# Chapter 3

# Calculus on Manifolds

This chapter describes the "kitchen" of differential geometry. We will discuss how one can operate with the various objects we have introduced so far. In particular, we will introduce several derivations of the various algebras of tensor fields, and we will also present the inverse operation of integration.

## 3.1 The lie derivative

### 3.1.1 *Flows on manifolds*

The notion of flow should be familiar to anyone who has had a course in ordinary differential equations. In this section we only want to describe some classical analytic facts in a geometric light. We strongly recommend [4] for more details, and excellent examples.

A neighborhood $\mathcal{N}$ of $\{0\} \times M$ in $\mathbb{R} \times M$ is called *balanced* if, $\forall m \in M$, there exists $r \in (0, \infty]$ such that

$$(\,\mathbb{R} \times \{m\}\,) \cap \mathcal{N} = (-r, r) \times \{m\}.$$

Note that any continuous function $f : M \to (0, \infty)$ defines a balanced open

$$\mathcal{N}_f := \big\{(t, m) \in \mathbb{R} \times M;\ \ |t| < f(m)\,\big\}.$$

**Definition 3.1.1.** A *local flow* is a smooth map $\Phi : \mathcal{N} \to M$, $(t, m) \mapsto \Phi^t(m)$, where $\mathcal{N}$ is a balanced neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that

(a) $\Phi^0(m) = m$, $\forall m \in M$.
(b) $\Phi^t(\Phi^s(m)) = \Phi^{t+s}(m)$ for all $s, t \in \mathbb{R}$, $m \in M$ such that

$$(s, m),\ (s + t, m),\ \ (t, \Phi^s(m)) \in \mathcal{N}.$$

When $\mathcal{N} = \mathbb{R} \times M$, $\Phi$ is called a *flow*. □

The conditions (a) and (b) above show that a flow is nothing but a left action of the additive (Lie) group $(\mathbb{R}, +)$ on $M$.



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**Example 3.1.2.** Let $A$ be an $n \times n$ real matrix. It generates a flow $\Phi_A^t$ on $\mathbb{R}^n$ by

$$\Phi_A^t x = e^{tA}x = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k\right)x. \qquad \square$$

**Definition 3.1.3.** Let $\Phi : \mathcal{N} \to M$ be a local flow on $M$. The *infinitesimal generator* of $\Phi$ is the vector field $X$ on $M$ defined by

$$X(p) = X_\Phi(p) := \frac{d}{dt}\,|_{t=0}\,\Phi^t(p), \quad \forall p \in M,$$

i.e., $X(p)$ is the tangent vector to the smooth path $t \mapsto \Phi^t(p)$ at $t = 0$. This path is called the *flow line* through $p$. $\square$

**Exercise 3.1.4.** Show that $X_\Phi$ is a *smooth* vector field. $\square$

**Example 3.1.5.** Consider the flow $e^{tA}$ on $\mathbb{R}^n$ generated by an $n \times n$ matrix $A$. Its generator is the vector field $X_A$ on $\mathbb{R}^n$ defined by

$$X_A(u) = \frac{d}{dt}\,|_{t=0}\,e^{tA}u = Au. \qquad \square$$

**Proposition 3.1.6.** *Let $M$ be a smooth $n$-dimensional manifold. The map*

$$X : \{\text{Local flows on } M\} \to \mathrm{Vect}\,(M), \quad \Phi \mapsto X_\Phi,$$

*is a surjection. Moreover, if $\Phi_i : \mathcal{N}_i \to M$ (i=1,2) are two local flows such that $X_{\Phi_1} = X_{\Phi_2}$, then $\Phi_1 = \Phi_2$ on $\mathcal{N}_1 \cap \mathcal{N}_2$.*

***Proof.*** *Surjectivity.* Let $X$ be a vector field on $M$. An *integral curve* for $X$ is a smooth curve $\gamma : (a, b) \to M$ such that

$$\dot{\gamma}(t) = X(\gamma(t)).$$

In local coordinates $(x^i)$ over on open subset $U \subset M$ this condition can be rewritten as

$$\dot{x}^i(t) = X^i\big(\,x^1(t), ..., x^n(t)\,\big), \;\; \forall i = 1, \dots, n, \tag{3.1.1}$$

where $\gamma(t) = (x^1(t), ..., x^n(t))$, and $X = X^i \frac{\partial}{\partial x^i}$. The above equality is a system of ordinary differential equations. Classical existence results (see e.g. [4, 43]) show that, for any precompact open subset $K \subset U$, there exists $\varepsilon > 0$ such that, for all $x \in K$, there exists a unique integral curve for $X$, $\gamma_x : (-\varepsilon, \varepsilon) \to M$ satisfying

$$\gamma_x(0) = x. \tag{3.1.2}$$

Moreover, as a consequence of the smooth dependence upon initial data we deduce that the map

$$\Phi_K : \mathcal{N}_K = (-\varepsilon, \varepsilon) \times K \to M, \;\; (x, t) \mapsto \gamma_x(t),$$

is smooth.

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Now we can cover $M$ by open, precompact, local coordinate neighborhoods $(K_\alpha)_{\alpha\in\mathcal{A}}$, and as above, we get smooth maps $\Phi_\alpha : \mathcal{N}_\alpha = (-\varepsilon_\alpha, \varepsilon_\alpha) \times K_\alpha \to M$ solving the initial value problem (3.1.1-2). Moreover, by uniqueness, we deduce

$$\Phi_\alpha = \Phi_\alpha \text{ on } \mathcal{N}_\alpha \cap \mathcal{N}_\beta.$$

Define

$$\mathcal{N} := \bigcup_{\alpha\in\mathcal{A}} \mathcal{N}_\alpha,$$

and set $\Phi : \mathcal{N} \to M$, $\Phi = \Phi_\alpha$ on $\mathcal{N}_\alpha$.

The uniqueness of solutions of initial value problems for ordinary differential equations implies that $\Phi$ satisfies all the conditions in the definition of a local flow. Tautologically, $X$ is the infinitesimal generator of $\Phi$. The second part of the proposition follows from the uniqueness in initial value problems. □

The family of local flows on $M$ with the same infinitesimal generator $X \in \mathrm{Vect}(M)$ is naturally ordered according to their domains,

$$(\Phi_1 : \mathcal{N}_1 \to M) \prec (\Phi_2 : \mathcal{N}_2 \to M)$$

if and only if $\mathcal{N}_1 \subset \mathcal{N}_2$. This family has a *unique* maximal element which is called the *local flow generated by* $X$, and it is denoted by $\Phi_X$.

**Exercise 3.1.7.** Consider the unit sphere

$$S^2 = \big\{ (x,y,z) \in \mathbb{R}^3;\ \ x^2+y^2+z^2 = 1 \big\}.$$

For every point $p \in S^2$ we denote by $X(p) \in T_p\mathbb{R}^3$, the orthogonal projection of the vector $\boldsymbol{k} = (0,0,1)$ onto $T_pS^2$.
(a) Prove that $p \mapsto X(p)$ is a smooth vector field on $S^2$, and then describe it in cylindrical coordinates $(z,\theta)$, where

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r = (x^2+y^2)^{1/2}.$$

(b) Describe explicitly the flow generated by $X$. □

### 3.1.2 *The Lie derivative*

Let $X$ be a vector field on the smooth $n$-dimensional manifold $M$ and denote by $\Phi = \Phi_X$ the local flow it generates. For simplicity, we assume $\Phi$ is actually a flow so its domain is $\mathbb{R} \times M$. The local flow situation is conceptually identical, but notationally more complicated.

For each $t \in \mathbb{R}$, the map $\Phi^t$ is a diffeomorphism of $M$ and so it induces a push-forward map on the space of tensor fields. If $S$ is a tensor field on $M$ we define its *Lie derivative* along the direction given by $X$ as

$$L_X S_m := -\lim_{t\to 0} \frac{1}{t}\left( (\Phi^t_* S)_m - S_m \right)\ \ \forall m \in M. \tag{3.1.3}$$

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Intuitively, $L_X S$ measures how fast is the flow $\Phi$ changing[1] the tensor $S$.

If the limit in (3.1.3) exists, then one sees that $L_X S$ is a tensor of the same type as $S$. To show that the limit exists, we will provide more explicit descriptions of this operation.

**Lemma 3.1.8.** *For any $X \in \mathrm{Vect}\,(M)$ and $f \in C^\infty(M)$ we have*

$$Xf := L_X f = \langle df, X\rangle = df(X).$$

*Above, $\langle \bullet, \bullet\rangle$ denotes the natural duality between $T^*M$ and $TM$,*

$$\langle \bullet, \bullet\rangle : C^\infty(T^*M) \times C^\infty(TM) \to C^\infty(M),$$

$$C^\infty(T^*M) \times C^\infty(TM) \ni (\alpha, X) \mapsto \alpha(X) \in C^\infty(M).$$

*In particular, $L_X$ is a derivation of $C^\infty(M)$.*

***Proof.*** Let $\Phi^t = \Phi^t_X$ be the local flow generated by $X$. Assume for simplicity that it is defined for all $t$. The map $\Phi^t$ acts on $C^\infty(M)$ by the pullback of its inverse, i.e.

$$\Phi^t_* = (\Phi^{-t})^*.$$

Hence, for point $p \in M$ we have

$$L_X f(p) = \lim_{t\to 0} \frac{1}{t}\big(f(p) - f(\Phi^{-t}p)\big) = -\frac{d}{dt}\,|_{t=0}\; f(\Phi^{-t}p) = \langle df, X\rangle_p. \qquad \square$$

**Exercise 3.1.9.** Prove that any derivation of the algebra $C^\infty(M)$ is of the form $L_X$ for some $X \in \mathrm{Vect}(M)$, i.e.

$$\mathrm{Der}\,(C^\infty(M)) \cong \mathrm{Vect}\,(M). \qquad \square$$

**Lemma 3.1.10.** *Let $X, Y \in \mathrm{Vect}\,(M)$. Then the Lie derivative of $Y$ along $X$ is a new vector field $L_X Y$ which, viewed as a derivation of $C^\infty(M)$, coincides with the commutator of the two derivations of $C^\infty(M)$ defined by $X$ and $Y$ i.e.*

$$L_X Y f = [X, Y]f, \;\; \forall\, f \in C^\infty(M).$$

*The vector field $[X, Y] = L_X Y$ is called the* Lie bracket *of $X$ and $Y$. In particular the Lie bracket induces a Lie algebra structure on* $\mathrm{Vect}\,(M)$.

[1] Arnold refers to the Lie derivative $L_X$ as the "*fisherman's derivative*". Here is the intuition behind this very suggestive terminology. We place an observer (fisherman) at a fixed point $p \in M$, and we let him keep track of the the sizes of the tensor $S$ carried by the flow at the point $p$. The Lie derivatives measures the rate of change in these sizes.

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***Proof.*** We will work in local coordinates $(x^i)$ near a point $m \in M$ so that

$$X = X^i \frac{\partial}{\partial x_i} \text{ and } Y = Y^j \frac{\partial}{\partial x_j}.$$

We first describe the commutator $[X, Y]$. If $f \in C^\infty(M)$, then

$$[X, Y]f = (X^i \frac{\partial}{\partial x_i})(Y^j \frac{\partial f}{\partial x^j}) - (Y^j \frac{\partial}{\partial x_j})(X^i \frac{\partial f}{\partial x^i})$$

$$= \left( X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} \right) - \left( X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \right),$$

so that the commutator of the two derivations is the derivation defined by the vector field

$$[X, Y] = \left( X^i \frac{\partial Y^k}{\partial x^i} - Y^j \frac{\partial X^k}{\partial x^j} \right) \frac{\partial}{\partial x_k}. \tag{3.1.4}$$

Note in particular that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, i.e., the basic vectors $\frac{\partial}{\partial x_i}$ commute as derivations.

So far we have not proved the vector field in (3.1.4) is independent of coordinates. We will achieve this by identifying it with the intrinsically defined vector field $L_X Y$.

Set $\gamma(t) = \Phi^t m$ so that we have a parametrization $\gamma(t) = (x^i(t))$ with $\dot{x}^i = X^i$. Then

$$\Phi^{-t} m = \gamma(-t) = \gamma(0) - \dot{\gamma}(0)t + O(t^2) = \left( x_0^i - tX^i + O(t^2) \right),$$

and

$$Y^j_{\gamma(-t)} = Y^j_m - tX^i \frac{\partial Y^j}{\partial x^i} + O(t^2). \tag{3.1.5}$$

Note that $\Phi_*^{-t} : T_{\gamma(0)}M \to T_{\gamma(-t)}M$ is the linearization of the map

$$(x^i) \mapsto \left( x_0^i - tX^i + O(t^2) \right),$$

so it has a matrix representation

$$\Phi_*^{-t} = \mathbb{1} - t \left( \frac{\partial X^i}{\partial x^j} \right)_{i,j} + O(t^2). \tag{3.1.6}$$

In particular, using the geometric series

$$(\mathbb{1} - A)^{-1} = \mathbb{1} + A + A^2 + \cdots,$$

where $A$ is a matrix of operator norm strictly less than 1, we deduce that the differential

$$\Phi_*^t = (\Phi_*^{-t})^{-1} : T_{\gamma(-t)}M \to T_{\gamma(0)}M,$$

has the matrix form

$$\Phi_*^t = \mathbb{1} + t \left( \frac{\partial X^i}{\partial x^j} \right)_{i,j} + O(t^2). \tag{3.1.7}$$

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Using (3.1.7) in (3.1.5) we deduce

$$Y_m^k - \left(\Phi_*^t Y_{\Phi^{-t} m}\right)^k = t\left( X^i \frac{\partial Y^k}{\partial x^i} - Y^j \frac{\partial X^k}{\partial x^j}\right) + O(t^2).$$

This concludes the proof of the lemma. □

**Lemma 3.1.11.** *For any differential form $\omega \in \Omega^1(M)$, and any vector fields $X, Y \in Vect(M)$ we have*

$$(L_X\omega)(Y) = L_X\left(\omega(Y)\right) - \omega([X,Y]), \tag{3.1.8}$$

*where $X \cdot \omega(Y)$ denotes the (Lie) derivative of the function $\omega(Y)$ along the vector field $X$.*

***Proof.*** Denote by $\Phi^t$ the local flow generated by $X$. We have $\Phi_*^t\omega = (\Phi^{-t})^*\omega$, i.e., for any $p \in M$, and any $Y \in \mathrm{Vect}\,(M)$, we have

$$(\Phi_*^t\omega)_p(Y_p) = \omega_{\Phi^{-t}p}\big(\,\Phi_*^{-t}Y_p\,\big).$$

Fix a point $p \in M$, and choose local coordinates $(x^i)$ near $p$. Then

$$\omega = \sum_i \omega^i dx^i,\ \ X = \sum_i X^i \frac{\partial}{\partial x^i},\ \ Y = \sum_i Y^i \frac{\partial}{\partial x^i}.$$

Denote by $\gamma(t)$ the path $t \mapsto \Phi^t(p)$. We set $\omega_i(t) = \omega_i(\gamma(t))$, $X_0^i = X^i(p)$, and $Y_0^i = Y^i(p)$. Using (3.1.6) we deduce

$$(\Phi_*^t\omega)_p(Y_p) = \sum_i \omega_i(-t)\cdot\Big(\,Y_0^i - t\sum_j \frac{\partial X^i}{\partial x^j}Y_0^j + O(t^2)\,\Big).$$

Hence

$$-(L_X\omega)Y = \frac{d}{dt}|_{t=0}(\Phi_*^t\omega)_p(Y_p) = -\sum_i \dot\omega_i(0)Y_0^i - \sum_{i,j}\omega_i(0)\frac{\partial X^i}{\partial x^j}Y_0^j.$$

On the other hand, we have

$$X\cdot\omega(Y) = \frac{d}{dt}|_{t=0}\sum_i \omega_i(t)Y^i(t) = \sum_i \dot\omega_i(0)Y_0^i + \sum_{i,j}\omega_i(0)X_0^j\frac{\partial Y^i}{\partial x^j}.$$

We deduce that

$$X\cdot\omega(Y) - (L_X\omega)Y = \sum_{i,j}\omega_i(0)\Big(\,X_0^j\frac{\partial Y^i}{\partial x^j} - \frac{\partial X^i}{\partial x^j}Y_0^j\,\Big) = \omega_p([X,Y]_p).$$ □

Observe that if $S, T$ are two tensor fields on $M$ such that both $L_XS$ and $L_XT$ exist, then using (3.1.3) we deduce that $L_X(S\otimes T)$ exists, and

$$L_X(S\otimes T) = L_XS\otimes T + S\otimes L_XT. \tag{3.1.9}$$

Since any tensor field is locally a linear combination of tensor monomials of the form

$$X_1\otimes\cdots\otimes X_r\otimes\omega_1\otimes\cdots\otimes\omega_s,\quad X_i\in\mathrm{Vect}\,(M),\ \ \omega_j\in\Omega^1(M),$$

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we deduce that the Lie derivative $L_X S$ exists for every $X \in \mathrm{Vect}\,(M)$, and any smooth tensor field $S$. We can now completely describe the Lie derivative on the algebra of tensor fields.

**Proposition 3.1.12.** *Let $X$ be a vector field on the smooth manifold $M$. Then the Lie derivative $L_X$ is the unique derivation of $\mathcal{T}^*_*(M)$ with the following properties.*
*(a) $L_X f = \langle df, X\rangle = Xf$, $\forall f \in C^\infty(M)$.*
*(b) $L_X Y = [X, Y]$, $\forall X, Y \in \mathrm{Vect}\,(M)$.*
*(c) $L_X$ commutes with the contraction $\mathrm{tr} : \mathcal{T}^{r+1}_{s+1}(M) \to \mathcal{T}^r_s(M)$.*

*Moreover, $L_X$ is a natural operation, i.e., for any diffeomorphism $\phi : M \to N$ we have $\phi_* \circ L_X = L_{\phi_* X} \circ \phi_*$, $\forall X \in \mathrm{Vect}\,(M)$, i.e., $\phi_*(L_X) = L_{\phi_* X}$.*

***Proof.*** The fact that $L_X$ is a derivation follows from (3.1.9). Properties (a) and (b) were proved above. As for part (c), in its simplest form, when $T = Y \otimes \omega$, where $Y \in \mathrm{Vect}\,(M)$, and $\omega \in \Omega^1(M)$, the equality

$$L_X\, \mathrm{tr}\, T = \mathrm{tr}\, L_X T$$

is equivalent to

$$L_X\,(\omega(Y)) = (L_X\omega)(Y) + \omega(L_X(Y), \tag{3.1.10}$$

which is precisely (3.1.8).

Since $L_X$ is a derivation of the algebra of tensor fields, its restriction to $C^\infty(M) \oplus \mathrm{Vect}\,(M) \oplus \Omega^1(M)$ uniquely determines the action on the entire algebra of tensor fields which is generated by the above subspace. The reader can check easily that the general case of property (c) follow from this observation coupled with the product rule (3.1.9).

The naturality of $L_X$ is another way of phrasing the coordinate independence of this operation. We leave the reader to fill in the routine details. □

**Corollary 3.1.13.** *For any $X, Y \in \mathrm{Vect}\,(M)$ we have*

$$[L_X, L_Y] = L_{[X,Y]},$$

*as derivations of the algebra of tensor fields on $M$. In particular, this says that* $\mathrm{Vect}\,(M)$ *as a space of derivations of $\mathcal{T}^*_*(M)$ is a Lie subalgebra of the Lie algebra of derivations.*

***Proof.*** The commutator $[L_X, L_Y]$ is a derivation (as a commutator of derivations). By Lemma 3.1.10, $[L_X, L_Y] = L_{[X,Y]}$ on $C^\infty(M)$. Also, a simple computation shows that

$$[L_X, L_Y]Z = L_{[X,Y]}Z, \;\; \forall Z \in \mathrm{Vect}\,(M),$$

so that $[L_X, L_Y] = L_{[X,Y]}$ on $\mathrm{Vect}\,(M)$. Finally, since the contraction commutes with both $L_X$ and $L_Y$ it obviously commutes with $L_X L_Y - L_Y L_X$. The corollary is proved. □

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**Exercise 3.1.14.** Prove that the map

$$\mathcal{D} : \mathrm{Vect}\,(M) \oplus \mathrm{End}\,(TM) \to Der(\mathcal{T}^*_*(M))$$

given by $\mathcal{D}(X, S) = L_X + S$ is well defined and is a linear isomorphism. Moreover,

$$[\mathcal{D}(X_1, S_1), \mathcal{D}(X_2, S_2)] = \mathcal{D}([X_1, X_2], [S_1, S_2]). \qquad \square$$

$L_X$ is a derivation of $\mathcal{T}^*_*$ with the remarkable property

$$L_X(\Omega^*(M)) \subset \Omega^*(M).$$

The wedge product makes $\Omega^*(M)$ an $s$-algebra, and it is natural to ask whether $L_X$ is an s-derivation with respect to this product.

