Contents

2 Cohomology of Sheaves.

Exercise 2.1. *a Let $X = \mathbb{A}^1$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.*

b More generally, let $Y \subseteq X = \mathbb{A}^n$ be the union of $n+1$ hyperplanes in suitable general position, and let $U = X - Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$, thus the result of (2.7) is the beset possible.

Solution. a The sheaf $\mathbb{Z}_U$ is a subsheaf of $\mathbb{Z}_X$ and so we get an exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z} \to 0$ where $i_{P*}\mathbb{Z}$ and $i_{Q*}\mathbb{Z}$ are the skyscraper sheaves at P and Q with value $\mathbb{Z}$. Taking cohomology gives a long exact sequence, one piece of which is $\cdots \to H^0(X, \mathbb{Z}_X) \to H^0(X, i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z}) \to H^1(X, \mathbb{Z}_U) \to \dots$, so if $H^1(X, \mathbb{Z}_U) = 0$, then $H^0(X, \mathbb{Z}_X) \to H^0(X, i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z})$ is surjective. But this is $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ which cannot be surjective.

Exercise 2.2. *Let $X = \mathbb{P}^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathscr{K} \to \mathscr{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.*

Solution. Since every pair of open subsets of X intersect nontrivially, every open subset is connected. So the constant sheaf $\mathscr{K}$ is actually the constant presheaf $\mathscr{K}$, and therefore flasque. To see that $\mathscr{K}/\mathcal{O}$ is flasque, write it as $\oplus_{P \in X} i_P(I_P)$ (Exercise II.1.21(d)). Exercise II.1.21(e) then tells us that applying the global sections functor we get an exact sequence, so $\Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0 \to \dots$ is exact, and since we can use this to calculate the cohomology, $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Exercise 2.3. Cohomology with Supports. *Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y.*

a *Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathscr{A}b(X)$ to $\mathscr{A}b$.*

b *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that*

$$0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'') \to 0$$

is exact.

c *Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X, \mathscr{F}) = 0$ for all $i > 0$.*

d *If $\mathscr{F}$ is flasque, show that the sequence*

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F}) \to 0$$

is exact.

e *Let $U = X - Y$. Show that for ny $\mathscr{F}$, there is a long exact sequence of cohomology groups*

$$\begin{aligned} 0 &\to H^0_Y(X, \mathscr{F}) \to H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) \to \\ &\to H^1_Y(X, \mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(U, \mathscr{F}|_U) \to \\ &\to H^2_Y(X, \mathscr{F}) \to \dots \end{aligned}$$

f Excision. *Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,*

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V)$$

Solution. a Let $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ be an exact sequence of sheaves of abelian groups on X. If $\Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F})$ is injective as a consequence of $\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ being injective. Similarly, the composition $\Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is zero as a consequence of $\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$ being zero. Consider a section $s \in \Gamma_Y(X, \mathscr{F})$ and suppose that it gets sent to zero in $\Gamma_Y(X, \mathscr{F}'')$. This implies that as an element of $\Gamma(X, \mathscr{F})$, the section s gets sent to zero in $\Gamma(X, \mathscr{F}'')$ and so is the image of some section $t \in \Gamma(X, \mathscr{F}')$. We just need to check that $t_x = 0$ for every $x \notin Y$. Let $x \in X - Y$ be such a point. Since $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact, we have an exact sequence of stalks $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$. The stalk of s_x is zero since $s \in \Gamma_Y(X, \mathscr{F})$ and therefore $t_x = 0$. Hence $t \in \Gamma_Y(X, \mathscr{F}')$.

b By part (a) we know that $\Gamma_Y(X, \cdot)$ is left exact so we just need to show that $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is surjective. Suppose that we have a section $s \in \Gamma_Y(X, \mathscr{F}'')$. This is a section of $\Gamma(X, \mathscr{F}'')$ and since $\mathscr{F}'$ is flasque, there is a section $t \in \Gamma(X, \mathscr{F})$ in its preimage (Exercise II.1.16(b)). This section does not necassarily have support in Y however. For every point $x \in X - Y$ consider the exact sequence of stalks $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$. The germ t_x gets sent to $s_x \in \mathscr{F}''_x$ which is zero since s has support in Y. So there is a germ $u_x \in \mathscr{F}'_x$ which gets sent to t_x. This means there is a neighbourhood U_i of x (which we can assume doesn't intersect Y) and a section u_i which gets sent to $t_i|_{U_i}$. In this way we get an open cover $\{U_i\}$ of $X - Y$ and for each i, a section u_i which gets sent to $t|_{U_i}$. Consider the intersections of the u_i. The sections $u_i|_{U_i \cap U_j} - u_j|_{U_i \cap U_j}$ get sent to $t|_{U_i \cap U_j} - t|_{U_i \cap U_j} = 0$ and since $\mathscr{F}' \to \mathscr{F}$ is injective, this means that $u_i|_{U_i \cap U_j} - u_j|_{U_i \cap U_j} = 0$ and so the u_i glue together to give a section $u' \in \mathscr{F}'(U)$ which gets sent to $t|_U$. Since $\mathscr{F}'$ is flasque, this lifts to a global section $u \in \Gamma(X, \mathscr{F}')$. Now consider $t - u \in \Gamma(X, \mathscr{F})$ this gets sent to $s \in \Gamma(X, \mathscr{F}'')$ since u came from $\mathscr{F}'$ and t got sent to s. Furthermore, for any point $x \in X - Y$, the germs of t and u agree since $t|_{U_i} = u|_{U_i} = u_i$ for every i in our cover above. Hence, we have found a global section $t - u \in \Gamma_Y(X, \mathscr{F})$ that gets sent to s.

c The proof from Proposition III.2.5 works. Embed $\mathscr{F}$ in an injective object $\mathscr{I}$ and let $\mathscr{G}$ be the quotient $\mathscr{F}/\mathscr{I}$. The sheaf $\mathscr{F}$ is flasque by hypothesis, and $\mathscr{I}$ is flasque by (2.4) so $\mathscr{G}$ is flasque by Exercise II.I.16(c). Since $\mathscr{F}$ is flasque, we have an exact sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{I}) \to \Gamma_Y(X, \mathscr{G}) \to 0$$

from part (b). On the other hand, $\mathscr{I}$ is injective and so $H^i_Y(X, \mathscr{I}) = 0$ for all $i > 0$. Thus, from the long exact sequence of cohomology, we get $H^1_Y(X, \mathscr{F}) = 0$ and $H^i_Y(X, \mathscr{F}) \cong H^{i-1}_Y(X, \mathscr{G})$ for each $i \geq 2$. But $\mathscr{G}$ is also flasque, and so by induction on i we get the result.

d This sequence is what you get if you apply the global sections functor to the sequence of Exercise II.1.20(b) so we just need to show that $\Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F})$ is surjective. But this is true, since $\mathscr{F}$ is flasque.

e To compute the cohomology of $\mathscr{F}$ we choose at the beginning an injective resolution $\mathscr{I}^i$ for $\mathscr{F}$. The functor $-|_U$ preserves injectives so we can use $\mathscr{I}^i|_U$ as an injective resolution to calculate the cohomology on U of $\mathscr{F}|_U$. Now injective sheaves are flasque by Lemma III.2.4 so for each i we have an exact sequence

$$0 \to \Gamma_Y(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}) \to \Gamma(U, \mathscr{I}|_U) \to 0$$

since $\Gamma(U, \mathscr{I}|_U) = \mathscr{I}(U)$. Now the long exact sequence is a consequence of the snake lemma.

f We use the espace étale of Exercse II.1.13 to show that there is an isomorphism of functors $\Gamma_Y(X, -) \to \Gamma_Y(V, -|_V)$. Given a sheaf $\mathscr{F}$ and an open subset $U \subset X$, using the espace étale we can consider $\mathscr{F}(U)$ as a set of continuous morphisms $U \to \text{Spé}\mathscr{F}$. Any section of $\Gamma_Y(X, \mathscr{F})$ takes the value $0 \in \mathscr{F}_x \subset \text{Spé}\mathscr{F}$ on any point x not in Y. So since $Y \subset V$, if two sections of $\Gamma_Y(X, \mathscr{F})$ agree on their restrictions to V, then they agree in $\Gamma_Y(X, \mathscr{F})$ so $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(U, \mathscr{F})$ is injective. On the other hand, if we have a section $s \in \Gamma_Y(U, \mathscr{F})$ we extend it to a section t in $\Gamma_Y(X, \mathscr{F})$ by sending $x \mapsto 0 \in \mathscr{F}_x$ for any point $x \in X - V$. This defines a function $X \to \text{Spé}\mathscr{F}$ which is a section but is not necassarily continuous. To see that it is continuous, consider the restriction to an open cover $\{U_i\}$ where for each i, either $U_i \subset V$ or $U_i \cap Y = \varnothing$ (or both). Since t came from a section s, for the i with $U_i \subset V$ we have $t|_{U_i} = s|_{U_i}$ and so these are continuous. For the i with $U_i \cap Y = \varnothing$, we have $t|_{U_i} = 0$, which is continuous by definition of the espace étale since these morphisms come from sections of $\mathscr{F}(U_i)$. So the restrictions of t to every element of an open cover of X are continuous, and therefore t is continuous, hence $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(V, \mathscr{F}|_V)$ is surjective.

Now as we mentioned in the previous part, if $\mathscr{I}^i$ is an injective resolution for $\mathscr{F}$, then $\mathscr{I}^i|_V$ is an injective resolution for $\mathscr{F}|_V$ and so the isomorphism $\Gamma_Y(X, -) \cong \Gamma_Y(V, -|_V)$ leads to the isomorphism of cohomology groups.

Exercise 2.4. Mayer-Vietoris Sequence. *Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports*

$$\cdots \to H^i_{Y_1 \cap Y_2}(X, \mathscr{F}) \to H^i_{Y_1}(X, \mathscr{F}) \oplus H^i_{Y_2}(X, \mathscr{F}) \to H^i_{Y_1 \cup Y_2}(X, \mathscr{F}) \to H^{i+1}_{Y_1 \cap Y_2} \to \ldots$$

Solution. Define $Y_{12} = Y_1 \cap Y_2$, $Y = Y_1 \cup Y_2$, $U_{12} = X - Y_{12}$, $U_i = X - Z_i$, $U = X - Y$ and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Y_{12}}(X, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1}(X, \mathscr{I}) \oplus \Gamma_{Y_2}(X, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma_Y(X, \mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{I}) \oplus \Gamma(X, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X, \mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(U_{12}, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma(U_1, \mathscr{I}) \oplus \Gamma(U_2, \mathscr{I}) \arrow[r] \arrow[d] & \Gamma(U, \mathscr{I}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the sheaf $\mathscr{I}$ is flasque, then the columns are exact. The lower two rows are exact (the lower one being exact as consequence of $\mathscr{I}$ being a sheaf) and so we can apply the Nine Lemma to find that the top row is exact. So if $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$, then we get an exact sequence of complexes

$$0 \to \Gamma_{Y_{12}}(X, \mathscr{I}^\bullet) \to \Gamma_{Y_1}(X, \mathscr{I}^\bullet) \oplus \Gamma_{Y_2}(X, \mathscr{I}^\bullet) \to \Gamma_Y(X, \mathscr{I}^\bullet) \to 0$$

The Snake Lemma applied to this exact sequence of complexes gives the desired long exact sequence.

Exercise 2.5. *Let X be a Zariski space. Show that for all i, $\mathscr{F}$, we have*

$$H^i_P(X, \mathscr{F}) = H^i_P(X_P, \mathscr{F}_P)$$

(see Hartshorne's statement of the exercise for notation).

Solution. We show a natural isomorphism $\Gamma_P(X, \mathscr{G}) \cong \Gamma_P(X_P, \mathscr{G}_P)$. By definition, $\Gamma(X_P, \mathscr{G}_P) = \varprojlim_{U \ni P} \mathscr{G}(U) = \mathscr{G}_P$ since $P \in U$ if and only if $U \supset X_P$, so there is a natural morphism $\Gamma(X, \mathscr{G}) \to \Gamma(X_P, \mathscr{G}_P)$ which induces a morphism $\Gamma_P(X, \mathscr{G}) \to \Gamma_P(X_P, \mathscr{G}_P)$. Injectivity: let s and t be two global sections with support on P. If they get sent to the same element in $\Gamma_P(X_P, \mathscr{G}_P)$ then the germs $s_P = t_P$ agree. But s and t have support in P so they are identically zero in every other stalk. Therefore they agree on every stalk and hence, $s = t$. Surjectivity: let $s \in \Gamma_P(X_P, \mathscr{G}_P) = \mathscr{G}_P$. Then there is an open neighbourhood of P and $s_U \in \mathscr{G}(U)$ which represents s. Since s has support in P we can choose U small enough so that $(s_U)_Q = 0$ for every point $Q \neq P$. Now consider $V = X - P$ and the zero section in $\mathscr{G}(U)$. Since the germ of s_U is zero on all points that aren't P, we have $s_U|_{U \cap V} = 0$ and so s_U and 0 glue together to give a global section with support in P. So the map is surjective.

Exercise 2.6. *Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective.*

Solution. For an open subset $U \subset X$ we define $\mathbb{Z}_U = i_! \mathbb{Z}_U$ where $\mathbb{Z}_U$ is the constant sheaf associated to the group $\mathbb{Z}$ and $i : U \to X$ is the inclusion.

Step 1. First we show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, and every map $f : \mathscr{R} \to \mathscr{I}$, there is an extension of f to a map of $\mathbb{Z}_U \to \mathscr{I}$.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R} \arrow[r] \arrow[dr] & \mathbb{Z}_U \arrow[d, dashed] \\ & & \mathscr{I} \end{tikzcd}$$

The direction ($\Rightarrow$) follows from the definition of an injective object. For the direction ($\Leftarrow$) we adapt the proof from the proof of Baer's Criterion (Theorem 2.3.1) in Weibel. Let $\mathscr{F} \subset \mathscr{G}$ be an injective morphism of sheaves, and suppose we have a morphism $\phi : \mathscr{F} \to \mathscr{I}$. Consider the poset of extensions of ϕ to a subsheaf $\mathscr{F}'$ of $\mathscr{G}$ containing $\mathscr{F}$, where the order is $\alpha \leq \alpha'$ if α' extends α. By Zorn's lemma this poset has a maximal element $\psi : \mathscr{F}' \to \mathscr{I}$ and so we just need to show that $\mathscr{F}' = \mathscr{G}$.

$$\begin{tikzcd} \mathscr{F} \arrow[r, hook] \arrow[dr] & \mathscr{F}' \arrow[r, hook] \arrow[d] & \mathscr{G} \\ & \mathscr{I} & \end{tikzcd}$$

Suppose that there is an open set U and a section $s \in \mathscr{G}(U)$ that is not in $\mathscr{F}'(U)$. This defines a morphism $\mathbb{Z}_U \to \mathscr{G}$ and the inclusion $\mathscr{F}' \to \mathscr{G}$ defines a subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$. Let $\mathscr{F}''$ be the subsheaf of $\mathscr{G}$ generated by $\mathscr{F}'$ and s. Then we can extend ψ to $\mathscr{F}''$ and so $\mathscr{F}' = \mathscr{G}$. Hence $\mathscr{I}$ is injective.

Step 2. Secondly, we show that any such subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ is finitely generated. Let $U = \coprod U_i$ be a decomposition of U into its connected components U_i. Since X is noetherian, the ascending chain $U_1 \subset U_1 \cup U_2 \subset \dots$ stabilizes (Exercise I.1.7(a)), say at n. So U is a finite union of connected open subsets. For each i we have subgroups $\mathscr{R}(U_i) \subset \mathbb{Z}_U(U_i) = \mathbb{Z}$, say that these are generated by $s_i \in \mathbb{Z}_U(U_i)$. Then these finitely many s_i generate $\mathscr{R}$.

Step 3. Let $s_i \in \mathscr{R}(U_i)$ be generating elements of $\mathscr{R}$ where $i = 0, \dots, n$. For any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$, the image of s_i is represented by some $t_i \in \mathscr{I}_{\alpha_i}(U_i)$ for some α_i. Due to the system being direct, there is an index β so that the image of s_i can be represented by $t_i' \in \mathscr{I}_\beta(U_i)$. Hence, the morphism factors as $\mathscr{R} \to \mathscr{I}_\beta \to \varinjlim \mathscr{I}_\alpha$. Now use the first part. For every open subset $U \subseteq X$, and subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, and map $f : \mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ the map f factors through some $f_\beta : \mathscr{R} \to \mathscr{I}_\beta$. Since $\mathscr{I}_\beta$ is injective, f_β extends to a map $\mathbb{Z}_U \to \mathscr{I}_\beta$ and so we get an extension $\mathbb{Z}_U \to \mathscr{I}_\beta \to \varinjlim \mathscr{I}_\alpha$ of f. Hence, by Step 1, $\varinjlim \mathscr{I}_\alpha$ is injective.

Exercise 2.7. *Let S^1 be the circle (with its usual topology), and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.*

- *a Show that $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$, using sheaf cohomology.*
- *b Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.*

3 Cohomology of a Noetherian Affine Scheme

Exercise 3.1. *Show that a noetherian scheme X is affine if and only if X_{red} is affine.*

Solution. If X is affine then $X_{red} = \operatorname{Spec}(A/N)$ where $A = \Gamma(X, \mathcal{O}_X)$ and N is the nilradical of A.

Conversely, suppose that X_{red} is affine. We want to show that X is affine by using Theorem 3.7 and induction on the dimension of X. If X has dimension 0 then affineness follows from the noetherian hypothesis since it must have finitely many points and each of these is contained in an affine neighbourhood. So suppose that result is true for noetherian schemes of dimension $< n$, and that $\dim X = n$. Let $\mathscr{N}$ be the sheaf of nilpotents on X and consider a coherent sheaf $\mathscr{F}$. For every integer i we have a short exact sequence

$$0 \to \mathscr{N}^{d+1} \cdot \mathscr{F} \to \mathscr{N}^d \cdot \mathscr{F} \to \mathscr{G}_d \to 0$$

where $\mathscr{G}_d$ is the appropriate quotient. This short exact sequence gives rise to a long exact sequence in cohomology:

$$\dots \to H^0(X, \mathscr{G}_d) \to H^1(X, \mathscr{N}^{d+1} \cdot \mathscr{F}) \to H^1(X, \mathscr{N}^d \cdot \mathscr{F}) \to H^1(X, \mathscr{G}_d) \to \dots$$

Since X is noetherian, there is some m for which $\mathscr{N}^d = 0$ for all $d \geq m$, so if we can show that $H^1(X, \mathscr{G}_d)$ is zero for each d, then the statement $H^1(X, \mathscr{F}) = 0$ will follow by induction and the long exact sequence above.

