Lecture 10

1 Theorem 8.5: Let K be a local field, then K is the completion of a global field.

Proof: Case 1: $|\cdot|$ archimedean.

$\mathbb{R}$ is completion of $\mathbb{Q}$ w.r.t. $|\cdot|_\infty$

$\mathbb{C}$ is " " $\mathbb{Q}(i)$ w.r.t. $|\cdot|_\infty$.

Case 2: $|\cdot|$ non-arch, equal char.

$K \cong \mathbb{F}_q((t))$, then K is completion of $\mathbb{F}_q(t)$ w.r.t. t-adic abs. value.

Case 3: $|\cdot|$ non-arch, mixed char.

$K = \mathbb{Q}_p(\alpha)$, α a root of a monic irreducible polynomial $f(X) \in \mathbb{Z}_p[X]$.

Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we choose $g(X) \in \mathbb{Z}[X]$ as in Prop. 8.4. Then $K = \mathbb{Q}_p(\beta)$, β a root of $g(X)$.

Since $\mathbb{Q}(\beta)$ dense in $\mathbb{Q}_p(\beta) = K$, K is the completion of $\mathbb{Q}(\beta)$.

IV Dedekind domain

2 Definition 9.1: A Dedekind domain is a ring R s.t.

ring R s...

(i) R is Noetherian integral domain.

(ii) R is integrally closed in $\mathrm{Frac}(R)$.

(iii) Every non-zero prime ideal is maximal.

Example: · The ring of integers in a number is a Dedekind domain.

· Any PID (hence DVR) is a Dedekind domain.

Theorem 9.2: A ring R is a DVR $\iff$ R is Dedekind domain with exactly one non-zero prime ideal.

Lemma 9.3: Let R be a Noetherian ring and $I \subseteq R$ non-zero ideal. Then there exists non-zero prime ideals $p_1, \dots, p_r \subseteq R$ s.t. $p_1 p_2 \dots p_r \subseteq I$.

Proof: Suppose not. Since R is Noetherian we may choose I maximal with this property. Then I is not prime, so $\exists x, y \in R \setminus$ s.t. $xy \in I$.

Let $I_1 = I + (x)$, $I_2 = I + (y)$.

Then by maximality of I, $\exists p_1, \dots, p_r$,

...

$q_1, \dots, q_s$ prime ideals s.t.

$$p_1 \cdots p_r \subseteq I_1, \quad q_1 \cdots q_s \subseteq I_2$$

$$\Rightarrow p_1 \cdots p_r q_1 \cdots q_s \subseteq I_1 I_2 \subseteq I \qquad \# \quad \square$$

Lemma 9.4: Let R be an integral domain which is integrally closed in $K = \mathrm{Frac}(R)$. Let $I \subseteq R$ be a non-zero finitely generated ideal and $x \in K$. Then if $xI \subseteq I$, we have $x \in R$.

Proof: Let $I = (c_1, \dots, c_n)$. We write

$$x c_i = \sum_{i=1}^{n} a_{ij} c_i \quad \text{for some } a_{ij} \in R.$$

Let A be the matrix $A = (a_{ij})_{1 \le i,j \le n}$

4 and set $B := x\,\mathrm{Id}_n - A \in M_{n \times n}(K)$.

Let $\mathrm{Adj}(B)$ be the adjugate matrix for B.

Then $B \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$ in K^n

$$\Rightarrow (\det B)\,\mathrm{Id}_n \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$$

$$\Rightarrow \det B = 0$$

But $\det B$ is a monic polynomial in x with coeff. in R. Thus x is integral over $R \Rightarrow x \in R$. $\square$

Proof of Theorem 9.2: "$\Rightarrow$" clear.

"$\Leftarrow$" We need to show R is PID.

The assumption implies R is a local ring with unique max. ideal m.

Step 1: m is principal.

Let $0 \neq x \in m$, By Lemma 9.3, $(x) \supseteq m^n$ some $n \geq 1$. Let n be minimal s.t. $(x) \supseteq m^n$, then we may choose $y \in m^{n-1} \setminus (x)$.