**Proposition 3.1.15.** *The Lie derivative along a vector field $X$ is an even s-derivation of $\Omega^*(M)$, i.e.*

$$L_X(\omega \wedge \eta) = (L_X\omega) \wedge \eta + \omega \wedge (L_X\eta), \;\; \forall \omega, \, \eta \in \Omega^*(M).$$

***Proof.*** As in Subsection 2.2.2, denote by $\mathcal{A}$ the anti-symmetrization operator $\mathcal{A} : (T^*M)^{\otimes k} \to \Omega^k(M)$. The statement in the proposition follows immediately from the straightforward observation that the Lie derivative commutes with this operator (which is a projector). We leave the reader to fill in the details. $\square$

**Exercise 3.1.16.** Let $M$ be a smooth manifold, and suppose that $\Phi, \Psi : \mathbb{R} \times M \to M$ are two smooth flows on $M$ with infinitesimal generators $X$ and respectively $Y$. We say that the two flows commute if

$$\Phi^t \circ \Psi^s = \Psi^s \circ \Psi^t, \;\; \forall s, t \in \mathbb{R}.$$

Prove that if

$$\big\{\, p \in M; \;\; X(p) = 0 \,\big\} = \big\{\, p \in M; \;\; Y(p) = 0 \,\big\},$$

the two flows commute if and only if $[X, Y] = 0$. $\square$

### 3.1.3 *Examples*

**Example 3.1.17.** Let $\omega = \omega_i dx^i$ be a 1-form on $\mathbb{R}^n$. If $X = X^j \frac{\partial}{\partial x^j}$ is a vector field on $\mathbb{R}^n$ then $L_X\omega = (L_X\omega)_k dx^k$ is defined by

$$(L_X\omega)_k = (L_X\omega)(\frac{\partial}{\partial x_k}) = X\omega(\frac{\partial}{\partial x_k}) - \omega(L_X\frac{\partial}{\partial x_k}) = X \cdot \omega_k + \omega\left(\frac{\partial X^i}{\partial x^k}\frac{\partial}{\partial x_i}\right).$$

Hence

$$L_X\omega = \left(X^j\frac{\partial \omega_k}{\partial x^j} + \omega_j\frac{\partial X^j}{\partial x^k}\right)dx^k.$$

In particular, if $X = \partial_{x^i} = \sum_j \delta^{ij}\partial_{x^j}$, then

$$L_X\omega = L_{\partial_{x^i}}\omega = \sum_{k=1}^n \frac{\partial \omega_k}{\partial x^i}dx^k.$$

If $X$ is the radial vector field $X = \sum_i x^i\partial_{x^i}$, then

$$L_X\omega = \sum_k (X \cdot \omega_k + \omega_k)dx^k. \qquad \square$$

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**Example 3.1.18.** Consider a smooth vector field $X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y} + H\frac{\partial}{\partial z}$ on $\mathbb{R}^3$. We want to compute $L_X dv$, where $dv$ is the volume form on $\mathbb{R}^3$, $dv = dx \wedge dy \wedge dz$. Since $L_X$ is an even s-derivation of $\Omega^*(M)$, we deduce

$$L_X(dx \wedge dy \wedge dz) = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz).$$

Using the computation in the previous example we get

$$L_X(dx) = dF := \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz, \;\; L_X(dy) = dG, \;\; L_X(dz) = dH$$

so that

$$L_X(dv) = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) dv = (\boldsymbol{div}\, X)dv.$$

In particular, we deduce that if $\boldsymbol{div}\, X = 0$, the local flow generated by $X$ preserves the form $dv$. We will get a better understanding of this statement once we learn integration on manifolds, later in this chapter. □

**Example 3.1.19. (The exponential map of a Lie group).** Consider a Lie group $G$. Any element $g \in G$ defines two diffeomorphisms of $G$: the left $(L_g)$, and the right translation $(R_g)$ on $G$,

$$L_g(h) = g \cdot hy, \;\; R_g(h) = h \cdot g, \;\; \forall h \in G.$$

A tensor field $T$ on $G$ is called *left* (respectively *right*) *invariant* if for any $g \in G$ $(L_g)_* T = T$ (respectively $(R_g)_* T = T$). The set of left invariant vector fields on $G$ is denoted by $\mathfrak{L}_G$. The naturality of the Lie bracket implies

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y],$$

so that $\forall X, Y \in \mathfrak{L}_G$, $[X, Y] \in \mathfrak{L}_G$. Hence $\mathfrak{L}_G$ is a Lie subalgebra of $\mathrm{Vect}\,(G)$. It is called called the *Lie algebra* of the group $G$.

**Fact 1.** $\dim \mathfrak{L}_G = \dim G$. Indeed, the left invariance implies that the restriction map $\mathfrak{L}_G \to T_1 G$, $X \mapsto X_1$ is an isomorphism (*Exercise*). We will often find it convenient to identify the Lie algebra of $G$ with the tangent space at 1.

**Fact 2.** Any $X \in \mathfrak{L}_G$ defines a local flow $\Phi_X^t$ on $G$ that is is defined for all $t \in \mathbb{R}$. In other wors, $\Phi_X^t$ is a flow. (*Exercise*) Set

$$\exp(tX) := \Phi_X^t(1).$$

We thus get a map

$$\exp : T_1 G \cong \mathfrak{L}_G \to G, \;\; X \mapsto \exp(X)$$

called the *exponential map of the group G*.

**Fact 3.** $\Phi_X^t(g) = g \cdot \exp(tX)$, i.e.,

$$\Phi_X^t = R_{\exp(tX)}.$$

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Indeed, it suffices to check that

$$\frac{d}{dt}\,|_{t=0}\,(g\cdot\exp(tX)) = X_g.$$

We can write $(g\cdot\exp(tX)) = L_g\exp(tX)$ so that

$$\frac{d}{dt}\,|_{t=0}\,(L_g\exp(tX)) = (L_g)_*\Big(\frac{d}{dt}\,|_{t=0}\exp(tX)\Big) = (L_g)_*X = X_g,\ \ \text{(left invariance)}.$$

The reason for the notation $\exp(tX)$ is that when $G = \mathrm{GL}(n,\mathbb{K})$, then the Lie algebra of $G$ is the Lie algebra $\underline{\mathrm{gl}}(n,\mathbb{K})$ of all $n\times n$ matrices with the bracket given by the commutator of two matrices, and for any $X\in\mathcal{L}_G$ we have (*Exercise*)

$$\exp(X) = e^X = \sum_{k\geq 0}\frac{1}{k!}X^k. \qquad \square$$

**Exercise 3.1.20.** Prove the statements left as exercises in the example above. $\square$

**Exercise 3.1.21.** Let $G$ be a matrix Lie group, i.e., a Lie subgroup of some general linear group $\mathrm{GL}(N,\mathbb{K})$. This means the tangent space $T_1G$ can be identified with a linear space of matrices. Let $X, Y\in T_1G$, and denote by $\exp(tX)$ and $\exp(tY)$ the 1-parameter groups with they generate, and set

$$g(s,t) = \exp(sX)\exp(tY)\exp(-sX)\exp(-tY).$$

(a) Show that

$$g_{s,t} = 1 + [X,Y]_{alg}st + O((s^2+t^2)^{3/2})\ \text{ as }\ s,t\to 0,$$

where the bracket $[X,Y]_{alg}$ (temporarily) denotes the commutator of the two matrices $X$ and $Y$.

(b) Denote (temporarily) by $[X,Y]_{geom}$ the Lie bracket of $X$ and $Y$ viewed as left invariant vector fields on $G$. Show that at $1\in G$

$$[X,Y]_{alg} = [X,Y]_{geom}.$$

(c) Show that $\underline{\mathrm{o}}(n)\subset\underline{\mathrm{gl}}(n,\mathbb{R})$ (defined in Section 1.2.2) is a Lie subalgebra with respect to the commutator $[\,\cdot\,,\,\cdot\,]$. Similarly, show that $\underline{\mathrm{u}}(n),\ \underline{\mathrm{su}}(n)\subset\underline{\mathrm{gl}}(n,\mathbb{C})$ are *real* Lie subalgebras of $\underline{\mathrm{gl}}(n,\mathbb{C})$, while $\underline{\mathrm{su}}(n,\mathbb{C})$ is even a *complex* Lie subalgebra of $\underline{\mathrm{gl}}(n,\mathbb{C})$.

(d) Prove that we have the following isomorphisms of *real* Lie algebras. $\mathcal{L}_{O(n)}\cong\underline{\mathrm{o}}(n)$, $\mathcal{L}_{U(n)}\cong\underline{\mathrm{u}}(n)$, $\mathcal{L}_{SU(n)}\cong\underline{\mathrm{su}}(n)$ and $\mathcal{L}_{SL(n,\mathbb{C})}\cong\underline{\mathrm{sl}}(n,\mathbb{C})$. $\square$

**Remark 3.1.22.** In general, in a non-commutative matrix Lie group $G$, the traditional equality

$$\exp(tX)\exp(tY) = \exp(t(X+Y))$$

no longer holds. Instead, one has the *Campbell-Hausdorff formula*

$$\exp(tX)\cdot\exp(tY) = \exp\big(\,td_1(X,Y) + t^2d_2(X,Y) + t^3d_3(X,Y) + \cdots\big),$$

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where $d_k$ are homogeneous polynomials of degree $k$ in $X$, and $Y$ with respect to the multiplication between $X$ and $Y$ *given by their bracket.* The $d_k$'s are usually known as Dynkin polynomials. For example,

$$d_1(X,Y) = X + Y, \;\; d_2(X,Y) = \frac{1}{2}[X,Y],$$

$$d_3(X,Y) = \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) \text{ etc.}$$

For more details we refer to [41, 84]. □

## 3.2 Derivations of $\Omega^\bullet(M)$

### 3.2.1 *The exterior derivative*

The super-algebra of exterior forms on a smooth manifold $M$ has additional structure, and in particular, its space of derivations has special features. This section is devoted precisely to these new features.

The Lie derivative along a vector field $X$ defines an even derivation in $\Omega^\bullet(M)$. The vector field $X$ also defines, via the contraction map, an odd derivation $i_X$, called the *interior derivation along* $X$, or the *contraction by* $X$,

$$i_X\omega := \mathrm{tr}\,(X \otimes \omega), \;\; \forall \omega \in \Omega^r(M).$$

More precisely, $i_X\omega$ is the $(r-1)$-form determined by

$$(i_X\omega)(X_1, \dots, X_{r-1}) = \omega(X, X_1, \dots, X_{r-1}), \;\; \forall X_1, \dots, X_{r-1} \in \mathrm{Vect}\,(M).$$

The fact that $i_X$ is an odd $s$-derivation is equivalent to

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (i_X\eta), \;\; \forall \omega, \eta \in \Omega^*(M).$$

Often the contraction by $X$ is denoted by $X \lrcorner$.

**Exercise 3.2.1.** Prove that the interior derivation along a vector field is a $s$-derivation. □

**Proposition 3.2.2.** *(a)* $[i_X, i_Y]_s = i_X i_Y + i_Y i_X = 0$.
*(b) The super-commutator of $L_X$ and $i_Y$ as s-derivations of $\Omega^*(M)$ is given by*

$$[L_X, i_Y]_s = L_X i_Y - i_Y L_X = i_{[X,Y]}. \qquad \square$$

The proof uses the fact that the Lie derivative commutes with the contraction operator, and it is left to the reader as an exercise.

The above $s$-derivations by no means exhaust the space of s-derivations of $\Omega^\bullet(M)$. In fact we have the following fundamental result.

**Proposition 3.2.3.** *There exists an odd s-derivation $d$ on the $s$-algebra of differential forms $\Omega^\bullet(\,\cdot\,)$ uniquely characterized by the following conditions.*

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*(a) For any smooth function* $f \in \Omega^0(M)$*,* $df$ *coincides with the differential of* $f$*.*
*(b)* $d^2 = 0$*.*
*(c)* $d$ *is natural, i.e., for any smooth function* $\phi : N \to M$*, and for any form* $\omega$ *on* $M$*, we have*

$$d\phi^*\omega = \phi^* d\omega (\Longleftrightarrow [\phi^*, d] = 0).$$

*The derivation* $d$ *is called the* exterior derivative*.*

***Proof.*** *Uniqueness.* Let $U$ be a local coordinate chart on $M^n$ with local coordinates $(x^1, \dots, x^n)$. Then, over $U$, any $r$-form $\omega$ can be described as

$$\omega = \sum_{1 \le i_1 < \cdots < i_r \le n} \omega_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}.$$

Since $d$ is an $s$-derivation, and $d(dx^i) = 0$, we deduce that, over $U$

$$d\omega = \sum_{1 \le i_1 < \cdots < i_r \le n} (d\omega_{i_1 \dots i_r}) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r})$$

$$= \sum_{1 \le i_1 < \cdots < i_r \le n} \left( \frac{\partial \omega_{i_1 \dots i_r}}{\partial x^i} dx^i \right) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_r}). \tag{3.2.1}$$

Thus, the form $d\omega$ is uniquely determined on any coordinate neighborhood, and this completes the proof of the uniqueness of $d$.

*Existence.* Consider an $r$-form $\omega$. For each coordinate neighborhood $U$ we define $d\omega\,|_U$ as in (3.2.1). To prove that this is a well defined operation we must show that, if $U$, $V$ are two coordinate neighborhoods, then

$$d\omega\,|_U = d\omega\,|_V \quad \text{on } U \cap V.$$

Denote by $(x^1, \dots, x^n)$ the local coordinates on $U$, and by $(y^1, \dots, y^n)$ the local coordinates along $V$, so that on the overlap $U \cap V$ we can describe the $y$'s as functions of the $x$'s. Over $U$ we have

$$\omega = \sum_{1 \le i_1 < \cdots < i_r \le n} \omega_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$d\omega = \sum_{1 \le i_1 < \cdots < i_r \le n} \left( \frac{\partial \omega_{i_1 \dots i_r}}{\partial x^i} dx^i \right) \wedge (dx^{i_1} \wedge \dots \wedge dx^{i_r}),$$

while over $V$ we have

$$\omega = \sum_{1 \le j_1 < \cdots < j_r \le n} \widehat{\omega}_{j_1 \dots j_r} dy^{j_1} \wedge \cdots \wedge dy^{j_r}$$

$$d\omega = \sum_{1 \le j_1 < \cdots < j_r \le n} \left( \frac{\partial \widehat{\omega}_{j_1 \dots j_r}}{\partial y^j} dy^j \right) (dy^{j_1} \wedge \cdots \wedge dy^{j_r}).$$

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The components $\omega_{i_1\dots i_r}$ and $\widehat{\omega}_{j_1\dots j_r}$ are skew-symmetric, i.e., $\forall \sigma \in \mathcal{S}_r$

$$\omega_{i_{\sigma(1)}\dots i_{\sigma(r)}} = \epsilon(\sigma)\omega_{i_1\dots i_r},$$

and similarly for the $\widehat{\omega}'$s. Since $\omega\,|_U = \omega\,|_V$ over $U \cap V$ we deduce

$$\omega_{i_1\dots i_r} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \widehat{\omega}_{j_1\dots j_r}.$$

Hence

$$\frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} = \sum_{k=1}^{r} \left( \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial^2 y^{j_k}}{\partial x^i \partial x^{i_k}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \widehat{\omega}_{j_1\dots j_r} + \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \frac{\partial \widehat{\omega}_{j_1\dots j_r}}{\partial x^i} \right),$$

where in the above equality we also sum over the indices $j_1, \dots, j_r$ according to Einstein's convention. We deduce

$$\sum_{1 \le i_1 < \cdots < i_r \le n} \frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$= \sum_i \sum_{k=1}^{r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial^2 x^{j_k}}{\partial x^i \partial x^{i_k}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \widehat{\omega}_{j_1\dots j_r} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$+ \sum_i \sum_{k=1}^{r} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \frac{\partial \widehat{\omega}_{j_1\dots j_r}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}. \tag{3.2.2}$$

Notice that

$$\frac{\partial^2}{\partial x^i \partial x^{i_k}} = \frac{\partial^2}{\partial x^{i_k} \partial x^i},$$

while $dx^i \wedge dx^{i_k} = -dx^{i_k} \wedge dx^i$ so that the first term in the right hand side of (3.2.2) vanishes. Consequently on $U \cap V$

$$\frac{\partial \omega_{i_1\dots i_r}}{\partial x^i} dx^i \wedge dx^{i_1} \cdots \wedge dx^{i_r} = \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_r}}{\partial x^{i_r}} \frac{\partial \widehat{\omega}_{j_1\dots j_r}}{\partial x^i} \wedge dx^i \wedge dx^{i_1} \cdots dx^{i_r}$$

$$= \left( \frac{\partial \widehat{\omega}_{j_1\dots j_r}}{\partial x^i} dx^i \right) \wedge \left( \frac{\partial y^{j_1}}{\partial x^{i_1}} dx^{i_1} \right) \wedge \cdots \wedge \left( \frac{\partial y^{j_r}}{\partial x^{i_r}} dx^{i_r} \right)$$

$$= (d\widehat{\omega}_{j_1\dots j_r}) \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_r} = \frac{\partial \widehat{\omega}_{j_1\dots j_r}}{\partial y^j} dy^j \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_r}.$$

This proves $d\omega\,|_U = d\omega\,|_V$ over $U \cap V$. We have thus constructed a well defined linear map

$$d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M).$$

To prove that $d$ is an odd $s$-derivation it suffices to work in local coordinates and show that the (super)product rule on monomials.

Let $\theta = f dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ and $\omega = g dx^{j_1} \wedge \cdots \wedge dx^{j_s}$. We set for simplicity

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_r} \ \text{ and } \ dx^J := dx^{j_1} \wedge \cdots \wedge dx^{j_s}.$$

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Then

$$d(\theta \wedge \omega) = d(fgdx^I \wedge dx^J) = d(fg) \wedge dx^I \wedge dx^J$$
$$= (df \cdot g + f \cdot dg) \wedge dx^I \wedge dx^J$$
$$= df \wedge dx^I \wedge dx^J + (-1)^r (f \wedge dx^I) \wedge (dg \wedge dx^J)$$
$$= d\theta \wedge \omega + (-1)^{\deg \theta} \theta \wedge d\omega.$$

We now prove $d^2 = 0$. We check this on monomials $fdx^I$ as above.

$$d^2(fdx^I) = d(df \wedge dx^I) = (d^2 f) \wedge dx^I.$$

Thus, it suffices to show $d^2 f = 0$ for all smooth functions $f$. We have

$$d^2 f = \frac{\partial f^2}{\partial x^i \partial x^j} dx^i \wedge dx^j.$$

The desired conclusion follows from the identities

$$\frac{\partial f^2}{\partial x^i \partial x^j} = \frac{\partial f^2}{\partial x^j \partial x^i} \text{ and } dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

Finally, let $\phi$ be a smooth map $N \to M$ and $\omega = \sum_I \omega_I dx^I$ be an $r$-form on $M$. Here $I$ runs through all ordered multi-indices $1 \leq i_1 < \cdots < i_r \leq \dim M$. We have

$$d_N(\phi^* \omega) = \sum_I \left( d_N(\phi^* \omega_I) \wedge \phi^*(dx^I) + \phi^* \omega^I \wedge d(\phi^* dx^I) \right).$$

For functions, the usual chain rule gives $d_N(\phi^* \omega_I) = \phi^*(d_M \omega_I)$. In terms of local coordinates $(x^i)$ the map $\phi$ looks like a collection of $n$ functions $\phi^i \in C^\infty(N)$ and we get

$$\phi^*(dx^I) = d\phi^I = d_N \phi^{i_1} \wedge \cdots \wedge d_N \phi^{i_r}.$$

In particular, $d_N(d\phi^I) = 0$. We put all the above together and we deduce

$$d_N(\phi^* \omega) = \phi^*(d_M \omega^I) \wedge d\phi^I = \phi^*(d_M \omega^I) \wedge \phi^* dx^I = \phi^*(d_M \omega).$$

The proposition is proved. □

**Proposition 3.2.4.** *The exterior derivative satisfies the following relations.*
*(a)* $[d, d]_s = 2d^2 = 0$.
*(b)* **(Cartan's homotopy formula)** $[d, i_X]_s = di_X + i_X d = L_X \ \forall X \in \mathrm{Vect}\,(M)$.
*(c)* $[d, L_X]_s = dL_X - L_X d = 0$, $\forall X \in \mathrm{Vect}\,(M)$.

*An immediate consequence of the homotopy formula is the following invariant description of the exterior derivative:*

$$(d\omega)(X_0, X_1, \ldots, X_r) = \sum_{i=0}^r (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r))$$
$$+ \sum_{0 \leq i < j \leq r} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r). \tag{3.2.3}$$

*Above, the hat indicates that the corresponding entry is missing, and* $[\ ,\ ]_s$ *denotes the super-commutator in the s-algebra of real endomorphisms of* $\Omega^\bullet(M)$.

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***Proof.*** To prove the homotopy formula set

$$\mathcal{D} := [d, i_X]_s = di_X + i_X d.$$

$\mathcal{D}$ is an even s-derivation of $\Omega^*(M)$. It is a local s-derivation, i.e., if $\omega \in \Omega^*(M)$ vanishes on some open set $U$ then $\mathcal{D}\omega$ vanishes on that open set as well. The reader can check easily by direct computation that $\mathcal{D}\omega = L_X\omega$, $\forall \omega \in \Omega^0(M) \oplus \Omega^1(M)$. The homotopy formula is now a consequence of the following technical result left to the reader as an exercise.

**Lemma 3.2.5.** *Let $\mathcal{D}$, $\mathcal{D}'$ be two local s-derivations of $\Omega^\bullet(M)$ which have the same parity, i.e., they are either both even or both odd. If $\mathcal{D} = \mathcal{D}'$ on $\Omega^0(M) \oplus \Omega^1(M)$, then $\mathcal{D} = \mathcal{D}'$ on $\Omega^\bullet(M)$.* □

Part (c) of the proposition is proved in a similar way. Equality (3.2.3) is a simple consequence of the homotopy formula. We prove it in two special case $r = 1$ and $r = 2$.