So the sheaf $\mathscr{G}_d = \mathscr{N}^d \cdot \mathscr{F} / \mathscr{N}^{d+1} \cdot \mathscr{F}$ on X. Recall that X_{red} has the same underlying topological space as X, but with sheaf of rings $\mathcal{O}_{X_{red}} = \mathcal{O}_X / \mathscr{N}$. So $\mathscr{G}_d$ is also a sheaf of $\mathcal{O}_{X_{red}}$-modules. Since cohomology is defined as cohomology of sheaves of abelian groups we have $H^1(X, \mathscr{G}_d) = H^1(X_{red}, \mathscr{G}_d)$ and so it follows from Theorem 3.7 that $H^1(X, \mathscr{G}_d) = 0$.

Exercise 3.2. *Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.*

Solution. If X is affine then every closed subscheme is affine (Exercise II.3.11(b)) and so every irreducible component is affine.

Conversely, suppose that each irreducible component is affine. Let Y_1, Y_2 be two closed subschemes of X and consider the coherent sheaves of ideals $\mathscr{I}_{Y_1}, \mathscr{I}_{Y_1 \cup Y_2}$. We have an exact sequence

$$0 \to \mathscr{I}_{Y_1 \cup Y_2} \to \mathscr{I}_{Y_1} \to \mathscr{F} \to 0$$

and it can be seen (reduced to the affine case) that $\mathscr{F} = i_* \mathscr{I}_{Y_1 \cap Y_2}$ where $i : Y_2 \to X$ is the closed imbedding. Let $Y = Y_1$ be an arbitrary closed subscheme. If $Z = Y_2$ is one of the irreducible components, it is affine and so $H^1(X, i_* \mathscr{I}_{Y \cap Z}) = H^1(Z, \mathscr{I}_{Y \cap Z}) = 0$. Hence, from the long exact sequence associated to the cohomology of the short exact sequence above, we see that $H^1(X, \mathscr{I}_{Y \cup Z}) \to H^1(X, \mathscr{I}_Y)$ is surjective.

Now let $Z_1, \dots, Z_n$ be the irreducible components of X. By induction we see that $H^1(X, \mathscr{I}_{Y \cup Z_1 \cup \dots \cup Z_n}) \to H^1(X, \mathscr{I}_Y)$ is surjectve. But $Y \cup Z_1 \cup \dots \cup Z_n = X$ and $\mathscr{I}_X = 0$ since X is reduced. Hence, $H^1(X, \mathscr{I}_Y)$ is zero and so it follows from Theorem III.3.7 that X is affine.

Exercise 3.3. *Let A be a noetherian ring and let $\mathfrak{a}$ be an ideal of A.*

a Show that $\Gamma_{\mathfrak{a}}(\cdot)$ is a left-exact functor from the category of A-modules to itself.

b Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M})$$

c For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

Solution. a Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Since $\Gamma_a(N) \subset N$ for any A-module, we know that

$$0 \to \Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M'')$$

is exact on the left, and the composition of the two rightmost morphisms is zero. So the only thing to show is that if m is in the kernel of $\Gamma_{\mathfrak{a}}(M) \to$

$\Gamma_{\mathfrak{a}}(M'')$, then it is in the image of $\Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M)$. By exactness of the original short exact sequence we know that there is a unique $n \in M'$ which gets sent to m. Since $m \in \Gamma_{\mathfrak{a}}(M)$ there is some i for which $\mathfrak{a}^i m = 0$. But $M' \to M$ is injective, and so $\mathfrak{a}^i n = 0$ and so $n \in \Gamma_{\mathfrak{a}}(M')$.

b Let $0 \to M \to I^\bullet$ be an injective resolution for M in the category of A-modules. Then we have an exact sequence of sheaves $0 \to \widetilde{M} \to \widetilde{I^\bullet}$ on X. Each $\widetilde{I^i}$ is flasque by (3.4) so we can use this resolution of $\widetilde{M}$ to calculate $H^i_Y(X, \widetilde{M})$ (Exercise III.2.3(c) and Proposition III.1.2A). The only thing left to show is that $\Gamma_{\mathfrak{a}}(\cdot) = \Gamma_Y(X, \widetilde{\cdot})$.

Consider $m \in \Gamma_{\mathfrak{a}}(M)$ for some arbitrary A-module M. Then by definition there is some n for which $\mathfrak{a}^n m = 0$. Let $\mathfrak{p}$ be a point of X not contained in Y. So $\mathfrak{p}$ doesn't contain $\mathfrak{a}$ and there is some $a \in \mathfrak{a}$ which is not in $\mathfrak{p}$. Then a^n is also not in $\mathfrak{p}$. Since $\mathfrak{a}^n m = 0$ we see that $a^n m = 0$ and so $m = 0$ in the localized module $M_{\mathfrak{p}}$. Hence, m is a global section of $\widetilde{M}$ with support in $Y = V(\mathfrak{a})$.

Conversely, let m be a global section of $\widetilde{M}$ with support in $Y = V(\mathfrak{a})$. So $Supp\ m \subseteq V(\mathfrak{a})$. By Exercise II.5.6(a) we have $Supp\ m = V(Ann\ m)$ and so $\sqrt{Ann\ m} \supseteq \sqrt{\mathfrak{a}}$ (Lemma II.2.1(c)). Since A is noetherian, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_n)$. For each i, there is n_i such that $f_i^{n_i} \in \sqrt{Ann\ m}$, and so there is some j_i such that $f_i^{n_i j_i} \in Ann\ m$. Set $N = \prod n_i j_i$ so that $f_i^N \in Ann\ m$ for all i. Then, there is some $N'(= nN)$ such that $\sum_{i=1}^n k_i \geq N' \Rightarrow k_i \geq N$ for some i where $k_i \geq 0$. So every element of $\mathfrak{a}^{N'}$ is a sum of elements which are divisible by f_i^N for some i. Hence, $\mathfrak{a}^{N'} \subseteq Ann\ m$, so $m \in \Gamma_{\mathfrak{a}}(M)$.

c By definition we know that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) \subseteq H^i_{\mathfrak{a}}(M)$. Consider $m \in H^i_{\mathfrak{a}}(M)$. So we have taken an injective resolution $0 \to M \to I^\bullet$ of M, we have $\dots \to \Gamma_{\mathfrak{a}}(I^{i-1}) \stackrel{d^i}{\to} \Gamma_{\mathfrak{a}}(I^i) \stackrel{d^{i+1}}{\to} \Gamma_{\mathfrak{a}}(I^{i+1}) \to \dots$ and m is an element of $\frac{\ker d^{i+1}}{\operatorname{im} d^i}$. In particular, it is represented by an element of $\ker d^{i+1} \subset \Gamma_{\mathfrak{a}}(I^i)$ and so there is some n for which $\mathfrak{a}^n m = 0$. Hence $H^i_{\mathfrak{a}}(M) \subseteq \Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M))$.

Exercise 3.4. *a Assume that A is noetherian, Show that if* $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, *then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated.*

b Show inductively, for M finitely generated, that for any $n \geq 0$, the following conditions are equivalent:

(a) $\operatorname{depth}_{\mathfrak{a}} M \geq n$*;*

(b) $H^i_{\mathfrak{a}}(M) = 0$ *for all $i < n$.*

For the converse to part (a) we use some Commutative Algebra results that can be found in Section 3.1 of Eisenbud's "Commutative Algebra".

Solution. a If $\operatorname{depth}_{\mathfrak{a}} M \geq 1$ then there is some $x \in \mathfrak{a}$ which is not a zero divisor of M. Let $m \in \Gamma_{\mathfrak{a}}(M)$. Then there is some n for which $\mathfrak{a}^n m = 0$, and so $x^n m = 0$. But x is not a zero divisor and so $m = 0$.

Conversely, suppose that M is finitely generated and that $\Gamma_{\mathfrak{a}}(M) = 0$. So for any (nonzero) $m \in M$ and $n \geq 0$ there is an $x \in \mathfrak{a}^n$ such that $xm \neq 0$. This means that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for any associated prime $\mathfrak{p}$ of M (i.e. primes $\mathfrak{p}$ such that $\mathfrak{p} = Ann(m)$ for some $m \in M$). So $\mathfrak{a} \not\subseteq \cup_{\mathfrak{p} \in Ass(M)} \mathfrak{p}$ [Eisenbud, Lemma 3.3, Theorem 3.1(a)]. The latter set is the set of zero divisors of M (including zero) [Eisenbud, Theorem 3.1(b)] and so we find that there is an element $x \in \mathfrak{a}$ that is not a zero divisor in M. Hence $\operatorname{depth}_{\mathfrak{a}} M \geq 1$.

b

Exercise 3.5. *Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:*

a *depth $\mathcal{O}_P \geq 2$;*

b *If U is any open neighbourhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.*

Solution. First note that we can assume U is affine, since given a point P and an open subscheme containing it, there is an open affine subscheme V of U containing P, and a section of $\mathcal{O}_X(U)$ is the same as giving a section of $\mathcal{O}_X(V)$ and a section of $\mathcal{O}_X(U - P)$ which agree on $V - P$ since $\mathcal{O}_X$ is a sheaf. So suppose that $U = X = \operatorname{Spec} A$ is an affine noetherian scheme.

Secondly, note that we have the long exact sequence of Exercise III.2.3(e):

$$\cdots \to H^0_P(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X) \to H^0(U - P, \mathcal{O}_X) \to H^1_P(X, \mathcal{O}_X) \to \dots$$

So the second statement in the problem is equivalent to showing that $H^i_P(X, \mathcal{O}_X) = 0$ for $i = 0, 1$. By Exercise III.3.3 this is the same as showing that $H^i_{\mathfrak{p}}(A) = 0$ for $i = 0, 1$ where $\mathfrak{p}$ is the prime ideal of A corresponding to the point P. Furthermore, by Exercise III.3.4(b) this is the same as showing that $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ since A noetherian implies that A is finitely generated.

So we have reduced the problem to showing that $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ if and only if $\operatorname{depth} A_{\mathfrak{p}} \geq 2$. If $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ then there are $x_1, x_2 \in \mathfrak{p}$ such that x_1 is not a zero divisor of A and x_2 is not a zero divisor of A/x_2. We can consider the x_i as elements of $A_{\mathfrak{p}}$ and so we get a regular sequence of length 2 of $A_{\mathfrak{p}}$. Conversely, if $\operatorname{depth} A_{\mathfrak{p}} \geq 2$ then there is a regular sequence $\frac{x_1}{s_1}, \frac{x_2}{s_2} \in \mathfrak{p} A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ where $x_1, x_2 \in \mathfrak{p}$ and $s_1, s_2 \in A \backslash \mathfrak{p}$. It can be seen that x_1, x_2 is then a regular sequence for A and so $\operatorname{depth}_{\mathfrak{p}} A \geq 2$.

Exercise 3.6. *Let X be a noetherian scheme.*

a *Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.*

b *Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.*

c *Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.*

Solution. a Recall that the sheaf $\mathscr{G}$ is constructed as follows. Cover X with a finite number of open affines $U_i = \operatorname{Spec} A_i$, and let $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Embed M_i in an injective A_i-module I_i. For each i let $f_i : U_i \to X$ be the inclusion, and let $\mathscr{G} = \oplus f_{i*}(\widetilde{I_i})$. Now suppose we have an inclusion of quasi-coherent sheaves $0 \to \mathscr{F}' \to \mathscr{F}$ and a morphism $\mathscr{F}' \to \mathscr{G}$. We want to show that this lifts to a morphism $\mathscr{F} \to \mathscr{G}$.

First notice that for each i, a morphism $\mathscr{F}' \to f_{i*}\widetilde{I}_i$ corresponds to a morphism $\mathscr{F}'|_{U_i} \to \widetilde{I}_i$ which lifts to $\mathscr{F}|_{U_i} \to \widetilde{I}_i$ (since I_i is injective), and this corresponds to a morphism $\mathscr{F} \to f_{i*}\widetilde{I}_i$. So each $f_{i*}\widetilde{I}_i$ is injective. Now notice that a direct sum $\oplus_{i=1}^n \mathscr{G}'_i$ of arbitrary injective objects $\mathscr{G}'_i$ is injective, since a morphism to $\oplus_{i=1}^n \mathscr{G}'_i$ is the same as an n-tuple of one morphism into each G'_i. Hence, $\mathscr{G}$ is injective, since it is a direct sum of injectives.

b By definition,the derived functors are calculated using injective resolutions. We have seen that the cohomology of a sheaf of abelian groups as it was defined in the text can be calculated using flasque resolutions. Hence, the derived functors of $\Gamma(X, \cdot)$ are the same as the cohomology groups $H^i(X, -)$.

Exercise 3.7. *Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.*

a For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \hom_A(\mathfrak{a}^n, M),$$

b Apply this in the case of an injective A-module I, to give another proof of (3.4).

Remark. A more general version of this is proved using a similar method in EGA I 6.9.17.

Solution. a First we define a morphism $\varinjlim_n \hom_A(\mathfrak{a}, M) \to \Gamma(U, \widetilde{M})$. Since A is noetherian, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_n)$. Furthermore, the basic opens $D(f_i)$ form a cover of U. This means that every section of $\Gamma(U, \widetilde{M})$ can be written as an element of $\oplus M_{f_i}$, and conversely, every element of $\oplus M_{f_i}$ which is in the kernel of $\oplus M_{f_i} \to \oplus M_{f_i f_j}$ defines a section of $\Gamma(U, \widetilde{M})$. So given a morphism $\phi : \mathfrak{a}^r \to M$ define a section by $\left(\frac{\phi(f_1^r)}{f_1^r}, \dots, \frac{\phi(f_n^r)}{f_n^r}\right)$. It can be checked fairly readily that this tuple actually does define a section (since $f_j^r \phi(f_i^r) - f_i^r \phi(f_j^r) = \phi(f_i^r f_j^r) - \phi(f_i^r f_j^r) = 0$) and furthermore, two representatives $\phi : \mathfrak{a}^r \to M$ and $\phi' : \mathfrak{a}^{r'} \to M$ of the same element of $\varinjlim_n \hom_A(\mathfrak{a}^r, M)$ give rise to the same section (since $\frac{\phi(f_i^{r+s})}{f_i^{r+s}} = \frac{f^s \phi(f_i^r)}{f_i^s f_i^r} = \frac{\phi(f_i^r)}{f_i^r}$). So we have a well-defined morphism

$$\varinjlim_n \hom_A(\mathfrak{a}^r, M) \to \Gamma(U, \widetilde{M})$$

This morphism is injective: if an element represented by $\phi : \mathfrak{a}^r \to M$ gets sent to zero, then $\frac{\phi(f_i^r)}{f_i^r} = 0 \in M_{f_i}$ for each i and so $f_i^{s_i}\phi(f_i^r) = 0 \in M$ for some s_i. Since there are finitely many f_i choose some $s > s_i$ so that we have $f_i^s\phi(f_i^r) = 0 \in M$ for all i. We then consider R big enough so that $\mathfrak{a}^R$ is generated by f_i^{s+r} (for example, $R > n(s+r)$) and the induced morphism $\phi : \mathfrak{a}^R \to M$ is consequently zero, since $\phi(f_i^{s+r}) = f_i^s\phi(f_i^r) = 0$.

Now to see that the morphism is surjective. Choose a section $f \in \Gamma(U, \widetilde{M})$. As already mentioned, this section gives rise to a tuple $(\frac{m_1}{f_1^{r_1}}, \dots, \frac{m_n}{f_n^{r_n}})$. By replacing $\frac{m_i}{f_i^{r_i}}$ with $\frac{f_i^{r-r_i}m_i}{f_i^r}$ where $r = \max r_i$ we can assume that all the r_i are the same. Since the tuple $(\frac{m_1}{f_1^{r_1}}, \dots, \frac{m_n}{f_n^{r_n}})$ came from a section $f \in \Gamma(U, \widetilde{M})$, for each i, j we have $(f_if_j)^{s_{ij}}(f_im_j - f_jm_i) = 0 \in M$ for some s_{ij}. Again, we can choose s big enough so that we can assume $s_{ij} = s$ for all i, j. Now define $m_i' = f_i^s m_i$. So we have $(f_i^{r+s}f - m_i')|_{D(f_j)} = (f_i^{r+s}\frac{m_j}{f_j^r} - f_i^s m_i) = \frac{1}{f_j^r}(f_i^{r+s}m_j - f_i^s f_j^r m_i) = \frac{1}{f_j^{r+s}}(f_if_j)^s(f_i^r m_j - f_j^r m_i) = 0$. The point of this is that since the $D(f_j)$ cover U, and we have $(f_i^{r+s}f - m_i')|_{D(f_j)} = 0$ for each j, we now have the relation

$$f_i^{r+s}f = m_i'$$

on U for each i.

