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TORIC SINGULARITIES

By Kazuya Kato

By toric singularity, we usually mean a singularity which appears on the ambient variety of a toroidal embedding. This notion has been considered for varieties over a base field. The aim of this note is to give a good definition of toric singularity with no reference to a base scheme. The "Jungian domain" defined by Abhyankar, which appeared at a key step of his work on the resolution of singularity of arithmetic surfaces, is an example of a toric singularity in our sense.

For our definition of toric singularity, we use "logarithmic structure" of Fontaine-Illusie, which can be regarded as "toric structure without base." When we talk about toric singularity, we always fix a logarithmic structure on our scheme. We shall define the regularity in the logarithmic sense, of a point on a scheme with a logarithmic structure. In our definition, toric singularity will be the same thing as the regularity in the logarithmic sense.

The principal line of this note is as follows. After a short introduction to the logarithmic structure in §1, we give in §2 our definition of toric singularity. The rest of this note is devoted to an attempt to generalize (i) results on classical regularity and (ii) results in the theory of toroidal embeddings, to our toric singularity. Concerning (i), we give logarithmic generalizations of the well known explicit description of a complete regular local ring (cf. §3), the fact that a closed immersion between regular schemes is a regular immersion (cf. §4), and the fact that a smooth scheme over a regular scheme is regular (cf. §8). Concerning (ii), we show that our toric singularity is Cohen-Macaulay (proven by Hochster [Ho] for the classical toric singularity) and normal (§4), and as the heart of this note, we show that some kind of cone decomposition is associated to a scheme with a log. str. having only toric singularities (§10) by which we can describe explicitly resolution of toric singularity (§10), dualizing sheaves (§11), dualizing complexes of some subschemes (§11), etc., as in the classical theory [KKMS] [Od] [Is] of toroidal embeddings. In the last section §12, we give comments on the method of Abhyankar on the desingularization of arithmetic surfaces.

The author expects that arithmetic algebraic geometry works sometimes more nicely with toric singularity than with classical regularity.

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1. Logarithmic Structures of Fontaine-Illusie.

(1.1) In this note, a ring (resp. a monoid) means a commutative ring (resp. monoid) having a unit element 1. A homomorphism of rings (resp. monoids) is assumed to preserve 1. A subring (resp. submonoid) is assumed to contain 1 of the total ring (resp. monoid). For a ring (resp. monoid) P, P^x denotes the group of invertible elements of P. For a monoid P, P^{gp} denotes the group $\{ab^{-1}; a, b \in P\}$ ($a_1 b_1^{-1} = a_2 b_2^{-1}$ if and only if $a_1 b_2 c = a_2 b_1 c$ for some $c \in P$).

A monoid P is said to be integral if $ab = ac$ implies $b = c$ in P, i.e. if $P \to P^{gp}$ is injective. A monoid P is said to be saturated if P is integral and has the following property (when regarded as a submonoid of P^{gp}): If $a \in P^{gp}$ and $a^n \in P$ for some $n \geq 1$, then $a \in P$.

For a ring R and a monoid P, $R[P]$ denotes the "monoid ring," and I_p denotes the ideal of $\mathbf{Z}[P]$ generated by $P \setminus P^x$. If $P^x = \{1\}$, $R[[P]]$ denotes $\varprojlim_n R[P]/I_P^n R[P]$.

We regard the set $\mathbf{N} = \{0, 1, 2, \ldots\}$ of natural numbers always as a monoid with respect to addition.

(1.2) Now we introduce the notion "logarithmic structure" found by Fontaine and Illusie. (There is a different but closely related formulation of logarithmic structures due to Deligne and to Faltings. Cf. [De] [Fa] [Ka1].)

A pre-logarithmic structure on a scheme X is a sheaf of monoids M on X endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$ where $\mathcal{O}_X$ is regarded as a monoid with respect to the multiplicative law.

A pre-log. str. (m, α) is called a log. str. if

$$\alpha^{-1}(\mathcal{O}_X^x) \overset{\simeq}{\to} \mathcal{O}_X^x \qquad \text{via } \alpha.$$

A scheme endowed with a log. str. is called a logarithmic scheme. A morphism of log. schemes is defined in a natural way.

The simplest log. str. $M = \mathcal{O}_X^x$ with the identity map α is called the trivial log. str. For a log. str. M, we regard $\mathcal{O}_X^x$ as a subsheaf of M by identifying it with $\alpha^{-1}(\mathcal{O}_X^x)$.

(1.3) For a pre-log. str. (M, α) on X, we define the log. str. $\tilde{M}$ associated to M to be the push out of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^x) & \overset{\subseteq}{\to} & M \\ \alpha \downarrow & & \\ \mathcal{O}_X^x & & \end{array}$$

in the category of sheaves of monoids on X, which is endowed with the map $\tilde{M} \to \mathcal{O}_X$ induced by $\alpha : M \to \mathcal{O}_X$.

(1.4) For a morphism of schemes $f : Y \to X$ and for a log. str. M on X, we define the inverse image f^*M to be the log. str. on Y associated to be the pre-log.

str. $f^{-1}(M)$. Here $f^{-1}(M)$ denotes the sheaf theoretic inverse image of M and is endowed with $f^{-1}(M) \to f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$.

(1.5) In this paper we mainly consider log. schemes (X, M) satisfying the following condition (S).

(S) : X is locally Noetherian, and there exists an open covering $\{U_\lambda\}_\lambda$ of X, finitely generated saturated monoids P_λ, and homomorphisms $h_\lambda : P_\lambda \to \mathcal{O}_{U_\lambda}$ (P_λ is regarded here as a constant sheaf) such that $M|_{U_\lambda}$ is isomorphic to the log. str. associated to the pre. log. str. (P_λ, h_λ) for each λ.

(1.6) LEMMA. *Let (X, M) be a log. scheme satisfying the condition (S). (In fact the locally Noetherian property of X in (S) is not necessary in this lemma.) Let $x \in X$ and let $P = M_x / \mathcal{O}^x_{X,x}$. Then:*

(1) *P is a finitely generated saturated monoid.*

(2) *There exists an open neighborhood U of x and a homomorphism $\varphi : P \to M_U$ such that the induced composite map $P \xrightarrow{\varphi_x} M_x \to P$ is the identity. Here M_U denotes the restriction of M to U.*

(3) *For U and φ as in (2), there exists an open neighborhood $V \subset U$ of x such that M_V is the log. str. associated to the pre-log. str. $P \xrightarrow{\varphi} M_V \to \mathcal{O}_V$.*

Proof. Exercise. (For the existence of φ, notice that P^{gp} is a free abelian group, and hence $M^{gp}_x \to P^{gp}$ has a section.)

(1.7) *Example.* Let k be a field, X a k-variety, and U an open subscheme of X. Define a log. str. M on X by

$$M = \{ f \in \mathcal{O}_X ; f \text{ is invertible on } U \}$$

with $\alpha : M \to \mathcal{O}_X$ the inclusion map. Then, $U \xhookrightarrow{} X$ is a toroidal embedding without self-intersection in the sense of [KKMS], Ch. II, if and only if: (X, M) satisfies the condition (S) and we can take P_λ in that condition so that the morphism $X \to Spec(k[P_\lambda])$ is etale for each λ.

(1.8) *Remark.* In this note, logarithmic structures are defined on the Zariski sites of schemes. We could consider also logarithmic structures on the etale sites as we did in [Ka1]. The etale site formulation is useful for example when we consider toroidal embeddings "with self-intersection," but would make the arguments in this note more complicated; so we do not adopt it in this note.

2. Definition of Toric Singularity.

(2.1) *Definition.* Let (X, M) be a log. scheme satisfying the condition (S) in (1.5). We say (X, M) is regular (or logarithmically regular to avoid confusion with the usual regularity) at x, or with (at worst) toric singularity at x, if the following

two conditions are satisfied. Let $I(x, M)$ be the ideal of $\mathcal{O}_{X,x}$ generated by the image $M_x \setminus \mathcal{O}_{X,x}^x$.

   (i) $\mathcal{O}_{X,x}/I(x, M)$ is a regular local ring.

   (ii) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/I(x, M)) + rank(M_x^{gp}/\mathcal{O}_{X,x}^x)$.

We say (X, M) is regular if (X, M) is regular at all $x \in X$.

   (2.2) *Example.*

   (1)   if M is the trivial log. str., then (X, M) is regular at x if and only if X is regular at x in the classical sense.

   (2)   If X is the ambient variety of a toroidal embedding without self-intersection over a field k, and if M is the corresponding log. str. (1.7), then (X, M) is regular.

   In (2.1)(ii), the inequality $\leq$ always holds. That is:

   (2.3) LEMMA. *Let (X, M) be a log. scheme satisfying the condition (S) and let $x \in X$. Then,*

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x}/I(x, M)) + rank(M_x^{gp}/\mathcal{O}_{X,x}^x).$$

   Proof. Write A, I, m_A, k instead of $\mathcal{O}_{X,x}$, $I(x, M)$, m_x, $\mathcal{O}_{X,x}/m_x$, respectively. Let $r = \dim(A/I)$ and take $a_1, \ldots, a_r \in m_A$ such that $A/(I, a_1, \ldots, a_r)$ is of finite length. Let $P = M_x/\mathcal{O}_{X,x}^x$ and take a section $\varphi : P \to M_x$ of $M_x \to P$. Let $\hat{A}$ be the completion of A. If A contains a field, a section of $\hat{A} \to k$ and the map $P \xrightarrow{by \ \varphi} \mathcal{O}_{X,x}$ induce a finite homomorphism

$$\psi : k[[P]][[T_1, \ldots, T_r]] \to \hat{A}; T_i \to a_i.$$

Hence we have

$$\dim(A) \leq \dim(k[[P]][[T_1, \ldots, T_r]]) = r + \dim(P^{gp}),$$

which proves (2.3) in this case. If A does not contain a field, take a complete discrete valuation ring R with residue field k in which the prime number $p = char(k)$ is a prime element, and take a lifting $R \to \hat{A}$ of $R \to k$. Then we obtain a finite R-homomorphism

$$\psi : R[[P]][[T_1, \ldots, T_r]] \to \hat{A}; T_i \to a_i.$$

We show that ψ is not injective. Then, we will have

$$\dim(A) \leq \dim(R[[P]][[T_1, \ldots, T_r]]) - 1 = r + rank(P^{gp}),$$

which will prove (2.3). Now if φ is injective, there exists a prime ideal $\mathfrak{p}$ of $\hat{A}$ such that $\psi^{-1}(\mathfrak{p}) = (P \setminus \{1\}, T_1, \ldots, T_r)$. Then $\mathfrak{p} \supset (I, a_1, \ldots, a_r)$ and hence $\mathfrak{p} = m_{\hat{A}}$. Thus we have $\psi^{-1}(\mathfrak{p}) = (m_R, P \setminus \{1\}, T_1, \ldots, T_r)$, a contradiction.