*The case $r = 1$.* Let $\omega$ be an 1-form and let $X, Y \in \mathrm{Vect}\,(M)$. We deduce from the homotopy formula

$$d\omega(X, Y) = (i_X d\omega)(Y) = (L_X\omega)(Y) - (d\omega(X))(Y).$$

On the other hand, since $L_X$ commutes with the contraction operator, we deduce

$$X\omega(Y) = L_X(\omega(Y)) = (L_X\omega)(Y) + \omega([X, Y]).$$

Hence

$$d\omega(X, Y) = X\omega(Y) - \omega([X, Y]) - (d\omega(X))(Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

This proves (3.2.3) in the case $r = 1$.

*The case $r = 2$.* Consider a 2-form $\omega$ and three vector fields $X$, $Y$ and $Z$. We deduce from the homotopy formula

$$(d\omega)(X, Y, Z) = (i_X d\omega)(Y, Z) = (L_X - di_X)\omega(Y, Z). \tag{3.2.4}$$

Since $L_X$ commutes with contractions we deduce

$$(L_X\omega)(Y, X) = X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z]). \tag{3.2.5}$$

We substitute (3.2.5) into (3.2.4) and we get

$$(d\omega)(X, Y, Z) = X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z]) - d(i_X\omega)(Y, X). \tag{3.2.6}$$

We apply now (3.2.3) for $r = 1$ to the 1-form $i_X\omega$. We get

$$\begin{aligned} d(i_X\omega)(Y, X) &= Y(i_X\omega(Z)) - Z(i_X\omega(Y)) - (i_X\omega)([Y, Z]) \\ &= Y\omega(X, Z) - Z\omega(X, Y) - \omega(X, [Y, Z]). \end{aligned} \tag{3.2.7}$$

If we use (3.2.7) in (3.2.6) we deduce

$$\begin{aligned} (d\omega)(X, Y, Z) &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &\quad -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned} \tag{3.2.8}$$

The general case in (3.2.3) can be proved by induction. The proof of the proposition is complete. □

**Exercise 3.2.6.** Prove Lemma 3.2.5. □

**Exercise 3.2.7.** Finish the proof of (3.2.3) in the general case. □

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### 3.2.2 *Examples*

**Example 3.2.8. (The exterior derivative in $\mathbb{R}^3$).**
(a) Let $f \in C^\infty(\mathbb{R}^3)$. Then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

The differential $df$ looks like the gradient of $f$.
(b) Let $\omega \in \Omega^1(\mathbb{R}^3)$, $\omega = Pdx + Qdy + Rdz$. Then

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx,$$

so that $d\omega$ looks very much like a curl.
(c) Let $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \in \Omega^2(\mathbb{R}^3)$. Then

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$

This looks very much like a divergence. □

**Example 3.2.9.** Let $G$ be a connected Lie group. In Example 3.1.11 we defined the Lie algebra $\mathfrak{L}_G$ of $G$ as the space of left invariant vector fields on $G$. Set

$$\Omega^r_{left}(G) = \text{left invariant } r\text{-forms on } G.$$

In particular, $\mathfrak{L}_G^* \cong \Omega^1_{left}(G)$. If we identify $\mathfrak{L}_G^* \cong T_1^*G$, then we get a natural isomorphism

$$\Omega^r_{left}(G) \cong \Lambda^r \mathfrak{L}_G^*.$$

The exterior derivative of a form in $\Omega^*_{left}$ can be described *only in terms of the algebraic structure of* $\mathfrak{L}_G$.

Indeed, let $\omega \in \mathfrak{L}_G^* = \Omega^1_{left}(G)$. For $X, Y \in \mathfrak{L}_G^*$ we have (see (3.2.3) )

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Since $\omega$, $X$ and $Y$ are left invariant, the scalars $\omega(X)$ and $\omega(Y)$ are constants. Thus, the first two terms in the above equality vanish so that

$$d\omega(X, Y) = -\omega([X, Y]).$$

More generally, if $\omega \in \Omega^r_{left}$, then the same arguments applied to (3.2.3) imply that for all $X_0, ..., X_r \in \mathfrak{L}_G$ we have

$$d\omega(X_0, X_1, ..., X_r) = \sum_{0 \le i < j \le r} (-1)^{i+j} \omega([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_r). \quad (3.2.9)$$

□

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## 3.3 Connections on vector bundles

### 3.3.1 *Covariant derivatives*

We learned several methods of differentiating tensor objects on manifolds. However, the tensor bundles are not the only vector bundles arising in geometry, and very often one is interested in measuring the "oscillations" of sections of vector bundles.

Let $E$ be a $\mathbb{K}$-vector bundle over the smooth manifold $M$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Technically, one would like to have a procedure of measuring the rate of change of a section $u$ of $E$ along a direction described by a vector field $X$. For such an arbitrary $E$, we encounter a problem which was not present in the case of tensor bundles. Namely, the local flow generated by the vector field $X$ on $M$ no longer induces bundle homomorphisms.

For tensor fields, the transport along a flow was a method of comparing objects in different fibers which otherwise are abstract linear spaces with no natural relationship between them.

To obtain something that looks like a derivation we need to formulate clearly what properties we should expect from such an operation.

(a) It should measure how fast is a given section changing along a direction given by a vector field $X$. Hence it has to be an operator

$$\nabla : \mathrm{Vect}\,(M) \times C^\infty(E) \to C^\infty(E), \;\; (X, u) \mapsto \nabla_X u.$$

(b) If we think of the usual directional derivative, we expect that after "rescaling" the direction $X$ the derivative along $X$ should only rescale by the same factor, i.e.,

$$\forall f \in C^\infty(M) : \;\; \nabla_{fX} u = f \nabla_X u.$$

(c) Since $\nabla$ is to be a derivation, it has to satisfy a sort of (Leibniz) product rule. The only product that exists on an abstract vector bundle is the multiplication of a section with a smooth function. Hence we require

$$\nabla_X(fu) = (Xf)u + f\nabla_X u, \;\; \forall f \in C^\infty(M), \; u \in C^\infty(E).$$

The conditions (a) and (b) can be rephrased as follows: for any $u \in C^\infty(E)$, the map

$$\nabla u : \mathrm{Vect}\,(M) \to C^\infty(E), \;\; X \mapsto \nabla_X u,$$

is $C^\infty(M)$-linear so that it defines a bundle map

$$\nabla u \in \mathrm{Hom}\,(TM, E) \cong C^\infty(T^*M \otimes E).$$

Summarizing, we can formulate the following definition.

**Definition 3.3.1.** A *covariant derivative* (or *linear connection*) on $E$ is a $\mathbb{K}$-linear map

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E),$$

such that, $\forall f \in C^\infty(M)$, and $\forall u \in C^\infty(E)$, we have

$$\nabla(fu) = df \otimes u + f\nabla u.$$

□

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**Example 3.3.2.** Let $\underline{\mathbb{K}}^r_M \cong \mathbb{K}^r \times M$ be the rank $r$ trivial vector bundle over $M$. The space $C^\infty(\underline{\mathbb{K}}_M)$ of smooth sections coincides with the space $C^\infty(M, \mathbb{K}^r)$ of $\mathbb{K}^r$-valued smooth functions on $M$. We can define

$$\nabla^0 : C^\infty(M, \mathbb{K}^r) \to C^\infty(M, T^*M \otimes \mathbb{K}^r)$$

$$\nabla^0(f_1, ..., f_r) = (df_1, ..., df_r).$$

One checks easily that $\nabla$ is a connection. This is called the *trivial connection.* □

**Remark 3.3.3.** Let $\nabla^0$, $\nabla^1$ be two connections on a vector bundle $E \to M$. Then for any $\alpha \in C^\infty(M)$ the map

$$\nabla = \alpha\nabla^1 + (1-\alpha)\nabla^0 : C^\infty(E) \to C^\infty(T^* \otimes E)$$

is again a connection. □

✍**Notation.** For any vector bundle $F$ over $M$ we set

$$\Omega^k(F) : = C^\infty(\Lambda^k T^*M \otimes F).$$

We will refer to these sections as *differential $k$-forms with coefficients in the vector bundle $F$.* □

**Proposition 3.3.4.** *Let $E$ be a vector bundle. The space $\mathcal{A}(E)$ of linear connections on $E$ is an affine space modeled on $\Omega^1(\mathrm{End}\,(E))$.*

***Proof.*** We first prove that $\mathcal{A}\,(E)$ is not empty. To see this, choose an open cover $\{U_\alpha\}$ of $M$ such that $E\,|_{U_\alpha}$ is trivial $\forall\alpha$. Next, pick a smooth partition of unity $(\mu_\beta)$ subordinated to this cover.

Since $E\,|_{U_\alpha}$ is trivial, it admits at least one connection, the trivial one, as in the above example. Denote such a connection by $\nabla^\alpha$. Now define

$$\nabla := \sum_{\alpha,\beta} \mu_\beta \nabla^\alpha.$$

One checks easily that $\nabla$ is a connection so that $\mathcal{A}(E)$ is nonempty. To check that $\mathcal{A}\,(E)$ is an affine space, consider two connections $\nabla^0$ and $\nabla^1$. Their difference $A = \nabla^1 - \nabla^0$ is an operator

$$A : C^\infty(E) \to C^\infty(T^*M \otimes E),$$

satisfying $A(fu) = fA(u)$, $\forall u \in C^\infty(E)$. Thus,

$$A \in C^\infty(\mathrm{Hom}\,(E, T^*M \otimes E)) \cong C^\infty(T^*M \otimes E^* \otimes E) \cong \Omega^1(E^* \otimes E) \cong \Omega^1(\mathrm{End}E).$$

Conversely, given $\nabla^0 \in \mathcal{A}(E)$ and $A \in \Omega^1(\mathrm{End}E)$ one can verify that the operator

$$\nabla^A = \nabla^0 + A : C^\infty(E) \to \Omega^1(E).$$

is a linear connection. This concludes the proof of the proposition. □

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The tensorial operations on vector bundles extend naturally to *vector bundles with connections*. The guiding principle behind this fact is the product formula. More precisely, if $E_i$ $(i = 1, 2)$ are two bundles with connections $\nabla^i$, then $E_1 \otimes E_2$ has a naturally induced connection $\nabla^{E_1 \otimes E_2}$ uniquely determined by the product rule,

$$\nabla^{E_1 \otimes E_2}(u_1 \otimes u_2) = (\nabla^1 u_1) \otimes u_2 + u_1 \otimes \nabla^2 u_2.$$

The dual bundle $E_1^*$ has a natural connection $\nabla^*$ defined by the identity

$$X\langle v, u\rangle = \langle \nabla^*_X v, u\rangle + \langle v, \nabla^1_X u\rangle, \quad \forall u \in C^\infty(E_1),\ v \in C^\infty(E_1^*),\ \ X \in \operatorname{Vect}(M),$$

where

$$\langle \bullet, \bullet\rangle : C^\infty(E_1^*) \times C^\infty(E_1) \to C^\infty(M)$$

is the pairing induced by the natural duality between the fibers of $E_1^*$ and $E_1$. In particular, any connection $\nabla^E$ on a vector bundle $E$ induces a connection $\nabla^{\operatorname{End}(E)}$ on $\operatorname{End}(E) \cong E^* \otimes E$ by

$$(\nabla^{\operatorname{End}(E)} T)(u) = \nabla^E(Tu) - T(\nabla^E u) = [\nabla^E, T]u, \tag{3.3.1}$$

$\forall T \in \operatorname{End}(E)$, $u \in C^\infty(E)$.

It is often useful to have a local description of a covariant derivative. This can be obtained using Cartan's moving frame method.

Let $E \to M$ be a $\mathbb{K}$-vector bundle of rank $r$ over the smooth manifold $M$. Pick a coordinate neighborhood $U$ such $E\,|_U$ is trivial. A *moving frame*[2] is a bundle isomorphism $\phi : \underline{\mathbb{K}}^r_U = \mathbb{K}^r \times M \to E\,|_U$.

Consider the sections $e_\alpha = \phi(\delta_\alpha)$, $\ \alpha = 1, ..., r$, where $\delta_\alpha$ are the natural basic sections of the trivial bundle $\underline{\mathbb{K}}^r_U$. As $x$ moves in $U$, the collection $(e_1(x), ..., e_r(x))$ describes a basis of the moving fiber $E_x$, whence the terminology moving frame. A section $u \in C^\infty(E\,|_U)$ can be written as a linear combination

$$u = u^\alpha e_\alpha \quad u^\alpha \in C^\infty(U, \mathbb{K}).$$

Hence, if $\nabla$ is a covariant derivative in $E$, we have

$$\nabla u = du^\alpha \otimes e_\alpha + u^\alpha \nabla e_\alpha.$$

Thus, the covariant derivative is completely described by its action on a moving frame.

To get a more concrete description, pick local coordinates $(x^i)$ over $U$. Then $\nabla e_\alpha \in \Omega^1(E\,|_U)$ so that we can write

$$\nabla e_\alpha = \Gamma^\beta_{i\alpha} dx^i \otimes e_\beta,\ \ \Gamma^\beta_{i\alpha} \in C^\infty(U, \mathbb{K}).$$

Thus, for any section $u^\alpha e_\alpha$ of $E\,|_U$ we have

$$\nabla u = du^\alpha \otimes e_\alpha + \Gamma^\beta_{i\alpha} u^\alpha dx^i \otimes e_\beta. \tag{3.3.2}$$

[2]A moving frame is what physicists call a choice of *local gauge*.

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It is convenient to view $\left(\Gamma_{i\alpha}^{\beta}\right)$ as an $r \times r$-matrix valued 1-form, and we write this as

$$\left(\Gamma_{i\alpha}^{\beta}\right) = dx^i \otimes \Gamma_i.$$

The form $\Gamma = dx^i \otimes \Gamma_i$ is called the *connection 1-form* associated to the choice of local gauge. A moving frame allows us to identify sections of $E\,|_U$ with $\mathbb{K}^r$-valued functions on $U$, and we can rewrite (3.3.2) as

$$\nabla u = du + \Gamma u. \tag{3.3.3}$$

A natural question arises: how does the connection 1-form changes with the change of the local gauge?

Let $\boldsymbol{f} = (f_\alpha)$ be another moving frame of $E\,|_U$. The correspondence $e_\alpha \mapsto f_\alpha$ defines an automorphism of $E\,|_U$. Using the local frame $\boldsymbol{e}$ we can identify this correspondence with a smooth map $g : U \to \mathrm{GL}(r;\mathbb{K})$. The map $g$ is called *the local gauge transformation* relating $\boldsymbol{e}$ to $\boldsymbol{f}$.

Let $\hat{\Gamma}$ denote the connection 1-form corresponding to the new moving frame, i.e.,

$$\nabla f_\alpha = \hat{\Gamma}_\alpha^\beta f_\beta.$$

Consider a section $\sigma$ of $E\,|_U$. With respect to the local frame $(e_\alpha)$ the section $\sigma$ has a decomposition

$$\sigma = u^\alpha e_\alpha,$$

while with respect to $(f_\beta)$ it has a decomposition

$$\sigma = \hat{u}^\beta f_\beta.$$

The two decompositions are related by

$$u = g\hat{u}. \tag{3.3.4}$$

Now, we can identify the $E$-valued 1-form $\nabla\sigma$ with a $\mathbb{K}^r$-valued 1-form in two ways: either using the frame $\boldsymbol{e}$, or using the frame $\boldsymbol{f}$. In the first case, the derivative $\nabla\sigma$ is identified with the $\mathbb{K}^r$-valued 1-form

$$du + \Gamma u,$$

while in the second case it is identified with

$$d\hat{u} + \hat{\Gamma}\hat{u}.$$

These two identifications are related by the same rule as in (3.3.4):

$$du + \Gamma u = g(d\hat{u} + \hat{\Gamma}\hat{u}).$$

Using (3.3.4) in the above equality we get

$$(dg)\hat{u} + gd\hat{u} + \Gamma g\hat{u} = gd\hat{u} + g\hat{\Gamma}\hat{u}.$$

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Hence

$$\hat{\Gamma} = g^{-1}dg + g^{-1}\Gamma g.$$

The above relation is the transition rule relating two local gauge descriptions of the same connection.

☞**A word of warning.** The identification

$$\{moving\ frames\} \cong \{local\ trivialization\}$$

should be treated carefully. These are like an object and its image in a mirror, and there is a great chance of confusing the right hand with the left hand.

More concretely, if $t_\alpha : E_\alpha \xrightarrow{\cong} \mathbb{K}^r \times U_\alpha$ (respectively $t_\beta : E_\beta \xrightarrow{\cong} \mathbb{K}^r \times U_\beta$) is a trivialization of a bundle $E$ over an open set $U_\alpha$ (respectively $U_\beta$), then the transition map "from $\alpha$ to $\beta$" over $U_\alpha \cap U_\beta$ is $g_{\beta\alpha} = t_\beta \circ t_\alpha^{-1}$. The standard basis in $\mathbb{K}^r$, denoted by $(\delta_i)$, induces two local moving frames on $E$:

$$\boldsymbol{e}_{\alpha,i} = t_\alpha^{-1}(\delta_i) \text{ and } \boldsymbol{e}_{\beta,i} = t_\beta^{-1}(\delta_i).$$

On the overlap $U_\alpha \cap U_\beta$ these two frames are related by the local gauge transformation

$$\boldsymbol{e}_{\beta,i} = g_{\beta\alpha}^{-1}\boldsymbol{e}_{\alpha,i} = g_{\alpha\beta}\boldsymbol{e}_{\alpha,i}.$$

This is precisely the opposite way the two trivializations are identified. □

The above arguments can be reversed producing the following global result.

**Proposition 3.3.5.** *Let $E \to M$ be a rank $r$ smooth vector bundle, and $(U_\alpha)$ a trivializing cover with transition maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r;\mathbb{K})$. Then any collection of matrix valued 1-forms $\Gamma_\alpha \in \Omega^1(\mathrm{End}\,\underline{\mathbb{K}}^r_{U_\alpha})$ satisfying*

$$\Gamma_\beta = (g_{\alpha\beta}^{-1}dg_{\alpha\beta}) + g_{\alpha\beta}^{-1}\Gamma_\alpha g_{\alpha\beta} = -(dg_{\beta\alpha})g_{\beta\alpha}^{-1} + g_{\beta\alpha}\Gamma_\alpha g_{\beta\alpha}^{-1} \text{ over } U_\alpha \cap U_\beta, \quad (3.3.5)$$

*uniquely defines a covariant derivative on $E$.* □

**Exercise 3.3.6.** Prove the above proposition. □

We can use the local description in Proposition 3.3.5 to define the notion of pullback of a connection. Suppose we are given the following data.

- A smooth map $f : N \to M$.
- A rank $r$ $\mathbb{K}$-vector bundle $E \to M$ defined by the open cover $(U_\alpha)$, and transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{K}^r)$.
- A connection $\nabla$ on $E$ defined by the 1-forms $\Gamma_\alpha \in \Omega^1(\mathrm{End}\,\underline{\mathbb{K}}^r_{U_\alpha})$ satisfying the gluing conditions (3.3.5).

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Then, these data define a connection $f^*\nabla$ on $f^*E$ described by the open cover $f^{-1}(U_\alpha)$, transition maps $g_{\beta\alpha}\circ f$ and 1-forms $f^*\Gamma_\alpha$. This connection is independent of the various choices and it is called the *pullback of* $\nabla$ by $f$.

**Example 3.3.7. (Complex line bundles).** Let $L \to M$ be a complex line bundle over the smooth manifold $M$. Let $\{U_\alpha\}$ be a trivializing cover with transition maps $z_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathrm{GL}(1,\mathbb{C})$. The bundle of endomorphisms of $L$, $\mathrm{End}\,(L) \cong L^*\otimes L$ is trivial since it can be defined by transition maps $(z_{\alpha\beta})^{-1}\otimes z_{\alpha\beta} = 1$. Thus, the space of connections on $L$, $\mathcal{A}\,(L)$ is an affine space modelled by the linear space of complex valued 1-forms. A connection on $L$ is simply a collection of $\mathbb{C}$-valued 1-forms $\omega^\alpha$ on $U_\alpha$ related on overlaps by

$$\omega^\beta = \frac{dz_{\alpha\beta}}{z_{\alpha\beta}} + \omega^\alpha = d\log z_{\alpha\beta} + \omega^\alpha. \qquad \square$$

### 3.3.2 *Parallel transport*

As we have already pointed out, the main reason we could not construct natural derivations on the space of sections of a vector bundle was the lack of a canonical procedure of identifying fibers at different points. We will see in this subsection that such a procedure is all we need to define covariant derivatives. More precisely, we will show that once a covariant derivative is chosen, it offers a simple way of identifying different fibers.

Let $E \to M$ be a rank $r$ $\mathbb{K}$-vector bundle and $\nabla$ a covariant derivative on $E$. For any smooth path $\gamma : [0,1] \to M$ we will define a *linear* isomorphism $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ called the *parallel transport* along $\gamma$. More exactly, we will construct an entire family of linear isomorphisms

$$T_t : E_{\gamma(0)} \to E_{\gamma(t)}.$$

One should think of this $T_t$ as identifying different fibers. In particular, if $u_0 \in E_{\gamma(0)}$ then the path $t \mapsto u_t = T_t u_0 \in E_{\gamma(t)}$ should be thought of as a “constant” path. The rigorous way of stating this “constancy” is via derivations: a quantity is “constant” if its derivatives are identically 0. Now, the only way we know how to derivate sections is via $\nabla$, i.e., $u_t$ should satisfy

$$\nabla_{\frac{d}{dt}} u_t = 0, \quad \text{where } \frac{d}{dt} = \dot\gamma.$$

The above equation suggests a way of defining $T_t$. For any $u_0 \in E_{\gamma(0)}$, and any $t \in [0,1]$, define $T_t u_0$ as the value at $t$ of the solution of the initial value problem

$$\begin{cases} \nabla_{\frac{d}{dt}} u(t) = 0 \\ u(0) \qquad = u_0 \end{cases}. \tag{3.3.6}$$

The equation (3.3.6) is a system of linear ordinary differential equations in disguise.