Now choose R big enough so that $\mathfrak{a}^R$ is generated by the f_i^{r+s} (for example $R > n(r+s)$) and define a morphism $\phi : \mathfrak{a}^R \to M$ by sending $\sum a_i f_i^{r+s}$ to $(\sum a_i m_i')|_U$ (note that $a_i \in A$ are global sections of $\Gamma(X, \mathcal{O}_X)$). We need to check that this is a well defined homorphism. Suppose that $\sum a_i f_i^{r+s} = 0$. Then we need $(\sum a_i m_i')|_U$ to be zero also. But we have $(\sum a_i m_i')|_U = \sum(a_i f_i^{r+s}f) = (\sum a_i f_i^{r+s})f = 0$ and so we really do have a well defined morphism. Moreover, the image of the morphism ϕ in $\Gamma(U, \widetilde{M})$ is $(\frac{m_1'}{f_1^{r+s}}, \dots, \frac{m_n'}{f_n^{r+s}}) = (\frac{f_1^s m_1}{f_1^{r+s}}, \dots, \frac{f_n^s m_n}{f_n^{r+s}}) = (\frac{m_1}{f_1^r}, \dots, \frac{m_n}{f_n^r}) = f$, the section we started with. So we have lifted $f \in \Gamma(U, \widetilde{M})$ to an element of $\varinjlim_n \hom_A(\mathfrak{a}, M)$ and consequently, the morphism $\varinjlim_n \hom_A(\mathfrak{a}, M) \to \Gamma(U, \widetilde{M})$ is surjective.

b Suppose I is an injective A-module. We want to show that for any two open subsets $V \subseteq U$, the restriction morphism $\Gamma(U, \widetilde{I}) \to \Gamma(V, \widetilde{I})$ is surjective. Using Deligne's formula, we can write the restriction as

$$\varinjlim_n \hom_A(\mathfrak{a}^n, I) \to \varinjlim_n \hom_A(\mathfrak{b}, I)$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are the (radical) ideals of the closed complements of U and V respectively. Since $V \subseteq U$, we have $V(\mathfrak{b}) \supseteq V(\mathfrak{a})$ and since we assumed $\mathfrak{a}$ and $\mathfrak{b}$ to be radical this implies $\mathfrak{b} \subseteq \mathfrak{a}$. The point is that this is an inclusion of A-modules, and so given a representative $\phi : \mathfrak{b}^n \to I$ is an element of $\varinjlim_n \hom_A(\mathfrak{b}, I)$, the fact that I is injective implies that there is a lifting to $\mathfrak{a}^n \to I$ Since $0 \to \mathfrak{b}^n \to \mathfrak{a}^n$ is an exact sequence of A-modules. Hence, the restriction homomorphism is surjective and so $\widetilde{I}$ is flasque.

Exercise 3.8. *Let $A = k[x_0, x_1, x_2, \dots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \dots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.*

Solution. Suppose that $I \to I_{x_0}$ is surjective. Then there is some $m \in I$ which gets sent to $\frac{1}{x_0}$. That is, $x_0^n(x_0 m - 1) = 0$ in I for some n. Multiplying by x_{n+1} and using the relation $x_{n+1}x_0^{n+1} = 0$ gives $x_0^n x_{n+1} = 0$ in A. But this is not true, and so we have a contradiction. Hence $I \to I_{x_0}$ is not surjective.

4 Čech Cohomology

Exercise 4.1. *Let $f : X \to Y$ be an affine morphism of noetherian separated schemes. Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geq 0$*

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

Solution. Let $\{V_i\}$ be an open affine cover of Y. Since f is an affine morphism the set of preimages $\{U_i = f^{-1}V_i\}$ form an open affine cover of X. Furthermore, since X and Y are separated, the intersections $V_{i_0,\dots,i_p}$ and $U_{i_0,\dots,i_p} = f^{-1}V_{i_0,\dots,i_p}$ are also affine. Let $U_{i_0,\dots,i_p} = \operatorname{Spec} A_{i_0,\dots,i_p}$ and $V_{i_0,\dots,i_p} = \operatorname{Spec} B_{i_0,\dots,i_p}$. As $\mathscr{F}$ is quasi-coherent its restrictions to each $U_{i_0,\dots,i_p}$ are of the form $\mathscr{F}|_{U_{i_0,\dots,i_p}} \cong \widetilde{M}_{i_0,\dots,i_p}$ where $M_{i_0,\dots,i_p}$ is an $A_{i_0,\dots,i_p}$-module. Proposition II.5.2(d) says that $f_*\mathscr{F}|_{V_{i_0,\dots,i_p}} = ({}_{B_{i_0,\dots,i_p}}M_{i_0,\dots,i_p})^\sim$.

Now consider the appropriate Čech complexes. In degree p we have

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} M_{i_0,\dots,i_p} \qquad \text{and} \qquad C^p(\mathfrak{V}, f_*\mathscr{F}) = \prod_{i_0 < \cdots < i_p} {}_{B_{i_0,\dots,i_p}}M_{i_0,\dots,i_p}$$

As complexes of abelian groups, these are identical and so their cohomology groups are the same. Since we can use Čech complexes of affine covers to compute the cohomology of quasi-coherent sheaves (Theorem III.4.5) we find the natural isomorphisms required.

Exercise 4.2. *Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.*

- a *Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomrophism at the generic point of Y.*

- b *For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.*

c *Now prove Chevalley's theorem. First use Exercise III.3.1 and Exercise III.3.2 to reduce to the case X and Y integral. Then use Theorem 3.7, Exercise 4.1, consider* $\ker\beta$ *and* $\operatorname{coker}\beta$*, and use noetherian induction on* Y.

Solution. a If we apply $\mathscr{H}om(\cdot,\mathscr{F})$ to α we get a morphism $\mathscr{H}om(f_*\mathscr{M},\mathscr{F}) \to \mathscr{H}om(\mathcal{O}_Y^r,\mathscr{F})$ which is an isomorphism at the generic point (to see this consider an affine neighbourhood of the generic point). We have an isomorphism $\mathscr{H}om(\mathcal{O}_Y^r,\mathscr{F}) \cong \mathscr{F}$ and by Exercise II.5.17, since $\mathscr{H}om(f_*\mathscr{M},\mathscr{F})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, there is a quasi-coherent $\mathcal{O}_X$-module $\mathscr{G}$ such that $\mathscr{H}om(f_*\mathscr{M},\mathscr{F}) \cong f_*\mathscr{G}$.

b The morphism $f : X \to Y$ induces a morphism $f_{red} : X_{red} \to Y_{red}$ which is still surjective (since the underlying topological spaces are the same) and still finite (since if a B-algebra A is finitely generated as a B-module then B_{red} is finitely generated as a A_{red}-module). Exercise III.3.1 says that Y_{red} is affine if and only if Y is and so we can assume that X and Y are reduced.

Now for each connected component Y' of Y, the induced morphism $f^{-1}Y' \to Y'$ is still surjective, and finite because the other three morphisms in the commutative square

$$\begin{array}{ccc} f^{-1}Y' & \rightarrowtail & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

are finite (closed immersions are finite). Since it is surjective, we can consider an irreducible component X' of $f^{-1}Y'$ that contains a point in the preimage of the generic point of Y'. The induced morphism $X' \to Y'$ is finite, since it is a composition of finite morphisms $X' \to f^{-1}Y' \to Y'$ (closed immersions are finite). Now since X is affine, each irreducible component is (since a closed subscheme of an affine scheme is affine (Exercise II.3.11(b)) and Exercise III.3.2 says that Y is affine if and only if each irreducible component is. So we can assume X and Y irreducible.

So we can assume X, Y integral. Now we use Theorem III.3.7 to show that Y is affine. The goal is to show that for any coherent sheaf of ideals $\mathscr{I}$ we have $H^1(Y,\mathscr{I}) = 0$. So let $\mathscr{I}$ be a coherent sheaf of ideals on Y. Then by part (b) we have a coherent sheaf $\mathscr{G}$ on X and a morphism $\beta : f_*\mathscr{G} \to \mathscr{I}^r$ which is an isomorphism at the generic point. This gives an exact sequence $0 \to \ker\beta \to f_*\mathscr{G} \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$ which we break up into to short exact sequences

$$0 \to \ker\beta \to f_*\mathscr{G} \to \operatorname{im}\beta \to 0 \qquad 0 \to \operatorname{im}\beta \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$$

which give rise to long exact sequences on cohomology. Since $H^i(Y, f_*\mathscr{G}) = H^i(X,\mathscr{G})$ (Exercise III.4.1) and X is affine, we have $H^i(Y, f_*\mathscr{G}) = 0$ for all $i > 0$ (Theorem III.3.7) and so $H^i(Y, \operatorname{im}\beta) \cong H^{i+1}(Y, \ker\beta)$ for $i > 0$.

On the other hand, since β is an isomorphism at the generic point, both $\ker\beta$ and $\operatorname{coker}\beta$ are zero at the generic point, and therefore have support in some closed subscheme, necassarily of smaller dimension than Y. That is, we have $\ker\beta = i_*i^*\ker\beta$ where $i : Z \to X$ is the closed immersion of the support and similarly for $\operatorname{coker}\beta$. By the inductive hyopothesis and Exercise III.4.1, we then have that $H^i(Y, \ker\beta) = H^i(X, i^*\ker\beta) = 0$ for $i > 0$ and similarly, for $\operatorname{coker}\beta$. Putting this together with the isomorphism $H^i(Y, \operatorname{im}\beta) \cong H^{i+1}(Y, \ker\beta)$ described above, we see that $H^i(Y, \operatorname{im}\beta) = 0$ for $i > 0$ as well and so putting these into the long exact sequence associated to the short exact sequence $0 \to \operatorname{im}\beta \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$ we obtain finally $H^i(Y, \mathscr{I}^r) = H^i(Y, \mathscr{I})^r = 0$ for $i > 0$. Hence, Y is affine by Theorem III.3.7.

Exercise 4.3. *Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^iy^j | i, j < 0\}$. In particular, it is infinite dimensional.*

Solution. Take the open cover $\{U_x = \operatorname{Spec} k[x, y, x^{-1}], U_y = \operatorname{Spec} k[x, y, y^{-1}]\}$. The intersection is $U_{xy} = U_x \cap U_y = \operatorname{Spec} k[x, y, x^{-1}, y^{-1}]$ and so the Čech complex of this cover is

$$0 \to k[x, y, x^{-1}] \oplus k[x, y, y^{-1}] \to k[x, y, x^{-1}, y^{-1}] \to 0 \to \dots$$

The first cohomology group of this complex is $k[x, y, x^{-1}, y^{-1}]$ over the image of the boundary morphism. This image consists of all polynomials which are linear combinations of monomials x^iy^j where at least one of i or j are not negative. Hence, the first cohomology group consists of linear combinations of monomials x^iy^j with $i, j < 0$.

Exercise 4.4. *a Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of the topological space X. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ (that is, a covering $\mathfrak{V} = \{V_j\}_{j\in J}$ together with a map $\lambda : J \to I$ of index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$), show that there is a natural induced map on the Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,*

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$$

b For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are compatible with the refinement maps above.

c Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

Solution. For each $p \geq 0$ and each tuple $(j_0, \ldots, j_p) \in J^{p+1}$ we have a morphism induced by the restriction morphisms $\mathscr{F}(U_{\lambda(j_0)\ldots\lambda(j_p)}) \to \mathscr{F}(V_{j_0\ldots j_p})$ which induces a morphism $C^p(\mathfrak{U}, \mathscr{F}) \to C^p(\mathfrak{V}, \mathscr{F})$. Since

$$\begin{aligned}
(\lambda^{i+1} d\alpha)_{j_0\ldots j_{p+1}} &= (d\alpha)_{\lambda(j_0)\ldots\lambda(j_{p+1})}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\ldots\lambda(\hat{j}_k)\ldots\lambda(j_{p+1})}|_{U_{\lambda(j_0)\ldots\lambda(j_{p+1})}}\Big|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\ldots\lambda(\hat{j}_k)\ldots\lambda(j_{p+1})}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\ldots\lambda(\hat{j}_k)\ldots\lambda(j_{p+1})}|_{V_{j_0\cdots\hat{j}_k\cdots j_{p+1}}}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k (\lambda^i \alpha)_{j_0\ldots\hat{j}_k\ldots j_{p+1}}|_{V_{j_0\cdots j_{p+1}}} \\
&= (d\lambda^i \alpha)_{j_0\ldots j_{p+1}}
\end{aligned}$$

we have commutative squares

$$\begin{tikzcd}
C^p(\mathfrak{U}, \mathscr{F}) \arrow[r, "d"] \arrow[d, "\lambda^i"] & C^{p+1}(\mathfrak{U}, \mathscr{F}) \arrow[d, "\lambda^{i+1}"] \\
C^p(\mathfrak{V}, \mathscr{F}) \arrow[r, "d"] & C^{p+1}(\mathfrak{V}, \mathscr{F})
\end{tikzcd}$$

and so we have a morphism of the Čech complexes $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F})$ induced by the restriction morphisms and λ. This induces a morphism on the Čech cohomology.

The maps $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ come from choosing an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of chain complexes and obtaining a map of chain complexes $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ unique up to homotopy. Our maps $\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$ from part (a) were induced by maps of chain complexes. Since the map $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ is unique up to homotopy, the map obtained as the composition $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F}) \to \mathscr{I}^\bullet$ is homotopic to $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ and therefore induces the same maps on cohomology. Therefore we have a commutative triangle

$$\begin{tikzcd}
\check{H}^i(\mathfrak{U}, \mathscr{F}) \arrow[r] \arrow[rr, bend left] & \check{H}^i(\mathfrak{V}, \mathscr{F}) \arrow[r] & H^i(X, \mathscr{F})
\end{tikzcd}$$

Exercise 4.5. *Show that* $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$ *for any ringed space* $(X, \mathcal{O}_X)$.

Solution. The map $\operatorname{Pic} X \to H^1(X, \mathcal{O}_X^*)$. Let $\mathscr{L}$ be an invertible sheaf on X. That is, a sheaf that is locally frem of rank one. By definition there is an open cover $\{U_i\}$ of X for which we have isomorphisms $\phi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$.

Restricting to the pairwise intersections we get isomorphisms $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \mathcal{O}_{U_{ij}} \to \mathscr{L}|_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ and on the triple intersections U_{ijk} the restriction of these isomorphisms satisfy the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ where ϕ_{ii} is the identity. Each of the isomorphisms ϕ_{ij} is determined by an element $\alpha_{ij} \in \mathcal{O}_X(U_{ij})$ (the image of the identity) which is a unit by consequence of the ϕ_{ij}'s being isomorphisms. The cocycle condition amounts to the relation $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ and $\alpha_{ii} = 1$. So the elements $\{\alpha_{ij}\}$ determine an element of $C^1(\{U_i\}, \mathcal{O}_X^*)$ which is a cocycle as a consequence of the cocycle conditions, as

$$(d\alpha)_{ijk} = \alpha_{jk}\alpha_{ik}^{-1}\alpha_{ij} = 1$$

So we have defined a map, of sets at least, from $\operatorname{Pic} X$ to $H^1(X, \mathcal{O}_X^*)$ via the morphisms $\check{H}^i(\mathfrak{U}, \mathcal{O}_X^*) \to H^i(X, \mathcal{O}_X^*)$.

Independance with respect to the ϕ_i. If we have chosen different isomorphisms $\phi_i' : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$ then we obtain isomorphisms $\psi_i = \phi_i^{-1} \circ \phi_i' : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ which correspond to elements $\gamma_i \in \mathcal{O}_X(U_i)^*$ (the image of the identity global section) as above. if $\alpha' = \{\alpha_{ij}'\}$ is the cocycle associated with the isomorphisms ϕ_i' then we have the relations $\alpha_{ij}^{-1}\alpha_{ij}' = \gamma_j^{-1}\gamma_i$ by the commutivity of the following diagram:

$$\begin{tikzcd}
\mathcal{O}_{U_{ij}} \arrow[r, "\phi_i'"] \arrow[d, "\psi_i"] & \mathscr{L}|_{U_i} \arrow[r, "\phi_j'^{-1}"] \arrow[d, equal] & \mathcal{O}_{U_{ij}} \arrow[d, "\psi_j"] \\
\mathcal{O}_{U_{ij}} \arrow[r, "\phi_i"] & \mathscr{L}|_{U_i} \arrow[r, "\phi_j^{-1}"] & \mathcal{O}_{U_{ij}}
\end{tikzcd}$$

Hence, $\alpha^{-1}\alpha'$ is a coboundary and so α and α' determine the same element in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$.

Compatibility with restriction (independence with respect to the cover). If $\mathscr{L}$, $\mathfrak{U}$ are as above and if $(\mathfrak{V}, \lambda)$ is a refinement of $\mathfrak{U}$ as in Exercise III.4.4, then by restricting a choice of isomorphism ϕ_i for $\mathfrak{U}$, we get isomorphisms for $\mathfrak{V}$ for which the corresponding cocycle $\beta = \{\beta_{k\ell}\}$ in $C^1(\mathfrak{V}, \mathcal{O}_X^*)$ is precisely the image of the cocycle $\alpha = \{\alpha_{ij}\}$ obtained from the ϕ_i under the morphism $C^1(\mathfrak{U}, \mathcal{O}_X^*) \to C^1(\mathfrak{V}, \mathcal{O}_X^*)$ described in Exercise III.4.4(a). So via Exercise III.4.4(b) we see that the image of $\mathscr{L}$ in $H^1(X, \mathcal{O}_X^*)$ is independent of the cover chosen.

Compatibilty with the group structure. If we have two invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ then choose a cover $\mathfrak{U} = \{U_i\}$ on which both sheaves are trivial. Then $\mathscr{L} \otimes \mathscr{M}$ is trivial on this cover as well, and we can take the isomorphisms $\phi_i : \mathcal{O}_{U_i} \cong \mathscr{L} \otimes \mathscr{M}|_{U_i}$ to be $\phi_{i,\mathscr{L}} \otimes \phi_{i,\mathscr{L}}$ where $\phi_{i,\mathscr{L}}$ and $\phi_{i,\mathscr{L}}$ are isomorphisms for $\mathscr{L}$ and $\mathscr{M}$ respectively. It is now straightforward to see that the cocycle for $\mathscr{L} \otimes \mathscr{M}$ and is the product of that for $\mathscr{L}$ and that for $\mathscr{M}$, so the map $\operatorname{Pic} X \to H^1(X, \mathcal{O}_X)$ is actually a group homomorphism.

The map is an isomorphism. To see that the map defined is an isomorphism we construct an inverse via the isomorphism $\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ discussed in Exercise III.4.4(c). This isomorphism implies that *every* element of $H^1(X, \mathscr{F})$ can be realized as an element of $\check{H}^1(\mathfrak{U}, \mathscr{F})$ for some cover $\mathfrak{U}$. So given an element of $H^1(X, \mathcal{O}_X^*)$ there is a cover $\mathfrak{U}$ for which the element is represented by a cocycle $\{\alpha_{ij}\} \in C^1(\mathfrak{U}, \mathcal{O}_X^*)$. By virtue of the fact that $\{\alpha_{ij}\}$ is a cocycle, these α_{ij} define isomorphisms $\mathcal{O}_{U_i}|_{U_{ij}} \to \mathcal{O}_{U_j}|_{U_{ij}}$ which satisfy the necassary condition for us to be able to glue the $\mathcal{O}_{U_i}$ together into an invertible sheaf (Exercise II.1.22). By construction it can be seen that this provides an inverse.