3. Explicit Description of Completed Toric Singularities. In this section, let (X, M) be a log. scheme satisfying (S) and let $x \in X$.

(3.1) THEOREM. (1) (X, M) *is regular at x if and only if there exists a complete regular local ring R, a finitely generated saturated monoid P such that $P^x = \{1\}$, and an isomorphism*

$$R[[P]]/(\theta) \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$$

with $\theta \in R[[P]]$ satisfying the following conditions (i) (ii).
   (i)   *The constant term of θ belongs to $m_R \setminus m_R^2$.*
   (ii)   *The inverse image of M on $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$ is induced by the map $P \to \hat{\mathcal{O}}_{X,x}$.*

(2) *Assume $\mathcal{O}_{X,x}$ contains a field. Then, (X, M) is regular at x if and only if there exists a field k, a finitely generated saturated monoid P such that $P^x = \{1\}$, and an isomorphism*

$$R[[P]][[T_1, \ldots, T_r]] \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$$

for some $r \geq 0$ satisfying the condition (ii) *in* (1).

A more precise version of (3.1) is as follows. Assume $\mathcal{O}_{X,x}/I(x, M)$ is regular. Take a finitely generated saturated monoid P and a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$ such that $\varphi^{-1}(\mathcal{O}_{X,x}^x) = \{1\}$ (so $P^x = \{1\}$) which induces M at x. Take $t_1, \ldots, t_r \in \mathcal{O}_{X,x}$ such that $(t_i \bmod I(x, M))_{1 \leq i \leq r}$ is a regular system of parameters of $\mathcal{O}_{X,x}/I(x, M)$.

(3.2) THEOREM. *Let the assumptions and the notation be as above.*

(1)   *Assume $\mathcal{O}_{X,x}$ contains a field and take a subfield k of $\hat{\mathcal{O}}_{X,x}$ such that $k \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}/\hat{m}_x$. Then, (X, M) is regular at x if and only if the surjective homormophism*

$$\psi : k[[P]][[T_1, \ldots, T_r]] \to \hat{\mathcal{O}}_{X,x}; T_i \mapsto t_i$$

is an isomorphism.

(2)   *Assume $\mathcal{O}_{X,x}$ does not contain a field. Take a complete discretion valuation ring R in which $p = char(\mathcal{O}_{X,x}/m_x)$ is a prime element, and a homomorphism $R \to \hat{\mathcal{O}}_{X,x}$ which induces $R/pR \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}/\hat{m}_x$. Then, (X, M) is regular at x if and only if the kernel of the surjective homomorphism*

$$\psi : R[[P]][[T_1, \ldots, T_r]] \to \hat{\mathcal{O}}_{X,x}$$

is generated by an element θ such that

$$\theta \equiv p \bmod (P \setminus \{1\}, T_1, \ldots, T_r).$$

(3.3) We prove Thm. (3.1) and Thm. (3.2). The "if" parts of these theorems are easily seen. We prove the "only if" parts. Assume (X, M) is regular at x, and take P, $\varphi : P \to \mathcal{O}_{X,x}$ and $t_1, \ldots, t_r$ as in (3.2). Assume first $\mathcal{O}_{X,x}$ contains a field and let $k \to \hat{\mathcal{O}}_{X,x}$ be as in (3.2) (1). Then, since

$$\dim (k[[P]][[T_1, \ldots, T_r]]) = \dim (P^{gp}) + r = \dim (\mathcal{O}_{X,x}) = \dim (\hat{\mathcal{O}}_{X,x}),$$

and since $k[[P]][[T_1, \ldots, T_r]]$ is an integral domain, the surjective map ψ in (3.2) (1) is an isomorphism.

Next assume that $\mathcal{O}_{X,x}$ does not contain a field, and let $R \to \hat{\mathcal{O}}_{X,x}$ be as in (3.2) (2). Since the map ψ sends the ideal $(P \setminus \{1\}, T_1, \ldots, T_r)$ of $R[[P]][[T_1, \ldots, T_r]]$ onto $\hat{m}_x$, there exists $\theta \in Ker(\psi)$ such that

$$\theta \equiv p \bmod (P \setminus \{1\}, T_1, \ldots, T_r).$$

The following lemma shows that $R[[P]][[T_1, \ldots, T_r]]/(\theta)$ is an integral domain, and hence by comparing the dimensions again, we see that ψ is an isomorphism.

Finally the "only if" parts of (3.1) follow from those of (3.2) easily.

(3.4) LEMMA. *Let R be a ring, π be a nonzero-divisor of R such that $R/(\pi)$ is an integral domain, let P be a finitely generated integral monoid such that $P^x = \{1\}$ and P^{gp} is torsion free. Let θ be an element of $R[[P]]$ such that $\theta \equiv \pi \bmod (P \setminus \{1\})$. Then, $R[[P]]/(\theta)$ is an integral domain.*

Proof. Take an injective homomorphism $P \overset{\subseteq}{\to} \mathbf{N}^s$ for some $s \geq 0$. Since $R[[\mathbf{N}^s]]/(\theta)$ is an integral domain (because $gr_{(\mathbf{N}^s \setminus \{1\})}(R[[\mathbf{N}^s]]/(\theta)) \cong (R/\pi R)[T_1, \ldots, T_r]$ is an integral domain), it suffices to show that $R[[P]]/(\theta) \to R[[\mathbf{N}^s]]/(\theta)$ is injective. Hence we are reduced to

(3.5) LEMMA. *Let the assumptions be as in (3.4) except that we do not need here the integrality of $R/(\pi)$, let Q be a finitely generated integral monoid such that $Q^x = \{1\}$, and let $P \to Q$ be an injective homomorphism. Then $R[[P]]/(\theta) \to R[[Q]]/(\theta)$ is injective.*

Proof. Assume

(*)
$$\left(\sum_{\alpha \in Q} x_\alpha \alpha \right) \theta = \sum_{\beta \in P} y_\beta \beta, \qquad x_\alpha, y_\beta \in R$$

in $R[[Q]]$ ($\sum$ can be infinite sums). Assume the set

$$\Phi = \{\alpha \in Q \setminus P; x_\alpha \neq 0\}$$

is nonempty, and let (α_0) be maximal in the set of ideals of $\mathbf{Z}[Q]$ of the form (α) such that $\alpha \in \Phi$. Then, by looking at the coefficients at α_0 of (*), we see

$$0 = \pi x_{\alpha_0} + \sum_{\substack{(\alpha,\beta) \in Q \times P \\ \alpha\beta = \alpha_0, \beta \neq 1}} x_\alpha z_\beta$$

for some $z_\beta \in R$. Since $\pi x_{\alpha_0} \neq 0$, we have $x_\alpha \neq 0$ for some $\alpha \in Q$ such that there exists $\beta \in P$, $\beta \neq 1$ satisfying $\alpha\beta = \alpha_0$. Then $\alpha \in \Phi$ and $(\alpha) \supsetneq (\alpha_0)$, a contradiction.

4. Some Properties of Toric Singularity. In this section, we prove the following two results.

(4.1) THEOREM. *Let (X, M) be a log. scheme satisfying (S) and assume (X, M) is regular. Then the scheme X is Cohen-Macaulay and is normal.*

(Cf. [Ho] for the Cohen-Macaulay property of the "classical toric singularities.")

(4.2) THEOREM. *Let (X, M), (X', M') be log. schemes satisfying (S), and assume that they are regular. Let $i : X' \to X$ be a closed immersion and assume $i^* M_X \cong M_{X'}$. Then i is a regular immersion.*

Proof of (4.1). By (3.1) (1), the Cohen-Macaulay property is reduced to the fact ([Ho]) that $k[P]$ is Cohen-Macaulay for any field k and for any finitely generated saturated monoid P.

For the normality, recall that a Cohen-Macaulay Noetherian ring is normal if it is regular in codimension ≤ 1. Let $x \in X$, $\dim (\mathcal{O}_{X,x}) \leq 1$. Then, $M_x/\mathcal{O}_{X,x}^x \cong \mathbf{N}$ or $\{1\}$, and hence $\mathcal{O}_{X,x}$ is regular.

Proof of (4.2). Let $x \in X'$, and let J be the kernel of $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x}$. Then, since $i^* M_X \cong M_{X'}$, we have

$$\mathcal{O}_{X,x}/(J + I(x, M_X)) \xrightarrow{\cong} \mathcal{O}_{X',x}/I(x, M_{X'}).$$

Since $\mathcal{O}_{X,x}/I(x, M_X)$ and $\mathcal{O}_{X',x}/I(x, M_{X'})$ are regular, we see that there exists $t_1, \ldots, t_r \in J$ whose images in $\mathcal{O}_{X,x}/I(x, M_X)$ form a part of a regular system of parameters and generate the kernel of $\mathcal{O}_{X,x}/I(x, M_X) \to \mathcal{O}_{X',x}/I(x, M_{X'})$. By (3.2), $t_1, \ldots, t_r$ form a regular sequence of $\mathcal{O}_{X,x}$.

5. Geometry of Monoids. (I).   In this section, we fix some terminologies, which we use in later sections, concerning the "commutative algebra for monoids."

(5.1) *Definition.* A subset I of a monoid P is called an ideal of P if $PI \subset I$. An ideal I of P is called a prime ideal if its complement $P \setminus I$ is a submonoid of P. We denote by $Spec(P)$ the set of all prime ideals of a monoid P.

For example, the empty set ϕ and the set $P \setminus P^x$ are prime ideals of P, but P itself is not a prime ideal of P.

For a homomorphism $h : P \to Q$ of monoids, we have a canonical map $Spec(Q) \to Spec(P); \mathfrak{p} \to h^{-1}(\mathfrak{p})$.

