = re^{i\theta_j}}$$

(4) Lemma: $\int_0^{2\pi} \log(1 - e^{i\theta})d\theta = 0$. These four things together prove the Jensen formula.

$\square$

For $n > 1$we define something called the Ronkin function. We have $f \in \mathcal{O}(\mathrm{Log}^{-1}(\Omega)), \Omega \subseteq \mathbb{R}^n$ a (convex) open set. Then the **Ronkin Function** is defined

$$N_f(x) = (\frac{1}{2\pi i})^n \int_{\mathrm{Log}^{-1}(x)} \mathrm{Log}\,|f(z_1, ..., z_n)| \frac{dz_1}{z_1} \vee ... \vee \frac{dz_n}{z_n}$$

Theorem 3.4. *(a) N_f is a convex $\mathcal{C}^0$-function*

(b) $\mathcal{A}_f = \mathrm{Log}(V(f)) \subseteq \Omega$ an Amoeba. For all $\mathcal{U} \subseteq \Omega$ open, connected $\mathcal{U} \cap \mathcal{A}_f = \emptyset \iff N_f|_{\mathcal{U}}$ affine linear.

(c) $x \in \Omega \setminus \mathcal{A}_f \implies \mathrm{grad}\, N_f(x) = (v_1, ..., v_n)$,

$$v_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \vee ... \vee \frac{dz_n}{z_n}.$$

Picture: $N_f(x) = \langle \alpha_1, x\rangle + c_1$

Proof. (sketch)

(a) $\log|f|$ is plurisubharmonic (i.e. is subharmonic (i.e. somehow less than harmonic functions on a circle) on each each holomorphic image of a disk). We have the following fact: if $h : \mathcal{U} \longrightarrow \mathbb{R}$ is subharmonic, $\mathcal{U} \subseteq \mathbb{C}$ a domain containing $\{|z| \leq R\}$, then $\varphi(r) = \int_{|z|=r=\exp(s)} h(x)dz$ is a convex function in $\log r = s$. Found this proof in a book of Runkin called “Introduction to the theory of entire functions,” page 84.

(b) Prove this next time

(c) $x \in \Omega \setminus \mathcal{A}_f$. Note:

$$\frac{\partial}{\partial x_j} \log|f| = \frac{1}{2} \frac{\partial}{\partial x_j} \log(f\overline{f}) = \mathrm{Re}\left(z_j \frac{\partial}{\partial z_j} \log f\overline{f} \right) = \mathrm{Re}\left(\frac{z_j \partial_j f}{f} \right).$$

$x \in \Omega \setminus \mathcal{A}_f$ implies that

$$\frac{\partial}{\partial x_j} N_f(x) = \mathrm{Re}\left(\frac{1}{2\pi i}^n \int_{\mathrm{Log}^{-1}} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n} \right).$$

Note: for all j, we have

$$\gamma_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(x)} \frac{z_j \partial_j f}{f} \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}.$$

This is a locally constant n-form on $\mathcal{U} \setminus A_f$ and is not defined on $\mathcal{A}_f$ since f is zero on $\mathcal{A}_f$. In fact, $\gamma_j \in \mathbb{Z} : \frac{1}{2\pi i} \int_{|z_j|=e^{x_j}} \frac{\partial_j f(z)}{f(z)} dz_j \in \mathbb{Z}$ by the argument principle.

Look at Passare, Rullgard “Amoebas, Monge – Ampere, measures and triangulations DMJ 2004” $\square$