Exercise 4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{I}$ be a sheaf of ideals with $\mathscr{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathscr{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,*

$$\dots \to H^1(X, \mathscr{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathscr{I}) \to \dots$$

Solution. Checking that the sequence is exact on stalks is fairly straightforward. As a consequence we have an exact sequence

$$\dots \to H^1(X, \mathscr{I}) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{O}_{X_0}^*) \to H^2(X, \mathscr{I}) \to \dots$$

and the required exact sequence then follows from Exercise III.4.5 above.

Exercise 4.7. *Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d (without assuming that f is irreducible). Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex*

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitely, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1$$

$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

Solution. There is a standard cover of $\mathbb{P}^2$ consisting of the opens $U_0 = \{x_0 \neq 0\}, U_1 = \{x_1 \neq 0\}, U_2 = \{x_2 \neq 0\}$ and so $\{U_0 \cap X, U_1 \cap X, U_2 \cap X\}$ is an open cover of X. Since closed subschemes of affine schemes are affine, this is an affine cover. The only point of $\mathbb{P}^2$ not in U_1 or U_2 is $(1, 0, 0)$ and since this is not in X, the open affine $U_0 \cap X$ can be removed from the set and it will still be an open affine cover.

The Čech complex is then

$$\begin{array}{ccc} \dfrac{k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]}{f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})} \oplus \dfrac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]}{f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)} & \to & \dfrac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]}{f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)} \\ (\overline{g}(\dfrac{x_0}{x_1}, \dfrac{x_2}{x_1}), \overline{h}(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2})) & \mapsto & \overline{g}(\dfrac{x_0}{x_2}\dfrac{x_2}{x_1}, \dfrac{x_2}{x_1}) - \overline{h}(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}) \end{array}$$

which is written more legibly as

$$\frac{k[u, v]}{f(u, 1, v)} \oplus \frac{k[x, y]}{f(x, y, 1)} \quad \to \quad \frac{k[x, y, y^{-1}]}{f(x, y, 1)}$$

$$(\overline{g}(u,v),\overline{h}(x,y)) \quad \mapsto \quad \overline{g}(xy^{-1},y^{-1}) - \overline{h}(x,y)$$

If $(\overline{g},\overline{h})$ is in the kernel of this morphism then $g-h$ is in the ideal generated by $f(x,y,1)$. So $g-h = f'f$ for some $f' \in k[x,y,y^{-1}]$. Now the assumption that $(1,0,0)$ is not a point implies $f(x_0,x_1,x_2) = \widetilde{f} + a_0x_0^d$ for some $\widetilde{f}$ and some nonzero a_0. Since scaling by units doesn't change the variety, we can assume that $a_0 = 1$. So we have $f(x,y,1) = \sum_{0\leq i\leq d, 0\leq j\leq d} a_{ij}x^iy^j$ with $a_{0d} = 1$. The polynomial f' in the expression $g-h = f'f$ is a linear combination of monomials. Write it as $f' = f_0 + f_1 + f_2$ where the f_k are linear combinations of monomials x^iy^j with $i \leq -d-j$ for f_0, with $j \geq 0$ for f_1, and with $j < 0$ and $i > -d-j$ for f_2. The point is that f_0f is in the image of $\frac{k[u,v]}{f(u,1,v)}$ and f_1f is in the image of $\frac{k[x,y]}{f(x,y,1)}$ and the monomials spanning these images overlap only on the constant term. So if we can show that f_2 is necassarily zero, then we necassarily have $g = f_0f + g_0$ and $h = -f_1f + h_0$ where g_0 and h_0 are constants. So it will imply that $(\overline{g},\overline{h})$ represents the same element as one where g and h are constant, and therefore equal.

To see that f_2 is necassarily zero consider a summand of it $a_{ij}x^i, y^j$ with i maximal and j minimal. Then $a_{ij}x^{i+d}y^j$ is a summand of f_2f. but f_2f is in the image of the boundary map of the Čech complex so either $i+d \leq -j$ or $j \geq 0$, both of which contradict our assumptions on f_2. Hence, $(\overline{g},\overline{h})$ represents the same element as one where g and h are constants, and therefore equal, so the kernel is (a,a) with $a \in k$ and therefore $\dim H^0(X,\mathcal{O}_X) = 1$.

Consider now the cokernel. Each element of the cokernel can be represented by a polynomial in $k[x,y,y^{-1}]$. Write it as a linear combination of monomials $\sum_{i\geq 0, j\in\mathbb{Z}} a_{ij}x^iy^j$. Any mononial with $j \geq 0$ represents zero in the cokernel as it is the image of $(0,x^iy^j))$. Similarly, any monomial with $j \geq i$ is the image of $(u^iv^{j-i},0)$. So we can represent an element of the cokernel with a polynomial $\sum_{j<0,j<i} a_{ij}x^iy^j$. Now the assumption that $(1,0,0)$ is not a point implies $f(x_0,x_1,x_2) = \widetilde{f} + a_0x_0^d$ for some $\widetilde{f}$ and some nonzero a_0. Since scaling by units doesn't change the variety, we can assume that $a_0 = 1$. Hence, in the ring $\frac{k[x,y,y^{-1}]}{f(x,y,1)}$ we have the relation $x^d = -\widetilde{f}(x,y,1)$ where $\widetilde{f}(x,y,1)$ linear combination of monomials x^iy^j with $0 \leq i < d$ and $0 \leq j$. Coming back to the cokernel, this means that every element of the cokernel can be represented by a polynomial of the form $\sum a_{ij}x^iy^j$ where $1 \leq i < d$ and $-i < j < 0$. So $\dim H^1(X,\mathcal{O}_X) \leq \frac{1}{2}(d-1)(d-2)$. To show that equality holds we need to show that polynomials of this form don't represent zero elements of the cokernel. Clearly, they are not in the image of the boundary map, by the argument already given, so we just need to show that they are not in the ideal generated by $f(x,y,1)$. But since $f(x,y,1) = x^d + \widetilde{f}(x,y,1)$ if they were, there would be a fact or of x with power $\geq d$. So we have equality.

Exercise 4.8. Cohomological Dimension. *Let X be a noetherian separated scheme.*

a *In the definition of $cd(X)$ show that it is sufficient to consider only coherent sheaves on X.*

b *If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.*

c *Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that* $\mathrm{cd}(X) \leq r$.

d *If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that* $\mathrm{cd}(X) \leq \dim X$.

e *Let Y be a set theoretic complete intersection of codimension r in $X = \mathbb{P}^n_k$. Show that* $\mathrm{cd}(X-Y) \leq r-1$.

Solution. a Suppose that $H^i(X, \mathscr{F}) = 0$ for all $i > n$ and all coherent sheaves $\mathscr{F}$. If $\mathscr{F}$ is a quasi-coherent sheaf then it is the union of its coherent subsheaves (Exercise II.5.15(a)), that is, $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$ where $\mathscr{F}_\alpha$ are the coherent subsheaves. Then by Proposition 2.9 for $i > n$ we have $H^i(X, \mathscr{F}) = H^i(X, \varinjlim \mathscr{F}_\alpha) \cong \varinjlim H^i(X, \mathscr{F}_\alpha) = \varinjlim 0 = 0$.

b By Proposition II.5.18 every coherent sheaf $\mathscr{F}$ can be written as a quotient of a finite rank locally free sheaf $\mathscr{E}$ so we have a short exact sequence $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$ which gives rise to an exact sequence

$$\cdots \to H^i(X, \mathscr{E}) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{G}) \to H^{i+1}(X, \mathscr{E}) \to \ldots$$

So if $H^i(X, \mathscr{E})$ for all locally free sheaves $\mathscr{E}$ and all $i > n$ then $H^i(X, \mathscr{F}) \cong H^{i+1}(X, \mathscr{G})$ for all $i > n$. Grothendieck's Theorem says that $H^i(X, \mathscr{G}) = 0$ for $i > \dim X$ so by induction, $H^i(X, \mathscr{F}) = 0$ for $i > n$.

c Since X is separated, we can use the Čech cohomology of an affine cover to calculate the cohomology of X. If there are only $r+1$ elements in the cover $\mathfrak{U}$ then for $p > r$ there are no p-tuples of indicies $(i_0, \ldots, i_p)$ with $i_0 < \cdots < i_p$ and so $C^p(\mathfrak{U}, \mathscr{F}) = 0$ and hence $H^i(X, \mathscr{F}) = 0$ for $p > r$ and therefore $cd(X) \leq r$.

d By definition if Y is a set-theoretic complete intersection of codimension r then it is the intersection of r hypersurfaces. The complement of each of these hypersurfaces is an affine variety (Proposition II.2.5) and so these r complements form an affine cover of $X - Y$ which is separated by virtue of it being projective (Theorem 4.9). So it follows from part (c) of this exercise that $cd(X-Y) \leq r-1$.

Exercise 4.9. *Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k, Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbb{P}^4_k$ is also not a set-theoretic complete intersection.*

Solution. If Y is a set theoretic complete intersection then $cd(X-Y) \leq 1$ (the same proof as for Exercise III.4.8(e) works). So to show that Y is not a complete intersection then we just need to show that $H^2(X-Y, \mathscr{F})$ for some quasi-coherent sheaf $\mathscr{F}$. Consider $\mathcal{O}_X$. We have the exact sequence from Exercise III.2.3:

$$\cdots \to H^2(X, \mathcal{O}_X) \to H^2(X-Y, \mathcal{O}_X) \to H^3_Y(X, \mathcal{O}_X) \to H^3(X, \mathcal{O}_X) \to \ldots \quad (1)$$

Since X is affine, we have $H^i(X, \mathcal{O}_X)$ for $i > 0$ and so $H^2(X-Y, \mathcal{O}_X) \to H^3_Y(X, \mathcal{O}_X)$ is an isomorphism. So our task is reduced to showing that $H^3_Y(X, \mathcal{O}_X) \neq 0$. Now consider the following exact sequence from Exercise III.2.4:

$$\begin{aligned} \ldots H^3_P(X, \mathcal{O}_X) &\to H^3_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \\ &\to H^3_Y(X, \mathcal{O}_X) \to H^4_P(X, \mathcal{O}_X) \to H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^4_{Y_2}(X, \mathcal{O}_X) \to \ldots \end{aligned} \quad (2)$$

Using a similar exact sequence to 1 we see that $H^i_{Y_j}(X, \mathcal{O}_X) \cong H^{i-1}(X-Y_j, \mathcal{O}_X)$ for $i = 3, 4$ and the later is zero since $X - Y_j$ is covered by the two open affines $x_{2j-1} \neq 0$ and $x_{2j} \neq 0$, and so the Čech complex is zero in these degrees (Exercise III.4.8(c)). Hence we have an isomorphism $H^3_Y(X, \mathcal{O}_X) \xrightarrow{\sim} H^4_P(X, \mathcal{O}_X)$ and so we want to show that $H^4_P(X, \mathcal{O}_X) \neq 0$.

Consider the exact sequence:

$$\cdots \to H^3(X, \mathcal{O}_X) \to H^3(X-P, \mathcal{O}_X) \to H^4_P(X, \mathcal{O}_X) \to H^4(X, \mathcal{O}_X) \to \ldots \quad (3)$$

Since X is affine, we have $H^i(X, \mathcal{O}_X)$ for $i > 0$ and so $H^3(X-P, \mathcal{O}_X) \to H^4_P(X, \mathcal{O}_X)$ is an isomorphism. Now we can calculate $H^3(X-P, \mathcal{O}_X)$ explicitely using the Čech complex of the cover $\mathfrak{U}$ consisting of the U_i with $x_i \neq 0$. We have $C^4(\mathfrak{U}, \mathcal{O}_X) = 0$ because there are four elements in the cover, so the cohomology group in question is the cokernel of $C^2(\mathfrak{U}, \mathcal{O}_X) \to C^3(\mathfrak{U}, \mathcal{O}_X)$. This morphism is

$$\bigoplus_{i=1}^{4} A_i \to k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}]$$

where A_i is $k[x_1, x_2, x_3, x_4]$ with x_j inverted for all $i \neq j$. The image of this morphism is spanned by all the monomials $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ such that at least one i_j is not negative. So the cokernel (and hence the cohomology) is spanned by monomials $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ with all $i_j < 0$. In particular, it is not zero.

So $H^3(X-P, \mathcal{O}_X) \neq 0$ and therefore $H^4_P(X, \mathcal{O}_X) \neq 0$ by 3 and so $H^3_Y(X, \mathcal{O}_X) \neq 0$ by 2 and therefore $H^2(X-Y, \mathcal{O}_X) \neq 0$ by 1. Hence, $cd(X-Y) > 1$ and so Y is not a set theoretic complete intersection.

Now if $\overline{Y}$ was a set theoretic complete intersection then we could restrict the two relavlent hypersurfaces to $\mathbb{A}^4$ and find that Y is a set theoretic complete intersection. But we have just proven that Y isn't, and so therefore, $\overline{Y}$ isn't either.

Exercise 4.10. *Let X be a nonsingular variety over an algebraically closed field k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathscr{F}$ up to isomorphism, and the group $H^1(X, \mathscr{F} \otimes \mathscr{T})$, where $\mathscr{T}$ is the tangent sheaf of X.*

Exercise 4.11. *Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = \{U_i\}$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$ that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geq 0$, the natural maps*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more generaly result.

Solution. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Consider the double complex $E_0^{p,q} = \prod_{i_0 < \cdots < i_p} \mathscr{I}^p(U_{i_0,\ldots,i_q})$. There are two spectral sequences associated to this double complex, one coming from the filtration of the total complex by columns and the other by rows.

Since for any open subset U and any i the sheaf $\mathscr{I}^i|_U$ is injective as a sheaf of abelian groups on U, the restriction $0 \to \mathscr{F}|_U \to \mathscr{I}^\bullet|_U$ is an injective resolution of $\mathscr{F}|_U$. So the "horizontal" cohomology groups $E_1^{p,q} \stackrel{def}{=} H^p(E_0^{\bullet,q})$ of this complex calculate the cohomology of $\mathscr{F}|_U$. By assumption, we then have

$$E_1^{p,q} = \begin{cases} C^q(\mathscr{F}, \mathfrak{U}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

The "vertical" differentials $E_0^{0,q} \to E_0^{0,q+1}$ induce the usual differentials on the complex $C^q(\mathscr{F}, \mathfrak{U})$ and so the "vertical" cohomology groups of E_1 are

$$E_2^{p,q} \stackrel{def}{=} H^q(E_1^{p,\bullet}) = \begin{cases} \check{H}^q(\mathscr{F}, \mathfrak{U}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now suppose we start with the vertical differentials first. So we define ${}'E_1^{p,q} \stackrel{def}{=} H^q(E_0^{p,\bullet})$. These calculate the Čech cohology of the sheaves $\mathscr{I}^p$. Since the $\mathscr{I}$ are flasque (Lemma III.2.4), their Čech cohomology vanishes in nonzero degree and so we have

$${}'E_1^{p,q} = \begin{cases} \Gamma(X, \mathscr{I}^p) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As above, the horizontal differentials induce the usual morphisms on the complex $\Gamma(X, \mathscr{I}^\bullet)$ and so we have

$${}'E_2^{p,q} \stackrel{def}{=} H^p({}'E_1^{\bullet}, q) = \begin{cases} H^q(X, \mathscr{F}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

So the cohomology of the total complex is isomorphic to both $H^\bullet(X, \mathscr{F})$ and $\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$, hence, they are isomorphic.

5 The Cohomology of Projective Space

Exercise 5.1. *Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. if*

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Solution. Consder the long exact sequence of cohomology

$$\dots \overset{\delta^{i-1}}{\to} H^i(X, \mathscr{F}') \overset{\phi^i}{\to} H^i(X, \mathscr{F}) \overset{\psi^i}{\to} H^i(X, \mathscr{F}'') \overset{\delta^i}{\to} H^{i+1}(X, \mathscr{F}') \to \dots$$

Since it is exact, we have (for example) $\dim H^i(X, \mathscr{F}) = \dim\ker\delta^i + \dim\ker\psi^i$. Now noting that $H^i(X, -)$ is zero for $i > \dim X = n$ (Grothendieck's Theorem) we can write

$$\begin{aligned}
\chi(\mathscr{F}) &= \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}) \\
&= \sum_{i=0}^{n} (-1)^i (\dim\ker\delta^i + \dim\ker\psi^i) \\
&= \sum_{i=0}^{n} (-1)^i \Big(\dim\ker\delta^i + \dim\ker\psi^i + \dim\ker\phi^i - \dim\ker\phi^i \Big) \\
&= \sum_{i=0}^{n} (-1)^i (\dim\ker\phi^i + \dim\ker\psi^i) \\
&\qquad\qquad + \sum_{i=0}^{n} (-1)^i (\dim\ker\delta^i - \dim\ker\phi^i) \\
&= \sum_{i=0}^{n} (-1)^i (\dim\ker\phi^i + \dim\ker\psi^i) \\
&\qquad\qquad + \sum_{i=0}^{n} (-1)^i (\dim\ker\delta^i + \dim\ker\phi^{i+1}) \\
&\qquad\qquad - \dim\ker\phi^{n+1} - \dim\ker\phi^0 \\
&= \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}') + \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}'') - 0 - 0 \\
&= \chi(\mathscr{F}') + \chi(\mathscr{F}'')
\end{aligned}$$

We have that $\dim\ker\phi^{n+1}$ is zero from Grothendiecks Theorem (since $H^{n+1}(X, -) = 0$) and $\dim\ker\phi^0$ is zero since $\phi^0 : \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ is injective.

Exercise 5.2. *a Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$.*

b Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \dots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (Chapter I, Section 7).