Set $\pi := \frac{x}{y}$. Then we have $ym \subseteq m^n \subseteq (x)$

$\Rightarrow \pi^{-1} m \subseteq R$.

If $\pi^{-1} m \subseteq m$, then $\pi^{-1} \in R$ by Lemma 9.4 and $y \in (x)$ $\#$

Hence $\pi^{-1} m = R \Rightarrow m = \pi R$ is principal.

Step 2: R is a PID.

Let $I \subseteq R$ be a non-zero ideal. Consider the sequence of fractional ideals.

$$I \subseteq \pi^{-1} I \subseteq \pi^{-2} I \subseteq \cdots \quad \text{in } K.$$

Then $\pi^{-k} I \neq \pi^{-(k+1)} I \quad \forall k$ by Lemma 9.4. Therefore since R is Noetherian, we may choose n maximal s.t. $\pi^{-n} I \subseteq R$

If $\pi^{-n} I \subseteq m = (\pi)$, then $\pi^{-(n+1)} I \subseteq R$ $\#$

Thus $\pi^{-n} I = R \Rightarrow I = (\pi^n)$. $\square$

Let R be an integral domain and $S \subseteq R$ a multicatively closed subset ($x, y \in S \Rightarrow xy \in S$, 1

The localization $S^{-1}R$ of R w.r.t. to S is

6 the ring

$$S^{-1}R = \{ \tfrac{r}{s} \mid r \in R, s \in S \} \subseteq \mathrm{Frac}(R).$$

If $\mathfrak{p}$ is a prime ideal in R, we write $R_{(\mathfrak{p})}$ for the localization w.r.t. $S = R \setminus \mathfrak{p}$.

Eg. $\cdot\ \mathfrak{p} = (0)$, $R_{(\mathfrak{p})} = \mathrm{Frac}(R)$.

$\cdot\ R = \mathbb{Z}$, $\mathbb{Z}_{(p)} = \left\{ \dfrac{a}{b} \;\middle|\; \begin{array}{l} a \in \mathbb{Z} \\ (b, p) = 1 \end{array} \right\}$.

Fact: $\cdot$ R Noetherian $\Rightarrow$ $S^{-1}R$ is Noetherian

$\cdot$ $\exists$ bijection

$$\{\text{prime ideals in } S^{-1}R\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{prime ideals } \mathfrak{p} \subseteq R \text{ s.t.} \\ \mathfrak{p} \cap S = \emptyset \end{array} \right\}$$

$$\mathfrak{p} S^{-1}R \leftrightarrow \mathfrak{p}$$

Corollary 9.5: Let R be a Dedekind domain, $\mathfrak{p} \subseteq R$ a non-zero prime ideal. Then $R_{(\mathfrak{p})}$ is a DVR.

Proof: By properties of localization, $R_{(\mathfrak{p})}$ is a Noetherian integral domain with a unique non-zero prime ideal $\mathfrak{p}R_{(\mathfrak{p})}$. It suffices to show that $R_{(\mathfrak{p})}$ is integrally closed in

7 $\mathrm{Frac}(R_{(\mathfrak{p})}) = \mathrm{Frac}(R)$

$\mathrm{Frac}(R_{(p)}) = \mathrm{Frac}(R)$.

(since then $R_{(p)}$ is Dedekind $\overset{\text{Thm } 9.2}{\Rightarrow}$ $R_{(p)}$ is a DVR)

Let $x \in \mathrm{Frac}(R)$ be integral over $R_{(p)}$.

Multiplying by denominators of a monic polynomial satisfied by x, we obtain

$$\underline{sx^n} + a_{n-1}x^{n-1} + \dots + a_0 = 0, \quad a_i \in R, s \in S.$$

Multiply by s^{n-1} $\Rightarrow$ xs integral over R.

$$\Rightarrow xs \in R$$

$$\Rightarrow x \in R_{(p)}. \qquad \square$$