(5.2) *Definition.* For a submonoid S of a monoid P, we define the monoid $S^{-1}P$ by $S^{-1}P = \{s^{-1}a; a \in P, s \in S\}$ where

$$s_1^{-1}a_1 = s_2^{-1}a_2 \Leftrightarrow t \in S; ts_1a_2 = ts_2a_1.$$

(5.3) For $\mathfrak{p} \in Spec(P)$, we define

$$P_{\mathfrak{p}} = S^{-1}P \text{ where } S = P \setminus \mathfrak{p}.$$

Then for $\mathfrak{p} \in Spec(P)$, the subset $\{q \in Spec(P); q \subset \mathfrak{p}\}$ of $Spec(P)$ is identified with $Spec(P_{\mathfrak{p}})$ whereas the subset $\{q \in Spec(P); q \supset \mathfrak{p}\}$ is identified with $Spec(P \setminus \mathfrak{p})$.

(5.4) *Definition.* For a monoid P, we define $\dim(P)$ to be the maximal length of a sequence of prime ideals $\mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \cdots \supsetneqq \mathfrak{p}_r$ of P. If such a maximum does not exist, we define $\dim(P) = \infty$.

(5.5) PROPOSITION. *Let P be a finitely generated monoid. Then:*

(1) $Spec(P)$ *is a finite set.*

(2) $\dim(P)$ *is finite and is equal to the rank of the $\mathbf{Z}$-module P^{gp}/P^x.*

(3)   *For $\mathfrak{p} \in Spec(P)$, we have*

$$\dim(P \setminus \mathfrak{p}) + \dim(P_{\mathfrak{p}}) = \dim(P).$$

This follows from the geometric interpretation that a prime ideal of P is nothing but the complement of a "face" ([KKMS], [Od]) of P.

(5.6) LEMMA. *For a finitely generated monoid P, any ideal I of P is finitely generated, i.e. there exists a finite family $a_1, \ldots, a_n$ of elements of I such that $I = \cup_{i=1}^{n} Pa_i$.*

This follows from the fact that $\mathbf{Z}[P]$ is a Noetherian ring.

(5.7) *Definition.* Let P be an integral monoid. Then, a subset I of P^{gp} is called a fractional ideal of P if $PI \subset I$ and $I \subset Pa$ for some $a \in P^{gp}$. For a fractional ideal I of P, let $I^{-1} = \{a \in P^{gp}; aI \subset P\}$. (Then, I^{-1} is also a fractional ideal of P.)

(5.8) PROPOSITION. *Let P be a finitely generated saturated monoid.*

(1)　*We have $P = \cap_{\mathfrak{p}} P_{\mathfrak{p}}$ where $\mathfrak{p}$ ranges over all prime ideals of P such that* $\dim(P_{\mathfrak{p}}) = 1$.

(2)　*More generally, for any fractional ideal I of P, we have $(I^{-1})^{-1} = \cap_{\mathfrak{p}} P_{\mathfrak{p}} I$ where $\mathfrak{p}$ ranges as in* (1).

Proof. Exercise.

6. Other Characterizations of Toric Singularities.

In this section, let (X, M) be a log. scheme satisfying (S) and let $x \in X$.

(6.1) THEOREM. *Let P be an integral monoid, let $\varphi : P \to \mathcal{O}_{X,x}$ be a homomorphism, and assume that φ induces M on $\mathrm{Spec}(\mathcal{O}_{X,x})$. Let I_φ be the ideal of $\mathbf{Z}[P]$ generated by $\{\alpha \in P; \varphi(\alpha) \notin \mathcal{O}_{X,x}^{\times}\}$. Assume $\mathcal{O}_{X,x}/I(x, M)$ is a regular local ring. Then the following five conditions* (i)–(v) *are equivalent.*

(i)　(X, M) *is regular at x.*

(ii)　*For any injective homomorphism $P \to Q$ with Q an integral monoid, and for any ideal J of Q, we have*

$$\mathrm{Tor}_i^{\mathbf{Z}[P]}(\mathbf{Z}[Q]/(J), \mathcal{O}_X) = 0 \text{ for all } i \geq 1.$$

(iii)　*For any ideal J of P, we have*

$$\mathrm{Tor}_i^{\mathbf{Z}[P]}(\mathbf{Z}[P]/(J), \mathcal{O}_{X,x}) = 0 \text{ for all } i \geq 1.$$

(iv)　$\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathbf{Z}[P]/I_\varphi, \mathcal{O}_{X,x}) = 0.$

(v)　$gr_{I_\varphi}(\mathbf{Z}[P]) \otimes_{\mathbf{Z}[P]/I_\varphi} \mathcal{O}_{X,x}/I(x, M) \xrightarrow{\cong} gr_{I(x,M)}(\mathcal{O}_{X,x}).$
(Note $I(x, M) = I_\varphi \mathcal{O}_{X,x}$.)

In this section, we prove the equivalence of the conditions (i), (iii), (iv) and (v) in (6.1). The equivalence with (ii) will be proved in the next section.

(6.2) THEOREM. *Let P be a finitely generated saturated monoid, let $\varphi : P \to \mathcal{O}_{X,x}$ be a homomorphism, and assume that φ induces M on $\mathrm{Spec}(\mathcal{O}_{X,x})$ and that $\varphi^{-1}(\mathcal{O}_{X,x}^{\times}) = \{1\}$. Assume $\mathcal{O}_{X,x}$ contains a field k. Then, (X, M) is regular at x if*

and only if $\mathcal{O}_{X,x}/I(x, M)$ is a regular local ring and the map $k[P] \to \mathcal{O}_{X,x}$ induced by φ is flat.

Proof of (6.1) *except the part* (ii). By replacing P by $P_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal $\varphi^{-1}(m_x)$ of P, we may assume that $P^x = \varphi^{-1}(\mathcal{O}_{X,x}^x)$. Then, $P/P^x \cong M_x/\mathcal{O}_{X,x}^x$ and from this we have $P = P^x \times P_1$ for some submonoid P_1 of P (indeed, take a section s of $P^{gp} \to P^{gp}/P^x$ and put $P_1 = s^{-1}(P/P^x)$). By replacing P by P_1, we may assume that P is a finitely generated saturated monoid and $\varphi^{-1}(\mathcal{O}_{X,x}^x) = \{1\}$ (so $P^x = \{1\}$). In the following we assume P satisfies these conditions. (In particular $I_\varphi = I_P =$ the ideal of $\mathbf{Z}[P]$ generated by $P \setminus P^x = P \setminus \{1\}$, cf. (1.1).)

We prove (i) $\Rightarrow$ (iii). By (3.1) and (3.2), $P \to \mathcal{O}_{X,x}$ is extended to an isomorphism $R[[P]]/(\theta) \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$ where R is a complete regular local ring and θ is an element of $R[[P]]$ such that $\theta \bmod (P \setminus \{1\})$ belongs to $m_R \setminus m_R^2$. In what follows, $\otimes^L$ means the tensor product in the derived category. For any ideal J of P, we have

$$
(\mathbf{Z}[P]/(J)) \overset{L}{\underset{\mathbf{Z}[P]}{\bigotimes}} \hat{\mathcal{O}}_{X,x} \cong (\mathbf{Z}[P]/(J)) \overset{L}{\underset{\mathbf{Z}[P]}{\bigotimes}} R[P] \overset{L}{\underset{R[P]}{\bigotimes}} \hat{\mathcal{O}}_{X,x}
$$

$$
\cong (R[P]/(J)) \overset{L}{\underset{R[P]}{\bigotimes}} \hat{\mathcal{O}}_{X,x}
$$

where the last isomorphism follows from

$$
(\mathbf{Z}[P]/(J)) \overset{L}{\underset{\mathbf{Z}[P]}{\bigotimes}} R[P] \cong (\mathbf{Z}[P]/(J)) \overset{L}{\underset{\mathbf{Z}[P]}{\bigotimes}} \mathbf{Z}[P] \overset{L}{\underset{\mathbf{Z}}{\bigotimes}} R
$$

$$
\cong (\mathbf{Z}[P]/(J)) \overset{L}{\underset{\mathbf{Z}}{\bigotimes}} R
$$

$$
\cong (R[P]/(J))
$$

(by the flatness of the $\mathbf{Z}$-module $\mathbf{Z}[P]/(J)$). Computing $R[P]/(J) \otimes_{R[P]}^L \hat{\mathcal{O}}_{X,x}$ by using the flat resolution

$$
0 \to R[[P]] \xrightarrow{\theta} R[[P]] \to \hat{\mathcal{O}}_{X,x} \to 0
$$

of the $R[P]$-module $\hat{\mathcal{O}}_{X,x}$, we are reduced to showing that the image of θ in $R[[P]]/(J)$ is a nonzero-divisor. Since $R[[P]]/(J)$ is the inverse limit of $R[P]/(I_P^n R[P] + (J))$, we are reduced to (6.3) below.

Next, the implication (iii) $\Rightarrow$ (iv) is clear.

We prove the implication (iv) $\Rightarrow$ (v). Since I_P^n/I_P^{n+1} are flat over $\mathbf{Z}[P]/I_P \cong \mathbf{Z}$, we have by (iv),

$$Tor_1^{\mathbf{Z}[P]}(I_P^n/I_P^{n+1}, \mathcal{O}_{X,x}) = 0 \text{ for all } n \geq 0.$$

Hence

$$Tor_1^{\mathbf{Z}[P]}(\mathbf{Z}[P]/I_P^n, \mathcal{O}_{X,x}) = 0 \text{ for all } n \geq 0.$$

From this and the exact sequence

$$0 \to I_P^n/I_P^{n+1} \to \mathbf{Z}[P]/I_P^{n+1} \to \mathbf{Z}[P]/I_P^n \to 0,$$

we have

$$(I_P^n/I_P^{n+1}) \bigotimes_{\mathbf{Z}[P]} \mathcal{O}_{X,x} \cong I_P^n \mathcal{O}_{X,x}/I_P^{n+1} \mathcal{O}_{X,x} \text{ for all } n \geq 0.$$

This implies (v).