Solution. a

b As a consequence of Theorem III.5.2(b), for each $n \geq n_0$ (where n_0 is the one from the statement of the theorem that depends on $\mathscr{F}$) and $i > 0$ we have $H^i(X, \mathscr{F}(n)) = 0$ and so $\chi(\mathscr{F}(n)) = \dim H^0(X, \mathscr{F}(n))$. That is, $P(n) = \dim M_n$, which is exactly the definition of the Hilbert function for M. Since this equality holds for $n \gg 0$, and $P(z)$ and $P_M(z)$ are both polynomials, it follows that $P(z) = P_M(z)$.

Exercise 5.3. Arithmetic Genus.

a *If X is integral, and k algebrically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that*

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$$

In particular, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

b *If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex 7.2), which apparently depended on the projective embedding.*

c *If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational.*

Solution. a As X is integral, it is isomorphic to a variety (Proposition II.4.10). So we can use Theorem I.3.4(a) to see that $H^0(X, \mathcal{O}_X) = k$. The desired result then follows from the definitions.

Exercise 5.4. a *Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very amply invertible sheaf on X. Show that there is a unique additive homomorphism*

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathscr{F}$ on X, $P(\gamma(\mathscr{F}))$ is the Hilbert polynomial of $\mathscr{F}$.

b *Now let $X = \mathbb{P}^r_k$. For each $i = 0, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that*

(a) *$K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0, \ldots, r\}$, and*

(b) *the map $P : K(X) \to \mathbb{Q}[z]$ is injective.*

Solution. a Since we have a map defined from the set of coherent sheaves (the free generators of $K(X)$) to $\mathbb{Q}[z]$ we just need to show that the map is compatible with the relations. That is, for every short exact sequence of coherent sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we want to show that $P(\gamma(\mathscr{F})) = P(\gamma(\mathscr{F}')) + P(\gamma(\mathscr{F}''))$. This follows immediately from the definition of the Hilbert polynomial and Exercise III.5.1.

b First suppose that (a) is indeed true and consider $P(\gamma(\mathcal{O}_{L_i}))$. We have $\mathcal{O}_{L_i} = i_* \mathcal{O}_{\mathbb{P}^i}$ for an appropriate linear embedding $i : \mathbb{P}^i \to \mathbb{P}^r$. We know the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^i}$ from the explicite calculations of Theorem 5.1 to be $\binom{i+z}{i}$. So an element $\sum a_i \gamma(\mathcal{O}_{L_i})$ of $K(X)$ gets sent to the polynomial $\sum a_i \binom{i+z}{i}$. If this is zero then by induction on the highest nonzero coefficient we see that each a_i is zero and so (b) is true.

Now having seen that (a) $\Rightarrow$ (b) we prove (a) and (b) together. The case $r = 0$ is trivially true so suppose that (a) and (b) are true for $\mathbb{P}^{r-1}$. By Exercise II.6.10 we have an exact sequence

$$K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1}) \to 0$$

where the first map is extension by zero. Suppose at the beginning we choose L_i such that $L_i \subseteq L_{r-1}$ for all $i < r$. The map P clearly factors through the first map of the exact sequence and so since the composition $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to \mathbb{Q}[z]$ is injective, we see that $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r)$ is injective. So $K(\mathbb{P}^r)$ has a subgroup $\mathbb{Z}^r$ with basis $\mathcal{O}_{L_i}$ for $i = 0, \dots, r-1$ and this subgroup is the kernel of the surjective morphism $K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1})$. The scheme $\mathbb{P}^r - \mathbb{P}^{r-1}$ is isomorphic to $\mathbb{A}^r$ and since $k[x_1, \dots, x_n]$ is a principle ideal domain $K(\mathbb{A}^r) = \mathbb{Z}$ generated by $\gamma(\mathcal{O}_{\mathbb{A}^r})$, which is in the image of $\gamma(\mathcal{O}_{\mathbb{P}^r})$ (see the proof of Exercise II.6.10). So $K(\mathbb{P}^r)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^r$. Since $\mathbb{Z}$ is projective, $Ext^1(\mathbb{Z}, \mathbb{Z}) = 0$ and so there are no nontrivial extensions and therefore, $K(\mathbb{P}^r) = \mathbb{Z}^{r+1}$, generated by $\gamma(\mathcal{O}_{L_i})$ for $i = 0, \dots, r$. We have already seen that (a) implies (b) and so (a) and (b) are both true for $\mathbb{P}^r$, completing the induction step.

Exercise 5.5. *Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection. Then:*

a for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

b Y is connected;

c $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

d $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

Solution. a

b We know from Theore III.5.1 that $H^0(X, \mathcal{O}_X) \cong k$ and from Theorem 5.2 that $H^0(Y, \mathcal{O}_Y)$ is a finitely generated k-algebra. From part (a) of this exercise, $H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is surjective and so $H^0(Y, \mathcal{O}_Y) \cong k$. If Y were to have more than one connected component then $\Gamma(Y, \mathcal{O}_Y)$ would have zero divisors. This is not the case and so Y is connected.

Exercise 5.6.

Exercise 5.7. *Let X (resp. Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.*

a If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

b $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{red} = \mathscr{L} \otimes \mathcal{O}_{X_{red}}$ is ample on X_{red}.

c Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

d Let $f : X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X.

Solution. a Let $\mathscr{F}$ be a coherent sheaf on Y. Then by Proposition III.5.3 there is some n_0 such that for each $n > n_0$ we have $H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) = 0$ for $i > 0$. Using the projection formula (Exercise II.5.1(d)) and Lemma III.2.10 we have $H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) = H^i(X, i_*(\mathscr{F} \otimes i^*\mathscr{L}^n)) = H^i(Y, \mathscr{F} \otimes i^*\mathscr{L}^n)$. Since $\mathscr{F}$ was arbitrary, it follows now from Proposition III.5.3 that $i^*\mathscr{L}$ is ample.

b Let $i : X_{red} \to X$ be the canonical closed immersion. Then we have $\mathscr{L} \otimes \mathcal{O}_{X_{red}} = i^*\mathscr{L}$ and so if $\mathscr{L}$ is ample, the ampleness of $i^*\mathscr{L}$ follows from part (a) of this exercise. Conversely, suppose that $i^*\mathscr{L}$ is ample. We use a similar strategy to Exercise III.3.1. We have a finite descending sequence

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots \supseteq 0$$

where $\mathscr{N}$ is the sheaf of nilpotents on X (finite since X is proper, and therefore finite type, over a noetherian base). At each d and n we have a short exact sequence $0 \to \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{G}_d \otimes \mathscr{L}^n \to 0$. giving rise to long exact sequences

$$\cdots \to H^i(X, \mathscr{G}_d \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{G}_d \otimes \mathscr{L}^n) \to$$

So if we can show that $H^i(X, \mathscr{G}_d \otimes \mathscr{L}^n) = 0$ for all i and all $n > n_0$ for some n_0, then it will follow by induction on d that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$ and $i > 0$.

Since $i^*\mathscr{L}$ is ample, and $\mathscr{G}_d$ are coherent sheaves on X_{red}, by Proposition III.5.3 and the finiteness of the filtration, there is some n_0 such that for all $n > n_0$ and $i > 0$ we have $H^i(X_{red}, i^*\mathscr{L}^n \otimes \mathscr{G}_d) = 0$. Since the $\mathscr{G}_d$

are already $\mathcal{O}_{X_{red}}$-modules, cohomology is defined via sheaves of abelian groups, and the fact that X and X_{red} share the same underlying topological space, we have $H^i(X, \mathscr{L}^n \otimes \mathscr{G}_d) = H^i(X_{red}, i^*\mathscr{L}^n \otimes \mathscr{G}_d) = 0$ for all $n > n_0$ and $i > 0$. Via the above mentioned long exact sequences, this shows that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$ and $i > 1$, and that there is a sequence of surjections

$$\dots \to H^1(X, \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^1(X, \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n) \to \dots$$

Since $\mathscr{N}^d = 0$ for some d big enough, this is enough to show also that $H^1(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$. Hence, it follows from Proposition III.5.3 that $\mathscr{L}$ is ample.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10. *Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \dots \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0 such that for all $n \geq n_0$, the sequence of global sections*

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \dots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact.

Solution. Proof by induction. In the case $r < 3$ there is nothing to prove. In the case $r = 3$, we have a short exact sequence $0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \mathscr{F}^3 \to 0$. By Proposition III.5.2 there is an integer n_0 for $\mathscr{F}^1$ such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathscr{F}^1(n)) = 0$. So considering the long exact sequence

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \Gamma(X, \mathscr{F}^3(n)) \to H^1(X, \mathscr{F}^1(n)) \to \dots$$

we see that for $n > n_0$ the sequence $0 \to \Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \Gamma(X, \mathscr{F}^3(n)) \to 0$ is exact.

Now suppose the result is true for $r - 1$. Given an exact sequence of sheaves $0 \to \mathscr{F}^1 \to \dots \to \mathscr{F}^{n-2} \xrightarrow{f} \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$ we obtain two exact sequences

$$0 \to \mathscr{F}^1 \to \dots \to \mathscr{F}^{n-2} \xrightarrow{f} \operatorname{coker} f \to 0$$

$$0 \to \operatorname{coker} f \to \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$$

Choose n_0 bigger than the two n_0 provided for both of these exact sequences by the induction hypothesis. Then for each $n > n_0$ we have exact sequences

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \dots \to \Gamma(X, \mathscr{F}^{n-2}(n)) \xrightarrow{f} \Gamma(X, \operatorname{coker} f(n)) \to 0$$

$$0 \to \Gamma(X, \operatorname{coker} f(n)) \to \Gamma(X, \mathscr{F}^{n-1}(n)) \to \Gamma(X, \mathscr{F}^n(n)) \to 0$$

which we can stick back together to get a long exact sequence

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \dots \to \Gamma(X, \mathscr{F}^{n-2}(n)) \to \Gamma(X, \mathscr{F}^{n-1}(n)) \to \Gamma(X, \mathscr{F}^n(n)) \to 0$$

6 Ext Groups and Sheaves

Exercise 6.1. *Show that there is a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and element of the group* $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

Solution. In Hartshorne's statement of the exercise we are given a map $E = \{\text{extensions of } \mathscr{F}'' \text{ by } \mathscr{F}'\} \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$; we construct an inverse. Let

$$0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0 \tag{4}$$

be an embedding of into an injective sheaf and $\mathscr{G}$ the cokernel. From this short exact sequence we get a long exact sequence and from this, a surjection $\hom(\mathscr{F}'', \mathscr{G}) \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to 0$ since $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{I}) = 0$ as a consequence of $\mathscr{I}$ being injective. So we can lift our element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ to a morphism in $\hom(\mathscr{F}'', \mathscr{G})$. We then define $\mathscr{F} = \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ (pullbacks exist in $\mathfrak{Mod}(X)$ since kernels and products do, and in fact are defined component wise in the sense that $(\mathscr{I} \times_{\mathscr{G}} \mathscr{F}'')(U) = \mathscr{I}(U) \times_{\mathscr{G}(U)} \mathscr{F}''(U)$). The two morphisms $\mathscr{F}' \to \mathscr{I}$ and $\mathscr{F}' \xrightarrow{0} \mathscr{F}''$ define a morphism $\mathscr{F}' \to \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ and so we get a sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \tag{5}$$

which turns out to be exact (it is exact even as a sequence of presheaves; this is straight forward to check since $\mathscr{F} = \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ is defined component-wise). So now we have two morphisms of sets $E \leftrightarrows \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ and we just need to check that they are actually inverses to each other.

Suppose we start with an element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$, construct an extension as above, and then look at what element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ this gives us. There is a morphism from sequence 5 to sequence 4 and therefore a morphism between the long exact sequences obtained through $\operatorname{Ext}^i(\mathscr{F}'', -)$. One square in this is

$$\begin{tikzcd}
\cdots \arrow[r] & \hom(\mathscr{F}'', \mathscr{F}'') \arrow[r] \arrow[d] & \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \hom(\mathscr{F}'', \mathscr{G}) \arrow[r] & \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \arrow[r] & \cdots
\end{tikzcd}$$

The image of the identity morphism $\mathscr{F}'' \xrightarrow{id} \mathscr{F}''$ in $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is what we are concerned with. Following $\mathscr{F}'' \xrightarrow{id} \mathscr{F}''$ down and to the right gives us back the element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ that we started with. Since the right vertical morphism is the identity, this shows that the composition $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to E \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is the identity.

Now we will show that $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$ is surjective, and this together with $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to E \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ being the identity shows that the two maps in question are inverses to each other. Embed $\mathscr{F}'$ in an injective $\mathscr{I}$ and let $\mathscr{G}$ be the cokernel so that we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0$. Then by construction, every short exact sequence in the image of $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$ is of the form $0 \to \mathscr{F}' \to \mathscr{I} \times_{\psi, \mathscr{G}, \phi} \mathscr{F}'' \to \mathscr{F}'' \to 0$ for some morphism $\phi : \mathscr{F}'' \to \mathscr{G}$. So given a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we have to show it is of this form. Since $\mathscr{I}$ is injective and $\mathscr{F}' \to \mathscr{F}$ injective, the identity $\mathscr{F}' = \mathscr{F}'$ lifts to a morphism $\mathscr{F} \to \mathscr{I}$, and then since $\hom(-, \mathscr{G})$ is right exact we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, "id"] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{I} \arrow[r, "\psi"] & \mathscr{G} \arrow[r] & 0 \end{tikzcd}$$

So we have a sequence $0 \to \mathscr{F} \to \mathscr{F}'' \oplus \mathscr{I} \overset{\phi-\psi}{\to} \mathscr{G}$ and if this sequence is exact, then $\mathscr{F} \cong \mathscr{F}'' \times_{\mathscr{G}} \mathscr{I}$ and so we are done. Consider the stalks at a point x, so we obtain diagrams of $\mathcal{O}_x$-modules. In the world of modules, we can chase elements around diagrams, and in this way prove that for every morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r, "g"] & C' \arrow[r] & 0 \end{tikzcd}$$

results in an isomorphism $B \cong B' \times_{g,C',f} C$. So for every point x, the sequence $0 \to \mathscr{F}_x \to \mathscr{F}''_x \oplus \mathscr{I}_x \overset{\phi_x-\psi_x}{\to} \mathscr{G}_x$ is exact. This implies that it is an exact sequence of sheaves. So we have our isomomorphism $\mathscr{F} \cong \mathscr{F}'' \times_{\mathscr{G}} \mathscr{I}$ and therefore $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is in the image of $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$.

Exercise 6.2. *Let $X = \mathbb{P}^1_k$ with k an infinite field.*

- a *Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$.*
- b *Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$.*

Solution. a Suppose that we have such a projective object, with such a surjection. Let $x \in U$ be a point in U and let $V \subset U$ be a neighbourhood of x strictly smaller than U, so $U \neq V$, and consider the surjection $\mathcal{O}_V \to k(x) \to 0$ where $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, $j : V \to X$ is the inclusion, and $k(x)$ is the skyscraper sheaf at x with value the stalk $\mathcal{O}_x$ of $\mathcal{O}_X$ at x. The composition $\mathscr{P} \to \mathcal{O}_X \to k(x)$ gives a surjection $\mathscr{P} \to k(x)$ which then lifts to $\mathscr{P} \to \mathcal{O}_V$ by the assumption that $\mathscr{P}$ is projective, so we have a commutative square

$$\begin{tikzcd} \mathscr{P} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\ \mathcal{O}_V \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

Evalutating at U, we see that $\mathscr{P}(U) \to k(x) = k(x)(U)$ factors through zero since $\mathcal{O}_V(U) = 0$, so for every section in $\mathscr{P}(U)$ the stalk at x is zero. Since U and x were arbitrarily chosen, we see that every section in $\mathscr{P}(U)$ for every open U is zero at every point $x \in U$ and so $\mathscr{P} = 0$. But this contradicts the existence of the surjection $\mathscr{P} \to \mathcal{O}_X$.

b

Exercise 6.3. *Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.*

- a *If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geq 0$.*
- b *If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geq 0$.*

Solution. a We immediately reduce to the affine case since by definition $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, the sheaf $\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})|_U = \mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U)$ (Proposition III.6.2) is coherent and similarly, for $\mathscr{F}$ and $\mathscr{G}$. So since $X = \operatorname{Spec} A$ is affine, the sheaves $\mathscr{F}$ and $\mathscr{G}$ correspond to finitely generated A-modules M and N. From Exercise III.6.7 we then have $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) = \operatorname{Ext}^i_A(M, N)^{\sim}$ so we know that $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})$ is at least quasi-coherent, so we have proven part (b). Now since M is finitely generated and A is noetherian, we can construct inductively a resolution of M by finite rank free A-modules $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$. We then have $\operatorname{Ext}^i_A(M, N) = h^i(\hom_A(A^{n_\bullet}, N)) = h^i(N^{n_\bullet})$. Since N is finitely generated, so are the N^{n_i} and consequently, so are the $h^i(N^{n_\bullet}) = \operatorname{Ext}^i_A(M, N)$. Hence, $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) = \operatorname{Ext}^i_A(M, N)^{\sim}$ is quasi-coherent.

b Was proven in part (a).

Exercise 6.4. *Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$ show that the δ-functor $\mathscr{E}xt^i(\cdot, \mathscr{G})$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$ is a contravariant universal δ-functor.*

Remark. We assume the hypothesis "every coherent sheaf on X is a quotient of a locally free sheaf of *finite rank*" was intended.

Solution. By Theorem III.1.3A we just need to show that $\mathscr{E}xt^i(\cdot, \mathscr{G})$ is coeffaceable. Since every coherent sheaf $\mathscr{F}$ is the quotient of a locally free sheaf *of finite rank*, $\mathscr{L} \to \mathscr{F} \to 0$, it is enough to show that $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ for $\mathscr{L}$ locally free *of finite rank*. From Proposition III.6.2, to see that $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ it is enough to show that $\mathscr{E}xt^i(\mathscr{L}|_U, \mathscr{G}|_U) = 0$ for every U in an open cover of X. Choose a cover such that for each U we have $U = \operatorname{Spec} A$ for some A and $\mathscr{L}|_U \cong \oplus_{i=1}^n \mathcal{O}_U$. Then we must show that $\mathscr{E}xt^i_{\mathcal{O}_U}(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{G}|_U) = 0$ for all $i > 0$. Take an injective resolution $0 \to \mathscr{G}|_U \to \mathscr{I}^\bullet$ of $\mathscr{G}|_U$. Then we have $\mathscr{E}xt^i(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{G}|_U) = h^i(\mathscr{H}om(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{I}^\bullet)) = h^i(\oplus_{i=1}^n \mathscr{H}om(\mathcal{O}_U, \mathscr{I}^\bullet)) = \oplus_{i=1}^n h^i(\mathscr{H}om(\mathcal{O}_U, \mathscr{I}^\bullet)) = \oplus_{i=1}^n h^i(\mathscr{I}^\bullet) = 0$ for $i > 0$.