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To see this, let us make the simplifying assumption that $\gamma(t)$ lies entirely in some coordinate neighborhood $U$ with coordinates $(x^1, ..., x^n)$, such that $E\,|_U$ is trivial. This is always happening, at least on every small portion of $\gamma$. Denote by $(e_\alpha)_{1\leq\alpha\leq r}$ a local moving frame trivializing $E|_U$ so that $u = u^\alpha e_\alpha$. The connection 1-form corresponding to this moving frame will be denoted by $\Gamma \in \Omega^1(\mathrm{End}\,(\underline{\mathbb{K}}^r))$. Equation (3.3.6) becomes (using Einstein's convention)

$$\begin{cases} \frac{du^\alpha}{dt} + \Gamma^\alpha_{t\beta} u^\beta = 0 \\ u^\alpha(0) = u^\alpha_0 \end{cases}, \tag{3.3.7}$$

where

$$\Gamma_t = \frac{d}{dt} \lrcorner\, \Gamma = \dot\gamma \lrcorner\, \Gamma \in \Omega^0(\mathrm{End}\,(\underline{\mathbb{K}}^r)) = \mathrm{End}\,(\underline{\mathbb{K}}^r).$$

More explicitly, if the path $\gamma(t)$ is given by the smooth map

$$t \mapsto \gamma(t) = \big(x^1(t), \dots, x^n(t)\big),$$

then $\Gamma_t$ is the endomorphism given by

$$\Gamma_t e_\beta = \dot x^i \Gamma^\alpha_{i\beta} e_\alpha.$$

The system (3.3.7) can be rewritten as

$$\begin{cases} \frac{du^\alpha}{dt} + \Gamma^\alpha_{i\beta} \dot x^i u^\beta = 0 \\ \\ u^\alpha(0) = u^\alpha_0. \end{cases} \tag{3.3.8}$$

This is obviously a system of linear ordinary differential equations whose solutions exist for any $t$. We deduce

$$\dot u(0) = -\Gamma_t u_0. \tag{3.3.9}$$

This gives a geometric interpretation for the connection 1-form $\Gamma$: for any vector field $X$, the contraction $-i_X\Gamma = -\Gamma(X) \in \mathrm{End}\,(E)$ describes the infinitesimal parallel transport along the direction prescribed by the vector field $X$, in the non-canonical identification of nearby fibers via a local moving frame.

In more intuitive terms, if $\gamma(t)$ is an integral curve for $X$, and $T_t$ denotes the parallel transport along $\gamma$ from $E_{\gamma(0)}$ to $E_{\gamma(t)}$, then, given a local moving frame for $E$ in a neighborhood of $\gamma(0)$, $T_t$ is identified with a $t$-dependent matrix which has a Taylor expansion of the form

$$T_t = \mathbb{1} - \Gamma_0 t + O(t^2), \quad t \text{ very small}, \tag{3.3.10}$$

with $\Gamma_0 = (i_X\Gamma)\,|_{\gamma(0)}$.

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### 3.3.3 *The curvature of a connection*

Consider a rank $r$ smooth $\mathbb{K}$-vector bundle $E \to M$ over the smooth manifold $M$, and let $\nabla : \Omega^0(E) \to \Omega^1(E)$ be a covariant derivative on $E$.

**Proposition 3.3.8.** *The connection $\nabla$ has a natural extension to an operator*

$$d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E)$$

*uniquely defined by the requirements,*

*(a)* $d^\nabla \mid_{\Omega^0(E)} = \nabla$,
*(b)* $\forall \omega \in \Omega^r(M)$, $\eta \in \Omega^s(E)$

$$d^\nabla(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d^\nabla \eta.$$

**Outline of the proof** *Existence.* For $\omega \in \Omega^r(M)$, $u \in \Omega^0(E)$ set

$$d^\nabla(\omega \otimes u) = d\omega \otimes u + (-1)^r \omega \nabla u. \tag{3.3.11}$$

Using a partition of unity one shows that any $\eta \in \Omega^r(E)$ is a locally finite combination of monomials as above so the above definition induces an operator $\Omega^r(E) \to \Omega^{r+1}(E)$. We let the reader check that this extension satisfies conditions (a) and (b) above.
*Uniqueness.* Any operator with the properties (a) and (b) acts on monomials as in (3.3.11) so it has to coincide with the operator described above using a given partition of unity. □

**Example 3.3.9.** The trivial bundle $\underline{\mathbb{K}}_M$ has a natural connection $\nabla^0$- the trivial connection. This coincides with the usual differential $d : \Omega^0(M) \otimes \mathbb{K} \to \Omega^1(M) \otimes \mathbb{K}$. The extension $d^{\nabla^0}$ is the usual exterior derivative. □

There is a major difference between the usual exterior derivative $d$, and an arbitrary $d^\nabla$. In the former case we have $d^2 = 0$, which is a consequence of the commutativity $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, where $(x^i)$ are local coordinates on $M$. In the second case, the equality $(d^\nabla)^2 = 0$ does not hold in general. Still, something very interesting happens.

**Lemma 3.3.10.** *For any smooth function $f \in C^\infty(M)$, and any $\omega \in \Omega^r(E)$ we have*

$$(d^\nabla)^2(f\omega) = f\{(d^\nabla)^2\omega\}.$$

*Hence $(d^\nabla)^2$ is a bundle morphism $\Lambda^r T^*M \otimes E \to \Lambda^{r+2} T^*M \otimes E$.*

***Proof.*** We compute

$$(d^\nabla)^2(f\omega) = d^\nabla(df \wedge \omega + f d^\nabla \omega)$$

$$= -df \wedge d^\nabla\omega + df \wedge d^\nabla \omega + f(d^\nabla)^2\omega = f(d^\nabla)^2\omega.$$ □

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As a map $\Omega^0(E) \to \Omega^2(E)$, the operator $(d^\nabla)^2$ can be identified with a section of

$$\mathrm{Hom}_{\mathbb{K}}(E, \Lambda^2 T^*M \otimes_{\mathbb{R}} E) \cong E^* \otimes \Lambda^2 T^*M \otimes_{\mathbb{R}} E \cong \Lambda^2 T^*M \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{K}}(E).$$

Thus, $(d^\nabla)^2$ is an $\mathrm{End}_{\mathbb{K}}(E)$-valued 2-form.

**Definition 3.3.11.** For any connection $\nabla$ on a smooth vector bundle $E \to M$, the object $(d^\nabla)^2 \in \Omega^2(\mathrm{End}_{\mathbb{K}}(E))$ is called the *curvature* of $\nabla$, and it is usually denoted by $F(\nabla)$. □

**Example 3.3.12.** Consider the trivial bundle $\underline{\mathbb{K}}^r_M$. The sections of this bundle are smooth $\mathbb{K}^r$-valued functions on $M$. The exterior derivative $d$ defines the trivial connection on $\underline{\mathbb{K}}^r_M$, and any other connection differs from $d$ by a $M_r(\mathbb{K})$-valued 1-form on $M$. If $A$ is such a form, then the curvature of the connection $d + A$ is the 2-form $F(A)$ defined by

$$F(A)u = (d + A)^2 u = (dA + A \wedge A)u, \quad \forall u \in C^\infty(M, \mathbb{K}^r).$$

The $\wedge$-operation above is defined for any vector bundle E as the bilinear map

$$\Omega^r(\mathrm{End}\,(E)) \times \Omega^s(\mathrm{End}\,(E)) \to \Omega^{r+s}(\mathrm{End}\,(E)),$$

uniquely determined by

$$(\omega^r \otimes A) \wedge (\eta^s \otimes B) = \omega^r \wedge \eta^s \otimes AB, \quad A, B \in \mathrm{End}\,(E).$$ □

We conclude this subsection with an alternate description of the curvature which hopefully will shed some light on its analytical significance.

Let $E \to M$ be a smooth vector bundle on $M$ and $\nabla$ a connection on it. Denote its curvature by $F = F(\nabla) \in \Omega^2(\mathrm{End}\,(E))$. For any $X, Y \in \mathrm{Vect}\,(M)$ the quantity $F(X, Y)$ is an endomorphism of $E$. In the remaining part of this section we will give a different description of this endomorphism.

For any vector field $Z$, we denote by $i_Z : \Omega^r(E) \to \Omega^{r-1}(E)$ the $C^\infty(M)$-*linear* operator defined by

$$i_Z(\omega \otimes u) = (i_Z\omega) \otimes u, \quad \forall \omega \in \Omega^r(M), \ u \in \Omega^0(E).$$

The covariant derivative $\nabla_Z$ extends naturally to elements of $\Omega^r(E)$ by

$$\nabla_Z(\omega \otimes u) := (L_Z\omega) \otimes u + \omega \otimes \nabla_Z u.$$

The operators $d^\nabla$, $i_Z$, $\nabla_Z$ satisfy the usual super-commutation identities.

$$i_Z d^\nabla + d^\nabla i_Z = \nabla_Z. \tag{3.3.12}$$

$$i_X i_Y + i_Y i_X = 0. \tag{3.3.13}$$

$$\nabla_X i_Y - i_Y \nabla_X = i_{[X,Y]}. \tag{3.3.14}$$

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For any $u \in \Omega^0(E)$ we compute using (3.3.12)-(3.3.14)

$$F(X,Y)u = i_Y i_X (d^\nabla)^2 u = i_Y (i_X d^\nabla)\nabla u$$

$$= i_Y(\nabla_X - d^\nabla i_X)\nabla u = (i_Y\nabla_X)\nabla u - (i_Y d^\nabla)\nabla_X u$$

$$= (\nabla_X i_Y - i_{[X,Y]})\nabla u - \nabla_Y\nabla_X u = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})u.$$

Hence

$$F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{3.3.15}$$

If in the above formula we take $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$, where $(x^i)$ are local coordinates on $M$, and we set $\nabla_i := \nabla_{\frac{\partial}{\partial x_i}}$, $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}}$, then we deduce

$$F_{ij} = -F_{ji} := F(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = [\nabla_i, \nabla_j]. \tag{3.3.16}$$

Thus, the endomorphism $F_{ij}$ measures the extent to which the partial derivatives $\nabla_i$, $\nabla_j$ fail to commute. This is in sharp contrast with the classical calculus and an analytically oriented reader may object to this by saying we were careless when we picked the connection. Maybe an intelligent choice will restore the classical commutativity of partial derivatives so we should concentrate from the very beginning to covariant derivatives $\nabla$ such that $F(\nabla) = 0$.

**Definition 3.3.13.** A connection $\nabla$ such that $F(\nabla) = 0$ is called *flat*. □

A natural question arises: given an arbitrary vector bundle $E \to M$ do there exist flat connections on $E$? If $E$ is trivial then the answer is obviously positive. In general, the answer is negative, and this has to do with the *global structure* of the bundle. In the second half of this book we will discuss in more detail this fact.

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### 3.3.4 *Holonomy*

The reader may ask a very legitimate question: why have we chosen to name curvature, the deviation from commutativity of a given connection. In this subsection we describe the geometric meaning of curvature, and maybe this will explain this choice of terminology. Throughout this subsection we will use Einstein's convention.

Let $E \to M$ be a rank $r$ smooth $\mathbb{K}$-vector bundle, and $\nabla$ a connection on it. Consider local coordinates $(x^1, ..., x^n)$ on an open subset $U \subset M$ such that $E\mid_U$ is trivial. Pick a moving frame $(e_1, ..., e_r)$ of $E$ over $U$. The connection 1-form associated to this moving frame is

$$\Gamma = \Gamma_i dx^i = (\Gamma^\alpha_{i\beta})dx^i, \;\; 1 \le \alpha, \beta \le r.$$

It is defined by the equalities ($\nabla_i := \nabla_{\frac{\partial}{\partial x_i}}$)

$$\nabla_i e_\beta = \Gamma^\alpha_{i\beta} e_\alpha. \tag{3.3.17}$$

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Using (3.3.16) we compute

$$F_{ij}e_\beta = (\nabla_i\nabla_j - \nabla_j\nabla_i)e_\beta = \nabla_i(\Gamma_j e_\beta) - \nabla_j(\Gamma_i e_\beta)$$

$$= \left( \frac{\partial \Gamma^\alpha_{j\beta}}{\partial x^i} - \frac{\partial \Gamma^\alpha_{j\beta}}{\partial x^j} \right) e_\alpha + \left( \Gamma^\gamma_{j\beta}\Gamma^\alpha_{i\gamma} - \Gamma^\gamma_{i\beta}\Gamma^\alpha_{j\gamma} \right) e_\alpha,$$

so that

$$F_{ij} = \left( \frac{\partial \Gamma_j}{\partial x^i} - \frac{\partial \Gamma_i}{\partial x^j} + \Gamma_i\Gamma_j - \Gamma_j\Gamma_i \right). \tag{3.3.18}$$

Though the above equation looks very complicated it will be the clue to understanding the geometric significance of curvature.

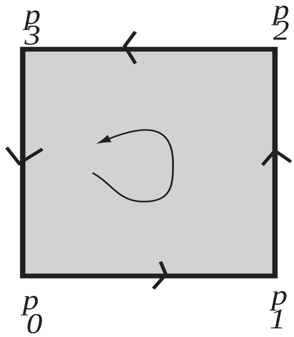


Fig. 3.1 *Parallel transport along a coordinate parallelogram.*

Assume for simplicity that the point of coordinates $(0, ..., 0)$ lies in $U$. Denote by $T_1^s$ the parallel transport (using the connection $\nabla$) from $(x^1, ..., x^n)$ to $(x^1 + s, x^2, ..., x^n)$ along the curve $\tau \mapsto (x^1 + \tau, x^2, ..., x^n)$. Define $T_2^t$ in a similar way using the coordinate $x^2$ instead of $x^1$.

Look at the parallelogram $P_{s,t}$ in the "plane" $(x^1, x^2)$ described in Figure 3.1, where

$$p_0 = (0, \dots, 0),\ \ p_1 = (s, 0, \dots, 0),\ \ p_2 = (s, t, 0, \dots, 0),\ \ p_3 = (0, t, 0, \dots, 0).$$

We now perform the counterclockwise parallel transport along the boundary of $P_{s,t}$. The outcome is a linear map $\mathfrak{T}_{s,t} : E_0 \to E_0$, where $E_0$ is the fiber of $E$ over $p_0$. Set $F_{12} := F(\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2})\,|_{(0,\dots,0)}$. $F_{12}$ is an endomorphism of $E_0$.

**Proposition 3.3.14.** *For any $u \in E_0$ we have*

$$F_{12}u = -\frac{\partial^2}{\partial s \partial t}\mathfrak{T}_{s,t}u.$$

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We see that the parallel transport of an element $u \in E_0$ along a closed path may not return it to itself. The curvature is an infinitesimal measure of this deviation.

***Proof.*** The parallel transport along $\partial P_{s,t}$ can be described as

$$\mathfrak{T}_{s,t} = T_2^{-t} T_1^{-s} T_2^t T_1^s.$$

The parallel transport $T_1^s : E_0 \to E_{p_1}$ can be approximated using (3.3.9)

$$u_1 = u_1(s,t) = T_1^s u_0 = u_0 - s\Gamma_1(p_0)u_0 + C_1 s^2 + O(s^3). \tag{3.3.19}$$

$C_1$ is a constant vector in $E_0$ whose exact form is not relevant to our computations. In the sequel the letter $C$ (eventually indexed) will denote constants.

$$u_2 = u_2(s,t) = T_2^t T_1^s u = T_2^t u_1 = u_1 - t\Gamma_2(p_1)u_1 + C_2 t^2 + O(t^3)$$

$$= u_0 - s\Gamma_1(p_0)u_0 - t\Gamma_2(p_1)(u_0 - s\Gamma_1(p_0)u_0) + C_1 s^2 + C_2 t^2 + O(3)$$

$$= \big\{ \mathbb{1} - s\Gamma_1(p_0) - t\Gamma_2(p_1) + ts\Gamma_2(p_1)\Gamma_1(p_0) \big\} u_0 + C_1 s^2 + C_2 t^2 + O(3).$$

$O(k)$ denotes an error $\leq C(s^2+t^2)^{k/2}$ as $s,t \to 0$. Now use the approximation

$$\Gamma_2(p_1) = \Gamma_2(p_0) + s\frac{\partial \Gamma_2}{\partial x^1}(p_0) + O(2),$$

to deduce

$$u_2 = \Big\{ \mathbb{1} - s\Gamma_1 - t\Gamma_2 - st\big( \frac{\partial\Gamma_2}{\partial x^1} - \Gamma_2\Gamma_1 \big) \Big\}|_{p_0}\, u_0$$

$$+ C_1 s^2 + C_2 t^2 + O(3). \tag{3.3.20}$$

Similarly, we have

$$u_3 = u_3(s,t) = T_1^{-s} T_2^t T_1^s u_0 = T_1^{-s} u_2 = u_2 + s\Gamma_1(p_2)u_2 + C_3 s^2 + O(3).$$

The $\Gamma$-term in the right-hand side of the above equality can be approximated as

$$\Gamma_1(p_2) = \Gamma_1(p_0) + s\frac{\partial\Gamma_1}{\partial x^1}(p_0) + t\frac{\partial\Gamma_1}{\partial x^2}(p_0) + O(2).$$

Using $u_2$ described as in (3.3.20) we get after an elementary computation

$$u_3 = u_3(s,t) = \Big\{ \mathbb{1} - t\Gamma_2 + st\Big( \frac{\partial\Gamma_1}{\partial x^2} - \frac{\partial\Gamma_2}{\partial x^1} + \Gamma_2\Gamma_1 - \Gamma_1\Gamma_2 \Big) \Big\}|_{p_0}\, u_0$$

$$+ C_4 s^2 + C_5 t^2 + O(3). \tag{3.3.21}$$

Finally, we have

$$u_4 = u_4(s,t) = T_2^{-t} = u_3 + t\Gamma_2(p_3)u_3 + C_6 t^2 + O(3),$$

with

$$\Gamma_2(p_3) = \Gamma_2(p_0) + t\frac{\partial\Gamma_2}{\partial x^2}(p_0) + C_7 t^2 + O(3).$$

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Using (3.3.21) we get

$$u_4(s,t) = u_0 + st\left(\frac{\partial\Gamma_1}{\partial x^2} - \frac{\partial\Gamma_2}{\partial x^1} + \Gamma_2\Gamma_1 - \Gamma_1\Gamma_2\right)|_{p_0}\, u_0$$

$$+C_8s^2 + C_9t^2 + O(3)$$

$$= u_0 - stF_{12}(p_0)u_0 + C_8s^2 + C_9t^2 + O(3).$$

Clearly $\frac{\partial^2 u_4}{\partial s \partial t} = -F_{12}(p_0)u_0$ as claimed. □

**Remark 3.3.15.** If we had kept track of the various constants in the above computation we would have arrived at the conclusion that $C_8 = C_9 = 0$ i.e.

$$\mathfrak{T}_{s,t} = \mathbb{1} - stF_{12} + O(3).$$

Alternatively, the constant $C_8$ is the second order correction in the Taylor expansion of $s \mapsto \mathfrak{T}_{s,0} \equiv$, so it has to be 0. The same goes for $C_9$. Thus we have

$$-F_{12} = \frac{d\mathfrak{T}_{s,t}}{d\mathrm{area} P_{s,t}} = \frac{d\mathfrak{T}_{\sqrt{s},\sqrt{s}}}{ds}.$$

Loosely speaking, the last equality states that the curvature is the the "amount of holonomy per unit of area". □

The result in the above proposition is usually formulated in terms of holonomy.

**Definition 3.3.16.** Let $E \to M$ be a vector bundle with a connection $\nabla$. The *holonomy* of $\nabla$ along a closed path $\gamma$ is the parallel transport along $\gamma$. □

We see that the curvature measures the holonomy along infinitesimal parallelograms. A connection can be viewed as an analytic way of trivializing a bundle. We can do so along paths starting at a fixed point, using the parallel transport, but using different paths ending at the same point we may wind up with trivializations which differ by a twist. The curvature provides an infinitesimal measure of that twist.

**Exercise 3.3.17.** Prove that any vector bundle $E$ over the Euclidean space $\mathbb{R}^n$ is trivializable.

**Hint:** Use the parallel transport defined by a connection on the vector bundle $E$ to produce a bundle isomorphism $E \to E_0 \times \mathbb{R}^n$, where $E_0$ is the fiber of $E$ over the origin. □

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### 3.3.5 *The Bianchi identities*

Consider a smooth $\mathbb{K}$-vector bundle $E \to M$ equipped with a connection $\nabla$. We have seen that the associated exterior derivative $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ does not satisfy the usual $(d^\nabla)^2 = 0$, and the curvature is to blame for this. The Bianchi identity describes one remarkable algebraic feature of the curvature.

Recall that $\nabla$ induces a connection in any tensor bundle constructed from $E$. In particular, it induces a connection in $E^* \otimes E \cong \mathrm{End}\,(E)$ which we continue to denote by $\nabla$. This extends to an "exterior derivative" $D : \Omega^p(\mathrm{End}\,(E)) \to \Omega^{p+1}(\mathrm{End}\,(E))$.

**Proposition 3.3.18. (The Bianchi identity)** *Let $E \to M$ and $\nabla$ as above. Then*

$$DF(\nabla) = 0.$$

Roughly speaking, the Bianchi identity states that $(d^\nabla)^3$ is 0.