Finally we show (v) $\Rightarrow$ (i). Take a surjective homomorphism

$$\psi : R[[P]]/(\theta) \to \hat{\mathcal{O}}_{X,x}$$

extending $\varphi : P \to \mathcal{O}_{X,x}$, which induces

$$R/(\bar{\theta}) \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}/I(x,M)\hat{\mathcal{O}}_{X,x},$$

where R is a complete regular local ring and where θ is an element of $R[[P]]$ such that $\bar{\theta} \overset{def}{=} \theta \bmod (P \setminus \{1\})$ is an element of $m_R \setminus m_R^2$. It suffices to prove that ψ is an isomorphism. But this is reduced to the bijectivity of the map induced by ψ

$$gr_{I_P R[[P]]}(R[[P]]/(\theta)) \to gr_{I_P \hat{\mathcal{O}}_{X,x}}(\hat{\mathcal{O}}_{X,x}).$$

which follows from (v).

(6.3) LEMMA. *Let P be a finitely generated integral monoid, let R be a ring, and let θ be an element of $R[P]$ such that $\theta \bmod I_P \in R[P]/I_P$ is the image of a nonzero divisor of R. Then, for any ideal J of P, the image of θ in $R[P]/JR[P]$ is a nonzero divisor.*

Proof. Assume

$$\left(\sum_{\alpha \in P} c_\alpha \alpha\right) \theta \in JR[P]. \qquad (c_\alpha \in R)$$

We show that the set $\Phi = \{\alpha \in P \backslash J ; c_\alpha \neq 0\}$ is empty. If Φ is not empty, let (α_0) $(\alpha_0 \in \Phi)$ be maximal in the set of ideals of $R[P]$ of the form (α) $(\alpha \in \Phi)$. By considering the coefficients of $(c_\alpha \alpha)\theta$ at α_0, we see that there exists $\alpha, \beta \in P$ such that $\alpha\beta = \alpha_0$, $c_\alpha \neq 0$, $\beta \in P \backslash P^x$. Then $\alpha \in \Phi$ and $(\alpha) \supsetneq (\alpha_0)$, a contradiction.

Proof of (6.2). This follows from the equivalence of (i) and (iii) of (6.1) by [GD] 0_{III} (10.2.2).

The following result will be used in §11.

(6.4) PROPOSITION. *Let (X, M) be a log. scheme satisfying (S) and assume (X, M) is regular. Let P be a finitely generated saturated monoid, and $P \to \mathcal{O}_X$ a homomorphism which induces M. Then, for any ideals I, J of P, we have $I\mathcal{O}_X \cap J\mathcal{O}_X = (I \cap J)\mathcal{O}_X$.*

This follows easily from (i) $\Rightarrow$ (iii) of (6.1).

7. Localization.

Let (X, M) be a log. scheme satisfying the condition (S). We prove the following two propositions, and complete the proof of (6.1).

(7.1) PROPOSITION. *Let $x \in X$, and assume that (X, M) is regular at x. Then for any $y \in X$ such that $x \in \overline{\{y\}}$, (X, M) is regular at y.*

(7.2) PROPOSITION. *Let $\mathfrak{p}$ be a prime ideal of M_x and endow $X' = Spec(\mathcal{O}_{X,x}/\mathfrak{p}\mathcal{O}_{X,x})$ with the log. str. associated to $M_x \backslash \mathfrak{p} \to \mathcal{O}_{X,x}/\mathfrak{p}\mathcal{O}_{X,x}$. Then, (X', M') satisfies (S) and is regular at $x \in X'$.*

(7.3) COROLLARY. *With the notation as in (7.2), $\mathfrak{p}\mathcal{O}_{X,x}$ is a prime ideal of height $\dim((M_x)_\mathfrak{p})$.*

(7.4) We prove (7.2) first. Take a finitely generated saturated monoid P with a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$ such that $\varphi^{-1}(\mathcal{O}_{X,x}^x) = \{1\}$ which induces M at x. Then, by (3.2), φ extends to an isomorphism $R[[P]]/(\theta) \xrightarrow{\cong} \hat{\mathcal{O}}_{X,x}$ for some complete regular local ring R and an element θ of $R[[P]]$ whose constant part belongs to $m_R \backslash m_R^2$. Let $\bar{\mathfrak{p}}$ be the prime ideal of P corresponding to $\mathfrak{p}$ via $P \xrightarrow{\cong} M_x/\mathcal{O}_{X,x}^x$. Then, $\hat{\mathcal{O}}_{X,x}/\mathfrak{p}\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}_{X,x}/\bar{\mathfrak{p}}\hat{\mathcal{O}}_{X,x}$ is isomorphic to $R[[P \backslash \bar{p}]]/(\bar{\theta})$ where $\bar{\theta} = \theta \mod \mathfrak{p}R[[P]]$. Note the log. str. M' is associated to $P \backslash \bar{\mathfrak{p}} \to \mathcal{O}_{X,x}$. By (3.1), we obtain (7.2).

(7.5) We prove (7.1). Take a finitely generated saturated monoid P and a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$, which induces M at x. Let $y \in X$, $x \in \overline{\{y\}}$, and let $\mathfrak{p} \in Spec(P)$ be the inverse image of the prime ideal of $\mathcal{O}_{X,x}$ corresponding to y. We shall prove (7.1) by induction on $\dim(P \backslash \mathfrak{p})$.

By the equivalence of (i) and (iii) of (6.1),

$$Tor_i^{\mathbf{Z}[P]}(\mathbf{Z}[P]/(\mathfrak{p}), \mathcal{O}_{X,x}) = 0 \text{ for } i \geq 1$$

and hence

$$Tor_i^{\mathbf{Z}[P_\mathfrak{p}]}(\mathbf{Z}[P_\mathfrak{p}]/(\mathfrak{p}), \mathcal{O}_{X,y}) = 0 \text{ for } i \geq 1.$$

Hence by the equivalence of (i) and (iv) of (6.1), to prove (7.1), it suffices to show that $\mathcal{O}_{X,y}/I(y,M)$ is a regular local ring. Recall that $\mathfrak{p}\mathcal{O}_{X,x}$ is a prime ideal of $\mathcal{O}_{X,x}$ by (7.2), and that $I(y,M) = \mathfrak{p}\mathcal{O}_{X,y}$. By replacing y by the point $\mathfrak{p}\mathcal{O}_{X,x} \in Spec(\mathcal{O}_{X,x})$, we may assume y is the point $\mathfrak{p}\mathcal{O}_{X,x} \in Spec(\mathcal{O}_{X,x})$. Note we may assume $\mathfrak{p} \neq \varphi^{-1}(m_x)$. Take $q \in Spec(P)$ such that $\varphi^{-1}(m_x) \supset q \supset \mathfrak{p}$ and $\dim(P \setminus q) = \dim(P \setminus \mathfrak{p}) - 1$. Let $z \in Spec(\mathcal{O}_{X,x})$ be the prime ideal $q\theta_{X,x}$. By induction on $\dim(P \setminus \mathfrak{p})$, (X,M) is regular at z. Hence by (7.2), the log. scheme $Spec(\mathcal{O}_{X,z}/\mathfrak{p}\mathcal{O}_{X,z})$ endowed with the log. str. associated to $P_q \setminus \mathfrak{p}P_q \rightarrow \mathcal{O}_{X,z}/\mathfrak{p}\mathcal{O}_{X,z}$ is regular at z. Since $\dim(P_q \setminus \mathfrak{p}P_q) = 1$, $(P_q \setminus \mathfrak{p}P_q)/(P_q \setminus \mathfrak{p}P_q)^x$ is isomorphic to $\mathbf{N}$ and from this we see that $\mathcal{O}_{X,z}/\mathfrak{p}\mathcal{O}_{X,z}$ is a regular local ring (in the classical sense). Hence $\mathcal{O}_{X,y}/\mathfrak{p}\mathcal{O}_{X,y}$ is regular as is desired.

(7.6) Now we complete the proof of (6.1). It remains to prove (i) $\Rightarrow$ (ii). To consider the vanishing of Tor_i ($i \geq 1$) at a prime ideal $\mathcal{B}$ of $\mathcal{O}_{X,x} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, we may replace P (resp. $\mathcal{O}_{x,x}$, resp. Q) by $P_\mathfrak{p}$ (resp. $(\mathcal{O}_{X,x})_{\mathfrak{p}}$, resp. $Q_{\mathfrak{p}''}$) where $\mathfrak{p}$ (resp. $\mathfrak{p}'$, resp. $\mathfrak{p}''$) is the inverse image of $\mathcal{B}$ in P (resp. $\mathcal{O}_{X,x}$, resp. Q). Here we can pass from $\mathcal{O}_{X,x}$ to $(\mathcal{O}_{X,x})_{\mathfrak{p}}$ by (7.1) (for this point, we had to postpone the part of the proof of (6.1) concerning (ii) until now). Furthermore, by replacing Q with the push out of

$$\begin{array}{c} Q^x \rightarrow Q \\ \downarrow \\ G \end{array}$$

for a divisible abelian group G containing Q^x, we may assume that there are submonoids $P_1 \subset P$ and $Q_1 \subset Q$ such that $P = P_1 \times P^x$, $Q = Q_1 \times Q^x$ and the image of P_1 in P is contained in Q_1. By replacing P and Q by P_1 and Q_1, respectively, we may assume $P^x = \{1\}$. In summary, we may assume that $\varphi^{-1}(\mathcal{O}_{X,x}^x) = \{1\}$ and the inverse image of Q^x in P is $\{1\}$, and it is sufficient to prove that Tor_i ($i \geq 1$) in problem vanish at prime ideals of $\mathbf{Z}[Q]$ including the image of I_P. Furthermore, by a limit argument, we may assume that Q is finitely generated (so that $R[Q]$ is Noetherian). Assuming these, with the notation as in the proof of (i) $\Rightarrow$ (iii) of (6.1) given in §6, the arguments there show that we are reduced to showing the following: the image of θ in the localization of $R[[P]] \otimes_{R[P]} R[Q]/(J)$ at a prime ideal containing I_P is a nonzero-divisor. So, it

is sufficient to show that the image of θ in

$$\left(R[[P]] \bigotimes_{R[P]} R[Q]/(J) \right) /(\text{the image of } I_P)^n = R[Q]/(I_P^n R[Q] + (J))$$

is a nonzero-divisor for any n. Hence we are reduced to Lemma (6.3).