Exercise 6.5. *Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees. Show*

- a *$\mathscr{F}$ is locally free if and only if $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;*
- b $\operatorname{hd}(\mathscr{F}) \leq n$ *if and only if $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;*

c $\operatorname{hd}(\mathscr{F}) = \sup_x \operatorname{pd}_{\mathcal{O}_X} \mathscr{F}_x$.

Remark. Again, we assume the hypothesis "every coherent sheaf on X is a quotient of a locally free sheaf of *finite rank*" was intended.

Solution. a If $\mathscr{F}$ is locally free *of finite rank* then by Proposition III.6.5 we have $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > 0$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Conversely, suppose that $\mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $i > 0$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Taking stalks, we have by Proposition III.6.8 that $0 = \mathscr{E}xt^1(\mathscr{F},\mathscr{G})_x \cong \operatorname{Ext}^1_{\mathcal{O}_x}(\mathscr{F}_x,\mathscr{G}_x)$ for $i > 0$. This is a criterion for $\mathscr{F}_x$ to be projective, and finitely generated modules over local rings are projective if and only if they are free (Proposition 6 at the end of this section). So $\mathscr{F}_x$ is free for each x. Hence, $\mathscr{F}$ is locally free (Exercise II.5.7(b)).

b First suppose that $\operatorname{hd}(\mathscr{F}) \leq n$. Then there exists a locally free resolution $\cdots \to 0 \to \mathscr{E}_n \to \mathscr{E}_{n-1} \to \cdots \to \mathscr{E}_0 \to \mathscr{F} \to 0$ of $\mathscr{F}$ of length n. We can use this to calculate $\mathscr{E}xt$ by Proposition III.6.5 and so we find that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$. We prove the converse by induction on n. The case $n = 0$ has the same proof as part (a) of this question. Consider $n > 0$ and suppose that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Express $\mathscr{F}$ as the quotient of a locally free sheaf $\mathscr{E}$ and consider the resulting short exact sequence $0 \to \mathscr{F}' \to \mathscr{E} \to \mathscr{F} \to 0$. This gives rise to a long exact sequence

$$\cdots \to \mathscr{E}xt^n(\mathscr{E},\mathscr{G}) \to \mathscr{E}xt^n(\mathscr{F}',\mathscr{G}) \to \mathscr{E}xt^{n+1}(\mathscr{F},\mathscr{G}) \to \mathscr{E}xt^{n+1}(\mathscr{E},\mathscr{G}) \to \dots$$

Since $\mathscr{E}$ is locally free and $n > 0$ part (a) of this exercise tells use that the two outer groups vanish and so we have an isomorphism $\mathscr{E}xt^n(\mathscr{F}',\mathscr{G}) \cong \mathscr{E}xt^{n+1}(\mathscr{F},\mathscr{G})$. By hypothesis $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and so we find that $\mathscr{E}xt^i(\mathscr{F}',\mathscr{G}) = 0$ for all $i > n-1$ which by the inductive hypothesis implies that $\operatorname{hd}\mathscr{F}' \leq n-1$. So there is a locally free resolution $\cdots \to 0 \to \mathscr{E}_{n-1} \to \cdots \to \mathscr{E}_0 \to \mathscr{F}' \to 0$ of $\mathscr{F}'$ of length $n-1$. The exact sequence $\cdots \to 0 \to \mathscr{E}_{n-1} \to \cdots \to \mathscr{E}_0 \to \mathscr{E} \to \mathscr{F} \to 0$ where $\mathscr{E}_0 \to \mathscr{E}$ is the composition $\mathscr{E}_0 \to \mathscr{F}' \to \mathscr{E}$ then gives us a resolution of length n and so $\operatorname{hd}\mathscr{F} \leq n$.

c Given a locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ of $\mathscr{F}$ of length n, taking stalks gives a free (and hence projective) resolution of length $\leq n$ of $\mathscr{F}_x$ for each point x, hence $\operatorname{hd}(\mathscr{F}) \geq \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x$. Suppose equality doesn't hold. Then for every point x we have $\operatorname{hd}(\mathscr{F}) > \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x$. By Proposition III.6.10A this means that $\operatorname{Ext}^i(\mathscr{F}_x, N) = 0$ for all points x, all $i \geq \operatorname{hd}\mathscr{F}$ and all $\mathcal{O}_x$-modules N. Using Proposition III.6.8 this says that $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})_x = 0$ for all points x, all $i \geq \operatorname{hd}\mathscr{F}$ and all $\mathcal{O}_X$-modules $\mathscr{G}$, and so $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i \geq \operatorname{hd}\mathscr{F}$ and all $\mathcal{O}_X$-modules $\mathscr{G}$. By part (b) of this exercise, this implies that $\operatorname{hd}(\mathscr{F}) < \operatorname{hd}(\mathscr{F})$ which is clearly a contradiction. Hence, the inequality $\operatorname{hd}(\mathscr{F}) \geq \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x$ is actually an equality.

Exercise 6.6. *Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.*

a *M is projective if and only if* $\operatorname{Ext}^i(M, A) = 0$ *for all* $i > 0$.

b *Use (a) to show that for any n,* $\operatorname{pd} M \leq n$ *if and only if* $\operatorname{Ext}^i(M, A) = 0$ *for all* $i > n$.

Solution. a If M is projective then $\operatorname{Ext}^i(M, A) = 0$ for $i > 0$ since $\operatorname{Ext}^i(M, A)$ can be defined as the ith left derived functor of $\hom(-, A)$. Conversely, suppose that $\operatorname{Ext}^i(M, A) = 0$ for all $i > 0$. Let N be a finitely generated A-module. Then we have an exact sequence $0 \to K \to A^n \to N \to 0$ and a corresponding long exact sequence

$$\cdots \to \underbrace{\operatorname{Ext}^{i-1}(M, A^n)}_{=0} \to \operatorname{Ext}^{i-1}(M, N) \to \operatorname{Ext}^i(M, K) \to \underbrace{\operatorname{Ext}^i(M, A^n)}_{=0} \to \dots$$

(one of the many ways to see $\operatorname{Ext}^i(M, A^n) = 0$ is by considering the long exact sequence associated to $0 \to A^{n-1} \to A^n \to A \to 0$ and using induction). So if the statement:

(S_i) $\operatorname{Ext}^i(M, N) = 0$ for all finitely generated A-modules N

is true with $i > 2$ then (S_{i-1}) is also true. The statement (S_i) for all $i > \dim A$ is true as a consequence of Proposition III.6.11A and so by induction we have the verity of (S_i) for all $i \geq 1$. In particular, consider the exact sequence $0 \to K \to A^n \to M \to 0$ and the corresponding exact sequence $\cdots \to \operatorname{Ext}^0(M, A^n) \to \operatorname{Ext}^0(M, M) \to \operatorname{Ext}^1(M, K) \to \dots$. Since $\operatorname{Ext}^1(M, K)$ is zero, the morphism $\operatorname{Ext}^0(M, A^n) \to \operatorname{Ext}^0(M, M)$ is surjective and so the identity $M \to M$ is a composition $M \to A^n \to M$. In otherwords, M is a direct summand of A^n. This is one criteria for M to be projective.

b If $\operatorname{pd} M \leq n$ then we can calculate $\operatorname{Ext}^i(M, A)$ using a projective resolution of M of length $\leq n$ which implies that $\operatorname{Ext}^i(M, A) = 0$ for $i > n$. Conversely, suppose that $\operatorname{Ext}^i(M, A) = 0$ for $i > n$, and suppose that $\operatorname{Ext}^i(M', A) = 0$ for $i > n - 1$ implies that $\operatorname{pd} M' \leq n - 1$. If $n = 0$ then we have $\operatorname{pd} M \leq 0$ by part (a). If not then take a finite set of generators of M and the associated short exact sequence $0 \to N \to A^k \to M \to 0$. This gives a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}(A^k, A) \to \operatorname{Ext}^{i-1}(N, A) \to \operatorname{Ext}^i(M, A) \to \operatorname{Ext}^i(A^k, A) \to \dots$$

Since A^k is already free, we have $\operatorname{Ext}^i(A^k, A) = 0$ for all $i > 0$ and so $\operatorname{Ext}^{i-1}(N, A) \cong \operatorname{Ext}^i(M, A)$ for $i > 1$. This means that $\operatorname{Ext}^i(N, A)$ for $i > n - 1$ and so by the inductive hypothesis $\operatorname{pd} N \leq n - 1$. So there exists a projective resolution $0 \to P_{n-1} \to \cdots \to P_0 \to N \to 0$ of length $n - 1$ and from this we obtain a projective resolution $0 \to P_{n-1} \to \cdots \to P_0 \to A^k \to M \to 0$ of length $n - 1$ where $P_0 \to A^k$ is the composition $P_0 \to N \to A^k$. Hence, $\operatorname{pd} M \leq n$.

Exercise 6.7. *Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then*

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$

and

$$\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)^\sim$$

Solution. Since M is finitely generated and A noetherian, we can find a resolution of M by finite rank free modules $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$. The A-modules $\operatorname{Ext}^i_A(M, N)$ can then be calculated as $h^i(\hom_A(A^{n_\bullet}, N))$. Now compare the two functors

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{\cdot}) \qquad\qquad h^i(\hom_A(A^{n_\bullet}, \cdot))$$

that map A-mod to A-mod. Since $(\cdot)^\sim$ is an exact equivalence between A-mod and $\mathfrak{Qco}(X)$ the functor $\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{\cdot})$ is a derived functor and therefore automatically a universaly δ-functor (Corollary III.1.4). Since $\hom_A(A^n, \cdot) \cong (\cdot)^n$ is exact for finite n, the functors $h^i(\hom_A(A^{n_\bullet}, \cdot))$ are also a δ-functor (use the Snake Lemma). Since A-mod has enough injectives, and $\hom_A(\cdot, I)$ is exact for any injective I, the functors $h^i(\hom_A(A^{n_\bullet}, \cdot))$ are effaceable for $i > 0$ and therefore form a universal δ-functor. Now $\operatorname{Ext}^0_X(\widetilde{M}, \widetilde{N}) \cong \hom_A(M, N)$ and $h^0(\hom_A(A^{n_\bullet}, N) = \hom_A(M, N)$. So the two sequences of functors are the isomorphic.

Now consider $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})$ and $\operatorname{Ext}^i_A(M, N)^\sim$. We use the same resolution $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$ and get a finite rank free resolution of $\widetilde{M}$ which can be used to calculate $\mathscr{E}xt$ by Proposition III.6.5 as $\mathscr{E}xt^i(\widetilde{M}, \widetilde{M}) \cong h^i(\mathscr{H}om(\widetilde{A^{n_\bullet}}, \widetilde{N}))$. Now since M is finitely generated and A noetherian, we have $\hom_A(M, N)^\sim \cong \mathscr{H}om(\widetilde{M}, \widetilde{N})$. [1] Hence we have

$$\begin{aligned}\mathscr{E}xt^i(\widetilde{M}, \widetilde{M}) &\cong h^i(\mathscr{H}om(\widetilde{A^{n_\bullet}}, \widetilde{N})) \cong h^i(\hom_A(A^{n_\bullet}, N)^\sim) \\ &\cong h^i(\hom_A(A^{n_\bullet}, N))^\sim = \operatorname{Ext}^i_A(M, N)^\sim\end{aligned}$$

Exercise 6.8. *Prove the following theorem of Kleiman: if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).*

- *a First show that open sets of the form X_s for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X.*
- *b Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.*

[1] This doesn't hold in general. Consider the values of these two sheaves on a basic open $D(f)$. On the left we have $(\hom_A(M, N))_f$ and on the rights $\hom_{A_f}(M_f, N_f)$. There is a clear morphism $(\hom_A(M, N))_f \to \hom_{A_f}(M_f, N_f)$ but in general this morphism is neither injective nor surjective (consider the ring $\oplus_{i=1}^\infty k[x, y]/(x^i)$ with $M = N = \oplus_{i=1}^\infty k[x]/(x^i)$ localized at $f = x$). If M is finitely generated though, the morphism is an isomorphism. It is also natural with respect to inclusions of basic open affines, and so the sheaves are isomorphic.

Solution. a We show that given a closed point $x \in X$ and an open neighbourhood U of x, there is an $\mathscr{L}$ and s such that $x \in X_s \subseteq U$. Let $Z = X - U$ and $Z = \cup_{i=1}^n Z_i$ be the decomposition of Z into its irreducible components. If the statement is true for each $U_i = X - Z_i$ then we can take the global section $s = s_1 \otimes \cdots \otimes s_n$ of $\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$. Then we have $(s_1 \otimes \cdots \otimes s_n)_y \notin \mathfrak{m}_y \mathscr{L}_y$ if and only if $(s_i)y \notin \mathfrak{m}_y \mathscr{L}_y$ for all i. So $X_s = \cap X_{s_i}$ and so $x \in X_s \subseteq \cap U_i = U$. So we can assume that Z is irreducible. Therefore it is a prime Weil divisor and by Proposition II.6.11 corresponds to a Cartier Divisor D. That is, a global section of $\mathscr{K}^*/\mathcal{O}^*$. This is represented by a (finite) cover $\{U_i\}$ and for each U_i an element $f_i \in K$ such that $f_i/f_j \in \mathcal{O}_X^*(U)$. By construction these f_i also satisfy: for any codimension one irreducible subscheme Z' we have $f_i \in \mathfrak{m}_{Z'}\mathcal{O}_{X,Z'}$ if and only if $Z' = Z$. We then have Proposition II.6.13 which gives us an invertible sheaf $\mathscr{L}(D)$, constructed as the sub-$\mathcal{O}_X$-module of $\mathscr{K}^*$ generated locally by f_i^{-1}. The local sections $f_i f_i^{-1} \in \Gamma(U_i, \mathscr{L}(D))$ then glue together to give a global section $s \in \Gamma(X, \mathscr{L}(D))$ such that under the isomorphisms $\Gamma(U_i, \mathscr{L}(D)) \cong \Gamma(U_i, \mathcal{O}_X)$ defined by $f f_i^{-1} \leftrightarrow f$ we have $s|_{U_i} \leftrightarrow f_i$ and so $X_s = U$. Hence we have found $\mathscr{L}, s$ such that $x \in X_s \subseteq U$.

b Let $\mathscr{F}$ be a coherent sheaf on X. Then there is a cover by open affines $U_i = \operatorname{Spec} A_i$, such that on U_i, we have $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for some finitely generated A_i-module M_i. This means that $\mathscr{F}_{U_i}$ is generated by finitely many sections $m_{ij} \in M_i = \Gamma(U_i, \mathscr{F}|_{U_i})$. Now take a refinement of this cover consisting of open set of the form $X_{s_{ik}} \subseteq U_i$ for some $s_{ik} \in \Gamma(X, \mathscr{L}_{ik})$ and some $\mathscr{L}_{ik}$. Then each m_{ij} is a section of $\Gamma(X_{s_{ik}}, \mathscr{F})$ and so by Lemma II.5.14 there is some n_{ij} such that $s_{ik}^{n_{ij}} m_{ij}$ extends to a global section of $\mathscr{L}^{n_{ik}} \otimes \mathscr{F}$. This global section defines a morphism $\mathcal{O}_X \to \mathscr{L}^{n_{ik}} \otimes \mathscr{F}$ and tensoring with $\mathscr{L}^{-n_{ik}}$ we obtain a morphism $\mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$. Take the direct sum of these morphisms $\bigoplus \mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$. On the open set $X_{s_{ik}}$ the section m_{ij} is in the image of the morphism $\mathscr{L}^{-n_{ik}} \to \mathscr{F}$ and so since the m_{ij} generate $\mathscr{F}$ locally, the morphism $\bigoplus \mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$ is surjective.

Exercise 6.9. *Let X be a noetherian, integral, separated, regular scheme. Show that the natural group homomorphism $\varepsilon : K_{vec}(X) \to K_{coh}(X)$ from the Grothendieck group of the category of locally free finite rank sheaves, to the Grothendieck group of the category of coherent sheaves is an isomorphism as follows.*

a *Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution*

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

b *For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum (-1)^i \gamma(\mathscr{E}_i)$ in $K_{vec}(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K_{coh}(X)$ to $K_{vec}(X)$, and finally, that it is an inverse to ε.*

Solution. a By Exercise III.6.8 every coherent sheaf is a quotient of a locally free sheaf of finite rank (regular implies locally factorial; this is a hard theorem [Matsumura Theorem 48, page 142]), and so $\mathfrak{Coh}(X)$ has enough locally frees. Hence, we can define the homological dimension of $\mathscr{F}$ and by Exercise III.6.5(c) we have $\operatorname{hd}\mathscr{F} = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x$. Since X is regular, each $\mathcal{O}_x$ is regular and so by Proposition III.6.11A we have $\operatorname{pd}\mathscr{F}_x \leq \dim \mathcal{O}_{X,x} \leq \dim X$. Hence $\operatorname{hd}\mathscr{F} = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x \leq \dim X$ and so there exists a finite locally free finite rank resolution of $\mathscr{F}$.

b

Remark. This proof mimicks that found in [Borel, Serre - Théorème de Riemann-Roch].