***Proof.*** We will use the identities (3.3.12)–(3.3.14). For any vector fields $X$, $Y$, $Z$ we have

$$i_X D = \nabla_X - D i_X.$$

Hence,

$$(DF)(X,Y,Z) = i_Z i_Y i_X DF = i_Z i_Y (\nabla_X - D i_X) F$$

$$= i_Z (\nabla_X i_Y - i_{[X,Y]}) F - i_Z (\nabla_Y - D i_Y) i_X F$$

$$= (\nabla_X i_Z i_Y - i_{[X,Z]} i_Y - i_Z i_{[X,Y]}) F - (\nabla_Y i_Z i_X - i_{[Y,Z]} i_X - \nabla_Z i_Y i_X) F$$

$$= (i_{[X,Y]} i_Z + i_{[Y,Z]} i_X + i_{[Z,X]} i_Y) F - (\nabla_X i_Y i_Z + \nabla_Y i_Z i_X + \nabla_Z i_X i_Y) F.$$

We compute immediately

$$i_{[X,Y]} i_Z F = F(Z,[X,Y]) = \left[\nabla_Z, \nabla_{[X,Y]}\right] - \nabla_{[Z,[X,Y]]}.$$

Also for any $u \in \Omega^0(E)$ we have

$$(\nabla_X i_Y i_Z F) u = \nabla_X (F(Z,Y)u) - F(Z,Y)\nabla_X u = [\nabla_X, F(Z,Y)]\, u$$

$$= \left[\nabla_X, \nabla_{[Y,Z]}\right] u - [\nabla_X, [\nabla_Y, \nabla_Z]]\, u.$$

The Bianchi identity now follows from the classical Jacobi identity for commutators. □

**Example 3.3.19.** Let $\underline{\mathbb{K}}$ be the trivial line bundle over a smooth manifold $M$. Any connection on $\underline{\mathbb{K}}$ has the form $\nabla^\omega = d + \omega$, where $d$ is the trivial connection, and $\omega$ is a $\mathbb{K}$-valued 1-form on $M$. The curvature of this connection is

$$F(\omega) = d\omega.$$

The Bianchi identity is in this case precisely the equality $d^2\omega = 0$. □

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### 3.3.6 *Connections on tangent bundles*

The tangent bundles are very special cases of vector bundles so the general theory of connections and parallel transport is applicable in this situation as well. However, the tangent bundles have some peculiar features which enrich the structure of a connection.

Recall that, when looking for a local description for a connection on a vector bundle, we have to first choose local coordinates on the manifolds, and then a local moving frame for the vector bundle. For an arbitrary vector bundle there is no correlation between these two choices.

For tangent bundles it happens that, once local coordinates $(x^i)$ are chosen, they automatically define a moving frame of the tangent bundle, $\left(\partial_i = \frac{\partial}{\partial x_i}\right)$, and it is thus very natural to work with this frame. Hence, let $\nabla$ be a connection on $TM$. With the above notations we set

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k \quad (\nabla_i = \nabla_{\partial_i}).$$

The coefficients $\Gamma_{ij}^k$ are usually known as the *Christoffel symbols* of the connection. As usual we construct the curvature tensor

$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \operatorname{End}(TM).$$

Still, this is not the only tensor naturally associated to $\nabla$.

**Lemma 3.3.20.** *For $X, Y \in \operatorname{Vect}(M)$ consider*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \in \operatorname{Vect}(M).$$

*Then $\forall f \in C^\infty(M)$*

$$T(fX, Y) = T(X, fY) = fT(X, Y),$$

*so that $T(\bullet, \bullet)$ is a tensor $T \in \Omega^2(TM)$, i.e., a 2-form whose coefficients are vector fields on $M$. The tensor $T$ is called the* torsion *of the connection $\nabla$.* □

The proof of this lemma is left to the reader as an exercise. In terms of Christoffel symbols, the torsion has the description

$$T(\partial_i, \partial_j) = (\Gamma_{ij}^k - \Gamma_{ji}^k)\partial_k.$$

**Definition 3.3.21.** A connection on $TM$ is said to be *symmetric* if $T = 0$. □

We guess by now the reader is wondering how the mathematicians came up with this object called torsion. In the remaining of this subsection we will try to sketch the geometrical meaning of torsion.

To find such an interpretation, we have to look at the finer structure of the tangent space at a point $x \in M$. It will be convenient to regard $T_xM$ as an affine space modeled by $\mathbb{R}^n$, $n = \dim M$. Thus, we will no longer think of the elements

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of $T_xM$ as vectors, but instead we will treat them as points. The tangent space $T_xM$ can be coordinatized using *affine frames*. These are pairs $(p; \boldsymbol{e})$, where $p$ is a point in $T_xM$, and $\boldsymbol{e}$ is a basis of the underlying vector space. A frame allows one to identify $T_xM$ with $\mathbb{R}^n$, where $p$ is thought of as the origin.

Suppose that $A$, $B$ are two affine spaces, both modelled by $\mathbb{R}^n$, and $(p; \boldsymbol{e})$ , $(p; \boldsymbol{f})$ are affine frames of $A$ and respectively $B$. Denote by $(x^i)$ the coordinates in $A$ induced by the frame $(p; \boldsymbol{e})$, and by $(y^j)$ the coordinates in $B$ induced by the frame $(q; \boldsymbol{f})$. An affine isomorphism $T : A \to B$ can then be described using these coordinates as

$$T : \mathbb{R}^n_x \to \mathbb{R}^n_y \quad x \mapsto y = Sx + v,$$

where $v$ is a vector in $\mathbb{R}^n$, and $S$ is an invertible $n \times n$ real matrix. Thus, an affine map is described by a "rotation" $S$, followed by a translation $v$. This vector measures the "drift" of the origin. We write $T = S \hat{+} x$

If now $(x^i)$ are local coordinates on $M$, then they define an affine frame $\mathcal{A}_x$ at each $x \in M$: $(\mathcal{A}_x = (0; (\partial_i))$. Given a connection $\nabla$ on $TM$, and a smooth path $\gamma : I \to M$, we will construct a family of affine isomorphisms $T_t : T_{\gamma(0)} \to T_{\gamma(t)}$ called the *affine transport of* $\nabla$ *along* $\gamma$. In fact, we will determine $T_t$ by imposing the initial condition $T_0 = \mathbb{1}$, and then describing $\dot{T}_t$.

This is equivalent to describing the infinitesimal affine transport at a given point $x_0 \in M$ along a direction given by a vector $X = X^i \partial_i \in T_{x_0}M$. The affine frame of $T_{x_0}M$ is $\mathcal{A}_{x_0} = (0; (\partial_i))$.

If $x_t$ is a point along the integral curve of $X$, close to $x_0$ then its coordinates satisfy

$$x^i_t = x^i_0 + tX^i + O(t^2).$$

This shows the origin $x_0$ of $\mathcal{A}_{x_0}$ "drifts" by $tX + O(t^2)$. The frame $(\partial_i)$ suffers a parallel transport measured as usual by $\mathbb{1} - ti_X\Gamma + O(t^2)$. The total affine transport will be

$$T_t = (\mathbb{1} - ti_X\Gamma) \hat{+} tX + O(t^2).$$

The holonomy of $\nabla$ along a closed path will be an affine transformation and as such it has two components: a "rotation"" and a translation. As in Proposition 3.3.14 one can show the torsion measures the translation component of the holonomy along an infinitesimal parallelogram, i.e., the "amount of drift per unit of area". Since we will not need this fact we will not include a proof of it.

**Exercise 3.3.22.** Consider the vector valued 1-form $\omega \in \Omega^1(TM)$ defined by

$$\omega(X) = X \quad \forall X \in \mathrm{Vect}\,(M).$$

Show that if $\nabla$ is a linear connection on $TM$, then $d^\nabla \omega = T^\nabla$, where $T^\nabla$ denotes the torsion of $\nabla$. □

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**Exercise 3.3.23.** Consider a smooth vector bundle $E \to M$ over the smooth manifold $M$. We assume that both $E$ and $TM$ are equipped with connections and moreover the connection on $TM$ is torsionless. Denote by $\hat{\nabla}$ the induced connection on $\Lambda^2 T^*M \otimes \operatorname{End}(E)$. Prove that $\forall X, Y, Z \in \operatorname{Vect}(M)$

$$\hat{\nabla}_X F(Y,Z) + \hat{\nabla}_Y F(Z,X) + \hat{\nabla}_Z F(X,Y) = 0. \qquad \square$$

## 3.4 Integration on manifolds

### 3.4.1 *Integration of 1-densities*

We spent a lot of time learning to differentiate geometrical objects but, just as in classical calculus, the story is only half complete without the reverse operation, integration.

Classically, integration requires a background measure, and in this subsection we will describe the differential geometric analogue of a measure, namely the notion of 1-density on a manifold.

Let $E \to M$ be a rank $k$, smooth real vector bundle over a manifold $M$ defined by an open cover $(U_\alpha)$ and transition maps $g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})$ satisfying the cocycle condition. For any $r \in \mathbb{R}$ we can form the real line bundle $|\Lambda|^r(E)$ defined by the same open cover and transition maps

$$t_{\beta\alpha} := |\det g_{\beta\alpha}|^{-r} = |\det g_{\alpha\beta}|^r : U_{\alpha\beta} \to \mathbb{R}_{>0} \hookrightarrow \mathrm{GL}(1, \mathbb{R}).$$

The fiber at $p \in M$ of this bundle consists of $r$-densities on $E_p$ (see Subsection 2.2.4).

**Definition 3.4.1.** Let $M$ be a smooth manifold and $r \geq 0$. The bundle of $r$-densities on $M$ is

$$|\Lambda|^r_M := |\Lambda|^r(TM).$$

When $r = 1$ we will use the notation $|\Lambda|_M = |\Lambda|^1_M$. We call $|\Lambda|_M$ the *density bundle* of $M$. $\square$

Denote by $C^\infty(|\Lambda|_M)$ the space of smooth sections of $|\Lambda|_M$, and by $C_0^\infty(|\Lambda|_M)$ its subspace consisting of compactly supported densities.

It helps to have local descriptions of densities. To this aim, pick an open cover of $M$ consisting of coordinate neighborhoods $(U_\alpha)$. Denote the local coordinates on $U_\alpha$ by $(x^i_\alpha)$. This choice of a cover produces a trivializing cover of $TM$ with transition maps

$$T_{\alpha\beta} = \left( \frac{\partial x^i_\alpha}{\partial x^j_\beta} \right)_{1 \leq i,j \leq n},$$

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where $n$ is the dimension of $M$. Set $\delta_{\alpha\beta} = |\det T_{\alpha\beta}|$. A 1-density on $M$ is then a collection of functions $\mu_\alpha \in C^\infty(U_\alpha)$ related by

$$\mu_\alpha = \delta_{\alpha\beta}^{-1}\mu_\beta.$$

It may help to think that for each point $p \in U_\alpha$ the basis $\frac{\partial}{\partial x_\alpha^1}, ..., \frac{\partial}{\partial x_\alpha^n}$ of $T_pM$ spans an infinitesimal parallelepiped and $\mu_\alpha(p)$ is its "volume". A change in coordinates should be thought of as a change in the measuring units. The gluing rules describe how the numerical value of the volume changes from one choice of units to another.

The densities on a manifold resemble in many respects the differential forms of maximal degree. Denote by $\det TM = \Lambda^{\dim M} TM$ the determinant line bundle of $TM$. A density is a map

$$\mu : C^\infty(\det TM) \to C^\infty(M),$$

such that $\mu(f\boldsymbol{e}) = |f|\mu(\boldsymbol{e})$, for all smooth functions $f : M \to \mathbb{R}$, and all $\boldsymbol{e} \in C^\infty(\det TM)$. In particular, any smooth map $\phi : M \to N$ between manifolds of the same dimension induces a pullback transformation

$$\phi^* : C^\infty(|\Lambda|_N) \to C^\infty(|\Lambda|_M),$$

described by

$$(\phi^*\mu)(\boldsymbol{e}) = \mu\big(\,(\det \phi_*)\cdot \boldsymbol{e}\,\big) = |\det \phi_*|\,\mu(\boldsymbol{e}),\ \ \forall \boldsymbol{e} \in C^\infty(\det TM).$$

**Example 3.4.2.** (a) Consider the special case $M = \mathbb{R}^n$. Denote by $e_1, ..., e_n$ the canonical basis. This extends to a trivialization of $T\mathbb{R}^n$ and, in particular, the bundle of densities comes with a natural trivialization. It has a nowhere vanishing section $|dv_n|$ defined by

$$|dv_n|(e_1 \wedge ... \wedge e_n) = 1.$$

In this case, any smooth density on $\mathbb{R}^n$ takes the form $\mu = f|dv_n|$, where $f$ is some smooth function on $\mathbb{R}^n$. The reader should think of $|dv_n|$ as the standard Lebesgue measure on $\mathbb{R}^n$.

If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, viewed as a collection of $n$ smooth functions

$$\phi_1 = \phi_1(x^1, ..., x^n), \ldots, \phi_n = \phi_n(x^1, ..., x^n),$$

then,

$$\phi^*(|dv_n|) = \left|\det\left(\frac{\partial \phi_i}{\partial x^j}\right)\right| \cdot |dv_n|.$$

(b) Suppose $M$ is a smooth manifold of dimension $m$. Then any top degree form $\omega \in \Omega^m(M)$ defines a density $|\omega|$ on $M$ which associates to each section $\boldsymbol{e} \in C^\infty(\det TM)$ the smooth function

$$x \mapsto |\omega_x(\,\boldsymbol{e}(x)\,)|$$

Observe that $|\omega| = |-\omega|$, so this map $\Omega^m(M) \to C^\infty(|\Lambda|_M)$ *is not linear.*

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(c) Suppose $g$ is a Riemann metric on the smooth manifold $M$. The *volume density* defined by $g$ is the density denoted by $|dV_g|$ which associates to each $\boldsymbol{e} \in C^\infty(\det TM)$ the pointwise length

$$x \mapsto |\boldsymbol{e}(x)|_g.$$

If $(U_\alpha, (x^i_\alpha))$ is an atlas of $M$, then on each $U_\alpha$ we have top degree forms

$$dx_\alpha := dx^1_\alpha \wedge \cdots \wedge dx^m_\alpha,$$

to which we associate the density $|dx_\alpha|$. In the coordinates $(x^i_\alpha)$ the metric $g$ can be described as

$$g = \sum_{i,j} g_{\alpha;ij} dx^i_\alpha \otimes dx^j_\alpha.$$

We denote by $|g_\alpha|$ the determinant of the symmetric matrix $g_\alpha = (g_{\alpha;ij})_{1 \leq i,j \leq m}$. Then the restriction of $|dV_g|$ to $U_\alpha$ has the description

$$|dV_g| = \sqrt{|g_\alpha|}\,|dx_\alpha|. \qquad \square$$

The importance of densities comes from the fact that they are exactly the objects that can be integrated. More precisely, we have the following abstract result.

**Proposition 3.4.3.** *There exists a natural way to associate to each smooth manifold $M$ a linear map*

$$\int_M : C^\infty_0(|\Lambda|_M) \to \mathbb{R}$$

*uniquely defined by the following conditions.*
*(a) $\int_M$ is invariant under diffeomorphisms, i.e., for any smooth manifolds $M$, $N$ of the same dimension $n$, any diffeomorphism $\phi : M \to N$, and any $\mu \in C^\infty_0(|\Lambda|_M)$, we have*

$$\int_M \phi^*\mu = \int_N \mu.$$

*(b) $\int_M$ is a local operation, i.e., for any open set $U \subset M$, and any $\mu \in C^\infty_0(|\Lambda|_M)$ with* $\operatorname{supp}\mu \subset U$*, we have*

$$\int_M \mu = \int_U \mu.$$

*(c) For any $\rho \in C^\infty_0(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \rho|dv_n| = \int_{\mathbb{R}^n} \rho(x)dx,$$

*where in the right-hand side stands the Lebesgue integral of the compactly supported function $\rho$. $\int_M$ is called the* integral *on $M$.*

***Proof.*** To establish the existence of an integral we associate to each manifold $M$ a collection of data as follows.

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(i) A smooth partition of unity $\mathcal{A} \subset C_0^\infty(M)$ such that $\forall \alpha \in \mathcal{A}$ the support $\operatorname{supp} \alpha$ lies entirely in some precompact coordinate neighborhood $U_\alpha$, and such that the cover $(U_\alpha)$ is locally finite.

(ii) For each $U_\alpha$ we pick a collection of local coordinates $(x^i_\alpha)$, and we denote by $|dx_\alpha|$ $(n = \dim M)$ the density on $U_\alpha$ defined by

$$|dx_\alpha| \left( \frac{\partial}{\partial x^1_\alpha} \wedge \ldots \wedge \frac{\partial}{\partial x^n_\alpha} \right) = 1.$$

For any $\mu \in C^\infty(|\Lambda|)$, the product $\alpha\mu$ is a density supported in $U_\alpha$, and can be written as

$$\alpha\mu = \mu_\alpha |dx_\alpha|,$$

where $\mu_\alpha$ is some smooth function compactly supported on $U_\alpha$. The local coordinates allow us to interpret $\mu_\alpha$ as a function on $\mathbb{R}^n$. Under this identification $|dx^i_\alpha|$ corresponds to the Lebesgue measure $|dv_n|$ on $\mathbb{R}^n$, and $\mu_\alpha$ is a compactly supported, smooth function. We set

$$\int_{U_\alpha} \alpha\mu := \int_{\mathbb{R}^n} \mu_\alpha |dx_\alpha|.$$

Finally, define

$$\int_M^{\mathcal{A}} \mu = \int_M \mu \stackrel{def}{=} \sum_{\alpha \in \mathcal{A}} \int_{U_\alpha} \alpha\mu.$$

The above sum contains only finitely many nonzero terms since $\operatorname{supp} \mu$ is compact, and thus it intersects only finitely many of the $U'_\alpha s$ which form a locally finite cover.

To prove property (a) we will first prove that the integral defined as above is independent of the various choices: the partition of unity $\mathcal{A} \subset C_0^\infty(M)$, and the local coordinates $(x^i_\alpha)_{\alpha \in \mathcal{A}}$.

• **Independence of coordinates.** Fix the partition of unity $\mathcal{A}$, and consider a new collection of local coordinates $(y^i_\alpha)$ on each $U_\alpha$. These determine two densities $|dx^i_\alpha|$ and respectively $|dy^j_\alpha|$. For each $\mu \in C_0^\infty(|\Lambda|_M)$ we have

$$\alpha\mu = \alpha\mu^x_\alpha |dx_\alpha| = \alpha\mu^y_\alpha |dy_\alpha|,$$

where $\mu^x_\alpha$, $\mu^y_\alpha \in C_0^\infty(U_\alpha)$ are related by

$$\mu^y_\alpha = \left| \det \left( \frac{\partial x^i_\alpha}{\partial y^j_\alpha} \right) \right| \mu^x_\alpha.$$

The equality

$$\int_{\mathbb{R}^n} \mu^x_\alpha |dx_\alpha| = \int_{\mathbb{R}^n} \mu^y_\alpha |dy_\alpha|$$

is the classical change in variables formula for the Lebesgue integral.

• **Independence of the partition of unity.** Let $\mathcal{A}, \mathcal{B} \subset C_0^\infty(M)$ two partitions of unity on $M$. We will show that

$$\int_M^{\mathcal{A}} = \int_M^{\mathcal{B}}.$$

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Form the partition of unity

$$\mathcal{A} * \mathcal{B} := \left\{ \alpha\beta \,;\ (\alpha, \beta) \in \mathcal{A} \times \mathcal{B} \right\} \subset C_0^\infty(M).$$

Note that $\operatorname{supp} \alpha\beta \subset U_{\alpha\beta} = U_\alpha \cap U_\beta$. We will prove

$$\int_M^{\mathcal{A}} = \int_M^{\mathcal{A}*\mathcal{B}} = \int_M^{\mathcal{B}}.$$

Let $\mu \in C_0^\infty(|\Lambda|_M)$. We can view $\alpha\mu$ as a compactly supported function on $\mathbb{R}^n$. We have

$$\int_{U_\alpha} \alpha\mu = \sum_\beta \int_{U_\alpha \subset \mathbb{R}^n} \beta\alpha\mu = \sum_\beta \int_{U_{\alpha\beta}} \alpha\beta\mu. \tag{3.4.1}$$

Similarly

$$\int_{U_\beta} \beta\mu = \sum_\alpha \int_{U_{\alpha\beta}} \alpha\beta\mu. \tag{3.4.2}$$

Summing (3.4.1) over $\alpha$ and (3.4.2) over $\beta$ we get the desired conclusion.

To prove property (a) for a diffeomorphism $\phi : M \to N$, consider a partition of unity $\mathcal{A} \subset C_0^\infty(N)$. From the classical change in variables formula we deduce that, for any coordinate neighborhood $U_\alpha$ containing the support of $\alpha \in \mathcal{A}$, and any $\mu \in C_0^\infty(|\Lambda|_N)$ we have

$$\int_{\phi^{-1}(U_\alpha)} (\phi^*\alpha)\phi^*\mu = \int_{U_\alpha} \alpha\mu.$$

The collection $\Big(\phi^*\alpha = \alpha \circ \phi\Big)_{\alpha \in \mathcal{A}}$ forms a partition of unity on $M$. Property (a) now follows by summing over $\alpha$ the above equality, and using the independence of the integral on partitions of unity.