8. Smooth Morphisms.

(8.1) Let (X, M) and (Y, N) be log. schemes satisfying (S), and let $f : (Y, N) \to (X, M)$ be a morphism. Then, the following two conditions (i), (ii) are equivalent. We say f is smooth (or logarithmically smooth to avoid confusion) if f satisfies these equivalent conditions. Such equivalence was proved in [Ka1] §5 for log. str.'s on etale sites, and the proof there works for the present situation (log. str.'s on Zariski sites), so we omit the proof here.

(i) Assume we are given a commutative diagram of log. schemes

$$
\begin{array}{ccc}
(T', L') & \xrightarrow{\ s\ } & (Y, N) \\
i \downarrow & & \downarrow f \\
(T, L) & \xrightarrow[\ t\ ]{} & (X, M)
\end{array}
$$

such that: (T, L) and (T', L') satisfy (S), the morphism of schemes $i : T' \to T$ is a closed immersion, T' is defined in T by a nilpotent ideal of $\mathcal{O}_T$ and $i^*L \to L'$ is an isomorphism. Then, locally on T, there is a morphism $h : (T, L) \to (Y, N)$ such that $fh = t$ and $hi = s$. Furthermore, the underlying morphism of schemes $Y \to X$ is locally of finite type.

(ii) Etale locally on X and Y, there exist finitely generated saturated monoids P, Q, an injective homomorphism $h : P \to Q$ such that the order of the torsion part of $Q^{gp}/h(P^{gp})$ is invertible on Y, and a commutative diagram of log. schemes of the form

$$
\begin{array}{ccc}
(Y, N) & \longrightarrow & (Spec(\mathbf{Z}[Q]), \tilde{Q}) \\
f \downarrow & & \downarrow \text{by } h \\
(X, M) & \longrightarrow & (Spec(\mathbf{Z}[P]), \tilde{P}),
\end{array}
$$

where $\tilde{P}$ (resp. $\tilde{Q}$) denotes the log. str. associated to $P \overset{\subseteq}{\to} \mathbf{Z}[P]$ (resp. $Q \overset{\subseteq}{\to} \mathbf{Z}[Q]$), such that the inverse image of $\tilde{P}$ on X is M, the inverse image of $\tilde{Q}$ on Y is N, and the induced morphism of schemes

$$Y \to X \times_{Spec(\mathbf{Z}[P])} Spec(\mathbf{Z}[Q])$$

is smooth in the classical sense.

(When we say "etale locally on X (resp. Y)," the etale schemes which cover X (resp. Y) are assumed to be endowed with the inverse images of M (resp. N)).

(8.2) THEOREM. *Let $f : (Y,N) \to (X,M)$ be a smooth morphism between log. schemes satisfying (S), and assume that (X,M) is regular. Then (Y,N) is regular.*

(8.3) PROPOSITION. *Let k be a field and let (X,M) be a log. scheme satisfying (S) such that X is a k-scheme locally of finite type. Endow $Spec(k)$ with the trivial log. str. k^x. Then:*

(1) *If (X,M) is smooth over $(Spec(k), k^x)$, then (X,M) is regular.*

(2) *The converse of (1) is true if k is perfect.*

Proof of (8.2). We may work etale locally, so we may assume that: there exist finitely generated saturated monoids P, Q and an injective homomorphism $h : P \to Q$ such that the order of the torsion part of $Q^{gp}/h(P^{gp})$ is invertible on Y, there exists a homomorphism $P \to \mathcal{O}_X$ which induces M, $Y = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, and N is induced by $Q \to \mathcal{O}_Y$. Let $y \in Y$ and $x = f(x) \in X$. To prove that (Y,N) is regular at y, we may assume further that the inverse image of $\mathcal{O}_{X,x}^x$ in P is P^x and the inverse image of $\mathcal{O}_{Y,y}^x$ in Q is Q^x. Assuming these;

(8.4) *Claim.* The ring $\mathcal{O}_{Y,y}/I(y,N)$ is a local ring of a smooth algebra in the classical sense over $\mathcal{O}_{X,x}/I(x,M)$.

Indeed, $\mathcal{O}_{Y,y}/I(y,N)$ is a local ring of

$$(\mathbf{Z}[Q]/I_Q) \bigotimes_{\mathbf{Z}[P]} \mathcal{O}_{X,x} \cong (\mathbf{Z}[Q]/I_Q) \bigotimes_{\mathbf{Z}[P]/I_P} \mathcal{O}_{X,x}/I(x,M)$$

$$\cong \mathbf{Z}[Q^x] \bigotimes_{\mathbf{Z}[P^x]} \mathcal{O}_{X,x}/I(x,M).$$

It is enough to notice that $\mathbf{Z}[\frac{1}{n}][P^x] \to \mathbf{Z}[\frac{1}{n}][Q^x]$ is smooth with n the order of the torsion part of $Q^x/h(P^x)$, and that the torsion part of $Q^x/h(P^x)$ injects into the torsion part of $Q^{gp}/h(P^{gp})$.

By (8.4), the ring $\mathcal{O}_{Y,y}/I(y,M)$ is regular. So, by (6.1), it suffices to prove that

$$Tor_i^{\mathbf{Z}[Q]}(\mathbf{Z}[Q]/(J), \mathcal{O}_{Y,y}) = 0$$

for any ideal J of Q and for any $i \geq 1$. But this is a consequence of (i) $\Rightarrow$ (ii) of (6.1).

Proof of (8.3). (1) follows from Thm. (8.2).

Assume k is perfect and (X,M) is regular. Let $x \in X$, and take a finitely generated saturated monoid P, and a homomorphism $\varphi : P \to \mathcal{O}_{X,x}$ which induces M at x such that $\varphi^{-1}(\mathcal{O}_{X,x}^x) = \{1\}$. It suffices to show that $\mathcal{O}_{X,x}$ is a local ring of

a smooth algebra over $k[P]$. But the map $k[P] \to \mathcal{O}_{X,x}$ is flat (6.2) and its "fiber"

$$k = k[P]/(P \setminus \{1\}) \to \mathcal{O}_{X,x}/I(x,M)$$

is smooth (for k is perfect and $\mathcal{O}_{X,x}/I(x,M)$ is regular).

9. Geometry of Commutative Monoids. (II).   In this section, we reformulate the theory of "fan" or "rational partial polyhedral decomposition" (cf. [Od], [KKMS]) into the form

$$\text{ring : scheme = monoid : fan.}$$

The definition of "fan" adopted below differs slightly from the known ones [Od] as is explained in (9.5). Our fan is essentially equivalent to "conical polyhedral complex with integral structure" defined in [KKMS] Ch. II Def. 5, Def. 6. Our formulation is very useful in later sections.

(9.1) *Definition.* A monoidal space (T, M_T) is a topological space T endowed with a sheaf of monoids M_T such that $M_{T,t}^x = \{1\}$ for any $t \in T$. A morphism of monoidal spaces $T' \to T$ is a pair (f, h) of a continuous map $f : T' \to T$ and a homomorphism $h : f^{-1}(M_T) \to M_{T'}$ such that the induced map $h_t : M_{T,f(t)} \to M_{T',t}$ satisfies $h_t^{-1}(\{1\}) = \{1\}$ for any $t \in T'$.

We first introduce the "affine fan" associated with a monoid.

(9.2) PROPOSITION. *Fix a monoid P. Then, the functor*

$$T \to Hom(P, \Gamma(T, M_T))$$

from the category of monoidal spaces to the category of sets is represented by the following monoidal space (F, M_F). As a set, $F = Spec(P)$ (5.1). The topology of F is the one for which the sets $D(f) = \{\mathfrak{p} \in Spec(P); f \notin \mathfrak{p}\}$ with $f \in P$ form a basis of open sets. The sheaf M_F is characterized by

$$M_F(D(f)) = S^{-1}P/(S^{-1}P)^x \text{ with } S = \{f^n; n \geq 0\}.$$

We denote the monoidal space (F, M_F) in (9.2) also by $Spec(P)$.
Now we give our definition of a "fan."

(9.3) *Definition.*

(1)   A monoidal space is called an affine fan if it is isomorphic to the monoidal space $Spec(P)$ for some monoid P.

(2)   A monoidal space is called a fan if it has an open covering consisting of affine fans.

As is easily seen, the category of monoids P such that $P^x = \{1\}$ is anti-equivalent to the category of affine fans via the functor $P \to Spec(P)$.

(9.4) We shall mainly consider fans F satisfying the following condition (S_{fan}).

(S_{fan}): There exists an open covering $\{U_\lambda\}_\lambda$ of F such that for each λ, $U_\lambda \cong Spec(P_\lambda)$ as a fan for a finitely generated saturated monoid P_λ.

(9.5) Let L be a finitely generated free abelian group. A fan in L in [Od] is essentially a set Δ of submonoids of L satisfying the following conditions (i)–(iii).

(i) If $P \in \Delta$, P is finitely generated and saturated, $P^x = \{1\}$, and L/P^{gp} is torsion free.

(ii) If $P \in \Delta$ and $\mathfrak{p} \in Spec(P)$, then $P \setminus \mathfrak{p} \in \Delta$.

(iii) If $P, Q \in \Delta$, then $P \cap Q = P \setminus \mathfrak{p} = Q \setminus \mathfrak{q}$ for some $\mathfrak{p} \in Spec(P)$ and $\mathfrak{q} \in Spec(Q)$.

It is easily checked that to give a Δ as above is equivalent to giving a fan in our sense F endowed with a homomorphism of sheaves $h : L^* \to M_F^{gp}$ where $L^* = Hom(L, \mathbf{Z})$, satisfying the following conditions (i)–(iii).

(i) F satisfies the condition (S_{fan}) in (9.4).

(ii) h is surjective.

(iii) The map $Mor(Spec(\mathbf{N}), F) \to L$ induced by h is injective. In fact, Δ and F reconstruct each other as follows. As a set, $\Delta = F$. For $x \in F$, denote by $U(x)$ the smallest open neighborhood $\{y \in F; x \in \overline{\{y\}}\}$ of x in F. Then, the element of Δ corresponding to $x \in F$ is the image of $Mor(Spec(\mathbf{N}), U(x))$ in L. On the other hand, if we are given a Δ first, the topology of $F = \Delta$ is defined by taking the sets $U(P) = \{P' \in \Delta; P' \subset P\}$ ($P \in \Delta$) as a basis of open sets of F, the sheaf M_F is defined by

$$M_F(U(P)) = Hom(P, \mathbf{N}),$$

and $L^* \to M_F^{gp}$ is induced by $P \overset{\subset}{\to} L$.