Lemma 1. *Suppose we have a diagram* $0 \leftarrow \mathscr{F} \leftarrow \mathscr{F}'' \rightarrow \mathscr{F}' \rightarrow 0$ *in the category of coherent sheaves. Then there is a commutative square with* $\mathscr{E}$ *locally free and all morphisms surjective*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[d] \\ \mathscr{F}' \arrow[r] & \mathscr{F} \end{tikzcd}$$

Proof. Let $\mathscr{G}$ be the kernel of the canonical morphism $\mathscr{F}' \oplus \mathscr{F}'' \rightarrow \mathscr{F}$. Since $\mathscr{F}' \rightarrow \mathscr{F}$ and $\mathscr{F}'' \rightarrow \mathscr{F}$ are both surjective, the same is true of the compositions with projections $\mathscr{G} \rightarrow \mathscr{F}' \oplus \mathscr{F}'' \rightarrow \mathscr{F}'$ and $\mathscr{G} \rightarrow \mathscr{F}' \oplus \mathscr{F}'' \rightarrow \mathscr{F}''$. So the two morphisms $\mathscr{G} \rightarrow \mathscr{F}'$ and $\mathscr{G} \rightarrow \mathscr{F}''$ are surjective. Then we express $\mathscr{G}$ as the quotient of a locally free sheaf $\mathscr{E} \rightarrow \mathscr{G}$ and take the compositions $\mathscr{E} \rightarrow \mathscr{G} \rightarrow \mathscr{F}' \oplus \mathscr{F}'' \rightarrow \mathscr{F}'$ and $\mathscr{E} \rightarrow \mathscr{G} \rightarrow \mathscr{F}' \oplus \mathscr{F}'' \rightarrow \mathscr{F}''$. ☐

Lemma 2. *Suppose we have the commutative exact diagram of solid arrows in the category of coherent sheaves with* $\mathscr{E}, \mathscr{E}'$ *locally free. Then we can find* $\mathscr{G}''$ *and* $\mathscr{E}''$ *with* $\mathscr{E}''$ *locally free and extend the diagram to a commutative exact diagram with the dashed arrows.*

$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & \mathscr{G}' \arrow[r] \arrow[u, dashed] & \mathscr{E}' \arrow[r] \arrow[u, dashed] & \mathscr{F}' \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r, dashed] & \mathscr{G}'' \arrow[r, dashed] \arrow[u, dashed] \arrow[d, dashed] & \mathscr{E}'' \arrow[r, dashed] \arrow[u, dashed] \arrow[d, dashed] & \mathscr{F}'' \arrow[r, dashed] \arrow[u] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, dashed] & \mathscr{E} \arrow[r] \arrow[d, dashed] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Proof. First we use Lemma 1 above to obtain the comutative square on the left with $\mathscr{E}_1$ locally free, and then again to obtain the commutative square in the center with $\mathscr{E}_2$ locally free. Note that this gives us the diagram on the right where all morphism are epimorphisms.

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{F} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E}_2 \arrow[rr] \arrow[d] & & \mathscr{E}' \arrow[d] \\ \mathscr{E}_1 \arrow[r] & \mathscr{F}'' \arrow[r] & \mathscr{F}' \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E}' \arrow[r] & \mathscr{F}' \\ \mathscr{E}_2 \arrow[u, "a"] \arrow[r, "c"] \arrow[d, "b"'] & \mathscr{F}'' \arrow[u] \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{F} \end{tikzcd}$$

Then we take expressions for $\mathscr{G}$ and $\mathscr{G}'$ as quotients of locally free sheaves $\mathscr{E}_3 \xrightarrow{d} \mathscr{G}$ and $\mathscr{E}_4 \xrightarrow{e} \mathscr{G}'$. Now we have a diagram

$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & \mathscr{G}' \arrow[u] \arrow[r, "h"] & \mathscr{E}' \arrow[u] \arrow[r] & \mathscr{F}' \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & \ker(c) \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a|_{\ker c}+0+e"'] \arrow[r] \arrow[d, "b|_{\ker c}+d+0"] & \mathscr{E}_2 \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a+0+he"'] \arrow[r, "c+0+0"] \arrow[d, "b+gd+0"] & \mathscr{F}'' \arrow[u] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{G} \arrow[r, "g"] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

which satisfies the requirements. □

Corollary 3. *For any two locally free resolutions* $\mathscr{E}_\bullet \xrightarrow{\varepsilon} \mathscr{F} \to 0$ *and* $\mathscr{E}'_\bullet \xrightarrow{\varepsilon'} \mathscr{F} \to 0$ *of a coherent sheaf* $\mathscr{F}$ *there is a third locally free resolution* $\mathscr{E}''_\bullet \xrightarrow{\varepsilon''} \mathscr{F} \to 0$ *together with a commutative diagram where the vertical morphisms are all surjective.*

$$\begin{tikzcd} \mathscr{E}'_\bullet \arrow[r, "\varepsilon'"] & \mathscr{F} \arrow[r] & 0 \\ \mathscr{E}''_\bullet \arrow[u] \arrow[d] \arrow[r, "\varepsilon''"] & \mathscr{F} \arrow[u, equal] \arrow[d, equal] \arrow[r] & 0 \\ \mathscr{E}_\bullet \arrow[r, "\varepsilon"] & \mathscr{F} \arrow[r] & 0 \end{tikzcd}$$

Proof. We construct $\mathscr{E}''_\bullet$ inductively. From Lemma 2 we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker\varepsilon' \arrow[r] & \mathscr{E}'_0 \arrow[r, "\varepsilon'"] & \mathscr{F} \arrow[r] & 0 \\ 0 \arrow[r] & \ker\varepsilon'' \arrow[u] \arrow[d] \arrow[r] & \mathscr{E}''_0 \arrow[u] \arrow[d] \arrow[r, "\varepsilon''"] & \mathscr{F} \arrow[u, equal] \arrow[d, equal] \arrow[r] & 0 \\ 0 \arrow[r] & \ker\varepsilon \arrow[r] & \mathscr{E}_0 \arrow[r, "\varepsilon"] & \mathscr{F} \arrow[r] & 0 \end{tikzcd}$$

with surjective vertical morphisms. For the inductive step we use Lemma 2 to get a the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker d_i' \arrow[r] & \mathscr{E}_i' \arrow[r, "d_i'"] & \ker d_{i-1}' \arrow[r] & 0 \\
0 \arrow[r] & \ker d_i'' \arrow[r] \arrow[u] \arrow[d] & \mathscr{E}_i'' \arrow[r, "d_i''"] \arrow[u] \arrow[d] & \ker d_{i-1} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & \ker d_i' \arrow[r] & \mathscr{E}_i \arrow[r, "d_i"] & \ker d_{i-1}'' \arrow[r] & 0
\end{tikzcd}$$

with surjective vertical morphisms. □

Proof of independence of the chosen resolution. Now we have the results we need to show that the class $\sum(-1)^i[\mathscr{E}_i]$ in $K_{vec}(X)$ is independent of the resolution chosen. Suppose that we have a second resolution $\mathscr{E}_\bullet' \to \mathscr{F} \to 0$ as in Corollary 3. Then we get a third resolution $\mathscr{E}_\bullet'' \to \mathscr{F} \to 0$ which "dominates" the other two and so we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
\cdots \arrow[r] & \mathscr{G}_1 \arrow[r] \arrow[d] & \mathscr{G}_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathscr{E}_1'' \arrow[r] \arrow[d] & \mathscr{E}_0'' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and and analogous one for $\mathscr{E}_\bullet'$ (denote the kernels in this analgogous diagram by $\mathscr{G}_i'$ instead of $\mathscr{G}_i$). If the $\mathscr{G}_i$ are locally free then we get

$$\begin{aligned}
\sum(-1)^i\mathscr{E}_i &= \sum(-1)^i(\mathscr{E}_i'' - \mathscr{G}_i) \\
&= \sum(-1)^i\mathscr{E}_i'' - \sum(-1)^i\mathscr{G}_i \\
&= \sum(-1)^i\mathscr{E}_i'' \\
&= \sum(-1)^i\mathscr{E}_i'' - \sum(-1)^i\mathscr{G}_i' \\
&= \sum(-1)^i\mathscr{E}_i'
\end{aligned}$$

in $K_{vec}(X)$ and so we just need to prove:

□

δ defines a morphism $K_{coh}(X) \to K_{vec}(X)$. We must show that formal sums of coherent sheaves that are zero in $K_{coh}(X)$ get sent to zero in

$K_{vec}(X)$. For this it is enough to show that for any short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves, we have $\delta(\mathscr{F}) = \delta(\mathscr{F}') + \delta(\mathscr{F}'')$ in $K_{vec}(X)$. To show this we will see that there exist resolutions for $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ that themselves form an exact sequence, so we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}'_\bullet \arrow[r] \arrow[d] & \mathscr{E}_\bullet \arrow[r] \arrow[d] & \mathscr{E}''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

As in the proof of Corollary 3 we build the sequences step by step. Each step uses the following lemma.

Lemma 4. Suppose that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of coherent sheaves. Then there is an exact sequence of locally free sheaves* $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ *together with a surjective morphism to the original sequence.*

Proof. Expressing $\mathscr{F}''$ as a quotient of a locally free sheaf $\mathscr{E}'' \to \mathscr{F}''$ we obtain $\mathscr{E}''$. Now use Lemma 1 to obtain a commutative diagram of surjective morphisms

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "a"] \arrow[d, "b"] & \mathscr{E}'' \arrow[d] \\
\mathscr{F} \arrow[r] & \mathscr{F}''
\end{tikzcd}$$

Express $\mathscr{F}'$ as a quotient of a locally free sheaf $\mathscr{E}' \xrightarrow{c} \mathscr{F}'$ and we end up with a diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker a \oplus \mathscr{E}' \arrow[r] \arrow[d, "b|_{\ker a}+c"] & \mathscr{G} \oplus \mathscr{E}' \arrow[r, "a+0"] \arrow[d, "b+dc"] & \mathscr{E}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r, "d"] & \mathscr{F} \arrow[r] & \mathscr{F}'' \arrow[r] & 0
\end{tikzcd}$$

with the desired properties. $\square$

Now using this lemma and given the ith step of the resolutions, we can construct the $(i+1)$th step by forming the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}'_{i+1} \arrow[r] \arrow[d] & \mathscr{E}_{i+1} \arrow[r] \arrow[d] & \mathscr{E}''_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker d'_i \arrow[r] & \ker d_i \arrow[r] & \ker d''_i \arrow[r] & 0
\end{tikzcd}$$

where $d_i : \mathscr{E}_i \to \mathscr{E}_{i-1}$ and similarly for d'_i and d''_i. Hence we get a commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}'_\bullet \arrow[r] \arrow[d] & \mathscr{E}_\bullet \arrow[r] \arrow[d] & \mathscr{E}''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

and so we have $\delta(\mathscr{F}) = \sum(-1)^i[\mathscr{E}_i] = \sum(-1)^i([\mathscr{E}'_i]+[\mathscr{E}''_i]) = \sum(-1)^i[\mathscr{E}'_i]+\sum(-1)^i[\mathscr{E}''_i] = \delta(\mathscr{F}') + \delta(\mathscr{F}'')$.

δ provides an inverse to ε. Clearly, if $\mathscr{E}$ is a locally free sheaf then we can take the resolution $\cdots \to 0 \to \mathscr{E} \to \mathscr{E} \to 0$ and so $\delta(\varepsilon(\mathscr{E})) = [\mathscr{E}]$. Conversely, for any counded exact sequence $0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_0 \to 0$ in $\mathfrak{Coh}(X)$ we have the relation $\sum(-1)^i[\mathscr{F}_i]$ in $K_{coh}(X)$ and so if $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ is a bounded resolution by locally free sheaves then $\varepsilon(\delta(\mathscr{F})) = \varepsilon(\sum(-1)^i[\mathscr{E}_i]) = \sum(-1)^i[\mathscr{E}_i] = [\mathscr{F}]$.

Exercise 6.10. Duality for a Finite Flat Morphism.

a *Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_Y$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{F}$.*

b *Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism*

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})$$

c *For each $i \geq 0$, there is a natural map*

$$\phi_i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G})$$

d *Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y. Show that ϕ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y.*

7 The Serre Duality Theorem

Exercise 7.1. *Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$.*

Exercise 7.2. *Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.*

a *Show that $f^!\omega_Y^\circ$ is a dualizing sheaf for X.*

b If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_*\omega_X \to \omega_Y$.

Exercise 7.3. *Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q, \leq n$.*

Solution. Consider the exact sequence of Theorem 8.13. From Exercise II.5.16(d) we have a filtration for each r

$$\wedge^r(\mathcal{O}(-1)^{n+1}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients $F^p/F^{p+1} \cong \Omega^p \otimes \wedge^{r-p}\mathcal{O}$. Since $\wedge^{r-p}\mathcal{O} \cong 0$ for $r - p \neq 0, 1$ and $\wedge^{r-p}\mathcal{O} \cong \mathcal{O}$ for $r - p = 0, 1$ we see that $F^p = F^{p+1}$ for $p \neq r, r-1$ so our filtration is $\wedge^r(\mathcal{O}(-1)^{n+1}) \supseteq F^r \supseteq F^{r+1} = 0$. The quotient $F^r/F^{r+1} = F^r$ is $\Omega^r \otimes \wedge^{r-r}\mathcal{O} \cong \Omega^r$ and the quotient $F^{r-1}/F^r = \wedge^r(\mathcal{O}(-1)^{n+1})/\Omega^r$ is $\Omega^{r-1} \otimes \wedge^{r-(r-1)}\mathcal{O} \cong \Omega^{r-1}$ so the filtration is actually an exact sequence:

$$0 \to \Omega^r \to \wedge^r(\mathcal{O}(-1)^{n+1}) \to \Omega^{r-1} \to 0$$

Now for any line bundle $\mathscr{L}$ on any ringed space we have $\wedge^r(\mathscr{L}^{\oplus m}) \cong (\mathscr{L}^{\otimes r})^{\oplus\binom{r}{m}}$ (one way of showing this is to take a trivializing cover, choose a local basis, and then look at the transition morphisms) and so our exact sequence is

$$0 \to \Omega^r \to \mathcal{O}(-r)^{\oplus N} \to \Omega^{r-1} \to 0$$

for suitable N that we don't care about. This gives rise to a long exact sequence on cohomology. Since $H^i(X, \mathcal{O}(-r)) = 0$ for $i < n$ or $r < n+1$ (Theorem III.5.1) we have isomorphisms $H^i(X, \Omega^r) \cong H^{i-1}(X, \Omega^{r-1})$ for $1 \leq i$ if $r < n+1$. If $r \geq n+1$ then we still have isomorphisms but only for $1 \leq i < n$.

Now we know that $H^0(X, \Omega^0) \cong H^0(X, \mathcal{O}_X) \cong k$ (Theorem III.5.1) and so using these isomorphisms we see that $H^i(X, \Omega^i) \cong k$ for $0 \leq i \leq n$. Again, using Theorem III.5.1 we know the cohomology of $\Omega^n \cong \mathcal{O}(-n-1)$, and in particular, that $H^i(X, \Omega^n) \cong 0$ for $i < n$. Using our isomorphisms above, this tells us that $H^i(X, \Omega^r) = 0$ in the region $i < r, 0 \leq r \leq n$. All that remains to show is the region $i > r, 0 \leq i \leq n$ and this follows from Corollary III.7.13.

Exercise 7.4.

8 Higher Direct Images of Sheaves

Exercise 8.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^if_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geq 0$,*

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

Solution. Take an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$ on X. Then $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ is an injective resolution of $f_*\mathscr{F}$ on Y. A priori, this complex is not necassarily exact but the hypothesis $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$ says that it is infact exact. By definition the cohomology of $\mathscr{F}$ is the cohomology of the complex $\Gamma(X, \mathscr{I}^\bullet)$ which is actually the same complex as $\Gamma(Y, f_*\mathscr{I}^\bullet)$. Hence, $H^i(X, \mathscr{F}) = H^i(Y, f_*\mathscr{F})$.

Exercise 8.2. *Let $f : X \to Y$ be an affine morphism of schemes, with X noetherian and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypothesis of Exercise III.8.1 are satisfied, and hence that $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geq 0$.*

Solution. By Proposition III.8.1 we know that $R^i f_*\mathscr{F}$ is the sheaf associated to $V \mapsto H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$. Since f is affine, $f^{-1}(V)$ is affine for every open subscheme V of Y (Exercise II.5.17). Theorem III.3.7 then tells us that $H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)}) = 0$ for $i > 0$. Hence, $R^i f_*\mathscr{F} = 0$ for $i > 0$.

Exercise 8.3. *Let $f : X \to Y$ be a morphism of ringed spaecs, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula*

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$$

Solution. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Using the natural isomorphisms from Exercise II.5.1(d) we get an isomorphism of chain complexes

$$f_*(\mathscr{I}^\bullet \otimes f^*\mathscr{E}) \cong f_*(\mathscr{I}^\bullet) \otimes \mathscr{E}$$

Consider the cohomology sheaves of these chain complexes. The pullback $f^*\mathscr{E}$ is locally free and so by Proposition III.6.7 $0 \to \mathscr{F} \otimes f^*\mathscr{E} \to \mathscr{I}^\bullet \otimes f^*\mathscr{E}$ is an injective resolution of $\mathscr{F} \otimes f^*\mathscr{E}$ and so can be used to calculate the right derived functors of f_* (tensoring with locally free sheaves is exact: check stalks). So the cohomology sheaves of $f_*(\mathscr{I}^\bullet \otimes f^*\mathscr{E})$ are $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E})$.

Now $R^i f_*(\mathscr{F})$ are the cohomology sheaves of $f_*\mathscr{I}^\bullet$. More explicitely, $R^i f_*(\mathscr{F}) = \operatorname{coker}(f_*\mathscr{I}^{i-1} \to \ker(f_*\mathscr{I}^i \to f_*\mathscr{I}^{i+1}))$. As tensoring with a locally free sheaf is exact, it follows that $R^i f_*(\mathscr{F}) \otimes \mathscr{E}$ are isomorphic to the cohomology sheaves of $f_*(\mathscr{I}^\bullet) \otimes \mathscr{E}$.

Hence, the isomorphisms of cohomology sheaves induced by our isomorphism of complexes above are the desired isomorphisms.