To prove property (b) on the local character of the integral, pick $U \subset M$, and then choose a partition of unity $\mathcal{B} \subset C_0^\infty(U)$ subordinated to the open cover $(V_\beta)_{\beta \in \mathcal{B}}$. For any partition of unity $\mathcal{A} \subset C_0^\infty(M)$ with associated cover $(V_\alpha)_{\alpha \in \mathcal{A}}$ we can form a new partition of unity $\mathcal{A} * \mathcal{B}$ of U with associated cover $V_{\alpha\beta} = V_\alpha \cap V_\beta$. We use this partition of unity to compute integrals over $U$. For any density $\mu$ on $M$ supported on $U$ we have

$$\int_M \mu = \sum_\alpha \int_{V_\alpha} \alpha\mu = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} \int_{V_{\alpha\beta}} \alpha\beta\mu = \sum_{\alpha\beta \in \mathcal{A}*\mathcal{B}} \int_{V_{\alpha\beta}} \alpha\beta\mu = \int_U \mu.$$

Property (c) is clear since, for $M = \mathbb{R}^n$, we can assume that all the local coordinates chosen are Cartesian. The uniqueness of the integral is immediate, and we leave the reader to fill in the details. $\square$

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### 3.4.2 *Orientability and integration of differential forms*

Under some mild restrictions on the manifold, the calculus with densities can be replaced with the richer calculus with differential forms. The mild restrictions referred to above have a *global nature*. More precisely, we have to require that the background manifold is *oriented*.

Roughly speaking, the oriented manifolds are the "2-sided manifolds", i.e., one can distinguish between an "inside face" and an "outside face" of the manifold. (Think of a 2-sphere in $\mathbb{R}^3$ (a soccer ball) which is naturally a "2-faced" surface.)

The 2-sidedness feature is such a frequent occurrence in the real world that for many years it was taken for granted. This explains the "big surprise" produced by the famous counter-example due to Möbius in the first half of the 19th century. He produced a 1-sided surface nowadays known as the Möbius band using paper and glue. More precisely, he glued the opposite sides of a paper rectangle attaching arrow to arrow as in Figure 3.2. The 2-sidedness can be formulated rigorously as

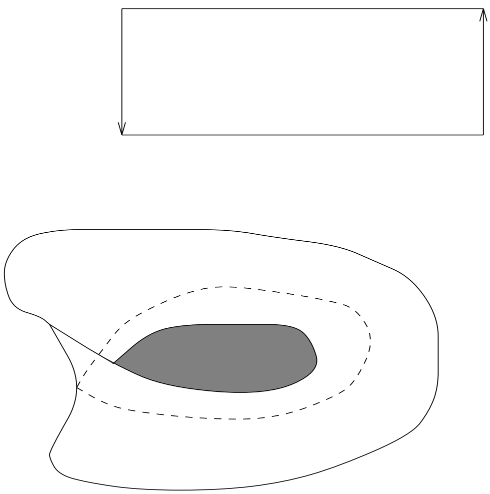

Fig. 3.2 *The Mobius band.*

follows.

**Definition 3.4.4.** A smooth manifold $M$ is said to be *orientable* if the determinant line bundle $\det TM$ (or equivalently $\det T^*M$) is trivializable. □

We see that $\det T^*M$ is trivializable if and only if it admits a nowhere vanishing section. Such a section is called a *volume form* on $M$. We say that two volume forms $\omega_1$ and $\omega_2$ are equivalent if there exists $f \in C^\infty(M)$ such that

$$\omega_2 = e^f \omega_1.$$

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This is indeed an equivalence relation, and an equivalence class of volume forms will be called an *orientation* of the manifold. We denote by $\boldsymbol{Or}(M)$ the set of orientations on the smooth manifold $M$. A pair (*orientable manifold, orientation*) is called an *oriented* manifold.

Let us observe that if $M$ is orientable, and thus $\boldsymbol{Or}(M) \neq \emptyset$, then for every point $p \in M$ we have a natural map

$$\boldsymbol{Or}(M) \to \boldsymbol{Or}(T_pM), \ \ \boldsymbol{Or}(M) \ni \boldsymbol{or} \mapsto \boldsymbol{or}_p \in \boldsymbol{Or}(T_pM),$$

defined as follows. If the orientation $\boldsymbol{or}$ on $M$ is defined by a volume form $\omega$, then $\omega_p \in \det T_p^*M$ is a nontrivial volume form on $T_pM$, which canonically defines an orientation $\boldsymbol{or}_{\omega,p}$ on $T_pM$. It is clear that if $\omega_1$ and $\omega_2$ are equivalent volume form then $\boldsymbol{or}_{\omega_1,p} = \boldsymbol{or}_{\omega_2,p}$.

This map is clearly a surjection because $\boldsymbol{or}_{-\omega,p} = -\boldsymbol{or}_{\omega,p}$, for any volume form $\omega$.

**Proposition 3.4.5.** *If $M$ is a connected, orientable smooth manifold $M$, then for every $p \in M$ the map*

$$\boldsymbol{Or}(M) \ni \boldsymbol{or} \mapsto \boldsymbol{or}_p \in \boldsymbol{Or}(T_pM)$$

*is a bijection.*

***Proof.*** Suppose $\boldsymbol{or}$ and $\boldsymbol{or}'$ are two orientations on $M$ such that $\boldsymbol{or}_p = \boldsymbol{or}'_p$. The function

$$M \ni q \mapsto \epsilon(q) = \boldsymbol{or}'_q / \boldsymbol{or}_q \in \{\pm 1\}$$

is *continuous*, and thus *constant*. In particular, $\epsilon(q) = \epsilon(p) = 1$, $\forall q \in M$.

If $\boldsymbol{or}$ is given by the volume form $\omega$ and $\boldsymbol{or}'$ is given by the volume form $\omega'$, then there exists a *nowhere vanishing smooth function* $\rho : M \to \mathbb{R}$ such that $\omega' = \rho\omega$. We deduce

$$\operatorname{sign} \rho(q) = \epsilon(q), \ \ \forall q \in M.$$

This shows that the two forms $\omega'$ and $\omega$ are equivalent and thus $\boldsymbol{or} = \boldsymbol{or}'$. □

The last proposition shows that on a *connected, orientable* manifold, a choice of an orientation of one of its tangent spaces uniquely determines an orientation of the manifold. A natural question arises.

*How can one decide whether a given manifold is orientable or not.*

We see this is just a special instance of the more general question we addressed in Chapter 2: how can one decide whether a given vector bundle is trivial or not. The orientability question can be given a very satisfactory answer using topological techniques. However, it is often convenient to decide the orientability issue using ad-hoc arguments. In the remaining part of this section we will describe several simple ways to detect orientability.

**Example 3.4.6.** If the tangent bundle of a manifold $M$ is trivial, then clearly $TM$ is orientable. In particular, all Lie groups are orientable. □

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**Example 3.4.7.** Suppose $M$ is a manifold such that the Whitney sum $\underline{\mathbb{R}}^k_M \oplus TM$ is trivial. Then $M$ is orientable. Indeed, we have

$$\det(\underline{\mathbb{R}}^k \oplus TM) = \det \underline{\mathbb{R}}^k \otimes \det TM.$$

Both $\det \underline{\mathbb{R}}^k$ and $\det(\underline{\mathbb{R}}^k \oplus TM)$ are trivial. We deduce $\det TM$ is trivial since

$$\det TM \cong \det(\underline{\mathbb{R}}^k \oplus TM) \otimes (\det \underline{\mathbb{R}}^k)^*.$$

This trick works for example when $M \cong S^n$. Indeed, let $\nu$ denote the normal line

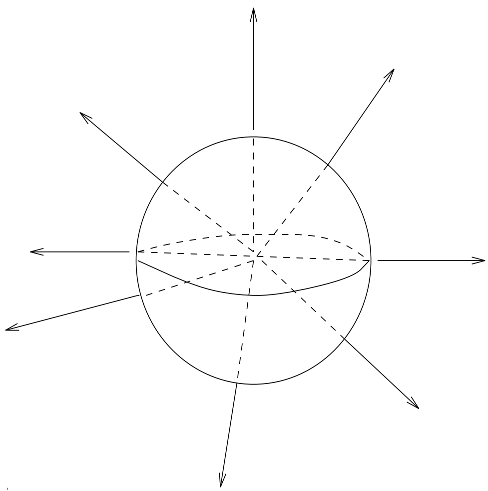

Fig. 3.3 *The normal line bundle to the round sphere.*

bundle. The fiber of $\nu$ at a point $p \in S^n$ is the 1-dimensional space spanned by the position vector of $p$ as a point in $\mathbb{R}^n$; (see Figure 3.3). This is clearly a trivial line bundle since it has a tautological nowhere vanishing section $p \mapsto p \in \nu_p$. The line bundle $\nu$ has a remarkable feature:

$$\nu \oplus TS^n = \underline{\mathbb{R}}^{n+1}.$$

Hence all spheres are orientable. □

☞**Important convention.** The *canonical orientation* on $\mathbb{R}^n$ is the orientation defined by the volume form $dx^1 \wedge \cdots \wedge dx^n$, where $x^1, ..., x^n$ are the canonical Cartesian coordinates.

The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is orientable. In the sequel we will exclusively deal with its *canonical orientation*. To describe this orientation it suffices to describe a positively oriented basis of $\det T_pM$ for some $p \in S^n$. To this aim we will use the relation

$$\mathbb{R}^{n+1} \cong \nu_p \oplus T_pS^n.$$

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An element $\omega \in \det T_pS^n$ defines the canonical orientation if $\vec{p} \wedge \omega \in \det \mathbb{R}^{n+1}$ defines the canonical orientation of $\mathbb{R}^{n+1}$. Above, by $\vec{p}$ we denoted the position vector of $p$ as a point inside the Euclidean space $\mathbb{R}^{n+1}$. We can think of $\vec{p}$ as the "outer" normal to the round sphere. We call this orientation *outer normal first*. When $n = 1$ it coincides with the counterclockwise orientation of the unit circle $S^1$. □

**Lemma 3.4.8.** *A smooth manifold $M$ is orientable if and only if there exists an open cover $(U_\alpha)_{\alpha \in \mathcal{A}}$, and local coordinates $(x^1_\alpha, ..., x^n_\alpha)$ on $U_\alpha$ such that*

$$\det\left(\frac{\partial x^i_\alpha}{\partial x^j_\beta}\right) > 0 \quad \text{on } U_\alpha \cap U_\beta. \tag{3.4.3}$$

***Proof.*** **1.** We assume that there exists an open cover with the properties in the lemma, and we will prove that $\det T^*M$ is trivial by proving that there exists a volume form.

Consider a partition of unity $\mathcal{B} \subset C^\infty_0(M)$ subordinated to the cover $(U_\alpha)_{\alpha \in \mathcal{A}}$, i.e., there exists a map $\varphi : \mathcal{B} \to \mathcal{A}$ such that

$$\operatorname{supp} \beta \subset U_{\varphi(\beta)} \ \ \forall \beta \in \mathcal{B}.$$

Define

$$\omega := \sum_\beta \beta \omega_{\varphi(\beta)},$$

where for all $\alpha \in \mathcal{A}$ we define $\omega_\alpha := dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$. The form $\omega$ is nowhere vanishing since condition (3.4.3) implies that on an overlap $U_{\alpha_1} \cap \cdots \cap U_{\alpha_m}$ the forms $\omega_{\alpha_1}, ..., \omega_{\alpha_m}$ differ by a *positive* multiplicative factor.

**2.** Conversely, let $\omega$ be a volume form on $M$ and consider an atlas $(U_\alpha; (x^i_\alpha))$. Then

$$\omega\,|_{U_\alpha} = \mu_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha,$$

where the smooth functions $\mu_\alpha$ are nowhere vanishing, and on the overlaps they satisfy the gluing condition

$$\Delta_{\alpha\beta} = \det\left(\frac{\partial x^i_\alpha}{\partial x^j_\beta}\right) = \frac{\mu_\beta}{\mu_\alpha}.$$

A permutation $\varphi$ of the variables $x^1_\alpha, ..., x^n_\alpha$ will change $dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$ by a factor $\epsilon(\varphi)$ so we can always arrange these variables in such an order so that $\mu_\alpha > 0$. This will insure the positivity condition

$$\Delta_{\alpha\beta} > 0.$$

The lemma is proved. □

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We can rephrase the result in the above lemma in a more conceptual way using the notion of *orientation bundle.* Suppose $E \to M$ is a *real* vector bundle of rank $r$ on the smooth manifold $M$ described by the open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$, and gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{R}).$$

The *orientation bundle associated to* $E$ is the real line bundle $\mathbf{\Theta}(E) \to M$ described by the open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$, and gluing cocycle

$$\epsilon_{\beta\alpha} := \operatorname{sign}\det g_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{R}^* = \mathrm{GL}(1, \mathbb{R}).$$

We define *orientation bundle* $\mathbf{\Theta}_M$ of a smooth manifold $M$ as the orientation bundle associated to the tangent bundle of $M$, $\mathbf{\Theta}_M := \mathbf{\Theta}(TM)$.

The statement in Lemma 3.4.8 can now be rephrased as follows.

**Corollary 3.4.9.** *A smooth manifold $M$ is orientable if and only if the orientation bundle $\mathbf{\Theta}_M$ is trivializable.* □

From Lemma 3.4.8 we deduce immediately the following consequence.

**Proposition 3.4.10.** *The connected sum of two orientable manifolds is an orientable manifold.* □

**Exercise 3.4.11.** Prove the above result. □

Using Lemma 3.4.8 and Proposition 2.2.76 we deduce the following result.

**Proposition 3.4.12.** *Any complex manifold is orientable. In particular, the complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^n)$ are orientable.* □

**Exercise 3.4.13.** Supply the details of the proof of Proposition 3.4.12. □

The reader can check immediately that the product of two orientable manifolds is again an orientable manifold. Using connected sums and products we can now produce many examples of manifolds. In particular, the connected sums of $g$ tori is an orientable manifold.

By now the reader may ask where does orientability interact with integration. The answer lies in Subsection 2.2.4 where we showed that an orientation $\boldsymbol{or}$ on a vector space $V$ induces a *canonical, linear isomorphism* $\imath_{\boldsymbol{or}} : \det V^* \to |\Lambda|_V$; see (2.2.13).

Similarly, an orientation $\boldsymbol{or}$ on a smooth manifold $M$ defines an isomorphism

$$\imath_{\boldsymbol{or}} : C^\infty(\det T^*M) \to C^\infty(|\Lambda|_M).$$

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For any compactly supported differential form $\omega$ on $M$ of maximal degree we define its integral by

$$\int_M \omega := \int_M \imath_{\boldsymbol{or}}\omega.$$

We want to emphasize that this definition *depends on the choice of orientation.*

We ought to pause and explain in more detail the isomorphism $\imath_{\boldsymbol{or}}$ : $C^\infty(\det T^*M) \to C^\infty(|\Lambda|_M)$. Since $M$ is oriented we can choose a coordinate atlas $\big(U_\alpha,\ (x^i_\alpha)\big)$ such that

$$\det\Big[\frac{\partial x^i_\beta}{\partial x^j_\alpha}\Big]_{1\leq,i,j\leq M} > 0,\ \ n = \dim M, \tag{3.4.4}$$

and on each coordinate patch $U_\alpha$ the orientation is given by the top degree from $dx_\alpha = dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$.

The differential form is described by a collection of forms

$$\omega_\alpha = \rho_\alpha dx_\alpha,\ \ \rho_\alpha \in C^\infty(U_\alpha),$$

and due to the condition (3.4.4) the collection of desities

$$\mu_\alpha = \rho_\alpha |dx_\alpha| \in C^\infty(U_\alpha, |\,|\Lambda|\,|_M)$$

satisfy $\mu_\alpha = \mu_\beta$ on the overlap $U_{\alpha\beta}$. Thus they glue together to a density on $M$, which is precisely $\imath_{\boldsymbol{or}}\omega$.

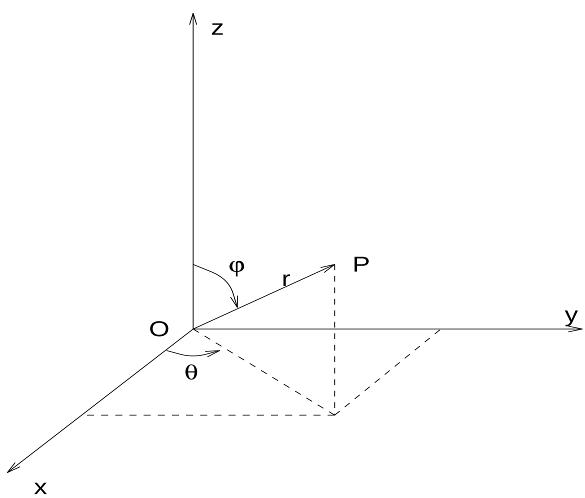


Fig. 3.4 *Spherical coordinates.*

**Example 3.4.14.** Consider the 2-form on $\mathbb{R}^3$, $\omega = xdy\wedge dz$, and let $S^2$ denote the unit sphere. We want to compute $\int_{S^2} \omega\,|_{S^2}$, where $S^2$ has the canonical orientation.

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To compute this integral we will use spherical coordinates $(r, \varphi, \theta)$. These are defined by (see Figure 3.4)

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}.$$

At the point $p = (1, 0, 0)$ we have

$$\partial_r = \partial_x = \vec{p}, \ \ \partial_\theta = \partial_y \ \ \partial_\varphi = -\partial_z,$$

so that the standard orientation on $S^2$ is given by $d\varphi \wedge d\theta$. On $S^2$ we have $r \equiv 1$ and $dr \equiv 0$ so that

$$xdy \wedge dz \mid_{S^2} = \sin\varphi\cos\theta\,(\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi)) \wedge (-\sin\varphi)d\varphi$$

$$= \sin^3\varphi\cos^2\theta d\varphi \wedge d\theta.$$

The standard orientation associates to this form de density $\sin^3\varphi\cos^2\theta|d\varphi d\theta|$, and we deduce

$$\int_{S^2}\omega = \int_{[0,\pi]\times[0,2\pi]} \sin^3\varphi\cos^2\theta|d\varphi d\theta| = \int_0^\pi \sin^3\varphi d\varphi \cdot \int_0^{2\pi}\cos^2\theta d\theta$$

$$= \frac{4\pi}{3} = \text{volume of the unit ball } B^3 \subset \mathbb{R}^3.$$

As we will see in the next subsection the above equality is no accident. □

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**Example 3.4.15. (Invariant integration on compact Lie groups).** Let $G$ be a compact, connected Lie group. Fix once and for all an orientation on the Lie algebra $\mathcal{L}_G$. Consider a positively oriented volume element $\omega \in \det\mathcal{L}_G^*$. We can extend $\omega$ by left translations to a left-invariant volume form on $G$ which we continue to denote by $\omega$. This defines an orientation, and in particular, by integration, we get a positive scalar

$$c = \int_G \omega.$$

Set $dV_G = \frac{1}{c}\omega$ so that

$$\int_G dV_G = 1. \tag{3.4.5}$$

The differential form $dV_G$ is the unique left-invariant $n$-form ($n = \dim G$) on $G$ satisfying (3.4.5) (assuming a fixed orientation on $G$). We claim $dV_G$ is also right invariant.

To prove this, consider the *modular function* $G \ni h \mapsto \Delta(h) \in \mathbb{R}$ defined by

$$R_h^*(dV_G) = \Delta(h)dV_G.$$

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The quantity $\Delta(h)$ is independent of $h$ because $R_h^* dV_G$ is a left invariant form, so it has to be a scalar multiple of $dV_G$. Since $(R_{h_1h_2})^* = (R_{h_2}R_{h_1})^* = R_{h_1}^* R_{h_2}^*$ we deduce

$$\Delta(h_1h_2) = \Delta(h_1)\Delta(h_2) \quad \forall h_1,\, h_2 \in G.$$

Hence $h \mapsto \Delta h$ is a *smooth* morphism

$$G \to (\mathbb{R} \setminus \{0\}, \cdot).$$

Since $G$ is connected $\Delta(G) \subset \mathbb{R}_+$, and since $G$ is compact, the set $\Delta(G)$ is bounded. If there exists $x \in G$ such that $\Delta(x) \neq 1$, then either $\Delta(x) > 1$, or $\Delta(x^{-1}) > 1$, and in particular, we would deduce the set $(\Delta(x^n))_{n\in\mathbf{Z}}$ is unbounded. Thus $\Delta \equiv 1$ which establishes the right invariance of $dV_G$.

The invariant measure $dV_G$ provides a very simple way of producing invariant objects on $G$. More precisely, if $T$ is tensor field on $G$, then for each $x \in G$ define

$$\overline{T}_x^{\ell} = \int_G ((L_g)_* T)_x dV_G(g).$$

Then $x \mapsto \overline{T}_x$ defines a smooth tensor field on $G$. We claim that $\overline{T}$ is left invariant. Indeed, for any $h \in G$ we have

$$(L_h)_*\overline{T}^{\ell} = \int_G (L_h)_*((L_g)_*T) dV_G(g) = \int_G ((L_{hg})_*T) dV_G(g)$$

$$\stackrel{u=hg}{=} \int_G (L_u)_* T L_{h^{-1}}^* dV_G(u) = \overline{T}^{\ell} \quad (\, L_{h^{-1}}^* dV_G = dV_G\,).$$

If we average once more on the right we get a tensor

$$G \ni x \mapsto \int_G \big(\,(R_g)_*\overline{T}^{\ell}\,\big)_x dV_G,$$

which is both left and right invariant. $\square$

**Exercise 3.4.16.** Let $G$ be a Lie group. For any $X \in \mathfrak{L}_G$ denote by $\operatorname{ad}(X)$ the linear map $\mathfrak{L}_G \to \mathfrak{L}_G$ defined by

$$\mathfrak{L}_G \ni Y \mapsto [X, Y] \in \mathfrak{L}_G.$$

(a) If $\omega$ denotes a left invariant volume form prove that $\forall X \in \mathfrak{L}_G$

$$L_X\omega = \operatorname{tr}\operatorname{ad}(X)\omega.$$

(b) Prove that if $G$ is a compact Lie group, then $\operatorname{tr}\operatorname{ad}(X) = 0$, for any $X \in \mathfrak{L}_G$. $\square$

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### 3.4.3 *Stokes' formula*

The Stokes' formula is the higher dimensional version of the fundamental theorem of calculus (Leibniz-Newton formula)

$$\int_a^b df = f(b) - f(a),$$

where $f : [a, b] \to \mathbb{R}$ is a smooth function and $df = f'(t)dt$. In fact, the higher dimensional formula will follow from the simplest 1-dimensional situation.