Note that if Δ corresponds to F in the above sense, we have

$$\bigcup_{P \in \Delta} P = Mor(Spec(\mathbf{N}), F) \qquad \text{in } L.$$

(9.6) *Definition.* Let F be a fan satisfying (S_{fan}) (cf. (9.4)). Then, a subdivision of F is a fan F' satisfying (S_{fan}) endowed with a morphism $f : F' \to F$ such that $M_{F,f(t)}^{gp} \to M_{F',t}^{gp}$ is surjective for any $t \in F'$ and such that the induced map

$$Mor(Spec(\mathbf{N}), F') \to Mor(Spec(\mathbf{N}), F)$$

is injective.

It is easily checked that if L is a finitely generated abelian group and F is a fan with $L^* \to M_F^{gp}$ which corresponds to some Δ as in (9.5), then a subdivision of F is equivalent to a subdivision of Δ in the sense of [KKMS].

(9.7) *Definition.* Let F be a fan satisfying (S_{fan}). We say a subdivision $f : F' \to F$ is proper if the following two conditions are satisfied.

(i) $f^{-1}(x)$ is a finite set for any $x \in F$.

(ii)   The map $Mor(Spec(\mathbf{N}), F') \to Mor(Spec(\mathbf{N}), F)$ is bijective.

The result on resolution of "singularity" of cones ([KKMS] Ch. I Thm. 11) is translated as follows.

(9.8) PROPOSITION. *For any finite fan F satisfying (S_{fan}), there exists a subdivision $f : F' \to F$ such that for each $x \in F'$,*

$$M_{F', x} \cong \mathbf{N}^{r(x)} \text{ for some } r(x) \geq 0.$$

(9.9) PROPOSITION. *Let F be a fan satisfying (S_{fan}), let (X, M) be a log. scheme with X locally Noetherian, and assume that we are given a morphism of monoidal spaces*

$$\pi : (X, M_X/\mathcal{O}_X^x) \to F$$

such that $\pi^{-1}(M_F) \xrightarrow{\cong} M_X/\mathcal{O}_X^x$. Then:

(1)   *(X, M) satisfies (S).*

(2)   *Let F' be a subdivision of F. Let $\mathcal{C}$ be the category of log. schemes (X', M') endowed with a morphism $f : (X', M') \to (X, M)$ of log. schemes and with a morphism of monoidal spaces $\pi' : (X', M'/\mathcal{O}_{X'}^x) \to F'$ such that the diagram*

$$
\begin{array}{ccc}
(X', M'/\mathcal{O}_{X'}^x) & \xrightarrow{\ \pi'\ } & F' \\
{\scriptstyle by\, f}\downarrow & & \downarrow \\
(X, M/\mathcal{O}_X^x) & \xrightarrow{\ \pi\ } & F
\end{array}
$$

is commutative. Then this category has a final object. Furthermore if (X', M') with (f, π') is a final object, then X' is locally of finite type over X (and hence locally Noetherian), and $\pi'^{-1}(M_{F'}) \xrightarrow{\cong} M_{X'}/\mathcal{O}_{X'}^x$ (hence (X', M') satisfies (S) and (1)).

Proof. We may work locally, so we may assume $F = Spec(P)$, $F' = Spec(P')$ with P, P' finitely generated saturated monoids such that $P^x = \{1\}$, $(P')^x = \{1\}$. Then, the map $P^{gp} \to (P')^{gp}$ is surjective. We may work locally so we may assume

that $P \to M/\mathcal{O}_X^x$ lifts to a homomorphism $P \to M$ such that M is associated to $P \to M \to \mathcal{O}_X$. Then, we obtain a final object (X', M') by

$$X' = X \times_{Spec(\mathbf{Z}[P])} Spec(\mathbf{Z}[Q])$$

where $Q = \{a \in P^{gp}$; the image of a in $(P')^{gp}$ belongs to $P'\}$, with M' the log. str. associated to $Q \to \mathcal{O}_{X'}$.

(9.10) *Definition.* In the situation of (9.9) (2), we denote a final object in that category symbolically by $(X, M) \times_F F'$ (this is abuse of notation, for we do not have a morphism $(X, M) \to F$).

(9.11) PROPOSITION. *In the situation of (9.9) (2), if $F' \to F$ is proper, then the underlying morphism of schemes of $(X, M) \times_F F' \to (X, M)$ is proper.*

Proof. Exercise.

10. The Associated Fans and Resolution of Toric Singularities.

In this section, let (X, M) be a log. scheme satisfying (S). We assume further that (X, M) is regular.

We define a fan (§9) associated with (X, M), and show that a resolution of singularity of X is given by a subdivision of this fan.

(10.1) PROPOSITION. *Let $F(X) = \{x \in X; I(x, M) = m_x\}$, and endow $F(X)$ with the inverse image of the topology of X and with the inverse image of the sheaf $M/\mathcal{O}_X^x$. Then $F(X)$ is a fan satisfying the condition (S_{fan}).*

Proof. Let $x \in X$, and take an open neighborhood U of x in X, a finitely generated saturated monoid P, and a homomorphism $\varphi : P \to \mathcal{O}_U$ which induces M on U such that $\varphi_x^{-1}(\mathcal{O}_{X,x}^x) = P^x$. Take an open neighborhood V of x in U such that the closed subschemes $Spec(\mathcal{O}_V/\mathfrak{p}\mathcal{O}_V)$ of V are integral for any $\mathfrak{p} \in Spec(P)$. This is possible since $Spec(\mathcal{O}_U/\mathfrak{p}\mathcal{O}_U)$ are locally integral by (7.3). Then, we see that $F(X) \cap V$ is isomorphic to the affine fan $Spec(P)$.

(10.2) We define a morphism of monoidal spaces

$$\pi : (X, M/\mathcal{O}_X^x) \to F(X)$$

as follows. For $x \in X$, let $\pi(x)$ be the point of X corresponding to the prime ideal $I(x, M)$ of $\mathcal{O}_{X,x}$. Then, it is easily seen that the map $\pi : X \to F(X)$ is continuous and open, and that there exists a canonical isomorphism $M/\mathcal{O}_X^x \cong \pi^{-1}(M_{F(X)})$ making π a morphism of monoidal spaces.

(10.3) PROPOSITION. *Let F' be a subdivision (9.6) of $F(X)$ and let $(X', M') = (X, M) \times_{F(X)} F'$ (9.10). Then, (X', M') is regular, F' is identified with the fan associated with (X', M'), and the morphism of schemes $X' \to X$ is birational.*

Proof. The regularity of (X', M') follows from the smoothness of $(X', M') \to (X, M)$ by (8.2). The rest is easy.

(10.4) Assume X is quasi-compact (then $F(X)$ is finite as is easily seen). We give a standard way to resolve the singularity of X. Take a subdivision F' of $F(X)$ which is proper over $F(X)$ such that for any $x \in F'$, there is an isomorphism $M_{F', x} \cong \mathbf{N}^{r(x)}$ for some $r(x) \geq 0$ (9.8). Let $(X', M') = (X, M) \times_{F(X)} F'$. Then (X', M') is regular (10.3) and for $x \in X'$, $M'_x / \mathcal{O}^x_{X', x} \cong \mathbf{N}^{r(x)}$ for some $r(x) \geq 0$. Hence X' is a regular scheme in the usual sense.

11. Coherent Ideals and Dualizing Sheaves.

In this section, let (X, M) be a log. scheme satisfying (S) and assume (X, M) is regular. Let $F(X)$ be the associated fan (10.1), and let $\pi : (X, M/\mathcal{O}^x_X) \to F(X)$ be the canonical morphism (10.2).

We show that some results in [KKMS] Ch. I §3 on dualizing sheaves and on higher direct images of some coherent ideals with respect to proper modifications are generalized to our toric singularities (Thm. (11.2), (11.3)). We state a generalization of a result of Ishida to our toric singularites (Thm. (11.4)) with a sketch of the proof, concerning the dualizing complexes of some subschemes of X. We also give simple descriptions (11.6) of M and M^{gp}.

(11.1) *Definition.* Let F be a fan satisfying the condition (S_{fan}).

(1) A subsheaf I of M_F (resp. M_F^{gp}) is called a coherent ideal (resp. a coherent fractional ideal) of F if it satisfies the following condition: For each $x \in F$, there exist an affine open neighborhood U of X in F and an ideal (resp. a fractional ideal (5.7)) J of $M_F(U)$ such that for any $y \in U$, I_y coincides with the ideal (resp. fractional ideal) of M_y generated by the image of J.

(2) For coherent fractional ideals I, J of F, IJ denotes the coherent fractional ideal $\{axy; a \in M_F, x \in I, y \in J\}$ of F. For a coherent fractional ideal I of F, I^{-1} denotes the coherent fractional ideal $\{a \in M_F^{gp}; aI \subset M_F\}$ of F.

(3) A coherent fractional ideal I of F is said to be saturated if it has the following property: If $a \in M_F^{gp}$ and $a^n \in I^n$ for some $n \geq 1$, then $a \in I$. (Here I^n means the product of n copies of I.) For a coherent fractional ideal I of F, I^{sat} denotes the saturated coherent fractional ideal $\{a \in M_T^{gp}; a^n \in I^n$ for some $n \geq 1\}$ of F. We call I^{sat} the saturation of I.

(4) Let (X, M) and $F(X)$ be as at the beginning of this section. For a coherent fractional ideal I of $F(X)$, by abuse of notation, we denote by $I\mathcal{O}_X$ the coherent fractional ideal of X generated by the image of local sections of M whose classes in $M^{gp}/\mathcal{O}^x$ belong to the image of $\pi^{-1}(I)$.

(11.2) THEOREM. *Let K be the coherent ideal of $F(X)$ defined by $K(U) = \{a \in M_{F(X)}(U);$ the image of a in $M_{F(X),\mathfrak{p}} \cong \mathbf{N}$ is ≥ 1 for any point of codimension one $\mathfrak{p}$ of $U\}$ for any open set U of $F(X)$. Then, $K\mathcal{O}_X$ is a dualizing sheaf of X.*

Note that we have already seen that X is Cohen-Macaulay (§4).