Exercise 8.4. *Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geq 1$. Let $X = \mathbb{P}(\mathscr{E})$, with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.*

- a *Then $\pi_*(\mathcal{O}(l)) \cong S^i(\mathscr{E})$ for $l \geq 0$, $\pi_*(\mathcal{O}(l)) = 0$ for $l < 0$; $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and $l \in \mathbb{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.*
- b *Show there is a natural exact sequence*

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0$$

and conclude that the relative canonical sheaf $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$

c *Now show, for any $i \in \mathbb{Z}$, that*

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee$$

d *Show that $p_a(X) = (-1)^n p_a(Y)$ and $p_g(X) = 0$,*

e *In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g, p_g = 0$, and irregularity g.*

Solution. a Let $\{U_i\}$ be a trivializing cover on X for $\mathscr{E}$ such that each U_i is affine, and consequently the spectrum of a noetherian ring A_i. So we have $\mathscr{E}(U_i) \cong \mathcal{O}_X^{n+1}$ for each U_i and hence $\pi^{-1}(U_i) \cong \mathbb{P}^n_{A_i}$. This means in particular that $H^j(\pi^{-1}U_i, \mathcal{O}(l)|_{\pi^{-1}U_i}) = H^j(\mathbb{P}^n_{A_i}, \mathcal{O}(l)|_{\pi^{-1}U_i})$ which is zero for $0 < j < n$ after Theorem III.5.1. As a consequence of this, $R^j\pi_*\mathcal{O}(l) = 0$ for $0 < j < n$ after Proposition III.8.1. By the same reasoning, $R^n\pi_*\mathcal{O}(l) = 0$ for $l > -n-1$ since $H^n(\mathbb{P}^n_{A_i}, \mathcal{O}(l)) = 0$ for $l > -n-1$.

b Part (b) of Theorem II.7.11 gives us a natural surjection $\pi^*\mathscr{E} \to \mathcal{O}(1)$. Consider exact sequence arising from the twist of this by $\mathcal{O}(-1)$

$$0 \to \mathscr{F} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0$$

Let $U = \operatorname{Spec} A$ be any open affine subscheme of Y on which $\mathscr{E}$ is isomorphic to $\mathcal{O}_Y^{n+1}$. Then $\pi^{-1}U \cong \mathbb{P}^n_A$ and the restriction of this exact sequence looks like

$$0 \to \mathscr{F}|_{\mathbb{P}^n_A} \to \mathcal{O}(-1)_{\mathbb{P}^n_A} \to \mathcal{O}|_{\mathbb{P}^n_A} \to 0$$

which is easily recognisable as the exact sequence from Theorem II.8.13. So we have isomorphisms $\mathscr{F}|_{\mathbb{P}^n_A} \cong \Omega_{\mathbb{P}^n_A/U}$. These isomorphisms are compatible with restrictions to smaller affine subsets and so we obtain a global isomorphism $\mathscr{F} \cong \Omega_{X/Y}$.

The isomorphism $\wedge^n\Omega_{X/Y} \cong (\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$ is a consequence of Exercise II.5.16. If we then cover X with open subets of the form $U_i = \mathbb{P}^n_{A_i}$ where $\operatorname{Spec} A_i$ are opens of Y on which $\mathscr{E} \cong \mathcal{O}_Y^{n+1}$ (and so $\pi^{-1}U \cong \mathbb{P}^n_A$), then restricting to these we get isomorphisms $\omega_{X/Y}|_{\pi^{-1}U} \cong \mathcal{O}_{\pi^{-1}U}(-n-1)$ via the isomorphisms just mentioned. So we have $R^n\pi_*(\omega_{X/Y})|_{\operatorname{Spec} A} \cong R^n\pi_*(\omega_{X/Y}|_{\mathbb{P}^n_A}) \cong H^n(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A/A})^\sim \cong A^\sim = \mathcal{O}_{\operatorname{Spec} A}$ (Corollary III.8.2, Proposition III.8.5, and Theorem III.5.1). Since these isomorphisms are all natural, we obtain the desired isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$.

c

d

e There is nothing to show.

9 Flat Morphisms

Exercise 9.1. *A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open.*

Solution. We need to show that for any open subscheme $U \subset X$ the image $f(U)$ is open in Y. Since the induced morphism $U \to Y$ is also of finite type we can restrict to the case when $U = X$. By Exercise II.3.18 we know that $f(X)$ is constructible, and so if it is closed under generization, then it will be open. That is, we need to show that given a generization $y' \in Y$ of a point $y \in f(X)$ there is some point $x' \in X$ whose image is y'. let $\operatorname{Spec} B$ be an open affine neighbourhood of y. The scheme $\operatorname{Spec} B$ also contains y', and the induced morphism $f^{-1} \operatorname{Spec} B \to \operatorname{Spec} B$ is still a flat morphism of finite type of noetherian schemes. Let x be a point whose image is y, and let $\operatorname{Spec} A$ be an open affine neighbourhood of y. By Proposition III.9.1A(d) A is a flat B-module.

So now we have a homomorphism $\phi : B \to A$ of noetherian rings where A is a finitely generated B-algebra and flat as a B-module. We have two primes $\mathfrak{p}' \subset \mathfrak{p}$ of B, a prime $\mathfrak{q}$ of A such that $\phi^{-1}\mathfrak{q} = \mathfrak{p}$ and we are looking for a prime $\mathfrak{q}' \subset \mathfrak{q}$ such that $\phi^{-1}\mathfrak{a}' = \mathfrak{p}'$. This is a commutative algebra result that can be found in Matsumara.

Exercise 9.2. *Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.*

Solution. The curve has parametric coordinates $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$ and projection is from the point $(0, 0, 1, 0)$. That is, we are considering the family of curves (t^3, t^2u, atu^2, u^3) projecting to the projective plane $z = 0$. We are interested in what happens at the cusp $(0, 0, 0, 1)$ of the projected curve so we only need to consider the affine space $w \neq 0$.

X_a has the parametric equations

$$\begin{cases} x = t^3 \\ y = t^2 \\ z = at \end{cases}$$

To get the ideal $I \subseteq k[a, x, y, z]$ of the total family $\overline{X}$ extended over all of $\mathbb{A}^1$ we eliminate t from the parametric equations, and make sure a is not a zero divisor in $k[a, x, y, z]/I$, so that $\overline{X}$ will be flat. We find

$$I = (y^3 - x^2, z^2 - a^2y, z^3 - a^3x, zy - ax, zx - ay^2)$$

Setting $a = 0$ we obtain the ideal $I_0 \subseteq k[x, y, z]$ of X_0 which is

$$I_0 = (y^3 - x^2, z^2, zx, zy)$$

So X_0 has support equal to the curve $x^2 = y^3$ in $\operatorname{Spec} k[x, y]$. Now at points where $\mathfrak{p}$ with $x \notin \mathfrak{p}$ we have $z \in \mathfrak{p}$ since $xz = 0 \in \mathfrak{p}$ and so these local rings are reduced. At the prime $\mathfrak{p} = (x, y)$ however, z is not zero and so $A_\mathfrak{p}$ has a nonzero nilpotent element.

Exercise 9.3. *Some examples of flatness and nonflatness.*

a *If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.*

b *Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and $X = \operatorname{Spec} k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.*

c *Again let $Y = \operatorname{Spec} k[x, y]$, but take $X = \operatorname{Spec} k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{red} \cong Y$, X has no embedded points, but that f is not flat.*

Solution. a

b Suppose x is the intersection point. The morphism is finite and so for it to be flat, $\mathcal{O}_{x,X}$ must be a finite rank free $\mathcal{O}_{f(x),Y}$-module (Proposition III.9.1A(f)). We have $\mathcal{O}_{x,X}/\mathfrak{m}_{f(x),Y}\mathcal{O}_{x,X} \cong k$ and so if $\mathcal{O}_{x,X}$ is a finite rank free $\mathcal{O}_{f(x),Y}$-module then it has rank one and therefore we would have an isomorphism $\mathcal{O}_{f(x),Y} \xrightarrow{\sim} \mathcal{O}_{x,X}$ as $\mathcal{O}_{f(x),Y}$-modules. Let $f \in \mathcal{O}_{x,X}$ be the image of 1 under this isomorphism. Then $z = gf$ for some $g \in \mathcal{O}_{f(x),Y}$. But z can't be expressed in this way in $\mathcal{O}_{x,X}$. Hence, the isomorphism doesn't exist and the morphism is not flat.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Exercise 9.7. *let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/(t^2)$ be the ring of dual numbers, and define an infinitesimal deformation of Y as a closed subscheme of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathscr{N}_{Y/X})$, where*

$$\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y)$$

Solution. First a lemma to make the affine case easier to deal with.

Lemma 5. *Consider ideals $I \subset A$ and $I' \subset A[t]$. Then $\operatorname{Spec} A[t]/I'$ is an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$ if and only if*

a $t^2 \in I'$;

b *under the map $A[t] \to A$ sending t to zero, the image of I' is I; and*

c *the kernel of the composite morphism $A \to A[t]/I' \xrightarrow{t} A[t]/I'$ is contained in I'.*

Proof. Condition (a) just says that $A[t]/I'$ is a D-algebra. Condition (b) is equivalent to saying that the composition $(\operatorname{Spec} A[t]/I') \otimes_D k \to A/I$ is an isomorphism. Condition (c) is equivalent to saying that $\operatorname{Spec} A[t]/I'$ is flat over D. To see this consider the criteria of Proposition III.9.1A(a). Since D has a unique nonzero ideal, we only need to test (t). Furthermore, by writing every element of $A[t]/I' \otimes_D (t)$ as $a \otimes t$ we reduce to showing that for $a \in A$, it holds that $at = 0$ implies $a \otimes t = 0$. Hence, the condition. $\square$

Now given a ring A, an ideal I, and a homomorphism $\phi \in \hom_{A/I}(I/I^2, A/I)$, define an ideal $I' \subset A[t]$ to be the set of polynomials $a_0 + a_1 t + \cdots + a_n t^n \in A[t]$ such that $a_0 \in I$ and $\phi(a_0) = a_1$ or 0 in A/I. It is fairly straightforward to check that the conditions of the lemma are fulfilled and so we have an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$. Conversely, given an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$, we can define a morphism $\phi \in \hom_{A/I}(I/I^2, A/I)$ as follows. Given an element $a \in I$, consider elements of the form $a + bt \in I'$. There must be at least one, for otherwise condition (b) of the lemma does not hold. Define $\phi(a) = b$. Note that if $a + b't \in I'$ is a different choice, then $(b' - b)t \in I'$, so $(b' - b) \in I'$ by condition (c), so $(b - b') \in I$ by condition (b) and so we end up with the same morphism $I/I^2 \to A/I$. We still need to show that ϕ is A/I-linear. That is, we must show that $\phi(ax + by) = a\phi(x) + b\phi(y)$ for $a, b \in A/I$ and $x, y \in I/I^2$. Given our definition of ϕ, this amounts to showing that for any elements $(ax + by) + zt$, $x + x't$ and $y + y't$ in I', we have $z - ax' - by' \in I$. We know that $ax + ax't$ and $by + by't$ are in I' and so $(ax + by) + zt - (ax + ax't) - (by + by't) = (z - ax' - by')t \in I'$ and this implies that $z - ax' - by' \in I$ using conditions (b) and (c) of the lemma. So we have given an isomorphism

$$\hom_{A/I}(I/I^2, A/I) \to \mathfrak{Inf}(\operatorname{Spec}(A/I)/\operatorname{Spec} A)$$

and its inverse where $\mathfrak{Inf}(Y/X)$ is the set of infinitesimal deformations of Y as a subscheme of X.

Now that the affine case is done, we prove the general case by glueing in the usual way by glueing. The first thing to notice is that if we have ideals $I \subset A$, $J \subset B$, and ring homomorphism $\psi : A \to B$ such that $\psi^{-1} J \subset I$ then we get a commutative square

$$\begin{tikzcd}
\hom_{A/I}(I/I^2, A/I) \arrow[r, "\sim"] \arrow[d] & \mathfrak{Inf}(\operatorname{Spec}(A/I)/\operatorname{Spec} A) \arrow[d] \\
\hom_{B/J}(J/J^2, B/J) \arrow[r, "\sim"] & \mathfrak{Inf}(\operatorname{Spec}(B/J)/\operatorname{Spec} B)
\end{tikzcd}$$

So in the general case, since both sides are sheaves, and we have natural isomorphisms for affine opens, we can glue to get a global isomorphism

$$\hom_{\mathcal{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y/\mathscr{I}_Y) \cong \mathfrak{Inf}(Y/X)$$

Exercise 9.8.

Exercise 9.9. *Let $A = k[x,y,z,w]/(x,y)\cap(z,w)$, and show that A is rigid.*

Solution. Let $P = k[x,y,z,w]$ and $J = (x,y)\cap(z,w)$ as in the previous exercise. By the previous exercise, we must show that the morphism

$$\hom_A(\Omega_{P/k}\otimes A, A) \to \hom_A(J/J^2, A)$$

is surjective. We do this explicitely.

We have $\Omega_{P/k} \cong P^4$ with basis dx, dy, dz, dw and so $\Omega_{P/k}\otimes A \cong A^4$ with the same basis and $\hom_A(\Omega_{P/k}\otimes A, A) \cong A^4$ with the dual basis dx^*, dy^*, dz^*, dw^*. The ideal J is generated by xz, xw, yz, yw as a P-module and since A is a quotient of A, these elements represent generators of the A-module J/J^2. So any morphism $\phi \in \hom_A(J/J^2, A)$ is determined by its value on xz, xw, yz, yw and in this way we get $\hom_A(J/J^2, A) \subset A^4$, by identifying a morphism with its value on xz, xw, yz, yw.

The morphism $J/J^2 \to \Omega_{P/k}\otimes A$ sends f to $df\otimes 1$ and so using

$$\hom_A(\Omega_{P/k}\otimes A, A) \cong A^4 \qquad \hom_A(J/J^2, A) \subset A^4$$

we can represent the morphism $\hom_A(\Omega_{P/k}\otimes A, A) \to \hom_A(J/J^2, A)$ as a matrix. The morphism in $\hom_A(\Omega_{P/k}\otimes A, A)$ that sends dx to 1 and all other generators to zero gets sent to $(z, w, 0, 0)$ in $\hom_A(J/J^2, A)$ since $d(xz) = zdx + xdz$, $d(xw) = \dots$. Continuing like this we find the matrix to be

$$\begin{pmatrix} z & w & 0 & 0 \\ 0 & 0 & z & w \\ x & 0 & y & 0 \\ 0 & x & 0 & y \end{pmatrix}$$

We want to show that the morphism induced by this matrix is surjective.

Consider an element (b_1, b_2, b_3, b_4) of $\hom_A(J/J^2, A) \subset A^4$ where, recall that b_1 (resp. b_2, b_3, b_4) is the image of xz (resp. xw, yz, yw). We have $yb_1 = xb_3$. Since xz, xw, yz, yw are all zero in A, multiplying by x or y kills all the terms with z or w in them, but "preserves" any terms without, x sending x^iy^j to $x^{i+1}y^j$ and y sending it to x^iy^{j+1}. So $b_1 = \frac{x}{y}b_3 + b_1'$ where $b_1' \in (z,w)k[z,w]$. Similarly, from the relation $wb_1 = zb_2$ we see that $b_1 = \frac{z}{w}b_2 + b_1''$ where $b_1'' \in (x,y)k[x,y]$. Putting these two together we see that $b_1 = \frac{z}{w}b_2 + \frac{x}{y}b_3$. We use a similar argument for b_2, b_3, b_4 to find that

$$\begin{aligned} b_1 &= \frac{z}{w}b_2 + \frac{x}{y}b_3 \\ b_2 &= \frac{x}{y}b_4 + \frac{w}{z}b_1 \\ b_3 &= \frac{y}{x}b_1 + \frac{z}{w}b_4 \\ b_4 &= \frac{y}{x}b_2 + \frac{w}{z}b_3 \end{aligned}$$

abd consequently, (b_1, b_2, b_3, b_4) is in the image of $\hom_A(\Omega_{P/k}\otimes A, A) \to \hom_A(J/J^2, A)$. Hence, it is surjective, and so $T^1(A) = 0$ and therefore, the k-algebra A is rigid.

Exercise 9.10. *a Show that $\mathbb{P}^1_k$ is rigid.*

b

c

Solution. a By (9.13.2) the infinitesimal deformations are classified by $H^1(X, \mathscr{T}_X)$. When $X = \mathbb{P}^1_k$ we know that $\Omega_{X,k} \cong \mathcal{O}(-2)$ and so $\mathscr{T}_X = \mathcal{O}(2)$ and we have already calculated the cohomology of this sheaf. We find that $H^1(X, \mathscr{T}_X) = H^1(X, \mathcal{O}(2)) = 0$. Hence, there are no infinitesimal deformations.

Proposition 6. *Let M be a finitely generated module over a local ring $(A, \mathfrak{m})$. Then M is projective if and only if M is free.*

Proof. For any module over any ring, free implies projective so we need only prove the converse. Since M is finitely generated $M/\mathfrak{m}M$ is a finite dimensional $(A/\mathfrak{m})$-vector space. Take a set of elements $m_1, \ldots, m_n$ in M whose image in $M/\mathfrak{m}M$ is a basis. Then by Nakayam's Lemma, the m_i generated M and so we get an exact sequence $0 \to N \to A^n \to M \to 0$. Since M is projective, this sequence splits and we see that $A^n \cong M \oplus N$. Now we have $A^n/\mathfrak{m}A^n \cong M/\mathfrak{m}M \oplus N/\mathfrak{m}N$. But both $A^n/\mathfrak{m}A^n$ and $M/\mathfrak{m}M$ are finite dimensional vector spaces of the same dimension. Hence $N/\mathfrak{m}N = 0$ which implies $\mathfrak{m}N = N$ and Nakayama's Lemma says that this implies $N = 0$. So $A^n \cong M$. □

Corollary 7. *If $\mathscr{E}_1 \to \mathscr{E}_0$ is a surjective morphism of locally free coherent sheaves then the kernel is also locally free.*

Proof. Let $\mathscr{G}$ be the kernel. At each point x we get an exact sequence of $\mathcal{O}_{X,x}$-modules, and since the $\mathscr{E}_i$ are locally free, this has the form $0 \to \mathscr{G}_x \to \mathcal{O}_{X,x}^n \to \mathcal{O}_{X,x}^m \to 0$. Since finite rank free modules are projective, the sequence splits and so $\mathscr{G}_x$ is a direct summand of the free module $\mathcal{O}_{X,x}^n$, and hence projective. But $\mathscr{G}_x$ is a finitely generated module over a local ring and so being projective is equivalent to being free (Proposition 6 above). □