We will spend most of the time finding the correct formulation of the general version, and this requires the concept of *manifold with boundary.* The standard example is the lower half-space

$$\mathbf{H}_-^n = \{ (x^1, ..., x^n) \in \mathbb{R}^n \ ; \ x^1 \leq 0 \}.$$

**Definition 3.4.17.** A smooth *manifold with boundary* of dimension $n$ is a topological space with the following properties.

(a) There exists a smooth $n$-dimensional manifold $\widetilde{M}$ that contains $M$ as a closed subset.
(b) The interior of $M$, denoted by $M^0$, is non empty.
(c) For each point $p \in \partial M := M \setminus M^0$, there exist smooth local coordinates $(x^1, ..., x^n)$ defined on an open neighborhood $\mathcal{N}$ of $p$ in $\widetilde{M}$ such that

($c_1$) $M^0 \cap \mathcal{N} = \{ q \in \mathcal{N}; \ \ x^1(q) < 0 \}$.
($c_2$) $\partial M \cap \mathcal{N} = \{ x^1 = 0 \}$.

The set $\partial M$ is called the *boundary* of $M$. A manifold with boundary $M$ is called orientable if its interior $M^0$ is orientable. □

**Example 3.4.18.** (a) A closed interval $I = [a, b]$ is a smooth 1-dimensional manifold with boundary $\partial I = \{a, b\}$. We can take $\widetilde{M} = \mathbb{R}$.

(b) The closed unit ball $B^3 \subset \mathbb{R}^3$ is an orientable manifold with boundary $\partial B^3 = S^2$. We can take $\widetilde{M} = \mathbb{R}^3$.

(c) Suppose $X$ is a smooth manifold, and $f : X \to \mathbb{R}$ is a smooth function such that $0 \in \mathbb{R}$ is a regular value of $f$, i.e.,

$$f(x) = 0 \Longrightarrow df(x) \neq 0.$$

Define $M := \{ x \in X; \ \ f(x) \leq 0 \}$. A simple application of the implicit function theorem shows that the pair $(X, M)$ defines a manifold with boundary. Note that examples (a) and (b) are special cases of this construction. In the case (a) we take $X = \mathbb{R}$, and $f(x) = (x - a)(x - b)$, while in the case (b) we take $X = \mathbb{R}^3$ and $f(x, y, z) = (x^2 + y^2 + z^2) - 1$. □

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**Definition 3.4.19.** Two manifolds with boundary $M_1 \subset \widetilde{M_1}$, and $M_2 \subset \widetilde{M_2}$ are said to be *diffeomorphic* if, for every $i = 1, 2$ there exists an open neighborhood $U_i$ of $M_i$ in $\widetilde{M_i}$, and a diffeomorphism $F : U_1 \to U_2$ such that $F(M_1) = M_2$. □

**Exercise 3.4.20.** Prove that any manifold with boundary is diffeomorphic to a manifold with boundary constructed via the process described in Example 3.4.18(c). □

**Proposition 3.4.21.** *Let $M$ be a smooth manifold with boundary. Then its boundary $\partial M$ is also a smooth manifold of dimension* $\dim \partial M = \dim M - 1$. *Moreover, if $M$ is orientable, then so is its boundary.* □

The proof is left to the reader as an exercise.

☞**Important convention.** Let $M$ be an orientable manifold with boundary. There is a (*non-canonical*) way to associate to an orientation on $M^0$ an orientation on the boundary $\partial M$. This will be the only way in which we will orient boundaries throughout this book. If we do not pay attention to this convention then our results may be off by a sign.

We now proceed to described this *induced orientation* on $\partial M$. For any $p \in \partial M$ choose local coordinates $(x^1, ..., x^n)$ as in Definition 3.4.17. Then the induced orientation of $T_p\partial M$ is defined by

$$\epsilon dx^2 \wedge \cdots \wedge dx^n \in \det T_p\partial M, \quad \epsilon = \pm 1,$$

where $\epsilon$ is chosen so that for $x^1 < 0$, i.e. inside $M$, the form

$$\epsilon dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

is positively oriented. The differential $dx^1$ is usually called an *outer conormal* since $x^1$ increases as we go towards the exterior of $M$. $-dx^1$ is then the *inner conormal* for analogous reasons. The rule by which we get the induced orientation on the boundary can be rephrased as

$$\{\text{outer conormal}\} \wedge \{\text{induced orientation on boundary}\} = \{\text{orientation in the interior}\}.$$

We may call this rule *"outer (co)normal first"* for obvious reasons. □

**Example 3.4.22.** The canonical orientation on $S^n \subset \mathbb{R}^{n+1}$ coincides with the induced orientation of $S^{n+1}$ as the boundary of the unit ball $B^{n+1}$. □

**Exercise 3.4.23.** Consider the hyperplane $H_i \subset \mathbb{R}^n$ defined by the equation $\{x^i = 0\}$. Prove that the induced orientation of $H_i$ as the boundary of the half-space $\mathbf{H}_+^{n,i} = \{x^i \geq 0\}$ is given by the $(n-1)$-form $(-1)^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n$ where, as usual, the hat indicates a missing term. □

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**Theorem 3.4.24 (Stokes formula).** *Let $M$ be an oriented $n$-dimensional manifold with boundary $\partial M$ and $\omega \in \Omega^{n-1}(M)$ a compactly supported form. Then*

$$\int_{M^0} d\omega = \int_{\partial M} \omega.$$

*In the above formula $d$ denotes the exterior derivative, and $\partial M$ has the induced orientation.*

***Proof.*** Via partitions of unity the verification is reduced to the following two situations.

**Case 1.** *The $(n-1)$-form $\omega$ is compactly supported in $\mathbb{R}^n$.* We have to show

$$\int_{\mathbb{R}^n} d\omega = 0.$$

It suffices to consider only the special case

$$\omega = f(x)dx^2 \wedge \cdots \wedge dx^n,$$

where $f(x)$ is a compactly supported smooth function. The general case is a linear combination of these special situations. We compute

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge dx^n = 0,$$

since

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x^1} dx^1 = f(\infty, x^2, ..., x^n) - f(-\infty, x^2, ..., x^n),$$

and $f$ has compact support.

**Case 2.** *The $(n-1)$-form $\omega$ is compactly supported in $\mathbf{H}^n_-$.* Let

$$\omega = \sum_i f_i(x) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Then

$$d\omega = \left( \sum_i (-1)^{i+1} \frac{\partial f}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n.$$

One verifies as in **Case 1** that

$$\int_{\mathbf{H}^n_-} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = 0 \quad \text{for } i \neq 1.$$

For $i = 1$ we have

$$\int_{\mathbf{H}^n_+} \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbf{R}^{n-1}} \left( \int_{-\infty}^0 \frac{\partial f}{\partial x^1} dx^1 \right) dx^2 \wedge \cdots \wedge dx^n$$

$$= \int_{\mathbb{R}^{n-1}} \big( f(0, x^2, ..., x^n) - f(-\infty, x^2, ..., x^n) \big) dx^2 \wedge \cdots \wedge dx^n$$

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$$= \int_{\mathbb{R}^{n-1}} f(0, x^2, ..., x^n) dx^2 \wedge \cdots \wedge dx^n = \int_{\partial \mathbf{H}^n_-} \omega.$$

The last equality follows from the fact that the induced orientation on $\partial \mathbf{H}^n_-$ is given by $dx^2 \wedge \cdots \wedge dx^n$. This concludes the proof of the Stokes formula. □

**Remark 3.4.25.** Stokes formula illustrates an interesting global phenomenon. It shows that the integral $\int_M d\omega$ is independent of the behavior of $\omega$ inside $M$. It only depends on the behavior of $\omega$ on the boundary. □

**Example 3.4.26.**

$$\int_{S^2} x dy \wedge dz = \int_{B^3} dx \wedge dy \wedge dz = \mathrm{vol}\,(B^3) = \frac{4\pi}{3}.$$ □

**Remark 3.4.27.** The above considerations extend easily to more singular situations. For example, when $M$ is the cube $[0,1]^n$ its topological boundary is no longer a smooth manifold. However, its singularities are inessential as far as integration is concerned. The Stokes formula continues to hold

$$\int_{I^n} d\omega = \int_{\partial I} \omega \quad \forall \omega \in \Omega^{n-1}(I^n).$$

The boundary is smooth outside a set of measure zero and is given the induced orientation: " outer (co)normal first". The above equality can be used to give an explanation for the terminology "exterior derivative" we use to call $d$. Indeed if $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ and $I_h = [0,h]$ then we deduce

$$d\omega\,|_{x=0} = \lim_{h \to 0} h^{-n} \int_{\partial I^n_h} \omega. \tag{3.4.6}$$

When $n = 1$ this is the usual definition of the derivative. □

**Example 3.4.28.** We now have sufficient technical background to describe an example of vector bundle which admits no flat connections, thus answering the question raised at the end of Section 3.3.3.

Consider the complex line bundle $L_n \to S^2$ constructed in Example 2.1.36. Recall that $L_n$ is described by the open cover

$$S^2 = U_0 \cup U_1, \;\; U_0 = S^2 \setminus \{\text{South pole}\}, \;\; U_1 = S^2 \setminus \{\text{North pole}\},$$

and gluing cocycle

$$g_{10} : U_0 \cap U_1 \to \mathbb{C}^*, \;\; g_{10}(z) = z^{-n} = g_{01}(z)^{-1},$$

where we identified the overlap $U_0 \cap U_1$ with the punctured complex line $\mathbb{C}^*$.

A connection on $L_n$ is a collection of two complex valued forms $\omega_0 \in \Omega^1(U_1) \otimes \mathbb{C}$, $\omega_1 \in \Omega^1(U_1) \otimes \mathbb{C}$, satisfying a gluing relation on the overlap (see Example 3.3.7)

$$\omega_1 = -\frac{dg_{10}}{g_{10}} + \omega_0 = n\frac{dz}{z} + \omega_0.$$

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If the connection is flat, then

$$d\omega_0 = 0 \ \text{ on } U_0 \ \text{ and } \ d\omega_1 = 0 \ \text{ on } \ U_1.$$

Let $E^+$ be the Equator equipped with the induced orientation as the boundary of the northern hemisphere, and $E^-$ the equator with the opposite orientation, as the boundary of the southern hemisphere. The orientation of $E^+$ coincides with the orientation given by the form $d\theta$, where $z = \exp(\boldsymbol{i}\theta)$.

We deduce from the Stokes formula (which works for complex valued forms as well) that

$$\int_{E^+} \omega_0 = 0 \quad \int_{E^+} \omega_1 = -\int_{E^-} \omega_1 = 0.$$

On the other hand, over the Equator, we have

$$\omega_1 - \omega_0 = n\frac{dz}{z} = n\boldsymbol{i}d\theta,$$

from which we deduce

$$0 = \int_{E^+} \omega_0 - \omega_1 = n\boldsymbol{i}\int_{E^+} d\theta = 2n\pi\boldsymbol{i} \ \ !!!$$

Thus there exist no flat connections on the line bundle $L_n$, $n \neq 0$, and at fault is the gluing cocycle defining $L$. In a future chapter we will quantify the measure in which the gluing data obstruct the existence of flat connections. □

### 3.4.4 *Representations and characters of compact Lie groups*

The invariant integration on compact Lie groups is a very powerful tool with many uses. Undoubtedly, one of the most spectacular application is Hermann Weyl's computation of the characters of representations of compact semi-simple Lie groups. The invariant integration occupies a central place in his solution to this problem.

We devote this subsection to the description of the most elementary aspects of the representation theory of compact Lie groups.

Let $G$ be a Lie group. Recall that a *(linear) representation* of $G$ is a left action on a (finite dimensional) vector space $V$

$$G \times V \to V \quad (g, v) \mapsto T(g)v \in V,$$

such that the map $T(g)$ is linear for any $g$. One also says that $V$ has a *structure of $G$-module*. If $V$ is a real (respectively complex) vector space, then it is said to be a real (respectively complex) $G$-module.

**Example 3.4.29.** Let $V = \mathbb{C}^n$. Then $G = \mathrm{GL}(n, \mathbb{C})$ acts linearly on $V$ in the tautological manner. Moreover $V^*$, $V^{\otimes k}$, $\Lambda^m V$ and $S^\ell V$ are complex $G$-modules. □

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**Example 3.4.30.** Suppose $G$ is a Lie group with Lie algebra $\mathfrak{L}_G - T_1G$. For every $g \in G$, the conjugation

$$\boldsymbol{C}_g : G \to G, \;\; h \mapsto \boldsymbol{C}_g(h) = ghg^{-1},$$

sends the identity $1 \in G$ to itself. We denote by $\mathrm{Ad}_g : \mathfrak{L}_G \to \mathfrak{L}$ the differential of $h \mapsto \boldsymbol{C}_g(h)$ at $h = 1$. The operator $\mathrm{Ad}_g$ is a linear isomorphism of $\mathfrak{L}_G$, and the resulting map

$$G \mapsto \mathrm{Aut}(\mathfrak{L}_G), \;\; g \mapsto \mathrm{Ad}_g,$$

is called *adjoint representation* of $G$.

For example, if $G = SO(n)$, then $\mathfrak{L}_G = \underline{\mathrm{so}}(n)$, and the adjoint representation $\mathrm{Ad} : SO(n) \to \mathrm{Aut}(\underline{\mathrm{so}}(n))$, can be given the more explicit description

$$\underline{\mathrm{so}}(n) \ni X \overset{\mathrm{Ad}(g)}{\longmapsto} gXg^{-1} \in \underline{\mathrm{so}}(n), \;\; \forall g \in SO(n). \qquad \square$$

**Exercise 3.4.31.** Let $G$ be a Lie group, with Lie algebra $\mathfrak{L}_G$. Prove that for any $X, Y \in \mathfrak{L}_G$ we have

$$\frac{d}{dt}|_{t=0}\, \mathrm{Ad}_{\exp(tX)}(Y) = \mathrm{ad}(X)Y = [X, Y]. \qquad \square$$

**Definition 3.4.32.** A *morphism of $G$-modules* $V_1$ and $V_2$ is a linear map $L : V_1 \to V_2$ such that, for any $g \in G$, the diagram below is commutative, i.e., $T_2(g)L = LT_1(g)$.

$$\begin{tikzcd} V_1 \arrow[r, "L"] \arrow[d, "T_1(g)"'] & V_2 \arrow[d, "T_2(g)"] \\ V_1 \arrow[r, "L"] & V_2 \end{tikzcd}$$

The space of morphisms of $G$-modules is denoted by $\mathrm{Hom}_G(V_1, V_2)$. The collection of isomorphisms classes of complex $G$-modules is denoted by $G - \boldsymbol{Mod}$. $\square$

If $V$ is a $G$-module, then an *invariant subspace* (or submodule) is a subspace $U \subset V$ such that $T(g)(U) \subset U$, $\forall g \in G$. A $G$-module is said to be *irreducible* if it has no invariant subspaces other than $\{0\}$ and $V$ itself.

**Proposition 3.4.33.** *The direct sum "$\oplus$", and the tensor product "$\otimes$" define a structure of semi-ring with* $1$ *on* $G-\boldsymbol{Mod}$. $0$ *is represented by the null representation* $\{0\}$, *while* $1$ *is represented by the trivial module* $G \to \mathrm{Aut}\,(\mathbb{C})$, $g \mapsto 1$. $\square$

The proof of this proposition is left to the reader.

**Example 3.4.34.** Let $T_i : G \to \mathrm{Aut}\,(U_i)$ $(i = 1, 2)$ be two complex $G$-modules. Then $U_1^*$ is a $G$-module given by $(g, u^*) \mapsto T_1(g^{-1})^\dagger\, u^*$. Hence $\mathrm{Hom}\,(U_1, U_2)$ is also a $G$-module. Explicitly, the action of $g \in G$ is given by

$$(g, L) \longmapsto T_2(g)LT_1(g^{-1}), \;\; \forall L \in \mathrm{Hom}(U_1, U_2).$$

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We see that $\operatorname{Hom}_G(U_1, U_2)$ can be identified with the linear subspace in $\operatorname{Hom}(U_1, U_2)$ consisting of the linear maps $U_1 \to U_2$ unchanged by the above action of $G$. □

**Proposition 3.4.35 (Weyl's unitary trick).** *Let $G$ be a* compact *Lie group, and $V$ a complex $G$-module. Then there exists a Hermitian metric $h$ on $V$ which is $G$-invariant, i.e., $h(gv_1, gv_2) = h(v_1, v_2)$, $\forall v_1, v_2 \in V$.*

***Proof.*** Let $h$ be an arbitrary Hermitian metric on $V$. Define its $G$-average by

$$\overline{h}(u, v) := \int_G h(gu, gv) dV_G(g),$$

where $dV_G(g)$ denotes the normalized bi-invariant measure on $G$. One can now check easily that $\overline{h}$ is $G$-invariant. □

In the sequel, $G$ will always denote a compact Lie group.

**Proposition 3.4.36.** *Let $V$ be a complex $G$-module and $h$ a $G$-invariant Hermitian metric. If $U$ is an invariant subspace of $V$ then so is $U^\perp$, where "$\perp$" denotes the orthogonal complement with respect to $h$.*

***Proof.*** Since $h$ is $G$-invariant it follows that, $\forall g \in G$, the operator $T(g)$ is unitary, $T(g)T^*(g) = \mathbb{1}_V$. Hence, $T^*(g) = T^{-1}(g) = T(g^{-1})$, $\forall g \in G$.

If $x \in U^\perp$, then for all $u \in U$, and $\forall g \in G$

$$h(T(g)x, u) = h(x, T^*(g)u) = h(x, T(g^{-1})u) = 0.$$

Thus $T(g)x \in U^\perp$, so that $U^\perp$ is $G$-invariant. □

**Corollary 3.4.37.** *Every $G$-module $V$ can be decomposed as a direct sum of irreducible ones.* □

If we denoted by $\boldsymbol{Irr}(G)$ the collection of isomorphism classes of irreducible $G$-modules, then we can rephrase the above corollary by saying that $\boldsymbol{Irr}(G)$ generates the semigroup $(G - \boldsymbol{Mod}, \oplus)$.

To gain a little more insight we need to use the following remarkable trick due to Isaac Schur.

**Lemma 3.4.38 (Schur lemma).** *Let $V_1$, $V_2$ be two irreducible* complex *$G$-modules. Then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V_1, V_2) = \begin{cases} 1 \text{ if } V_1 \cong V_2 \\ 0 \text{ if } V_1 \ncong V_2 \end{cases}.$$

***Proof.*** Let $L \in \operatorname{Hom}_G(V_1, V_2)$. Then $\ker L \subset V_1$ is an invariant subspace of $V_1$. Similarly, $\operatorname{Range}(L) \subset V_2$ is an invariant subspace of $V_2$. Thus, either $\ker L = 0$ or $\ker L = V_1$.

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The first situation forces $\operatorname{Range} L \neq 0$ and, since $V_2$ is irreducible, we conclude $\operatorname{Range} L = V_2$. Hence, $L$ has to be in isomorphism of $G$-modules. We deduce that, if $V_1$ and $V_2$ are not isomorphic as $G$-modules, then $\operatorname{Hom}_G(V_1, V_2) = \{0\}$.

Assume now that $V_1 \cong V_2$ and $S : V_1 \to V_2$ is an isomorphism of $G$-modules. According to the previous discussion, any other nontrivial $G$-morphism $L : V_1 \to V_2$ has to be an isomorphism. Consider the automorphism $T = S^{-1}L : V_1 \to V_1$. Since $V_1$ is a *complex* vector space $T$ admits at least one (non-zero) eigenvalue $\lambda$.

The map $\lambda \mathbb{1}_{V_1} - T$ is an endomorphism of $G$-modules, and $\ker(\lambda \mathbb{1}_{V_1} - T) \neq 0$. Invoking again the above discussion we deduce $T \equiv \lambda \mathbb{1}_{V_1}$, i.e. $L \equiv \lambda S$. This shows $\dim \operatorname{Hom}_G(V_1, V_2) = 1$. □

Schur's lemma is powerful enough to completely characterize $S^1 - \boldsymbol{Mod}$, the representations of $S^1$.

**Example 3.4.39. (The irreducible (complex) representations of $S^1$).** Let $V$ be a complex irreducible $S^1$-module

$$S^1 \times V \ni (e^{\boldsymbol{i}\theta}, v) \longmapsto T_\theta v \in V,$$

where $T_{\theta_1} \cdot T_{\theta_2} = T_{\theta_1 + \theta_2 \bmod 2\pi}$. In particular, this implies that each $T_\theta$ is an $S^1$-automorphism since it obviously commutes with the action of this group. Hence $T_\theta = \lambda(\theta)\mathbb{1}_V$ which shows that $\dim V = 1$ since any 1-dimensional subspace of $V$ is $S^1$-invariant. We have thus obtained a smooth map

$$\lambda : S^1 \to \mathbb{C}^*,$$

such that

$$\lambda(e^{\boldsymbol{i}\theta} \cdot e^{\boldsymbol{i}\tau}) = \lambda(e^{\boldsymbol{i}\theta})\lambda(e^{\boldsymbol{i}\theta}).$$

Hence $\lambda : S^1 \to \mathbb{C}^*$ is a group morphism. As in the discussion of the modular function we deduce that $|\lambda| \equiv 1$. Thus, $\lambda$ looks like an exponential, i.e., there exists $\alpha \in \mathbb{R}$ such that (*verify!*)

$$\lambda(e^{\boldsymbol{i}\theta}) = \exp(\boldsymbol{i}\alpha\theta), \quad \forall \theta \in \mathbb{R}.$$

Moreover, $\exp(2\pi \boldsymbol{i}\alpha) = 1$, so that $\alpha \in \mathbb{Z}$.