Proof of (11.2). By localization and completion, we may assume that X is the Spec of a complete Noetherian local ring. Then, $X = Spec(R[[P]]/(\theta))$ with R, θ as in (3.1) (1) and M is associated to the canonical map $P \to R[[P]]/(\theta)$. We are reduced easily to the case $X = R[[P]]$, and then to the case $X = Spec(R[P])$ (with the log. str. associated to $P \xrightarrow{\subseteq} R[P]$). As a general fact for Cohen-Macaulay schemes, we have that for any dualizing sheaf $\mathcal{F}$ on X and for any open subscheme U of X containing all points of codimension one of X, the map $\mathcal{F} \to j_*(\mathcal{F}|_U)$ is an isomorphism where j is the canonical inclusion map $U \to X$. Take as $\mathcal{F}$ the dualizing sheaf of X relative to R, and as U the biggest open subscheme of X which is smooth in the classical sense over R. Then, $\mathcal{F}|_U = \Omega^r_{U/R}$ with $r = rank(P^{gp})$. If D denotes the divisor $\sum_\mathfrak{p} \mathfrak{p}$ on U where $\mathfrak{p}$ ranges over all points of codimension one of $F(X) \subset X$, we have a canonical isomorphism

$$\mathcal{O}_U \bigotimes_{\mathbf{Z}} \Lambda^r P^{gp} \xrightarrow{\cong} \Omega^r_{U/R}(D);$$
$$a \otimes (b_1 \Lambda \ldots \Lambda b_r) \to ad\log(b_1)\Lambda \ldots \Lambda d\log(b_r).$$

Hence $\mathcal{F} = j_*(\mathcal{O}_U(-D)) \otimes_{\mathbf{Z}} \Lambda^r P^{gp}$ and this sheaf is just the sheaf $K\mathcal{O}_X \otimes_{\mathbf{Z}} \Lambda^r P^{gp} \cong K\mathcal{O}_X$ by (11.8) below.

(11.3) THEOREM. *Let F' be a subdivision of $F(X)$ which is proper over $F(X)$, let $(X', M') = (X, M) \times_{F(X)} F'$, and let $f : X' \to X$ be the canonical morphism. Let the coherent ideal K of $F(X)$ be as in (11.2), and let K' be the coherent ideal of $F' = F(X')$ defined in the same lay. Then for any saturated coherent fractional ideal I of $F(X)$, we have*

$$I\mathcal{O}_X \xrightarrow{\cong} Rf_*(I'\mathcal{O}_{X'}),$$
$$(IK)^{sat}\mathcal{O}_X \xrightarrow{\cong} Rf_*((I'K')^{sat}\mathcal{O}_{X'}),$$

where I' is the saturation of the coherent fractional ideal $IM_{F(X')}$ of $F(X')$.

Proof. Locally on X, there exist a finitely generated saturated monoid P and a homomorphism $P \to \mathcal{O}_X$ which induces M such that if Y denotes $Spec(\mathbf{Z}[P])$ endowed with the log. str. associated to $P \xrightarrow{\subseteq} \mathbf{Z}[P]$, the morphism $(X, M) \to (Y, N)$ induces an isomorphism of fans $F(X) \xrightarrow{\cong} F(Y) = Spec(P)$. The subdivision $f : F' \to F(X)$ defines the corresponding log. scheme $(Y', N') = (Y, N) \times_{F(X)} F'$, and we have $X' = X \times_Y Y'$. By the vanishing of Tor_i ($i \geq 1$) in (6.1) (ii), we are

reduced to the case $(X, M) = (Y, N)$. But this case is proved in [KKMS] Ch. I §3 Cor. 1 (in fact this reference treats the case of $k[P]$ for a field k, but the arguments work for $\mathbf{Z}[P]$).

(11.4) THEOREM. (cf. [Is] Thm. (3.3). *Let I be a coherent ideal of $F(X)$ satisfying the condition that if $a \in M_{F(X)}$ and $a^n \in I$ for some $n \geq 1$, then $a \in I$. Let Y be the closed subscheme $Spec(\mathcal{O}_X/I\mathcal{O}_X)$ of X. Then, the complex $C^{\cdot}$ defined as below is a dualizing complex of Y. If $I = \phi$ (then $Y = X$), $C^{\cdot}$ is a quasi-isomorphic to $K\mathcal{O}_X$ put in degree 0 with K as in (11.2).*

The definition of $C^{\cdot}$ is as follows. For $i \in \mathbf{Z}$, C^i is the direct sum of

$$Hom_{\mathbf{Z}}(\Lambda^i M^{gp}_{F(X),x}, \mathcal{O}_{\overline{\{x\}}})$$

where x ranges over all points of $F(X)$ of codimension i contained in Y, and $\overline{\{x\}}$ is the closure of x in X endowed with the reduced scheme structure. (Note $M^{gp}_{F(X),x} \cong \mathbf{Z}^i$ for a point x of codimension i of $F(X)$.) The map $C^i \to C^{i+1}$ is the direct sum of its (x, y)-components $\delta_{x,y}$ defined as follows, where $x, y \in F(X) \cap Y$, x of codimension i in $F(X)$ and y of codimension $i + 1$ in $F(X)$. If $y \notin \overline{\{x\}}$, then $\delta_{x,y}$ is zero. If $y \in \overline{\{x\}}$, $\delta_{x,y}$ is the map induced by the isomorphism

$$\Lambda^{i+1} M^{gp}_{F(X),y} \xrightarrow{\cong} \Lambda^i M^{gp}_{F(X),x}$$

which is characterized by

$$a_1 \Lambda \ldots \Lambda a_i \Lambda e \to \bar{a}_1 \Lambda \ldots \Lambda \bar{a}_i$$

where $a_j \in M_{F(X),y}$, $\bar{a}_j$ the canonical image of a_j in $M_{F(X),x}$, and e is the generator of $Ker(M^{gp}_{F(X),y} \to M^{gp}_{F(X),x}) \cong \mathbf{Z}$ which belongs to $M_{F(X),y}$.

(11.5) *Remark.* The scheme Y in (11.4) is reduced. Indeed, for any affine open fan U of $F(X)$, $I(U)$ is the intersection of a finite number of prime ideals of $M_{F(X)}(U)$ (the proof of this is the same with the proof of its analogy for ring theory that an ideal I of a Noetherian ring R such that R/I is reduced is the intersection of a finite number of prime ideals), and so we can apply (6.4) and (7.3).

We sketch the proof of (11.4). By an argument similar to that in the proof of (11.2), we are reduced to the case $X = Spec(R[P])$ for a regular local ring R and for a finitely generated saturated monoid P endowed with the log. str. associated to $P \xrightarrow{\subseteq} R[P]$. The proof in the case R is a field is given in [Is]. The general case is proved by modifying slightly the proof there.

(11.6) THEOREM. *Let U be the biggest open subscheme of X such that the restriction of M to U is trivial, and let $j : U \to X$ be the inclusion morphism. Then, we have isomorphisms of sheaves*

$$M^{gp} \xrightarrow{\cong} j_* \mathcal{O}_U^x$$
$$M \xrightarrow{\cong} \mathcal{O}_X \cap j_* \mathcal{O}_U^x.$$

Proof. We show first that the canonical map $\alpha : M \to \mathcal{O}_X$ is injective. Assume α is not injective at $x \in X$, and let $\bar{M}$ be the log. str. on $Spec(\mathcal{O}_{X,x})$ associated to $Image(\alpha_x) \to \mathcal{O}_{X,x}$. Then $rank(\bar{M}_x/\mathcal{O}_{X,x}^x) < rank(M_x/\mathcal{O}_{X,x}^x)$ and hence

$$\dim(\mathcal{O}_{X,x}) > \dim(\bar{M}_x/\mathcal{O}_{X,x}^x) + \dim(\mathcal{O}_{X,x}/I(x, \bar{M})).$$

This contradicts (2.3).

We next prove the bijectivity of $M^{gp} \to j_* \mathcal{O}_U^x$. It suffices to prove the surjectivity. Let h be a section of $j_* \mathcal{O}_U^x$. Then, the divisor $div(h)$ of h has the form $\sum m(\mathfrak{p})\mathfrak{p}$ $(m(\mathfrak{p}) \in \mathbf{Z})$ where $\mathfrak{p}$ ranges over points of codimension one of $F(X)$. Let I be the coherent fractional ideal of $F(X)$ defined by

$$I(U) = \{a \in M_{F(X)}^{gp}(U); \text{ the image of } a \text{ in } M_{F(X),\mathfrak{p}}^{gp} \cong \mathbf{Z} \text{ is } \geq m(\mathfrak{p})$$
$$\text{for any point of codimension one } \mathfrak{p} \text{ of } U\}$$

for any open U of $F(X)$. Then, by (11.8) below, we have $I\mathcal{O}_X = h\mathcal{O}_X$. Hence at each point x of X, there exists $a \in I_{\pi(x)}$ such that h_x mod $\mathcal{O}_{X,x}^X = a$. This shows that h belongs to the image of $M^{gp} \to j_* \mathcal{O}_U^x$.

Finally the surjectivity of $M \to \mathcal{O}_X \cap j_* \mathcal{O}_U^x$ follows from that of $M^{gp} \to j_* \mathcal{O}_U^x$ and from (5.8) (1).

(11.7) LEMMA. *For a coherent fractional ideal I of $F(X)$, we have*

$$(I\mathcal{O}_X)^{-1} = I^{-1}\mathcal{O}_X.$$

Here the left hand side is the usual notation expressing the inverse of a coherent fractional ideal of X.

Proof. By working locally, we may assume that $I = \cup_{i=1}^r M_{F(X)}\alpha_i$ for some sections $\alpha_1, \ldots, \alpha_r$ of $M_{F(X)}^{gp}$. Then, $I^{-1} = \cap_{i=1}^r M_{F(X)}\alpha_i^{-1}$ and $(I\mathcal{O}_X)^{-1} = \cap_{i=1}^r \alpha_i^{-1}\mathcal{O}_X$. Hence we are reduced to (6.4).

(11.8) COROLLARY. *Let S be the set of all points of codimension one of $F(X)$ and let $m : S \to \mathbf{Z}$ be a function such that the intersection of the support of m with*

*any affine open fan of $F(X)$ is finite. Let I be the coherent fractional ideal of $F(X)$
defined by*

$$I(U) = \{a \in M_{F(X)}^{gp}(U); \text{ the image of } a \text{ in } M_{F(X),\mathfrak{p}}^{gp} \cong \mathbf{Z} \text{ is } \geq m(\mathfrak{p})$$
$$\text{ for any } \mathfrak{p} \in S \cap U\}$$

*for any open set U of $F(X)$. Then, the coherent fractional ideal $I\mathcal{O}_X$ of X coincides
with $\mathcal{O}_X(-D)$ where D is the divisor $\sum_{\mathfrak{p} \in S} m(\mathfrak{p}) \mathfrak{p}$ on X.*

Proof. By commutative algebra, $((I\mathcal{O}_X)^{-1})^{-1}(U)$ coincides with $\mathcal{O}_X(-D)$
with D as above. But $((I\mathcal{O}_X)^{-1})^{-1} = (I^{-1})^{-1}\mathcal{O}_X$ by (11.7) and $(I^{-1})^{-1} = I$ as is
easily seen.

12. Comments on a Method of Abhyankar.

In this section, we show first
that a "Jungian domain" of Abhyankar is logarithmically regular with respect to
a suitable log. str. (see (12.1) (12.2)). We next review (12.3) some key ideas in
the work of Abhyankar [Ab2] on resolution of singularity of arithmetic surfaces,
and lastly propose some problems related to the desingularization of arithmetic
schemes of higher dimensions (12.4).

(12.1) We do not review here the definition of the Jungian domain in [Ab3],
but give the following two properties (i) (ii) of a Jungian domain A which are
sufficient for the proof of the logarithmic regularity of A.

(i) A is a normal Noetherian two dimensional local ring.

(ii)   There exists a finitely generated multiplicative submonoid P of $A \setminus \{0\}$
such that $P \cap A^x = P^x$, $P^{gp}/P^x \cong \mathbf{Z}^2$, and the image of $P \setminus P^x$ in A generates the
maximal ideal m_A.

We endow $X = Spec(A)$ with the log. str. M associated to $P \to A$. Then, by
Prop. (12.2) below, we see that P is saturated and (X, M) is regular.

(12.2) Let P be a finitely generated integral monoid, A a Noetherian local
ring, $\varphi : P \to A$ a homomorphism such that $\varphi^{-1}(A^x) = P^x$, and let M be the log.
str. on $X = Spec(A)$ associated to $P \to A$. Assume that the following (i)–(iii) are
satisfied.

(i) P^{gp}/P^x is torsion free.

(ii) $A/I_P A$ is a regular local ring (I_P denotes $(P \setminus P^x)$ as before).

(iii) $\dim(A/I_P) + rank(P^{gp}/P^x) = \dim(A)$.

Note that if P is furthermore saturated, (i) is automatically true, and (ii) and
(iii) mean that (X, M) is regular.

Under these assumptions, we have:

(12.2) PROPOSITION. *Let P^{sat} be the saturation of P, i.e.*

$$P = \{a \in P^{gp}; a^n \in P \text{ for some } n \geq 1\},$$

let $B = \mathbf{Z}[P^{sat}] \otimes_{\mathbf{Z}[P]} A$ and endow $Y = Spec(B)$ with the log. str. N associated to $P^{sat} \to B$. Then:

(1) *(Y, N) is regular.*

(2) *B is the normalization of A, and is a local ring.*

(3) *If A is already normal, then P is saturated.*

Proof. The assertion (1) is proved easily and (2) follows from (1) by (4.1).

We prove (3). Assume A is normal. Then, $B = A$ by (1), that is, $A \xrightarrow{\cong} \mathbf{Z}[P^{sat}] \otimes_{\mathbf{Z}[P]} A$. Consider the induced isomorphism

$$(*) \qquad A/I_P A \xrightarrow{\cong} (\mathbf{Z}[P^{sat}]/I_P \mathbf{Z}[P^{sat}]) \bigotimes_{\mathbf{Z}[P]/I_P \mathbf{Z}[P]} A/I_P A.$$

Since $\mathbf{Z}[P]/I_P \cong \mathbf{Z}[P^x]$ and $\mathbf{Z}[P^{sat}]/I_P \mathbf{Z}[P^{sat}]$ is a free module over $\mathbf{Z}[P^x]$, the isomorphism $(*)$ shows that

$$\mathbf{Z}[P^x] \xrightarrow{\cong} \mathbf{Z}[P^{sat}]/I_P \mathbf{Z}[P^{sat}].$$

Hence the following $(**)$ holds.

$(**)$ If $\alpha \in P^{sat}$ and $\alpha \notin P^x$, then there exist $\beta \in P \setminus P^x$ and $\gamma \in P^{sat}$ such that $\alpha = \beta\gamma$.

If $P \neq P^{sat}$, let (α_0) be maximal in the set of ideals of P^{sat} of the form (α) with $\alpha \in P^{sat} \setminus P$. Write $\alpha_0 = \beta\gamma$ with $\beta \in P \setminus P^x$ and $\gamma \in P^{sat}$. Then $\gamma \in P^{sat} \setminus P$ and $(\alpha_0) \subsetneq (\gamma)$, a contradiction. Hence we have $P = P^{sat}$.

(12.3) The method of Abhyankar in [Ab2] [Ab3] to resolve a singularity of a two dimensional excellent scheme is, roughly speaking, the following. By using some reduction steps, he finds that it is sufficient to prove the following fact:

"Let X be a two dimensional excellent connected scheme, and assume X is regular. Let K be the function field of X, L a normal extension of K of degree a prime number, and let Y be the integral closure of X in L. Then, the singularity of Y can be resolved."

Now his method to resolve this Y is the following. For simplicity assume L is separable over K, so L is a cyclic extension of K of prime degree. Then, by repeating to blow up X, we obtain the following situation: There exists a divisor D on X with simple normal crossings such that the covering $Y \to X$ is unramified outside D. He further repeats to blow up X, and he shows that we can reach the following situation: All local rings at points of codimension two of Y are Jungian domains. He then shows that the singularity of a Jungian domain can be resolved.

By our "toric interpretation" of Jungian domain, we are naturally led to the following questions.

(12.4.1) *Question.* Let X be an excellent regular scheme, let U be a dense open subscheme of X, and let $V \to U$ be an etale finite covering. Then, is there

a proper birational morphism $f : X' \to X$ satisfying the following conditions (i)–(iii)? Let $U' = f^{-1}(U)$.

(i) $(X' \setminus U')_{red}$ is a divisor with simple normal crossings on X' $(( )_{red}$ means the reduced part).

(ii) $U' \xrightarrow{\cong} U$ via f.

(iii)   (This is the key point.) Let Y be the integral closure of X' in V (V is regarded as a scheme over U' via $V' \to U. \cong U'$), and let N be the subsheaf of $\mathcal{O}_Y$ defined by

$$N = \{h \in \mathcal{O}_Y; h \text{ is invertible on } V\}.$$

Then, the log. scheme (Y, N) satisfies the condition (S) and is regular.

(12.4.2) *Question.* Is the answer of (12.4.1) yes if we assume V to be a cyclic covering of degree a prime of U?

If X is a scheme over $\mathbf{Q}$, these questions have affirmative answers: By Hironaka's desingularization theorem [Hi], there exists $f : X' \to X$ satisfying (i) (ii). Furthermore, for schemes over $\mathbf{Q}$, (iii) holds whenever (i) (ii) hold. However in the mixed or positive characteristic, (iii) does not hold in general even if (i) (ii) hold; the wild ramification in $Y \to X'$ prevents (Y, N) from being regular.

In the case $\dim(X) = 2$, (12.4.2) is solved affirmatively by Abhyankar [Ab3] as I roughly introduced in (12.3). (Strictly speaking, he makes some mild assumptions on X such as all points of codimension two have perfect residue fields. But these assumptions can be removed as we see in [Ka2] Part II.) I do not know the answer of (12.4.1) even in the case $\dim(X) = 2$.

As a psychological support in higher dimensions, [Ka2] Part I Prop. (1.13) shows that an analogue of the question (12.4.2) in the theory of $\mathcal{D}$-modules has an affirmative answer (cf. [Ka2] Part II for the explanation of the analogy).

Note that it is impossible to require Y in (12.4.2) to be regular in the classical sense. This is known as the impossibility of the simultaneous desingulization ([Ab1]).

I expect that the study of the question (12.4.2) will be useful for the desingularization of three dimensional arithmetic schemes.

REFERENCES

[Ab1]   S. S. Abhyankar, Simultaneous resolution for algebraic surfaces, *Amer. J. Math.* **78** (1956) 761–790.
[Ab2]   _____, Resolution of singularity of arithmetical surfaces, *Arithmetical Algebraic Geometry*, Harper & Row 1963, pp. 111–152.

[Ab3]   _____, Uniformization in a p-cyclic extension of a two dimensional regular local domain of residue field characteristic p, *Wissenschaftliche Abhandlungen des Landes Nordhein-Westfalen* **33** (1966) 243–317.

[GD]   A. Grothendieck, and J. Dieudonné, Eléments de géométrie algébrique, III, volume 1, *Inst. Hautes Études Sci. Publ. Math.* **11** (1961).

[Hi]   H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, *Ann. of Math.* **79** (1964) 109–326.

[Ho]   M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, *Ann. on Math.* **96** (1972) 318–337.

[Is]   M. Ishida, Torus embeddings and dualizing complexes, *Tohoku Math. J.* **32** (1980) 111–146.

[Ka1]   K. Kato, Logarithmic structures of Fontaine-Illusie, *Algebraic Analysis, Geometry, and Number Theory*, The Johns Hopkins Univ. Press, Baltimore, pp. 191–224.

[Ka2]   _____, Class field theory, $\mathcal{D}$-modules, and ramification on higher dimensional schemes, in preparation.

[KKMS]   G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings I*, *Lecture Notes in Math.* vol. 339, Springer-Verlag, New York, 1973.

[Od]   T. Oda, *Convex Bodies and Algebraic Geometry*, Springer-Verlag, New York, 1988.