Conversely, for any integer $n \in \mathbb{Z}$ we have a representation

$$S^1 \xrightarrow{\rho_n} \operatorname{Aut}(\mathbb{C}) \quad (e^{\boldsymbol{i}\theta}, z) \mapsto e^{in\theta} z.$$

The exponentials $\exp(\boldsymbol{i}n\theta)$ are called the *characters* of the representations $\rho_n$. □

**Exercise 3.4.40.** Describe the irreducible representations of $T^n$-the $n$-dimensional torus. □

**Definition 3.4.41.** (a) Let $V$ be a complex $G$-module, $g \mapsto T(g) \in \operatorname{Aut}(V)$. The *character* of $V$ is the smooth function

$$\boldsymbol{\chi}_V : G \to \mathbb{C}, \quad \boldsymbol{\chi}_V(g) := \operatorname{tr} T(g).$$

(b) A *class function* is a continuous function $f : G \to \mathbb{C}$ such that

$$f(hgh^{-1}) = f(g) \quad \forall g, h \in G.$$

(The character of a representation is an example of class function.) □

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**Theorem 3.4.42.** *Let $G$ be a compact Lie group, $U_1$, $U_2$ complex $G$-modules and $\boldsymbol{\chi}_{U_i}$ their characters. Then the following hold.*
*(a)* $\boldsymbol{\chi}_{U_1\oplus U_2} = \boldsymbol{\chi}_{U_1} + \boldsymbol{\chi}_{U_2}$, $\boldsymbol{\chi}_{U_1\otimes U_2} = \boldsymbol{\chi}_{U_1}\cdot\boldsymbol{\chi}_{U_1}$.
*(b)* $\boldsymbol{\chi}_{U_i}(1) = \dim U_i$.
*(c)* $\boldsymbol{\chi}_{U_i^*} = \overline{\boldsymbol{\chi}}_{U_i}$*-the complex conjugate of* $\boldsymbol{\chi}_{U_i}$.
*(d)*

$$\int_G \boldsymbol{\chi}_{U_i}(g)dV_G(g) = \dim U_i^G,$$

*where $U_i^G$ denotes the space of $G$-invariant elements of $U_i$,*

$$U_i^G = \{x \in U_i \ ;\ x = T_i(g)x \ \ \forall g \in G\}.$$

(e)

$$\int_G \boldsymbol{\chi}_{U_1}(g)\cdot\overline{\boldsymbol{\chi}}_{U_2}(g)dV_G(g) = \dim \operatorname{Hom}_G(U_2, U_1).$$

***Proof.*** The parts (a) and (b) are left to the reader. To prove (c), fix an invariant Hermitian metric on $U = U_i$. Thus, each $T(g)$ is a unitary operator on $U$. The action of $G$ on $U^*$ is given by $T(g^{-1})^\dagger$. Since $T(g)$ is unitary, we have $T(g^{-1})^\dagger = \overline{T(g)}$. This proves (c).

*Proof of (d).* Consider

$$P : U \to U, \quad Pu = \int_G T(g)u\, dV_G(g).$$

Note that $PT(h) = T(h)P$, $\forall h \in G$, i.e., $P \in \operatorname{Hom}_G(U, U)$. We now compute

$$T(h)Pu = \int_G T(hg)u\, dV_G(g) = \int_G T(\gamma)u\, R^*_{h^{-1}}dV_G(\gamma),$$

$$\int_G T(\gamma)u\, dV_G(\gamma) = Pu.$$

Thus, each $Pu$ is $G$-invariant. Conversely, if $x \in U$ is $G$-invariant, then

$$Px = \int_G T(g)xdg = \int_G x\, dV_G(g) = x,$$

i.e., $U^G = \operatorname{Range} P$. Note also that $P$ is a projector, i.e., $P^2 = P$. Indeed,

$$P^2u = \int_G T(g)Pu\, dV_G(g) = \int_G Pu\, dV_G(g) = Pu.$$

Hence $P$ is a projection onto $U^G$, and in particular

$$\dim_{\mathbb{C}} U^G = \operatorname{tr} P = \int_G \operatorname{tr} T(g)\, dV_G(g) = \int_G \boldsymbol{\chi}_U(g)\, dV_G(g).$$

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*Proof of (e).*

$$\int_G \chi_{U_1} \cdot \overline{\chi}_{U_2} dV_G(g) = \int_G \chi_{U_1} \cdot \chi_{U_2^*} dV_G(g) = \int_G \chi_{U_1 \otimes U_2^*} dV_G(g) = \int_G \chi_{\mathrm{Hom}(U_2, U_1)}$$
$$= \dim_{\mathbb{C}} \left(\mathrm{Hom}\left(U_2, U_1\right)\right)^G = \dim_{\mathbb{C}} \mathrm{Hom}_G(U_2, U_1),$$

since $\mathrm{Hom}_G$ coincides with the space of $G$-invariant morphisms. □

**Corollary 3.4.43.** *Let $U$, $V$ be irreducible $G$-modules. Then*

$$(\chi_U, \chi_V) = \int_G \chi_U \cdot \overline{\chi}_V dg = \delta_{UV} = \begin{cases} 1 \,, \; U \cong V \\ 0 \,, \; U \ncong V \end{cases}.$$

***Proof.*** Follows from Theorem 3.4.42 using Schur's lemma. □

**Corollary 3.4.44.** *Let $U$, $V$ be two $G$-modules. Then $U \cong V$ if and only if $\chi_U = \chi_V$.*

***Proof.*** Decompose $U$ and $V$ as direct sums of irreducible $G$-modules

$$U = \oplus_1^m (m_i U_i) \quad V = \oplus_1^\ell (n_j V_j).$$

Hence $\chi_U = \sum m_i \chi_{V_i}$ and $\chi_V = \sum n_j \chi_{V_j}$. The equivalence "representation" $\Longleftrightarrow$ "characters" stated by this corollary now follows immediately from Schur's lemma and the previous corollary. □

Thus, the problem of describing the representations of a compact Lie group boils down to describing the characters of its irreducible representations. This problem was completely solved by Hermann Weyl, but its solution requires a lot more work that goes beyond the scope of this book. We will spend the remaining part of this subsection analyzing the equality (d) in Theorem 3.4.42.

Describing the invariants of a group action was a very fashionable problem in the second half of the nineteenth century. Formula (d) mentioned above is a truly remarkable result. It allows (in principle) to compute the maximum number of linearly independent invariant elements.

Let $V$ be a complex $G$-module and denote by $\chi_V$ its character. The complex exterior algebra $\Lambda_c^\bullet V^*$ is a complex $G$-module, as the space of complex multi-linear skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{C}.$$

Denote by $b_k^c(V)$ the complex dimension of the space of $G$-invariant elements in $\Lambda_c^k V^*$. One has the equality

$$b_k^c(V) = \int_G \chi_{\Lambda_c^k V^*} dV_G(g).$$

These facts can be presented coherently by considering the $\mathbb{Z}$-graded vector space

$$\mathcal{I}_c^\bullet(V) := \bigoplus_k \Lambda_{inv}^k V^*.$$

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Its Poincaré polynomial is

$$P_{\mathcal{I}^\bullet_c(V)}(t) = \sum t^k b^c_k(V) = \int_G t^k \boldsymbol{\chi}_{\Lambda^k_c V^*} dV_G(g).$$

To obtain a more concentrated formulation of the above equality we need to recall some elementary facts of linear algebra.

For each endomorphism $A$ of $V$ denote by $\sigma_k(A)$ the trace of the endomorphism

$$\Lambda^k A : \Lambda^k V \to \Lambda^k V.$$

Equivalently, (see Exercise 2.2.25) the number $\sigma_k(A)$ is the coefficient of $t^k$ in the characteristic polynomial

$$\sigma_t(A) = \det(\mathbb{1}_V + tA).$$

Explicitly, $\sigma_k(A)$ is given by the sum

$$\sigma_k(A) = \sum_{1 \le i_1 \cdots i_k \le n} \det\left(a_{i_\alpha i_\beta}\right) \quad (n = \dim V).$$

If $g \in G$ acts on $V$ by $T(g)$, then $g$ acts on $\Lambda^k V^*$ by $\Lambda^k T(g^{-1})^\dagger = \Lambda^k \overline{T(g)}$. (We implicitly assumed that each $T(g)$ is unitary with respect to some $G$-invariant metric on $V$.) Hence

$$\boldsymbol{\chi}_{\Lambda^k_c V^*} = \sigma_k\big(\,\overline{T(g)}\,\big). \tag{3.4.7}$$

We conclude that

$$P_{\mathcal{I}^\bullet_c(V)}(t) = \int_G \sum t^k \sigma_k\big(\,\overline{T(g)}\,\big) dV_G(g) = \int_G \det\big(\,\mathbb{1}_V + t\,\overline{T(g)}\,\big)\, dV_G(g). \tag{3.4.8}$$

Consider now the following situation. Let $V$ be a *complex* $G$-module. Denote by $\Lambda^\bullet_r V$ the space of $\mathbb{R}$-multi-linear, skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{R}.$$

The vector space $\Lambda^\bullet_r V^*$ is a *real* $G$-module. We complexify it, so that $\Lambda^\bullet_r V \otimes \mathbb{C}$ is the space of $\mathbb{R}$-multi-linear, skew-symmetric maps

$$V \times \cdots \times V \to \mathbb{C},$$

and as such, it is a *complex* $G$-module. The *real* dimension of the subspace $\mathcal{I}^k_r(V)$ of $G$-invariant elements in $\Lambda^k_r V^*$ will be denoted by $b^r_k(V)$, so that the Poincaré polynomial of $\mathcal{I}^\bullet_r(V) = \oplus_k \mathcal{I}^k_r$ is

$$P_{\mathcal{I}^\bullet_r(V)}(t) = \sum_k t^k b^r_k(V).$$

On the other hand, $b^r_k(V)$ is equal to the *complex* dimension of $\Lambda^k_r V^* \otimes \mathbb{C}$. Using the results of Subsection 2.2.5 we deduce

$$\Lambda^\bullet_r V \otimes \mathbb{C} \cong \Lambda^\bullet_c V^* \otimes_{\mathbb{C}} \Lambda^\bullet_c \overline{V}^* = \bigoplus_k \left(\oplus_{i+j=k} \Lambda^i_c V^* \otimes \Lambda^j_c \overline{V}^*\right). \tag{3.4.9}$$

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Each of the above summands is a $G$-invariant subspace. Using (3.4.7) and (3.4.9) we deduce

$$\begin{aligned} P_{\mathcal{J}_r^\bullet(V)}(t) &= \sum_k \int_G \sum_{i+j=k} \sigma_i\big(T(g)\big)\sigma_j\big(\overline{T(g)}\big)t^{i+j}\,dV_G(g) \\ &= \int_G \det\big(\mathbb{1}_V + tT(g)\big)\det\big(\mathbb{1}_{\overline{V}} + t\,\overline{T}(g)\big)\,dV_G(g) \\ &= \int_G \big|\det\big(\mathbb{1}_V + tT(g)\big)\big|^2\,dV_G(g). \end{aligned} \tag{3.4.10}$$

We will have the chance to use this result in computing topological invariants of manifolds with a "high degree of symmetry" like, e.g., the complex Grassmannians.

### 3.4.5 *Fibered calculus*

In the previous section we have described the calculus associated to objects defined on a *single manifold*. The aim of this subsection is to discuss what happens when we deal with an entire family of objects parameterized by some smooth manifold. We will discuss only the fibered version of integration. The exterior derivative also has a fibered version but its true meaning can only be grasped by referring to Leray's spectral sequence of a fibration and so we will not deal with it. The interested reader can learn more about this operation from [40], Chapter 3, Sec.5.

Assume now that, instead of a single manifold $F$, we have an entire (smooth) family of them $(F_b)_{b\in B}$. In more rigorous terms this means that we are given a smooth fiber bundle $p : E \to B$ with standard fiber $F$.

On the total space $E$ we will always work with *split coordinates* $(x^i; y^j)$, where $(x^i)$ are local coordinates on the standard fiber $F$, and $(y^j)$ are local coordinates on the base $B$ (the parameter space).

The model situation is the bundle

$$E = \mathbb{R}^k \times \mathbb{R}^m \xrightarrow{p} \mathbb{R}^m = B,\ \ (x, y) \overset{p}{\longmapsto} y.$$

We will first define a fibered version of integration. This requires a fibered version of orientability.

**Definition 3.4.45.** Let $p : E \to B$ be a smooth bundle with standard fiber $F$. The bundle is said to be *orientable* if the following hold.

(a) The manifold $F$ is *orientable*;

(b) There exists an open cover $(U_\alpha)$, and trivializations $p^{-1}(U_\alpha) \xrightarrow{\Psi_\alpha} F \times U_\alpha$, such that the gluing maps

$$\Psi_\beta \circ \Psi_\alpha^{-1} : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta} \quad (U_{\alpha\beta} = U_\alpha \cap U_\beta)$$

are fiberwise orientation preserving, i.e., for each $y \in U_{\alpha\beta}$, the diffeomorphism

$$F \ni f \mapsto \Psi_{\alpha\beta}(f, y) \in F$$

preserves any orientation on $F$. □

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**Exercise 3.4.46.** If the base $B$ of an orientable bundle $p : E \to B$ is orientable, then so is the total space $E$ (as an abstract smooth manifold). $\square$

☞**Important convention.** Let $p : E \to B$ be an orientable bundle with oriented basis $B$. The *natural orientation* of the total space $E$ is defined as follows.

If $E = F \times B$ then the orientation of the tangent space $T_{(f,b)}E$ is given by $\Omega_F \times \omega_B$, where $\omega_F \in \det T_f F$ (respectively $\omega_B \in \det T_b B$) defines the orientation of $T_f F$ (respectively $T_b B$).

The general case reduces to this one since any bundle is locally a product, and the gluing maps are fiberwise orientation preserving. This convention can be briefly described as

$$\text{orientation total space} = \text{orientation fiber} \wedge \text{orientation base}.$$

The natural orientation can thus be called the *fiber-first* orientation. In the sequel all orientable bundles will be given the fiber-first orientation. $\square$

Let $p : E \to B$ be an orientable fiber bundle with standard fiber $F$.

**Proposition 3.4.47.** *There exists a linear operator*

$$p_* = \int_{E/B} : \Omega^{\bullet}_{cpt}(E) \to \Omega^{\bullet - r}_{cpt}(B), \quad r = \dim F,$$

*uniquely defined by its action on forms supported on domains $D$ of split coordinates*

$$D \cong \mathbb{R}^r \times \mathbb{R}^m \xrightarrow{p} \mathbb{R}^m, \quad (x; y) \mapsto y.$$

*If $\omega = f dx^I \wedge dy^J$, $f \in C^{\infty}_{cpt}(\mathbb{R}^{r+m})$, then*

$$\int_{E/B} = \begin{cases} 0 \, , & |I| \neq r \\ \left( \int_{\mathbb{R}^r} f dx^I \right) dy^J \, , & |I| = r \end{cases}.$$

*The operator $\int_{E/B}$ is called the* integration-along-fibers *operator.* $\square$

The proof goes *exactly* as in the non-parametric case, i.e., when $B$ is a point. One shows using partitions of unity that these local definitions can be patched together to produce a well defined map

$$\int_{E/B} : \Omega^{\bullet}_{cpt}(E) \to \Omega^{\bullet - r}_{cpt}(B).$$

The details are left to the reader.

**Proposition 3.4.48.** *Let $p : E \to B$ be an orientable bundle with an $r$-dimensional standard fiber $F$. Then for any $\omega \in \Omega^*_{cpt}(E)$ and $\eta \in \omega^*_{cpt}(B)$ such that* $\deg \omega + \deg \eta = \dim E$ *we have*

$$\int_{E/B} d_E \omega = (-1)^r d_B \int_{E/B} \omega.$$

*If $B$ is oriented and $\omega$, $\eta$ are as above then*

$$\int_E \omega \wedge p^*(\eta) = \int_B \left( \int_{E/B} \omega \right) \wedge \eta. \qquad \text{(Fubini)}$$

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The last equality implies immediately the *projection formula*

$$p_*(\omega \wedge p^*\eta) = p_*\omega \wedge \eta. \tag{3.4.11}$$

***Proof.*** It suffices to consider only the model case

$$p : E = \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m = B, \ \ (x; y) \xrightarrow{p} y,$$

and $\omega = f dx^I \wedge dy^J$. Then

$$d_E\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge dy^J + (-1)^{|I|} \sum_j \frac{\partial f}{\partial y^j} dx^I \wedge dy^j \wedge dy^J.$$

$$\int_{E/B} d_E\omega = \left( \int_{\mathbb{R}^r} \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) + (-1)^{|I|} \left( \int_{\mathbf{R}^r} \sum_j \frac{\partial f}{\partial y^j} dx^I \right) \wedge dy^j \wedge dy^J.$$

The above integrals are defined to be zero if the corresponding forms do not have degree $r$. Stokes' formula shows that the first integral is always zero. Hence

$$\int_{E/B} d_E\omega = (-1)^{|I|} \frac{\partial}{\partial y^j} \left( \int_{\mathbf{R}^r} \sum_j dx^I \right) \wedge dy^j \wedge dy^J = (-1)^r d_B \int_{E/B} \omega.$$

The second equality is left to the reader as an exercise. □

**Exercise 3.4.49. (Gelfand-Leray).** Suppose that $p : E \to B$ is an oriented fibration, $\omega_E$ is a volume form on $E$, and $\omega_B$ is a volume form on $B$.

(a) Prove that, for every $b \in B$, there exists a unique volume form $\omega_{E/B}$ on $E_b = p^{-1}(b)$ with the property that, for every $x \in E_b$, we have

$$\omega_E(x) = \omega_{E/B}(x) \wedge (p^*\omega_B)(x) \in \Lambda^{\dim E} T_x^* E.$$

This form is called the *Gelfand-Leray residue* of $\omega_E$ rel $p$.

(b) Prove that for every compactly supported smooth function $f : E \to \mathbb{R}$ we have

$$\big(p_*(f\omega_E)\big)_b = \left( \int_{E_b} f\omega_{E/B} \right) \omega_B(b), \forall b \in B, \ \ \int_E f\omega_E = \int_B \left( \int_{E_b} f\omega_{E/B} \right) \omega_B.$$

(c) Consider the fibration $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \overset{p}{\mapsto} t = ax + by$, $a^2 + b^2 \neq 0$. Compute the Gelfand-Leray residue $\frac{dx \wedge dy}{dt}$ along the fiber $p(x, y) = 0$. □

**Definition 3.4.50.** A $\partial$-bundle is a collection $(E, \partial E, p, B)$ consisting of the following.

(a) A smooth manifold $E$ with boundary $\partial E$.
(ii) A smooth map $p : E \to B$ such that the restrictions $p : \operatorname{Int} E \to B$ and $p : \partial E \to B$ are smooth bundles.

The standard fiber of $p : \operatorname{Int} E \to B$ is called the *interior fiber*. □

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One can think of a $\partial$-bundle as a smooth family of manifolds with boundary.

**Example 3.4.51.** The projection

$$p : [0,1] \times M \to M \quad (t; m) \mapsto m,$$

defines a $\partial$-bundle. The interior fiber is the open interval $(0,1)$. The fiber of $p : \partial(I \times M) \to M$ is the disjoint union of two points. □

**Standard Models** A $\partial$-bundle is obtained by gluing two types of local models.

- *Interior models* $\mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^m$
- *Boundary models* $\mathbf{H}_+^r \times \mathbb{R}^m \to \mathbb{R}^m$, where

$$\mathbf{H}_+^r := \left\{ (x^1, \cdots, x^r) \in \mathbb{R}^r;\ \ x^1 \geq 0 \right\}.$$

**Remark 3.4.52.** Let $p : (E, \partial E) \to B$ be a $\partial$-bundle. If $p : \operatorname{Int} E \to B$ is orientable and the basis $B$ is oriented as well, then on $\partial E$ one can define two orientations.

(i) The fiber-first orientation as the total space of an oriented bundle $\partial E \to B$.
(ii) The induced orientation as the boundary of $E$.

These two orientations coincide. □

**Exercise 3.4.53.** Prove that the above orientations on $\partial E$ coincide. □

**Theorem 3.4.54.** *Let $p : (E, \partial E) \to B$ be an orientable $\partial$-bundle with an $r$-dimensional interior fiber. Then for any $\omega \in \Omega^\bullet_{cpt}(E)$ we have*

$$\int_{\partial E/B} \omega = \int_{E/B} d_E \omega - (-1)^r d_B \int_{E/B} \omega \quad \text{(Homotopy formula)}. \qquad \square$$

The last equality can be formulated as

$$\int_{\partial E/B} = \int_{E/B} d_E - (-1)^r d_B \int_{E/B}.$$

This is “the mother of all homotopy formulæ”. It will play a crucial part in Chapter 7 when we embark on the study of DeRham cohomology.

**Exercise 3.4.55.** Prove the above theorem. □

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